On seven dimensional quaternionic contact solvable Lie groups

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Abstract

We answer in the affirmative a question posed by Ivanov and Vassilev [13] on the existence of a seven dimensional quaternionic contact manifold with closed fundamental 4-form and non-vanishing torsion endomorphism. Moreover, we show an approach to the classification of seven dimensional solvable Lie groups having an integrable left invariant quaternionic contact structure. In particular, we prove that the unique seven dimensional nilpotent Lie group admitting such a structure is the quaternionic Heisenberg group.

1   Introduction

The notion of quaternionic contact (qc) structure was introduced by Biquard in [3], and it is the natural geometrical structure that appears on the $(4n + 3)$-sphere as the conformal infinity of the quaternionic hyperbolic space. Such structures have been considered in connection with the quaternionic contact Yamabe problem [10, 11, 14, 15]. Results about the CR-structure on the twistor space of a qc manifold were given in [2, 4, 8, 7].

In general, a qc structure on a differentiable manifold of dimension (4n+3) is a distribution H of codimension 3 on M, called the horizontal space, such that there exists a metric g on H and a triplet (η_1, η_2, η_3) of locally defined differential 1-forms vanishing on H and such that the restrictions $d\eta_r|_H$ to H of the 2-forms $d\eta_r$ $(1 \leq r \leq 3)$ are the local Kähler 2-forms of an almost quaternion Hermitian structure on H.

The triplet of 1-forms (η_1, η_2, η_3) is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$. Therefore, H is equipped with a conformal class $[g]$ of Riemannian metrics and a rank 3 bundle (the quaternionic bundle) $\mathbb{Q}$ of endomorphisms of H such that $\mathbb{Q}$ is locally generated by almost complex structures (I_1, I_2, I_3) satisfying the quaternion relations. The 2-sphere bundle of one forms determines uniquely the associated metric and a conformal change of the metric is equivalent to a conformal change of the one forms.

Biquard in [3] shows that if M is a qc manifold of dimension greater than 7, to every metric in the fixed conformal class $[g]$, one can associate a unique complementary distribution V of H in the tangent bundle TM such that V is

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locally generated by vector fields ξ_1, ξ_2, ξ_3 satisfying certain relations (see (2) in section 2). Using these vector fields ξ_r, we extend the metric g on H to a metric on M by requiring Span $\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_r, \xi_k) = \delta_{rk}$. Moreover, in [3] it is also proved that M has a canonical linear connection ∇ preserving the qc structure and the splitting $TM = H \oplus V$. This connection is known as the Biquard connection.

However, if the dimension of M is seven, there might be no vector fields satisfying (2). Duchemin shows in [8] that if there are vector fields ξ_r ($1 \leq r \leq 3$) satisfying the relations (2) before mentioned, then the Biquard connection is also defined on M. In this case, the qc structure on the 7-manifold is said to be *integrable*. In this paper, we assume the integrability of the qc structure when we refer to a 7-dimensional qc manifold.

If M is a qc manifold with horizontal space H, the restriction to H of the Ricci tensor of (g, ∇) gives rise, on the one hand, to the qc-scalar curvature S and, on the other hand, to two symmetric trace-free (0,2) tensor fields T^0 and U defined on the distribution H (see also Section 4). The tensors T^0 and U determine the trace-free part of the Ricci tensor restricted to H and can also be expressed in terms of the torsion endomorphisms of the Biquard connection [11]. Moreover, the vanishing of the torsion endomorphisms of the Biquard connection is equivalent to $T^0 = U = 0$ and if the dimension is at least eleven, then the function S has to be constant. For any 7-dimensional qc manifold, in [3, 4] it is proved that $U = 0$ and in [11, 13] it is shown that S is constant if the torsion endomorphism vanishes and the distribution V is integrable (that is, $[V, V] \subset V$).

Associated to the $Sp(n)Sp(1)$ structure on the distribution H of a qc structure, one has the fundamental 4-form Ω defined (globally) on H by

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3, \tag{1}$$

where ω_r, $1 \leq r \leq 3$, are the local Kähler 2-forms of the almost quaternion structure on H. In [6] it is proved that for a seven dimensional manifold with an integrable qc structure, the vertical space V is integrable if and only if the fundamental 4-form Ω is closed. Ivanov and Vassilev in [13] prove that when the dimension of the manifold is greater than seven, the 4-form form Ω is closed if and only if the torsion endomorphism of the Biquard connection vanishes. They raise the question of the existence of a seven dimensional qc manifold with a closed fundamental four form and a non-vanishing torsion endomorphism. In this article, we answer the question in the affirmative by proving the following.

Theorem 1.1. *There are seven dimensional manifolds with an integrable qc structure such that the fundamental four form Ω is closed but the torsion endomorphism does not vanish.*

Examples of qc manifolds can be found in [3, 4, 11, 9]. The compact homogeneous model is the sphere S^{4n+3}, considered as the boundary at infinity of quaternionic projective $(n+1)$-space, while the non-compact homogeneous model is the quaternionic Heisenberg group $G(\mathbb{H}) = \mathbb{H}^n \times Im(\mathbb{H})$ endowed with its natural qc structure; in fact, $G(\mathbb{H})$ is isomorphic as a qc manifold to S^{4n+3} minus a point, via the quaternionic Cayley transform. Moreover, an extensively studied class of examples of quaternionic contact structures are provided by the 3-Sasakian manifolds. We recall that a $(4n+3)$-dimensional Riemannian manifold (M, g) is called 3-Sasakian if the cone metric $g_c = t^2 g + dt^2$ on $C = M \times \mathbb{R}^+$

is a hyper Kähler metric, namely, it has holonomy contained in $Sp(n+1)$ [5]. For any 3-Sasakian manifold, it was shown in [11] that the torsion endomorphism vanishes, and the converse is true if in addition the qc scalar curvature (see (22)) is a positive constant. Explicit examples of seven dimensional qc manifolds with zero or non-zero torsion endomorphism were recently given in [6]. Nevertheless, the fundamental 4-form is non-closed on the seven dimensional qc manifold with non-zero torsion endomorphism presented in [6].

To prove Theorem 1.1, we consider solvable 7-dimensional Lie algebras $\mathfrak{g}$ with a *normal ascending flag*, that is, the dual space $\mathfrak{g}^*$ has a flag

$$V^0 \subsetneq \cdots \subsetneq V^7 = \mathfrak{g}^*,$$

V^i being an i-dimensional subspace of $\mathfrak{g}$ such that $dV^i \subset \Lambda^2 V^i$, where d is the Chevalley-Eilenberg differential on $\mathfrak{g}^*$. Firstly, in Section 2, we prove that if $\mathfrak{g}$ has an integrable qc structure, then e^1, e^2, e^3 and e^4 are in V^6 for any $\{e^1, \cdots, e^7\}$ basis of $\mathfrak{g}^*$ *adapted* to the qc structure and for any normal ascending flag of $\mathfrak{g}$. In Section 3, we show that the unique solvable 7-dimensional Lie algebras having a normal ascending flag and an integrable qc structure are the Lie algebra of the quaternionic Heisenberg group and the two Lie algebras defined by

$$de^1 = 0, \qquad de^2 = (1+\mu)e^{12} - \mu e^{15} + \mu e^{34} - \mu e^{46},$$
$$de^3 = -(1+\mu)e^{13} - (2+3\mu)e^{24} - \mu e^{16} + \mu e^{45}, \quad de^4 = 2\mu e^{14},$$
$$de^5 = e^{12} + e^{34} - e^{46},$$
$$de^6 = e^{13} + e^{42} + e^{45},$$
$$de^7 = e^{14} + e^{23} + \mu e^{56},$$

where $\mu = -1, -\frac{1}{3}$. For convenience in the notation, in Section 4, we shall denote by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ the Lie algebras defined by $\mu = -1$ and $\mu = -\frac{1}{3}$, respectively. We consider the simply connected solvable Lie group G_s with Lie algebra $\mathfrak{g}_s$ $(s = 1, 2)$, and define an integrable qc structure on each G_s. We prove (see Theorems 4.2 and 4.3) that the fundamental 4-form on G_s is closed but the torsion endomorphism of the Biquard connection is non-zero. Finally, we notice that the qc structure on G_s is not locally qc conformal to the standard flat qc structure on the quaternionic Heisenberg group $G(\mathbb{H})$.

2 Solvable 7-dimensional Lie algebras with a normal ascending flag

In this section, we consider solvable 7-dimensional Lie algebras with a *normal ascending flag* and an *integrable* qc structure. For such a Lie algebra $\mathfrak{g}$, we study the behaviour of a coframe $\{e^1, \cdots, e^7\}$ *adapted* to the qc structure. First, we need some definitions and results about qc manifolds and the Biquard connection.

Let $(M, g, \mathbb{Q})$ be a *qc manifold* of dimension $4n + 3$, that is, M has a horizontal distribution H of dimension $4n$ with a metric g satisfying: i) H is locally determined by the kernel of three differential 1-forms η_r $(1 \leq r \leq 3)$ on M; ii) H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a rank-three bundle

$\mathbb{Q}$ consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1I_2 = -I_2I_1 = I_3$, $I_1I_2I_3 = -id_{|H}$, which are Hermitian with respect to the metric g, i.e. $g(I_r \cdot, I_r \cdot) = g(\cdot, \cdot)$; and iii) the following compatibility conditions hold: $2g(I_rX, Y) = d\eta_r(X, Y)$, for $1 \leq r \leq 3$ and for any $X, Y \in H$.

Biquard in [3] proves that if M is a qc manifold of dimension $(4n + 3) > 7$, there exists a canonical connection on M. In the following theorem, we recall the properties that distinguish that connection.

Theorem 2.1. *[3] Let $(M, g, \mathbb{Q})$ be a qc manifold of dimension $4n + 3 > 7$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

i) *∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H, i.e. $\nabla g = 0, \nabla \sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$;*

ii) *the torsion T on H is given by $T(X, Y) = -[X, Y]_{|V}$;*

iii) *for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;*

iv) *the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla \varphi = 0$.*

In the part iii), the inner product $\langle, \rangle$ of $End(H)$ is given by $\langle A, B \rangle = \sum_{i=1}^{4n} g(A(e_i), B(e_i))$, for $A, B \in End(H)$.

We shall call the above connection *the Biquard connection*. Biquard [3] also described the supplementary subspace V, namely, V is (locally) generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0,$$
$$(\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \tag{2}$$

where $\lrcorner$ denotes the interior multiplication.

If the dimension of M is seven, there might be no vector fields satisfying (2). Duchemin shows in [8] that if we assume, in addition, the existence of vector fields $\{\xi_1, \xi_2, \xi_3\}$ as in (2), then an analogue of Theorem 2.1 holds. In this case, the qc structure on the 7-manifold is called *integrable*.

From now on, given a 7-dimensional Lie algebra $\mathfrak{g}$ whose dual space is spanned by $\{e^1, \ldots, e^7\}$, we will write $e^{ij} = e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so forth. Moreover, let us fix some language. On a Lie algebra $\mathfrak{g}$, an *integrable qc structure* can be characterized by the existence of a coframe $e^1, \ldots, e^7$ with

$$de^5 = e^{12} + e^{34} + f_2 \wedge e^7 - f_3 \wedge e^6 \quad \mod \text{Span}\left\{e^{56}, e^{57}, e^{67}\right\},$$
$$de^6 = e^{13} + e^{42} + f_3 \wedge e^5 - f_1 \wedge e^7 \quad \mod \text{Span}\left\{e^{56}, e^{57}, e^{67}\right\}, \tag{3}$$
$$de^7 = e^{14} + e^{23} + f_1 \wedge e^6 - f_2 \wedge e^5 \quad \mod \text{Span}\left\{e^{56}, e^{57}, e^{67}\right\},$$

where the f_i are in $\text{Span}\left\{e^1, e^2, e^3, e^4\right\}$. This condition is invariant under the action of $\mathbb{R}^* \times SO(4)$, where the relevant representation of $SO(4)$ is

$$\mathbb{R}^7 = \mathbb{R}^4 \oplus \Lambda_+^2(\mathbb{R}^4),$$

and $\lambda \in \mathbb{R}^*$ acts as

$$\text{diag}(\lambda, \lambda, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2).$$

Definition 2.2. Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with an integrable qc structure, and let $\{e^1, \ldots, e^7\}$ be a basis of the dual space $\mathfrak{g}^*$. We say that $\{e^1, \ldots, e^7\}$ is an *adapted coframe to the qc structure on $\mathfrak{g}$ if with respect to that basis, $\mathfrak{g}$ is defined by equations as* (3).

Furthermore, we consider solvable 7-dimensional Lie algebras $\mathfrak{g}$ that admit a *normal ascending flag*. This means that there is a flag

$$\mathfrak{g}_0 \subsetneq \cdots \subsetneq \mathfrak{g}_7 = \mathfrak{g},$$

where $\mathfrak{g}_k$ is a k-dimensional ideal of $\mathfrak{g}$. In particular this implies that $\mathfrak{g}$ is solvable since $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i-1}$; moreover, taking annihilators, we obtain a dual flag

$$V^0 \subsetneq \cdots \subsetneq V^7 = \mathfrak{g}^*, \quad dV^i \subset \Lambda^2 V^i. \tag{4}$$

The following straightforward result will be useful in the sequel.

Lemma 2.3. *Let $\mathfrak{g}$ be a Lie algebra with a normal ascending flag, and let α be an element of $\mathfrak{g}^*$.*

- *If $(d\alpha)^k \neq 0$, then $\alpha \notin V^i$, $i < 2k$. If in addition*

$$\alpha \in V^{2k}, \quad (d\alpha)^k = \eta^1 \wedge \cdots \wedge \eta^{2k},$$

  then $V^{2k} = \operatorname{Span}\left\{\eta^1, \cdots, \eta^{2k}\right\}$.

- *If $\alpha \wedge (d\alpha)^k \neq 0$, then $\alpha \notin V^i$, $i < 2k+1$. If in addition*

$$\alpha \in V^{2k+1}, \quad \alpha \wedge (d\alpha)^k = \eta^1 \wedge \cdots \wedge \eta^{2k+1},$$

  then $V^{2k} = \operatorname{Span}\left\{\eta^1, \cdots, \eta^{2k+1}\right\}$.

Lemma 2.4. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a normal ascending flag and an integrable qc structure. Fix an adapted coframe $\{e^1, \ldots, e^7\}$ to the qc structure and a flag as in* (4). *Then,*

$$\dim \operatorname{Span}\left\{e^5, e^6, e^7\right\} \cap V^4 = 0,$$
$$\dim \operatorname{Span}\left\{e^5, e^6, e^7\right\} \cap V^5 = 1,$$
$$\dim \operatorname{Span}\left\{e^5, e^6, e^7\right\} \cap V^6 = 2.$$

Proof. Observe that $\operatorname{SU}(2)_+ \subset \operatorname{SO}(4)$ acts on $\operatorname{Span}\left\{e^5, e^6, e^7\right\}$ as $\operatorname{SO}(3)$ acts on $\mathbb{R}^3$. Hence, if we had a nonzero element of $\operatorname{Span}\left\{e^5, e^6, e^7\right\} \cap V^4$, we could assume it is e^5. Since $e^5 \wedge (de^5)^2$ is non-zero, this contradicts Lemma 2.3.

It follows that
$$\dim \operatorname{Span}\left\{e^5, e^6, e^7\right\} \cap V^5 \leq 1,$$

because the intersection with V^4 is trivial, and V^4 has codimension one in V^5. On the other hand, equality must hold, because V^5 has codimension two in V^7.

The last equality is proved in the same way. $\qquad\square$

Lemma 2.5. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a normal ascending flag and an integrable qc structure. Fix an adapted coframe $e^1, \ldots, e^7$ and a flag as in* (4). *Then e^1, e^2, e^3 and e^4 are in V^6.*

Proof. By Lemma 2.4, we can act by an element of SO(4) and obtain that e^5 is in V^5 and e^6 is in V^6. Moreover by dimension count

$$\dim \operatorname{Span}\left\{e^1, e^2, e^3, e^4\right\} \cap V^6 \geq 3.$$

Thus, up to SO(4) action we can assume that e^1, e^2, e^3, e^5, e^6 lie in V^6. Hence, by Lemma 2.4, e^7 is not in V^6.

We must show that e^4 is also in V^6. Suppose otherwise. Then there is some $a \in \mathbb{R}$ with $e^4 + ae^7$ in V^6. Up to $\mathbb{R}^*$ action we can assume $a = 1$, that is

$$V^6 = \operatorname{Span}\left\{e^1, e^2, e^3, e^4 + e^7, e^5, e^6\right\}.$$

Therefore, (3) gives

$$de^{56} = (e^{36} - f_2 \wedge e^6 + e^{25} - f_1 \wedge e^5) \wedge e^7 \mod \Lambda^3 V^6 + \operatorname{Span}\left\{e^{567}\right\};$$

and, on the other hand, e^{56} is in $\Lambda^2 V^6$, so $f_2 = e^3$ and $f_1 = e^2$, and

$$de^7 = e^{14} + e^{23} + e^{26} - e^{35} \mod \operatorname{Span}\left\{e^{56}, e^{57}, e^{67}\right\}. \tag{5}$$

Write

$$f_3 = f_3' + \lambda e^4, \quad f_3' \in \operatorname{Span}\left\{e^1, e^2, e^3\right\}.$$

Then

$$de^5 = e^{12} + e^3(e^4 + e^7) - f_3' \wedge e^6 - \lambda e^{46} \mod \operatorname{Span}\left\{e^{56}, e^{57}, e^{67}\right\},$$

but by construction de^5 has to be in $\Lambda^2 V^6$, so

$$de^5 = e^{12} + e^3(e^4 + e^7) - f_3' \wedge e^6 + \lambda e^6 \wedge (e^4 + e^7) \mod \operatorname{Span}\left\{e^{56}\right\}.$$

Imposing $(de^5)^3 = 0$ we get

$$de^5 = e^{12} + e^3(e^4 + e^7) + \lambda e^6 \wedge (e^4 + e^7) - f_3' \wedge e^6.$$

If

$$f_3' = \mu_1 e^1 + \mu_2 e^2 + \mu_3 e^3,$$

then $(de^5)^2 \wedge e^5$ should be equal to $2E^{1234} \wedge e^5$, where

$$E^1 = e^1 + \mu_2 e^6, \quad E^2 = e^2 - \mu_1 e^6, \quad E^3 = e^3 + \lambda e^6, \quad E^4 = e^4 + e^7 - \mu_3 e^6.$$

Indeed,

$$de^5 = E^{12} + E^{34}.$$

By Lemma 2.3, it follows that

$$V^5 = \operatorname{Span}\left\{E^1, E^2, E^3, E^4, e^5\right\}.$$

We can rewrite (5) as

$$de^7 = e^{14} + e^{23} + e^{26} - e^{35} + ae^{57} + be^{67} + ce^{56}.$$

Then computing $d^2 e^7 \mod \Lambda^3 V^6$ we find

$$0 = d(-e^1 + ae^5 + be^6) = d(-E^1 + ae^5 + (\mu_2 + b)e^6). \tag{6}$$

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We claim that
$$\mu_2 + b = 0.$$
Otherwise,
$$de^6 = \frac{1}{\mu_2 + b}(dE^1 - ade^5) \in \Lambda^2 V^5,$$
and on the other hand
$$de^6 = e^{13} + e^{42} + f_3' \wedge e^5 + \lambda e^{45} - e^{27} \mod \operatorname{Span}\left\{e^{56}, e^{57}, e^{67}\right\}; \quad (7)$$
thus, α_3 is zero. Now
$$d^2 e^7 = d((ae^5 + (\mu_2 + b)e^6 - E^1)e^7 + ce^{56}) \mod \Lambda^3 V^5 dd,$$
which is not zero because e^{236} is not in $\Lambda^3 V^5$.

We can therefore assume that $b = -\mu_3$; then
$$dE^1 = ade^5 = a(E^{12} + E^{34}). \quad (8)$$
Imposing that de^6 is in $\Lambda^2 V^6$, (7) becomes
$$de^6 = e^{13} + f_3' \wedge e^5 + \lambda(e^4 + e^7)e^5 - e^2(e^4 + e^7) + xe^{56},$$
for some real x. In order to simplify the notation, we shall think of E^1, E^2, E^3, E^4 as a coframe on a 4-dimensional vector space, defining an SU(2)-structure; in particular a scalar product is defined, allowing us to take interior products of forms, as well as three complex structures. Explicitly, we set
$$J_1 = \cdot\lrcorner\,(E^{12} + E^{34}), \quad J_2 = \cdot\lrcorner\,(E^{13} + E^{42}), \quad J_3 = \cdot\lrcorner\,(E^{14} + E^{23}).$$
We denote by β the projection of f_3 on $\operatorname{Span}\left\{E^1, E^2, E^3, E^4\right\}$. Thus,
$$de^6 = E^{13} + E^{42} + \beta \wedge e^5 + (J_3\beta + xe^5)e^6.$$

Now define two derivations δ, γ on $\Lambda \operatorname{Span}\left\{E^1, E^2, E^3, E^4\right\}$, of degrees one and zero respectively, by the rule
$$d\eta = \delta\eta + e^5 \wedge \gamma\eta.$$
Then $d^2 = 0$ implies
$$\delta\gamma = \gamma\delta, \quad \delta^2\eta + (E^{12} + E^{34}) \wedge \gamma\eta = 0,$$
and so δ determines γ via
$$\gamma\eta = J_1 * \delta^2\eta, \quad (9)$$
for any 1-form η.

In particular, (8) gives
$$\delta E^1 = a(E^{12} + E^{34}), \quad \gamma E^1 = 0,$$
which is consistent with (9) because $d^2 e^5 = 0$ implies
$$\delta(E^{12} + E^{34}) = 0, \quad \gamma(E^{12} + E^{34}) = 0.$$

Due to (9), the latter can be rewritten as

$$0 = \sum_{i=1}^{4} E^i \wedge *\delta^2 E^i = \sum *(E_i \lrcorner \delta^2 E^i). \qquad (10)$$

Taking $d^2 e^6$ and separating the components we find

$$\gamma(J_3\beta) = 0, \quad \delta(J_3\beta) + x(E^{12} + E^{34}) = 0,$$
$$\delta(E^{13} + E^{42}) = \beta \wedge (E^{12} + E^{34}) + J_3\beta \wedge (E^{13} + E^{42}) = 2\beta \wedge (E^{12} + E^{34}),$$
$$\gamma(E^{13} + E^{42}) = -\delta\beta + x(E^{13} + E^{42}) + J_3\beta \wedge \beta.$$

Similarly,

$$de^7 = E^{14} + E^{23} - E^3 \wedge e^5 + (E^2 - J_2\beta + (c-\lambda)e^5) \wedge e^6 + (ae^5 - E^1) \wedge e^7;$$

taking d again and separating the components we get

$$\delta(E^{14} + E^{23}) = -3E^{123} - J_2\beta \wedge (E^{13} + E^{42}),$$
$$\gamma(E^{14} + E^{23}) + \beta \wedge (E^2 - J_2\beta) - (c-\lambda)(E^{13} + E^{42}) + a(E^{14} + E^{23}) = 0,$$
$$\delta(E^2 - J_2\beta) + (c-\lambda)(E^{12} + E^{34}) + (E^2 - J_2\beta)(-J_3\beta - E^1) = 0,$$
$$\gamma(E^2 - J_2\beta) + (c-\lambda)(-J_3\beta - E^1) - (a-x)(E^2 - J_2\beta) = 0.$$

In order to make use of these equations we need $E_2 - J_2\beta$ to be nonzero, so let us assume first

$$\beta = E^4, \qquad (11)$$

and obtain a contradiction. Indeed, in this case $c = \lambda$ and $a = -x = 0$, for

$$0 = (E^{12} + E^{34}) \wedge \gamma(E^{12} + E^{34}) = (E^{13} + E^{42}) \wedge \gamma(E^{14} + E^{23}).$$

Hence, we have

$$\delta E^1 = 0, \quad \delta(E^{12} + E^{34}) = 0, \quad \gamma(E^{12} + E^{34}) = 0,$$
$$\delta(E^{13} + E^{42}) = 2E^{124}, \quad \gamma(E^{13} + E^{42}) = -\delta E^4 - E^{14},$$
$$\delta(E^{14} + E^{23}) = -2E^{123}, \quad \gamma(E^{14} + E^{23}) = 0.$$

In particular, since γ is a derivation, we see that $\gamma(E^{13} + E^{42})$ gives zero on wedging with

$$E^{12} + E^{34}, \quad E^{13} + E^{42}, \quad E^{14} + E^{23}, \qquad (12)$$

and so the same holds of $\delta E^4 + E^{14}$. Thus δE^4 has the form

$$\delta E^4 = -\frac{1}{2}(E^{14} + E^{23}) + x(E^{12} - E^{34}) + y(E^{13} - E^{42}) + z(E^{14} - E^{23}). \quad (13)$$

Wedging E^1 with the forms (12) and applying δ we obtain

$$\delta E^{134} = 0 = \delta E^{123} = \delta E^{124}.$$

Similarly,

$$\delta E^{234} = \delta(E^{14} + E^{23}) \wedge E^4 + (E^{14} + E^{23}) \wedge \delta E^4 = -3E^{1234}.$$

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Using (13) we get

$$\delta^2 E^4 \wedge E^4 = E^{1234} + (x\delta(E^{12} - E^{34}) + y\delta(E^{13} - E^{42}) + z\delta(E^{14} - E^{23})) \wedge E^4$$
$$= E^{1234} + x\delta E^{124} - x(E^{12} - E^{34}) \wedge \delta E^4 + y\delta E^{134} - y(E^{13} - E^{42}) \wedge \delta E^4 - zdE^{234} - z(E^{14} - E^{23}) \wedge \delta E^4$$
$$= E^{1234}(1 + 2x^2 + 2y^2 + 3z + 2z^2) \quad (14)$$

Observe that

$$\delta E^{14} = -E^1 \wedge \delta E^4 = E^1 \wedge \gamma(E^{13} + E^{42}) = -\gamma E^{124},$$

so

$$\delta^2 E^4 = -\delta E^{14} - 2\gamma E^{124} = \delta E^{14}.$$

By Equation (13),

$$\delta E^{14} \wedge E^4 = -E^{14} \wedge \delta E^4 = (z + \frac{1}{2})E^{1234};$$

comparing with (14) we obtain

$$2x^2 + 2y^2 + 2z^2 + 2z + \frac{1}{2} = 0$$

and therefore

$$\delta E^4 = -E^{14}.$$

In particular, we see that $\gamma = 0$ and δ defines a 4-dimensional Lie algebra characterized by the equations

$$\delta E^1 = 0, \quad \delta E^4 = -E^{14}, \quad \delta(E^{12} + E^{34}) = 0, \quad \delta(E^{13} + E^{42}) = 2E^{124}, \quad \delta E^{23} = -2E^{123}.$$

It is easy to check that no such Lie algebra exists. Indeed these linear conditions on δ imply

$$\delta E^2 = p(E^{12} - E^{34}) + q(E^{13} - E^{42}) - \frac{3}{2}(E^{12} + E^{34}),$$
$$\delta E^3 = q(E^{12} - E^{34}) - p(E^{13} - E^{42}) - \frac{1}{2}(E^{13} + E^{42})$$

for some p, q; but then δ^2 is not zero. Summing up, (11) leads to a contradiction.

We can therefore assume that

$$0 \neq \tilde{E}^1 = E^1 + J_3\beta.$$

Then

$$\tilde{E}^1, \quad \tilde{E}^2 = J_1\tilde{E}^1 = E^2 - J_2\beta, \quad \tilde{E}^3 = J_2\tilde{E}^1 = E^3 + J_1\beta, \quad \tilde{E}^4 = J_3\tilde{E}^1 = E^4 - \beta$$

is an orthonormal basis up to a scale factor, hence it also satisfies (10). We compute

$$\delta^2 \tilde{E}^2 = -(c - \lambda)(E^{12} + E^{34})\tilde{E}^1 - (a - x)\tilde{E}^2(E^{12} + E^{34}), \quad (15)$$

and therefore

$$\tilde{E}^2 \lrcorner \delta^2 \tilde{E}^2 = -(a - x)\tilde{E}^{34}.$$

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Now (10) implies that $a - x = 0$ and

$$\tilde{E}^3 \wedge *\delta^2 \tilde{E}^3 + \tilde{E}^4 \wedge \delta^2 \tilde{E}^4 = 0,$$

so in particular $*\delta^2 \tilde{E}^3$ and $*\delta^2 \tilde{E}^3$ are in the span of $\tilde{E}^3$ and $\tilde{E}^4$.

Using (9), we find

$$J_1 \gamma(\tilde{E}^{14} + \tilde{E}^{23}) = -\tilde{E}^1 \wedge *\delta^2 \tilde{E}^4 - *\delta^2 \tilde{E}^2 \wedge \tilde{E}^3 - \tilde{E}^2 \wedge *\delta^2 \tilde{E}^3$$

On the other hand, we know that

$$J_1 \gamma(E^{14} + E^{23}) - J_1 \beta \wedge \tilde{E}^1 + (c - \lambda)(E^{13} + E^{42}) - a(E^{14} + E^{23}) = 0.$$

Comparing the two expressions and using (15), we find

$$-\tilde{E}^1 \wedge *\delta^2 \tilde{E}^4 - \tilde{E}^2 \wedge *\delta^2 \tilde{E}^3 + (c-\lambda)\tilde{E}^{23} - \left\|\tilde{E}^1\right\|^2 J_1 \beta \wedge \tilde{E}^1 + (c-\lambda)(\tilde{E}^{13} + \tilde{E}^{42}) - a(\tilde{E}^{14} + \tilde{E}^{23}) = 0.$$

This shows that $J_1 \beta$ has no component along $\tilde{E}^2$, so

$$\left\|\tilde{E}^1\right\|^2 J_1 \beta = \beta \lrcorner \tilde{E}^{34} \mod \tilde{E}^1;$$

using the fact that $*\delta^2 \tilde{E}^3$ and $*\delta^2 \tilde{E}^4$ are in the span of $\tilde{E}^3$ and $\tilde{E}^4$, we deduce

$$*\delta^2 \tilde{E}^3 = -(c - \lambda)\tilde{E}^4 - (a - c + \lambda)\tilde{E}^3, \quad *\delta^2 \tilde{E}^4 = \beta \lrcorner \tilde{E}^{34} + (c - \lambda)\tilde{E}^3 - a\tilde{E}^4,$$

and therefore by (10)

$$*\delta^2 \tilde{E}^4 = -(c - \lambda)\tilde{E}^3 \mod \tilde{E}^4;$$

it follows that

$$\beta \lrcorner \tilde{E}^{34} = -2(c - \lambda)\tilde{E}^3 \mod \tilde{E}^4. \tag{16}$$

Similarly, we compute

$$J_1 \gamma(\tilde{E}^{13} + \tilde{E}^{42}) = -\tilde{E}^2 \wedge *\delta^2 \tilde{E}^3 + *\delta^2 \tilde{E}^4 \wedge \tilde{E}^1 + \tilde{E}^3 \wedge *\delta^2 \tilde{E}^2$$

$$= -(c - \lambda)(\tilde{E}^{13} + \tilde{E}^{42}) + a(\tilde{E}^{14} + \tilde{E}^{23}) + \left\|\tilde{E}^1\right\|^2 J_1 \beta \wedge \tilde{E}^1.$$

Comparing with

$$\gamma(E^{13} + E^{42}) = -\delta\beta + a(E^{13} + E^{42}) + J_3 \beta \wedge \beta,$$

we obtain

$$(c - \lambda - a)(E^{13} + E^{42}) - a(E^{14} + E^{23}) + \beta \wedge (J_3 \beta - \tilde{E}^2) + \delta\beta = 0.$$

Taking δ,

$$aE^{123} + (c - \lambda - 3a)(E^{13} + E^{42}) \wedge (\tilde{E}^2) + a(E^{14} + E^{23}) \wedge (-\tilde{E}^2) - \beta \wedge \tilde{E}^{12} + \delta^2\beta = 0,$$

and therefore taking $*$

$$a\beta - (c - \lambda - 4a)\tilde{E}^4 - a\tilde{E}^3 - \beta \lrcorner \tilde{E}^{34} + *\delta^2\beta = 0. \tag{17}$$

So we have two cases.

i) If $a = c - \lambda$, write

$$\beta = r\tilde{E}^3 + s\tilde{E}^4 \quad \mathrm{mod} \ \tilde{E}^2;$$

working mod $\tilde{E}^2$, (17) gives

$$0 = \tilde{E}^3(ar) + \tilde{E}^4(as + 2a - ar) + (s-1)(\beta \lrcorner \tilde{E}^{34} + a\tilde{E}^3 - a\tilde{E}^4)$$
$$= \tilde{E}^3(ar - as + a) + \tilde{E}^4(3a - ar + (s-1)r\left\|\tilde{E}^3\right\|^2),$$

which implies

$$a = 0 = r = s,$$

since $s = \frac{2a}{\left\|\tilde{E}_1\right\|^2}$ by (16). Then β is a multiple of $\tilde{E}^2$, and γ, δ^2 are zero. In particular,

$$\delta\beta = J_3\beta \wedge \beta$$

is linearly dependent on

$$\delta\tilde{E}^2 = \tilde{E}^{12},$$

implying that $\beta = 0$. Summing up, δ defines a 4-dimensional Lie algebra characterized by the equations

$$\delta E^1 = 0, \quad \delta E^2 = E^{12}, \quad \delta E^{34} = 0 = \delta(E^{13} + E^{42}), \quad \delta(E^{14} + E^{23}) = -3E^{123}.$$

Much like in the case that $\beta = E^4$, one verifies that no such Lie algebra exists, for these linear conditions on δ imply

$$\delta E^3 = p(E^{13} - E^{42}) + q(E^{14} - E^{23}) + \frac{1}{2}(E^{13} + E^{42}),$$
$$\delta E^4 = q(E^{13} - E^{42}) - (p + 5/2)(E^{14} - E^{23}) + 2(E^{14} + E^{23}),$$

where p and q are real numbers; but then δ^2 is not zero.

ii) Suppose $a \neq c - \lambda$; then (17) implies that β lies in the span of $\tilde{E}^3$ and $\tilde{E}^4$, so $J_3\beta$ lines in the span of $\tilde{E}^1$ and $\tilde{E}^2$. Thus $\tilde{E}^1$ and $J_3\beta$ are linearly dependent, for otherwise

$$\delta\tilde{E}^2 \in \mathrm{Span}\left\{E^{12} + E^{34}\right\},$$

which is absurd. It follows that $J_3\beta$ is a multiple of $\tilde{E}^1$, say $\beta = s\tilde{E}^4$. Then

$$1 = \left\|E^4\right\|^2 = \left\|\tilde{E}^4 + \beta\right\|^2 = (1 + s)^2 \left\|\tilde{E}^4\right\|,$$

so

$$c - \lambda = \frac{s}{2(1+s)^2},$$

and

$$*\delta^2\tilde{E}^3 = -\frac{s}{2(1+s)^2}\tilde{E}^4 + \left(\frac{s}{2(1+s)^2} - a\right)\tilde{E}^3, \quad *\delta^2\tilde{E}^4 = -\frac{s}{2(1+s)^2}\tilde{E}^3 - a\tilde{E}^4.$$

Then (17) gives

$$-\left(\frac{s}{2(1+s)^2} - 4a\right)\tilde{E}^4 + \left(\frac{s^2}{2(1+s)^2} - a\right)\tilde{E}^3 = 0,$$

so both $c - \lambda$ and a are zero, which is absurd. $\qquad\square$

3 Classification of 7-dimensional qc Lie algebras with a normal ascending flag

In this section we carry out the classification of 7-dimensional Lie algebras with an integrable qc structure and a normal ascending flag.

Proposition 3.1. *There are exactly three non-isomorphic Lie algebras of dimension 7, with an integrable qc structure and a normal ascending flag, namely*

$$(0, 0, 0, 0, 12 + 34, 13 + 42, 14 + 23)$$

and

$$
\begin{aligned}
de^1 &= 0 \\
de^2 &= (1 + \mu)e^{12} - \mu e^{15} + \mu e^{34} - \mu e^{46} \\
de^3 &= -(1 + \mu)e^{13} - (2 + 3\mu)e^{24} - \mu e^{16} + \mu e^{45} \\
de^4 &= 2\mu e^{14} \\
de^5 &= e^{12} + e^{34} - e^{46} \\
de^6 &= e^{13} + e^{42} + e^{45} \\
de^7 &= e^{14} + e^{23} + \mu e^{56}
\end{aligned}
\tag{18}
$$

where $\mu = -1, -1/3$.

Proof. Let $\mathfrak{g}$ be a Lie algebra with an integrable qc structure and a fixed flag V^i as in (4). By Lemma 2.5, we know that e^1, e^2, e^3, e^4 are in V^6; moreover, by the argument in the proof of that same lemma, we can assume that e^5 is in V^5 and e^6 is in V^6.

The characterization of the V^i implies $\alpha_1 = 0 = \alpha_2$, and

$$
\begin{aligned}
de^5 &= e^{12} + e^{34} - \alpha_3 \wedge e^6 \quad \text{mod Span} \left\{ e^{56} \right\}, \\
de^6 &= e^{13} + e^{42} + \alpha_3 \wedge e^5 \quad \text{mod Span} \left\{ e^{56} \right\}, \\
de^7 &= e^{14} + e^{23} \quad \text{mod Span} \left\{ e^{56}, e^{57}, e^{67} \right\}.
\end{aligned}
$$

We claim that

$$de^7 = e^{14} + e^{23} \quad \text{mod Span} \left\{ e^{56} \right\}. \tag{19}$$

Indeed de^{56} and $d(e^{14} + e^{23})$ are in $\Lambda^3 V^6$, whereas

$$de^{57} = e^{127} + e^{347} - \alpha_3 \wedge e^{67} \quad \text{mod } (\Lambda^3 V^6 + \text{Span} \left\{ e^{567} \right\}),$$

$$de^{67} = e^{137} + e^{427} + \alpha_3 \wedge e^{57} \quad \text{mod } (\Lambda^3 V^6 + \text{Span} \left\{ e^{567} \right\}).$$

Thus, $d^2 e^7 = 0$ implies (19).

Let us consider the splitting

$$\Lambda^h \mathfrak{g}^* = \bigoplus_{p+q=h} \Lambda^{p,q} = \bigoplus_{p+q=h} \Lambda^p \text{Span} \left\{ e^1, e^2, e^3, e^4 \right\} \wedge \Lambda^q q \text{Span} \left\{ e^5, e^6, e^7 \right\}.$$

Observe that

$$de^{56} = e^{126} + e^{346} - e^{135} - e^{425};$$

in particular de^{56} has no (1,2)-component. Thus (19) implies that $d(e^{14} + e^{23})$ has no (1,2)-component either, whence

$$d\Lambda^{1,0} \subset \Lambda^{2,0} \oplus \Lambda^{1,1}. \tag{20}$$

Then

$$0 = (d^2 e^5)^{1,2} = -d(\alpha_3 \wedge e^6)^{1,2};$$

and the same holds of $\alpha_3 \wedge e^5$.

We know that de^5 equals $e^{12} + e^{34} - \alpha_3 \wedge e^6$ plus a multiple ϵ of e^{56}, but if this multiple were nonzero, then

$$(de^5)^3 = 6\epsilon e^{123456} \neq 0,$$

contradicting Lemma 2.3 and the assumption $e^5 \in V^5$. So, for some constants $\lambda, \mu \in \mathbb{R}$, we have

$$de^5 = e^{12} + e^{34} - \alpha_3 \wedge e^6,$$
$$de^6 = e^{13} + e^{42} + \alpha_3 \wedge e^5 + \lambda e^{56},$$
$$de^7 = e^{14} + e^{23} + \mu e^{56}.$$

Then

$$0 = d(\alpha_3 \wedge e^6)^{1,2} = d\alpha_3^{1,1} \wedge e^6 - \lambda \alpha_3 \wedge e^{56}, \quad 0 = d(\alpha_3 \wedge e^5)^{1,2} = d\alpha_3^{1,1} \wedge e^5,$$

whence

$$(d\alpha_3)^{1,1} = \lambda \alpha_3 \wedge e^5.$$

We have two cases, according to whether α_3 is zero or not.

a) If $\alpha_3 = 0$, we know that e^5 is in V^5, and $e^5 \wedge (de^5)^2 = 2e^{12345}$; hence,

$$V^5 = \text{Span}\left\{e^1, e^2, e^3, e^4, e^5\right\}.$$

Then

$$0 = d^2 e^6 = \lambda(e^{126} + e^{346}) \mod \Lambda^3 V^5,$$
$$0 = d^2 e^7 = \mu(e^{126} + e^{346}) \mod \Lambda^3 V^5.$$

These equations imply that $\lambda = \mu = 0$.

Now $\text{Span}\left\{e^1, \ldots, e^4\right\}$ intersects V^4 in a space of dimension at least three, so up to $SO(4)$ action we can assume

$$V^4 = \text{Span}\left\{e^1, e^2, e^3, e^4 + ae^5\right\}, \quad a \in \mathbb{R},$$

whence

$$de^4 = -ade^5 = -ae^{34} = a^2 e^{35} \mod \Lambda^2 V^4.$$

Therefore

$$0 = d^2 e^5 = -ade^3 \wedge e^5 \mod \Lambda^3 V^4,$$
$$0 = d^2 e^6 = a^2 e^{235} + ae^5 \wedge de^2 \mod \Lambda^3 V^4,$$
$$0 = d^2 e^7 = -a^2 e^{135} - ae^5 \wedge de^1 \mod \Lambda^3 V^4.$$

We claim that $a = 0$. In fact if $a \neq 0$, we see that de^1, de^2 and de^3 are in $\Lambda^2 \operatorname{Span}\{e^1, e^2, e^3\}$, and so, taking d^2 of e^5, e^6, e^7, it follows that de^4 must also be in $\Lambda^2 \operatorname{Span}\{e^1, e^2, e^3, e^4\}$. Denoting by $e_1, \ldots, e_7$ the basis of $\mathfrak{g}$ dual to $e^1, \ldots, e^7$, we see that $\operatorname{Span}\{e_5, e_6, e_7\}$ is an ideal, and $\mathfrak{g}$ projects onto a hyperkähler 4-dimensional, solvable algebra. This has to be abelian, because the corresponding Lie group is a homogeneous Ricci-flat manifold, hence flat by [1]. This implies that e^4 is closed, which contradicts the assumption $a \neq 0$. Consequently, $a = 0$. This implies that $\operatorname{Span}\{e_5, e_6, e_7\}$ is an ideal, so by the same argument as above, $\operatorname{Span}\{e_1, e_2, e_3, e_4\}$ is abelian, and

$$\mathfrak{g} = (0, 0, 0, 0, 12 + 34, 13 + 42, 14 + 23).$$

b) Suppose α_3 is non-zero. In this case, up to $SO(4)$ action, we can assume that α_3 is a multiple of e^4, and up to $\mathbb{R}^*$ action, we obtain $\alpha_3 = e^4$. The equations become

$$de^5 = e^{12} + e^{34} - e^{46}$$
$$de^6 = e^{13} + e^{42} + e^{45} + \lambda e^{56}$$
$$de^7 = e^{14} + e^{23} + \mu e^{56}$$
$$(de^4)^{1,1} = \lambda e^{45}.$$

In particular, looking at $(de^5)^2 \wedge e^5$ we compute

$$V^5 = \operatorname{Span}\{e^1, e^2, e^3 + e^6, e^4, e^5, \}.$$

Thus

$$de^3 + de^6 = \gamma_3 \wedge e^5 + \beta_3 \wedge (e^3 + e^6) + h(e^{35} + e^{65}) \quad \mod \Lambda^2 \operatorname{Span}\{e^1, e^2, e^4\}$$

where from now on the γ_i, β_i are in $\operatorname{Span}\{e^1, e^2, e^4\}$. We can determine h by

$$de^3 = -e^{13} - e^{45} - \lambda e^{56} + \gamma_3 \wedge e^5 + \beta_3 \wedge (e^3 + e^6) + h(e^{35} + e^{65}) \quad \mod \Lambda^2 \operatorname{Span}\{e^1, e^2, e^4\}$$

which by (20) implies $h = -\lambda$. Similarly,

$$de^1 = \gamma_1 \wedge e^5 + \beta_1 \wedge (e^3 + e^6) \quad \mod \Lambda^2 \operatorname{Span}\{e^1, e^2, e^4\},$$
$$de^2 = \gamma_2 \wedge e^5 + \beta_2 \wedge (e^3 + e^6) \quad \mod \Lambda^2 \operatorname{Span}\{e^1, e^2, e^4\}.$$

Now observe that

$$(d^2 e^7)^{2,1} = (de^{14})^{2,1} + (de^2)^{1,1} \wedge e^3 - e^2 \wedge (de^3)^{1,1} + \mu(e^{126} + e^{346} - e^{135} - e^{425});$$

therefore

$$0 = (de^2)^{1,1} \wedge e^3 + \mu(e^{346} - e^{135}) + \lambda e^{235} \quad \mod \Lambda^3 \operatorname{Span}\{e^1, e^2, e^4, e^5, e^6\},$$

so the $(1,1)$-component of de^2 is $-\mu e^{46} - \mu e^{15} + \lambda e^{25}$, and

$$de^2 = -\mu e^{15} + \lambda e^{25} - \mu e^4 \wedge (e^3 + e^6) \quad \mod \Lambda^2 \operatorname{Span}\{e^1, e^2, e^4\}.$$

Now

$$0 = (d^2 e^6)^{2,1} = (de^1)^{1,1} \wedge e^3 - e^1 \wedge (de^3)^{1,1} + \lambda e^{346} - \lambda e^{135} \quad \mod \Lambda^3 \operatorname{Span}\{e^1, e^2, e^4, e^5, e^6\}$$
$$= (de^1)^{1,1} \wedge e^3 + \lambda e^{346} \quad \mod \Lambda^3 \operatorname{Span}\{e^1, e^2, e^4, e^5, e^6\}$$

so
$$(de^1)^{1,1} = -\lambda e^{46}.$$

On the other hand

$$(d^2 e^5)^{2,1} = (de^{12})^{2,1} + (de^3 + de^6)^{1,1} \wedge e^4 - \lambda e^{345} - e^6 \wedge (de^4)^{2,0}$$
$$= -\lambda e^{125} - \gamma_3 \wedge e^{45} \mod \Lambda^3 \text{ Span} \left\{ e^1, e^2, e^4, e^6 \right\},$$

which implies that $\lambda = 0$ and $\gamma_3 = ke^4$ for some $k \in \mathbb{R}$, i.e.

$$\begin{aligned}
de^1 &= 0 & \mod \Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}, \\
de^2 &= -\mu e^{15} - \mu e^4 \wedge (e^3 + e^6) & \mod \Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}, \\
de^3 + de^6 &= ke^{45} + \beta_3 \wedge (e^3 + e^6) & \mod \Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}, \\
de^4 &= 0 & \mod \Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}, \\
de^5 &= e^{12} + e^{34} - e^{46}, \\
de^6 &= e^{13} + e^{42} + e^{45}, \\
de^7 &= e^{14} + e^{23} + \mu e^{56}.
\end{aligned}$$

Assume first that $\mu = 0$. Then

$$\begin{aligned}
d^2 e^5 &= -e^3 \wedge de^4 + \beta_3 \wedge e^{34} & \mod \Lambda^3 \text{ Span} \left\{ e^1, e^2, e^4, e^5, e^6 \right\}, \\
d^2 e^6 &= de^1 \wedge e^3 - e^1 \wedge \beta_3 \wedge e^3 & \mod \Lambda^3 \text{ Span} \left\{ e^1, e^2, e^4, e^5, e^6 \right\}, \\
d^2 e^7 &= de^2 \wedge e^3 - e^2 \wedge \beta_3 \wedge e^3 - e^{123} & \mod \Lambda^3 \text{ Span} \left\{ e^1, e^2, e^4, e^5, e^6 \right\}.
\end{aligned}$$

With an appropriate change in the definition of k, we obtain

$$\begin{aligned}
de^1 &= e^1 \wedge \beta_3, & de^2 &= e^2 \wedge \beta_3 + e^{12}, \\
de^3 &= -e^{13} + ke^{45} + \beta_3 \wedge (e^3 + e^6) & & \mod \Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}, \\
de^4 &= -\beta_3 \wedge e^4, & de^5 &= e^{12} + e^{34} - e^{46}, \\
de^6 &= e^{13} + e^{42} + e^{45}, & de^7 &= e^{14} + e^{23}.
\end{aligned}$$

Thus $d\beta_3 = \beta_3(e_2)e^{12}$. Moreover,

$$0 = d^2 e^4 = -d\beta_3 \wedge e^4,$$

so $d\beta_3 = 0$. Thus
$$0 = d^2 e^2 = e^{12} \wedge \beta_3 - 2e^{12} \wedge \beta_3,$$

which implies β_3 is a multiple of e^1. More precisely,

$$0 = d^2 e^6 = -ke^{145} - \beta_3 \wedge e^{45} + 2(\beta_3 - e^1)e^{24}$$

implies $\beta_3 = e^1$. But then

$$d^2 e^7 = -e^2 \wedge de^3 \neq 0,$$

which is absurd.

Thus $\mu \neq 0$. Then de^{12} and de^{24} are linearly independent (mod e^{124}), and in consequence the exact forms in $\Lambda^2 \text{ Span} \left\{ e^1, e^2, e^4 \right\}$ are multiples of e^{14}; in particular,

$$de^1, de^4 \in \text{Span} \left\{ e^{14} \right\}.$$

By Lemma 2.3, V^4 contains no linear combination of the form $e^5 + ae^1$, $e^5 + ae^4$, which means that e^1, e^4 are in V^4. So a linear combination of e^1, e^4 is in V^3.

Now
$$(d^2 e^6)^{2,1} = (1 - k - \mu)e^{145} + e^{16} \wedge \beta_3 + de^4 \wedge e^5$$

shows that β_3 is a multiple of e^1 and
$$de^4 = (k + \mu - 1)e^{14}.$$

Similarly,
$$(d^2 e^5)^{2,1} = (de^{12})^{2,1} + (de^3 + de^6)^{1,1} \wedge e^4 - e^6 \wedge de^4$$
$$= (\mu e^{14} - \beta_3 \wedge e^4 - (k + \mu - 1)e^{14}) \wedge e^6$$

shows that
$$\beta_3 = (1 - k)e^1.$$

Finally, we have
$$(d^2 e^7)^{2,1} = (1 - k)e^{245} + (1 - k)e^{126} + \mu e^{126} + \mu e^{245}.$$

Consequently, $k = \mu + 1$ and

$$
\begin{aligned}
de^1 &= 0 & &\text{mod } e^{14}, \\
de^2 &= -\mu e^{15} - \mu e^4 \wedge (e^3 + e^6) & &\text{mod } \Lambda^2 \operatorname{Span}\left\{e^1, e^2, e^4\right\}, \\
de^3 + de^6 &= (\mu + 1)e^{45} - \mu e^1 \wedge (e^3 + e^6) & &\text{mod } \Lambda^2 \operatorname{Span}\left\{e^1, e^2, e^4\right\}, \\
de^4 &= 2\mu e^{14}, \\
de^5 &= e^{12} + e^{34} - e^{46}, \\
de^6 &= e^{13} + e^{42} + e^{45}, \\
de^7 &= e^{14} + e^{23} + \mu e^{56}.
\end{aligned}
$$

Imposing $d^2 = 0$, a straightforward computation leads to (18), with $\mu = -1, -1/3$. Notice that the two resulting Lie algebras are non-isomorphic because their second cohomology groups $H^2(\mathfrak{g}^*)$ are different. In fact, $H^2(\mathfrak{g}^*)$ is 2-dimensional if $\mu = -1$ but it is zero if $\mu = -1/3$. One can check that a normal ascending flag exists in both cases by setting

$$V^1 = \operatorname{Span}\left\{e^1\right\}, \quad V^2 = \operatorname{Span}\left\{e^1, e^4\right\}, \quad V^3 = \operatorname{Span}\left\{e^1, e^4, e^2 - \mu e^5\right\},$$
$$V^4 = \operatorname{Span}\left\{e^1, e^4, e^2 - \mu e^5, e^3 + e^6\right\}. \qquad \square$$

4 7-dimensional qc manifolds with non-vanishing torsion endomorphism and closed fundamental 4-form

The purpose of this section is to prove Theorem 1.1. For this, we consider the simply connected solvable Lie group G_s ($s = 1, 2$) of dimension 7 whose Lie algebra $\mathfrak{g}_s$ is defined by (18) considering there $\mu = -1$ for $\mathfrak{g}_1$, and $\mu = -\frac{1}{3}$ for $\mathfrak{g}_2$. We show that G_s has an integrable left invariant qc structure such that

the fundamental 4-form is closed, but the torsion endomorphism of the Biquard connection is non-zero. Firstly, we need some definitions and results about the torsion endomorphism of the Biquard connection on a qc manifold.

Let M be a manifold of dimension $4n + 3$ with a qc structure that we suppose integrable when $n = 1$. According to Section 2, we know that M has a distribution H of dimension $4n$, locally determined by the kernel of three differential 1-forms η_r on M, and such that there is an almost quaternion Hermitian structure (g, I_1, I_2, I_3) on H satisfying $2g(I_r X, Y) = d\eta_r(X, Y)$, for $r = 1, 2, 3$ and for any $X, Y \in H$. Let us consider the local vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M satisfying (2) for $n \geq 1$ [8]. Using these vector fields ξ_r, we extend the metric g on H to a metric on M (that we also write with the same letter g) by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_r, \xi_k) = \delta_{rk}$.

Since the Biquard connection ∇ on M is metric, it is related to the Levi-Civita connection ∇^g of the metric g on M by

$$g(\nabla_A B, C) = g(\nabla_A^g B, C) + \frac{1}{2}\Big[g(T(A, B), C) - g(T(B, C), A) + g(T(C, A), B)\Big], \tag{21}$$

where A, B, C are arbitrary vector fields on M and T is the torsion of ∇.

Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. We denote the curvature tensor of type $(0,4)$ by the same letter, $R(A, B, C, D) = g(R(A, B)C, D)$, for any vector fields A, B, C, D on M. The *qc-Ricci 2-forms* ρ_r $(r = 1, 2, 3)$ *and the* normalized qc-scalar curvature S of the Biquard connection are defined by

$$4n\rho_r(A, B) = R(A, B, e_a, I_r e_a), \quad 8n(n+2)S = R(e_b, e_a, e_a, e_b), \tag{22}$$

where $\{e_1, \ldots, e_{4n}\}$ is a local orthonormal basis of the distribution H.

Regarding the torsion endomorphism $T_\xi = T(\xi, \cdot) : H \to H$, $\xi \in V$, Biquard shows in [3] that it is completely trace-free, i.e. $tr\, T_\xi = tr\, T_\xi \circ I_r = 0$, and for 7-dimensional qc manifolds the skew-symmetric part of $T_\xi : H \to H$ vanishes, so T_ξ only has symmetric part.

Now, we consider the 2-tensor T^0 on H defined by

$$T^0(X, Y) = g((T_{\xi_1} I_1 + T_{\xi_2} I_2 + T_{\xi_3} I_3)X, Y),$$

for $X, Y \in H$. In [11] it is proved

$$T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0. \tag{23}$$

Moreover, taking into account [12, Proposition 2.3], on a *seven dimensional* qc manifold, the torsion endomorphism satisfies the following relations

$$4g(T_{\xi_r}(I_r X), Y) = T^0(X, Y) - T^0(I_r X, I_r Y), \quad r = 1, 2, 3. \tag{24}$$

In order to determine the torsion endomorphism of the Biquard connection on M, we need know the differential 1-forms α_r such that

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j,$$

where from now on (i, j, k) is an arbitrary cyclic permutation of $(1, 2, 3)$. The 1-forms α_r are called the $sp(1)$-connection forms. Biquard in [3] shows that on H they are expressed by

$$\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V, \tag{25}$$

while on the distribution V they are given by (see [11])

$$\alpha_i(\xi_s) = d\eta_s(\xi_j, \xi_k) - \delta_{is}\left(\frac{S}{2} + \frac{1}{2}\left(d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2)\right)\right),$$
(26)

where S is the *normalized* qc scalar curvature defined by (22). We notice that in [11] it is proved that the vanishing of the $sp(1)$-connection 1-forms on H implies the vanishing of the torsion endomorphism of the Biquard connection.

The qc Ricci 2-forms are determined by the $sp(1)$-connection 1-forms α_r as follows

$$2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B),$$
(27)

for any vector fields A, B on M. Moreover (see below (28)), the qc Ricci forms restricted to H can be expressed in terms of the endomorphism torsion of the Biquard connection. We collect the necessary facts from [14, Theorem 4.3.5] for 7-dimensional qc manifolds, so the torsion endomorphism T_ξ only has symmetric part.

Theorem 4.1. *[11] On a 7-dimensional qc manifold $(M, \eta, \mathbb{Q})$ the following formulas hold:*

$$\begin{aligned}
\rho_r(X, Y) &= \frac{1}{2}\Big[T^0(X, I_rY) - T^0(I_rX, Y)\Big] - S\omega_r(X, Y), \\
T(\xi_i, \xi_j) &= -S\xi_k - [\xi_i, \xi_j]_H, \qquad S = -g(T(\xi_1, \xi_2), \xi_3),
\end{aligned}$$
(28)

where $r = 1, 2, 3$ and $X, Y \in H$.

4.1 Example 1 ($\mu = -1$)

Consider the simply connected solvable (non-nilpotent) Lie group G_1 of dimension 7 whose Lie algebra is defined by the equations

$$\begin{aligned}
de^1 &= 0, \\
de^2 &= (1/2)e^{15} - e^{34} + (1/2)e^{46}, \\
de^3 &= (1/2)e^{16} + e^{24} - (1/2)e^{45}, \\
de^4 &= -2e^{14}, \\
de^5 &= 2(e^{12} + e^{34}) - e^{46}, \\
de^6 &= 2(e^{13} + e^{42}) + e^{45}, \\
de^7 &= 2(e^{14} + e^{23}) - (1/2)e^{56}.
\end{aligned}$$
(29)

We must notice that this Lie algebra is isomorphic to the Lie algebra $\mathfrak{g_1}$ defined by (18) for $\mu = -1$. In fact, considering the basis $\{f^j; 1 \le j \le 7\}$ of $\mathfrak{g_1}^*$ given by $f^j = e^j$ for $1 \le j \le 4$, and $f^j = 2e^j$ for $5 \le j \le 7$, equations (18) with $\mu = -1$ become (29), where we write e^j instead of f^j.

Let $\{e_j; 1 \le j \le 7\}$ be the basis of left invariant vector fields on G_1 dual to $\{e^j, 1 \le j \le 7\}$. We define a global qc structure on the Lie group G_1 by

$$\begin{aligned}
\eta_1 &= e^5, \quad \eta_2 = e^6, \quad \eta_3 = e^7, \quad \xi_1 = e_5, \quad \xi_2 = e_6, \quad \xi_3 = e_7, \\
\mathbb{H} &= Span\{e^1, \dots, e^4\}, \\
\omega_1 &= e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23}.
\end{aligned}$$
(30)

It follows from (29) that the triplet $\{\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7\}$ defined by (30) are vector fields on G_1 satisfying (2). Therefore, the qc structure on G_1 is integrable, and so the Biquard connection exists.

Theorem 4.2. *The left invariant qc structure defined by (30) on G_1 is such that the torsion endomorphism of the Biquard connection is non-zero, the fundamental 4-form is closed and the normalized qc scalar curvature is $S = -\frac{1}{2}$.*

Proof. Clearly, (29) and (30) imply that the fundamental 4-form Ω on G_1 defined by (1) is such that $d\Omega = 2e^{1234} = 0$. The closedness of Ω can also be seen as a consequence of the fact that the vertical distribution is integrable. Indeed, a result in [6, Theorem 4.7] states that *for a qc structure in dimension 7, the fundamental four form is closed if and only if the vertical distribution is integrable.*

We determine the connection 1-forms α_r of the Biquard connection on G_1. The structure equations (29) together with (25) and (26) imply

$$\alpha_1 = -\frac{1}{2}(S - \frac{1}{2})e^5, \quad \alpha_2 = -\frac{1}{2}(S - \frac{1}{2})e^6, \quad \alpha_3 = -e^4 - \frac{1}{2}(S + \frac{1}{2})e^7. \quad (31)$$

Now, (27), (29) and (31) yield

$$\begin{aligned}
\rho_1(X,Y) &= -\frac{1}{2}(S - \frac{1}{2})\omega_1(X,Y), \\
\rho_2(X,Y) &= -\frac{1}{2}(S - \frac{1}{2})\omega_2(X,Y), \\
\rho_3(X,Y) &= e^{14}(X,Y) - \frac{1}{2}(S + \frac{1}{2})\omega_3(X,Y) \\
&= \frac{1}{2}(e^{14} - e^{23})(X,Y) - \frac{1}{2}(S - \frac{1}{2})\omega_3(X,Y),
\end{aligned} \quad (32)$$

for $X, Y \in H$. Comparing (32) with (28) we conclude

$$\begin{aligned}
T^0(X, I_1 Y) - T^0(I_1 X, Y) = 0, \qquad S = -\frac{1}{2}, \\
T^0(X, I_2 Y) - T^0(I_2 X, Y) = 0, \\
T^0(X, I_3 Y) - T^0(I_3 X, Y) = (e^{14} - e^{23})(X, Y),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
T^0(I_1 X, I_1 Y) + T^0(X, Y) = 0, \qquad S = -\frac{1}{2}, \\
T^0(I_2 X, I_2 Y) + T^0(X, Y) = 0, \\
T^0(I_3 X, I_3 Y) + T^0(X, Y) = -(e^{14} - e^{23})(X, I_3 Y).
\end{aligned} \quad (33)$$

From equations (33) and (24) we have $T_{\xi_3} = 0$ and

$$T^0(X,Y) = -\frac{1}{2}(e^{14} - e^{23})(X, I_3 Y), \quad g(T(\xi_r, X), Y) = \frac{1}{4}(e^{14} - e^{23})(I_r X, I_3 Y), \quad (34)$$

for $r = 1, 2$. Equations (34) imply that the endomorphism torsion is non-zero. In fact, we have $T(e_5, e_1) = T_{\xi_1}(e_1) = -\frac{1}{4}e_2 \neq 0$ which completes the proof. $\square$

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Now, following [12], we consider the *qc conformal curvature tensor* W^{qc} of a seven dimensional qc manifold with distribution H, that is, the tensor on H of type $(0,4)$ given by

$$W^{qc}(X,Y,Z,V) = R(X,Y,Z,V) + (g \oslash L_0))(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \oslash I_s L_0)(X,Y,Z,V)$$

$$-\frac{1}{2}\sum_{s=1}^{3}\Big[\omega_s(X,Y)\big\{T^0(Z,I_sV) - T^0(I_sZ,V)\big\} + \omega_s(Z,V)\big\{T^0(X,I_sY) - T^0(I_sX,Y)\big\}\Big]$$

$$+\frac{S}{4}\Big[(g \oslash g)(X,Y,Z,V) + \sum_{s=1}^{3}\Big((\omega_s \oslash \omega_s)(X,Y,Z,V) + 4\omega_s(X,Y)\omega_s(Z,V)\Big)\Big], \tag{35}$$

where $X,Y,Z,V \in H$, $L_0 = \frac{1}{2}T^0$, $I_s L_0(X,Y) = -L_0(X,I_sY)$ $(s=1,2,3)$, and $\oslash$ is the Kulkarni-Nomizu product of 2-tensors, which is defined as follows. If μ and ν are 2-tensors on H, then $\mu \oslash \nu$ is the 4-tensor given by

$$(\mu \oslash \nu)(X,Y,Z,V) := \mu(X,Z)\nu(Y,V) + \mu(Y,V)\nu(X,Z) - \mu(Y,Z)\nu(X,V)$$
$$- \mu(X,V)\nu(Y,Z), \quad (36)$$

for any $X,Y,Z,V \in H$.

The tensor W^{qc} is the obstruction for a qc structure to be locally qc conformal to the flat structure on the quaternionic Heisenberg group.

Theorem 4.3. *[12, Theorem 4.4] A qc structure on a $(4n+3)$-dimensional smooth manifold is locally qc conformal to the standard flat qc structure on the quaternionic Heisenberg group $G(\mathbb{H})$ if and only if $W^{qc} = 0$. In this case, the qc structure is said to be a qc conformally flat structure.*

Proposition 4.4. *The left invariant qc structure defined by (30) on G_1 is not locally qc conformally flat.*

Proof. Using (34) and (36) we see that the tensor W^{qc} given by (35) satisfies

$$W^{qc}(e_1,e_2,e_1,e_2) = R(e_1,e_2,e_1,e_2) + \frac{S}{4}\Big[(g \oslash g)(e_1,e_2,e_1,e_2)$$
$$+ \sum_{r=1}^{3}\Big((\omega_r \oslash \omega_r)(e_1,e_2,e_1,e_2) + 4\omega_r(e_1,e_2)\omega_r(e_1,e_2)\Big)\Big], \quad (37)$$

since other terms on the right hand side of (35) vanish on the quadruplet (e_1,e_2,e_1,e_2). It is straightforward to check from (21), (29), (30), (34) and (36) that

$$R(e_1,e_2,e_1,e_2) = \frac{1}{2}, \quad (g \oslash g)(e_1,e_2,e_1,e_2) = 2,$$

$$\sum_{s=1}^{3}((\omega_s \oslash \omega_s)(e_1,e_2,e_1,e_2) + 4\omega_s(e_1,e_2)\omega_s(e_1,e_2)) = 6.$$

Substituting these equalities in (37) we obtain $W^{qc}(e_1,e_2,e_1,e_2) = -\frac{1}{2} \neq 0$, which completes the proof according to Theorem 4.3. $\square$

4.2 Example 2 ($\mu = -\frac{1}{3}$)

Next, we consider the simply connected solvable (non-nilpotent) Lie group G_2 of dimension 7 whose Lie algebra is defined by

$$
\begin{aligned}
de^1 &= 0, \\
de^2 &= \frac{2}{3}e^{12} + \frac{1}{6}e^{15} - \frac{1}{3}e^{34} + \frac{1}{6}e^{46}, \\
de^3 &= -\frac{2}{3}e^{13} + \frac{1}{6}e^{16} - e^{24} - \frac{1}{6}e^{45}, \\
de^4 &= -\frac{2}{3}e^{14}, \\
de^5 &= 2(e^{12} + e^{34}) - e^{46}, \\
de^6 &= 2(e^{13} + e^{42}) + e^{45}, \\
de^7 &= 2(e^{14} + e^{23}) - \frac{1}{6}e^{56}.
\end{aligned}
\tag{38}
$$

This Lie algebra is isomorphic to the Lie algebra $\mathfrak{g}_2$ obtained in Proposition 3.1 for $\mu = -1/3$. Indeed, taking the basis $\{f^j; 1 \leq j \leq 7\}$ of $\mathfrak{g}_2{}^*$ defined by $f^j = e^j$ for $1 \leq j \leq 4$, and $f^j = 2e^j$ for $5 \leq j \leq 7$, equations (18) with $\mu = -1/3$ become (38), where we write e^j instead of f^j.

Let $\{e_j; 1 \leq j \leq 7\}$ be the basis of left invariant vector fields on G_2 dual to $\{e^j, 1 \leq j \leq 7\}$. We define a global qc structure on the Lie group G_2 by

$$
\begin{aligned}
\eta_1 &= e^5, \quad \eta_2 = e^6, \quad \eta_3 = e^7, \quad \xi_1 = e_5, \quad \xi_2 = e_6, \quad \xi_3 = e_7, \\
\mathbb{H} &= Span\{e^1, \ldots, e^4\}, \\
\omega_1 &= e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42} \quad \omega_3 = e^{14} + e^{23}.
\end{aligned}
\tag{39}
$$

From (38) we have that the triplet $\{\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7\}$ of vector fields on G_2 defined by (39) satisfy (2). Therefore the Biquard connection does exist.

Theorem 4.5. *The left invariant qc structure defined by* (39) *on the simply connected solvable Lie group G_2 is such that the torsion endomorphism of the Biquard connection is non-zero, the fundamental 4-form is closed and the normalized qc scalar curvature is $S = -\frac{1}{6}$.*

Proof. Using (38), (39) and Theorem 4.7 in [6] we get that the fundamental 4-form Ω on G_2, defined by (1), is closed since the vertical distribution of the qc structure is integrable.

On the other hand, the structure equations (38) together with (25) and (26) imply

$$
\alpha_1 = -\frac{1}{2}(S - \frac{1}{6})e^5, \quad \alpha_2 = -\frac{1}{2}(S - \frac{1}{6})e^6, \quad \alpha_3 = -e^4 - \frac{1}{2}(S + \frac{1}{6})e^7. \tag{40}
$$

Now, from (27), (38) and (40), we get

$$
\begin{aligned}
\rho_1(X,Y) &= -\frac{1}{2}(S - \frac{1}{6})\omega_1(X,Y), \\
\rho_2(X,Y) &= -\frac{1}{2}(S - \frac{1}{6})\omega_2(X,Y), \\
\rho_3(X,Y) &= -\frac{1}{2}(S - \frac{1}{6})\omega_3(X,Y) + \frac{1}{6}(e^{14} - e^{23})(X,Y),
\end{aligned}
\tag{41}
$$

for $X, Y \in H$. Comparing (41) with (28) we get

$$T^0(X, I_1 Y) - T^0(I_1 X, Y) = 0, \qquad S = -\frac{1}{6},$$
$$T^0(X, I_2 Y) - T^0(I_2 X, Y) = 0,$$
$$T^0(X, I_3 Y) - T^0(I_3 X, Y) = \frac{1}{3}(e^{14} - e^{23})(X, Y),$$

or, equivalently,

$$T^0(I_1 X, I_1 Y) + T^0(X, Y) = 0,$$
$$T^0(I_2 X, I_2 Y) + T^0(X, Y) = 0, \qquad (42)$$
$$T^0(I_3 X, I_3 Y) + T^0(X, Y) = -\frac{1}{3}(e^{14} - e^{23})(X, I_3 Y),$$

for $X, Y \in H$. From equations (42) and taking into account (23) and (24), we have $T_{\xi_3} = 0$ and

$$T^0(X, Y) = -\frac{1}{6}(e^{14} - e^{23})(X, I_3 Y), \quad g(T(\xi_r, X), Y) = \frac{1}{12}(e^{14} - e^{23})(I_r X, I_3 Y),$$
(43)

for $r = 1, 2$. Equations (43) imply that the endomorphism torsion is non-zero. In fact, $T(e_5, e_1) = T_{\xi_1}(e_1) = -\frac{1}{12} e_2 \neq 0$. $\qquad \square$

Proposition 4.6. *The left invariant qc structure defined by (39) on G_2 is not locally qc conformally flat.*

Proof. Using (35), (36) and (43) we have that the expression of $W^{qc}(e_1, e_2, e_1, e_2)$ becomes as (37). From (21), (36), (38), (39) and (43) we obtain

$$R(e_1, e_2, e_1, e_2) = \frac{11}{18}, \quad (g \oslash g)(e_1, e_2, e_1, e_2) = 2,$$

$$\sum_{r=1}^{3} ((\omega_r \oslash \omega_r)(e_1, e_2, e_1, e_2) + 4\omega_r(e_1, e_2)\omega_r(e_1, e_2)) = 6.$$

Therefore, $W^{qc}(e_1, e_2, e_1, e_2) = -\frac{5}{18} \neq 0$. The result follows from Theorem 4.3. $\qquad \square$

References

[1] D. V. Alekseevskiĭ, B. N. Kimel′fel′d, Structure of homogeneous Riemannian spaces with zero Ricci curvature. (Russian) Funkcional. Anal. i PriloŽen. **9**(2) (1975).

[2] J. Alt, On the twistor space of a quaternionic contact manifold, *J. Geom. Phys.* **61** (2011), 1783-1788.

[3] O. Biquard, Quaternionic contact structures, *Quaternionic structures in mathematics and physics* (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999.

[4] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000).

[5] Ch. Boyer, K. Galicki, B. Mann, The geometry and topology of 3-Sasakian manifolds, *J. Reine Angew. Math.* **455** (1994), 183–220.

[6] L. C. de Andrés, M. Fernández, S. Ivanov, J.A. Santisteban, L. Ugarte, D. Vassilev, Quaternionic Kähler and Spin(7) metrics arising from quaternionic contact Einstein structures. Preprint 2010, arXiv:1009.2745.

[7] J. Davidov, S. Ivanov, I. Minchev, The twistor space of a quaternionic contact manifold. To appear in *Quart. J. Math.* Preprint 2010, arXiv:1010.4994.

[8] D. Duchemin, Quaternionic contact structures in dimension 7, *Ann. Inst. Fourier* **56** (4) (2006), 851–885.

[9] D. Duchemin, Quaternionic contact hypersurfaces. Preprint 2010, arXiv:math/0604147.

[10] S. Ivanov, I. Minchev, D. Vassilev, Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem, *J. Eur. Math. Soc.* **12** (4) (2010), 1041–1067.

[11] S. Ivanov, I. Minchev, D. Vassilev, Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem. Preprint 2010, arXiv:math/0611658.

[12] S. Ivanov, D. Vassilev, Conformal quaternionic contact curvature and the local sphere theorem, *J. Math. Pures Appl.* **93** (2010), 277–307.

[13] S. Ivanov, D. Vassilev, Quaternionic contact manifolds with a closed fundamental 4-form, *Bull. London Math. Soc.* **42** (2010), 1021–1030.

[14] S. Ivanov, D. Vassilev, *Extremals for the Sovolev inequality and the quaternionic contact Yamabe problem*, World Scientific Publish., 2011.

[15] W. Wang, The Yamabe problem on quaternionic contact manifolds, *Ann. Mat. Pura Appl.* **186** (2) (2007), 359–380.

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