CONVERGENCE IN MEASURE UNDER FINITE ADDITIVITY

GIANLUCA CASSESE

Abstract. We investigate the possibility of replacing the topology of convergence in probability with convergence in L^1, upon a change of the underlying measure under finite additivity. We establish conditions for the continuity of linear operators and convergence of measurable sequences, including a finitely additive analogue of Komlós Lemma. We also prove several topological implications. Eventually, a characterization of continuous linear functionals on the space of measurable functions is obtained.

1. Introduction and Notation

This paper investigates some properties of the space $L^0(\lambda)$ of λ-measurable, real valued functions on some set Ω, where λ is a bounded, finitely additive set function defined on an algebra $\mathscr{A}$ of subsets of Ω, i.e. $\lambda \in ba(\mathscr{A})$. We first characterize in section 2 the dual space of $L^0(\lambda)$ and study some of its properties, particularly positivity. In the following section 3 we investigate several boundedness conditions and, in sections 4 and 5, we develop some topological implications including conditions for continuity of linear operators. Eventually, in section 6 we study convergence properties of sequences. Section 2 is quite independent from the following ones.

Although being an entirely standard and widely used concept in probability and mathematical statistics, convergence in measure is much less popular in analysis, even assuming countable additivity. It is known that the corresponding topology is completely metrizable but, in general, not separable; moreover, it is not linear so that some useful tools such as separation theorems are not available. Actually, even a characterization of continuous linear functionals is missing. Finite additivity introduces additional complications inducing, e.g., incompleteness.

The main idea of this paper is to show that some of the techniques developed in the classical setting are still available under finite additivity, by a change of the given measure. In particular we show that, upon replacing the original measure λ with another suitably chosen but *near* to it, μ, the topology of convergence in λ-measure may be replaced by the $L^1(\mu)$ topology. Our analysis focuses on bounded, convex sets of measurable functions. Convexity is a crucial property for our technique but is delicate as the topology of convergence in measure is not locally convex. We prove in Theorem 3 that $L^0(\lambda)$ is a locally convex topological vector space if and only if λ is strongly discontinuous, a property connected with Sobczyk and Hammer decomposition. The main result, Theorem 4, shows that bounded, convex subsets of $L^0(\lambda)$ which admit a lower bound are actually

Date: June 10, 2013.

2010 *Mathematics Subject Classification.* Primary 28A20, Secondary 46E30.

Key words and phrases. Riesz representation, L^0 bounded sets, convergence in measure, Komlós Lemma.

bounded in $L^1(\mu)$. We draw from this conclusion a number of implications. In Theorem 5 we obtain a set for which L^0 and L^1 closures coincide while in Corollaries 4 and 5 we establish conditions under which continuous, $L^0(\lambda)$ valued operators are continuous as $L^1(\mu)$ valued maps. Likewise, Theorem 8 proves, that a λ-convergent sequence admits a subsequence converging in $L^1(\mu)$. We also obtain in Theorem 10 a partial, finitely additive analogue of the celebrated lemma of Komlós. We make use of some results developed in a related paper, [5].

Some of the results obtained here have a countably additive counterpart and, as always, a possible approach would then be to pass through the Stone space representation (see e.g. [11]). We find, however, that even when this possibility is available, working directly under finite additivity is preferable as it gives explicit constructions rather than isomorphic ones.

In the notation as well as in the terminology on finitely additive measures and integrals we mainly follow Dunford and Schwartz [8]. We prefer, though, the symbol $|\mu|$ of [3] to denote the total variation measure associated with $\mu \in ba(\mathscr{A})$. The space $ba(\mathscr{A})$ is endowed with the usual lattice structure described, e.g., in [3] and we thus use the lattice symbols $\mu^+ = \mu \vee 0$ and $\mu^- = -(\mu \wedge 0)$ and the fact that $|\mu| = \mu^+ + \mu^-$. The integral of $f \in L^1(\mu)$ will be denoted at will as $\int f d\mu$ or $\mu(f)$ but always as μ_f when considered itself as a set function.

We consider some special subfamilies of $ba(\mathscr{A})$, in particular the family $ba_0(\mathscr{A})$ of set functions on $\mathscr{A}$ with finite range and $ba(\lambda) = \{\mu \in ba(\mathscr{A}) : \mu \ll \lambda\}$. Moreover, we denote by (*i*) $ba_0(\lambda)$, (*ii*) $ba_\infty(\lambda)$ and (*iii*) $ba_*(\lambda)$ the classes of those set functions $\mu \in ba(\lambda)$ such that (*i*) μ has finite range, (*ii*) $|\mu| \le c|\lambda|$ for some $c \in \mathbb{R}_+$ and (*iii*) $\mu \in ba_\infty(\lambda)$ and $|\mu|(A) = 0$ if and only if $|\lambda|(A) = 0$, respectively. In the above defined families the symbol ba will be replaced by $\mathbb{P}$ to indicate the intersection of the corresponding family with the set $\mathbb{P}(\mathscr{A})$ of finitely additive probability measures on $\mathscr{A}$.

The linear space of $\mathscr{A}$-simple functions, generated by the indicators of sets in $\mathscr{A}$, will be indicated by $\mathscr{S}(\mathscr{A})$ and, when considered as a normed space, will always be endowed with the supremum norm. A sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $L^0(\lambda)$ λ-converges to $f \in \mathbb{R}^\Omega$ if $\lim_n |\lambda|^*(|f_n - f| > c) = 0$ for any $c > 0$, in which case $f \in L^0(\lambda)$ too. In fact $f \in L^0(\lambda)$ if and only if there exists a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ which λ-converges to f. As in [8, II.1.11], a (not necessarily measurable) function $f : \Omega \to \mathbb{R}$ is said to be λ-null if $|\lambda|^*(|f| > c) = 0$ for all $c > 0$ and a subset of Ω is λ-null when its indicator function is null. A function $f : \Omega \to \mathbb{R}$ possesses some property λ-a.s. – e.g. $f \ge 0$ λ-a.s. – if there exists a λ-null function g such that $f + g$ possesses that property. Given its use in the sequel, we say that f is a λ-a.s. lower bound of a set $\mathcal{K}$ if for each $c > 0$ and $k \in \mathcal{K}$ we have $|\lambda|^*(k < f - c) = 0$.

The set $L^0(\lambda)$ of measurable functions is endowed with the metric

$$d(f, g) = \inf \{c + |\lambda|^*(|f - g| \ge c) : c > 0\} \tag{1}$$

(or equivalently with $\rho(f, g) = \int (|f - g| \wedge 1) d|\lambda|$). By a bounded subset $\mathcal{K}$ of $L^0(\lambda)$ we will always mean a subset which, upon delation, is contained in any ball around the origin. This definition turns out to be equivalent to the condition

$$\lim_{c\to\infty}\sup_{f\in\mathcal{K}}|\lambda|^*(|f|>c)=0 \tag{2}$$

Of course, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are convex, bounded subsets of $L^0(\lambda)$ then from

$$|\lambda|^*(|af+(1-a)g|>c)\le|\lambda|^*(|f|>c)+|\lambda|^*(|g|>c)$$

we deduce that the convex hull $\mathrm{co}(\mathcal{K}_1\cup\mathcal{K}_2)$ of $\mathcal{K}_1\cup\mathcal{K}_2$ is itself bounded. Any space X of measurable functions mentioned in this paper, including $L^0(\lambda)$ and $\mathscr{S}(\mathscr{A})$, will be endowed with pointwise ordering in terms of which $f\ge g$ is synonymous to $f(\omega)\ge g(\omega)$ for all $\omega\in\Omega$. The symbol X_+ should be interpreted accordingly. The closure of a set H in $L^p(\lambda)$ will be denoted as $\overline{H}^{L^p(\lambda)}$.

We will use repeatedly the following, finitely additive version of Tchebycheff inequality where $f\in L^1(\lambda)_+$:

$$|\lambda|^*(f>c)=\sup m(f>c)\le c^{-1}\sup m(f)\le c^{-1}|\lambda|(f) \tag{3}$$

where the supremum is computed over all $m\in ba(\sigma\mathscr{A})_+$ which are extensions of $|\lambda|$, see [3, 3.3.3].

Eventually, if X and Y are vector lattices, a linear map $T:X\to Y$ defines an order bounded operator if for all sets of the form, $U=\{x\in X:x_1\le x\le x_2\}$ there exists $y\in Y$ such that $|Tx|\le|y|$ for all $x\in U$.

2. Linear Functionals on $L^0(\lambda)$

λ belongs to $ba_0(\mathscr{A})$ if and only if it may be written as a finite sum $\lambda=\sum_{n=1}^N\alpha_n\lambda_n$ where, $a_1,\ldots,a_N\in\mathbb{R}$, $\lambda_1,\ldots,\lambda_N\in ba(\mathscr{A})$ take their values in the set $\{0,1\}$ and there exists a disjoint collection $\{A_1,\ldots,A_N\}\subset\mathscr{A}$ such that $\lambda_n(A_n^c)=0$, [3, Lemma 11.1.3]. Other useful properties are proved in the next

Lemma 1. *The following properties are equivalent: (i) $\lambda\in ba_0(\mathscr{A})$, (ii) $|\lambda|\in ba_0(\mathscr{A})$, (iii) there exists $\eta>0$ such that $A\in\mathscr{A}$ and $|\lambda|(A)>0$ imply $|\lambda|(A)>\eta$, (iv) there exists $c>0$ such that $A\in\mathscr{A}$ and $|\lambda(A)|>0$ imply $|\lambda(A)|>c$.*

Proof. By construction, the range of λ^+ (resp. λ^-) is contained in that of λ (resp. $-\lambda$) so that the range of $|\lambda|=\lambda^++\lambda^-$ is finite if that of λ is so. The implication (*ii*)⇒(*iii*) is obvious. Let η be as in (*iii*) and let $A,B\in\mathscr{A}$ be such that $\lambda^+(A)>0$ and that $\lambda^+(B^c)+\lambda^-(B)<(\lambda^+(A)\wedge\eta)/2$. Then, $|\lambda|(A\cap B)\ge\lambda^+(A)-\lambda^+(B^c)\ge\lambda^+(A)/2>0$ so that $|\lambda|(A\cap B)>\eta$, by (*iii*), and so

$$\lambda^+(A)\ge\lambda^+(A\cap B)\ge|\lambda|(A\cap B)-\lambda^-(B)>\eta/2$$

But then, by [3, Proposition 11.1.5], both $|\lambda|$ and λ^+ have finite range and the same must be true of λ, which implies (*iv*). The implication (*iv*)⇒(*i*) is again a consequence of [3, Proposition 11.1.5]. □

Thus $ba_0(\mathscr{A})$ is a vector sublattice of $ba(\mathscr{A})$. Moreover,

Lemma 2. *(i) $ba_0(\mathscr{A})_+$ is a convex, extreme subset of $ba(\mathscr{A})_+$, (ii) $\mu \ll m$ and $m \in ba_0(\mathscr{A})$ imply $\mu \in ba_0(\mathscr{A})$.*

Proof. Convexity of $ba_0(\mathscr{A})_+$ is obvious. To prove that $ba_0(\mathscr{A})_+$ is an extreme subset of $ba(\mathscr{A})_+$, choose $\lambda_1, \lambda_2 \in ba(\mathscr{A})_+$ and $0 < t < 1$ such that $\mu = t\lambda_1 + (1-t)\lambda_2 \in ba_0(\mathscr{A})_+$. Then $\lambda_1, \lambda_2 \ll \mu$ so that we only need to prove (ii). Let $m, \mu \in ba(\mathscr{A})$ be such that $\mu \ll m$ and $m \in ba_0(\mathscr{A})$. By Lemma 1 we can choose $m \geq 0$. Let $\{A_1, \ldots, A_N\} \subset \mathscr{A}$ be the disjoint collection supporting m and such that $B \in \mathscr{A}$ implies either $m(A_n \cap B) = 0$ or $m(A_n \cap B^c) = 0$. If $A \in \mathscr{A}$ and $|\mu|(A) > 0$ then $|\mu|(A \cap A_{n_0}) > 0$ for some $1 \leq n_0 \leq N$. But then $m(A \cap A_{n_0}) > 0$ and thus $|\mu|(A^c \cap A_{n_0}) = m(A^c \cap A_{n_0}) = 0$. We conclude that $|\mu|(A) \geq |\mu|(A \cap A_{n_0}) = |\mu|(A_{n_0}) \geq \inf_{\{1 \leq i \leq N : |\mu|(A_i) > 0\}} |\mu|(A_i) > 0$. □

Lemma 3. *Continuous linear functionals on $L^0(\lambda)$ form a vector lattice.*

Proof. Indeed, if $f \in L^0(\lambda)$ the set $\mathscr{U}(f) = \{g \in L^0(\lambda) : |g| \leq |f|\}$ is bounded in $L^0(\lambda)$ and so is any order bounded set $[h, f] = \{g \in L^0(\lambda) : h \leq g \leq f\}$ given the inclusion $[h, f] \subset h + \mathscr{U}(f - h)$. Any continuous linear functional ϕ on $L^0(\lambda)$ is thus order bounded and the claim follows from [1, Theorem 1.13]. □

Theorem 1. *There exists a linear isomorphism between the space of continuous linear functionals on $L^0(\lambda)$ and the space $ba_0(\lambda)$ and this is defined implicitly via the identity*

$$\phi(f) = \int f d\mu \qquad f \in L^0(\lambda) \tag{4}$$

Proof. By Lemma 3 there is no loss of generality in assuming, as we shall do henceforth, that ϕ is positive. By [4, Theorem 1] we have the representation

$$\phi(f) = \phi^\perp(f) + \int f d\mu \tag{5}$$

where $\phi^\perp$ is a positive linear functional on $L^0(\lambda)$ with $\phi^\perp(1) = 0$ and $\mu \in ba(\mathscr{A})_+$ is such that $L^0(\lambda) \subset L^1(\mu)$. Thus $\phi^\perp(f \wedge n) = 0$ for all $f \in L^0(\lambda)_+$ so that

$$\phi(f) = \lim_n \phi(f \wedge n) = \lim_n \int (f \wedge n) d\mu = \int f d\mu$$

This follows from the fact that $f \wedge n$ converges to f in $L^0(\lambda)$ and, since $f \in L^1(\mu)$, in $L^1(\mu)$ as well. $\mu \ll \lambda$ is a consequence of ϕ being continuous. Suppose that for each $n \in \mathbb{N}$ there is $H_n \in \mathscr{A}$ such that $0 < \mu(H_n) \leq 2^{-n}$ and let $G_k \in \mathscr{A}$ be such that $\mu(G_k^c) + \lambda_\mu^\perp(G_k) < 2^{-k}$, with $\lambda_\mu^\perp$ being the part of λ orthogonal to μ emerging from Lebesgue decomposition. Then, choosing the integer k_n large enough and $H'_n = H_n \cap G_{k_n}$, we have $0 < \mu(H_n) - 2^{-k_n} \leq \mu(H'_n) \leq 2^{-n}$ and $\lim_n \lambda(H'_n) = 0$. If $f_n = \mu(H'_n)^{-1} \mathbf{1}_{H'_n}$ then $\langle f_n \rangle_{n \in \mathbb{N}}$ λ-converges to 0 but $\phi(f_n) = \int f_n d\mu = 1$, a contradiction. We conclude that for n large enough $\mu(A) \leq 2^{-n}$ implies $\mu(A) = 0$, i.e. $\mu \in ba_0(\lambda)$. Conversely, assume that $\mu \in ba_0(\lambda)$ and that $U \subset L^0(\lambda)$ is bounded in $L^0(\lambda)$. Then, choosing δ accurately, $|\mu|^*(|f| > \delta) = 0$ for all $f \in U$ so that $\sup_{f \in U} \left| \int f d\mu \right| \leq \delta \|\mu\|$ and thus the right hand side of (4) defines a bounded linear functional on $L^0(\lambda)$. □

By Theorem 1, $L^0(\lambda) \subset L^1(\mu)$ upon a change of the underlying measure. The inclusion $\mu \in ba_0(\lambda)$ implies also that a set bounded in $L^0(\lambda)$ is necessarily bounded in $L^1(\mu)$ or even in $L^\infty(\mu)$. Moreover, if m is countably additive, then so is μ. This notwithstanding, λ and μ may be very far from one another, e.g. for what concerns null sets.

A linear functional ϕ on $L^0(\lambda)$ is strictly positive if it is positive and if $f \in L^0(\lambda)_+$ and $|\lambda|^*(f > c) > 0$ for some $c > 0$ imply $\phi(f) > 0$.

Corollary 1. *$L^0(\lambda)$ admits a continuous, strictly positive linear functional if and only if $\lambda \in ba_0(\mathscr{A})$.*

Proof. In fact, if ϕ and μ are related via (4) then in order for ϕ to be strictly positive one should have $\mu \geq 0$ and $|\lambda|(A) = 0$ if and only if $\mu(A) = 0$. But this together with $\mu \in ba_0(\lambda)$ implies the existence of $\delta > 0$ such that $|\lambda|(A) < \delta$ implies $\mu(A) = 0$ and thus $|\lambda|(A) = 0$. We conclude that $\lambda \in ba_0(\mathscr{A})$, by Lemma 1. On the other hand, if $\lambda \in ba_0(\mathscr{A})$ then the integral $\int f d|\lambda|$ is well defined for all $f \in L^0(\lambda)$ and strictly positive as $f \in L^0(\lambda)_+$ and $\int f d|\lambda| = 0$ imply $|\lambda|^*(f > c) = 0$ for all $c > 0$, i.e. f is λ null. $\square$

Thus if $\lambda \notin ba_0(\mathscr{A})$ there does not exist any strictly positive linear functional, so that if μ is as in Theorem 1 one necessarily has sets $A \in \mathscr{A}$ such that $\lambda(A) > 0 = \mu(A)$. The goal of the next section will be to find $\mu \in ba(\mathscr{A})$ that guarantees the integrability of some subset of $L^0(\lambda)$ without affecting null sets.

We provide, as a last result, a different proof of [14, Theorem 1][1].

Corollary 2 (Mukherjee and Summers)**.** *Let $\mathscr{A}$ be a σ-algebra, $\lambda \in ca(\mathscr{A})$ and let ϕ be a continuous linear functional on $L^0(\lambda)$. Then either $\phi = 0$ or λ has atoms.*

Proof. Let $\mu \in ba_0(\lambda)_+$, $\mu \neq 0$. Under the current assumptions, μ admits a Radon Nikodym derivative $f_\mu \in L^1(\lambda)$ with respect to $|\lambda|$, moreover μ has atoms. Let $\eta > 0$ be such that $A \in \mathscr{A}$ and $\mu(A) < \eta$ imply $\mu(A) = 0$. Let also $c < \eta/\|\lambda\|$. Then $\mu(f_\mu < c) \leq c|\lambda|(f_\mu < c) < \eta$ so that $\mu(f_\mu < c) = 0$. If $A' \in \mathscr{A}$ is an atom of μ then so is $A = A' \cap \{f_\mu \geq c\}$. If $B \in \mathscr{A}$ and $B \subset A$ then $|\lambda|(B) \leq c^{-1}\mu(B)$ so that either $|\lambda|(B) = 0$ or $|\lambda|(A \backslash B) = 0$. Thus A is an atom for λ too and the claim follows from Theorem 1. $\square$

Mukherjee and Summers prove this claim using the fact that if $\lambda \in ca(\mathscr{A})$ is atomless then its range is convex, a fact which is simply not true under finite additivity, see the examples in [3, p. 143]. To prove a corresponding finitely additive version we need the decomposition of Sobczyk and Hammer, see [3, Theorem 5.2.7 and Remark 5.2.8]: each $\lambda \in ba(\mathscr{A})$ decomposes uniquely as

$$\lambda = \lambda_0 + \sum_n a_n \lambda_n \tag{6}$$

where λ_0 is strongly continuous (i.e. such that for each $\varepsilon > 0$ there exists a finite partition $\{A_1, \ldots, A_N\} \subset \mathscr{A}$ with $\sup_n |\lambda_0|(A_n) < \varepsilon$), the λ_n's are distinct and $\{0,1\}$-valued, $a_n \neq 0$ and

[1] I am in debt with an anonymous referee for calling my attention on the paper of Mukherjee and Summers [14].

$\sum_n |a_n| < \infty$. The set function $\sum_n \alpha_n \lambda_n$ in (6) is easily seen to be orthogonal to any strongly continuous element of $ba(\mathscr{A})$ and may thus be rightfully called strongly discontinuous. One may thus rephrase the result of Sobczyk and Hammer by saying that each $\lambda \in ba(\mathscr{A})$ decomposes uniquely as the sum of a strongly continuous and a strongly discontinuous part. Strongly discontinuous set functions will play a role later on.

Theorem 2. *$ba_0(\lambda) \neq \{0\}$ if and only λ is not strongly continuous.*

Proof. If λ is not strongly continuous, $ba_0(\lambda)$ contains each $\{0,1\}$-valued component of λ in the decomposition (6). Conversely, if λ is strongly continuous and $\mu \in ba_0(\lambda)$, then for $\varepsilon > 0$ small enough so that $0 < |\lambda|(A) < \varepsilon$ implies $|\mu|(A) = 0$ we find an $\mathscr{A}$-measurable finite partition of Ω on which $|\mu|$ vanishes. We conclude that $\mu = 0$. □

3. Bounded Subsets of $L^0(\lambda)$

In this section we provide conditions under which bounded subsets of $L^0(\lambda)$ are bounded in L^1 under a change of the given measure. The technique of changing the underlying probability measure is rather popular in stochastic analysis, e.g. in the study of semimartingale topologies, see [13]. It is also widely used in mathematical finance where the new probability measure is referred to as the *risk-neutral* measure, see e.g. [12] or [6].

Lemma 4. *Let $\mathcal{K} \subset L^1(\lambda)$ be convex with $\emptyset \in \mathcal{K}$. If $\mathcal{K}$ is bounded in $L^0(\lambda)$ then there exists $\mu \in \mathbb{P}_*(\lambda)$ such that $\mathcal{K} \subset L^1(\mu)$ and $\sup_{k\in\mathcal{K}} \int k d\mu < \infty$.*

Proof. Let $\mathcal{C} = \mathcal{K} - \mathscr{S}(\mathscr{A})_+$, pick $A \in \mathscr{A}$ such that $|\lambda|(A) > 0$ and fix $x > 0$. Suppose that $2x\mathbf{1}_A \in \overline{\mathcal{C}}^{L^1(\lambda)}$. For each $n \in \mathbb{N}$ there exist then $k_n \in \mathcal{K}$ and $h_n \in \mathcal{C}$ such that $k_n \geq h_n$ and $|\lambda|(|h_n - 2x\mathbf{1}_A|) < 2^{-n}$. Thus,

$$
\begin{aligned}
|\lambda|^*(k_n > x) &\geq |\lambda|^*(h_n > x) \\
&\geq |\lambda|^*(|h_n - 2x| < x) \\
&\geq |\lambda|(A) - |\lambda|^*(A \cap \{|h_n - 2x| \geq x\}) \\
&\geq |\lambda|(A) - |\lambda|^*(|h_n - 2x\mathbf{1}_A| \geq x) \\
&\geq |\lambda|(A) - x^{-1}|\lambda|(|h_n - 2x\mathbf{1}_A|) \\
&\geq |\lambda|(A) - x^{-1}2^{-n}
\end{aligned}
\tag{7}
$$

i.e. $\sup_{k\in\mathcal{K}} |\lambda|^*(|k| > x) \geq |\lambda|(A)$. Thus, for x sufficiently high, $2x\mathbf{1}_A \notin \overline{\mathcal{C}}^{L^1(\lambda)}$. By ordinary separation theorems there exists a continuous linear functional ϕ on $L^1(\lambda)$ such that $\sup_{h\in\mathcal{C}} \phi(h) < c_A < \phi(2x\mathbf{1}_A)$ for some $c_A > 0$. By the inclusion $-\mathscr{S}(\mathscr{A})_+ \subset \mathcal{C}$, the functional ϕ is positive and, by [4, Theorem 2], it admits the representation as an integral with respect to some $\mu_A \in ba(\lambda)_+$ such that $\mu_A(A) > 0$. Moreover, since ϕ is bounded on any bounded subset of $L^1(\lambda)$ there exists $d_A > 0$ such that $\mu_A \leq d_A|\lambda|$. By normalization we may assume $d_A \leq 1$ and $c_A \leq 1$. By a finitely additive version of Halmos-Savage theorem [5, Theorem 6], the corresponding collection

$\{\mu_A : A \in \mathscr{A}, |\lambda|(A) > 0\}$ contains a countable subcollection $\{\mu_{A_n} : n \in \mathbb{N}\}$ such that, letting $\bar{\mu} = \sum_n 2^{-n}\mu_{A_n}$, then $\bar{\mu} \gg \mu_A$ for all $A \in \mathscr{A}$ with $|\lambda|(A) > 0$ and, therefore, that $\bar{\mu}(A) = 0$ if and only if $|\lambda|(A) = 0$. Moreover $\bar{\mu} \leq |\lambda|$ and if $k \in \mathcal{K}$ then $\bar{\mu}(k) = \sum_n 2^{-n}\mu_{A_n}(k) \leq 1$. It is then enough to put $\mu = \bar{\mu}/\|\bar{\mu}\|$. □

Remark 1. *The proof of Lemma 4 may be adapted to the case in which λ is real valued and additive but not necessarily bounded provided that each $A \in \mathscr{A}$ with $|\lambda|(A) = \infty$ admits some $B \in \mathscr{A}$ such that $B \subset A$ and $|\lambda|(B) < \infty$. To see this, it is enough to rewrite the proof upon choosing $A \in \mathscr{A}$ such that $0 < |\lambda|(A) < \infty$. One easily sees that* (7) *still holds as well as the separation argument invoked. We would obtain a collection $\{\mu_A : A \in \mathscr{A}, 0 < |\lambda|(A) < \infty\}$ and, from it, $\bar{\mu} = \sum_n 2^{-n}\mu_{A_n}$. Then $\bar{\mu} \ll \lambda$ while $B \in \mathscr{A}$ and $\mu(B) = 0$ imply $\lambda(B \cap A) = 0$ for all $A \in \mathscr{A}$ with $0 < |\lambda|(A) < \infty$, i.e. $|\lambda|(B) = 0$.*

Let us remark that Lemma 4 requires that $\mathcal{K}$ is convex. This assumption is necessary due to the important fact that the convex hull of a bounded subset of $L^0(\lambda)$ need not be itself bounded. This is a crucial remark as it implies that, generally speaking, the topology of convergence in measure fails to be locally convex – and actually not even linear. This implication makes some useful tools, such as separation theorems, simply unavailable.

The next example considers the case of an unbounded set function.

Example 1. *Let $\Omega = \mathbb{N} \times \mathbb{R}_+$, fix $f : [0,1] \to \mathbb{R}_+$ with $\sup_{0\leq x\leq 1} f(x) = \infty$ and $\inf_{0\leq x\leq 1} f(x) = 1$ and define $f_n : \Omega \to \mathbb{R}_+$ by letting $f_k(n,x) = f(x)$ if $n = k$ or else 0. Clearly, $\{f_n > c\} = \{n\} \times \{f > c\}$. Define also*

$$\mathscr{A}_0 = \left\{\bigcup_{i=1}^{I}\{f_{n_i} \geq c_i\} : n_1, \dots, n_I \in \mathbb{N},\ c_1, \dots, c_I \in \mathbb{R}_+,\ I \in \mathbb{N}\right\}$$

and

$$m\left(\bigcup_{i=1}^{I}\{f_{n_i} \geq c_i\}\right) = \sum_{i=1}^{I}(1 \vee c_i)^{-1/2} \quad \text{and} \quad m(\varnothing) = 0$$

Each pair $A^1, A^2 \in \mathscr{A}_0$ admits the representation $A^j = \bigcup_{i=1}^{I}\{n_i\} \times \{f \geq c_i(j)\}$ for $j = 1, 2$, where $c_1(j), \dots, c_I(j) \in \mathbb{R}_+ \cup \{\infty\}$. It is then easy to see that

$$\begin{aligned} m(A^1 \cup A^2) + m(A^1 \cap A^2) &= \sum_{i=1}^{I}(1 \vee (c_i(1) \wedge c_i(2)))^{-1/2} + \sum_{i=1}^{I}(1 \vee (c_i(1) \vee c_i(2)))^{-1/2} \\ &= \sum_{i=1}^{I}(1 \vee c_i(1))^{-1/2} + \sum_{i=1}^{I}(1 \vee c_i(2))^{-1/2} \\ &= m(A^1) + m(A^2) \end{aligned}$$

Thus, by [3, Theorems 3.1.6 and 3.2.5]*, m admits an extension (still denoted by m) as an additive set function on the algebra $\mathscr{A}$ generated by $\mathscr{A}_0$. The set $\mathcal{K} = \{f_n : n \in \mathbb{N}\}$ is clearly bounded in*

$L^0(m)$ as $m(f_n > c) = (1 \vee c)^{-1/2}$. However, since the f_n's have disjoint support then if $c > 1$

$$m\left(\frac{1}{N}\sum_{n=1}^{N} f_n \geq c\right) = \sum_{n=1}^{N} m\left(f_n > cN\right) = \frac{1}{\sqrt{c}}\sum_{n=1}^{N}\frac{1}{\sqrt{N}} = \sqrt{N/c}$$

so that $\mathrm{co}(\mathcal{K})$ is not bounded in $L^0(m)$.

For the case of a bounded additive set function we have a general result that relates some important topological properties of $L^0(\lambda)$ with the measure theoretic properties of λ.

Theorem 3. *The following properties are equivalent: (i) if $\mathcal{K}$ is bounded in $L^0(\lambda)$ then so is $\mathrm{co}(\mathcal{K})$, (ii) λ is strongly discontinuous, (iii) $L^0(\lambda)$ is a locally convex topological vector space.*

Proof. *(i)$\Rightarrow$(ii).* Let $|\lambda|$ have a strongly continuous part, $|\lambda|_c$. By orthogonality, fix a sequence $\langle E_k\rangle_{k\in\mathbb{N}}$ of sets in $\mathscr{A}$ with $|\lambda|_c(E_k^c) + |\lambda|_d(E_k) < 2^{-k-1}$ and, for each k, let $\pi(k)$ be a finite $\mathscr{A}$ partition of E_k such that $|\pi(k)| > |\pi(k-1)|$ and $\sup_{A\in\pi(k)} |\lambda|_c(A) < 2^{-k-1}$ (see [3, 5.2.4]). Then, $\sup_{A\in\pi(k)} |\lambda|(A) < 2^{-k}$. Define $J(r) = \sum_{k=1}^{r} |\pi(k)|$ and $k_n = \inf\{k : J(k) > n\}$. Write each $\pi(k)$ as $\{A_k^i : i = 1, \ldots, I_k\}$ and for each $n \in \mathbb{N}$ define $A(n) = A_{k_n}^i$ with $i = n - J(k_n - 1)$. It is then clear that $\{A(n) : n \in \mathbb{N}\}$ is an enumeration of $\{A_k^i : i = 1, \ldots, I_k,\ k \in \mathbb{N}\}$. Define $f_n = |\pi(k_n)|^{p+1}\mathbf{1}_{A(n)}$, with $p > 0$. Observe that $\lim_n |\lambda|(A(n)) = 0$, as $A(n) \in \pi(k_n)$, and that $\lim_n |\pi(k_n)| = \infty$. If n_0 is large enough so that $\sup_{m>n_0} |\lambda|(A(m)) < \varepsilon$ and $c > \sup_{i\leq n_0} |\pi(k_i)|^{p+1}$, then $\sup_n |\lambda|(f_n > c) < \varepsilon$. Thus, $\mathcal{K} = \{f_n : n \in \mathbb{N}\}$ is bounded in $L^0(\lambda)$. However,

$$|\pi(k_n)|^p \mathbf{1}_{E_{k_n}} = \sum_{A\in\pi(k(n))} |\pi(k_n)|^p \mathbf{1}_A = \sum_{i=1+J(k_n-1)}^{J(k_n)} \frac{f_i}{J(k_n) - J(k_n-1)} \in \mathrm{co}(\mathcal{K}) \tag{8}$$

so that $\mathrm{co}(\mathcal{K})$ is not bounded in $L^0(\lambda)$.

(ii)$\Rightarrow$(i). Let λ be strongly discontinuous, i.e. let $|\lambda|$ be of the form $|\lambda| = \sum_{n\geq 1} a_n\lambda_n$ with the λ_n's being $\{0,1\}$-valued and distinct, $a_n > 0$ and $\sum_{n\geq 1} a_n < \infty$. Observe that

$$|\lambda|^*(B) = \inf_{\{A\in\mathscr{A}:B\subset A\}} |\lambda|(A) = \inf_{\{A\in\mathscr{A}:B\subset A\}} \sum_{n\geq 1} a_n\lambda_n(A) \geq \sum_{n\geq 1} a_n\lambda_n^*(B)$$

On the other hand, for each $N \in \mathbb{N}$ there exists a finite partition $\{F_1, \ldots, F_N\} \subset \mathscr{A}$ such that $\lambda_n(F_n) = 1$ for $n = 1, \ldots, N$, [3, Proposition 5.2.2]. Thus if we choose N such that $\sum_{n>N} a_n < \varepsilon$, and if $B \subset A_n$ and $A_n \in \mathscr{A}$ we have $B \subset \bigcup_{n=1}^{N} A_n \cap F_n$ and so

$$\sum_{n\leq N} a_n\lambda_n(A_n) = \sum_{n\leq N} a_n\lambda_n\left(\bigcup_{n=1}^{N} A_n \cap F_n\right) \geq |\lambda|\left(\bigcup_{n=1}^{N} A_n \cap F_n\right) - \varepsilon \geq |\lambda|^*(B) - \varepsilon \tag{9}$$

Therefore, $|\lambda|^* = \sum_n a_n\lambda_n^*$. Take $\mathcal{K}$ to be bounded in $L^0(\lambda)$, and thus in $L^0(\lambda_n)$ for each $n \in \mathbb{N}$. Given that each λ_n is purely atomic and that $\lambda_n \ll |\lambda|$, there exists $c_n > 0$ sufficiently high so that $\sup_{f\in\mathcal{K}} \lambda_n^*(f > c) = 0$ whenever $c > c_n$. Take $\sum_{i=1}^{I} b_i f_i \in \mathrm{co}(\mathcal{K})$ with $f_1, \ldots, f_I \in \mathcal{K}$, $b_1, \ldots, b_I \geq 0$ and $\sum_{i=1}^{I} b_i = 1$. Then, when $c > c_n$ we have

$$\lambda_n^*\left(\sum_{i=1}^{I} b_i f_i > c\right) \leq \lambda_n^*\left(\bigcup_{i=1}^{I}\{f_i > c\}\right) \leq \sum_{i=1}^{I} \lambda_n^*\left(f_i > c\right) = 0$$

If N is such that $\sum_{n>N} a_n < \varepsilon$ and $c > \sup_{n\le N} c_n$ then from (9) we conclude

$$\lambda^*\left(\sum_{i=1}^I b_i f_i > c\right) = \sum_{n>1} a_n \lambda_n^*\left(\sum_{i=1}^I b_i f_i > c\right) = \sum_{n>N} a_n \lambda_n^*\left(\sum_{i=1}^I b_i f_i > c\right) < \varepsilon$$

so that $\mathrm{co}(\mathcal{K})$ is bounded in $L^0(\lambda)$.

$(ii)\Rightarrow(iii)$. Let $\mathcal{K}_\varepsilon = \{f \in L^0(\lambda) : \int |f| \wedge 1 d\lambda < \varepsilon\}$. Under (ii) the collection $\{\mathrm{co}(\mathcal{K}_\varepsilon) : \varepsilon > 0\}$ forms a base of absolutely convex, absorbing sets at the origin. Its translates constitute then a base for a corresponding locally convex, linear topology. Since $\mathcal{K}_\varepsilon \subset \mathrm{co}(\mathcal{K}_\varepsilon)$, this topology is weaker than the topology of λ-convergence. However, given that $\mathrm{co}(\mathcal{K}_\varepsilon)$ is $L^0(\lambda)$ bounded, the converse is also true.

$(iii)\Rightarrow(i)$. Let $L^0(\lambda)$ be a locally convex topological vector space and let $\mathcal{K}$ be bounded in $L^0(\lambda)$. There will then be a convex open set C around the origin and $\kappa > 0$ such that $\mathcal{K} \subset \kappa C$ and thus that $\mathrm{co}(\mathcal{K}) \subset \kappa C$ so that $\mathrm{co}(\mathcal{K})$ is bounded in $L^0(\lambda)$. □

For convenience of future reference, let us introduce the class

$$\mathbb{P}_*(\lambda;\mathcal{K}) = \left\{\mu \in \mathbb{P}_*(\lambda) : \mathcal{K} \text{ is bounded in } L^1(\mu)\right\} \tag{10}$$

The most important consequence of the preceding Lemma 4 is the following:

Theorem 4. *Let $\mathcal{K} \subset L^0(\lambda)$ be convex and admit a λ-a.s. lower bound. If $\mathcal{K}$ is bounded in $L^0(\lambda)$ then $\mathbb{P}_*(\lambda;\mathcal{K})$ is non empty.*

Proof. Let $f \in L^0(\lambda)$ be a λ-a.s. lower bound for $\mathcal{K}$ and define the sets

$$\mathcal{K}_0 = \{\alpha(k-f) + \beta|f| : k \in \mathcal{K},\ \alpha,\beta \ge 0,\ \alpha+\beta \le 1\} \quad \text{and} \quad \mathcal{K}_1 = \{h \wedge k : h \in L^1(\lambda)_+, k \in \mathcal{K}_0\}$$

Observe that $\mathcal{K}_1$ is a convex subset of $L^1(\lambda)_+$ with $\emptyset \in \mathcal{K}_1$; moreover, $\mathcal{K}_1$ is bounded in $L^0(\lambda)$. We deduce from Lemma 4 the existence of $\mu \in \mathbb{P}_*(\lambda)$ such that $\mathcal{K}_1 \subset L^1(\mu)$ and $\sup_{h\in\mathcal{K}_1} \int h d\mu < \infty$. If $k \in \mathcal{K}$ and $c > 0$, then the following inequality holds λ-a.s.:

$$|k| \wedge n = |(k-(f-c)) + (f-c)| \wedge n \le ((k-f)\wedge n) + (|f|\wedge n) + 2c$$

Given that $k-f, |f| \in \mathcal{K}_0$ we conclude that $\int |k| d\mu = \lim_n \int (|k|\wedge n) d\mu \le 2[\sup_{h\in\mathcal{K}_1} \int h d\mu + c]$ and the claim follows from the fact that c was chosen arbitrarily. □

Of course there are cases in which the claim of Theorem 4 is rather trivial. The following are two easy examples.

Example 2. *Let $\Omega = \mathbb{N}$ and $\mathscr{A} = 2^{\mathbb{N}}$. Define $\lambda^c, \lambda^\perp \in ba(\mathscr{A})$ implicitly by letting*

$$\lambda^c(A) = \sum_{i\in A} 2^{-i} \quad \textit{and} \quad \lambda^\perp(A) = \operatorname*{LIM}_i \mathbf{1}_A(i) \qquad A \in \mathscr{A}$$

where LIM *denotes the Banach limit. Let $\lambda = \lambda^c + \lambda^\perp$. Of course, λ^c and $\lambda^\perp$ are the countably additive and the purely finitely additive components of λ. Define the function*

$$f_n(i) = \exp\left(\frac{n}{1+|n-i|}\right) \qquad i,n \in \mathbb{N}$$

and let $\mathcal{K} = \mathrm{co}(\{f_n : n \in \mathbb{N}\})$*. Observe that*

$$\int f_n d\lambda \geq 2^{-n} f_n(n) = 2^{-n} \exp(n)$$

so that $\sup_n \int f_n d\lambda = \infty$*. Fix* $c > 1$ *and observe that the inequality* $f_n(i) > \exp(c)$ *implies* $n, i > c$*. Thus if* $g = \sum_{i=1}^{I} a_i f_{n_i} \in \mathcal{K}$ *then*

$$\{j : g(j) > \exp(c)\} \subset \bigcup_{i=1}^{I} \{j : f_{n_i}(j) > \exp(c)\} \subset \{j : j > c\}$$

so that $\lambda^c(i > c) \leq 2^{-c}$*. On the other hand* $\lambda^{\perp}$ *does not charge any finite set so that* $\sup_{n,\varepsilon} \lambda^{\perp}(f_n > \varepsilon) = 0$*. The set* $\mathcal{K}$ *then meets the conditions of Theorem 4. Let* $z(i) = e^{-i}$ *and observe that* $zf_n \leq 1$ *so that, letting* $\mu = \lambda_z$*,*[2] *we obtain* $\sup_{k \in \mathcal{K}} \int k d\mu \leq 1$*. Moreover,* $z(i) \leq 1$ *so that, upon normalization,* $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$*. Observe also that* $(\mu + \lambda^{\perp})/2$ *is another element of* $\mathbb{P}_*(\lambda; \mathcal{K})$*.*

In the preceding example the set $\mathcal{K}$ actually admits a finite supremum. The following example shows that if the underlying space is countable then the existence of a finite supremum is somehow unavoidable under countable additivity, a fact that motivates interest for finite additivity.

Example 3. *Let* Ω *and* $\mathscr{A}$ *be as in the previous example and let* $\lambda \in ba(\mathscr{A})_+$ *be such that* $\lambda^c \neq 0$*. Observe that for each* $A \in \mathscr{A}$*,*

$$\begin{aligned} \lambda(A) &= \lim_n \lambda(A \cap \{i \leq n\}) + \lim_n \lambda(A \cap \{i > n\}) \\ &= \sum_{i \in A} \lambda(\{i\}) + \lim_n \lambda(A \cap \{i > n\}) \qquad (11) \\ &= \lambda^c(A) + \lambda^{\perp}(A) \end{aligned}$$

Let $\mathcal{K}$ *be the convex hull of a set* $\{f_n : n \in \mathbb{N}\}$ *of functions* $f_n : \mathbb{N} \to \mathbb{R}_+$ *and define* $f^* = \sup_n f_n$*. For* $\mathcal{K}$ *to be bounded in* $L^0(\lambda)$ *it is necessary that* $\lambda^c(f^* = \infty) = 0$*. Suppose not. Then there exists* $i \in \mathbb{N}$ *such that* $\lambda(\{i\}) > 0$ *and for each* j *there exists* $n_j \in \mathbb{N}$ *such that* $f_{n_j}(i) > 2^j$*. But then, if* μ *is as in the statement of Theorem 4 one has* $\mu(\{i\}) \leq 2^{-j} \int f_{n_j} d\mu \leq 2^{-j} \sup_{k \in \mathcal{K}} \int k d\mu$ *so that* $\mu(\{i\}) = 0$ *contradicting the inclusion* $\mu \in \mathbb{P}_*(\lambda)$*. Let now*

$$f_n(i) = \frac{2^{2n}}{1 + |i - n|}$$

It is then obvious that $f_n(i) < f_{n+1}(i)$ *and that* $f^*(i) = \infty$ *for each* $i \in \mathbb{N}$*. However,* $\mathcal{K}$ *is bounded in* $L^1(\mu)$ *if and only if* μ *is purely finitely additive. In fact for any such* μ *and* $N \in \mathbb{N}$ *one has* $\mu(\{1, \ldots, N\})) = 0$ *so that* $\sup_{n,\varepsilon} \mu(f_n > \varepsilon) = 0$ *and thus* $\int f_n d\mu = 0$*. On the other hand, as shown above, if the integrals* $\int f_n d\mu$ *are uniformly bounded this implies* $\mu(\{i\}) = 0$ *and, by (11),* $\mu^c = 0$*.*

The following result further contributes to understand the role of convexity.

Corollary 3. *Let* $\mathcal{K} \subset L^0(\lambda)_+$ *and assume that* $\lambda \in ca(\mathscr{A})$*. Then* $\mathbb{P}_*(\lambda; \mathcal{K}) \neq \varnothing$ *if and only if* $\mathrm{co}(\mathcal{K})$ *is bounded in* $L^0(\lambda)$*.*

[2] That is $\mu(A) = \lambda(z \mathbf{1}_A)$

Proof. If $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$ then $\mathcal{K}$ is a bounded subset of $L^1(\mu)$ and so is its convex hull $\mathrm{co}(\mathcal{K})$ which is then bounded in $L^0(\mu)$ too. However, under the assumption that λ is countably additive, $\mu \in \mathbb{P}_*(\lambda)$ implies $\lambda \ll \mu$ from which follows that $\mathrm{co}(\mathcal{K})$ is bounded in $L^0(\lambda)$. The converse implication follows easily from Theorem 4. $\square$

4. Some Topological Implications

Theorem 4 implies that some subsets of $L^0(\lambda)$ are closed in the $L^1(\mu)$ topology with $\mu \in \mathbb{P}_*(\lambda)$. A first implication of Theorem 4 is the following:

Theorem 5. *Let $\mathcal{K} \subset L^0(\lambda)_+$ be convex and bounded in $L^0(\lambda)$ and define*

$$\mathcal{C} = \left\{ f \in L^0(\lambda)_+ : f \le g \textit{ for some } g \in \mathcal{K} \right\} \tag{12}$$

Then,

$$\overline{\mathcal{C}}^{L^0(\lambda)} \subset \overline{\mathcal{C}}^{L^0(\mu)} = \overline{\mathcal{C}}^{L^1(\mu)} \qquad \mu \in \mathbb{P}_*(\lambda; \mathcal{C}) \tag{13}$$

If $\lambda \in ca(\mathscr{A})$, then $\overline{\mathcal{C}}^{L^0(\lambda)} = \overline{\mathcal{C}}^{L^1(\mu)}$ for every $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$.

Proof. By Theorem 4 we can choose $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$. Then $\mathcal{C}$ is a bounded subset of $L^1(\mu)_+$ and thus of $L^0(\mu)$. A relative comparison of the corresponding topologies shows that $\overline{\mathcal{C}}^{L^0(\lambda)} \subset \overline{\mathcal{C}}^{L^0(\mu)}$ and that $\overline{\mathcal{C}}^{L^1(\mu)} \subset \overline{\mathcal{C}}^{L^0(\mu)}$. It remains to prove that $\overline{\mathcal{C}}^{L^0(\mu)} \subset \overline{\mathcal{C}}^{L^1(\mu)}$. Fix $f \in \overline{\mathcal{C}}^{L^0(\mu)}$. Then $f \ge 0$ μ-a.s. as $\mu^*(|f - h| > c) \ge \mu^*(f < -\varepsilon)$ for $c < \varepsilon$ and $h \in \mathcal{C}$. There is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{C}$ that μ-converges to f and thus such that $|f_n - f| \wedge k$ converges to 0 in $L^1(\mu)$ for all $k > 0$. The inequality $f_n \wedge k - f \wedge k \le |f_n - f| \wedge k$ implies that the sequence $\langle f_n \wedge k \rangle_{n \in \mathbb{N}}$ converges to $f \wedge k$ in $L^1(\mu)$. Thus $f \wedge k \in \overline{\mathcal{C}}^{L^1(\mu)}$ and, since $\mathcal{C}$ is bounded in $L^1(\mu)$, $f \in \overline{\mathcal{C}}^{L^1(\mu)}$. If $\lambda \in ca(\mathscr{A})$ and $\mu \in \mathbb{P}_*(\lambda)$ then $\lambda \ll \mu$ so that $\overline{\mathcal{C}}^{L^0(\mu)} = \overline{\mathcal{C}}^{L^0(\lambda)}$. $\square$

The coincidence of the $L^0(\mu)$ (or even $L^0(\lambda)$) and the $L^1(\mu)$ closures may be useful in applications such as the separation of sets, a problem which is generally difficult to deal with in $L^0(\mu)$ due to the non linear nature of the induced topology.

Theorem 6. *Let $\mathscr{A}$ be a σ algebra and $\lambda \in ca(\mathscr{A})_+$. Let $\mathcal{K}$ and $\mathcal{C}$ be as in Theorem 5, define $\mathcal{D} = \mathcal{C} \cap L^\infty(\lambda)$ and designate by $\overline{\mathcal{D}}^*$ the closure of $\mathcal{D}$ in the weak* topology of $L^\infty(\lambda)$. Then,*

$$\overline{\mathcal{D}}^* = \overline{\mathcal{D}}^{L^1(\mu)} \cap L^\infty(\lambda) \qquad \mu \in \mathbb{P}_*(\lambda; \mathcal{K}) \tag{14}$$

Proof. Fix $f \in \overline{\mathcal{D}}^*$. Then $f \ge 0$ λ-a.s. as otherwise $\inf_{h \in \mathcal{D}} \int_{\{f < -\varepsilon\}} (h - f) d\lambda > \varepsilon \lambda(f < -\varepsilon)$ which is contradictory. Let $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$ and denote by Z its density with respect to λ. If $f \in \overline{\mathcal{D}}^*$ and $\varepsilon > 0$, then there exists $h \in \mathcal{D}$ such that $\varepsilon \ge \int Z(f - h) d\lambda = \int f d\mu - \int h d\mu$, so that, $\mathcal{D}$ being bounded in $L^1(\mu)$, the same must be true of $\overline{\mathcal{D}}^*$. By standard arguments, [8, V.3.13], $\overline{\mathcal{D}}^{L^1(\mu)}$ is closed in the weak topology of $L^1(\mu)$, i.e. the topology induced by $L^\infty(\mu)$. Given the inclusion $L^\infty(\mu) \subset L^1(\lambda)$, the restriction of the weak topology of $L^1(\mu)$ to $L^\infty(\lambda)$ is weaker than the weak* topology of $L^\infty(\lambda)$, we conclude that $\overline{\mathcal{D}}^* \subset \overline{\mathcal{D}}^{L^1(\mu)}$. Conversely, for each $f \in \overline{\mathcal{D}}^{L^1(\mu)} \cap L^\infty(\lambda)$, there exists a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C}$ that converges to $f \in L^\infty(\lambda)$ in the norm of $L^1(\mu)$ and is thus λ-convergent. Upon passing to a subsequence and letting $\bar{h}_n = h_n \wedge \|f\|_{L^\infty(\lambda)}$, we conclude that $\langle \bar{h}_n\rangle_{n\in\mathbb{N}}$ converges λ-a.s. to f. Observe that $\bar{h}_n \in \mathcal{D}$ and that, for $g \in L^1(\lambda)$, Lebesgue dominated convergence implies $\lim_n \int g\bar{h}_n d\lambda = \int gf d\lambda$. We conclude that $f \in \overline{\mathcal{D}}^*$. □

Theorem 6 thus implies that the weak* topology on $\mathcal{D}$ is metrizable and may thus, e.g., be described in terms of sequences. Let us also mention that the situation described in the statement is crucial in many problems in mathematical finance and was first considered by Delbaen and Schachermayer [6, Theorem 2.1] who exploited it to establish a special version of the no arbitrage principle.

5. Implications for $L^0(\lambda)$-Valued Operators.

We establish here some results on $L^1(\mu)$ continuity of $L^0(\lambda)$ valued operators.

Corollary 4. *Let X be a locally convex topological vector space, $V \subset X$ a convex neighborhood of the origin and $T : X \to L^0(\lambda)$ a continuous linear operator such that $T[V] = \{T(x) : x \in V\}$ admits a λ-a.s. lower bound. There exists then $\mu \in \mathbb{P}_*(\lambda)$ such that $T[X] \subset L^1(\mu)$ and $T : X \to L^1(\mu)$ is continuous.*

Proof. Let $\mathcal{K} = T[V]$. Then, $\mathcal{K}$ is convex, bounded in $L^0(\lambda)$ by continuity and lower bounded by assumption. By Theorem 4, there exists $\mu \in \mathbb{P}_*(\lambda;\mathcal{K})$. Given that each neighborhood of the origin is absorbing, this implies that $T[X] \subset L^1(\mu)$. Moreover, $T : X \to L^1(\mu)$ is bounded on a neighborhood of the origin and it is thus continuous. □

A susbet U of a vector lattice X is solid if $x \in U$, $y \in X$ and $|y| \le |x|$ imply $y \in U$.

Corollary 5. *Let X be a vector lattice with a convex, solid topological basis. A positive, continuous operator $T : X \to L^0(\lambda)$ admits $\mu \in \mathbb{P}_*(\lambda)$ such that $T[X] \subset L^1(\mu)$ and $T : X \to L^1(\mu)$ is continuous.*

Proof. Let V be a convex, solid neighborhood of the origin on which T is bounded, $V_+ = V \cap X_+$ and let $\mathcal{K} = T[V_+]$. Choose, by Theorem 4, $\mu \in \mathbb{P}_*(\lambda;\mathcal{K})$. If $x \in V$, then $|T(x)| \le T(|x|) \in \mathcal{K}$, as $|x| \in V_+$ whenever $x \in V$. Thus $T[V]$ is a bounded susbet of $L^1(\mu)$. □

This last Corollary applies, e.g., to the space $X = \mathfrak{B}(S)$ of bounded functions on some set S (endowed with the supremum norm).

Corollary 6. *Any positive linear operator $T : \mathfrak{B}(S) \to L^0(\lambda)$ admits $\mu \in \mathbb{P}_*(\lambda)$ such that $T[\mathfrak{B}(S)] \subset L^1(\mu)$ and that $T : \mathfrak{B}(S) \to L^1(\mu)$ is continuous. If $\lambda \in ca(\mathscr{A})$ then $T : \mathfrak{B}(S) \to L^0(\lambda)$ is continuous too.*

Proof. The unit ball V of $\mathfrak{B}(S)$ around the origin is mapped into the set $T[V] \subset [-T(1), T(1)]$ which is bounded in $L^0(\lambda)$ and admits $-T(1)$ as a lower bound. By Theorem 4 there is $\mu \in \mathbb{P}_*(\lambda;T[V])$

so that $T : \mathfrak{B}(S) \to L^1(\mu)$ is continuous. If $\lambda \in ca(\mathscr{A})$, then $T[V]$, being bounded in $L^1(\mu)$, is also bounded in $L^0(\mu)$ and thus in $L^0(\lambda)$ as μ and λ are equivalent. □

Example 4. *Let Σ be an algebra on a given non empty set S and $\gamma \in ba(\Sigma)_+$. Consider a map $T : \Omega \times S \to \mathbb{R}_+$ and define its ω-section $T_\omega : S \to \mathbb{R}_+$ by letting $T_\omega(s) = T(\omega, s)$. Assume that (i) $T_\omega \in L^1(\gamma)$ for λ-almost all $\omega \in \Omega$ and (ii) $\int_D T_\omega d\gamma \in L^0(\lambda)$ for each $D \in \Sigma$. Then T induces the positive linear operator $\Psi : \mathfrak{B}(\Sigma) \to L^0(\lambda)$ defined by letting*

$$\Psi(b) = \int bT_\omega d\gamma \qquad b \in \mathfrak{B}(\Sigma) \tag{15}$$

By Corollary 6, there exists $\mu \in \mathbb{P}_*(\lambda)$ such that $\Psi : \mathfrak{B}(\Sigma) \to L^1(\mu)$ and is continuous in the corresponding topology. Of course, the map $b \to \int \Psi(b)d\mu$ is then a continuous, positive linear functional on $\mathfrak{B}(\Sigma)$ and admits, by standard results, the representation as $\int bd\nu$ with $\nu \in ba(\Sigma)_+$.

Example 4 easily extends from the random quantities T_ω to the induced vector measure $\int T_\omega d\gamma$.

Theorem 7. *Let Σ be an algebra of subsets of some non empty set S and $\mathscr{S}(\Sigma, \mathscr{A})$ the space of Σ-simple functions with coefficients in $\mathscr{S}(\mathscr{A})$ endowed with the norm $\|f\| = \sup_{\omega,s} |f(\omega, s)|$. Let also $F : \Sigma \to L^0(\lambda)$ be a vector measure. If the expression*

$$\int fdF = \sum_{n=1}^{N} f_n F(H_n) \qquad f = \sum_{n=1}^{N} f_n \mathbf{1}_{H_n} \in \mathscr{S}(\Sigma, \mathscr{A}) \tag{16}$$

implicitly defines a continuous linear map of $\mathscr{S}(\Sigma, \mathscr{A})$ into $L^0(\lambda)$ then there exists $\mu \in \mathbb{P}_*(\lambda)$ such that the integral $\int fdF$ is a continuous linear mapping of $\mathscr{S}(\Sigma, \mathscr{A})$ into $L^1(\mu)$.

Proof. $\int fdF : \mathscr{S}(\Sigma, \mathscr{A}) \to L^0(\lambda)$ is a continuous linear map if and only if the set

$$I = \left\{ \int fdF : f \in \mathscr{S}(\Sigma, \mathscr{A}),\ \|f\| \le 1 \right\}$$

is bounded in $L^0(\lambda)$. Observe that $J = \mathrm{co}\{|F(H)| : H \in \Sigma\} \subset I$. By Theorem 4 there is $\nu \in \mathbb{P}_*(\lambda; J)$. We claim that $I \subset L^1(\nu)$. In fact, each $f \in \mathscr{S}(\Sigma, \mathscr{A})$ admits the canonical representation $\sum_{n=1}^N f_n \mathbf{1}_{H_n}$ where the sets H_n being pairwise disjoint. Thus, if $f \in I$ the canonical representation is such that $\sup_n |f_n| \le 1$. We conclude that $N^{-1} \left|\int fdF\right| \le N^{-1} \sum_{n=1}^N |f_n||F(H_n)| \le N^{-1} \sum_{n=1}^N |F(H_n)| \in J$. In addition I is bounded in $L^0(\nu)$ so that, by Lemma 4, there is $\mu \in \mathbb{P}_*(\nu; I) \subset \mathbb{P}_*(\lambda)$, as claimed. □

A classical example of an operator mapping (a subspace of) $\mathfrak{B}(S)$ into $L^0(\lambda)$ is of course the stochastic integral $\int hdS$ when S is a λ semimartingale and λ a classical probability. The preceding Corollaries thus seem to suggest that a meaningful definition of a semimartingale, which is beyond the scope of the present paper, may perhaps be obtained even when λ fails to be countably additive.

6. λ-Convergence of Sequences.

The same measure change technique exploited above will be applied in this section to sequences[3]. In particular we are interested in the possibility of replacing convergence in measure with L^1 convergence. The next Theorem 8 establishes a finitely additive version of a beautiful result of Memin [13, Lemma I.4], widely used in the theory of stochastic integration. Its proof is based on the following Lemma, perhaps of its own interest.

Lemma 5. *Every sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $L^0(\lambda)$ that λ-converges to 0 admits a subsequence $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ such that the following set is bounded in $L^0(\lambda)$:*

$$\mathcal{K}=\left\{\sum_{k=1}^{K}\alpha_k 2^k|f_{n_k}|:\alpha_1,\ldots,\alpha_K\geq 0,\ \sum_{k=1}^{K}\alpha_k\leq 1,\ K\in\mathbb{N}\right\} \tag{17}$$

Proof. Choose iteratively $n_k>n_{k-1}$ such that $\sup_p|\lambda|^*\left(|f_{n_k+p}|>2^{-k}\right)\leq 2^{-k}$ and put $g_k=2^k|f_{n_k}|$. Fix $c>1$ and let $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ be a sequence of positive numbers with finitely many non null terms and $\sum_k\alpha_k\leq 1$. Exploiting the subadditivity of the set function $|\lambda|^*$ we obtain the following inequality:

$$\begin{aligned}|\lambda|^*\left(\sum_k\alpha_k g_k>2c\right)&\leq|\lambda|^*\left(\sum_{k<k_0}\alpha_k g_k>c\right)+|\lambda|^*\left(\sum_{k\geq k_0}\alpha_k g_k>c\right)\\&\leq|\lambda|^*\left(\sum_{k<k_0}\alpha_k|f_{n_k}|>2^{-k_0}c\right)+\sum_{k\geq k_0}|\lambda|^*\left(|f_{n_k}|>2^{-k}\right)\\&\leq|\lambda|^*\left(\sup_{k<k_0}|f_{n_k}|>2^{-k_0}c\right)+2^{-k_0+1}\end{aligned}$$

If k_0 and c are large enough so that $2^{-k_0+1}<\varepsilon/2$ and $|\lambda|^*(\sup_{k<k_0}|f_{n_k}|>2^{-k_0}c)<\varepsilon/2$, then, $|\lambda|^*\left(\sum_k\alpha_k g_k>2c\right)<\varepsilon$. □

We say that a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ is λ-Cauchy if $f_n\in L^0(\lambda)$ for every $n\in\mathbb{N}$ and

$$\lim_n\sup_{p,q}|\lambda|^*\left(|f_{n+p}-f_{n+q}|>c\right)=0\qquad c>0 \tag{18}$$

Theorem 8. *For each $i\in\mathbb{N}$, let $\langle f^i_n\rangle_{n\in\mathbb{N}}$ be λ-Cauchy and $\langle h^i_n\rangle_{n\in\mathbb{N}}$ λ-convergent to 0. Let $\mathcal{K}_1$ be a convex, bounded subset of $L^0(\lambda)_+$. There exists $\mu\in\mathbb{P}_*(\lambda;\mathcal{K}_1)$ and a sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ of positive integers increasing to ∞ such that*

$$\lim_k\sup_{p,I\in\mathbb{N}}\int\sum_{i=1}^{I}\sum_{j=k}^{k+p}\left[\left|f^i_{n_j}-f^i_{n_{j+1}}\right|+\left|h^i_{n_j}\right|\right]d\mu=0 \tag{19}$$

Proof. By a diagonal argument, it is possible to fix $\langle n_k\rangle_{k\in\mathbb{N}}$ so that

$$\sup_{p,q}|\lambda|^*\left(\sum_{i\leq k}\left[\left|f^i_{n_{k+p}}-f^i_{n_k+q}\right|+\left|h^i_{n_{k+p}}\right|\right]>2^{-k}\right)\leq 2^{-k} \tag{20}$$

[3] After this paper was completed I came across the work of Kardaras and Žitković [9] that treats some of the topics addressed here but only for the countably additive case.

Let $\hat{g}_k = \sum_{i\le k}[|f^i_{n_k} - f^i_{n_{k+1}}| + |h^i_{n_k}|]$. The sequence $\langle \hat{g}_k\rangle_{k\in\mathbb{N}}$ is λ-convergent to 0 so that, by Lemma 5 and by letting $g_k = 2^k\hat{g}_k$, the set $\mathcal{K}_2$ of finite sums of the form $\sum_{k=1}^K \alpha_k g_k$ with $\alpha_1,\ldots,\alpha_K \ge 0$ and $\sum_{k=1}^K \alpha_k \le 1$ is bounded in $L^0(\lambda)_+$. Given that $\mathcal{K}_1$ and $\mathcal{K}_2$ are bounded and convex, then so is $\mathcal{K} = \mathrm{co}(\mathcal{K}_1 \cup \mathcal{K}_2)$, as remarked in the introduction. But then, Theorem 4 implies the existence of $\mu \in \mathbb{P}_*(\lambda, \mathcal{K})$. Then, from

$$\sum_{j=k}^{k+p} \hat{g}_j = 2^{-(k-1)} \sum_{j=k}^{k+p} 2^{-(j-k+1)} g_j \quad \text{and} \quad \sum_{j=k}^{k+p} 2^{-(j-k+1)} g_j \in \mathcal{K}$$

we conclude that $\lim_k \sup_p \int \sum_{j=k}^{k+p} \hat{g}_j d\mu \le \lim_k 2^{-k} \sup_{h\in\mathcal{K}} \mu(h) = 0$. The proof is complete upon noting that $\sum_{j=k}^{k+p}[|f^i_{n_j} - f^i_{n_{j+1}}| + |h^i_{n_j}|] \le \sum_{j=k}^{k+p} \hat{g}_j$, for $i = 1,\ldots,k$. $\square$

One should note that the sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in the claim does not depend on $i \in \mathbb{N}$. Observe also that each sequence $\langle f^i_{n_k}\rangle_{k\in\mathbb{N}}$ is Cauchy in $L^1(\mu)$ and each $\langle h^i_{n_k}\rangle_{k\in\mathbb{N}}$ is convergent in $L^1(\mu)$. Due to incompleteness of L^p spaces under finite additivity, the existence of a sequence which is Cauchy in $L^1(\mu)$ may appear an unsatisfactory conclusion. Incompleteness is amended, however, if we replace each $f_n \in L^1(\mu)$ with its isomorphic image in $ba(\lambda)$, as the sequence $\langle \mu_{f_n}\rangle_{n\in\mathbb{N}}$ converges in norm to some $m \in ba(\mu) \subset ba(\lambda)$ although m may not be representable as a μ integral.

In the classical theory of stochastic processes this result has a number of applications. If, e.g., $(M_t : t \in \mathbb{R}_+)$ is a non negative supermartingale on some filtration $(\mathscr{A}_t : t \in \mathbb{R}_+)$ of sub σ algebras of $\mathscr{A}$, then, by Doob's convergence Theorem, M converges to a λ-a.s. finite limit M_∞. By Theorem 8, we can replace λ with an equivalent probability measure μ such that M converges to M_∞ in $L^1(\mu)$ and is therefore uniformly integrable with respect to μ.

This conclusion is based on the strict interplay between convergence in measure and pointwise convergence which is a distinguishing feature of countable additivity. Under finite additivity, however, the situation may be more complex. The following example, making use of the notation employed in the proof of Theorem 3, illustrates some possible pathologies.

Example 5. *Assume that λ is not strongly discontinuous and borrow from the proof of Theorem 3 the definition of $\langle E_k\rangle_{k\in\mathbb{N}}$, $\langle \pi(k)\rangle_{k\in\mathbb{N}}$ and $\langle A(n)\rangle_{n\in\mathbb{N}}$. Let $\langle g_n\rangle_{n\in\mathbb{N}}$ be a sequence that λ-converges to g (and therefore bounded in $L^0(\lambda)$) and let $f_n = |\pi(k_n)|\mathbf{1}_{A(n)} g_{|\pi(k_n)|}$. Then, $\langle f_n\rangle_{n\in\mathbb{N}}$ is λ-convergent to 0 while*

$$h_k = \sum_{i=J(k-1)+1}^{J(k)} \frac{f_i}{J(k) - J(k-1)} = \sum_{i=J(k-1)+1}^{J(k)} \mathbf{1}_{A(i)} g_{|\pi(k_i)|} = g_{|\pi(k)|}\mathbf{1}_{E_k} \tag{21}$$

Then $h_k \in \mathrm{co}\{f_k, f_{k+1}, \ldots\}$; moreover, $\langle h_k\rangle_{k\in\mathbb{N}}$ is λ-Cauchy but does not λ-converge to 0. In fact if λ is strongly continuous – and so $E_k = \Omega$ – then $\langle h_k\rangle_{k\in\mathbb{N}}$ λ-converges to g. If λ has a strongly discontinuous part then it may well not converge at all. Take the case in which $E_k \subset E_{k+1} \uparrow \Omega$ and $g_n = g$. Then h_k converges pointwise to g but $|\lambda|^*(|g - h_k| > c) \ge |\lambda|(E_k^c) \ge \lambda_d(\Omega) - 2^{-k}$.

In the countably additive setting, Kardaras and Žitković [9, Example 1.2] construct the example of a sequence converging in measure from which it is possible to extract via convex combinations further sequences which converge in measure to any, preassigned measurable function.

Theorem 8 allows to replace measure convergence with L^1 convergence. We can also obtain conditions under which a λ-convergent sequence also converges λ-a.s..

We start proving the following preliminary result.

Lemma 6. *Let $f, f_n \in L^0(\lambda)$ for $n = 1, 2, \ldots$ be such that*

$$\lim_k |\lambda|^* \left(\inf_{n>k} f_n < f - c \right) = 0 \qquad c > 0 \tag{22}$$

Then $\liminf_n f_n \geq f$, λ-a.s..

Proof. Assume (22), fix $c > 0$ and let $g_k = \sum_{n \leq k} 2^n (f - c - f_n)^+$ and $g = \sum_n 2^n (f - c - f_n)^+$. Then, $\{|g - g_k| > c\} \subset \bigcup_{n>k} \{f_n < f - c\} = \{\inf_{n>k} f_n < f - c\}$. By assumption, $\langle g_k \rangle_{k \in \mathbb{N}}$ λ-converges to $g \in L^0(\lambda)$ so that $|\lambda|^*(g = \infty) = 0$. Moreover, since g_k converges to g monotonically too then $f_n \geq f - c - 2^{-n} g$ so that $\{\liminf_n f_n < f - c\} \subset \{g = \infty\}$. □

Lemma 6 provides a sufficient criterion for the existence of a measurable lower bound to a sequence. It also provides a sufficient condition for λ-a.s. convergence:

Theorem 9. *Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence in $L^0(\lambda)$ and define $g_k = \inf_{n,m>k} (f_n - f_m)$. If $\langle g_k \rangle_{k \in \mathbb{N}}$ λ-converges to 0, then $\liminf_n f_n = \limsup_n f_n$, λ-a.s..*

Proof. Fix $c > 0$. By assumption, $\lim_j |\lambda|^*(g_j < -c) = 0$. Lemma 6 thus implies that $\liminf_n f_n - \limsup_m f_m = \liminf_j g_j \geq 0$, λ-a.s.. □

It is important to remark that, contrary to the classical case, the random quantity g_k in the claim is not generally measurable and so neither is the λ-a.s. limit of the sequence $\langle f_n \rangle_{n \in \mathbb{N}}$. The need to consider convergence properties of non measurable elements arises also in other parts of probability, see [2] for an illustration and references.

It is also easily seen that in the classical case any λ-convergent sequence admits a subsequence that meets the criterion of Theorem 9 which may thus may be regarded as a partial, finitely additive version of the classical property by which each λ-converging sequence admits a subsequence converging λ-a.s..

To conclude, in the following Theorem 10 we prove a finitely additive version of a subsequence principle that is often useful in applications. It is related to a well known result of Komlós [10]. It proves that it is possible, given any λ-bounded sequence, to build a sequence which is λ-Cauchy – although not necessarily λ-convergent.

If $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence, denote by $\Gamma(f_1, f_2, \ldots)$ the family of all those sequences $\langle h_n \rangle_{n \in \mathbb{N}}$ such that $h_n \in \text{co}\{f_n, f_{n+1}, \ldots\}$ for all $n \in \mathbb{N}$.

Theorem 10. *Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in a convex subset $\mathcal{K}$ of $L^0(\lambda)_+$. (i) If $\mathcal{K}$ is bounded in $L^1(\lambda)$ then $\Gamma(f_1, f_2, \ldots)$ contains a λ-Cauchy sequence; (ii) if $\mathcal{K}$ is bounded in $L^0(\lambda)$ then $\Gamma(f_1, f_2, \ldots)$ contains a sequence which is Cauchy in $L^1(\mu)$ for some $\mu \in \mathbb{P}_*(\lambda; \mathcal{K})$.*

Proof. With no loss of generality assume $\lambda \geq 0$ and let $\mathcal{K}$ be bounded in $L^1(\lambda)$. Consider the sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ with $\lambda_n = \lambda_{f_n}$. By [5, Theorem 5] there exists $\mu_n \in \mathrm{co}\{\lambda_n, \lambda_{n+1}, \ldots\}$ such that the sequence $\langle\mu_n \wedge a\lambda\rangle_{n\in\mathbb{N}}$ is norm convergent for all $a \in \mathbb{R}_+$. Let $h_n \in \mathrm{co}\{f_n, f_{n+1}, \ldots\}$ be such that $\mu_n = \lambda_{h_n}$. Clearly, $\lambda_{h_n\wedge a} \leq \lambda_{h_n} \wedge a\lambda$. In fact if $\langle h_{n,r}\rangle_{r\in\mathbb{N}}$ is a sequence in $\mathscr{S}(\mathscr{A})$ converging to h_n in $L^1(\lambda)$, then, using norm convergence,

$$\lambda_{h_n} \wedge a\lambda = \lim_r(\lambda_{h_{n,r}} \wedge a\lambda) = \lim_r \lambda_{h_{n,r}\wedge a} = \lambda_{h_n\wedge a}$$

the last line following from the inequality $|x_1\wedge a - x_2\wedge a| \leq |x_1 - x_2|$. Thus the sequence $\langle h_n \wedge a\rangle_{n\in\mathbb{N}}$ is Cauchy in $L^1(\lambda)$ for all $a \in \mathbb{R}_+$ so that

$$\begin{aligned}\lambda^*(|h_n \wedge a - h_m \wedge a| > c) &\geq \lambda^*(|h_n - h_m| > c; h_m \vee h_n \leq a)\\ &\geq \lambda^*(|h_n - h_m| > c) - |\lambda|^*(h_m \geq a) - \lambda^*(h_n \geq a)\end{aligned}$$

and thus $\lambda^*(|h_n - h_m| > c) \leq 2a^{-1}\sup_{k\in\mathcal{K}}\int kd\lambda + c^{-1}\int |h_n \wedge a - h_m \wedge a|$. We can then choose the sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $n_k \geq k$ and that

$$\sup_{p,q}\lambda^*\left(|h_{n_{k+p}} \wedge a - h_{n_{k+q}} \wedge a| > 2^{-k}\right) \leq 2^{-k}$$

The subsequence $\langle h_{n_k}\rangle_{k\in\mathbb{N}}$ is thus λ-Cauchy. If, $\mathcal{K}$ is just bounded in $L^0(\lambda)$, then (ii) follows from Theorem 8 upon passing to a further subsequence, still denoted by $\langle h_{n_k}\rangle_{k\in\mathbb{N}}$ for convenience. The proof is complete if we let $g_k = h_{n_k}$ upon noting that indeed $\langle g_k\rangle_{k\in\mathbb{N}} \in \Gamma(f_1, f_2, \ldots)$. □

Claim (*ii*) of Theorem 10 becomes considerably stronger under countable additivity, when completeness of L^p spaces may be invoked. The sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ would then converge in $L^1(\mu)$ and, upon passing to a subsequence if necessary, a.s. too. The statement asserting that, by taking convex combinations, it is possible to extract from a sequence of positive, measurable functions another sequence that converges a.s., is often referred to as Komlós lemma (see [10, Theorem 1]) and has become widely used in the literature. The interplay between convergence in measure and a.s. convergence is crucial to this end and requires countable additivity. When λ is just finitely additive, Theorem 10 may be useful to obtain from a sequence converging a.s. a further sequence that converges a.s. and is Cauchy in measure.

As a final application of Theorem 10 we obtain the following:

Corollary 7. *Let $\varphi : L^1(\lambda) \to \mathbb{R}$ be uniformly continuous and $\mathcal{K}$ a convex, uniformly integrable subset of $L^1(\lambda)_+$. For each sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ there exists a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $\Gamma(f_1, f_2, \ldots)$ such that $\varphi(bh_n)$ converges for every $b \in \mathfrak{B}(\mathscr{A})$.*

Proof. Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be the λ-Cauchy sequence of Theorem 10. By uniform integrability, $\lim_{a\to\infty}\sup_{f\in\mathcal{K}}\|f-(f\wedge a)\|_{L^1(\lambda)}=0$; by continuity, the limit $\lim_{a\to\infty}\varphi(f\wedge a)$ exists uniformly in $f\in\mathcal{K}$. Thus, for each $b\in\mathfrak{B}(\mathscr{A})$ we obtain

$$\begin{aligned}\limsup_{n,m}|\varphi(bh_n)-\varphi(bh_m)| &= \limsup_{n,m}\lim_{a\to\infty}|\varphi(b(h_n\wedge a))-\varphi(b(h_m\wedge a))|\\ &= \lim_{a\to\infty}\limsup_{n,m}|\varphi(b(h_n\wedge a))-\varphi(b(h_m\wedge a))|\\ &= 0\end{aligned}$$

where we exploited [8, I.7.6], the inequality $|(h_{n+p}\wedge a)-(h_{n+q}\wedge a)|\le|h_{n+p}-h_{n+q}|\wedge a$ and the fact that as $|h_{n+p}-h_{n+q}|\wedge a$ tends to 0 in $L^1(\mu)$ as n approaches ∞. □

Acknowledgement: *I am deeply grateful to an anonymous referee for a number of helpful suggestions and for pointing out several mistakes in a previous draft.*

References

[1] C. D. Aliprantis, O. Burkinshaw: *Positive Operators*, Academic Press, Orlando, 1985.

[2] P. Berti., P. Rigo (2004), *Convergence in Distribution of Nonmeasurable Random Elements*, Ann. Probab. **32**, 365-379.

[3] K. P. S. Bhaskara Rao, M. Bhaskara Rao: *Theory of Charges*, Academic Press, London, 1983.

[4] G. Cassese: *Sure Wins, Separating Probabilities and the Representation of Linear Functionals*, J. Math. Anal. Appl. **354** (2009), 558-563.

[5] G. Cassese: *Some Implications of Lebesgue Decomposition*, (2012), arXiv:1203.1192v2.

[6] F. Delbaen, W. Schachermayer: *A general version of the fundamental theorem of asset pricing*, Math. Ann. **300** (1994), 463-520.

[7] C. Dellacherie, P. A. Meyer: *Probabilities and Potential B*, North-Holland, Amsterdam, 1982.

[8] N. Dunford, J. Schwartz: *Linear Operators. General Theory*, Wiley, New York, 1988.

[9] C. Kardaras, G. Žitković: *Forward-Convex Convergence in Probability of Sequences of Nonnegative Random Variables*, Proc. Amer. Math. Soc., **141** (2013), 919-929.

[10] J. Komlós: *A Generalization of a Problem of Steinhaus*, Acta Math. Acad. Sci. Hung. **18** (1967), 217-229.

[11] C. Fefferman: *L_p Spaces over Finitely Additive Measures*, Pacific J. Math. **26** (1968), 265-271.

[12] J. M. Harrison, D. M. Kreps: *Martingales and Arbitrage in Multiperiod Securities Markets*, J. Econ. Theory **20** (1979), 381-408.

[13] J. Memin: *Espace de Semi Martingales et Changement de Probabilité*, Z. Wahrsch. Verw. Gebiete **52** (1980), 9-39.

[14] T. K. Mukherjee, W. H. Summers: *Functionals Arising from Convergence in Measure*, Amer. Math. Month. **81** (1974), 63-66.

Università Milano Bicocca and University of Lugano

E-mail address: gianluca.cassese@unimib.it

Current address: Department of Economics, Statistics and Management, Building U7, Room 2097, via Bicocca degli Arcimboldi 8, 20126 Milano - Italy