G-expectations in infinite dimensional spaces and related PDEs

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1 Introduction

1.1 Aim and description of the work

This thesis is devoted to a study of the theory of G-expectations in infinite dimensions. The theory in finite dimensions was invented and developed by Peng [47, 68, 69, 70, 71, 72], and it quickly attracted the interest of many researchers. In the past few years a large number of papers was devoted to G-expectations, both developing the general theory and investigating new applications.
Actually, G-expectation is a special case of a sublinear expectation (a monotone, sublinear and constant preserving functional defined on a linear space of random variables) which in many cases can be represented as a supremum of a family of ordinary linear expectations. So, a sublinear expectations can be seen as a tool to model uncertainty, when the actual probabilistic model that governs a given phenomenon is not entirely known. Also, this notion provides a robust way to measure a risk loss and it is therefore of great interest in financial applications. G-expectations are also of interest for their applications to the theory of partial differential equations (PDEs for short), as they can be used to construct or represent solutions to a large class of fully non-linear PDEs.
In spite of the fact that the theory of G-expectations is now considered an important mathematical tool, so far no results have been proved in the infinite dimensional framework. The present thesis is the first attempt to fill this gap: starting from the finite dimensional case, in the present work we are going to extend the theory of G-expectations, and some of its applications, to infinite dimensions.

Thus, the first aim of the present thesis is to introduce basic objects and notions of the theory of G-expectations in a Hilbert space. A second aim is the study of fully nonlinear parabolic PDEs in Hilbert spaces. There is an extensive literature on PDEs in infinite dimensions (see, i.e. [4, 5, 11, 12, 31, 32, 34, 56, 78]) but only a relatively small part is devoted to the fully nonlinear case. Here, making use of G-expectations as a probabilistic tool, we will study equations of the form:

$$\begin{cases} \partial_t u + \langle Ax, D_x u \rangle + G(D^2_{xx} u) = 0\,, \quad t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{1}$$

We call this equation a G-PDE, because of the occurrence of the nonlinear coefficient G. G is a certain sublinear functional which is connected to a G-expectation $\mathbb{E}$ by the formula $G(\cdot) = \frac{1}{2}\mathbb{E}\big[\langle \cdot X, X\rangle\big]$. The term A in the PDE is a given generator of a C_0-semigroup $\left(e^{tA}\right)$: the occurrence of this unbounded, not everywhere defined term is important for the applications, but it requires to face additional difficulties.

The solution to equation (1) will be understood in the sense of viscosity solutions. The theory of viscosity solutions for the finite dimensional case now is well developed and the reader can consult, for instance, [6, 7, 23, 24]. Treating viscosity solutions in the infinite dimensions requires to overcome special difficulties (see, i.e., [25, 26, 27, 28, 29]).

Święch (see [56, 78]) was the first author to include the “unbounded” term $\langle Ax, D_x u\rangle$ in the second order PDE. Together with Kelome (see [55, 56]) he proved a comparison principle and existence and uniqueness results for a nonlinear second order PDE. We will make use of their results on uniqueness of the solution to equation (1). In order to prove existence we will use a probabilistic representation which is entirely different from the method of Kelome and Święch. The probabilistic representation of the solution to equation (1) is formally analogous to the classical case. To this aim we consider the associated stochastic differential equation:

$$\begin{cases} dX_\tau = AX_\tau + dB_\tau\,, \quad \tau \in (t,T] \subset [0,T]\ ; \\ X_t = x, \end{cases} \tag{2}$$

where, however, B_τ is a so called G-Brownian motion in the Hilbert space H, i.e. a Brownian motion related to a G-expectation that we introduce in an appropriate way.

The solution to equation (2) is the following process, formally analogue to the Ornstein-Uhlenbeck process:

$$X_\tau := X_\tau^{t,x} = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A} dB_\sigma.$$

We will see that the formula $u(t,x) := \mathbb{E}\big[f(X_T^{t,x})\big]$ gives the required representation of the unique viscosity solution to equation (1).
In the definition of X_τ we are naturally led to considering a stochastic integral, which can be of the more general form

$$\int_0^\tau \Phi(\sigma)\, dB_\sigma.$$

So, some parts of the thesis will be devoted to the definition of the stochastic integral with respect to a G-Brownian motion and the investigation of related properties and results. This is a third aim of the present work, of independent interest. In particular, special attention will be devoted to the identifying a suitable class of integrand processes Φ, with values in an appropriate space of linear operators that will be introduced to this purpose.
We finally mention that some of the obtained results have already been presented at international workshops and conferences in Germany, Morocco, France, Italy, Romania and Ukraine.

1.2 Plan and main results

In order to orient the reader, in this subsection we are going to describe the contents of the chapters that follow, summarizing some of the most important results.
It should be said at the outset that, although we will shortly recall some notations and basic definitions, all the results in the present thesis are new. The present work is not aimed at a complete exposition of theory of G-expectation and in particular it does not include known arguments and proofs. Some parts of the standard, finite-dimensional part of the theory admit a straightforward generalization to the infinite-dimensional case, but most of the extensions need a completely new approach.
In chapter 2 we start our plan by introducing a class of functionals, generically denoted by the symbol G. In the finite dimensional theory the functional G is defined starting from a given sublinear expectation $\mathbb{E}$ and a random variable X by means of the formula:

$$G(A) = \frac{1}{2}\mathbb{E}\big[\langle AX, X\rangle\big], \tag{3}$$

and it follows easily that G is a continuous sublinear functional, admitting the following representation:

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} \mathrm{Tr}\big[A \cdot B\big]. \tag{4}$$

The extension of these results to the infinite dimensional case requires some effort. In particular, new issues arise concerning the continuity properties of G, as several topologies can be introduced on spaces of linear operators. We will start by introducing an appropriate notion of G-functional, a sublinear functional continuous in the uniform topology. Then we will be able to prove that, if defined on the subspace of compact symmetric linear operators, it admits a representation of the form (4), for an appropriate set of operators Σ. We will also prove that Σ is determined by G: more precisely we will show that $\Sigma = \ker G^*$, where G^* is the Legendre transform of G. So, we will establish a one-to-one correspondence between a G-functional G and the set Σ.
We will then introduce the class of Hilbert space valued, G-normal distributed random variables X, which are related to a G-functional by formula (3). Since the G-functional also admits the representation (4), we will use the notation

$$X \sim N_G(0, \Sigma),$$

and we will treat the set Σ as a covariance set for the random variable X. This is justified by some other properties that will be established, for instance the moment estimate

$$c_m \cdot \sup_{Q \in \Sigma} \mathrm{Tr}\big[Q^m\big] \leqslant \mathbb{E}\|X\|_{\mathsf{H}}^{2m} \leqslant C_m \cdot \sup_{Q \in \Sigma} \Big(\mathrm{Tr}\big[Q\big]\Big)^m,$$

or the fact that, if $Z = SX$ where S is a bounded linear operator, then $Z \sim N_G(0, \Sigma_Z)$ with a covariance set $\Sigma_Z = \big\{SQS^* \big| Q \in \Sigma\big\}$.

In Chapter 3 we will first recall some basic facts on the theory of viscosity solutions in infinite dimensions, using the framework developed by Kelome and Święch (see [55, 56, 78]). The main result here is the following: if the coefficient G in equation (1) is a G-functional (as defined in the previous chapter) then it satisfies all the conditions required in the theory of Kelome and Święch in order to have a uniqueness result for the viscosity solution to equation (1).

In chapter 4 we describe one of the main concepts of the theory, the notion of a G-expectation, which is in fact a special case of a sublinear expectation as introduced earlier in chapter 2. Generalizing the construction in Peng [72] we also introduce the notion of Hilbert space valued G-Brownian motion: it is the analogue of the classical Brownian motion process, where in particular increments are G-normal distributed. In order to deal with some technical points (for instance, generalizations of the Burkholder-Davis-Gundy inequality), we also need to develop an extension of the theory of upper expectation as described in the paper of Denis, Hu and Peng [36]. In particular we identify a space of Hilbert space valued random variables, denoted ${}^{\mathsf{H}}L_G^p(\Omega)$, where the upper expectation $\bar{\mathbb{E}}$ and the G-expectation $\mathbb{E}$ coincide. It is important for applications and actual computations of G-expectations that that we are able to describe ${}^{\mathsf{H}}L_G^p(\Omega)$ as a space of random variables and not only as an abstract completion with respect to an appropriate norm.
After these preparations, in this chapter we are in position to find a solution to equation (1) in the special case $A = 0$, namely we prove that its (unique) viscosity solution is just the function

$$u(t, x) = \mathbb{E}\big[f(x + B_{T-t})\big].$$

The section "basic space constructions" in chapter 4 makes an essential connection with the subsequent chapter 5 devoted to the stochastic integral. The aim is to introduce a new Banach space of linear operators Φ, denoted L_2^Σ, endowed with an appropriate norm $\|\Phi\|^2_{L_2^\Sigma} := \sup_{Q \in \Sigma} \mathrm{Tr}\big[\Phi Q \Phi^*\big] = \sup_{Q \in \Sigma} \|\Phi Q^{1/2}\|^2_{L_2(\mathsf{U},\mathsf{H})}$. L_2^Σ will be shown to be the natural state space for integrand processes in the stochastic integral with respect to G-Brownian motion.

The definition of the stochastic integral with respect to the classical Brownian motion with values in a Hilbert space is described, for instance, by Da Prato and Zabczyk [32]. In chapter 5 we extend this construction to the framework of G-expectation. We define a stochastic integral with respect to G-Brownian motion, including a description of the natural space of integrand processes. In our case the classical Itô isometry has to be replaced by an inequality. We also prove an extension of the Burkholder-

Davis-Gundy inequality, a version of the stochastic Fubini theorem and continuity properties of the stochastic convolution.

Finally in chapter 6 we introduce the generalized Ornstein-Uhlenbeck process as the solution to equation (2) and we are able to provide an existence result for the (unique) viscosity solution to equation (1), overcoming the difficulties mentioned before related to the unbounded term $\langle Ax, D_x u\rangle$.

1.3 Conclusions and comments

The present thesis is the first work on the G-expectation theory in infinite dimensions.
The main aims were the extension of the theory of G-expectation to infinite dimensions and the probabilistic representation of viscosity solutions to some parabolic PDEs in a Hilbert space. In order to achieve these aims we proved new results regarding the representation of G-functionals, the construction and characterization of the space of integrand processes for a stochastic integral with respect to the G-Brownian motion, the Itô isometry (or rather inequality) and the Burkholder-Davis-Gundy inequalities.
Several extensions of the results presented in this thesis seem natural and interesting. For instance, instead of G-Brownian motion, one can try to define the notion of a cylindrical G-Brownian motion or of a G-martingale in infinite dimensions. The corresponding theory of stochastic integration needs to be developed and may lead to interesting applications. It is also natural to consider and try to solve a more general PDE of the form

$$\begin{cases} \partial_t u + \langle Ax + F(x), D_x u\rangle + G(H(x)\cdot D^2_{xx}u \cdot H^*(x)) = 0\,, \quad t\in[0,T]\,,\ x\in \mathsf{H}; \\ u(T,x) = f(x). \end{cases}$$

An easy conjecture is that the solution should be represented by a more general stochastic differential equation of the obvious form; this would extend our results beyond the case of the generalized Ornstein-Uhlenbeck process.
Finally, the abstract theory of stochastic evolution equations driven by G-Brownian motion can be applied to stochastic partial differential equations driven by the G-Brownian motion.

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2 Sublinear functionals and distributions

2.1 G-functional

The notion of G-functional we need in order to use some sublinear functionals in infinite dimensions. Mainly it applies to the second order term in the heat equation

$$\begin{cases} \partial_t u + G(D^2_{xx} u) = 0\,, & t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{5}$$

Also, in the G-expectation theory G-functional plays a very important role as a tool of characterization G-normal distributed random variables.

So, let us consider H is a real separable Hilbert space and $\{e_i\,, i \geqslant 1\}$ be an orthonormal basis on it.
Keeping the standard notations, define the following sets in this way:
$L(\mathsf{H}) := \big\{ A : \mathsf{H} \to \mathsf{H} \mid A$ – linear, continuous in $L(\mathsf{H})$-topology$\big\}$;
$K(\mathsf{H}) := \big\{ A \in L(\mathsf{H}) \mid A$ – compact$\big\}$;
$L_S(\mathsf{H}) := \big\{ A \in L(\mathsf{H}) \mid A = A^* \big\}$ and $K_S(\mathsf{H}) := \big\{ A \in K(\mathsf{H}) \mid A = A^* \big\}$.

Definition 2.1. *A monotone, sublinear, continuous (in the operator norm) functional* $G : D \subset L_S(\mathsf{H}) \to \mathbb{R}$ *is said to be* **G *-functional*.**
That is, $G(\,\cdot\,)$ *is required to satisfy the following conditions:*
1) $A \geqslant \bar{A} \;\Rightarrow\; G(A) \geqslant G(\bar{A})$.
2) $G(A + \bar{A}) \leqslant G(A) + G(\bar{A})$;
3) $G(\lambda A) = \lambda G(A),\ \lambda \geqslant 0$;
4) G *is* $L(\mathsf{H})$ *-continuous.*

The following theorem is the analogous result from the finite-dimensional case (see [72, Th.2.1]). Note that some proves of the results in this thesis are just adjusted the finite dimensional case. There are some pieces that can be easily kept, but there are some that have troubles when we pass from infinite to finite dimensions. Naturally we will note if the some part of the proof coincides with the proof in finite dimensions.

Theorem 2.1. *Let* X *be a linear space.*
$F : \mathsf{X} \to \mathbb{R}$ *is a sublinear functional, i.e.*
1) $F(x+y) \leqslant F(x) + F(y)$ *;*
2) $F(\lambda x) = \lambda F(x)$, $\lambda \geqslant 0$ *.*
Then there exists a family of linear functionals $\{f_\theta : \mathsf{X} \to \mathbb{R},\ \theta \in \Theta\}$ *, such that:*

$$F(x) = \sup_{\theta\in\Theta} f_\theta(x),\ \ x \in \mathsf{X}. \tag{6}$$

Moreover, (a) If F *is continuous* $\Rightarrow$ f_θ *in* (6) *are continuous.*
(b) If F *is a monotone, sublinear functional, that preserves constants (such functional we call* ***sublinear expectation****)*
$\Rightarrow$ f_θ *in* (6) *are linear expectations.*

Proof.
In order to get (6) and the part **(b)** there are no essential differences with a finite dimensional case (see [72, Th.2.1]). But there are some moments which we will use to prove the part **(a)**.

Let Q be a family of linear functionals $\{f_\theta : \mathsf{X} \to \mathbb{R},\ \theta \in \Theta\}$, such that $F(x) \geqslant f_\theta(x)$ for every $x \in \mathsf{X}$.
For fixed element $x \in \mathsf{X}$ we define a set $L := \{ax\,,\ a \in \mathbb{R}\}$, which is a subspace of X.
Define a mapping $I : L \to \mathbb{R}$ in the following way $I(ax) := aF(x)$, $a \in \mathbb{R}$.
It is clear that I is linear on X.
Also $I \leqslant F$ on L. In fact, if we take $a > 0$ then $F(ax) = aF(x) = I(ax)$, and $F(-ax) = aF(-x) \geqslant -aF(x) = I(-ax)$.
By the Hahn-Banach (in analytic form) there exists a linear functional
$f : \mathsf{X} \to \mathbb{R}$, such that $f = I$ on L and $f \leqslant F$ on X. $(\,*\,)$
At the point x: $F(x) = I(x) = f(x) =: f_\theta(x)$, where θ depends of x.
Take the set $\Theta := \{\theta = \theta(x)\,,\ x \in \mathsf{X}\}$ and consider a functional

$$F_\Theta(x) := \sup_{\theta\in\Theta} f_\theta(x).$$

From the one hand, surely $F_\Theta(x) \leqslant F(x)$.
But from the other hand, for every x there exists $\theta = \theta(x)$, such that $F(x) = f_{\theta(x)}(x)$, but $f_{\theta(x)}(x) \leqslant \sup_{\theta\in\Theta} f_\theta(x) = F_\Theta(x)$.
It means that F_Θ coincides with F on X, i.e. $F(x) = \sup_{\theta\in\Theta} f_\theta(x)\ \ \forall x \in \mathsf{X}$.

It remains just to show **(a)**:
Let F be continuous, it follows that I is continuous on L, and f in $(*)$ is also continuous (and linear) on L.
So, we have that: $f : \mathsf{X} \to \mathbb{R}$ is linear ;
$f\big|_L$ is linear and continuous on the linear subspace L;
$F : \mathsf{X} \to \mathbb{R}$ is sublinear and continuous;
$f \leqslant F$ on X.
Then following the proof of the Hahn-Banach theorem in the classical form (i.e., there exists a linear continuous functional g on X, such that $g = f$ on L, and $\|g\| = \|f\|$): following the proof of this theorem (see e.g. [62, Th.4]) in the same way (as with a functional g) we can construct a linear continuous functional $\widetilde{f}$, such that:
$\widetilde{f} = f$ on L, and $\widetilde{f} \leqslant F$ on X.
From $(*)$ it follows that: $\widetilde{f} = I$ on L, and $\widetilde{f}$ on X.
Proceeding the same reasonings we can conclude that $F(x) = \sup\limits_{\theta\in\Theta} \widetilde{f}_\theta(x)$, where $\left\{\widetilde{f}_\theta : \mathsf{X} \to \mathbb{R},\ \theta \in \Theta\right\}$ is a family of linear continuous functionals.

□

Definition 2.2. *The von Neumann-Schatten classes of operators are defined as follows:*

$$C_p(\mathsf{H}) := \left\{A \in L(\mathsf{H}) \mid \sum_{j\geqslant 1} \left|\langle Ae_j, e_j\rangle\right|^p < \infty\right\}, \quad 1 \leqslant p < \infty;$$
$$C_\infty(\mathsf{H}) := L(\mathsf{H}).$$

Introducing the norm for $A \in C_p(\mathsf{H}),\quad 1 \leqslant p \leqslant \infty$:

$$\|A\|_{C_p} := \left[\mathrm{Tr}\big(A\cdot A^*\big)^{p/2}\right]^{1/p},\ 1 \leqslant p < \infty,$$
$$\|A\|_{C_\infty} := \|A\|_{L(\mathsf{H})};$$

Also we know that $\big(C_p(\mathsf{H}), \|\cdot\|_{C_p}\big),\ 1 \leqslant p \leqslant \infty$ is a Banach space (see [75, 2.1]).

Remark 2.1. *In the sequel we will call the classes $C_1(\mathsf{H})$ and $C_2(\mathsf{H})$ as **trace-class** and **Hilbert-Schmidt class** of operators respectively, and denote them in this way:*

$$C_1(\mathsf{H}) \equiv L_1(\mathsf{H}) := \left\{A \in L(\mathsf{H}) \mid \mathrm{Tr}\big[(A\cdot A^*)^{1/2}\big] < \infty\right\};$$
$$C_2(\mathsf{H}) \equiv L_2(\mathsf{H}) := \left\{A \in L(\mathsf{H}) \mid \mathrm{Tr}\big[A\cdot A^*\big] < \infty\right\}.$$

Now we would like to give the representation of the G-functional defined on the set of compact symmetric operators on H. Actually, we can't give the same representation result in the general case with the domain of linear bounded operators. But afterwards we discuss about extension G-functional on $L_S(\mathsf{H})$.

Theorem 2.2. *Let $G : K_S(\mathsf{H}) \to \mathbb{R}$ be a G-functional.*
Then there exists the set Σ such that:
1) $\Sigma \subset C_1(\mathsf{H})$*;*
2) $\forall\, B \in \Sigma \;\Rightarrow\; B = B^*,\; B \geqslant 0$*;*
3) Σ *is convex;*
4) Σ *is closed subspace of* $C_1(\mathsf{H})$*;*
5) $G(A) = \dfrac{1}{2} \sup\limits_{B \in \Sigma} \mathrm{Tr}\big[A \cdot B\big], \quad \forall\, A \in K_S(\mathsf{H})$.

Proof.
For every sublinear continuous functional $F : K(\mathsf{H}) \to \mathbb{R}$ according to **Th.2.1**, there exist a family of linear continuous (in $L(\mathsf{H})$-topology) functionals $\big\{f_\theta,\ \theta \in \Theta\big\}$, such that $F(A) = \sup\limits_{\theta \in \Theta} f_\theta(A)$, $A \in K(\mathsf{H})$.

Let us fix θ and consider $f = f_\theta \in L\big(K(\mathsf{H}), \mathbb{R}\big)$.

For $x, y \in \mathsf{H}$ we define a bounded linear operator $x \otimes y$ on H as follows: $(x \otimes y)z := \langle z, y\rangle x$, $z \in \mathsf{H}$.

It is clear that $\mathrm{rk}(x \otimes y) = 1$, if $x \neq 0,\ y \neq 0$, and
$\mathrm{Tr}[x \otimes y] = \sum\limits_{j \geqslant 1} \big\langle (x \otimes y)e_j, e_j \big\rangle = \sum\limits_{j \geqslant 1} \langle e_j, y\rangle \langle x, e_j\rangle = \langle x, y\rangle$, and
the norm $\|x \otimes y\|_{L(\mathsf{H})} := \|x\|_{\mathsf{H}} \cdot \|y\|_{\mathsf{H}}$, $1 \leqslant p \leqslant \infty$.

The bilinear form $L(x, y) := f(x \otimes y)$ satisfies
$\big|L(x, y)\big| = \big|f(x \otimes y)\big| \leqslant \|f\|_{L(L(\mathsf{H}),\mathsf{H})} \cdot \|x \otimes y\|_{L(\mathsf{H})} \leqslant \|f\|_{L(L(\mathsf{H}),\mathsf{H})} \cdot \|x\|_{\mathsf{H}} \cdot \|y\|_{\mathsf{H}}$,
So that, it is bounded.

But for every bounded bilinear form there exists a unique $B \in L(\mathsf{H})$ such that $L(x, y) = \langle Bx, y\rangle$ (see [75, Th.1.7.1]).

It follows that
$$f(x \otimes y) = \langle Bx, y\rangle = \langle x, B^* y\rangle = \mathrm{Tr}[x \otimes B^* y] = \mathrm{Tr}\big[(x \otimes y)B\big],$$
because $(x \otimes B^* y)z = \langle z, B^* y\rangle x = \langle Bz, y\rangle x = (x \otimes y)Bz$.

So, we can conclude that $f(A) = \mathrm{Tr}\big[AB\big]$, where $A := x \otimes y$. $\quad (*)$

Denote $\mathcal{F}$ to be the set of all operators of finite rank on H.
But every $A \in \mathcal{F}$ can be represented in the form $A = \sum_{j=1}^{m} \alpha_j \psi_j \otimes \varphi_j$ (see [75, Th.1.9.3]), where (α_j) is uniquely determined sequence of the real elements, and $(\psi_j), (\varphi_j)$ are two orthonormal systems in H.
For this reason $(*)$ holds for every $A \in \mathcal{F}$.
In order to show that $B \in C_1(\mathsf{H})$ we use the following lemma:

Lemma 2.1 (Ringrose, [75], Lm.2.3.7)**.**
Let $1 \leqslant p \leqslant \infty$ *, and* q *be a* ***conjugate exponent*** *of* p *(i.e.* $\frac{1}{p} + \frac{1}{q} = 1$ *).*
Then $T \in C_q(\mathsf{H})$ *if and only if* $\sup\Big\{\big|\operatorname{Tr}[ST]\big| : \ S \in \mathcal{F}, \ \|S\|_{C_p} \leqslant 1\Big\} < \infty$.
And in such a case the value of the supremum is equal to $\|T\|_{C_q}$.

According to this lemma we get that

$$\sup\Big\{\big|\operatorname{Tr}[AB]\big| : \ B \in \mathcal{F}, \ \|B\|_{C_\infty} \leqslant 1\Big\} \leqslant \|f\| < \infty;$$

So that for every $A \in \mathcal{F}$ we have $f(A) = \operatorname{Tr}\big[AB\big]$, $B \in C_1(\mathsf{H})$.
Both sides in this equality are $L(\mathsf{H})$-continuous, and since the set $\mathcal{F}$ is dense in $K(\mathsf{H})$, this implies that for every $A \in K(\mathsf{H})$ we have also $f(A) = \operatorname{Tr}\big[AB\big]$, $B \in C_1(\mathsf{H})$.
In the case when $A \in K_S(\mathsf{H})$ therefore there exists $B_0 \in C_1(\mathsf{H})$, such that $f(A) = \operatorname{Tr}\big[AB_0\big]$.
Consider an operator $B := \dfrac{B_0 + B_0^*}{2}$, which is symmetric and trace-class;
Hence:

$$\begin{aligned}
\operatorname{Tr}\big[AB\big] &= \operatorname{Tr}\left[\frac{A(B_0 + B_0^*)}{2}\right] = \frac{1}{2}\operatorname{Tr}\big[AB_0\big] + \frac{1}{2}\operatorname{Tr}\big[AB_0^*\big] \\
&= \frac{1}{2}\operatorname{Tr}\big[AB_0\big] + \frac{1}{2}\operatorname{Tr}\big[B_0A^*\big] = \frac{1}{2}\operatorname{Tr}\big[AB_0\big] + \frac{1}{2}\operatorname{Tr}\big[B_0A\big] \\
&= \frac{1}{2}\operatorname{Tr}\big[AB_0\big] + \frac{1}{2}\operatorname{Tr}\big[AB_0\big] = \operatorname{Tr}\big[AB_0\big] = f(A).
\end{aligned}$$

So we can conclude that for every $A \in K_S(\mathsf{H})$ we have $f(A) = \operatorname{Tr}\big[AB\big]$, $B = B^* \in C_1(\mathsf{H})$.

Now let us take the following sublinear functional $F(A) := 2\,G(A)$, defined on the set $K_S(\mathsf{H})$.

Applying **Th.2.1** we have
$F(A) = \sup_{\theta\in\Theta} f_\theta(A) = \sup_{\theta\in\Theta} \mathrm{Tr}\big[AB_\theta\big]\,, \quad B_\theta = B_\theta^* \in C_1(\mathsf{H})\,,\ \theta \in \Theta\,;$
Hence there exists a set of operators $\big\{B_\theta,\, \theta \in \Theta \big| \, B_\theta = B_\theta^* \in C_1(\mathsf{H})\big\}$, such that $G(A) = \dfrac{1}{2}\sup_{\theta\in\Theta} \mathrm{Tr}\big[AB_\theta\big]\,,\ A \in K_S(\mathsf{H})\,.$
G is monotone. For this reason we can only take that B_θ which are positive: if $x \in \mathsf{H}$ then $\langle B_\theta x, x\rangle \geqslant 0\,.$
Indeed, by the definition of supremum we have that:

$$\forall\,\varepsilon > 0\ \ \exists\,\theta = \theta_\varepsilon \in \Theta :\ \ \frac{1}{2}\mathrm{Tr}\big[AB_\theta\big] \leqslant G(A) < \frac{1}{2}\mathrm{Tr}\big[AB_\theta\big]+\varepsilon\,. \qquad (\,\#\,)$$

Assume that there exists a basis vector e_i, such that $\langle B_\theta e_i, e_i\rangle := \beta_\varepsilon < 0\,.$
then consider an operator A_0 such that $\langle A_0 e_j, e_j\rangle = \mathbb{1}_{\{j=i\}}\,,\ j \geqslant 1\,.$
It is clear that A_0 is symmetric and compact.
Also if $A_0 \geqslant 0$ yields $G(A_0) \geqslant G(0) = 0\,.$
From $(\,\#\,)$ we have $\dfrac{1}{2}\beta_\theta \leqslant G(A) < \dfrac{1}{2}\beta_\theta + \varepsilon\,.$
And passing to the limit $\varepsilon \to 0$: gives $G(A) = \dfrac{1}{2}\beta_0 < 0\,,$ a contradiction.
And finally it may be concluded that $G(A) = \dfrac{1}{2}\sup_{\theta\in\Theta} \mathrm{Tr}\big[AB_\theta\big]\,,\ A \in K_S(\mathsf{H})\,.$
where operators $\big\{B_\theta\,,\ \theta \in \Theta\big\}$ satisfy required properties:

$$B_\theta = B_\theta^* \geqslant 0\,,\ B_\theta \in C_1(\mathsf{H})\,.$$

And now we are going to finish the proof by the construction of the set Σ:
Let us define the following sets:
$\widetilde{\Sigma} := \big\{B \in C_1(\mathsf{H}) \mid B = B^* \geqslant 0\big\}\,;$
$\Sigma' = \Sigma'_G := \big\{B \in \widetilde{\Sigma} \mid \forall\,\varepsilon > 0\ \ \dfrac{1}{2}\mathrm{Tr}\big[AB\big] \leqslant G(A) < \dfrac{1}{2}\mathrm{Tr}\big[AB\big] + \varepsilon\big\}\,;$
Take $\Sigma := \overline{\mathrm{conv}(\Sigma')}$ (closure in C_1-norm).
Since $\sup_{\theta\in\Theta} \mathrm{Tr}\big[AB_\theta\big] = \sup_{B\in\Sigma'} \mathrm{Tr}\big[AB\big] = \sup_{B\in\Sigma} \mathrm{Tr}\big[AB\big]\,,$ and $\Sigma \subseteq \widetilde{\Sigma} \subset C_1(\mathsf{H})\,,$
we can set $G(A) := \dfrac{1}{2}\sup_{B\in\Sigma} \mathrm{Tr}\big[A \cdot B\big],\ \ \forall\, A \in K_S(\mathsf{H})\,.$
Moreover, by a such construction one can see that G defines the Σ-set uniquely.

□

Remark 2.2. *For every functional* $f_B(A) : L(\mathsf{H}) \to \mathbb{R}$, $B \in C_1(\mathsf{H})$, *such that* $f_B(A) = \mathrm{Tr}\big[AB\big]$ *we have* $f_B \in L\big(L(H), \mathbb{R}\big)$.
Moreover the mapping $B \to f_B$ *defines an isometric isomorphism from* $C_1(\mathsf{H})$ *to* $\big(K(\mathsf{H})\big)^*$.

Proof.
We have $AB \in C_1(\mathsf{H})$ and $\mathrm{Tr}\big[AB\big] = \|AB\|_{C_1} \leqslant \|A\|_{C_\infty}\|B\|_{C_1}$ (see [75, Th.2.3.10]).
So that, f_B is a linear continuous functional.
Moreover, $\|f_B\| \leqslant \|B\|_{C_1}$;
But according to **Lm.2.1**:

$$\|f_B\| \geqslant \sup\Big\{\big|\, f_B(A)\,\big| : \ A \in \mathcal{F},\ \|A\|_{C_\infty} \leqslant 1\Big\}$$
$$= \sup\Big\{\big|\,\mathrm{Tr}[AB]\,\big| : \ A \in \mathcal{F},\ \|A\|_{C_\infty} \leqslant 1\Big\} = \|B\|_{C_1};$$

whence it follows that $\|f_B\| = \|B\|_{C_1}$.

□

It is easy to see that if Σ is a convex, closed (in $C_1(\mathsf{H})$-topology) set of symmetrical, non-negative, trace-class operators, then functional $G(A) := \dfrac{1}{2}\sup\limits_{B\in\Sigma}\mathrm{Tr}\big[A\cdot B\big]$ is a G-functional, $A \in K_S(\mathsf{H})$.
Furthermore, we see that between G and Σ is settled a one-to-one correspondence, such that from the one we can get another one: $G \leftrightarrow \Sigma$.

Proposition 2.1. *For the* G*-functional* $G(A) = \dfrac{1}{2}\sup\limits_{B\in\Sigma}\mathrm{Tr}\big[A\cdot B\big]$ *the set* $\Sigma \equiv \ker G^*$.

Proof.
Consider the indicator functional $f(B) := \begin{cases} 0, & B \in \Sigma; \\ \infty, & B \in C_1(\mathsf{H}) \setminus \Sigma. \end{cases}$
According to the topological properties of Σ (closed, convex and nonempty set) it follows that Σ is a proper, convex and lower semicontinuous set.
For every $A \in K_S(\mathsf{H})$ we have

$$G(A) = \frac{1}{2}\sup_{B\in\Sigma}\mathrm{Tr}\big[AB\big] = \frac{1}{2}\sup_{B\in C_1(\mathsf{H})}\Big\{\mathrm{Tr}\big[AB\big] - f(B)\Big\} = \frac{1}{2}f^*(B),$$

where f^* is the Legendre transform of f.

We thus get $2G = f^*$ and hence $2G^* = f^{**} = f$ by the Fenchel-Moreau theorem.
Therefore $\Sigma = \left\{B \in \Sigma \middle| \, G^*(B) = 0\right\} = \ker G^*$.

$\square$

2.2 Some remarks regarding the extension of the G-functional to $L_S(\mathsf{H})$

In general case the extension of G to the space $L_S(\mathsf{H})$ (of linear, bounded and symmetric operators) is not unique. To see it we consider such a functional $\widetilde{G}(A) := \frac{1}{2} \sup_{B \in \Sigma} \operatorname{Tr}\big[A \cdot B\big] + \rho(A)$, $A \in L_S(\mathsf{H})$;
where $\rho(A) = \max_{\lambda \in \sigma_{ess}(A)} \lambda$, the maximal point of essential spectrum $\sigma_{ess}(A)$, which is defined as $\sigma_{ess}(A) = \big\{\lambda \in \sigma(A) \mid \forall \varepsilon > 0 \quad \exists \mu \in \sigma(A) : |\lambda - \mu| < \varepsilon\big\}$.
That is $\sigma_{ess}(A)$ consists of all unisolated points of the spectrum $\sigma(A)$.
If $A \in K_S(\mathsf{H})$ yields that $\rho(A) = 0$,
because in such a case every $\lambda \in \sigma(A) \setminus \{0\}$ is an isolated point,
i.e. $\lambda \notin \sigma_{ess}(A)$ (see [22, Claim 7.6]).
It follows that $\widetilde{G}(A) - G(A) = \rho(A) \cdot \mathbb{1}_{\{A \in L_S(\mathsf{H}) \setminus K_S(\mathsf{H})\}}$.
But in order to to be sure that $\widetilde{G}(A)$ is G-functional indeed we set the following lemma:

Lemma 2.2. $\rho(A)$ *is a continuous, sublinear and monotone functional.*

Proof.
Let $\rho(A) = \lambda_0$ and $\lambda_0 \in \sigma_{ess}(A)$. Since A is self-adjoint then $\|A\| = |\lambda_0|$. Now we perturb operator A by an operator B with a small norm $\|B\| < \varepsilon$. Then $\|A + B\| \leqslant |\lambda_0| + \varepsilon$. It means that essential spectrum A is within the limits $|\lambda_0| + \varepsilon$. What do the continuity of max of essential spectrum A means in fact.
If $\lambda_0 \notin \sigma_{ess}(A)$. Then by the spectral theorem (see [22, IX,Th.2.2]) operator A has the following representation $A = \int_{-\infty}^{+\infty} x \, d\mu(x)$, where μ is a spectral measure.

Let $A_0 = \int\limits_{\{|\lambda_0|\leqslant x\}} x\,d\mu(x)$ and $A_1 = \int\limits_{\{|\lambda_0|>x\}} x\,d\mu(x)$, so that $A = A_0 + A_1$.
Then we have that A_0 is such as described above, and A_1 is compact.
But as we have already seen that the essential spectrum does not change by the compact perturbation, so it follows that $\rho(A) = \rho(A_0)$ which is continuous.

In the same way we deduce the sublinearity of $\rho(A)$, which follows from the sublinearity of the norm.

In order to obtain the monotonicity let $A \geqslant B$.
Assume that the maximum of essential spectrum B is greater than the maximum of essential spectrum A. Therefore that part of the spectrum A which is greater of the maximum of essential spectrum B consists of only with eigenvalues of finite multiplicity. Then we can decompose operator A as follows $A = A_0 + A_1$, where the spectrum A_0 is completely on the left part from the maximum of essential spectrum B, and A_1 is a finite rank operator. Then we can use the Weyl criterion (see [81, Lm.6.17]): there exists an orthonormal sequence $\{x_n\}$, such that $\|Bx_n - \lambda_B x_n\| \xrightarrow[n\to 0]{} 0$, where λ_B is the maximum of essential spectrum B. It follows that $\|Bx_n\| \xrightarrow[n\to 0]{} \lambda_B$. $\qquad (\,*\,)$
Then for such a sequence $\{x_n\}$ we have that $A_1 x_n \xrightarrow[n\to 0]{} 0$, because operator A_1 has only finite number of the eigenvalues of finite multiplicity. Also $\|A_0\| < \lambda_B$, because the spectrum A_0 is on the left from λ_B. Therefore for enough large n we have $\|Ax_n\| < \lambda_B \|x_n\|$. And using $(\,*\,)$ we can conclude that for enough large n $\|Ax_n\| < \|Bx_n\|$, which contradicts that $A \geqslant B$.

□

Now let us call a **canonical extension** of G the following G-functional on $L_S(\mathsf{H})$:
According to **Th.2.2** the given G defines Σ;
For $A \in L_S(\mathsf{H})$ we can define $\overline{G}(A) = \overline{G}_\Sigma(A) := \dfrac{1}{2} \sup\limits_{B\in\Sigma} \mathrm{Tr}\big[A \cdot B\big]$, the canonical extension of G.

2.3 Sublinear expectation

In this chapter we discuss the notion of sublinear expectation. The material mainly was taken from [72], which we have generalized to infinite dimensional case.
Let $(\Omega, \mathcal{F}, P)$ is an ordinary probability space and $(\mathsf{X}, \|\cdot\|_{\mathsf{X}})$, $(\mathsf{Y}, \|\cdot\|_{\mathsf{Y}})$ are normed space.
Let us define the class of Lipschitz functions with a polynomial growth as follows:

$$\mathbf{C}_{p.Lip}(\mathsf{X}, \mathsf{Y}) := \Big\{ \varphi : \mathsf{X} \to \mathsf{Y} \;\Big|\; \|\varphi(x) - \varphi(y)\|_{\mathsf{Y}} \leqslant C \cdot \big(1 + \|x\|_{\mathsf{X}}^m + \|y\|_{\mathsf{X}}^m\big) \cdot \|x - y\|_{\mathsf{X}} \Big\}.$$

$$\mathbf{C}_{p.Lip}(\mathsf{X}) := \mathbf{C}_{p.Lip}(\mathsf{X}, \mathbb{R}).$$

Definition 2.3. *We define the class $\mathcal{H}^0$ to be a linear space that satisfies the following conditions:*
1) If $c \in \mathbb{R}$ then $c \in \mathcal{H}^0$;
2) Each $\xi \in \mathcal{H}^0$ is a random variable on a $(\Omega, \mathcal{F}, \mathbb{P})$;
3) If $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}^0$ then $\varphi(\xi_1, \xi_2, \dots, \xi_n) \in \mathcal{H}^0$ for every $\varphi \in \mathbf{C}_{p.Lip}(\mathbb{R}^n)$.

Definition 2.4. *Let us set*
$\mathcal{H} := \big\{ X : \Omega \to \mathsf{X}$ *– r.v. on* $(\Omega, \mathcal{F}, \mathbb{P}) \mid \psi(X) \in \mathcal{H}^0 \;\; \forall \psi \in \mathbf{C}_{p.Lip}(\mathsf{X}) \big\}$.

Remark 2.3. *If $X \in \mathcal{H}$ then $\|X\|^m \in \mathcal{H}^0$, $m \geqslant 1$*

Proof.
The proof follows from the following elementary inequality:

$$\Big| \|X\|^m - \|Y\|^m \Big| \leqslant \Big| \|X\| - \|Y\| \Big| \cdot C \Big(1 + \|X\|^{m-1} + \|Y\|^{m-1}\Big).$$

□

Definition 2.5. *A functional $\mathbb{E} : \mathcal{H}^0 \to \mathbb{R}$ is called a* ***sublinear expectation*** *if it satisfies the following conditions:*
1) ***Monotonicity****: if $X \geqslant Y$ then $\mathbb{E}[X] \geqslant \mathbb{E}[Y]$;*
2) ***Constant preserving****: $c \in \mathbb{R}$ then $\mathbb{E}[c] = c$;*
3) ***Sub-additivity****: $\mathbb{E}[X + Y] \leqslant \mathbb{E}[X] + \mathbb{E}[Y]$;*
4) ***Positive homogeneity****: $\lambda \geqslant 0$ then $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.*

A triple $(\Omega, \mathcal{H}, \mathbb{E})$ we shall call a **sublinear expectation space**.

Definition 2.6. *A functional* $\mathbf{F}_X[\varphi] := \mathbb{E}[\varphi(X)]$ *is said to be a* ***distribution*** *of* $X \in \mathcal{H}$, *for* $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{X})$.

Proposition 2.2. *(Some elementary properties of* $\mathbb{E}$*)*

1) $\mathbb{E}[\alpha X] = \alpha^+\mathbb{E}[X] + \alpha^-\mathbb{E}[-X], \quad \Big(\alpha := \alpha^+ - \alpha^-\Big)$;

2.a) $-\mathbb{E}[X] \leqslant \mathbb{E}[-X]$;

2.b) $\big|\mathbb{E}[X]\big| \leqslant \mathbb{E}[\big|X\big|]$;

2.c) $\mathbb{E}[Z] - \mathbb{E}[X] \leqslant \mathbb{E}[Z - X]$;

2.d) $\big|\mathbb{E}[Z] - \mathbb{E}[X]\big| \leqslant \big|\mathbb{E}[Z - X]\big| \leqslant \mathbb{E}[\big|Z - X\big|]$;

3.a) $X \geqslant 0 \Rightarrow \ \mathbb{E}[X] \geqslant 0$,

3.b) $X \leqslant 0 \Rightarrow \ \mathbb{E}[X] \leqslant 0$;

4) $\lambda \leqslant 0 \quad \Rightarrow \quad \mathbb{E}[\lambda X] \geqslant \lambda\mathbb{E}[X]$;

5.a) $c \in \mathbb{R} \quad \Rightarrow \quad \mathbb{E}[X + c] = \mathbb{E}[X] + c$;

5.b) $Y : \mathbb{E}[Y] = \mathbb{E}[-Y] = 0 \quad \Rightarrow \quad \mathbb{E}[X + Y] = \mathbb{E}[X]$;

5.c) $Y : \mathbb{E}[Y] = \mathbb{E}[-Y] \quad \Rightarrow \quad \mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha\,\mathbb{E}[Y]$;

Proof.
All these properties are trivial consequences of the definition of the sublinear expectation, but we put some notes to be more precise.

1) Obvious.

2) Follows from **Def.2.5, 3)**;

And $\big|\mathbb{E}[X]\big| \leqslant -\mathbb{E}[-X] \leqslant \mathbb{E}[X] \leqslant \mathbb{E}[\big|X\big|]$.

3) Follows from **Def.2.5, 1)**.

4) $\mathbb{E}[\lambda X] \overset{\textbf{Prop.2.2, 2)}}{\geqslant} -\mathbb{E}[-\lambda X] \overset{\textbf{Def.2.5, 4)}}{=} \lambda\mathbb{E}[X]$.

5), (c) $\forall\alpha \ \ \mathbb{E}[\alpha Y] \overset{\textbf{Prop.2.2, 4), Def.2.5, 4)}}{\geqslant} \alpha\,\mathbb{E}[Y] \overset{\textbf{Prop.2.2, 2)}}{\geqslant} -\alpha\,\mathbb{E}[-Y]$;

$\mathbb{E}[\alpha Y] = \alpha^+\mathbb{E}[Y] + \alpha^-\mathbb{E}[-Y] = \alpha^+\mathbb{E}[Y] + \alpha^-\mathbb{E}[Y] = \alpha\,E[Y]$;

$$\mathbb{E}[X + \alpha Y] \leqslant \mathbb{E}[X] + \mathbb{E}[\alpha] = \mathbb{E}[X] + \alpha\mathbb{E}[Y] \leqslant \mathbb{E}[X] - \alpha\mathbb{E}[-Y] \leqslant \mathbb{E}[X + \alpha Y];$$

So that $\mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha\,\mathbb{E}[Y]$.

□

Remark 2.4. *Note the following false implications for a sublinear expectation:*

$$\mathbb{E}[X + Y] = \mathbb{E}[X] \quad \not\Rightarrow \quad \mathbb{E}[Y] = 0 \quad \not\Rightarrow \quad \mathbb{E}[-Y] = 0.$$

Proposition 2.3 (Cauchy–Bunyakovsky–Schwarz inequality).
Let $X, Y \in \mathcal{H}^0$, *then*

$$\mathbb{E}\big[XY\big] \leqslant \Big(\mathbb{E}\big[X^2\big]\cdot\mathbb{E}\big[Y^2\big]\Big)^{\frac{1}{2}}.$$

Proof.
The proof is trivial and based on the classical CBS inequality and on representation theorem for sublinear functional **Th.2.1**:

$$\mathbb{E}\big[XY\big] = \sup_{\theta\in\Theta} E_\theta\big[XY\big] \leqslant \sup_{\theta\in\Theta}\Big(\mathbb{E}_\theta\big[X^2\big]\cdot\mathbb{E}_\theta\big[Y^2\big]\Big)^{\frac{1}{2}}$$

$$= \left\{\sup_{\theta\in\Theta}\Big(\mathbb{E}_\theta\big[X^2\big]\cdot\mathbb{E}_\theta\big[Y^2\big]\Big)\right\}^{\frac{1}{2}} \leqslant \left\{\sup_{\theta_1\in\Theta}\mathbb{E}_{\theta_1}\big[X^2\big]\cdot\sup_{\theta_2\in\Theta}\mathbb{E}_{\theta_2}\big[Y^2\big]\right\}^{\frac{1}{2}}$$

$$= \Big(\mathbb{E}\big[X^2\big]\cdot\mathbb{E}\big[Y^2\big]\Big)^{\frac{1}{2}}.$$

□

2.4 G-normal distribution

We again refer us to Peng [72] in order to introduce a notion of the G-normal distribution. All the definitions can be carried just from the 1-dimensional case to the infinite dimensional case.
Note that in this chapter and later on we imply that all the used random variables are defined on the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$.

Definition 2.7.
We say that random variables X *and* Y *have* ***identical distribution*** *and denote* $X \sim Y$ *if their distributions coincide,*
i.e., for every $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{H})$ $\quad \mathbb{E}\big[\varphi(X)\big] = \mathbb{E}\big[\varphi(Y)\big]$.

We say that random variables Y *is* ***independent*** *from random variable* X *and denote* $Y \perp\!\!\!\perp X$ *if they satisfy the following equality:*

$$\mathbb{E}\big[\varphi(X,Y)\big] = \mathbb{E}\Big[\mathbb{E}\big[\varphi(x,Y)\big]_{x=X}\Big], \quad \varphi \in \mathbf{C}_{p.Lip}(\mathsf{H}\times\mathsf{H}).$$

Remark 2.5. *Peng has already mentioned that* $Y \perp\!\!\!\perp X$ *does not imply that* $X \perp\!\!\!\perp Y$ *(see [72, Rem.3.12]).*

In the our turn when $Y \perp X$ we put some obvious properties which we shall use later:

1) $\mathbb{E}[\varphi_1(X) + \varphi_2(Y)] = \mathbb{E}[\varphi_1(X)] + \mathbb{E}[\varphi_2(Y)], \quad \varphi \in \mathbf{C}_{p.Lip}(\mathsf{H})$.

2) $\mathbb{E}[XY] = \mathbb{E}\Big[\mathbb{E}[x, Y]_{x=X}\Big] = \mathbb{E}\Big[X^+ \cdot \mathbb{E}[Y] + X^- \cdot \mathbb{E}[-Y]\Big]$
$= \mathbb{E}[X^+] \cdot \mathbb{E}[Y]^+ + \mathbb{E}[-X^+] \cdot \mathbb{E}[Y]^- + \mathbb{E}[X^-] \cdot \mathbb{E}[-Y]^+ + \mathbb{E}[-X^-] \cdot \mathbb{E}[-Y]^-$.

Definition 2.8. *Random variable X on the $(\Omega, \mathcal{H}, \mathbb{E})$ is said to be* **G*-normal distributed*** *if for every $\bar{X}$ which is and independent copy of X (i.e. has identical distribution):*

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}\; X, \text{ where } a, b > 0.$$

Let us consider a G-functional defined on the $K_S(\mathsf{H})$ in such a way:

$$G(A) := \frac{1}{2}\,\mathbb{E}\big[\langle AX, X\rangle\big] \tag{7}$$

In fact, it is clear that the sublinear expectation provides the fulfillment of all G-functional's properties.
Assume that X is a G-normal distributed random variable with respect to the sublinear expectation $\mathbb{E}$, and $G(\cdot)$ is defined in (7).
Later (**Th.4.2**) we will see that for every G-functional $\widetilde{G}(\cdot)$ there exists G-normal distributed random variable $\widetilde{X}$ and a sublinear expectation $\widetilde{\mathbb{E}}$, such that $\widetilde{G}(A) = \frac{1}{2}\,\widetilde{\mathbb{E}}\big[\langle A\widetilde{X}, \widetilde{X}\rangle\big]$.
In view of such fact we shortly make a following notation

$$X \sim N_G(0, \Sigma).$$

According to **Prop.2.1** G defines a Σ-set, which we will call a **covariance set**, about what we will discuss more precisely in the chapter 2.5.

Remark 2.6. *If random variables have the same distribution then their G-functionals coincide, i.e.: if $X \sim Y \sim N_G(0, \Sigma)$ then $G_X = G_Y$.*

Proof.
$G_X(A) = \frac{1}{2} \sup_{B \in \Sigma} \operatorname{Tr}[A \cdot B] = G_Y(A)$.

□

Proposition 2.4. *For a G-normal distributed random variable $X \sim N_G(0,\Sigma)$ we have the following estimation of the moments for $m \geqslant 1$:*

$$c_m \cdot \sup_{Q\in\Sigma} \mathrm{Tr}\big[Q^m\big] \leqslant \mathbb{E}\big[\|X\|_{\mathsf{H}}^{2m}\big] \leqslant C_m \cdot \sup_{Q\in\Sigma}\Big(\mathrm{Tr}\big[Q\big]\Big)^m ,$$

where c_m are real constants dependent just of m.
In particular, if $m=1$ then $c_m = 1$ and $\mathbb{E}\big[\|X\|_{\mathsf{H}}^2\big] = \sup_{Q\in\Sigma}\mathrm{Tr}\big[Q\big]$.

Proof.
Fix $Q\in\Sigma$ and let $\{\lambda_i\,,\ i\geqslant 1\}$ be eigenvalues of Q.
Consider a gaussian measure $\mu := N_Q$.

Then according to **Th.2.1** we have

$$\mathbb{E}\big[\|X\|_{\mathsf{H}}^{2m}\big] = \sup_{Q\in\Sigma} E_Q\big[\|X\|_{\mathsf{H}}^{2m}\big] = \sup_{Q\in\Sigma}\int_{\mathsf{H}} |x|^{2m}\mu(dx)\,. \qquad (\,*\,)$$

Define $J_m := \int_{\mathsf{H}} |x|^{2m}\mu(dx)$ and $F(\varepsilon) := \int_{\mathsf{H}} e^{\frac{\varepsilon}{2}|x|^2}\mu(dx)$.

Then $F^{(m)}(\varepsilon) = \int_{\mathsf{H}} e^{\frac{\varepsilon}{2}|x|^2}\cdot\frac{|x|^{2m}}{2^m}\mu(dx)$.

And therefore $J_m = 2^m F^{(m)}(0)$. $\qquad (\,**\,)$

Take ε small, namely $\varepsilon < \min_k \frac{1}{\lambda_k}$, so we have:

$$F(\varepsilon) = \int_{\mathsf{H}} e^{\frac{\varepsilon}{2}|x|^2}\mu(dx) = \prod_{k\geqslant 1}\int_{-\infty}^{+\infty} e^{\frac{\varepsilon}{2}x_k^2}\, N_{\lambda_k}(dx) = \prod_{k\geqslant 1}\int_{-\infty}^{+\infty} e^{\frac{\varepsilon}{2}x_k^2}\frac{1}{\sqrt{2\pi\lambda_k}}\, e^{\frac{-x_k^2}{2\lambda_k}}\, dx$$

$$= \prod_{k\geqslant 1}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\lambda_k}} e^{-x_k^2/\left(\frac{2\lambda_k}{1-\varepsilon_k\lambda_k}\right)}\, dx \overset{\text{Euler-Poisson integral}}{=} \prod_{k\geqslant 1}\frac{1}{\sqrt{2\pi\lambda_k}}\sqrt{\frac{2\lambda_k}{1-\varepsilon_k\lambda_k}}\sqrt{\pi}$$

$$= \prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k)^{-\frac12} = \Big(\prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k)\Big)^{-\frac12} = \Big[\det(1-\varepsilon Q)\Big]^{-\frac12}.$$

Recall the following elementary formula:

$$\Big(\prod_{k\geqslant 1} f_k\Big)' = f_1'\prod_{k\geqslant 2} f_k + f_1\Big(\prod_{k\geqslant 2} f_k\Big)' = \frac{f_1'}{f_1}\prod_{k\geqslant 1} f_k + f_1 f_2'\prod_{k\geqslant 3} f_k + f_1 f_2\Big(\prod_{k\geqslant 1} f_k\Big)'$$

$$= \frac{f_1'}{f_1}\prod_{k\geqslant 1} f_k + \frac{f_2'}{f_2}\prod_{k\geqslant 1} f_k + \frac{f_3'}{f_3}\prod_{k\geqslant 1} f_k + \ldots = \prod_{k\geqslant 1} f_k\cdot\sum_{k\geqslant 1}\frac{f_k'}{f_k}\,.$$

Therefore $F'(\varepsilon) = \left(\left(\prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k)\right)^{-\frac{1}{2}}\right)'$

$$= -\frac{1}{2}\left(\prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k)\right)^{-\frac{3}{2}} \cdot \prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k) \cdot \sum_{k\geqslant 1}\frac{-\lambda_k}{1-\varepsilon_k\lambda_k}$$

$$= \frac{1}{2}\left(\prod_{k\geqslant 1}(1-\varepsilon_k\lambda_k)\right)^{-\frac{1}{2}} \cdot \sum_{k\geqslant 1}\underbrace{\frac{\lambda_k}{1-\varepsilon_k\lambda_k}}_{=:g_k(\varepsilon)} = \frac{1}{2}\sum_{k\geqslant 1}F(\varepsilon)g_k(\varepsilon)\,.$$

We define $G(\varepsilon) := \sum_{k\geqslant 1}F(\varepsilon)g_k(\varepsilon)\,.$

Then let us compute the m-th derivative of G, we have

$$G^{(m)}(\varepsilon) := \sum_{k\geqslant 1}\big(F(\varepsilon)g_k(\varepsilon)\big)^{(m)} = \sum_{k\geqslant 1}\sum_{j=0}^{m}\binom{m}{j}F^{(j)}(\varepsilon)g_k^{(m-j)}(\varepsilon)$$

$$= \sum_{k\geqslant 1}\sum_{j=0}^{m}\frac{m!}{(m-j)!j!}\,(m-j)!\,F^{(j)}(\varepsilon)\,g_k^{m+1-j}(\varepsilon)\,,$$

since $g_k^{(p)}(\varepsilon) = \left(\frac{\lambda_k}{1-\varepsilon_k\lambda_k}\right)^{(p)} = \left(\lambda_k\cdot\frac{\lambda_k}{(1-\varepsilon_k\lambda_k)^2}\right)^{(p-1)}$

$= \left(2\lambda_k\cdot\frac{\lambda_k^2}{(1-\varepsilon_k\lambda_k)^3}\right)^{(p-2)} = \ldots = p!\cdot\frac{\lambda_k^{p+1}}{(1-\varepsilon_k\lambda_k)^{p+1}} = p!\cdot g_k^{p+1}(\varepsilon)\,,$

then $F^{(m)}(0) = \frac{1}{2}\sum_{k\geqslant 1}\sum_{j=0}^{m-1}\frac{(m-1)!}{j!}\,F^{(j)}(0)\,g_k^{m-j}(0)$

$$= \frac{(m-1)!}{2}\cdot\sum_{j=0}^{m-1}\frac{F^{(j)}(0)}{j!}\cdot\sum_{k\geqslant 1}\lambda_k^{m-j} = \frac{(m-1)!}{2}\cdot\sum_{j=0}^{m-1}\frac{F^{(j)}(0)}{j!}\cdot\mathrm{Tr}\big[Q^{m-j}\big]\,.$$

Since $F(0)=1$ it follows that $F^{(1)}(0) = \frac{1}{2}\cdot\mathrm{Tr}\big[Q\big]$.

Hence $c_1\cdot\mathrm{Tr}\big[Q^1\big] \leqslant F^{(1)}(0) \leqslant c_1\cdot\mathrm{Tr}\big[Q\big]^1$.

To finish the proof we use the induction method:
Let for all $j<m$ required estimation $c_j\cdot\mathrm{Tr}\big[Q^j\big] \leqslant F^{(1)}(0) \leqslant c_j\cdot\mathrm{Tr}\big[Q\big]^j$ holds.

Since $\mathrm{Tr}\big[Q^m\big] \leqslant \mathrm{Tr}\big[Q^j\big]\mathrm{Tr}\big[Q^{m-j}\big] \leqslant \mathrm{Tr}\big[Q\big]^j\mathrm{Tr}\big[Q^{m-j}\big] \leqslant \mathrm{Tr}\big[Q\big]^m$,
we have

$$c_m\cdot\mathrm{Tr}\big[Q^m\big] \leqslant F^{(m)}(0) \leqslant c_m\cdot\mathrm{Tr}\big[Q\big]^m.$$

But according to $(\,*\,)$ and $(\,**\,)$ this gives us what we need.

□

Remark 2.7. *If we consider a canonical extension on $L_S(\mathsf{H})$ for a such particular $G(\cdot) = \frac{1}{2}\mathbb{E}\big[\langle \cdot X, X\rangle\big]$, there emerges a question if we are able to show that such G satisfies the representation given in **Th.2.2**, i.e. $\exists$ certain set Σ, s.t.* $\quad G(A) = \frac{1}{2}\sup_{B\in\Sigma}\mathrm{Tr}\big[A\cdot B\big], \quad A \in L_S(\mathsf{H})$ *?*

According to (**2.2**) in the general case we do not have uniqueness.
But, it is clear that for every $A \in L_S(\mathsf{H})$ there exist $A_n \in K_S(\mathsf{H})$ such that $\Big|\langle AX, X\rangle - \langle A_nX, X\rangle\Big| \xrightarrow[n\to\infty]{} 0$. $\qquad (\,*\,)$

To show this we take a projection operator $P_n x := \sum_{i=1}^{n}\langle x, e_i\rangle e_i$,
where $x \in \mathsf{H}$ and (e_i) is a basis of H.
Taking $A_n := \dfrac{AP_n + P_nA}{2}$ which is actually compact (since has a finite range) and symmetric.

And $(\,*\,)$ may be concluded from the following convergence and Cauchy-Bunyakovsky-Schwarz inequality:

$$\|Ax - A_nx\|_{\mathsf{H}} \leqslant \frac{1}{2}\Big(\|Ax - AP_nx\|_{\mathsf{H}} + \|Ax - P_nAx\|_{\mathsf{H}}\Big)$$
$$\leqslant \frac{1}{2}\Big(\|A\|_{L(\mathsf{H})}\cdot\|x - P_nx\|_{\mathsf{H}} + \|Ax - P_nAx\|_{\mathsf{H}}\Big)$$
$$\leqslant \frac{1}{2}\Big(\|A\|_{L(\mathsf{H})}\cdot\underbrace{\Big\|\sum_{i=n+1}^{\infty}\langle x, e_i\rangle e_i\Big\|_{\mathsf{H}}}_{\searrow 0} + \underbrace{\Big\|\sum_{i=n+1}^{\infty}\langle Ax, e_i\rangle e_i\Big\|_{\mathsf{H}}}_{\searrow 0}\Big) \xrightarrow[n\to\infty]{} 0.$$

So, the question remains just to understand under which conditions the following expression tends to zero:

$$\Big|\mathbb{E}\big[\langle AX, X\rangle\big] - \sup_{B\in\Sigma}\mathrm{Tr}\big[A_n\cdot B\big]\Big| = \Big|\mathbb{E}\big[\langle AX, X\rangle\big] - \mathbb{E}\big[\langle A_nX, X\rangle\big]\Big|$$
$$\leqslant \mathbb{E}\Big[\Big|\langle AX, X\rangle - \langle A_nX, X\rangle\Big|\Big] \xrightarrow[n\to\infty]{?} 0.$$

2.5 Covariance set under sublinear expectation

Let us describe the notion of the covariance operators for a random variable $X \sim N_G(0, \Sigma)$ under the sublinear expectation $\mathbb{E}$. Actually, we will see that Σ is a set of operators.

If $\mathbb{E}$ is linear (denote it as E), then:
$Cov(X) = Q$, where Q is defined as: $\langle Qh, k\rangle = E[\langle X, h\rangle\langle X, k\rangle]$.
Now we fix the family of linear functionals $\Big\{E_\theta,\ \theta \in \Theta\Big\}$ for given sublinear functional $\mathbb{E}$, and let $\{e_i, i \geqslant 1\}$ be a basis in H. Then we have:

$$\mathbb{E}\Big[\langle AX, X\rangle\Big] = \mathbb{E}\Big[\sum_{i\geqslant 1}\langle AX, e_i\rangle\langle X, e_i\rangle\Big] = \sup_{\theta\in\Theta} E_\theta\Big[\sum_{i\geqslant 1}\langle AX, e_i\rangle\langle X, e_i\rangle\Big]$$

We can change the order of integration in the last term. For check it we formulate the following lemma.

Lemma 2.3. *For A a linear bounded operator and square-integrable r.v. X under the linear expectation E, i.e. $E\Big[\|X\|^2_{\mathsf{H}}\Big] < \infty$, the following equality holds:*

$$E\Big[\sum_{i\geqslant 1}\langle AX, e_i\rangle\langle X, e_i\rangle\Big] = \sum_{i\geqslant 1} E\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big],$$

where $\{e_i, i \geqslant 1\}$ is a basis in Hilbert space H.

Proof.
We know that we can change the finite sums with a linear expectation, i.e.
$E\Big[\sum_{i=1}^{N}\langle AX, e_i\rangle\langle X, e_i\rangle\Big] = \sum_{i=1}^{N} E\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big]$.
So, we need to take just limits when $N \to \infty$ from the left and right part respectively, and see that they coincide.

(a) Define S_N as a partial sum of the given series.
$S_N := \sum_{i=1}^{N}\langle AX, e_i\rangle\langle X, e_i\rangle$.

We have that for every $\omega \in \Omega \quad S_N \xrightarrow[N\to\infty]{} \sum_{i=1}^{\infty}\langle AX, e_i\rangle\langle X, e_i\rangle = \langle AX, X\rangle$.

Take the projection operator $P_N := Proj(e_1, .., e_N)$, that is it is noting else but $P_N = P_N^* = P_N^2$ and $\|P_N\|_{\mathsf{H}} = 1$.

Then we have

$$S_N = \sum_{i=1}^{N}\langle AX, e_i\rangle\langle X, e_i\rangle = \sum_{i=1}^{N}\langle AX, P_N e_i\rangle\langle X, P_N e_i\rangle = \sum_{i=1}^{\infty}\langle AX, P_N e_i\rangle\langle X, P_N e_i\rangle$$
$$= \sum_{i=1}^{\infty}\langle P_N AX, e_i\rangle\langle P_N X, e_i\rangle = \langle P_N AX, P_N X\rangle = \langle P_N^2 AX, X\rangle = \langle P_N AX, X\rangle.$$

And we get that $|S_N| \leqslant \|P_N\|_{L(\mathsf{H})}\cdot\|A\|_{L(\mathsf{H})}\cdot\|X\|_{\mathsf{H}}^2 = \|A\|_{L(\mathsf{H})}\cdot\|X\|_{\mathsf{H}}^2 \in L^1(P)$, where P is a probability for the integral that E.

And it follows by the dominated convergence theorem that

$$E\big[S_N\big] \xrightarrow[N\to\infty]{} E\big[\langle AX, X\rangle\big].$$

(b) Let Q be a covariance set of operators of X (Q is a trace-class operator), then $\sum_{i=1}^{N} E\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big] = \sum_{i=1}^{N}\langle QA^* e_i, e_i\rangle$.

The last term has a limit $\sum_{i=1}^{\infty}\langle QA^* e_i, e_i\rangle \equiv \mathrm{Tr}\big[QA^*\big]$,

because $\sum_{i=1}^{\infty}\big|\langle QA^* e_i, e_i\rangle\big| < \infty$, since QA^* is also a trace-class operator.

So that,

$$\sum_{i=1}^{N} E\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big] \xrightarrow[N\to\infty]{} \sum_{i=1}^{\infty}\langle QA^* e_i, e_i\rangle = \sum_{i=1}^{\infty} E\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big].$$

□

Surely, we can use **Lm.2.3** in our case, because by **Prop.2.4** and **Th.2.1** we get that

$$\infty > \mathbb{E}\big[\|X\|_{\mathsf{H}}^2\big] = \sup_{\theta\in\Theta} E_\theta\big[\|X\|_{\mathsf{H}}^2\big].$$

It means that,

$$\mathbb{E}\big[\langle AX, X\rangle\big] = \sup_{\theta\in\Theta}\sum_{i\geqslant 1} E_\theta\Big[\langle AX, e_i\rangle\langle X, e_i\rangle\Big] = \sup_{\theta\in\Theta}\sum_{i\geqslant 1}\langle Q_\theta A e_i, e_i\rangle$$
$$= \sup_{\theta\in\Theta}\ \mathrm{Tr}[Q_\theta\cdot A].$$

In the same manner we get that

$$\mathbb{E}\big[\langle X, h\rangle\langle X, k\rangle\big] = \mathbb{E}\Big[\sum_{i\geqslant 1}\langle X, e_i\rangle\langle h, e_i\rangle\sum_{j\geqslant 1}\langle X, e_j\rangle\langle k, e_j\rangle\Big]$$
$$= \sup_{\theta\in\Theta}\ E_\theta\Big[\sum_{i,j\geqslant 1}\langle X, e_i\rangle\langle h, e_i\rangle\langle X, e_j\rangle\langle k, e_j\rangle\Big]$$

Using the same idea as in the proof of **Lm.2.3** permits us also to change the integration sums and we obtain that

$$\mathbb{E}\Big[\langle X,h\rangle\langle X,k\rangle\Big] = \sup_{\theta\in\Theta}\sum_{i,j\geqslant 1} E_\theta\Big[\langle X,e_i\rangle\langle X,e_j\rangle\Big]\langle h,e_i\rangle\langle k,e_j\rangle$$
$$= \sup_{\theta\in\Theta}\sum_{i,j\geqslant 1}\langle Q_\theta e_i,e_j\rangle\langle h,e_i\rangle\langle k,e_j\rangle = \sup_{\theta\in\Theta}\langle Q_\theta h,k\rangle.$$

So, we have that

$$\Sigma := \{Q_\theta - \text{covariation of } X \text{ under } E_\theta\ ,\ \theta\in\Theta\} = Cov(X) \tag{8}$$

$$\boxed{\mathbb{E}\Big[\langle X,h\rangle\langle X,k\rangle\Big] = \sup_{Q\in\Sigma}\langle Qh,k\rangle} \tag{9}$$

Remark 2.8. *In order to better understand the nature of the covariance set of operators under sublinear expectation, we list the following obvious properties:*

1) $\mathbb{E}\Big[-\langle AX,X\rangle\Big] = \sup_{Q\in\Sigma}\mathrm{Tr}[-AQ] = \sup_{Q\in\Sigma^-}\mathrm{Tr}[AQ]\,,$

where $\Sigma^- := \Big\{-A\Big|\ A\in\Sigma\Big\}\,.$

2) $\mathbb{E}\Big[-\langle AX,X\rangle\Big] \geqslant -\mathbb{E}\Big[\langle AX,X\rangle\Big]\,.$

3) $\mathbb{E}\Big[-\langle X,h\rangle\langle X,k\rangle\Big] = \sup_{Q\in\Sigma}\langle -Qh,k\rangle = \sup_{Q\in\Sigma^-}\langle Qh,k\rangle\,.$

4) $X_1\sim N_G(0,\Sigma),\ X_2\sim N_G(0,\Sigma^-) \quad\Rightarrow\quad G_{X_1}(A) = G_{X_2}(-A)\,.$

Also we can show that one-dimensional projection of the G-normal distributed random variable in the Hilbert space is also G-normal distributed.

Proposition 2.5. *Let X be a G-normal distributed random variable in the Hilbert space H, then for every $h\in\mathsf{H}$ $\langle X,h\rangle$ is G_h-normal distributed, where* $G_h(\alpha) = \dfrac{1}{2}\big(\alpha^+\,\overline{\sigma}^2(h) - \alpha^-\,\underline{\sigma}^2(h)\big)$;

$\overline{\sigma}^2(h) = \mathbb{E}[\langle X,h\rangle^2] = 2G(h\cdot h^T),$

$\underline{\sigma}^2(h) = -\mathbb{E}[-\langle X,h\rangle^2] = -2G(-h\cdot h^T)\,.$

So, we keep the notation and can write that $X\sim N_G(0,[\underline{\sigma}^2(h),\overline{\sigma}^2(h)])\,.$

Proof.
We use the definition of the one-dimensional G-normal distribution given by Peng (see[72, 2.1]), which actually we have generalized to infinite-dimensional case. So, by the definition we have

$$a\langle X,h\rangle + b\langle \bar{X},h\rangle = \langle aX + b\bar{X},h\rangle \sim \langle \sqrt{a^2+b^2}\,X,h\rangle = \sqrt{a^2+b^2}\,\langle X,h\rangle .$$

It follows that $\langle X,h\rangle$ is G-normal distributed.

Let us compute 1-dimensional G-functional for $\langle X,h\rangle$:

$$G_h(\alpha) = \frac{1}{2}\,\mathbb{E}\Big[\alpha\langle h,X\rangle^2\Big] = \frac{1}{2}\left(\alpha^+\mathbb{E}\Big[\langle X,h\rangle^2\Big] - \alpha^-\mathbb{E}\Big[-\langle X,h\rangle^2\Big]\right)$$
$$= \frac{1}{2}\Big(\alpha^+\,\overline{\sigma}^2(h) - \alpha^-\,\underline{\sigma}^2(h)\Big).$$

And $\quad 2\,G(h\cdot h^T) = \mathbb{E}\langle h\cdot h^T\cdot X,X\rangle = \mathbb{E}\left\{\sum\limits_{i\geqslant 1}\Big[\big(\sum\limits_{j\geqslant 1} h_ih_jX_j\big)X_i\Big]\right\}$

$$= \mathbb{E}\Big[\sum_{i\geqslant 1}(h_iX_i)^2\Big] = \mathbb{E}\Big[\langle X,h\rangle^2\Big].$$

□

Remark 2.9. *We also recall the following obvious fact settled in the finite-dimensions:*

if $X\in\mathcal{H}^0$ *then* $\mathbb{E}[X^2] := \overline{\sigma}^2 \geqslant \underline{\sigma}^2 =: -\mathbb{E}[-X^2] \geqslant 0\,.$

Some algebraical properties of the G-normal distributed random variables are listed below.

Proposition 2.6.
1) Let $X\sim N_G(0,\Sigma)$ *then* $aX\sim N_G(0,a^2\Sigma)\,,\ a\in\mathbb{R}\,;$
2) Let $Y = X_1+X_2$ *, where* $X_i\sim N_G(0,\Sigma_i)$ *are reciprocally independent, then* $Y\sim N_G(0,\Sigma_Y)$ *with a covariance set*

$$\Sigma_Y := \big\{Q_1+Q_2\big|\,Q_i\in\Sigma_i\,,\ i=\overline{1,2}\big\}\,;$$

3) Let $Z = SX$ *,* $S\in L(\mathsf{U},\mathsf{H})$*,* $Z\in\mathsf{H}$*,* $X\in\mathsf{U}$*,* $X\sim N_G(0,\Sigma)$ *then* $Z\sim N_G(0,\Sigma_Z)$ *with a covariance set* $\Sigma_Z := \big\{SQS^*\big|\,Q\in\Sigma\big\}$.

Proof.
1) By **Def.2.8** it follows that aX is G-normal distributed.
Also it is clear that:
$G_X(A) = \dfrac{1}{a^2} G_{aX}(A) = \dfrac{1}{2} \sup\limits_{Q \in \Sigma} \mathrm{Tr}[A \cdot Q]$.
From the other hand we have:
$G_{aX}(A) = \dfrac{1}{2} a^2 \sup\limits_{Q \in \Sigma} \mathrm{Tr}[A \cdot Q] = \dfrac{1}{2} \sup\limits_{Q \in \Sigma} \mathrm{Tr}[A(a^2 Q)] = \dfrac{1}{2} \sup\limits_{Q_1 \in a^2 \Sigma} \mathrm{Tr}[A \cdot Q_1]$.
So, we can conclude that $Cov(aX) \equiv a^2 \Sigma$.

2) By **Def.2.8** we get that random variable Y is G-normal distributed, because:
$a(X_1 + X_2) + b(\bar{X}_1 + \bar{X}_2) \sim \sqrt{a^2 + b^2}\ (X_1 + X_2)$.
Also AX_1 and X_2 are independent random variables.
In fact, let $\varphi(Ax, y) = \widetilde{\varphi}(x, y)$, so we have that
$$\mathbb{E}\big[\varphi(AX_1, X_2)\big] = \mathbb{E}\big[, \widetilde{\varphi}(X_1, X_2)\big] = \mathbb{E}\Big[\mathbb{E}\big[\widetilde{\varphi}(X_1, x_2)\big]_{x_2 = X_2}\Big]$$
$$= \mathbb{E}\Big[\mathbb{E}\big[\varphi(AX_1, x_2)\big]_{x_2 = X_2}\Big].$$
In the same way we can conclude that AX_2 and X_1 are also independent.
Now we compute the G-functional for the random variable Y:
$$G_Y(A) = G_{X_1 + X_2}(A) = \frac{1}{2} \mathbb{E}\langle A(X_1 + X_2), (X_1 + X_2)\rangle$$
$$= \frac{1}{2} \mathbb{E}\Big[\langle AX_1, X_1\rangle + \langle AX_2, X_2\rangle + \langle AX_1, X_2\rangle + \langle AX_2, X_1\rangle\Big]$$
$$= \frac{1}{2} \mathbb{E}\Big[\langle AX_1, X_1\rangle + \langle AX_2, X_2\rangle\Big] \overset{\textbf{Rem.2.5}}{=} G_{X_1}(A) + G_{X_2}(A).$$
Hence,
$$\sup_{Q \in \Sigma} \mathrm{Tr}[A \cdot Q] = 2 G_{X_1 + X_2}(A) = 2(G_{X_1}(A) + G_{X_2}(A))$$
$$= \sup_{Q_1 \in \Sigma_1} \mathrm{Tr}[A \cdot Q_1] + \sup_{Q_2 \in \Sigma_2} \mathrm{Tr}[A \cdot Q_2] = \sup_{\substack{Q_1 \in \Sigma_1 \\ Q_2 \in \Sigma_2}} \mathrm{Tr}\big[A(Q_1 + Q_2)\big] = \sup_{Q \in \Sigma_Y} \mathrm{Tr}\big[AQ\big].$$
And it means that $Y \sim N_G(0, \Sigma_Y)$.

3) By **Def.2.8** we obtain that random variable Z is G-normal distributed, because:
$a(SX) + b(S\bar{X}) \sim \sqrt{a^2 + b^2}\ (SX)$.

Then let us compute a G-functional for the random variable Z:

$$G_Z(A) = G_{SX}(A) = \frac{1}{2}\,\mathbb{E}\langle ASX, SX\rangle_{\mathsf{H}} = \frac{1}{2}\,\mathbb{E}\langle S^*ASX, X\rangle_{\mathsf{U}} = G_X(S^*AS)$$
$$= \frac{1}{2}\sup_{Q\in\Sigma}\mathrm{Tr}\big[S^*AS\cdot Q\big].$$

Consider $\quad \mathrm{Tr}\big[S^*ASQ\big] = \sum_{i\geqslant 1}\langle S^*ASe_i, Qe_i\rangle_{\mathsf{U}} = \sum_{i\geqslant 1}\langle ASe_i, SQe_i\rangle_{\mathsf{H}}$

$$= \sum_{i\geqslant 1}\sum_{j\geqslant 1}\langle ASe_i, f_j\rangle_{\mathsf{H}}\cdot\langle SQe_i, f_j\rangle_{\mathsf{H}} = \sum_{i\geqslant 1}\sum_{j\geqslant 1}\langle e_i, S^*A^*f_j\rangle_{\mathsf{U}}\cdot\langle e_i, Q^*S^*f_j\rangle_{\mathsf{U}}$$
$$= \sum_{j\geqslant 1}\langle Q^*S^*f_j, S^*A^*f_j\rangle_{\mathsf{U}} = \sum_{j\geqslant 1}\langle ASQ^*S^*f_j, f_j\rangle_{\mathsf{H}} = \mathrm{Tr}\big[ASQ^*S^*\big]$$
$$= \mathrm{Tr}\big[ASQS^*\big].$$

So that $\quad G_Z(A) = \frac{1}{2}\sup_{Q\in\Sigma}\mathrm{Tr}\big[ASQS^*\big] = \frac{1}{2}\sup_{Q\in\Sigma_Z}\mathrm{Tr}\big[AQ\big].$

And it means that $\quad Z\sim N_G(0,\Sigma_Z)$. $\square$

3 Viscosity solutions

In this chapter we describe the notion of the viscosity solution for a fully nonlinear infinite-dimensional parabolic PDEs. Mainly the material (definitions and results) was taken from Kelome [55]. In infinite-dimensions for a viscosity solutions Kelome uses a particular notion of B-continuity which we also describe below. We apply his results of comparison principle and uniqueness of viscosity solution to our theory where in the following chapters we will solve parabolic PDEs in infinite dimensions with a probabilistic tools of sublinear expectation.

3.1 B-continuity

Consider a fully nonlinear infinite-dimensional parabolic PDE:

$$\begin{cases} \partial_t u + \langle Ax, D_x u\rangle + G(D^2_{xx}u) = 0\,, \quad t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{P}$$

$u : [0,T] \times \mathsf{H} \to \mathbb{R}\,;$
$f \in \mathbf{C}_{p.Lip}(\mathsf{H})\,;$
$G : L_S(\mathsf{H}) \to \mathbb{R}$ is a canonical extension of a G-functional defined on $K_S(\mathsf{H})$ and denoted by the same symbol G;
$A : D(A) \to \mathsf{H}$ is a generator of C_0-semigroup $\left(e^{tA}\right)$.

Recall that $\mathbf{C}_{p.Lip}$ is a space of Lipschitz functions with a polynomial growth (see 2.3), and $K_S(\mathsf{H})$ is a space of compact symmetric operators (see 2.1).

The following condition on the operator A need to be held (see [55, 2]):

Condition. *There exists* $B \in L_S(\mathsf{H})$ *such that:*
1) $B > 0$*;*
2) $A^*B \in L(\mathsf{H})$*;*
2) $-A^*B + c_0 B \geqslant I$*, for some* $c_0 > 0$*.*

Remark 3.1. *The* $im(B)$ *should belong to the set* $D(A^*)$*. If it happens that* $D(A^*) \subset \mathsf{H}$ *compactly, then it is necessarily* B *be a compact operator.*

Proof.
In fact, let $\{x_n\,,\ n \geqslant 1\} \subset \mathsf{H}$ and $\|x_n\|_{\mathsf{H}} \leqslant c \ \ \forall n \geqslant 1$.
So then $\|A^*Bx_n\|_{\mathsf{H}} \leqslant \|A^*B\|_{L(\mathsf{H})} \cdot \|x_n\|_{\mathsf{H}} \leqslant c \cdot \|A^*B\|_{L(\mathsf{H})}$.
Since we assume that $D(A^*)$ is a compact embedding in H, we have $\{Bx_n\,,\ n \geqslant\}$ is bounded in $D(A^*)$.
Thus there exists a subsequence $\{x_{n_k}\,,\ k \geqslant 1\}$, such that $\{Bx_{n_k}\,,\ n \geqslant 1\}$ is convergent in H.
And we conclude that $B \in K(\mathsf{H})$.

□

Remark 3.2. *If* A *is a self-adjoint, maximal dissipative operator then we can take* $B := (I - A)^{-1}$ *with* $c_0 := 1$ *which satisfies the condition imposed above.*
Usually, in applications $A = \Delta$*, so such condition for finding the correspondent* B *is not too strict.*

Later we need a space H_{-1} which is defined to be the completion of H under the norm $\|x\|_{-1}^2 := \langle Bx, x\rangle = \langle B^{\frac{1}{2}}x, B^{\frac{1}{2}}x\rangle = \|B^{\frac{1}{2}}x\|_{\mathsf{H}}^2$.
Fix $\{\widetilde{e}_j\,,\ j \geqslant 1\}$ to be a basis of H_{-1} made of elements of H.
(Hence in such a case $\{B^{\frac{1}{2}}\widetilde{e}_j\,,\ j \geqslant 1\}$ is a basis of H).
Define $\mathsf{H}_N := span\{\widetilde{e}_1, .., \widetilde{e}_N\}$, $N \geqslant 1$.
And let P_N be an orthonormal projection H_{-1} onto H:

$$P_N x := \sum_{j=1}^{N} \widetilde{e}_j \langle x, \widetilde{e}_j\rangle_{-1}\,,\ x \in \mathsf{H}_{-1}\,.$$

Also we define the following operator $Q_N := I - P_N$.

Definition 3.1. *Let* $u, v : [0,T] \times \mathsf{H} \to \mathbb{R}$.
u *is said to be* B*-l.s.c. (*B*-lower semicontinuous)*

$$\text{if} \quad u(t,x) \leqslant \varliminf_{n\to\infty} u(t_n, x_n)\,;$$

And v *is said to be* B*-u.s.c. (*B*-upper semicontinuous)*

$$\text{if} \quad u(t,x) \geqslant \varlimsup_{n\to\infty} u(t_n, x_n)\,,$$

$$\text{whenever } x_n \xrightarrow{w} x\,,\ t_n \to t\,,\ Bx_n \xrightarrow{s} Bx\,.$$

Definition 3.2. *A function which is B-l.s.c. and B-u.s.c. simultaneously is called B-continuous.*

Remark 3.3. *Note that B-continuity means that function $u(t,x)$ is continuous on the bounded sets of $[0,T]\times\mathsf{H}$ for the $[0,T]\times\mathsf{H}_{-1}$-topology.*

Definition 3.3. *A function $u(t,x)$ is locally uniformly B-continuous if it is uniformly continuous on the bounded sets of $[0,T]\times\mathsf{H}$ for the $[0,T]\times\mathsf{H}_{-1}$-topology.*

Remark 3.4. *In some cases B is a compact operator.*
If it is so then from the convergence $x_n \xrightarrow{w} x$ it follows that $Bx_n \xrightarrow{s} Bx$. And notions "B-continuity", "locally uniformly B-continuity" and "weak continuity" are the same.

3.2 Test functions and viscosity solutions

Definition 3.4. *A function $\psi:(0,T)\times\mathsf{H}\to\mathbb{R}$ is said to be a test function if it admits a representation $\psi=\varphi+\chi$, such that:*

1) $\varphi\in C^{1,2}\Big((0,T)\times\mathsf{H}\to\mathbb{R}\Big)$;
φ *is B-continuous;*
$\Big\{\partial_t\varphi,\ A^*D_x\varphi,\ D_x\varphi,\ D^2_{xx}\varphi\Big\}$ *are locally uniformly continuous*[1] *on* $(0,T)\times\mathsf{H}$;

2) $\chi:(0,T)\times\mathsf{H}\to\mathbb{R}$ *and has the following representation* $\chi(t,x)=\xi(t)\cdot\eta(x)$, *such that:*
$\xi\in C^1\Big((0,T)\to(0,+\infty)\Big)$;
$\eta\uparrow,\ \eta\in C^2_p(\mathsf{H})$ – *i.e. the derivatives have polynomial growth:*

$$\|D\eta\|_{\mathsf{H}},\|D^2\eta\|_{L(\mathsf{H})}\leqslant C\big(1+|x|^m\big);$$

$\eta(x)=\eta(y)$ *whenever* $|x|=|y|$;
$\Big\{D\eta,\ D^2\eta\Big\}$ *are locally uniformly continuous on $(0,T)\times\mathsf{H}$ and have polynomial growth.*

[1] i.e., uniformly continuous on the bounded sets (but not necessary on the compact ones)

Definition 3.5. *Let* $u, v : [0,T] \times \mathsf{H} \to \mathbb{R}$ *.*
u *is said to be a* ***viscosity subsolution*** *of* (P) *at the point* (t_0, x_0) *if:*
1) u *is* B*-u.s.c. on* $[0,T] \times \mathsf{H}$*;*
2) for every test function ψ*:*

$$u \leqslant \psi\,;$$
$$u(t_0, x_0) = \psi(t_0, x_0)\,;$$
$$\Big[\partial_t \psi + \langle x, A^* D_x \varphi \rangle + G(D^2_{xx}\psi)\Big](t_0, x_0) \geqslant 0\,;$$
$$u(T, x) \leqslant f(x)\,.$$

Analogously, v *is said to be a* ***viscosity supersolution*** *of* (P) *at the point* (t_0, x_0) *if:*
1) u *is* B*-l.s.c. on* $[0,T] \times \mathsf{H}$*;*
2) $\forall$ *test function* ψ*:*

$$v \geqslant \psi\,;$$
$$v(t_0, x_0) = \psi(t_0, x_0)\,;$$
$$\Big[\partial_t \psi + \langle x, A^* D_x \varphi \rangle + G(D^2_{xx}\psi)\Big](t_0, x_0) \leqslant 0\,;$$
$$v(T, x) \geqslant f(x)\,.$$

Definition 3.6. *A function which at the point* (t_0, x_0) *is viscosity sub- and supersolution simultaneously is called viscosity solution.*

Remark 3.5. *Note that in the definition we imply that* $D_x \varphi \in D(A^*)$ *.*

Remark 3.6. *Actually the functional* G *in equation* (P) *can be considered only on a compact set of operators. Because when we solve this equation the compactness of the operator* $D^2_{xx}\psi$ *of test function is constrained only by such a thing that we are looking for only such functions as solutions which have compact second Fréchet derivative, i.e. on the domain of* G*-functional. This fact is subjected to only the above described requirement. In fact, the functions* φ *and* χ *are built in the following way (see [55, p.14]):*

we take a test function $\widetilde{\psi} = \widetilde{\varphi} + \widetilde{\chi}$ *defined on a* $(0,T) \times \widetilde{\mathsf{H}}_N$ *and* $\widetilde{\varphi}$, $\widetilde{\chi}$ *are bounded.* $\widetilde{\mathsf{H}}_N$ *is defined as a space* H_N *with* H_{-1}*-topology. Note that* $dim \mathsf{H}_N < \infty$*. And for a test function we take* $\varphi(t,x) := \widetilde{\varphi}(t, P_N x)$;

$$\chi(t,x) := \widetilde{\chi}(t, P_N x)\,.$$

It is clear that such φ *and* χ *have a compact second derivative, so a test function* $\psi = \varphi + \chi$ *satisfies required condition.*

3.3 Comparison principle

The following result is called a comparison principle which is obtained by Kelome and Święch (original formulation one can see in [55, Th.3.1]).

Theorem 3.1 (Comparison principle)**.**
Let u and v be respectively sub- and super- viscosity solutions of the (P)*, such that:*

1) there exists a positive M, such that
$u \leqslant M$,
$v \geqslant -M$;

2) f is bounded, locally uniformly B-continuous;

3) G satisfies the following conditions:

(i) if $A_1 \geqslant A_2$ then $G(A_1) \geqslant G(A_2)$;

(ii) there exists a radial, increasing, linearly growing function $\mu \in C^2(\mathsf{H} \to \mathbb{R})$ with bounded first and second derivatives, such that for all $\alpha > 0$: $\left|G\big(A+\alpha D^2\mu(x)\big)-G\big(A\big)\right| \leqslant C\big(1+|x|\big)\cdot\alpha$;

(iii) $$\sup\left\{\left|G\big(A+\lambda BQ_N\big)-G\big(A\big)\right| : \ \|A\|_{L(\mathsf{H})} < p,\ |\lambda| < p,\ A = P_N^* A P_N,\ p \in \mathbb{R}\right\} \xrightarrow[N\to\infty]{} 0.$$

Then $u \leqslant v$.

Remark 3.7. *If G is a G-functional then condition **3)** of **Th.3.1** holds.*

Proof.
(i) Obvious, by the definition.
(ii) Let us take $\mu(x) := \sqrt{1+\|x\|^2}$.

Hence $$D^2\mu(x) = -\frac{1}{\sqrt{1+\|x\|^2}}\left(I+\frac{x\otimes x}{1+\|x\|^2}\right).$$

And we have $\left\|D^2\mu(x)\right\|_{L(\mathsf{H})} \leqslant 2$.

Thus $$\left|G\big(A+\alpha D^2\mu(x)\big)-G\big(A\big)\right| \leqslant \alpha\cdot G\big(D^2\mu(x)\big) = \frac{\alpha}{2}\sup_{S\in\Sigma}\mathrm{Tr}\Big[D^2\mu(x)S\Big]$$
$$\leqslant \frac{\alpha}{2}\sup_{S\in\Sigma}\mathrm{Tr}\big[S\big]\cdot\|D^2\mu(x42)\|_{L(\mathsf{H})} \leqslant \alpha\cdot C \leqslant \alpha\cdot C\big(1+|x|\big).$$

(iii) $$\left|G\big(A+\lambda BQ_N\big)-G\big(A\big)\right| \leqslant \lambda\cdot G\big(BQ_N\big) = \frac{\lambda}{2}\sup_{S\in\Sigma}\mathrm{Tr}\Big[BQ_NS\Big].$$

But $\operatorname{Tr}\Big[BQ_NS\Big] = \sum_{k\geqslant 1}\Big\langle BQ_NS{\cdot}B^{1/2}e_k, B^{1/2}e_k\Big\rangle = \sum_{k\geqslant 1}\Big\langle B^{1/2}Q_NSB^{1/2}e_k, e_k\Big\rangle_{-1}$

$= \operatorname{Tr}_{(-1)}\Big[B^{1/2}Q_NSB^{1/2}\Big] = \operatorname{Tr}_{(-1)}\Big[Q_NSB\Big] = \operatorname{Tr}_{(-1)}\Big[SBQ_N\Big] = \sum_{k=N+1}^{\infty}\Big\langle BSBe_k, e_k\Big\rangle.$

Let us define $f_N(S) := \sum_{k=N+1}^{\infty}\Big\langle BSBe_k, e_k\Big\rangle, \quad S \in \Sigma$ – compact.

Hence $f_N \in C(\Sigma)$;

$f_N \downarrow$;

$f_N(S_0) \to 0$ for every $S_0 \in \Sigma$, since f_N is a tail of convergent series.

By the Dini theorem yields $f_N \underset{N\to\infty}{\rightrightarrows} 0$ on Σ.

That is $\sup_{S\in\Sigma} f_N \xrightarrow[N\to 0]{} 0$.

So we we can conclude that **(iii)** holds.

□

The following proposition is a comparison principle for the functions with a polynomial growth. In fact, we will use this result applied to the functions of the $\mathbf{C}_{p.Lip}$ class.

Proposition 3.1. *Let u is a sub- and v is a super- viscosity solution to the PDE (P), such that for every $t \in [0,T]$ and for every $x \in \mathsf{H}$ there satisfy:*

$$\begin{aligned} u(t,x) &\leqslant C(1+|x|^m);\\ -v(t,x) &\leqslant C(1+|x|^m);\\ |f(x)| &\leqslant C(1+|x|^m). \end{aligned} \qquad (\#)$$

And the following conditions hold:

(i) there exists continuous $\delta_\varepsilon : [0,T] \to [0,\infty)$, such that: at the every point $t \in [0,T]$: $\delta_\varepsilon(t) \xrightarrow[\varepsilon\downarrow 0]{} 0$;

(ii) there exists a radial, increasing, function g_0, such that:

g_0, Dg_0, D^2g_0 are locally uniformly continuous,

$\varliminf_{|x|\to\infty} \dfrac{g_0(x)}{|x|^{m+1}} > 0$;

(iii) $\delta_\varepsilon'(t)\cdot g_0(x) + \Big|G\big(A+\delta_\varepsilon(t)\cdot D^2g_0(x)\big) - G\big(A\big)\Big| \leqslant 0$.

Then the functions

$u_\varepsilon(t,x) := u(t,x) - \delta_\varepsilon(t)\cdot g_0(x)\,;$

$v_\varepsilon(s,y) := v(s,y) + \delta_\varepsilon(s)\cdot g_0(y)\,;$

are respectively sub- and super- viscosity solutions of the PDE (P)*, such that* $u_\varepsilon \leqslant v_\varepsilon$ *.*

Moreover, if $\varepsilon \to 0$ *then* $u \leqslant v$ *.*

Proof.
Fix a point $(t,x)\in[0,T]\times\mathsf{H}$.

Let ψ be a test function, such that: $u_\varepsilon \leqslant \psi\,;$
$$u_\varepsilon(t,x) = \psi(t,x)\,.$$

For the viscosity subsolution $u(s,y) = u_\varepsilon(s,y) + \delta_\varepsilon(s)\cdot g_0(y)$ we have that there exists a test function $\widetilde{\psi}$, such that

$u \leqslant \widetilde{\psi}\,;$

$u(t,x) = \widetilde{\psi}(t,x)\,,$

from this we can get that $\widetilde{\psi}(s,y) = \psi(s,y) + \delta_\varepsilon(s)\cdot g_0(y)\,.$

Hence
$$R := \partial_t\psi(t,x) + \delta'_\varepsilon(t)\cdot g_0(x) + \langle x, A^*D_x\varphi(t,x)\rangle + G\big(D^2_{xx}\psi(t,x) + \delta_\varepsilon(t)\cdot D^2 g_0(x)\big) \geqslant 0\,.$$

From another hand:
$$\begin{aligned}
&\partial_t\psi(t,x) + \langle x, A^*D_x\varphi(t,x)\rangle + G\big(D^2_{xx}\psi(t,x)\big)\\
&\overset{\textbf{(iii)}}{\geqslant} \partial_t\psi(t,x) + \langle x, A^*D_x\varphi(t,x)\rangle + G\big(D^2_{xx}\psi(t,x)\big) + \delta'_\varepsilon(t)\cdot g_0(x)\\
&\qquad + \Big|G\big(D^2_{xx}\psi(t,x) + \delta_\varepsilon(t)\cdot D^2 g_0(x)\big) - G\big(D^2_{xx}\psi(t,x)\big)\Big|\\
&\geqslant \partial_t\psi(t,x) + \langle x, A^*D_x\varphi(t,x)\rangle + G\big(D^2_{xx}\psi(t,x)\big) + \delta'_\varepsilon(t)\cdot g_0(x)\\
&\qquad + G\big(D^2_{xx}\psi(t,x) + \delta_\varepsilon(t)\cdot D^2 g_0(x)\big) - G\big(D^2_{xx}\psi(t,x)\big)\\
&= R \geqslant 0\,.
\end{aligned}$$

In the same time,
$$u_\varepsilon(T,x) = u(T,x) - \delta_\varepsilon(T)\cdot g_0(x) \leqslant f(x) - \delta_\varepsilon(T)\cdot g_0(x) \overset{\textbf{(i)},\textbf{(ii)}}{\geqslant} f(x)\,.$$

Also, from **(ii)** it follows that there exists a positive M such that
$$g_0(x) \leqslant M(1+|x|^{m+1})\,.$$

And $u_\varepsilon(t,x) = u(t,x) - \delta_\varepsilon(t)\cdot g_0(x) \leqslant C(1+|x|^m) - \delta_\varepsilon(t)\cdot g_0(x)\,.$

So, since $\underline{\lim}_{|x|\to\infty} u_\varepsilon(t,x) = -\infty$, therefore there exists a positive K such that $u_\varepsilon(t,x) \leqslant K$.

In the same way we can show that $v_\varepsilon(t,x)$ is a viscosity supersolution which is also bounded from beneath.

Using the comparison principle **Th.3.1** we deduce that $u_\varepsilon \leqslant v_\varepsilon$.

□

3.4 Uniqueness of viscosity solution

Theorem 3.2. *If PDE (P) has a B-continuous viscosity solution on $[0,T]\times\mathsf{H}$ that has a polynomial growth, then such a solution is unique.*

Proof.
Note that $f \in \mathbf{C}_{p.Lip}(\mathsf{H})$ means the f has a polynomial growth,

$$|f(x)| \leqslant C(1+|x|^m).$$

Let u, v be given two viscosity solutions of the PDE (P).
Assume that there is a point (t_0,x_0) such that $u(t_0,x_0) \geqslant v(t_0,x_0)$.
So, we can treat that u and v are respectively viscosity sub- and super-solutions of the PDE (P).

Take $\delta_\varepsilon(t) := \varepsilon\cdot e^{-\alpha t}$;
$g_0(x) := 1 + \|x\|^{m+1}$.

Then we have:
$D^2 g_0(x) = (m+1)\|x\|^{m-1}\cdot I + (m^2-1)\|x\|^{m-3} x\otimes x$;
$\|D^2 g_0(x)\|_{L(\mathsf{H})} \leqslant (m^2+m)\|x\|^{m-1}$.

So, the pair $\delta_\varepsilon(t)$ and $g_0(x)$ satisfies the conditions **(i)-(ii)** of **Prop.3.1**.

There only remains just to check the condition **(iii)**:

$$\delta'_\varepsilon(t)\cdot g_0(x) + \Big|G\big(A+\delta_\varepsilon(t)\cdot D^2 g_0(x)\big) - G\big(A\big)\Big|$$
$$\leqslant \delta'_\varepsilon(t)\cdot g_0(x) + \Big|G\big(\delta_\varepsilon(t)\cdot D^2 g_0(x)\big)\Big|$$
$$\leqslant -\alpha\,\varepsilon\cdot e^{-\alpha t}\big(1+\|x\|^{m+1}\big) + \frac{1}{2}\sup_{S\in\Sigma}\mathrm{Tr}\big[S\big]\cdot\varepsilon\cdot e^{-\alpha t}\big\|D^2 g_0(x)\big\|_{L(\mathsf{H})}$$
$$\leqslant -\varepsilon\cdot e^{-\alpha t}\Big(\alpha\big(1+\|x\|^{m+1}\big) - \frac{1}{2}\sup_{S\in\Sigma}\mathrm{Tr}\big[S\big]\cdot(m^2+m)\cdot\|x\|^{m-1}\Big).$$

Since, $\dfrac{\sup\limits_{S\in\Sigma} \mathrm{Tr}\big[S\big] \cdot (m^2+m) \cdot \|x\|^{m-1}}{1+\|x\|^{m+1}} \xrightarrow[\|x\|\to\infty]{} 0\,;$

And, $\dfrac{\sup\limits_{S\in\Sigma} \mathrm{Tr}\big[S\big] \cdot (m^2+m) \cdot \|x\|^{m-1}}{1+\|x\|^{m+1}} \xrightarrow[\|x\|\to 0]{} 0\,;$

So we have there exists a positive α such that for every $x \in \mathsf{H}$:

$$\alpha_0 \geqslant \frac{1}{2} \cdot \frac{\sup\limits_{S\in\Sigma} \mathrm{Tr}\big[S\big] \cdot (m^2+m) \cdot \|x\|^{m-1}}{1+\|x\|^{m+1}}\,.$$

Then we conclude that the condition **(iii)** holds for $\alpha := \alpha_0$.

Therefore, we have conditions **(i)-(iii)** of **Prop.3.1** hold.

Set $u_\varepsilon(t,x) := u(t,x) - \delta_\varepsilon(t) \cdot g_0^N(x)\,;$

$v_\varepsilon(t,x) := v(t,x) - \delta_\varepsilon(t) \cdot g_0^N(x)\,.$

Then from **Prop.3.1** we get $u_\varepsilon(t_0,x_0) \leqslant v_\varepsilon(t_0,x_0)$.

Letting $\varepsilon \to 0$ yields $u(t_0,x_0) \leqslant v(t_0,x_0)$.

So that $u(t_0,x_0) = v(t_0,x_0)$.

□

4 G-expectations

In this chapter we describe a main notion of the theory, the notion of a G-expectation, the special case of a sublinear expectation. It was introduced by Peng [72] and we by analogy carry it to the infinite-dimensions. Also we touch upon such objects as G-Brownian motion, capacity and upper expectation.

4.1 G-Brownian motion

Definition 4.1. $X_t : \Omega \times \mathbb{R}_+ \to \mathsf{X}$ *is called a stochastic process if* X_t *is random variable on a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$ *for every nonnegative* t.

Definition 4.2. *Stochastic process is called a G-Brownian motion if:*

1) $B_0 = 0$;
2) for all $t, s \geqslant 0 \quad (B_{t+s} - B_t) \sim N_G(0, s\Sigma)$;
3) for all $t, s \geqslant 0 \quad (B_{t+s} - B_t)$ *is independent from* $(B_{t_1}, .., B_{t_n})$
for every $n \in \mathbb{N},\ 0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t$.

Remark 4.1. $\left(B_{t_{k+1}} - B_{t_k}\right)$ *is independent from* $\varphi\left(B_{t_1}, .., B_{t_k}\right)$,
$0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant t_{k+1}$, $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{X}^k \to \mathbb{R})$.

Proof.

$$\mathbb{E}\Big[\psi\Big(B_{t_{k+1}} - B_{t_k}, \varphi\big(B_{t_1}, .., B_{t_k}\big)\Big)\Big] = \mathbb{E}\Big[\zeta\Big(B_{t_{k+1}} - B_{t_k}, \big(B_{t_1}, .., B_{t_k}\big)\Big)\Big]$$
$$\overset{\textbf{Def.2.7}}{=} \mathbb{E}\Big[\mathbb{E}\big[\zeta\big(B_{t_{k+1}} - B_{t_k}, \vec{v}\big)\big]_{\vec{v}=(B_{t_1},..,B_{t_k})}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\psi\big(B_{t_{k+1}} - B_{t_k}, \varphi(\vec{v})\big)\big]_{\vec{v}=(B_{t_1},..,B_{t_k})}\Big].$$

ψ, ζ are corresponding functions of $\mathbf{C}_{p.Lip}$ class.

□

Proposition 4.1. *Let* B_t *be a stochastic process and:*

$B_t \sim \sqrt{t} B_1$;

B_1 *is* G*-norm. distributed.*

Then B_1 *is* G*-Brownian motion*

Proof.
1) $B_0 = 0$;
2) $B_{t+s} - B_t \sim \sqrt{t+s}B_1 - \sqrt{t}B_1 \sim \sqrt{s}B_1 \sim N_G\Big(0,\ s\Sigma\Big)$;
3) $B_{t+s} - B_t \sim \sqrt{s}B_1$;
$\big(B_{t_1}, \ldots, B_{t_n}\big) \sim \big(\sqrt{t_1}B_1^{(1)}, \ldots, \sqrt{t_n}B_1^{(n)}\big)$, such that $B_1,\ B_1^{(1)}, \ldots, B_1^{(n)}$ are independent copies.
So that B_1 is independent form $\big(B_1^{(1)}, \ldots, B_1^{(n)}\big)$.
Therefore $\big(B_{t+s} - B_t\big)$ is independent form $\big(B_{t_1}, \ldots, B_{t_n}\big)$.

□

Corollary 4.1. *$B_t^h := \langle B_t, h\rangle$ is a 1-dimensional G-B.m.*

Proof.
It follows from **Prop.4.1** and **Prop.2.5**.

□

Remark 4.2. *If B_t is a G-Brownian motion then $G(A) = \dfrac{1}{2}\mathbb{E}\big[\langle AB_1, B_1\rangle\big]$.*
Moreover:

1) $\dfrac{1}{2}\mathbb{E}\big[\langle AB_t, B_t\rangle\big] = t\,G(A)$.
2) *for every* k, r $\quad \mathbb{E}\big[B_t^k \cdot B_t^r\big] = t\,\mathbb{E}\big[B_1^k \cdot B_1^r\big]$.

Proof.
1) For fixed t, s consider random variables $X := B_t$ and $Y := B_{t+s} - B_s$.
Since $X \sim Y \sim N_G(0, t\,\Sigma)$ then by **Rem.2.6** $G_X = G_Y$.
Hence $\mathbb{E}\big[\langle A(B_{t+s} - B_s), B_{t+s} - B_s\rangle\big] = \mathbb{E}\big[\langle AB_t, B_t\rangle\big]$.

Let $b(t) := \mathbb{E}\big[\langle AB_t, B_t\rangle\big]$.
Then let us compute

$$
\begin{aligned}
b(t+s) &= \mathbb{E}\big[\langle AB_{t+s}, B_{t+s}\rangle\big] \\
&= \mathbb{E}\big[\langle A(B_{t+s} - B_s + B_s), B_{t+s} - B_s + B_s\rangle\big] \\
&= \mathbb{E}\big[\langle A(B_{t+s} - B_s), B_{t+s} - B_s\rangle + \langle AB_s, B_s\rangle + \langle A(B_{t+s} - B_s), B_s\rangle \\
&\qquad + \langle AB_s, B_{t+s} - B_s\rangle\big] \\
&\overset{\textbf{Prop.2.2.5).(b), Rem.2.5}}{=} \mathbb{E}\big[\langle A(B_{t+s} - B_s), B_{t+s} - B_s\rangle\big] + \mathbb{E}\big[\langle AB_s, B_s\rangle\big] \\
&= b(t) + b(s).
\end{aligned}
$$

So it may be concluded that $\mathbb{E}\big[\langle AB_t, B_t\rangle\big] = b(t) = t{\cdot}b(1) = t\,\mathbb{E}\big[\langle AB_1, B_1\rangle\big]$.

2) Let $B_t \sim N_G(0, \Sigma_t)$ then $B_1 \sim N_G(0, \Sigma_1)$ and $B_t \sim \sqrt{t} B_1$.
From the **1)** we have $\Sigma_t = \Big\{ t \cdot s \Big| \; s \in \Sigma_1 \Big\}$.

Then $\mathbb{E}\big[B_t^k \cdot B_t^r\big] = \mathbb{E}\big[\langle B_t, e_k\rangle\langle B_t, e_r\rangle\big] \overset{(9)}{=} \sup\limits_{Q_t \in \Sigma_t} \langle Q_t e_k, e_r\rangle;$

$$\mathbb{E}\big[B_1^k \cdot B_1^r\big] = \mathbb{E}\big[\langle B_1, e_k\rangle\langle B_1, e_r\rangle\big] \overset{(9)}{=} \sup_{Q_1 \in \Sigma_1} \langle Q_1 e_k, e_r\rangle;$$

Therefore $\mathbb{E}\big[B_t^k \cdot B_t^r\big] = \sup\limits_{t\,Q_0 \in \Sigma_t} \langle t\,Q_0 e_k, e_r\rangle = t \sup\limits_{Q_0 \in \Sigma_t} \langle Q_0 e_k, e_r\rangle = t\,\mathbb{E}\big[B_1^k \cdot B_1^r\big]$.

□

4.2 Capacity and upper expectation

The notion of capacity, i.e. a supremum measure, was introduced by Choquet [21]. Since in our framework there are no fixed probability measures, so we will use exactly this object. The classical notion "almost surely" becomes "quasi surely".
So, let $\big(\Omega, \mathcal{B}(\Omega)\big)$ be a complete separable metric space with a Borel σ-algebra on it. $\mathcal{M}$ is the collection of all probability measures on $\big(\Omega, \mathcal{B}(\Omega)\big)$. Take a fixed $\mathcal{P} \subseteq \mathcal{M}$.

Definition 4.3. *Let us define capacity to be*

$$c(A) = c_{\mathcal{P}}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega). \tag{10}$$

Remark 4.3. *It it obvious that $c(A)$ is a Choquet capacity (see [21]), i.e.*

(1) $c(A) \in [0,1], \quad A \in \Omega$.

(2) if $A \subset B$ *then* $c(A) \leqslant c(B)$.

(3) if $\big\{A_n, \; n \geqslant 1\big\} \subset \mathcal{B}(\Omega)$ *then* $c(\bigcup\limits_{n \geqslant 1} A_n) \leqslant \sum\limits_{n \geqslant 1} c(A_n)$.

(4) if $\big\{A_n, \; n \geqslant 1\big\} \subset \mathcal{B}(\Omega): \; A_n \uparrow A = \bigcup\limits_{n \geqslant 1} A_n$

then $c(\bigcup\limits_{n \geqslant 1} A_n) = \lim\limits_{n \to \infty} c(A_n)$.

Definition 4.4. *A set A is called polar if $c(A) = 0$.*
The property holds ***quasi surely*** *(q.s.) if it holds outside a polar set.*

Definition 4.5. *We define an* ***upper expectation*** *as follows:*

$$\bar{\mathbb{E}}[\,\cdot\,] := \sup_{P \in \mathcal{P}} E_P[\,\cdot\,].$$

Lemma 4.1. *If $\bar{\mathbb{E}}\big[\|X\|^p\big] = 0$ then $X = 0$ quasi surely.*

Proof.
$\bar{\mathbb{E}}\big[\|X\|^p\big] = 0$ that is for every $P \in \mathcal{P}$ $E_P\big[\|X\|^p\big] = 0$ hence $X = 0$ P-a.s. for every $P \in \mathcal{P}$.
And finally $c\big(\{X = 0\}\big) = \sup_{P \in \mathcal{P}} P\big(\{X = 0\}\big) = 1 > 0$.

□

4.3 Solving the fully nonlinear heat equation

Let B_t be a G-Brownian motion with a corresponding G-functional $G(\cdot)$, such that $G(A) = \dfrac{1}{2t}\,\mathbb{E}\Big[\langle AB_t, B_t\rangle\Big]$.
Consider equation (P) with $A = 0$, i.e. the following parabolic PDE:

$$\begin{cases} \partial_t u + G(D^2_{xx} u) = 0\,, \quad t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{P0}$$

In order to proof the existence of a viscosity solution to equation (P0) we need Taylor's expansion for the function $u(t,x)$. Such a trivial statement will show us a following lemma:

Lemma 4.2 (Taylor's formula)**.**
Let $\psi \in C^2(\mathbb{R} \times \mathsf{H} \to \mathbb{R})$, $(\delta, \Delta x)$ and (t,x) are in $(\mathbb{R}, \mathsf{H})$, such that $(t + \delta, x + \Delta x) \in D(\psi)$.
Then:

$$\psi\big(t+\delta, x+\Delta x\big) = \psi(t,x) + \delta\cdot\partial_t\psi(t,x) + \langle D_x\psi(t,x), \Delta x\rangle + \frac{1}{2}\,\delta^2\cdot\partial^2_{tt}\,\psi(t,x)$$
$$+\delta\cdot\partial_t\big[\langle D_x\psi(t,x), \Delta x\rangle\big] + \frac{1}{2}\langle D^2_{xx}\psi(t,x)\Delta x, \Delta x\rangle + o(\delta^2 + \|\Delta x^2\|)\,.$$

Proof.
Consider $p(s,\theta) = \psi\big(t_0 + s(t-t_0), x_0 + \theta(x-x_0)\big)$, $p \in C^2(\mathbb{R}\times\mathbb{R} \to \mathbb{R})$.
For $f \in C^2(\mathsf{H})$ we have

$$f(x) = f(a) + (x-a)^\top Df(a) + \frac{1}{2} + (x-a)^\top D^2 f(a)(x-a) + o(\|x-a\|^2).$$

Then
$$p(s,\theta) = p(0,0) + s\,p_s(0,0) + \theta\,p_\theta(0,0) + \frac{1}{2}s^2\,p_{ss}(0,0) + s\theta\,p_{s\theta}(0,0) + \frac{1}{2}\theta^2\,p_{\theta\theta}(0,0) + o(s^2+\theta^2).$$

If we denote the variables in the following way:
$t := t_0$, $x := x_0$, $\delta := s(t-t_0)$, $\Delta x := \theta(x-x_0)$, we can easily see that what wee need to be proved.

□

Theorem 4.1. *Let f is a B-continuous of $\mathbf{C}_{p.Lip}(\mathsf{H})$-class real function. Then $u(t,x) := \mathbb{E}\big[f(x+B_{T-t})\big]$ is a unique viscosity solution to equation (P0):*

$$\begin{cases} \partial_t u + G(D^2_{xx}u) = 0\,, & t\in[0,T)\,,\ x\in\mathsf{H}; \\ u(T,x) = f(x)\,, \end{cases} \tag{P0}$$

where B_t is a G-Brownian motion with a corresponding G-functional $G(\,\cdot\,)$.

Proof.
Let ψ be a test function, and for every fixed point $(t,x)\in[0,T]\times\mathsf{H}$ we have: $u \leqslant \psi$;
$u(t,x) = \psi(t,x)$.

Taking a small enough δ yields:

$$\psi(t,x) = u(t,x) = \mathbb{E}[f(x+B_{T-t})] = \mathbb{E}[f(x + \underbrace{B_\delta + B_{T-(t+\delta)}}_{\text{indep.}})]$$

$$= \mathbb{E}\left[\mathbb{E}\Big[f(x+\beta+B_{T-(t+\delta)})\Big]_{\beta=B_\delta}\right] = \mathbb{E}\Big[u(t+\delta,x+\beta)\Big|_{\beta=B_\delta}\Big]$$
$$= \mathbb{E}\big[u(t+\delta,x+B_\delta)\big] \leqslant \mathbb{E}\big[\psi(t+\delta,x+B_\delta)\big].$$

Using the Taylor formula (**Lm.4.2**):

$$\psi(t+\delta, x+B_\delta) = \psi(t,x) + \delta\cdot\partial_t\psi(t,x) + \big\langle D_x\psi(t,x), B_\delta\big\rangle + \frac{1}{2}\,\delta^2\cdot\partial^2_{tt}\psi(t,x)$$
$$+\delta\cdot\partial_t\Big[\big\langle D_x\psi(t,x), B_\delta\big\rangle\Big] + \frac{1}{2}\,\big\langle D^2_{xx}\psi(t,x)B_\delta, B_\delta\big\rangle + o\big(\delta^2 + \|B_\delta\|^2\big)\,.$$

Then we have

$$0 \leqslant \mathbb{E}\Big[\psi(t+\delta, x+B_\delta)\Big] - \psi(t,x) \leqslant \mathbb{E}\Big[\psi(t+\delta, x+B_\delta) - \psi(t,x)\Big]$$
$$= \delta\,\partial_t\psi(t,x) + \frac{1}{2}\,\delta^2\,\partial^2_{tt}\psi(t,x) + \delta\cdot G(D^2_{xx}\psi)(t,x) + o\big(\delta^2 + \|B_\delta\|^2\big)\,.$$

Letting $\delta\to 0$ yields $\Big[\partial_t\psi + G(D^2_{xx}\psi)\Big](t,x) \geqslant 0\,.$

Note, that u is continuous at $(t,x)\in[0,T]\times\mathsf{H}$.
In fact, let us show that $\mathbb{E}\big[f(x+B_s)\big] \xrightarrow[s\to t]{} \mathbb{E}\big[f(x+B_t)\big]\,,\quad t\in[0,T]\;:$

$$0 \leqslant \Big|\mathbb{E}\big[f(x+B_s)\big] - \mathbb{E}\big[f(x+B_t)\big]\Big| \leqslant \mathbb{E}\Big[\big|f(x+B_s) - f(x+B_t)\big|\Big]$$
$$\leqslant \mathbb{E}\Big[\big(1 + \|x+B_s\|^m + \|x+B_t\|^m\big)\cdot\|B_s - B_t\|\Big]$$
$$\leqslant \left(\mathbb{E}\Big[\big(1 + \|x+B_s\|^m + \|x+B_t\|^m\big)^2\Big]\cdot\mathbb{E}\Big[\|B_s-B_t\|^2\Big]\right)^{\frac{1}{2}}$$
$$\leqslant \Bigg(\underbrace{\mathbb{E}\Big[\big(1 + 2m\|x\|^m + m\cdot s^{\frac{m}{2}}\|\bar{X}\|^m + m\cdot t^{\frac{m}{2}}\|\bar{\bar{X}}\|^m\big)^2\Big]}_{\text{bdd. by } \textbf{Prop.2.4}}\cdot(s-t)\,\underbrace{\mathbb{E}\Big[\|X\|^2\Big]}_{\text{bdd.}}\Bigg)^{\frac{1}{2}} \underset{s\to t}{\searrow} 0,$$

where X $\bar{X}$, $\bar{\bar{X}}$ are independent copies of B_1.

$u(T,x) = \mathbb{E}\big[f(x)\big] = f(x) \leqslant f(x)\,.$

So we see that u is a viscosity subsolution to equation (P0).
In the same way one can prove that u is a viscosity supersolution, and the existence is proved.

It is clear that if f is B-continuous and has a polynomial growth that u is also B-continuous and has a polynomial growth, because a sublinear expectation $\mathbb{E}$ does not influence on it. So we can conclude that u is a unique viscosity solution by **Th.3.2**.

□

4.4 Basic space constructions

Let U, H be Hilbert spaces, and $\{e_i\}$ and $\{f_j\}$ are systems of orthonormal bases on them respectively. We consider G-Brownian motion B_t with values in U. Assume that random variable $B_1 \sim N_G(0, \Sigma)$ with a covariance set Σ. Recall that Σ is a convex set of linear bounded non-negative symmetric trace-class operators.

Now we are going to define a Banach space of operators Φ with values in H and defined in an appropriate space of U, such that $dom\,\Phi \subset \mathsf{U}$, endowed with the norm $\|\Phi\|^2_{L_2^\Sigma} := \sup\limits_{Q\in\Sigma} \mathrm{Tr}\big[\Phi Q \Phi^*\big] = \sup\limits_{Q\in\Sigma} \|\Phi Q^{1/2}\|^2_{L_2(\mathsf{U},\mathsf{H})}$.

First of all, we formulate a trivial statement that makes us sure that $\|\cdot\|^2_{L_2^\Sigma}$ is a norm indeed.

Proposition 4.2. *$\|\cdot\|^2_{L_2^\Sigma}$ is a norm.*

Proof.

1) $\|a\Phi\|_{L_2^\Sigma} = \sup\limits_{Q\in\Sigma} \|a\,\Phi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} = |a| \cdot \sup\limits_{Q\in\Sigma} \|\Phi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} = |a| \cdot \|\Phi\|_{L_2^\Sigma}$.

2)
$$\|\Phi + \Psi\|^2_{L_2^\Sigma} = \sup_{Q\in\Sigma} \|(\Phi+\Psi)Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant \sup_{Q\in\Sigma} \|\Phi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} + \sup_{Q\in\Sigma} \|\Psi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} = \|\Phi\|^2_{L_2^\Sigma} + \|\Psi\|^2_{L_2^\Sigma}.$$

3) If $\|\Phi\|^2_{L_2^\Sigma} = 0$ then $\sup\limits_{Q\in\Sigma} \|\Phi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} = 0$.

So, it follows that for every $Q \in \Sigma$ $\|\Phi Q^{1/2}\|_{L_2(Q^{1/2}(\mathsf{U}),\mathsf{H})} = 0$.

And we can conclude that for every $Q \in \Sigma$ and every $x \in Q^{1/2}(\mathsf{U})$ we have that $\Phi x = 0$.

□

It needs that $dom\,\Phi \supset \bigcup\limits_{Q\in\Sigma} Q^{1/2}(\mathsf{U})$.

Take
$$dom\,\Phi = \mathsf{U}_\Sigma := \Big\{u \in \mathsf{U} \;\Big|\; \exists u_i \in Q_i^{1/2}(\mathsf{U}),\ Q_i \in \Sigma:\ u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum_{i=1}^{\infty} u_i,\ \sum_{i=1}^{\infty} \|u_i\|_{Q_i^{1/2}(\mathsf{U})} < \infty\Big\}.$$

and
$$\|u\|_{\mathsf{U}_\Sigma} := \inf\Big\{\sum_{i=1}^{m} \|u_i\|_{Q_i^{1/2}(\mathsf{U})} \;\Big|\; u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum_{i=1}^{\infty} u_i,\ u_i \in Q_i^{1/2}(\mathsf{U}),\ Q_i \in \Sigma\Big\}.$$

Note that $\|u_i\|_{Q_i^{1/2}(\mathsf{U})} = \|Q_i^{-1/2}u_i\|_{\mathsf{U}},\ u_i \in Q_i^{1/2}(\mathsf{U}),\ Q_i \in \Sigma\,;$
and it is known that $\left(Q_i^{1/2}(\mathsf{U}),\ \|\cdot\|_{Q_i^{1/2}(\mathsf{U})}\right)$ is a Banach space (see [32, 4.2]).

Remark 4.4. *In the definition of* U_Σ *:*
from $u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum\limits_{i=1}^{\infty} u_i$ *follows* $u \overset{\|\cdot\|_{\mathsf{U}_\Sigma}}{=} \sum\limits_{i=1}^{\infty} u_i\,.$

Proof.

$$\|u - \sum_{i=1}^{K} u^i\|_{U_\Sigma} = \|\sum_{i=K+1}^{\infty} u^i\|_{U_\Sigma} \leqslant \sum_{i=K+1}^{\infty} \|u_i\|_{Q_i^{1/2}(\mathsf{U})} \xrightarrow[K\to\infty]{} 0\,.$$

□

Lemma 4.3. $\mathsf{U}_\Sigma \hookrightarrow \mathsf{U}$ *(* U_Σ *is a continuously embedded in* U*), i.e.:*
$\mathsf{U}_\Sigma \subset \mathsf{U}\,;$
$\|\cdot\|_{\mathsf{U}} \leqslant C\cdot\|\cdot\|_{\mathsf{U}_\Sigma}\,.$
In particular $\forall Q \in \Sigma,\quad Q^{1/2}(\mathsf{U}) \hookrightarrow \mathsf{U}\,.$

Proof.
Let $Q \in \Sigma$.
It is well known that $Q^{1/2} \in L_2(\mathsf{U})$ (see [73, Prop.2.3.4]).
Therefore $Q^{1/2}(\mathsf{U}) \subset \mathsf{U}$.
So that for all $u \in Q^{1/2}$: $\|u\|_{\mathsf{U}} = \|Q^{1/2}Q^{-1/2}u\|_{\mathsf{U}} \leqslant \|Q^{1/2}\|_{\mathsf{U}}\cdot\|u\|_{Q^{1/2}(\mathsf{U})}\,.$ $(\,*\,)$

Every $Q_i \in \Sigma$ is bounded in $L(\mathsf{U})$ and let $\|Q_i\|_{\mathsf{U}} \leqslant C^2$.

Also we have that $C^2 \geqslant \|Q_i\|_{\mathsf{U}} = \|Q_i^{1/2}Q_i^{1/2}\|_{\mathsf{U}} = \|Q_i^{1/2}\|_{\mathsf{U}}^2\,.$
From $(\,*\,)$ it follows that $\|u\|_{\mathsf{U}} \leqslant C\cdot\|u\|_{Q_i^{1/2}(\mathsf{U})}$
Therefore $Q^{1/2}(\mathsf{U}) \hookrightarrow \mathsf{U}\,,\quad \forall Q_i \in \Sigma\,.$ $(\,\#\,)$

Also it is clear that $\mathsf{U}_\Sigma \subset \mathsf{U}$.
Hence for all $u \in \mathsf{U}_\Sigma,\ u = \sum\limits_{i\geqslant 1} u_i$:

$$\|u\|_{\mathsf{U}} = \|\sum_{i\geqslant 1} u_i\|_{\mathsf{U}} \leqslant \sum_{i\geqslant 1}\|u_i\|_{\mathsf{U}} \overset{(\#)}{\leqslant} C\cdot\sum_{i\geqslant 1}\|u_i\|_{Q_i^{1/2}(\mathsf{U})}\,.$$

Taking inf we obtain $\|u\|_{\mathsf{U}} \leqslant C\cdot\|u\|_{\mathsf{U}_\Sigma}\,.$

So we have $\mathsf{U}_\Sigma \hookrightarrow \mathsf{U}\,.$

□

Lemma 4.4. $L(\mathsf{U},\Sigma) \hookrightarrow L(\mathsf{U}_\Sigma,\mathsf{H})$.

Proof.
Since $\|\Phi u\|_{\mathsf{H}} \leqslant \|\Phi\|_{L(\mathsf{U},\mathsf{H})} \cdot \|u\|_{\mathsf{U}} \overset{\textbf{Lm.4.3}}{\leqslant} C \cdot \|\Phi\|_{L(\mathsf{U},\mathsf{H})} \cdot \|u\|_{\mathsf{U}_\Sigma}$ and $\mathsf{U}_\Sigma \subset \mathsf{U}$, then we have $L(\mathsf{U},\Sigma) \subset L(\mathsf{U}_\Sigma,\mathsf{H})$ and $\|\Phi\|_{L(\mathsf{U}_\Sigma,\mathsf{H})} \leqslant C \cdot \|\Phi\|_{L(\mathsf{U},\mathsf{H})}$.
□

Below we will use a criterion of completeness of a normed space (see [8, Lm.2.2.1]) which we have formulated in the following lemma

Lemma 4.5.
Normed space A is complete if and only if when from $\sum\limits_{n\geqslant 1} \|a_n\|_A < \infty$ it follows that there exists an element $a \in A$ such that $a \overset{\|\cdot\|_A}{=} \sum\limits_{n\geqslant 1} a_n$.

Proposition 4.3. $\left(\mathsf{U}_\Sigma, \|\cdot\|_{\mathsf{U}_\Sigma}\right)$ *is a Banach space.*

Proof.
Let a sequence $\{u^n\} \subset \mathsf{U}_\Sigma$ such that $\sum\limits_{n=1}^{\infty} \|u^n\|_{\mathsf{U}_\Sigma} < \infty$.
For every element u^n there exists a subsequence $u^n_{i_n} \in Q^{1/2}_{i_n}(\mathsf{U})$, where $Q_{i_n} \in \Sigma$, such that $u^n = \sum\limits_{i_n=1}^{\infty} u^n_{i_n}$.
Since Σ is no countable, but we can always reduce to the case with a countable number of $Q \in \Sigma$:
Taking $\Sigma' := \bigcup\limits_{n\geqslant 1} \{Q_{i_n}\} = \{Q_i\}_{i\geqslant 1}$, we can see that it is countable.
So we can use here the set Σ' instead of Σ, and in the same time we can suppose that $u^n = \sum\limits_{i=1}^{\infty} u^n_i$, $u^n_i \in Q^{1/2}_i(\mathsf{U})$, where $u^n_i = u^n_{i_n} \cdot \mathbb{1}_{\{i=i_n\}}$, i.e. we renumber the elements and putting some zeros. It means that for every u^n there exists $u^n_i \in Q^{1/2}_i(\mathsf{U})$, where $Q_i \in \Sigma$ such that $u^n = \sum\limits_{i=1}^{\infty} u^n_i$.
So that, there exists a sequence $\{\varepsilon_n\}$ with $\sum\limits_{n\geqslant 1} \varepsilon_n < \infty$, such that:

$$\varepsilon_n + \|u^n\|_{\mathsf{U}_\Sigma} \geqslant \sum_{i=1}^{\infty} \|u^n_i\|_{Q^{1/2}_i(\mathsf{U})} \geqslant \|u^n_i\|_{Q^{1/2}_i(\mathsf{U})} . \qquad (*)$$

Therefore $\sum\limits_{n=1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} < \infty\,,\ u_i^n \in Q_i^{1/2}(\mathsf{U})$ – Banach.

Then by **Lm.4.5** it follows that there exists $v_i \in Q_i^{\frac{1}{2}}(\mathsf{U})$, such that $v_i \overset{\|\cdot\|_{Q_i^{1/2}(\mathsf{U})}}{=} \sum\limits_{n=1}^{\infty} u_i^n\,,$ for every i.

So that $v_i \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum\limits_{n=1}^{\infty} u_i^n\,.$

Taking $v \overset{\|\cdot\|_{\mathsf{U}}}{:=} \sum\limits_{i=1}^{\infty} v^i$, let us show that this series really converges in the U-norm:

From $(*)$ we have:

$$\infty > \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} \overset{\textbf{Lm.4.3}}{\geqslant} C\cdot \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|u_i^n\|_{\mathsf{U}}$$
$$\geqslant C\cdot \sum_{i=1}^{\infty} \|v_i\|_{\mathsf{U}}\,.$$

By **Lm.4.5** we deduce that $\sum\limits_{i=1}^{\infty} v^i$ converges in $\|\cdot\|_{\mathsf{U}}$.

Also we can show that $\sum\limits_{i=1}^{\infty} \|v^i\|_{Q_i^{1/2}(\mathsf{U})}$:

In fact, from $(*)$ follows:

$$\infty > \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} \geqslant \sum_{i=1}^{\infty} \|v_i\|_{Q_i^{1/2}(\mathsf{U})}\,.$$

Therefore $v \in \mathsf{U}_\Sigma$.

Consider $\|v - \sum\limits_{n=1}^{K} u^n\|_{U_\Sigma} = \|\sum\limits_{i=1}^{\infty} v_i - \sum\limits_{i=1}^{\infty}\sum\limits_{n=1}^{K} u_i^n\|_{U_\Sigma} \leqslant \sum\limits_{i=1}^{\infty} \|\sum\limits_{n=K+1}^{\infty} u_i^n\|_{Q_i^{1/2}(\mathsf{U})}$

$\leqslant \sum\limits_{i=1}^{\infty}\sum\limits_{n=K+1}^{\infty} \|u_i^n\|_{Q_i^{1/2}(\mathsf{U})} \xrightarrow[K\to\infty]{} 0\,,$ by the monotone convergence theorem.

Hence $v \overset{\|\cdot\|_{\mathsf{U}_\Sigma}}{=} \sum\limits_{n=1}^{\infty} u^n\,.$

According to **Lm.4.5** we can conclude that $\left(\mathsf{U}_\Sigma, \|\cdot\|_{\mathsf{U}_\Sigma}\right)$ is complete.

□

In the very beginning of this chapter we would like to describe a Banach space with the norm $\|\cdot\|^2_{L_2^\Sigma}$. So, we are ready to define it. This space is closely related to the construction of stochastic integral in the following chapter 5.

Definition 4.6. $L_2^\Sigma = \left\{\Phi \in L(\mathsf{U}_\Sigma, \mathsf{H}) \;\middle|\; \|\Phi\|_{L_2^\Sigma} < \infty\right\}.$

Remark 4.5. *From the definition above follows that if* $Q \in \Sigma$ *then* $\Phi Q^{1/2} \in L_2(\mathsf{U}, \mathsf{H})$

Lemma 4.6. $L_2^\Sigma \hookrightarrow L(\mathsf{U}_\Sigma, \mathsf{H})$*.*
Moreover $\|\cdot\|_{L(\mathsf{U}_\Sigma,\mathsf{H})} \leqslant \|\cdot\|_{L_2^\Sigma}$ *(note that here a constant* $C = 1$*).*

Proof.
Let $\Phi \in L_2^\Sigma$ then $\Phi \in L(\mathsf{U}_\Sigma, \mathsf{H})$.

$u \in \mathsf{U}_\Sigma$ it means that $u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum_{i=1}^{\infty} u_i,\ u_i \in Q_i^{1/2}(\mathsf{U}),\ Q_i \in \Sigma$.

Note also that $\|\Phi\|_{L_2^\Sigma} \geqslant c_i := \|\Phi Q_i^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})}$.

Therefore $\|\Phi u\|_{\mathsf{H}} = \|\Phi(\sum_{i=1}^{\infty} u_i)\|_{\mathsf{H}} = \|\sum_{i=1}^{\infty} \Phi u_i\|_{\mathsf{H}} = \|\sum_{i=1}^{\infty} \Phi Q_i^{1/2} Q_i^{-1/2} u_i\|_{\mathsf{H}}$

$\leqslant \sum_{i=1}^{\infty} \|\Phi Q_i^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \|Q_i^{-1/2} u_i\|_{\mathsf{U}} = \sum_{i=1}^{\infty} c_i \|u_i\|_{Q_i^{1/2}(\mathsf{U})} \leqslant \sum_{i=1}^{\infty} \|\Phi\|_{L_2^\Sigma} \|u_i\|_{Q_i^{1/2}(\mathsf{U})}$.

Taking $\inf$ for all such u_i we get $\|\Phi u\|_{\mathsf{H}} \leqslant \|\Phi\|_{L_2^\Sigma} \|u\|_{\mathsf{U}_\Sigma}$.

Hence $\|\Phi\|_{L(\mathsf{U}_\Sigma,\mathsf{H})} \leqslant \|\Phi\|_{L_2^\Sigma}$.

So we can deduce that $L_2^\Sigma \hookrightarrow L(\mathsf{U}_\Sigma, \mathsf{H})$.

□

Proposition 4.4. $\left(L_2^\Sigma, \|\cdot\|_{L_2^\Sigma}\right)$ *is a Banach space.*

Proof.
Let $\{\Phi_n\} \subset L_2^\Sigma$ be a Cauchy sequence: $\|\Phi_n - \Phi_m\|_{L_2^\Sigma} \xrightarrow[n,m\to\infty]{} 0$.

Or we can rewrite it as $\sup_{Q \in \Sigma} \|\Phi_n Q^{1/2} - \Phi_m Q^{1/2}\|^2_{L_2(\mathsf{U},\mathsf{H})} \xrightarrow[n,m\to\infty]{} 0$.

Let $Q \in \Sigma$ be an arbitrary fixed.
In the classical case the space $L_2^0 = L_2\big(Q^{1/2}(\mathsf{U}), \mathsf{H}\big)$ with the norm $\|\Phi\|^2_{L_2^0} := \mathrm{Tr}\big[\Phi Q \Phi^*\big] = \|\Phi Q^{1/2}\|^2_{L_2(\mathsf{U},\mathsf{H})}$ is Banach (see [32, 4.2]).

It implies that $\Phi_n \xrightarrow[n\to\infty]{\|\cdot\|_{L_2^0}} \Phi^Q$, i.e.: $\|\Phi_n Q^{1/2} - \Phi^Q Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \xrightarrow[n\to\infty]{} 0$.

Therefore $\sup\limits_{Q\in\Sigma} \|\Phi^Q Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant C_0$,

and also $\sup\limits_{Q\in\Sigma} \|\Phi_n Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant C_0$. $\qquad (\,*\,)$

If $u \in \mathsf{U}_\Sigma$ then by the definition we have:

$u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum\limits_{i=1}^{\infty} u_i$, $u_i \in Q_i^{1/2}(\mathsf{U}), Q_i \in \Sigma,$ $\sum\limits_{i=1}^{\infty} \|u_i\|_{Q_i^{1/2}(\mathsf{U})} < \infty$.

It follows that $\Phi_n u_i \xrightarrow[n\to\infty]{\|\cdot\|_{\mathsf{H}}} \Phi^{Q_i} u_i$, and we can get that

$$\|\Phi_n u_i\|_{\mathsf{H}} \xrightarrow[n\to\infty]{} \|\Phi^{Q_i} u_i\|_{\mathsf{H}}.$$

Define $\Phi u \overset{\|\cdot\|_{\mathsf{H}}}{:=} \sum\limits_{i=1}^{\infty} \Phi^{Q_i} u_i$.

For this reason we need to show that the series converges in the H-norm and that Φu is well defined:

(a) Consider $\sum\limits_{i=1}^{\infty} \|\Phi^{Q_i} u_i\|_{\mathsf{H}} \overset{\textbf{Lm. Fatou}}{\leqslant} \varliminf\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \|\Phi_n u_i\|_{\mathsf{H}}$

$= \varliminf\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \|\Phi_n Q_i^{1/2} Q_i^{-1/2} u_i\|_{\mathsf{H}} \leqslant \varliminf\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \|\Phi_n Q_i^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \cdot \|Q_i^{-1/2} u_i\|_{\mathsf{U}}$

$$\overset{(\,*\,)}{\leqslant} C_0 \cdot \sum_{i=1}^{\infty} \|u_i\|_{Q_i^{1/2}(\mathsf{U})} < \infty.$$

Taking $\inf$ we get that $\|\Phi u\|_{\mathsf{H}} \leqslant C_0 \cdot \|u\|_{\mathsf{U}_\Sigma}$.

(b) Let $u \overset{\|\cdot\|_{\mathsf{U}}}{=} \sum\limits_{i=1}^{\infty} v_i$ is another representation.

Note that $\sum\limits_{i=1}^{\infty} (u_i - v_i) = 0$ in the U-norm,

and by **Rem.4.4** the series also converges to zero in the U_Σ-norm.

It means that $\|\sum\limits_{i=1}^{n} (u_i - v_i)\|_{\mathsf{U}_\Sigma} = \|\sum\limits_{i=n+1}^{\infty} (u_i - v_i)\|_{\mathsf{U}_\Sigma} \xrightarrow[n\to\infty]{} 0$. $\qquad (\,\#\,)$

Consider $\|\sum\limits_{i=1}^{\infty} \Phi^{Q_i} u_i - \sum\limits_{i=1}^{\infty} \Phi^{Q_i} v_i\|_{\mathsf{H}} = \|\sum\limits_{i=1}^{\infty} \Phi^{Q_i} (u_i - v_i)\|_{\mathsf{H}}$

$$\leqslant \underbrace{\sum_{i=n+1}^{\infty} \|\Phi^{Q_i}(u_i - v_i)\|_{\mathsf{H}}}_{\textbf{(a)}:\ \searrow 0,\ n\to\infty} + \|\sum_{i=1}^{n} \Phi^{Q_i}(u_i - v_i)\|_{\mathsf{H}} \leqslant \varepsilon + \lim_{m\to\infty} \|\sum_{i=1}^{n} \Phi_m (u_i - v_i)\|_{\mathsf{H}}$$

$$\leqslant \varepsilon + \sup_{m\geqslant 1} \|\Phi_m\|_{L(\mathsf{U}_\Sigma,\mathsf{H})} \cdot \| \sum_{i=1}^{n} (u_i - v_i) \|_{\mathsf{U}_\Sigma} \overset{\textbf{Lm.4.6},\ (\#)}{\leqslant} \varepsilon + \sup_{m\geqslant 1} \|\Phi_m\|_{L_2^\Sigma} \cdot \varepsilon \leqslant \varepsilon + C \cdot \varepsilon\,.$$

So we have $\sum_{i=1}^{\infty} \Phi^{Q_i} u_i = \sum_{i=1}^{\infty} \Phi^{Q_i} v_i$, and Φu does not depend on the representation.

(c) Also $\Phi\big|_{Q^{1/2}(\mathsf{U})} = \Phi^Q$.

Because, if $v \in Q_1^{1/2}(\mathsf{U}) \cap Q_2^{1/2}(\mathsf{U})$, $Q_1, Q_2 \in \Sigma$ then

$$\Phi^{Q_1} v = \lim_{n\to\infty} \Phi_n v = \Phi^{Q_2} v\,.$$

So we can conclude that for every $u_i \in Q_i^{1/2}(\mathsf{U})$: $\Phi^{Q_i} u_i = \Phi\, u_i$.

And we have that Φ is defined correctly.
It is also clear that Φ is linear, and boundedness follows from **(a)**.

Now let us turn back to the Cauchy sequence $\{\Phi_n\}$, so that:
$\forall \varepsilon > 0\ \exists N\ \ \forall n, m > N$: $\|\Phi_n Q^{1/2} - \Phi_m Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} < \varepsilon\ \ \forall Q \in \Sigma$.
Letting $m \to \infty$ yields $\|\Phi_n Q^{1/2} - \Phi^Q Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant \varepsilon\ \ \forall Q \in \Sigma$.
Or we can rewrite and obtain that:
$\forall \varepsilon > 0\ \exists N\ \forall n, m > N$: $\sup_{Q\in\Sigma} \|\Phi_n Q^{1/2} - \Phi^Q Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant \varepsilon$.
What implies that:
$\forall \varepsilon > 0\ \exists N\ \forall n, m > N$: $\sup_{Q\in\Sigma} \|\Phi_n Q^{1/2} - \Phi Q^{1/2}\|_{L_2(\mathsf{U},\mathsf{H})} \leqslant \varepsilon$.
And it is the same as $\|\Phi_n - \Phi\|_{L_2^\Sigma} \xrightarrow[n\to\infty]{} 0$.

□

4.5 Existence of G-normal distribution

Consider the G-PDE (P0). For the following calculations we change a time $t \longrightarrow T - t$ and obtain the analogous G-PDE.

$$\begin{cases} \partial_t u - G(D^2_{xx} u) = 0\,, & t \in (0, T]\,,\ x \in \mathsf{H}; \\ u(0, x) = f(x)\,. \end{cases} \tag{P'0}$$

Note that all results for viscosity solution to the G-PDE (P0) are also the same as for the (P′0). So, we can fix $u = u^f(t, x)$ as a unique viscosity solution to (P′0).

Lemma 4.7 (Some properties of the viscosity solutions of (P′0))**.**
1) $u^{u^f(s,\cdot)}(t,x) = u^f(t+s,x)\,;$
2) if $f \equiv C$ *then* $u^f \equiv C\,;$
3) $u^{f(\bar{x}+\sqrt{\lambda}\,\circ)}(1,0) = u^{f\,(\sqrt{\lambda}\,\circ)}(1,\frac{\bar{x}}{\sqrt{\lambda}}) = u^{f\,(\circ)}(\lambda,\bar{x})\,, \quad \lambda \geqslant 0\,;$
4) $u^{\lambda f} = \lambda u^f\,, \quad \lambda \geqslant 0\,;$
5) if $f \leqslant g$ *then* $u^f \leqslant u^g\,;$
6) $u^{f+g} \leqslant u^f + u^g\,.$

Proof.
We will prove just the **1)**, because the rest items can be straightforwardly checked implying that $u^f(t,x) = \mathbb{E}[(x+B_t)]\,.$
Let us consider a boundary condition $u(0,x) = u^f(s,x)\,.$
We can carry initial point $''0''$ to $''t''$ so that $u(t,x) = u^f(s+t,x)\,.$ On the other hand $u(t,x) = u^{u^f(s,\cdot)}(t,x)\,.$
What implies that $u^{u^f(s,\cdot)}(t,x) = u^f(t+s,x)\,.$

□

Theorem 4.2. *Let* $G(\cdot)$ *is a given* G*-functional.*
Then:
1) There exists $\xi \sim N_G(0,\Sigma)\,;$
2) There exist a sequence $\{\xi_i\,,\ i \geqslant 1\}$ *such that for every* i
$$\xi_i \sim N_G(0,\Sigma) \quad \textit{and} \quad \xi_{i+1} \perp\!\!\!\perp (\xi_1,\ldots,\xi_i)\,.$$

Proof.
1) This part of the proof is similar to 1-dimensional case (see [72, II.2]), however there are some specific moments, which compel us to bring it with all detail.

Let us consider the following spaces:
$\widetilde{\Omega} := \mathsf{H}\times\mathsf{H}$, $\widetilde{\mathcal{H}} := \mathbf{C}_{p.Lip}(\mathsf{H}\times\mathsf{H})$ and $\omega := (x,y) \in \mathsf{H}\times\mathsf{H}\,.$
Define a functional $\widetilde{\mathbb{E}}[\,\cdot\,]$ in the following way:
for a fixed $\psi \in \mathbf{C}_{p.Lip}(\mathsf{H}\times\mathsf{H})$ and for an element $X(\omega) := \psi(x,y) \in \widetilde{\mathcal{H}}$ we have:
$$\widetilde{\mathbb{E}}[X] := u^{\bar{\psi}(\cdot)}(1,0), \quad \bar{\psi}(x) = \psi^{(x,\cdot)}(1,0).$$
It is clear that a functional $\widetilde{\mathbb{E}}[X]$ is a sublinear expectation:
(a): $X \geqslant Y \quad \Rightarrow \quad \varphi \geqslant \psi \quad \Rightarrow \quad u^{\varphi} \geqslant u^{\psi} \quad \Rightarrow \quad \widetilde{\mathbb{E}}[X] \geqslant \widetilde{\mathbb{E}}[Y]\,;$

(b): $X = C \quad \Rightarrow \quad \varphi = C \quad \Rightarrow \quad u^{\varphi} = C \quad \Rightarrow \quad \widetilde{\mathbb{E}}[X] = C\,;$
(c): $\widetilde{\mathbb{E}}[X+Y] = u^{\varphi+\psi} \leqslant u^{\varphi} + u^{\psi} = \widetilde{\mathbb{E}}[X] + \widetilde{\mathbb{E}}[Y]\,;$
(d): $\widetilde{\mathbb{E}}[\lambda X] = u^{\lambda\varphi} = \lambda u^{\varphi} = \lambda\widetilde{\mathbb{E}}[X]\,, \quad \lambda \geqslant 0\,.$

Consider $\xi(\omega) = \xi(x,y) := x\,;$
$\eta(\omega) = \eta(x,y) := y\,;$

Using **Lm.4.7,2)** we have $\widetilde{\mathbb{E}}\big[\varphi(\xi)\big] = u^{\varphi}(1,0) = \widetilde{\mathbb{E}}\big[\varphi(\eta)\big]\,.$
Hence $\xi \sim \eta\,.$
Also $\widetilde{\mathbb{E}}[\psi(\xi,\eta)] = u^{\left[u^{\varphi(x,\cdot)}(1,0)\right](\cdot)}(1,0) = \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{E}}[\psi(\alpha,\eta)]_{\alpha=\xi}\Big]$
Therefore $\xi \overset{d}{=} \eta\,.$

For a fixed function $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{H})$ consider $v(t,x) := u^{\varphi}(\lambda t, \bar{x} + \sqrt{\lambda} x)\,,$ where $\lambda > 0,\ \bar{x} \in \mathsf{H}\,.$

It implies that $v(0,x) = u^{\varphi}(0, \bar{x} + \lambda x) = \varphi(\bar{x} + \lambda x) = \varphi(\bar{x} + \sqrt{\lambda}\, x)\,,$
and $\partial_t v - G(D^2_{xx} v) = \lambda \cdot \partial_t u^{\varphi} - G(\lambda D^2_{xx} u^{\varphi}) = \lambda\big(\partial_t u^{\varphi} - G(D^2_{xx} u^{\varphi})\big)\,.$
I.e., $v(t,x)$ is a viscosity solution to the equation

$$\begin{cases} \partial_t v - G(D^2_{xx} v) = 0\,, & t \in (0,T]\,,\ x \in \mathsf{H}; \\ v(0,x) = \varphi(\bar{x} + \sqrt{\lambda}\, x)\,. \end{cases}$$

By **Lm.4.7,3)** we have $\widetilde{\mathbb{E}}[\varphi(\bar{x} + \sqrt{\lambda}\xi)] = u^{\varphi(\bar{x}+\sqrt{\lambda}\,\circ)}(1,0) = u^{\varphi\,(\circ)}(\lambda, \bar{x})\,.$

And $\widetilde{\mathbb{E}}[\varphi(\sqrt{t}\,\xi + \sqrt{s}\,\eta)] = \widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{E}}[\varphi(\sqrt{t}\,x + \sqrt{s}\,\eta)]_{x=\xi}\Big] = \widetilde{\mathbb{E}}\Big[u^{\varphi}(s, \sqrt{t}\,x)\big|_{x=\xi}\Big]$
$= \widetilde{\mathbb{E}}[u^{\varphi}(s, \sqrt{t}\,\xi)] = u^{u^{\varphi(s,\cdot)}}(t,0) = u^{\varphi}(t+s,0) = \widetilde{\mathbb{E}}[\varphi(\sqrt{t+s}\xi)]\,.$
And we can conclude $\sqrt{t}\,\xi + \sqrt{s}\,\eta \overset{d}{=} \sqrt{t+s}\,\xi\,.$
So that ξ, η are G-normal distributed.
2) The case $i = 2$ is already proved in **1)**. Assume that it holds also for some $i \geqslant 2$ and let us pass to $i+1$.
I.e., let there exist $(\xi_1, \ldots, \xi_i)$, such that $\xi_j \sim N_G(0, \Sigma)$ and $\xi_j \perp\!\!\!\perp (\xi_1, \ldots, \xi_{j-1})\,.$
We denote the next spaces as follows:
$$\widetilde{\Omega}^{(i)} := \underbrace{\mathsf{H}\times, \ldots \times\mathsf{H}}_{i}, \quad \widetilde{\mathcal{H}} := \mathbf{C}_{p.Lip}(\widetilde{\Omega}^{(i)})\,.$$

Take a space $\widetilde{\Omega}^{(i+1)}$, $\omega := (x,y) \in \mathsf{H} \times \widetilde{\Omega}^{(i)}$.
And defining in the same way $\xi(\omega) = \xi(x,y) := x \in \mathsf{H}$;
$\eta(\omega) = \eta(x,y) := y \in \widetilde{\Omega}^{(i)}$,

we can proceed the same reasonings like in **1)**, we obtain what we need, namely, that there exists a $\xi_{i+1} := \xi$ and $(\xi_1, \ldots, \xi_i) := \eta$, such that $\xi_{i+1} \sim N_G(0, \Sigma)$ and $\xi_{i+1} \perp\!\!\!\perp (\xi_1, \ldots, \xi_i)$.

□

Theorem 4.3. *If* G *– is a given* G*-functional and* $u(t,x) := \mathbb{E}f(x + \sqrt{T-t}X)$ *is a viscosity solution to equation* (P0)*. Then* $X \sim N_G(0, \Sigma_G)$ *.*

Proof.
Since u is a unique viscosity solution to equation (P0) by **Th.4.1** then $u(t,x) = \mathbb{E}f(x + B_{T-t}) = \mathbb{E}f(x + \sqrt{T-t}X)$, where B_t is a G-Brownian motion with a covariation set Σ_G. Note that $\sqrt{T-t}X \sim B_{T-t}$, where $X \sim B_1 \sim N_G(0, \Sigma_G)$.

□

Remark 4.6.
Let $X \sim N_G\big(0, \Sigma\big)$ *, then* $(-X) \sim X$ *.*

Proof.
$G_X(A) = \dfrac{1}{2}\mathbb{E}[\langle AX, X\rangle] = G_{-X}(A)$, and it means that

$$\mathbf{F}_{-X}[\varphi] = u(1,0) = \mathbf{F}_X[\varphi] .$$

□

4.6 Existence of G-Brownian motion and notion of G-expectation

Let U, H will be Banach spaces. Recall (see 2.3) that

$$\mathbf{C}_{p.Lip}(\mathsf{U},\mathsf{H}) := \Big\{\varphi : \mathsf{U} \to \mathsf{H} \;\Big|\; \|\varphi(x) - \varphi(y)\|_{\mathsf{H}} \leqslant C \cdot \big(1 + \|x\|^m_{\mathsf{U}} + \|y\|^m_{\mathsf{U}}\big) \cdot \|x - y\|_{\mathsf{U}}\Big\} .$$

Later on we will need a bounded subspace:
$\mathbf{C}_{b,\,p.Lip}(\mathsf{U},\mathsf{H}) := \mathbf{C}_{p.Lip}(\mathsf{U},\mathsf{H}) \cap \mathbf{C}_b(\mathsf{U},\mathsf{H})$.

Let $\Omega = \big\{\omega_t : [0, +\infty) \to \mathsf{U} \mid \omega_0 = 0,\ \omega_t\text{–continuous}\big\}$;
$G(\,\cdot\,):\ K_S(\mathsf{H}) \to \mathbb{R}$ is a G-functional.
$B_t(\omega_s) := \omega_s\big|_{s=t}$ is a canonical process;
$Lip(\Omega_t) := \Big\{\varphi(B_{t_1}, .., B_{t_n}) \mid n \geqslant 1,\ t_1, .., t_n \in [0, t],\ \varphi \in \mathbf{C}_{p.Lip}\,(\mathsf{U}^n, L_2^\Sigma)\Big\}$;
$Lip(\Omega) := \bigcup\limits_{n\geqslant 1} Lip(\Omega_t)$.

Remark 4.7. *If* $X \in Lip(\Omega)$ *then* $\mathbb{E}\big[\|X\|^p_{L_2^\Sigma}\big] < \infty$, $p > 0$.

Proof.
We will use such a trivial fact:
$\forall a > 0 \quad \forall m > 0 \quad \exists M > m \quad \exists C > 0\ :\quad a(1 + a^m) \leqslant C(1 + a^M)$.
It means if $\varphi \in \mathbf{C}_{p.Lip}\,(\mathsf{U}^n, L_2^\Sigma)$ then

$$\|\varphi(x_1, .., x_n)\|_{L_2^\Sigma} \leqslant C \cdot \big(1 + \|(x_1, .., x_n)\|^M_{\mathsf{U}^n}\big) = C \cdot \Big(1 + \big(\sum_{k=1}^n \|x_k\|^2_{\mathsf{U}}\big)^{M/2}\Big) \leqslant C \cdot \Big(1 + \sum_{k=1}^n \|x_k\|^M_{\mathsf{U}}\Big).$$

So that, φ is a polynomial growth function.

We also have

$$X = \varphi(B_{t_1}, .., B_{t_n}) = \varphi\big(\sqrt{t_1}X^{(1)}, \ldots, \sqrt{t_n}X^{(n)}\big) = \widetilde{\varphi}\big(X^{(1)}, \ldots, X^{(n)}\big),$$
where $B_1 \sim X^{(k)} \sim N_G(0, \Sigma)$.

Hence $\|X\|^p_{L_2^\Sigma} = \|\widetilde{\varphi}\|^p_{L_2^\Sigma}\big(X^{(1)}, \ldots, X^{(n)}\big)$,
and $\|\widetilde{\varphi}\|^p_{L_2^\Sigma} : \mathsf{U}^n \to \mathbb{R}$ is a polynomial growth function.

Therefore

$$\mathbb{E}\big[\|X\|^p_{L_2^\Sigma}\big] \leqslant C \cdot \mathbb{E}\Big[\big(1 + \sum_{k=1}^n \|X^{(k)}\|^M_{\mathsf{U}}\big)\Big] \leqslant C \cdot \big(1 + \sum_{k=1}^n \mathbb{E}\big[\|X^{(k)}\|^M_{\mathsf{U}}\big]\big) \overset{\textbf{Prop.2.4}}{<} \infty.$$

□

Proposition 4.5. *Under settled above conditions there exists a sublinear expectation* $\mathbb{E}\big[\psi(\,\cdot\,)\big] = \mathbf{F}.\big[\psi\big] : Lip(\Omega) \to \mathbb{R}$, *such that* B_t *is a* G*-Brownian motion on* $\big(\Omega,\ Lip(\Omega),\ \mathbb{E}\big)$, *where* $\psi \in \mathbf{C}_{p.Lip}\,(L_2^\Sigma)$.

Proof.
Let $\big(\tilde{\Omega},\ \tilde{\mathcal{H}},\ \tilde{\mathbb{E}}\big)$ is a fixed sublinear expectation space.
According to **Th.4.2** we can construct on this space the sequence of G-normal distributed random variables $\big\{\xi_i\,,\ i \geqslant 1\big\}$, such that ξ_{i+1} is independent from $(\xi_1, .., \xi_i)$.
If $\forall X \in Lip(\Omega)$ then it can be represented as follows:
$X = \tilde{\varphi}(B_{t_1}, .., B_{t_n}) = \varphi(B_{t_1} - B_{t_0}, .., B_{t_n} - B_{t_{n-1}})$.
for some $\varphi, \tilde{\varphi} \in \mathbf{C}_{p.Lip}(\mathsf{U}^n, L_2^{\Sigma})$, $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n < \infty$.

For every $\psi \in \mathbf{C}_{p.Lip}(L_2^{\Sigma})$ define

$$\mathbb{E}[\psi(X)] = \mathbb{E}\big[\psi \circ \varphi(B_{t_1} - B_{t_0}, .., B_{t_n} - B_{t_{n-1}})\big] := \tilde{\mathbb{E}}\big[\psi \circ \varphi(\sqrt{t_1 - t_0}\,\xi_1, .., \sqrt{t_n - t_{n-1}}\,\xi_n)\big].$$

Since $\tilde{\mathbb{E}}$ is sublinear expectation then $\mathbb{E}$ also satisfies all the properties of sublinear expectation.

In order to prove that B_t is a G-Brownian motion we use **Prop.4.1**:

$$G_{B_t}(A) = \mathbb{E}\langle AB_t, B_t\rangle = \tilde{\mathbb{E}}\langle A\sqrt{t}\,\xi_1, \sqrt{t}\,\xi_1\rangle = t\,\tilde{\mathbb{E}}\langle A\xi_1, \xi_1\rangle = t\,\mathbb{E}\langle AB_1, B_1\rangle = \mathbb{E}\langle A(\sqrt{t}\,B_1), \sqrt{t}\,B_1\rangle = G_{\sqrt{t}\,B_1}(A).$$

So that $B_t \sim \sqrt{t}B_1$.
Consider $u(t,x) := \mathbb{E}\big[f(x + B_{T-t})\big] = \tilde{\mathbb{E}}\big[f(x + \sqrt{T-t}\,\xi_1)\big]$.
Since ξ_1 is G-normal distributed then $u(t,x)$ is a viscosity solution to the equation

$$\begin{cases} \partial_t u + G_{\xi_1}(D^2_{xx}u) = 0\,, & t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases}$$

But $G_{\xi_1}(A) = \tilde{\mathbb{E}}\langle A\xi_1, \xi_1\rangle = \mathbb{E}\langle AB_1, B_1\rangle = G_{B_1}(A)$.
It follows that $u(t,x)$ is a viscosity solution to the equation

$$\begin{cases} \partial_t u + G_{B_1}(D^2_{xx}u) = 0\,, & t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases}$$

And we can conclude that B_1 is G-distributed.
So, using **Prop.4.1** it implies that B_t is a G-Brownian motion.

□

Definition 4.7. *Such sublinear expectation $\mathbb{E}$ will be called G-expectation.*

In the same manner we can define related conditional G-expectation with respect to Ω_t:
For $X \in Lip(\Omega)$ which can be represented in the following form
$X = \varphi(B_{t_1} - B_{t_0}, .., B_{t_n} - B_{t_{n-1}})$, where $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^n, L_2^\Sigma)$,
$0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n < \infty$,
we have the next definition:

Definition 4.8. *Conditional G-expectation $\mathbb{E}[\,\cdot \mid \Omega_{t_j}]$ is defined as*

$$\mathbb{E}[\psi(X) \mid \Omega_{t_j}] := \chi(B_{t_1} - B_{t_0}, .., B_{t_j} - B_{t_{j-1}}),$$

where $\chi(x_1, .., x_j) = \tilde{\mathbb{E}}\big[\psi \circ \varphi(x_1, .., x_j, \sqrt{t_{j+1} - t_j}\,\xi_j, .., \sqrt{t_n - t_{n-1}}\,\xi_n)\big]$,
$\psi \in \mathbf{C}_{p.Lip}(L_2^\Sigma)$.

Remark 4.8. *Since φ and ψ are Lipschitz with a polynomial growth then χ is also Lipschitz with a polynomial growth.*
Hence $\chi(B_{t_1} - B_{t_0}, .., B_{t_j} - B_{t_{j-1}}) \in \mathcal{H}_0$.
Let us define a space $L_G^1(\Omega)$ to be the completion of $\mathcal{H}$ under the norm $\|\cdot\| = \mathbb{E}[\,\cdot\,]$.
Thus we will consider an unconditional G-expectation with values in the complete space:
$\mathbb{E}[\psi(X) \mid \Omega_{t_j}] : Lip(\Omega) \to L_G^1(\Omega)$.

Remark 4.9. *Actually, we apply the definitions of conditional and unconditional G-expectation only when $\psi = \|\cdot\|^p_{L_2^\Sigma}$.*

Let us define the following space:

$$Lip(\Omega^t) := \Big\{\varphi(B_{t_2} - B_{t_1}, .., B_{t_{n+1}} - B_{t_n}) \mid n \geqslant 1,\ t_1, .., t_n \in \mathbb{R}^+, \\ \varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^n, L_2^\Sigma)\Big\}.$$

Since a conditional G-expectation $\mathbb{E}[\,\cdot \mid \Omega_t]$ relates to a sublinear expectation $\tilde{\mathbb{E}}[\cdot]$, we can conclude that $\mathbb{E}[\,\cdot \mid \Omega_t]$ is also a sublinear expectation. The following properties are trivial consequences of the definition:

Proposition 4.6. *Let* $X \in Lip(\Omega)$, $Z_1 \in Lip(\Omega_t)$, $Z_2 \in Lip(\Omega^t)$,
$\varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^n, L_2^\Sigma)$.

Then the following equalities hold:

(1) $\mathbb{E}\Big[\mathbb{E}[\varphi(X) \mid \Omega_t] \mid \Omega_s\Big] = \mathbb{E}[\varphi(X) \mid \Omega_{t\wedge s}]$;
$\mathbb{E}\Big[\mathbb{E}[\varphi(X) \mid \Omega_t]\Big] = \mathbb{E}[\varphi(X)]$;

(2) $\mathbb{E}[\varphi(Z_1) \mid \Omega_t] = \varphi(Z_1)$;

(3) $\mathbb{E}[\varphi(Z_2) \mid \Omega_t] = \mathbb{E}[\varphi(Z_2)]$.

Remark 4.10. *Due to this* ***Prop.4.6*** *and its properties* (2) *and* (3) *respectively we will call the variable* Z_1 *as* Ω_t***-measurable****, and* Z_2 *as* ***independent from*** Ω_t.

Now we are going to describe an extension of the G-expectation on the completion of the space $Lip(\Omega)$. We will make such completion under the norm $\|X\|^p_{\Sigma,p} = \mathbb{E}\Big[\|X\|^p_{L_2^\Sigma}\Big]$, since this norm on $Lip(\Omega)$ is finite by **Rem.4.7**.
Denote ${}^{\mathsf{H}}L^p_G(\Omega) = \text{compl}\Big(Lip(\Omega), \mathbb{E}\big[\|\cdot\|^p_{L_2^\Sigma}\big]\Big)$.

Proposition 4.7. *G-expectation* $\mathbb{E}\big[\|\cdot\|^p_{L_2^\Sigma}\big] : Lip(\Omega) \to \mathbb{R}$ *as well as conditional one* $\mathbb{E}\big[\|\cdot\|^p_{L_2^\Sigma} \mid \Omega_t\big] : Lip(\Omega) \to L^1_G(\Omega)$ *can be continuously extended to the space* ${}^{\mathsf{H}}L^p_G(\Omega)$.

Proof.
If $X \in {}^{\mathsf{H}}L^p_G(\Omega)$ then there exist $X_n \in Lip(\Omega)$, such that
$\mathbb{E}\Big[\|X - X_n\|^p_{L_2^\Sigma}\Big] \xrightarrow[n\to\infty]{} 0$.
Here we take $p \geqslant 2$ because the case $p = 1$ is trivial.

1) Firstly we show the following convergence (which we use for the extension):

$$\Big|\mathbb{E}\Big[\|X\|^p_{L_2^\Sigma}\Big] - \mathbb{E}\Big[\|X_n\|^p_{L_2^\Sigma}\Big]\Big| \leqslant \mathbb{E}\Big[\Big|\|X\|^p_{L_2^\Sigma} - \|X_n\|^p_{L_2^\Sigma}\Big|\Big]$$
$$\leqslant \mathbb{E}\Big[\|X - X_n\|_{L_2^\Sigma} \cdot \big(\|X\|^{p-1}_{L_2^\Sigma} + \ldots + \|X_n\|^{p-1}_{L_2^\Sigma}\big)\Big]$$

$$\leqslant \left(\mathbb{E}\Big[\|X - X_n\|^2_{L_2^\Sigma}\Big]\right)^{1/2} \cdot C \cdot \left(\mathbb{E}\Big[\big(\|X\|^{2(p-1)}_{L_2^\Sigma} + \|X_n\|^{2(p-1)}_{L_2^\Sigma}\big)\Big]\right)^{1/2}$$

$$\leqslant \widetilde{C} \cdot \left(\mathbb{E}\Big[\|X - X_n\|^p_{L_2^\Sigma}\Big]\right)^{1/p} \xrightarrow[n\to\infty]{} 0\,.$$

2) Now it is enough to only take the more general case with a conditional G-expectation. We consider an operator T on the space $Lip(\Omega)$, such that $TX_n = \mathbb{E}\big[\|X_n\|^p_{L_2^\Sigma} \mid \Omega_t\big]$.

Let us define $TX := \lim\limits_{n\to\infty} TX_n$ if $\mathbb{E}\Big[\big|TX - TX_n\big|\Big] \xrightarrow[n\to\infty]{} 0$.

Now we are going to show that the sequence $\big(TX_n\big)$ is Cauchy in $L^1_G(\Omega)$.

Consider $\mathbb{E}\Big[\Big|\mathbb{E}\big[\|X_n\|^p_{L_2^\Sigma} \mid \Omega_t\big] - \mathbb{E}\big[\|X_m\|^p_{L_2^\Sigma} \mid \Omega_t\big]\Big|\Big]$

$$\leqslant \mathbb{E}\left[\mathbb{E}\Big[\big|\|X_n\|^p_{L_2^\Sigma} - \|X_m\|^p_{L_2^\Sigma}\big| \mid \Omega_t\Big]\right] \overset{\textbf{Prop.4.6,1)}}{=} \mathbb{E}\Big[\big|\|X_n\|^p_{L_2^\Sigma} - \|X_m\|^p_{L_2^\Sigma}\big|\Big]$$

$$\overset{\textbf{1)}}{\leqslant} \widetilde{C} \cdot \left(\mathbb{E}\Big[\|X_n - X_m\|^p_{L_2^\Sigma}\Big]\right)^{1/p} \xrightarrow[n,m\to\infty]{} 0\,,$$

since the sequence (X_n) is convergent in ${}^{\mathsf{H}}L^p_G\big(\Omega\big)$.

So, we have that $\lim\limits_{n\to\infty} TX_n$ exists, because $L^1_G(\Omega)$ is complete.

If we take another sequence (Y_n) of elements from $Lip(\Omega)$ such that (TY_n) converges to TX, then we have

$\mathbb{E}\Big[\big|TX_n - TY_n\big|\Big] \leqslant \mathbb{E}\Big[\big|TX_n - TX\big|\Big] + \mathbb{E}\Big[\big|TY_n - TX\big|\Big] \xrightarrow[n\to\infty]{} 0$.

And by **Lm.4.1** we see that $TX_n = TY_n$ q.s.

Thus we have an extension of conditional G expectation to ${}^{\mathsf{H}}L^p_G\big(\Omega\big)$:

$\mathbb{E}\big[\|X\|^p_{L_2^\Sigma} \mid \Omega_t\big] := \lim\limits_{n\to\infty} \mathbb{E}\big[\|X_n\|^p_{L_2^\Sigma} \mid \Omega_t\big]$ in $L^1_G(\Omega)$ norm.

□

Remark 4.11. *From the definition of the operator T in the proof of* ***Prop.4.7*** *and by passing to the limit, one can easily seen that* ***Prop.4.6*** *holds for extended conditional G-expectation on ${}^{\mathsf{H}}L^p_G\big(\Omega\big)$ for $\varphi = \|\cdot\|^p_{L_2^\Sigma}$.*

4.7 G-expectation and upper expectation

In this chapter we consider a notion of an upper expectation and compare it with a G-expectation in order to see how they are related to one with another. Later on we deduce see that they coincide on a considered above Banach space ${}^{\mathsf{H}}L_G^p(\Omega)$. The material for this theory is taken mainly from [36], where was considered only one-dimensional case. So, we follow this reference in order to give detailed description of the material in infinite dimensions. The proofs will be given only those ones that differ from the one-dimensional case, in the same time we will only point out those ones that can be just repeated.

In the next chapter dedicated to the construction of stochastic integral with respect to G-Brownian motion we widely use such a space ${}^{\mathsf{H}}L_G^p(\Omega)$ for it. There for a correct representation we shall be sure that ${}^{\mathsf{H}}L_G^p(\Omega)$ is not just an abstract completion of $Lip(\Omega)$ under the norm $\mathbb{E}\big[\|\cdot\|^p_{L_2^\Sigma}\big]$, but a space of random variables.

Let $(\Omega, \mathcal{F}, P)$ be a probability space,
where $\Omega = \big\{\omega_t : [0,+\infty) \to \mathsf{U} \mid \omega_0 = 0,\ \omega_t\text{–continuous}\big\}$, the space of continuous trajectories as mentioned above. W_t is a cylindrical Wiener process in U under the measure P. $\mathcal{F}_t := \sigma\{W_u,\ 0 \leqslant u \leqslant t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of P-null subsets.

Fixing Σ-set we define a G-functional, i.e. $G(A) = \dfrac{1}{2}\sup\limits_{Q\in\Sigma} \mathrm{Tr}\big[A\cdot Q\big]$.

It follows that there exists a bounded closed set Θ of Hilbert-Schmidt operators, such that $\Sigma = \big\{Q \mid Q = \gamma\cdot\gamma^T,\ \gamma\in\Theta\big\}$ and

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Theta}\mathrm{Tr}\big[\gamma\gamma^T\cdot A\big].$$

Let us define a following set of random processes
$\mathcal{A}^\Theta_{t,T} := \big\{\theta_s : \Omega \to \Theta \mid s\in[t,T]\subset[0,\infty),\ \theta_s - (\mathcal{F}_s)\text{-adopted}\big\}$.

Since $\theta_s \in \Theta$ it follows that $\int\limits_t^T \mathrm{Tr}[\theta_s\,\theta_s^T]\,ds < \infty$.

For a B-continuous function $\varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^n)$ let us consider such a function $v(t,x) := \sup\limits_{\theta\in\mathcal{A}^\Theta_{t,T}} E_P\big[\varphi(x + B_T^{t,\theta})\big]$, where $B_T^{t,\theta} := \int\limits_t^T \theta_s\,dW_s$, $\theta_s \in \mathcal{A}^\Theta_{t,T}$.

Remark 4.12. *Let* P_θ *be a law of the process* $B_t^{0,\theta} = \int\limits_0^t \theta_s \, dW_s$ *.* P_θ *is a probability on* $\Omega = \{\omega_t : [0, +\infty) \to \mathsf{U} \mid \omega_0 = 0, \ \omega_t\text{–continuous}\}$ *.*
Elements of Ω *are* $B_t(\omega) = \omega_t$ *.*
So, we have

$$E_P\big[\varphi(B_{t_1}^{0,\theta}, \dots, B_{t_n}^{0,\theta})\big] = E_{P_\theta}\big[\varphi(B_{t_1}, \dots, B_{t_n})\big] \quad .$$

Note also that $B_t^{s,\theta} = B_t^{0,\theta} - B_s^{0,\theta}$ *by definition. Also in the same manner we denote that* $B_t^s := B_t - B_s$ *.*

Theorem 4.4. v *is a viscosity solution to the G-heat equation:*

$$\begin{cases} \partial_t v + G(D_{xx}^2 v) = 0 \, ; \\ v(T, x) = \varphi(x) \, . \end{cases} \tag{11}$$

Proof.
The proof completely coincide with the finite dimensional case (see [36, Th.47]).

□

Here we have that:
$v(0, x) = \sup\limits_{\theta \in \mathcal{A}_{0,T}^{\Theta}} E_P\big[\varphi(x + B_T^{0,\theta})\big] = \sup\limits_{\theta \in \mathcal{A}_{t,T}^{\Theta}} E_{P_\theta}\big[\varphi(x + B_T)\big] \equiv \mathbb{E}\big[\varphi(x + B_T)\big]$,
since such a viscosity solution is unique by **Th.3.2**.
For the future work we need the following auxiliary result that generalizes dynamical programming principle:

Lemma 4.8. *Let* $\zeta \in L^2(\Omega, \mathcal{F}_s, P; \mathsf{H})$ *,* $0 \leqslant s \leqslant t \leqslant T$ *.*
Then $\forall \varphi \in \mathbf{C}_{p.Lip}(\mathsf{H} \times \mathsf{U} \times \mathsf{U} \to \mathbb{R})$ *:*

$$ess \sup_{\theta \in \mathcal{A}_{s,T}^{\Theta}} E_P\big[\varphi(\zeta, B_t^{s,\theta}, B_T^{t,\theta}) \big| \mathcal{F}_s\big] = ess \sup_{\theta \in \mathcal{A}_{s,t}^{\Theta}} E_P\big[\psi(\zeta, B_t^{s,\theta}) | \mathcal{F}_s\big] \quad ,$$

where $\psi(x, y) := ess \sup\limits_{\overline{\theta} \in \mathcal{A}_{t,T}^{\Theta}} E_P\big[\varphi(x, y, B_T^{t,\overline{\theta}}) \big| \mathcal{F}_t\big] = \sup\limits_{\overline{\theta} \in \mathcal{A}_{t,T}^{\Theta}} E_P\big[\varphi(x, y, B_T^{t,\overline{\theta}})\big]$ *.*

Proof.
If $\varphi \in \mathbf{C}_{b,\,p.Lip}$ then the proof of this result is completely according to the finite dimensional case, see [36, Th.44].
Let $\varphi \in \mathbf{C}_{p.Lip}$. Hence, the truncation $\varphi_n := \varphi \cdot \mathbb{1}_{\{|\varphi|\leqslant n\}} + n \cdot \dfrac{\varphi}{|\varphi|} \cdot \mathbb{1}_{\{|\varphi|>n\}}$ satisfies the statement of the lemma. Also, it is clear that $\varphi \in L_1$ w.r.t. P (because of the gaussianity of $B_t^{s,\theta}$). $\quad (\,*\,)$

And it remains just to estimate the following expression:

$$\Big|E_P\big[\varphi(\,\cdot\,)\big|\mathcal{F}_s\big] - E_P\big[\varphi_n(\,\cdot\,)\big|\mathcal{F}_s\big]\Big| \leqslant E_P\Big[\Big|\varphi(\,\cdot\,) - \varphi_n(\,\cdot\,)\Big|\,\Big|\mathcal{F}_s\Big]$$
$$= E_P\Big[\big(|\varphi|(\,\cdot\,) - n\big)\cdot \mathbb{1}_{\{|\varphi|>n\}}\big|\mathcal{F}_s\Big] \leqslant E_P\Big[|\varphi|(\,\cdot\,)\cdot \mathbb{1}_{\{|\varphi|>n\}}\big|\mathcal{F}_s\Big] \xrightarrow[n\to\infty]{(\,*\,)} 0\,,$$ by the dominated convergence theorem.

$\square$

Proposition 4.8. *According to the imposed above notations there holds*

$$\mathbb{E}\big[\varphi(B^0_{t_1},\dots,B^{t_n-1}_{t_n})\big] = \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_P\big[\varphi(B^{0,\theta}_{t_1},\dots,B^{t_{n-1},\theta}_{t_n})\big]$$
$$= \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\big[\varphi(B^0_{t_1},\dots,B^{t_n-1}_{t_n})\big]\,,$$

Proof.
Firstly, we know that $v(t,0) = \mathbb{E}\big[\varphi(B^t_T)\big] = \sup\limits_{\theta\in\mathcal{A}^{\Theta}_{t,T}} E_{P_\theta}\big[\varphi(B^t_T)\big]$.

Now we increase the number of points:

1) $$\sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_2}} E_{P_\theta}\big[\varphi(B^0_{t_1},B^{t_1}_{t_2})\big] = \sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_2}} E_P\big[\varphi(B^{0,\theta}_{t_1},B^{t_1,\theta}_{t_2})\big]$$
$$= \sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_2}} E_P\big[\varphi(B^{0,\theta}_{t_1},B^{t_1,\theta}_{t_2})\big|\mathcal{F}_0\big] \overset{\textbf{Lm.4.8}}{=} \sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_1}} E_P\Big[\sup_{\overline{\theta}\in\mathcal{A}^{\Theta}_{t_1,t_2}} E_P\big[\varphi(x,B^{t_1,\overline{\theta}}_{t_2})\big]\Big|_{x=B^{0,\theta}_{t_1}}\Big]$$
$$= \mathbb{E}\Big[\mathbb{E}\big[\varphi(x,B^{t_1}_{t_2})\big]\Big|_{x=B^0_{t_1}}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\varphi(B^0_{t_1},B^{t_1}_{t_2})\big]\Big].$$

2) $$\sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_3}} E_{P_\theta}\big[\varphi(B^0_{t_1},B^{t_1}_{t_2},B^{t_3}_{t_2})\big] = \sup_{\theta\in\mathcal{A}^{\Theta}_{0,t_3}} E_P\big[\varphi(B^{0,\theta}_{t_1},B^{t_1,\theta}_{t_2},B^{t_2,\theta}_{t_3})\big]$$
$$= \Big\{\theta = [\theta',\theta''] := \theta'\,\mathbb{1}_{[0,t_1]} + \theta''\,\mathbb{1}_{[t_1,t_3]}\Big\}$$

$$= \sup_{\theta' \in \mathcal{A}^{\Theta}_{0,t_1}} \sup_{\theta'' \in \mathcal{A}^{\Theta}_{t_1,t_3}} E_P\Big[E_P\big[\varphi(B^{0,\theta}_{t_1}, B^{t_1,\theta}_{t_2}, B^{t_2,\theta}_{t_3}) \big| \mathcal{F}_{t_1}\big]\Big]$$

$$= \sup_{\theta' \in \mathcal{A}^{\Theta}_{0,t_1}} E_P\Big[\sup_{\theta'' \in \mathcal{A}^{\Theta}_{t_1,t_3}} E_P\big[\varphi(B^{0,\theta}_{t_1}, B^{t_1,\theta}_{t_2}, B^{t_2,\theta}_{t_3}) \big| \mathcal{F}_{t_1}\big]\Big]$$

$$= \left\{ \begin{array}{l} [t_1,t_2] \in [s,T]\,, \quad B^{t_1,\theta}_{t_2} = \int_{t_1}^{t_2} \theta_s\, dW_s \\ \Rightarrow \sup_{\theta \in \mathcal{A}^{\Theta}_{s,T}} E_P\big[\varphi(B^{t_1,\theta}_{t_2})\big] = \sup_{\widetilde{\theta} \in \mathcal{A}^{\Theta}_{t_1,t_2}} E_P\big[\varphi(B^{t_1,\widetilde{\theta}}_{t_2})\big] \end{array} \right\}$$

$$\overset{\text{Lm.4.8}}{=} \sup_{\theta' \in \mathcal{A}^{\Theta}_{0,t_1}} E_P\left[\sup_{\theta'' \in \mathcal{A}^{\Theta}_{t_1,t_2}} E_P\Big[\sup_{\overline{\theta''} \in \mathcal{A}^{\Theta}_{t_2,t_3}} E_P\big[\varphi(x,y,B^{t_2,\overline{\theta''}}_{t_3})\big]\Big|_{\substack{x = B^{0,\theta'}_{t_1} \\ y = B^{t_1,\theta''}_{t_2}}} \Big]\right]$$

$$= \sup_{\theta' \in \mathcal{A}^{\Theta}_{0,t_1}} E_P\left[\sup_{\theta'' \in \mathcal{A}^{\Theta}_{t_1,t_2}} E_P\Big[\mathbb{E}\big[\varphi(x,y,B^{t_2}_{t_3})\big]\Big|_{\substack{x = B^{0,\theta'}_{t_1} \\ y = B^{t_1,\theta''}_{t_2}}} \Big]\right]$$

$$= \sup_{\theta' \in \mathcal{A}^{\Theta}_{0,t_1}} E_P\left[\mathbb{E}\Big[\mathbb{E}\big[\varphi(x,B^{t_1}_{t_2},B^{t_2}_{t_3})\big]\Big|_{x = B^{0,\theta'}_{t_1}} \Big]\right] = \mathbb{E}\Big[\mathbb{E}\big[\varphi(B^0_{t_1}, B^{t_1}_{t_2}, B^{t_2}_{t_3})\big]\Big].$$

3) So, up to now we saw that the statement of the proposition holds for $n = 1, 2, 3$.

Assume that it is OK till t_{n-1} and let us pass to t_n:

$$\sup_{\theta \in \mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\big[\varphi(B^0_{t_1},\ldots,B^{t_n-1}_{t_n})\big] = \sup_{\theta \in \mathcal{A}^{\Theta}_{0,t_n}} E_P\big[\varphi(B^{0,\theta}_{t_1},\ldots,B^{t_{n-1},\theta}_{t_n})\big]$$

$$= \left\{ \zeta := (B^{0,\theta}_{t_1},\ldots,B^{t_{n-3},\theta}_{t_{n-2}}) \in \mathcal{F}_{n-2} \right\} = \sup_{\theta \in \mathcal{A}^{\Theta}_{0,t_n}} E_P\big[\varphi(\zeta, B^{t_{n-2},\theta}_{t_{n-1}}, B^{t_{n-1},\theta}_{t_n})\big]$$

$$= \overset{\text{the same when } n=3}{\cdots} = \sup_{\theta \in \mathcal{A}^{\Theta}_{0,t_{n-2}}} E_P\Big[\mathbb{E}\big[\varphi(x, B^{t_{n-2},\theta}_{t_{n-1}}, B^{t_{n-1},\theta}_{t_n})\big]\Big|_{x=\zeta\equiv(B^{0,\theta}_{t_1},\ldots,B^{t_{n-3},\theta}_{t_{n-2}})}\Big]$$

$$= \left\{ \widetilde{\varphi}(\zeta) := \mathbb{E}\big[\varphi(\zeta, B^{t_{n-2}}_{t_{n-1}}, B^{t_{n-1}}_{t_n})\big] \right\} = \sup_{\theta \in \mathcal{A}^{\Theta}_{0,t_{n-2}}} E_P\big[\widetilde{\varphi}(B^{0,\theta}_{t_1},\ldots,B^{t_{n-3},\theta}_{t_{n-2}})\big]$$

$$\overset{\text{induction}}{=} \mathbb{E}\big[\widetilde{\varphi}(B^0_{t_1},\ldots,B^{t_{n-3}}_{t_{n-2}})\big] = \mathbb{E}\Big[\mathbb{E}\big[\varphi(x, B^{t_{n-2}}_{t_{n-1}}, B^{t_{n-1}}_{t_n})\big]\Big|_{x=\zeta}\Big]$$

$$= \mathbb{E}\big[\varphi(B^0_{t_1},\ldots,B^{t_n-1}_{t_n})\big].$$

□

Proposition 4.9. *The family of probability measures* $\{P_\theta\,,\ \theta \in \mathcal{A}^{\Theta}_{0,\infty}\}$ *is tight.*

Proof.

The proof is the same as in finite-dimensional case (see [36, Prop.49]). □

Now let us consider the following spaces:

- $L^0(\Omega):$ the space of $\mathcal{B}(\Omega)$-measurable real functions;
- $L^0_{L^\Sigma_2}(\Omega):$ the space of $\mathcal{B}(L^\Sigma_2)/\mathcal{B}(\Omega)$-measurable $\Omega \to L^\Sigma_2$ mappings;
- $B_{L^\Sigma_2}(\Omega):$ all bounded mappings in $L^0_{L^\Sigma_2}(\Omega)$;
- $CB_{L^\Sigma_2}(\Omega):$ all bounded and continuous mappings in $L^0_{L^\Sigma_2}(\Omega)$.

Definition 4.9. *For a random variable* $X \in L^0(\Omega)$ *such that a linear (classical) expectation* E_P *exists for all* $P \in \mathcal{P}$ *an* ***upper expectation*** *of* $\mathcal{P}$ *is defined as follows:*

$$\bar{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} E_P[X].$$

Later on we consider such a set of probability measures

$$\mathcal{P} := \{P_\theta\,,\ \theta \in \mathcal{A}^{\Theta}_{0,\infty}\}\,.$$

Remark 4.13. *It is clear that on the expectation space* $(\Omega, B_{L^\Sigma_2}(\Omega), \bar{\mathbb{E}})$ *as well as on the* $(\Omega, CB_{L^\Sigma_2}(\Omega), \bar{\mathbb{E}})$ *the upper expectation* $\bar{\mathbb{E}}[\,\cdot\,]$ *is a sublinear expectation.*

Remark 4.14. *For a* G*-expectation* $\mathbb{E}$ *by* ***Prop.4.8*** *we have that*

$$\mathbb{E}[\psi(X)] = \bar{\mathbb{E}}[\psi(X)]\,, \quad X \in Lip(\Omega)\,,\ \psi \in \mathbf{C}_{p.Lip}(L^\Sigma_2)\ .$$

For $X \in L^0_{L^\Sigma_2}(\Omega)$ *we define the norm* $\overline{\|X\|}^p_{\Sigma,p} := \bar{\mathbb{E}}\big[\|X\|^p_{L^\Sigma_2}\big]$.

Therefore $\|X\|^p_{\Sigma,p} := \mathbb{E}\big[\|X\|^p_{L^\Sigma_2}\big] = \bar{\mathbb{E}}\big[\|X\|^p_{L^\Sigma_2}\big] = \overline{\|X\|}^p_{\Sigma,p}\ ,\ X \in Lip(\Omega)\,.$

Define the following spaces:

- $\mathcal{L}^p := \left\{X \in L^0_{L_2^\Sigma}(\Omega) \mid \bar{\mathbb{E}}\|X\|^p_{L_2^\Sigma} < \infty\right\}$;
- $\mathcal{N} := \left\{X \in L^0_{L_2^\Sigma}(\Omega) \mid X = 0 \quad c\text{ - q.s.}\right\}$;
- $\mathbb{L}^p_{L_2^\Sigma} := \mathcal{L}^p/\mathcal{N}$;
- $\mathbb{L}^p_{B,L_2^\Sigma}$ is the completion of $B_{L_2^\Sigma}(\Omega)/\mathcal{N}$ under $\overline{\|\cdot\|}_{\Sigma,p}$;
- $\mathbb{L}^p_{CB,L_2^\Sigma}$ is the completion of $CB_{L_2^\Sigma}(\Omega)/\mathcal{N}$ under $\overline{\|\cdot\|}_{\Sigma,p}$.

Remark 4.15. *Similar to the classical arguments of L^p-theory we can conclude that $\left(\mathbb{L}^p_{L_2^\Sigma}, \overline{\|\cdot\|}_{\Sigma,p}\right)$ is a Banach space.*

Since $CB_{L_2^\Sigma}(\Omega) \subset B_{L_2^\Sigma}(\Omega) \subset \mathcal{L}^p \quad \Rightarrow \quad CB_{L_2^\Sigma}(\Omega)/\mathcal{N} \subset B_{L_2^\Sigma}(\Omega)/\mathcal{N} \subset \mathbb{L}^p_{L_2^\Sigma}$. So we have the following inclusions: $\mathbb{L}^p_{CB,L_2^\Sigma} \subset \mathbb{L}^p_{B,L_2^\Sigma} \subset \mathbb{L}^p_{L_2^\Sigma}$.

Proposition 4.10.

$$\mathbb{L}^p_{B,L_2^\Sigma} = \left\{X \in \mathbb{L}^p_{L_2^\Sigma}(\Omega) \mid \lim_{n\to\infty} \bar{\mathbb{E}}\left[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\right] = 0\right\}.$$

Proof.
There is no changes with a proof for a finite-dimensional case (see [36, Prop.18]).

□

Definition 4.10. *A mapping X on Ω with values in a topological space is said to be **quasi-continuous** if for every $\varepsilon > 0$ there exists an open set O with capacity $c(O) < \varepsilon$, such that the restriction of mapping X to the complement O^c is continuous.*

Definition 4.11. *We say that mapping X on Ω has a quasi-continuous version if there exists a quasi-continuous mapping Y on Ω, such that $X = Y$ quasi surely.*

Proposition 4.11. *If* $X \in \mathbb{L}^p_{CB,L_2^\Sigma}$ *then* X *has a quasi-continuous version.*

Proof.
There is also no changes with a proof for a finite-dimensional case (see [36, Prop.24]).

□

Proposition 4.12.

$$\mathbb{L}^p_{CB,L_2^\Sigma} = \Big\{X \in \mathbb{L}^p_{L_2^\Sigma}(\Omega) \mid X \textit{ has a q.c. vers.}, \lim_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0\Big\}.$$

Proof.
For this proof we follows [36, Prop.25], but here there are some difficulties when we pass to infinite-dimensions. So here we give the whole proof with all details.
Let us denote a set
$A := \Big\{X \in \mathbb{L}^0_{L_2^\Sigma}(\Omega) \mid X \text{ has a q.c. vers.}, \lim_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0\Big\}$.
1) If $X \in \mathbb{L}^p_{CB,L_2^\Sigma}$ then by **Prop.4.11** we have that X has a quasi-continuous version.
Also for $X \in \mathbb{L}^p_{B,L_2^\Sigma}$ by **Prop.4.10** we have $\lim_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0$.
So we can conclude that $X \in A$.

2) Every $X \in A$ is quasi-continuous.
Take the truncation

$$X_n^t := \mathrm{Trunc}(X, n) \equiv X \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} \leqslant n\}} + n \cdot \frac{X}{\|X\|_{L_2^\Sigma}} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}.$$

Hence

$$\bar{\mathbb{E}}\big[\|X - X_n^t\|^p_{L_2^\Sigma}\big] = \bar{\mathbb{E}}\Big[\|X\|^p_{L_2^\Sigma} \cdot \big(1 - \frac{n}{\|X\|_{L_2^\Sigma}}\big)^p \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\Big]$$
$$\leqslant \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] \xrightarrow[n\to\infty]{} 0.$$

Since X_n^t is quasi-continuous then there exists a closed set A_n, such that $c(A_n^c) < \frac{1}{n^{p+1}}$ and X_n is continuous on A_n.
For the following step we need to use the Dugundji theorem (the infinite-dimensional version of the Tietze extension theorem):

Theorem 4.5 (Dugundji). *[64, Th.1.2.2.].*
Let L be a locally convex linear space and let $C \subseteq L$ be convex. Then for every space Ω with closed subspace A, every continuous function $f : A \to C$ can be extended to a continuous function $\bar{f} : \Omega \to C$.

As is known that every normable space is locally convex it follows that the Banach space L_2^Σ is also locally convex.

We have $\|X_n^t\|_{L_2^\Sigma} \leqslant \|X\|_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} \leqslant n\}} + n \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}} \leqslant n$.
Define a set $C_n := \{Y \in L_2^\Sigma \mid \|Y\|_{L_2^\Sigma} \leqslant n\}$.
So that $X_n^t\big|_{A_n} : A_n \to C_n$ is continuous.
By the Dugundji theorem (**Th.4.5**) it follows that there exists a continuous extension $\widetilde{X}_n : \Omega \to C_n$, which satisfies the following:
$\widetilde{X}_n \in CB_{L_2^\Sigma}(\Omega)$,
$X_n^t = \widetilde{X}_n$ on A_n,
$\|\widetilde{X}_n\|_{L_2^\Sigma} \leqslant n$.

Hence
$$\bar{\mathbb{E}}\big[\|X_n^t - \widetilde{X}_n\|_{L_2^\Sigma}^p\big] \leqslant \bar{\mathbb{E}}\Big[\Big(\|X_n^t\|_{L_2^\Sigma}^p + \|\widetilde{X}_n\|_{L_2^\Sigma}^p\Big) \cdot \mathbb{1}_{\{C_n \setminus A_n\}}\Big]$$
$$\leqslant 2n^p \cdot c\big(A_n^c\big) = \frac{2n^p}{n^{p+1}} = \frac{2}{n} \xrightarrow[n\to\infty]{} 0.$$

And we have
$$\bar{\mathbb{E}}\big[\|X - \widetilde{X}_n\|_{L_2^\Sigma}^p\big] \leqslant 2^{p-1}\Big(\bar{\mathbb{E}}\big[\|X - X_n^t\|_{L_2^\Sigma}^p\big] + \bar{\mathbb{E}}\big[\|X_n^t - \widetilde{X}_n\|_{L_2^\Sigma}^p\big]\Big) \xrightarrow[n\to\infty]{} 0.$$
So that $X \in \mathbb{L}^p_{CB, L_2^\Sigma}$.

□

5 Stochastic Integral with respect to G-Brownian motion

5.1 Definition of the stochastic integral for elementary integrand processes

Recall that
$Lip(\Omega_T) := \Big\{\varphi(B_{t_1},..,B_{t_n}) \mid n \geqslant 1,\ t_1,..,t_n \in [0,T],\ \varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^n, L_2^\Sigma)\Big\}$.
For $p \geqslant 1$ the space ${}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$ is the completion of $Lip(\Omega_T)$ under the norm $\|X\|_{\Sigma,p}^p = \mathbb{E}\big[\|X\|_{L_2^\Sigma}^p\big]$.
Hold the notation $\|\cdot\|_\Sigma := \|X\|_{\Sigma,2}$.
Since ${}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$ is an abstract completion we need to show that it is a space of random variables what provides the following theorem:

Remark 5.1. *Let us denote the following subspace of* $Lip(\Omega_T)$ *:*
$BLip(\Omega_T) := \Big\{\varphi(B_{t_1},..,B_{t_n}) \mid n \geqslant 1,\ t_1,..,t_n \in [0,T],\ \varphi \in \mathbf{C}_{b,\,p.Lip}(\mathsf{U}^n, L_2^\Sigma)\Big\}$.
And the space ${}^{\mathsf{H}}BL_G^p\big(\Omega_T\big)$ *be the completion of* $BLip(\Omega_T)$ *under the norm* $\|X\|_{\Sigma,p}^p = \mathbb{E}\big[\|X\|_{L_2^\Sigma}^p\big]$.
Then we have that ${}^{\mathsf{H}}BL_G^p\big(\Omega_T\big) = {}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$.

Proof.
If $X \in {}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$ then there exists a sequence $(X_k) \subset Lip(\Omega_T)$, such that $\mathbb{E}\big[\|X - X_k\|_{L_2^\Sigma}^p\big] \xrightarrow[n\to\infty]{} 0$ and $\mathbb{E}\big[\|X_k\|_{L_2^\Sigma}^p\big] < \infty$.
Take $X_k^n := \mathrm{Trunc}(X_k, n) \in BLip(\Omega_T)$.
Therefore $\|X - X_k^n\|_{\Sigma,p}^p = \mathbb{E}\big[\|X - X_k^n\|_{L_2^\Sigma}^p\big] \leqslant \mathbb{E}\big[\|X\|_{L_2^\Sigma}^p \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] \xrightarrow[n\to\infty]{} 0$.
So we can deduce that $\mathrm{compl}\Big(BLip(\Omega_T),\ \|\cdot\|_{\Sigma,p}\Big) = {}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$.

□

Theorem 5.1.
$${}^{\mathsf{H}}L_G^p\big(\Omega_T\big) = \big\{X \in L^0_{L_2^\Sigma}(\Omega_T) \mid X \text{ has a q.c. vers.}, \lim_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|_{L_2^\Sigma}^p \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0\big\}.$$
Moreover, if $X \in {}^{\mathsf{H}}L_G^p\big(\Omega_T\big)$ *then* $\mathbb{E}\big[\|X\|_{L_2^\Sigma}^p\big] = \bar{\mathbb{E}}\big[\|X\|_{L_2^\Sigma}^p\big]$.

Proof.

Let $\Phi \in CB_{L_2^\Sigma}(\Omega_T)$, i.e. $\Phi : \Omega_T \to L_2^\Sigma$ is bounded and continuous L_2^Σ-valued random variable.

Take a compact set $K \subset \Omega_T = \big\{\omega_t : [0,T] \to \mathsf{U} \mid \omega_0 = 0,\ \omega_t\text{–continuous}\big\}$.

We claim that Φ be uniformly continuous on K.

In fact, since Φ is bounded and continuous on the compact set K then for every $\omega \in K$ there exists a subsequence $\omega_{n_j} \xrightarrow[j\to\infty]{} \omega$, such that

$$\Phi(\omega_{n_j}) \xrightarrow[j\to\infty]{\|\cdot\|_{L_2^\Sigma}} \Phi(\omega)\,. \qquad (*)$$

If Φ is not uniformly continuous on K then $\exists \varepsilon > 0 \quad \forall \delta_n \downarrow 0 \quad \forall \omega_n, \omega'_n \in \Omega_T$, such that $\|\omega_n - \omega'_n\|_\Omega < \delta$ and it follows $\|\Phi(\omega_n) - \Phi(\omega'_n)\|_{L_2^\Sigma} < \varepsilon$.

Let us take two different subsequences: $\omega_{n_j} \xrightarrow[j\to\infty]{} \omega$ and $\omega'_{n_j} \xrightarrow[j\to\infty]{} \omega$.

So that $\exists\, \bar{j} \quad \forall j > \bar{j}: \quad \|\omega_{n_j} - \omega'_{n_j}\|_\Omega < \delta_n$,

Hence $\varepsilon < \|\Phi(\omega_{n_j}) - \Phi(\omega'_{n_j})\|_{L_2^\Sigma}$

$$\leqslant \|\Phi(\omega_{n_j}) - \Phi(\omega)\|_{L_2^\Sigma} + \|\Phi(\omega'_{n_j}) - \Phi(\omega)\|_{L_2^\Sigma} \xrightarrow[j\to\infty]{(*)} 0\,, \text{ a contradiction.}$$

Therefore $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \omega, \omega' \in \Omega_T$, such that $\|\omega - \omega'\|_\Omega < \delta$ and it follows $\|\Phi(\omega) - \Phi(\omega')\|_{L_2^\Sigma} < \varepsilon$.

Take a partition $\pi_n = \big\{0 = t_1^n < \ldots < t_{N_n}^n = T\big\}$, $|\pi_n| := \max\limits_{1\leqslant i\leqslant N_n} |t_{i+1}^n - t_i^n|$.

Then for every $\omega \in K$ $\pi_n\omega$ is a linear approximation of ω at the points of the given partion π_n, i.e. $\pi_n\omega(t_i^n) = \omega(t_i^n)$.

Define $\Phi_n(\omega) := \Phi(\pi_n\omega) = \widetilde{\Phi}(t_1^n, \ldots, t_{N_n}^n) \in BLip(\Omega_T)$.

Owning to the Arzelà-Ascolì theorem we have that K is an equicontinuous set: $\forall \omega \in K \quad \forall \delta > 0 \quad \exists \varkappa > 0 \quad \forall t, t' \in [0,T]$, such that $|t - t'| < \varkappa$ and it follows $\|\omega(t) - \omega(t')\|_{\mathsf{U}} < \delta$. $\qquad (\#)$

Now for a fixed $t \in (t_i, t_{i+1})$ we have

$$\pi_n\omega(t) = \frac{t - t_i}{t_{i+1} - t_i} \cdot \omega(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \cdot \omega(t_i)\,.$$

Let $|\pi_n| := \eta < \varkappa$ then let us calculate

$$\|\pi_n\omega(t) - \omega(t)\|_{\mathsf{U}}$$
$$= \Big\| \frac{t - t_i}{t_{i+1} - t_i} \cdot \omega(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \cdot \omega(t_i) - \frac{t - t_i}{t_{i+1} - t_i} \cdot \omega(t) - \frac{t_{i+1} - t}{t_{i+1} - t_i} \cdot \omega(t) \Big\|_{\mathsf{U}}$$

$$\leqslant \frac{t-t_i}{t_{i+1}-t_i}\cdot\|\omega(t_{i+1})-\omega(t)\|_{\mathsf{U}}+\frac{t_{i+1}-t}{t_{i+1}-t_i}\cdot\|\omega(t_i)-\omega(t)\|_{\mathsf{U}}$$

$$\overset{(\#)}{<}\frac{t-t_i}{t_{i+1}-t_i}\cdot\delta+\frac{t_{i+1}-t}{t_{i+1}-t_i}\cdot\delta=\delta\,.$$

Hence $\|\pi_n\omega-\omega\|_\Omega=\max\limits_{\substack{t\in(t_i,t_{i+1})\\1\leqslant i\leqslant N_n}}\|\pi_n\omega(t)-\omega(t)\|_{\mathsf{U}}<\delta\,.$

It means, $\forall\varepsilon>0\quad\exists\eta>0$, such that $|\pi_n|<\eta$ and it follows

$$\|\Phi(\pi_n\omega)-\Phi(\omega)\|_{L_2^\Sigma}<\varepsilon\,.$$

Since $|\pi_n|\xrightarrow[n\to\infty]{}0$ then $\sup\limits_{\omega\in K}\|\Phi(\omega)-\Phi_n(\omega)\|_{L_2^\Sigma}\xrightarrow[n\to\infty]{}0\,.$

Recall that $\bar{\mathbb{E}}[\,\cdot\,]:=\sup\limits_{P\in\mathcal{P}}E_P[\,\cdot\,]$ and $\mathcal{P}$ is tight (**Prop.4.9**).

It follows that for every $n\geqslant 1$ there exists a compact set $K_n\subset\Omega_n$, such that $c(K_n^c)<\dfrac{1}{n}$.

Therefore $\forall\Phi\in CB_{L_2^\Sigma}(\Omega_T)\quad\exists\Phi_n\in BLip(\Omega_T)$, such that

$$\sup_{\omega\in K_n}\|\Phi(\omega)-\Phi_n(\omega)\|_{L_2^\Sigma}<\frac{1}{n}\,.$$

So that $\bar{\mathbb{E}}\big[\|\Phi-\Phi_n\|^p_{L_2^\Sigma}\big]\leqslant\sup\limits_{\omega\in\Omega}\underbrace{\|\Phi(\omega)-\Phi_n(\omega)\|^p_{L_2^\Sigma}}_{<\infty}\cdot c(K_n^c)+\dfrac{1}{n}\cdot c(K_n)$

$$\leqslant\frac{1}{n}\cdot(C_0+1)\xrightarrow[n\to\infty]{}0\,.$$

Hence $\Phi\in{}^{\mathsf{H}}L_G^p\big(\Omega_T\big)\,.$

It is clear that $BLip(\Omega_T)\subset CB_{L_2^\Sigma}(\Omega_T)\,.$

So we have the following inclusions:

$$BLip(\Omega_T)\subset CB_{L_2^\Sigma}(\Omega_T)\subset{}^{\mathsf{H}}L_G^p\big(\Omega_T\big)\,.\qquad(\circ)$$

Recall that $\mathbb{L}^p_{CB,L_2^\Sigma}=\mathrm{compl}\big(CB_{L_2^\Sigma}/\mathcal{N},\,\overline{\|\cdot\|}_{\Sigma,p}\big)$

$\overset{\textbf{Prop.4.12}}{=}\big\{X\in\mathbb{L}^p_{L_2^\Sigma}(\Omega)\mid X$ has a q.c. vers., $\lim\limits_{n\to\infty}\bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma}\cdot\mathbb{1}_{\{\|X\|_{L_2^\Sigma}>n\}}\big]=0\big\}\,;$

$\mathbb{L}^p_{L_2^\Sigma}=\mathcal{L}^p/\mathcal{N}=\big\{X\in L^0_{L_2^\Sigma}(\Omega)\mid\bar{\mathbb{E}}\|X\|^p_{L_2^\Sigma}<\infty\big\}/\mathcal{N}\,.$

But $\bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma}\cdot\mathbb{1}_{\{\|X\|_{L_2^\Sigma}>n\}}\big]\xrightarrow[n\to\infty]{}0$ so that $\bar{\mathbb{E}}\|X\|^p_{L_2^\Sigma}<\infty\,.$

Then by $(\circ)$ we have ${}^{\mathsf{H}}L_G^p(\Omega_T) \equiv \mathrm{compl}\big(CB_{L_2^\Sigma}/\mathcal{N},\, \overline{\|\cdot\|}_{\Sigma,p}\big)$
$= \big\{X \in \mathcal{L}^p \mid X$ has q.c. version, $\lim\limits_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0\big\}$
$= \big\{X \in L^0_{L_2^\Sigma}(\Omega_T) \mid X$ has q.c. version, $\lim\limits_{n\to\infty} \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma} \cdot \mathbb{1}_{\{\|X\|_{L_2^\Sigma} > n\}}\big] = 0\big\}$.

If $X \in Lip(\Omega)$ then $\mathbb{E}\big[\|X\|^p_{L_2^\Sigma}\big] = \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma}\big]$.
Since ${}^{\mathsf{H}}L_G^p(\Omega_T) = \mathrm{compl}\Big(BLip(\Omega_T),\, \mathbb{E}\big[\|\cdot\|^p_{L_2^\Sigma}\big]\Big)$

$$= \mathrm{compl}\Big(BLip(\Omega_T),\, \bar{\mathbb{E}}\big[\|\cdot\|^p_{L_2^\Sigma}\big]\Big),$$

It follows that for every $X \in {}^{\mathsf{H}}L_G^p(\Omega_T)$: $\mathbb{E}\big[\|X\|^p_{L_2^\Sigma}\big] = \bar{\mathbb{E}}\big[\|X\|^p_{L_2^\Sigma}\big]$.

□

Now we need speak a couple of words about convergence in ${}^{\mathsf{H}}L_G^p(\Omega_T)$ under the G-expectation. The biggest problem is that a dominated convergence theorem is not true in a given framework. But there is a result of a monotone convergence theorem (see [36, Th.31]). It has a trivial infinite dimensional extension which we will represent as the following theorem:

Theorem 5.2. *Let a sequence $\{X_n\} \subset {}^{\mathsf{H}}L_G^p(\Omega_T)$ be such that $X_n \downarrow X$ q.s. Then also holds $\mathbb{E}\big[\|X_n\|^p_{L_2^\Sigma}\big] \downarrow \mathbb{E}\big[\|X\|^p_{L_2^\Sigma}\big]$.*

So we are ready to define a space of the integrand processes:

Let ${}^{\mathsf{H}}M_G^{p,0}(0,T) := \Bigg\{\Phi(t) = \sum\limits_{k=0}^{N-1} \Phi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t) \mid$

$$\Phi_k(\omega) \in {}^{\mathsf{H}}L_G^p(\Omega_{t_k}),\ 0 = t_0 < t_1 \ldots < t_N = T\Bigg\}.$$

For an elementary process $\Phi \in {}^{\mathsf{H}}M_G^{p,0}(0,T)$ the stochastic integral we define as follows:

$$I_T(\Phi) := \int_0^T \Phi(t)dB_t = \sum_{k=0}^{N-1} \Phi_k(B_{t_{k+1}} - B_{t_k}) \quad .$$

Remark 5.2. *It is clear that* $\int\limits_0^T \Phi(t)dB_t = \int\limits_0^T \Phi(t)dB_{t-a}\,, \quad a \in \mathbb{R}$.

Remark 5.3. *For a fixed t we have that $\Phi(t) \in L^0_{L_2^\Sigma}(\Omega)$ then we can see that $I_T(\Phi)$ is an H-valued random variable.*

5.2 Itô's isometry and Burkholder–Davis–Gundy inequalities

Let B_t is a given G-Brownian motion with corresponding G-expectation $\mathbb{E}$ which coincides on the space $L^0_{L_2^\Sigma}$ with an upper expectation defined for the family of gaussian probability measures $\mathcal{P}$:

$$\mathbb{E}[X] = \bar{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X] \,.$$

If we fix a measure $P \in \mathcal{P}$ then we can define $\mathbb{E}_P[X] := \sup_{P \in \mathcal{P}_P} E_P[X]$, where $\mathcal{P}_P := \{P\}$.
Actually, $\mathbb{E}_P[X]$ is a classical linear expectation, but we could treat it also as sublinear with the all properties for a sublinear expectation.
So, we can define the G-functional for this expectation:

$$G_P(A) := \frac{1}{2}\,\mathbb{E}_P\big[\langle AX, X\rangle\big] = \frac{1}{2}\mathrm{Tr}[A \cdot Q_P],$$

for some non-negative symmetric trace-class operator Q_P, because functional $\mathbb{E}_P[X]$ is linear.
On the other hand $G_P(A) = \dfrac{1}{2} \sup\limits_{Q \in \Sigma_P} \mathrm{Tr}[A \cdot Q]$, where $\Sigma_P \subset \Sigma$.
Therefore $\Sigma_P \equiv \{Q_P\}$ and B_t is a classical Q_P-Wiener process under $P \in \mathcal{P}$, $Q_P \in \Sigma$.

Theorem 5.3 (Itô's isometry inequality)**.**
Let $\Phi \in {}^{\mathsf{H}}M_G^{2,0}(0,T)$ *then*

$$\mathbb{E}\Big[\Big\| \int_0^T \Phi(t)\, dB_t \Big\|_{\mathsf{H}}^2\Big] \leqslant \mathbb{E}\Big[\int_0^T \|\Phi(t)\|^2_{L_2^\Sigma}\, dt\Big]. \tag{12}$$

Proof.
1) Firstly, we are going to prove a "weaker" version of the Itô's isometry inequality, namely that

$$\mathbb{E}\Big[\Big\| \int_0^T \Phi(t)\, dB_t \Big\|_{\mathsf{H}}^2\Big] \leqslant \int_0^T \mathbb{E}\Big[\|\Phi(t)\|^2_{L_2^\Sigma}\Big]\, dt. \tag{13}$$

In fact,

$$\mathbb{E}\Big[\Big\| \int_0^T \Phi(t)\, dB_t \Big\|_{\mathsf{H}}^2\Big] = \mathbb{E}\Big[\Big\| \sum_{k=0}^{N-1} \Phi_k\, \Delta B_{t_k} \Big\|_{\mathsf{H}}^2\Big]$$

$$\leqslant \sum_{k=0}^{N-1} \mathbb{E}\|\Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^2 + 2 \sum_{k<n=1}^{N-1} \mathbb{E}\langle \Phi_k\, \Delta B_{t_k}, \Phi_n\, \Delta B_{t_n}\rangle_{\mathsf{H}} =: K\,.$$

(a) k < n:

Note that in this case for fixed n and k we have that $\Phi_k\, \Delta B_{t_k}$ and ΔB_{t_n} are independent from Ω_{t_n}, but Φ_n is Ω_{t_n}-measurable.

Therefore $\mathbb{E}\langle \Phi_k\, \Delta B_{t_k}, \Phi_n\, \Delta B_{t_n}\rangle_{\mathsf{H}} = \mathbb{E}\Big[\mathbb{E}\langle \Phi_k\, \Delta B_{t_k}, \Phi_n\, \Delta B_{t_n}\rangle_{\mathsf{H}} \Big| \Omega_{t_n}\Big]$

$$= \mathbb{E}\langle \Phi_k\, \Delta B_{t_k}, \Phi_n \cdot \underbrace{\mathbb{E}[\Delta B_{t_n}]}_{=0}\rangle_{\mathsf{H}} = 0.$$

(b) k = n:

Analogously, ΔB_{t_k} is independent out of Ω_{t_k}, but Φ_k is Ω_{t_k} measurable.

Therefore $\mathbb{E}\big[\|\Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^2\big] = \mathbb{E}\Big[\mathbb{E}\big[\|\Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^2 \big| \Omega_{t_k}\big]\Big]$

$$\overset{\substack{\textbf{Prop.4.6,1)} \\ \textbf{Rem.4.11} \\ \textbf{Prop.2.6, 3)}}}{=} \mathbb{E}\Big[(t_{k+1} - t_k) \cdot \sup_{Q \in \Sigma} \mathrm{Tr}\big[\Phi_k Q \Phi_k^*\big]\Big].$$

So we have $K = \sum_{k=0}^{N-1} \mathbb{E}\Big[(t_{k+1} - t_k) \cdot \|\Phi_k\|_{L_2^\Sigma}^2\Big] = \int_0^T \mathbb{E}\Big[\|\Phi(t)\|_{L_2^\Sigma}^2\Big] dt\,.$

2) If $\Phi \in {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)$ then $\Phi(t) = \sum_{k=0}^{N-1} \Phi_k(\omega) \mathbb{1}_{[t_k, t_{k+1})}(t)$,

$$0 = t_0 < t_1 \ldots < t_N = T$$

and for every k $\Phi_k \in {}^{\mathsf{H}}L_G^2(\Omega_{t_k})$ then there exists a sequence $\big\{\Phi_k^{(n)}\big\} \subset Lip(\Omega_{t_k})$, such that $\big\|\Phi_k - \Phi_k^{(n)}\big\|_\Sigma \xrightarrow[n\to\infty]{} 0\,.$

Define $\Phi^{(n)}(t) := \sum_{k=0}^{N-1} \Phi_k^{(n)}(\omega) \mathbb{1}_{[t_k, t_{k+1})}(t)\,.$

Since $\Phi_k^{(n)} \in Lip(\Omega_{t_k})$ then we can consider

$$\Phi_k^{(n)} = \varphi_k^{(n)}(B_{u_1^k}, .., B_{u_{m_k}^k}) \in L_2^\Sigma\,, \quad 0 \leqslant u_1^k \leqslant \ldots \leqslant u_{m_k}^k = t_k\,.$$

Therefore $I_T(\Phi_k^{(n)}) = \int_0^T \Phi^{(n)}(t)\, dB_t = \sum_{k=0}^{N-1} \Phi_k^{(n)}(B_{t_{k+1}} - B_{t_k})$

$$= \varphi(B_{u_1^0}, .., B_{u_{m_N}^N}) \in \mathsf{H}\,, \quad \varphi \in \mathbf{C}_{p.Lip}(\mathsf{U}^{m_N}, \mathsf{H})\,.$$

So, we have

$$\mathbb{E}\Big[\big\|\int_0^T \Phi^{(n)}(t)\,dB_t\big\|_{\mathsf{H}}^2\Big] = \mathbb{E}\Big[\|\varphi\|_{\mathsf{H}}^2(B_{u_1^0},..,B_{u_{m_N}^N})\Big] = \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\|\varphi\|_{\mathsf{H}}^2(B_{u_1^0},..,B_{u_{m_N}^N})\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\big\|\int_0^T \Phi^{(n)}(t)\,dB_t\big\|_{\mathsf{H}}^2\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\big\|\sum_{k=0}^{N-1}\varphi_k^{(n)}(B_{u_1^k},..,B_{u_{m_k}^k})\cdot(B_{t_{k+1}}-B_{t_k})\big\|_{\mathsf{H}}^2\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P}\Big[\big\|\sum_{k=0}^{N-1}\varphi_k^{(n)}(B_{u_1^k}^{0,\theta},..,B_{u_{m_k}^k}^{0,\theta})\cdot(B_{t_{k+1}}^{0,\theta}-B_{t_k}^{0,\theta})\big\|_{\mathsf{H}}^2\Big]$$

$$= \Big\{\Phi_*^{(n)}(t) := \sum_{k=0}^{N-1}\varphi_k^{(n)}(B_{u_1^k}^{0,\theta},..,B_{u_{m_k}^k}^{0,\theta})\cdot(t_{k+1}-t_k)\Big\}$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P}\Big[\big\|\int_0^T \Phi_*^{(n)}(t)\,dB_t^{0,\theta}\big\|_{\mathsf{H}}^2\Big] = \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P}\Big[\big\|\int_0^T \Phi_*^{(n)}(t)\theta_t\,dW_t\big\|_{\mathsf{H}}^2\Big]$$

$$\overset{\text{classical Itô's isometry}}{=} \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P}\Big[\int_0^T \big\|\Phi_*^{(n)}(t)\cdot\theta_t\big\|_{L_2(\mathsf{U},\mathsf{H})}^2 dt\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\int_0^T \big\|\Phi^{(n)}(t)\cdot\theta_t\big\|_{L_2(\mathsf{U},\mathsf{H})}^2 dt\Big] \leqslant \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\int_0^T \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}}\big\|\Phi^{(n)}(t)\cdot\theta\big\|_{L_2(\mathsf{U},\mathsf{H})}^2 dt\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\int_0^T \sup_{\gamma\in\Theta}\big\|\Phi^{(n)}(t)\cdot\gamma\big\|_{L_2(\mathsf{U},\mathsf{H})}^2 dt\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^{\Theta}} E_{P_\theta}\Big[\int_0^T \sup_{Q\in\Sigma}\big\|\Phi^{(n)}(t)\cdot Q^{1/2}\big\|_{L_2(\mathsf{U},\mathsf{H})}^2 dt\Big] = \bar{\mathbb{E}}\Big[\int_0^T \|\Phi^{(n)}(t)\|_{L_2^\Sigma}^2\,dt\Big]$$

$$= \mathbb{E}\Big[\int_0^T \|\Phi^{(n)}(t)\|_{L_2^\Sigma}^2\,dt\Big].$$

For the finishing of proof we need to pass to the limit:

(a): $$\left|\left(\mathbb{E}\Big[\big\|\int_0^T \Phi(t)\,dB_t\big\|_{\mathsf{H}}^2\Big]\right)^{\frac{1}{2}} - \left(\mathbb{E}\Big[\big\|\int_0^T \Phi^{(n)}(t)\,dB_t\big\|_{\mathsf{H}}^2\Big]\right)^{\frac{1}{2}}\right|$$

$$\overset{\text{Minkowski ineq.}}{\leqslant} \left(\mathbb{E}\Big[\big\|\int_0^T \Phi(t)\,dB_t - \int_0^T \Phi^{(n)}(t)\,dB_t\big\|_{\mathsf{H}}^2\Big]\right)^{\frac{1}{2}}$$

$$= \left(\mathbb{E}\Big[\Big\| \int\limits_0^T \big(\Phi(t) - \Phi^{(n)}(t)\big)\, dB_t \Big\|_{\mathsf{H}}^2\Big]\right)^{\frac{1}{2}} \overset{(13)}{\leqslant} \left(\int\limits_0^T \mathbb{E}\Big[\big\|\Phi(t) - \Phi^{(n)}(t)\big\|_{L_2^\Sigma}^2\Big]\, dt\right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} \Big[\underbrace{\big\|\Phi_k - \Phi_k^{(n)}\big\|_\Sigma^2}_{\searrow 0} \cdot (t_{k+1} - t_k)\Big]\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0\,.$$

(b): $$\left|\left(\mathbb{E}\Big[\int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt\Big]\right)^{\frac{1}{2}} - \left(\mathbb{E}\Big[\int\limits_0^T \|\Phi^{(n)}(t)\|_{L_2^\Sigma}^2\, dt\Big]\right)^{\frac{1}{2}}\right|$$

$$\overset{\text{Minkowski ineq.}}{\leqslant} \left(\mathbb{E}\left[\left|\Big(\int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt\Big)^{\frac{1}{2}} - \Big(\int\limits_0^T \|\Phi^{(n)}(t)\|_{L_2^\Sigma}^2\, dt\Big)^{\frac{1}{2}}\right|^2\right]\right)^{\frac{1}{2}}$$

$$\overset{\text{Minkowski ineq.}}{\leqslant} \left(\mathbb{E}\Big[\int\limits_0^T \|\Phi(t) - \Phi^{(n)}(t)\|_{L_2^\Sigma}^2\, dt\Big]\right)^{\frac{1}{2}} \leqslant \left(\int\limits_0^T \mathbb{E}\Big[\|\Phi(t) - \Phi^{(n)}(t)\|_{L_2^\Sigma}^2\Big]\, dt\right)^{\frac{1}{2}}$$

$$= \left(\int\limits_0^T \|\Phi(t) - \Phi^{(n)}(t)\|_\Sigma^2\, dt\right)^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} \Big[\underbrace{\big\|\Phi_k - \Phi_k^{(n)}\big\|_\Sigma^2}_{\searrow 0} \cdot (t_{k+1} - t_k)\Big]\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0\,.$$

□

The following result (so called the Burkholder-Davis-Gundy inequality) is a generalization of **Th.5.3**.

Theorem 5.4 (BDG inequality).
Let $\Phi \in {}^{\mathsf{H}}M_G^{p,0}(0,T)$ *then*

$$\mathbb{E}\Big[\Big\| \int\limits_0^T \Phi(t)\, dB_t \Big\|_{\mathsf{H}}^p\Big] \leqslant C_p \cdot \mathbb{E}\left[\Big(\int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt\Big)^{\frac{p}{2}}\right], \tag{14}$$

where $C_p > 0\,,\ p \geqslant 2\,.$

Proof.
The proof of the theorem is based on the proof of **Th.5.3** and BDG inequality in the classical case, described for instance in [32, Lm.7.2]). When $p = 2$ and $C_p = 1$ we just have the Itô isometry inequality.

As in the proof of **Th.5.3** we hold the same notations:

$\Phi \in {}^{\mathsf{H}}M_G^{p,0}\big(0,T\big)$ then $\Phi(t) = \sum\limits_{k=0}^{N-1} \Phi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t),$

$$0 = t_0 < t_1 \ldots < t_N = T$$

and for every k $\Phi_k \in {}^{\mathsf{H}}L_G^p(\Omega_{t_k})$ then there exists a sequence $\big\{\Phi_k^{(n)}\big\} \subset Lip(\Omega_{t_k})$, such that $\|\Phi_k - \Phi_k^{(n)}\|_{\Sigma,p} \xrightarrow[n\to\infty]{} 0$.

$\Phi^{(n)}(t) := \sum\limits_{k=0}^{N-1} \Phi_k^{(n)}(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t)$.

Denote $I_T := \int\limits_0^T \Phi(t)\, dB_t$ and $I_T^{(n)} := \int\limits_0^T \Phi^{(n)}(t)\, dB_t$.

So, we need to show that $\mathbb{E}\Big[\|I_T\|_{\mathsf{H}}^p\Big] \leqslant C_p \cdot \mathbb{E}\Big(\int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt\Big)^{\frac{p}{2}}$. $\qquad (*)$

Firstly we show that $\forall m \geqslant 2$ $\mathbb{E}\Big[\|I_T\|_{\mathsf{H}}^m\Big] < \infty$:

$$\mathbb{E}\Big[\|I_T\|_{\mathsf{H}}^m\Big] = \mathbb{E}\Big[\|\int\limits_0^T \Phi(t)\, dB_t\|_{\mathsf{H}}^m\Big] = \mathbb{E}\Big[\|\sum_{k=0}^{N-1} \Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^m\Big]$$

$$\leqslant C_m^1 \cdot \sum_{k=0}^{N-1} \mathbb{E}\Big[\|\Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^m\Big] = C_m^1 \cdot \sum_{k=0}^{N-1} \mathbb{E}\Big[\mathbb{E}\|\Phi_k\, \Delta B_{t_k}\|_{\mathsf{H}}^m \Big|\, \Omega_{t_k}\Big]$$

$$\overset{\substack{\textbf{Prop.2.4}\\ \textbf{Prop.2.6, 2)}}}{\leqslant} C_m^1 \cdot \sum_{k=0}^{N-1} \mathbb{E}\Big[(t_{k+1}-t_k)^{\frac{m}{2}} \cdot C_m^2 \cdot \sup_{Q\in\Sigma}\Big(\mathrm{Tr}\big[\Phi_k Q \Phi_k^*\big]\Big)^{\frac{m}{2}}\Big]$$

$$\leqslant C_m^0 \cdot \sum_{k=0}^{N-1} \mathbb{E}\Bigg[\Big((t_{k+1}-t_k) \cdot \|\Phi(t)\|_{L_2^\Sigma}^2\Big)^{\frac{m}{2}}\Bigg] < \infty\,.$$

It is clear also that $\mathbb{E}\Big[\|I_T^{(n)}\|_{\mathsf{H}}^m\Big] < \infty$.

Then according to the proof of **Th.5.3** we can get that

$$\mathbb{E}\Big[\|\int\limits_0^T \Phi^{(n)}(t)\, dB_t\|_{\mathsf{H}}^p\Big] = \mathbb{E}\Big[\|\varphi\|_{\mathsf{H}}^p(B_{u_1^0},..,B_{u_{m_N}^N})\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P_\theta}\Big[\|\varphi\|_{\mathsf{H}}^p(B_{u_1^0},..,B_{u_{m_N}^N})\Big] = \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P_\theta}\Big[\|\int\limits_0^T \Phi^{(n)}(t)\, dB_t\|_{\mathsf{H}}^p\Big]$$

$$= \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P_\theta}\Big[\|\sum_{k=0}^{N-1} \varphi_k^{(n)}(B_{u_1^k},..,B_{u_{m_k}^k}) \cdot (B_{t_{k+1}} - B_{t_k})\|_{\mathsf{H}}^p\Big]$$

$$= \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_P\Big[\big\|\sum_{k=0}^{N-1}\varphi^{(n)}_k(B^{0,\theta}_{u^k_1},..,B^{0,\theta}_{u^k_{m_k}})\cdot(B^{0,\theta}_{t_{k+1}}-B^{0,\theta}_{t_k})\big\|^p_{\mathsf{H}}\Big]$$

$$= \Big\{\Phi^{(n)}_*(t) := \sum_{k=0}^{N-1}\varphi^{(n)}_k(B^{0,\theta}_{u^k_1},..,B^{0,\theta}_{u^k_{m_k}})\cdot(t_{k+1}-t_k)\Big\}$$

$$= \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_P\Big[\big\|\int_0^T\Phi^{(n)}_*(t)\,dB^{0,\theta}_t\big\|^p_{\mathsf{H}}\Big] = \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_P\Big[\big\|\int_0^T\Phi^{(n)}_*(t)\theta_t\,dW_t\big\|^p_{\mathsf{H}}\Big]$$

$$\overset{\text{classical BDG inequality}}{=} C_p\cdot\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_P\Big[\Big(\int_0^T\big\|\Phi^{(n)}_*(t)\cdot\theta_t\big\|^2_{L_2(\mathsf{U},\mathsf{H})}dt\Big)^{\frac{p}{2}}\Big]$$

$$= C_p\cdot\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\Big[\Big(\int_0^T\big\|\Phi^{(n)}(t)\cdot\theta_t\big\|^2_{L_2(\mathsf{U},\mathsf{H})}dt\Big)^{\frac{p}{2}}\Big]$$

$$\leqslant C_p\cdot\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\Big[\Big(\int_0^T\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}}\big\|\Phi^{(n)}(t)\cdot\theta\big\|^2_{L_2(\mathsf{U},\mathsf{H})}dt\Big)^{\frac{p}{2}}\Big]$$

$$\leqslant C_p\cdot\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\Big[\Big(\int_0^T\sup_{\gamma\in\Theta}\big\|\Phi^{(n)}(t)\cdot\gamma\big\|^2_{L_2(\mathsf{U},\mathsf{H})}dt\Big)^{\frac{p}{2}}\Big]$$

$$= C_p\cdot\sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{P_\theta}\Big[\Big(\int_0^T\sup_{Q\in\Sigma}\big\|\Phi^{(n)}(t)\cdot Q^{1/2}\big\|^2_{L_2(\mathsf{U},\mathsf{H})}dt\Big)^{\frac{p}{2}}\Big]$$

$$= C_p\cdot\bar{\mathbb{E}}\Big[\Big(\int_0^T\|\Phi^{(n)}(t)\|^2_{L_2^{\Sigma}}\,dt\Big)^{\frac{p}{2}}\Big] = C_p\cdot\mathbb{E}\Big[\Big(\int_0^T\|\Phi^{(n)}(t)\|^2_{L_2^{\Sigma}}\,dt\Big)^{\frac{p}{2}}\Big].$$

And now we pass to the limit.

(a): $$\Big|\mathbb{E}\Big[\big\|I_T\big\|^p_{\mathsf{H}}\Big]-\mathbb{E}\Big[\big\|I^{(n)}_T\big\|^p_{\mathsf{H}}\Big]\Big| \leqslant \mathbb{E}\Big[\Big|\big\|I_T\big\|^p_{\mathsf{H}}-\big\|I^{(n)}_T\big\|^p_{\mathsf{H}}\Big|\Big]$$

$$\leqslant \mathbb{E}\Big[\big\|I_T-I^{(n)}_T\big\|_{\mathsf{H}}\cdot\Big(\big\|I_T\big\|^{p-1}_{\mathsf{H}}+\big\|I_T\big\|^{p-2}_{\mathsf{H}}\cdot\big\|I^{(n)}_T\big\|_{\mathsf{H}}+\ldots+\big\|I^{(n)}_T\big\|^{p-1}_{\mathsf{H}}\Big)\Big]$$

$$\overset{\textbf{Prop.2.3}}{\leqslant}\Big(\mathbb{E}\Big[\big\|I_T-I^{(n)}_T\big\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}\times$$

$$\times\underbrace{\Big(\mathbb{E}\Big[\Big(\big\|I_T\big\|^{p-1}_{\mathsf{H}}+\big\|I_T\big\|^{p-2}_{\mathsf{H}}\cdot\big\|I^{(n)}_T\big\|_{\mathsf{H}}+\ldots+\big\|I^{(n)}_T\big\|^{p-1}_{\mathsf{H}}\Big)^2\Big]\Big)^{\frac{1}{2}}}_{<\infty}$$

$$\overset{\textbf{Th.5.3}}{\leqslant} C \cdot \| \Phi - \Phi^{(n)} \|_T = C \cdot \left(\mathbb{E}\Big[\int_0^T \| \Phi(t) - \Phi^{(n)}(t) \|^2_{L_2^\Sigma} \, dt \Big] \right)^{\frac{1}{2}}$$

$$\leqslant C \cdot \left(\int_0^T \mathbb{E}\Big[\| \Phi(t) - \Phi^{(n)}(t) \|^2_{L_2^\Sigma} \Big] \, dt \right)^{\frac{1}{2}} = C \cdot \left(\int_0^T \| \Phi(t) - \Phi^{(n)}(t) \|^2_{\Sigma} \, dt \right)^{\frac{1}{2}}$$

$$= C \cdot \left(\sum_{k=0}^{N-1} \Big[\underbrace{\| \Phi_k - \Phi_k^{(n)} \|^2_{\Sigma}}_{\searrow 0} \cdot (t_{k+1} - t_k) \Big] \right)^{\frac{1}{2}} \xrightarrow[n \to \infty]{} 0 \, .$$

(b): $$\left| \left(\mathbb{E}\Big[\Big(\int_0^T \| \Phi(t) \|^2_{L_2^\Sigma} \, dt \Big)^{\frac{p}{2}} \Big] \right)^{\frac{1}{p}} - \left(\mathbb{E}\Big[\Big(\int_0^T \| \Phi^{(n)}(t) \|^2_{L_2^\Sigma} \, dt \Big)^{\frac{p}{2}} \Big] \right)^{\frac{1}{p}} \right|$$

$$\overset{\text{Minkowski ineq.}}{\leqslant} \left| \left(\mathbb{E}\left[\left(\Big(\int_0^T \| \Phi(t) \|^2_{L_2^\Sigma} \, dt \Big)^{\frac{1}{2}} - \Big(\int_0^T \| \Phi^{(n)}(t) \|^2_{L_2^\Sigma} \, dt \Big)^{\frac{1}{2}} \right)^p \right] \right)^{\frac{1}{p}} \right|$$

$$\overset{\text{Minkowski ineq.}}{\leqslant} \left(\mathbb{E}\Big[\Big(\int_0^T \| \Phi(t) - \Phi^{(n)}(t) \|^2_{L_2^\Sigma} \, dt \Big)^{\frac{p}{2}} \Big] \right)^{\frac{1}{p}}$$

$$\leqslant \left(\mathbb{E}\Big[\int_0^T \| \Phi(t) - \Phi^{(n)}(t) \|^p_{L_2^\Sigma} \, dt \Big] \right)^{\frac{1}{p}} \leqslant \left(\int_0^T \mathbb{E}\Big[\| \Phi(t) - \Phi^{(n)}(t) \|^p_{L_2^\Sigma} \Big] dt \right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \| \Phi(t) - \Phi^{(n)}(t) \|^p_{\Sigma, p} \, dt \right)^{\frac{1}{p}}$$

$$= \left(\sum_{k=0}^{N-1} \Big[\underbrace{\| \Phi_k - \Phi_k^{(n)} \|^p_{\Sigma, p}}_{\searrow 0} \cdot (t_{k+1} - t_k) \Big] \right)^{\frac{1}{p}} \xrightarrow[n \to \infty]{} 0 \, .$$

$\square$

Define ${}^{\mathsf{H}}M_G^p\big(0,T\big)$ as the completion of ${}^{\mathsf{H}}M_G^{p,0}\big(0,T\big)$ under the norm

$$|||\Phi|||_{T,p}^p := \mathbb{E}\Bigg[\Big(\int_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\,dt\Big)^{\frac{p}{2}}\Bigg].$$

Actually, such a norm is finite on ${}^{\mathsf{H}}M_G^{p,0}\big(0,T\big)$, because

$$|||\Phi|||_{T,p}^p = \mathbb{E}\Bigg[\Big(\sum_{k=0}^{N-1}(t_k - t_{k+1})\cdot\|\Phi(t)\|_{L_2^\Sigma}^2\Big)^{\frac{p}{2}}\Bigg]$$
$$\leqslant C_{p,N}\cdot\sum_{k=0}^{N-1}\Big((t_k - t_{k+1})^{\frac{p}{2}}\cdot\underbrace{\mathbb{E}\Big[\|\Phi(t)\|_{L_2^\Sigma}^p\Big]}_{<\infty}\Big) < \infty\,.$$

Also we denote $|||\Phi|||_{T,2}^2 := |||\Phi|||_T^2 = \mathbb{E}\Big[\int_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\,dt\Big]$.

Theorem 5.5. *$I_T = \int_0^T \Phi(t)\,dB_t$ can be extended on ${}^{\mathsf{H}}M_G^p\big(0,T\big)$.*
And for $\Phi \in {}^{\mathsf{H}}M_G^p\big(0,T\big)$ the BDG inequality (14) holds.
In particular, if $p=2$ the Itô's isometry inequality (12) holds.

Proof.
We have if $\Phi \in {}^{\mathsf{H}}M_G^p\big(0,T\big)$ then there exists a sequence $\big\{\Phi^{(n)},\ n \geqslant 1\big\} \subset {}^{\mathsf{H}}M_G^{p,0}\big(0,T\big)$, such that $|||\Phi - \Phi^{(n)}|||_{T,p} \xrightarrow[n\to\infty]{} 0$.
Let us define the norm $\|I_T\|_{\Omega_T,p}^p := \mathbb{E}\big[\|I_T\|_{\mathsf{H}}^p\big]$.
And we have

$$\|I_T(\Phi^{(n)}) - I_T(\Phi^{(m)})\|_{\Omega_T,p} = \|I_T(\Phi^{(n)} - \Phi^{(m)})\|_{\Omega_T,p} \overset{\textbf{Th.5.4}}{\leqslant} C_p\cdot|||\Phi^{(n)} - \Phi^{(m)}|||_{T,p}\,.$$

So that, we can extend I_T on ${}^{\mathsf{H}}M_G^p\big(0,T\big)$ as a continuous mapping.
Define $I_T(\Phi) := \lim_{n\to\infty} I_T(\Phi^{(n)})$.
Letting $m\to\infty$ yields

$$0 \leqslant \|I_T(\Phi^{(n)}) - I_T(\Phi)\|_{\Omega_T,p} \leqslant C_p\cdot|||\Phi^{(n)} - \Phi|||_{T,p} \xrightarrow[n\to\infty]{} 0\,.$$

Therefore $\|I_T(\Phi^{(n)})\|_{\Omega_T,p} \xrightarrow[n\to\infty]{} \|I_T(\Phi)\|_{\Omega_T,p}$ and $|||\Phi^{(n)}|||_{T,p} \xrightarrow[n\to\infty]{} |||\Phi|||_{T,p}$.
So we can conclude that $\|I_T(\Phi)\|_{\Omega_T,p} \leqslant C_p\cdot|||\Phi|||_{T,p}$.

□

5.3 Characterization of the space of integrand processes ${}^{\mathsf{H}}M_G^2\big(0,T\big)$

Proposition 5.1. $\Phi \in {}^{\mathsf{H}}M_G^2\big(0,T\big) \quad \Leftrightarrow \quad$ *1)* $|||\Phi|||_T < \infty$ *;*
2) $\Phi(t) \in {}^{\mathsf{H}}L_G^2\big(\Omega_t\big)$ *for almost all* t *.*

Proof.

$(\Rightarrow)$

If $\Phi \in {}^{\mathsf{H}}M_G^2\big(0,T\big)$ then $|||\Phi|||_T < \infty$ and there exists a sequence $\big\{\Phi_n,\ n \geqslant 1\big\} \subset {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)$, such that $|||\Phi - \Phi_n|||_T \xrightarrow[n\to\infty]{} 0$.
It follows that for almost all t we have $\|\Phi - \Phi_n\|_\Sigma \xrightarrow[n\to\infty]{} 0$.
(Recall that the norm $\|\cdot\|_\Sigma$ is introduced in 5.1).
For such a fixed $t = t'$ implies that $\Phi_n(t') = Const \in Lip\big(\Omega_t\big)$.
Since $\Big({}^{\mathsf{H}}L_G^2\big(\Omega_t\big), \|\cdot\|_\Sigma\Big)$ is a Banach space, then for almost all t we have that $\Phi(t) \in {}^{\mathsf{H}}L_G^2\big(\Omega_t\big)$.

$(\Leftarrow)$

1) Let for almost all t $\Phi(t) \in Lip\big(\Omega_t\big)$ be continuous.
Take partition of $[0,T]$: $\lambda_n = \{0 = t_0^n < t_1^n < \ldots < t_N^n = T\}$,
$$d(\lambda_n) \xrightarrow[n\to\infty]{} 0\,, \ N = N(n) \xrightarrow[n\to\infty]{} 0\,.$$
Define $\Phi_n := \sum\limits_{k=0}^{N-1} \Phi(t_k^n)(\omega)\mathbb{1}_{[t_k^n, t_{k+1}^n)}(t) \in {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)$.
We have $\Phi_n(t) \xrightarrow[n\to\infty]{} \Phi(t)$ for all t.
Let us calculate
$$|||\Phi - \Phi_n|||_T^2 = \mathbb{E}\Big[\Big\|\int_0^T \big(\Phi(t) - \Phi_n(t)\big)\, dB_t\Big\|_{\mathsf{H}}^2\Big] \leqslant \mathbb{E}\Big[\int_0^T \big\|\big(\Phi(t) - \Phi_n(t)\big)\big\|_{L_2^\Sigma}^2\, dt\Big]\,,$$
using Itô's inequality according to **Th.5.5**.
Then there exists a point $t_\theta \in [0,T]$ such that
$$\mathbb{E}\Big[\int_0^T \big\|\big(\Phi(t) - \Phi_n(t)\big)\big\|_{L_2^\Sigma}^2\, dt\Big] = T\cdot\mathbb{E}\Big[\big\|\big(\Phi(t_\theta,\omega) - \Phi_n(t_\theta,\omega)\big)\big\|_{L_2^\Sigma}^2\Big]\,.$$
But the last term tends to 0 according to **Th.5.2**.
So, we have that $|||\Phi - \Phi_n|||_T \xrightarrow[n\to\infty]{} 0$.

2) Let for almost all t $\Phi(t) \in Lip\big(\Omega_t\big)$ (it is not necessary continuous).

Define $\Phi_\varepsilon(t) := \dfrac{1}{\varepsilon}\int\limits_{-\infty}^{+\infty} \rho(\frac{t-s-\varepsilon}{\varepsilon})\Phi(s)ds = \dfrac{1}{\varepsilon}\int\limits_{t-2\varepsilon}^{t} \rho(\frac{t-s-\varepsilon}{\varepsilon})\Phi(s)ds$

$$= \int\limits_{-1}^{1} \rho(v)\Phi(t-\varepsilon v-\varepsilon)dv\,.$$

Hence for all t $\Phi_\varepsilon(t+) = \Phi_\varepsilon(t-) = \Phi_\varepsilon(t)$, so that $\Phi_\varepsilon(\,\cdot\,)$ is continuous.

Consider $A_\varepsilon := |\!|\!|\Phi - \Phi_\varepsilon|\!|\!|_T^2 = \int\limits_0^T \|\Phi(t) - \Phi_\varepsilon(t)\|_\Sigma^2\, dt$

$$= \int\limits_0^T \| \int\limits_{-1}^{1} \rho(v)\big(\Phi(t) - \Phi(t-\varepsilon v-\varepsilon)\big)dv\|_\Sigma^2\, dt$$

$$\leqslant \int\limits_0^T \Big(\int\limits_{-1}^{1} \rho(v)\|\Phi(t) - \Phi(t-\varepsilon v-\varepsilon)\|_\Sigma\, dv\Big)^2 dt$$

$$\leqslant \int\limits_0^T \Big(\int\limits_{-1}^{1} \rho(v)dv \cdot \int\limits_{-1}^{1} \rho(v)\|\Phi(t) - \Phi(t-\varepsilon v-\varepsilon)\|_\Sigma^2\, dv\Big)dt\,.$$

But for every separable Banach space B we have such a dense inclusion: $C\big([0,T],\,B\big) \subset L_2\big([0,T],\,B\big)$,

because every $f \in L_2\big([0,T],\,B\big)$ can be approximated by $\sum\limits_{i=1}^{m} f_i(t)b_i$,

where $(b_i)_{i\geqslant 1} \subset B$ - densely, $f_i \in L_2\big([0,T],\,B\big)$;

and such $f_i \in L_2\big([0,T],\,B\big)$ can be approximated by $g_i \in C\big([0,T],\,B\big)$, because it is well-known that $C\big([0,T],\,B\big) \subset L_2\big([0,T],\,B\big)$ densely.

So, f can be approximated by $\sum\limits_{i=1}^{m} g_i(t)b_i \in C\big([0,T],\,B\big)$.

Using this fact we have that:

For $\Phi : [0,T] \to {}^{\mathsf{H}}L_G^2\big(\Omega_T\big) =: B$, where $\big(B, \|\cdot\|_B\big)$ is a Banach separable space and $\int\limits_0^T \|\Phi(t)\|_B^2 dt < \infty$, there exists $\Psi_\delta \in C\big([0,T],\,B\big)$, such that $\int\limits_0^T \|\Phi(t) - \Psi_\delta(t)\|_B^2 dt < \delta$.

Therefore $\left|A_\varepsilon\right| \leqslant \int\limits_0^T \int\limits_{-1}^1 \rho(v)\|\Phi(t) - \Phi(t-\varepsilon v-\varepsilon)\|_\Sigma^2\, dvdt$

$$\leqslant 3\int\limits_0^T \int\limits_{-1}^1 \rho(v)\|\Phi(t) - \Psi_\delta(t)\|_\Sigma^2\, dvdt + 3\int\limits_0^T \int\limits_{-1}^1 \rho(v)\|\Phi(t-\varepsilon v-\varepsilon) - \Psi_\delta(t-\varepsilon v-\varepsilon)\|_\Sigma^2\, dvdt + 3\int\limits_0^T \int\limits_{-1}^1 \rho(v)\|\Psi_\delta(t) - \Psi_\delta(t-\varepsilon v-\varepsilon)\|_\Sigma^2\, dvdt$$

$$\leqslant 6\delta + 3\int\limits_0^T \int\limits_{-1}^1 \rho(v)\|\Psi_\delta(t) - \Psi_\delta(t-\varepsilon v-\varepsilon)\|_\Sigma^2\, dvdt\,.$$

Hence $\overline{\lim\limits_{\varepsilon\to 0}}\left|A_\varepsilon\right| \leqslant 6\delta\,.$

So that $\|\!|\Phi - \Phi_\varepsilon|\!\|_T \xrightarrow[\varepsilon\to 0]{} 0\,.$

3) Let for almost all t $\Phi(t) \in {}^{\mathsf{H}}L_G^2\big(\Omega_t\big)\,.$

By the definition there exists a sequence $\big\{\Phi_m,\ m \geqslant 1\big\} \subset Lip\big(\Omega_t\big)\,,$ such that $\|\Phi - \Phi_m\|_\Sigma \xrightarrow[n\to\infty]{} 0$ for almost all $t\,.$

As in the **1)** part we can get that

$$\|\!|\Phi - \Phi_m|\!\|_T^2 \leqslant T\cdot\mathbb{E}\Big[\Big\|\big(\Phi(t_\theta,\omega) - \Phi_n(t_\theta,\omega)\big)\Big\|_{L_2^\Sigma}^2\Big] \xrightarrow[n\to\infty]{} 0\,.$$

Hence $\|\!|\Phi - \Phi_m|\!\|_T \leqslant \|\!|\Phi - (\Phi_m)^n|\!\|_T + \|\!|\Phi_m - (\Phi_m)^n|\!\|_T\,,$

where $(\Phi_m)^n \in {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)\,.$

From **2)** it follows that $\|\!|\Phi - \Phi_m|\!\|_T \xrightarrow[m,n\to\infty]{} 0\,.$

So that $\Phi_m \in {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)\,.$

□

Remark 5.4. *If $\Phi(t)$ is nonrandom then condition **2)** of **Prop.5.1** can be omitted:*

$$\|\!|\Phi|\!\|_T < \infty \quad\Leftrightarrow\quad \Phi \in {}^{\mathsf{H}}M_G^2\big(0,T\big).$$

Proof.

$\|\!|\Phi|\!\|_T = \int\limits_0^T \|\Phi(t)\|_\Sigma^2\, dt = \int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt < \infty$ then $\|\Phi(t)\|_{L_2^\Sigma}^2 < \infty$

for almost all $t\,.$

From here we can conclude that $\Phi(t) \in L_2^\Sigma\,,$ hence that $\Phi(t) \in Lip\big(\Omega_t\big)$ (since $\Phi(t)$ is nonrandom) and finally that $\Phi(t) \in {}^{\mathsf{H}}L_G^2\big(\Omega_t\big)$

for almost all $t\,.$

And by **Prop.5.1** we get that $\Phi(t) \in {}^{\mathsf{H}}M_G^2\big(0,T\big)\,.$

□

5.4 Fubini theorem

Let $(\mathcal{X}, \mathcal{E}, \mu)$ is a measurable space, $\mu(\mathcal{X}) < \infty$.

Consider $^{\mathsf{H}}M_G^{2,0}\big(0, T; \mathcal{X}\big) := \left\{ \Phi(t,x) = \sum\limits_{k=0}^{N-1} \sum\limits_{j=0}^{M-1} \Phi_{kj}(\omega) \mathbb{1}_{[t_k, t_{k+1})}(t) \mathbb{1}_{A_j}(x) \;\middle|\; \Phi_k(\omega) \in {^{\mathsf{H}}L_G^2}(\Omega_{t_k}),\ 0 = t_0 < t_1 \ldots < t_N = T,\ A_j \in \mathcal{E} \right\}$.

Let $^{\mathsf{H}}M_G^2\big(0, T; \mathcal{X}\big)$ be the completion of $^{\mathsf{H}}M_G^{2,0}\big(0, T; \mathcal{X}\big)$ under the norm $|||\Phi|||_{T,\mathcal{X}} := \int\limits_{\mathcal{X}} |||\Phi(\,\cdot\,, x)|||_T \,\mu(dx)$.

Theorem 5.6. *If* $\Phi(t,x) \in {^{\mathsf{H}}M_G^2}\big(0, T; X\big)$, *then:*

$$\int\limits_{\mathcal{X}} \int\limits_0^T \Phi(t,x)\, dB_t\, \mu(dx) = \int\limits_0^T \int\limits_{\mathcal{X}} \Phi(t,x)\, \mu(dx)\, dB_t \quad q.s.$$

Proof.
Let $\big\{\Phi_n,\ n \geqslant 1\big\} \subset {^{\mathsf{H}}M_G^{2,0}}\big(0, T; \mathcal{X}\big)$.
Then in the same way as in **Prop.5.1** using **Th.5.5** and **Th.5.2** we can conclude that $|||\Phi(\,\cdot\,, x) - \Phi_n(\,\cdot\,, x)|||_T \xrightarrow[n\to\infty]{} 0$.
And due to the dominated convergence theorem have that

$$\int\limits_{\mathcal{X}} |||\Phi(\,\cdot\,, x) - \Phi_n(\,\cdot\,, x)|||_T \,\mu(dx) \xrightarrow[n\to\infty]{} 0.$$

For every n set $\Phi_n(t,x) := \sum\limits_{k=0}^{N_n-1} \sum\limits_{j=0}^{M_n-1} \Phi_{kj}^n(\omega) \mathbb{1}_{[t_k^n, t_{k+1}^n)}(t) \mathbb{1}_{A_j^n}(x)$.
And define the following random variables:
$\xi_n(x) := \int\limits_0^T \Phi_n(t,x)\, dB_t\,,\quad \xi(x) := \int\limits_0^T \Phi(t,x)\, dB_t\,.$
$\eta_n(t) := \int\limits_{\mathcal{X}} \Phi_n(t,x)\, \mu(dx)\,,\quad \eta(t) := \int\limits_{\mathcal{X}} \Phi(t,x)\, \mu(dx)\,.$
Then $\int\limits_{\mathcal{X}} \xi_n(x)\, \mu(dx) = \int\limits_0^T \eta_n(t)\, dB_t = \sum\limits_{k=0}^{N_n-1} \sum\limits_{j=0}^{M_n-1} \Phi_{kj}^n (B_{t_{k+1}^n} - B_{t_k^n}) \mu(A_j^n)$.
For the later calculations we will use a Cauchy-Schwarz-Bunyakovsky inequality (**Prop.2.3**) in a following form $\mathbb{E}[X] \leqslant \big(\mathbb{E}[X^2]\big)^{\frac{1}{2}}$:

(a) $\mathbb{E}\Big[\big\| \int\limits_{\mathcal{X}} \xi(x)\,\mu(dx) - \int\limits_{\mathcal{X}} \xi_n(x)\,\mu(dx)\big\|_{\mathsf{H}}\Big]$

$$= \mathbb{E}\Big[\big\| \int\limits_{\mathcal{X}} \int\limits_0^T \big(\Phi(t,x) - \Phi_n(t,x)\big)\, dB_t\, \mu(dx)\big\|_{\mathsf{H}}\Big]$$

$$\leqslant \int\limits_{\mathcal{X}} \mathbb{E}\Big[\big\| \int\limits_0^T \big(\Phi(t,x) - \Phi_n(t,x)\big)\, dB_t\big\|_{\mathsf{H}}\Big]\mu(dx)$$

$$\leqslant \int\limits_{\mathcal{X}} \Big(\mathbb{E}\Big[\big\| \int\limits_0^T \big(\Phi(t,x) - \Phi_n(t,x)\big)\, dB_t\big\|_{\mathsf{H}}^2\Big]\Big)^{\frac{1}{2}} \mu(dx)$$

$$= \int\limits_{\mathcal{X}} \|\!|\Phi(\,\cdot\,,x) - \Phi_n(\,\cdot\,,x)\|\!|_T\, \mu(dx) \xrightarrow[n\to\infty]{} 0\,.$$

(b) $\mathbb{E}\Big[\big\| \int\limits_0^T \eta(t)\, dB_t - \int\limits_{\mathcal{X}} \xi_n(x)\,\mu(dx)\big\|_{\mathsf{H}}\Big]$

$$= \mathbb{E}\Big[\big\| \int\limits_0^T \eta(t)\, dB_t - \int\limits_0^T \eta_n(t)\, dB_t\big\|_{\mathsf{H}}\Big]$$

$$= \mathbb{E}\Big[\big\| \int\limits_0^T \int\limits_{\mathcal{X}} \big(\Phi(t,x) - \Phi_n(t,x)\big)\, \mu(dx)\, dB_t\big\|_{\mathsf{H}}\Big]$$

$$\leqslant \|\!| \int\limits_{\mathcal{X}} \big(\Phi(\,\cdot\,,x) - \Phi_n(\,\cdot\,,x)\big)\|\!|_T\, \mu(dx)$$

$$\leqslant \int\limits_{\mathcal{X}} \|\!|\Phi(\,\cdot\,,x) - \Phi_n(\,\cdot\,,x)\|\!|_T\, \mu(dx) \xrightarrow[n\to\infty]{} 0\,.$$

Therefore $\mathbb{E}\big\| \int\limits_{\mathcal{X}} \xi(x)\,\mu(dx) - \int\limits_0^T \eta(t)\, dB_t\big\|_{\mathsf{H}}\,.$

And by **Lm.4.1** we get that $\int\limits_{\mathcal{X}} \xi(x)\,\mu(dx) = \int\limits_0^T \eta(t)\, dB_t$ quasi surely.

□

5.5 Distribution of the stochastic integral with non-random integrand

Let us consider a stochastic integral $I(\Phi) = \int\limits_0^T \Phi(t) dB_t\,,$ with respect to G-Brownian motion $B_t \sim N_G\big(0, t\cdot\Sigma\big)\,.$

Assume that Φ is non-random, then $\Phi : [0,T] \to L_2^{\Sigma}\,.$

Note that also in such a case ${}^{\mathsf{H}}L_G^2\big(\Omega\big) \equiv Lip(\Omega) \equiv L_2^{\Sigma}\,.$

1) Firstly we consider a case with elementary processes:

If $\Phi \in {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)$ then $\Phi(t) = \sum\limits_{k=0}^{N-1} \Phi_k \mathbb{1}_{[t_k,t_{k+1})}(t)$, $\Phi_k \in L_2^\Sigma$, and the norm $\|\!|\Phi|\!\|_T^2 = \int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2 dt < \infty$.

By the definition we have $I(\Phi) = \sum\limits_{k=0}^{N-1} \Phi_k(B_{t_{k+1}} - B_{t_k})$.

We know that a random variable $\dfrac{B_{t_{k+1}} - B_{t_k}}{\sqrt{t_{k+1} - t_k}} \sim B_1 \sim N_G\big(0,\Sigma\big)$.

Then using **Prop.2.6** we get that $I(\Phi) \sim N_G\big(0,\Sigma_I\big)$,

$$\text{where}\, \Sigma_I = \Big\{ \sum_{k=0}^{N-1} (t_{k+1} - t_k)\Phi_k Q \Phi_k^* \Big|\, Q \in \Sigma \Big\}.$$

Note that in this case $\sum\limits_{k=0}^{N-1} (t_{k+1} - t_k)\Phi_k Q \Phi_k^* = \int\limits_0^T \Phi(t) Q \Phi^*(t) dt$.

2) In a general case we have that for $\Phi \in {}^{\mathsf{H}}M_G^2\big(0,T\big)$ there exists a sequence $\big\{\Phi_n,\ n \geqslant 1\big\} \subset {}^{\mathsf{H}}M_G^{2,0}\big(0,T\big)$, such that $\|\!|\Phi - \Phi_n|\!\|_T \xrightarrow[n\to\infty]{} 0$.

And by **Th.5.5** we get that $\|I(\Phi) - I(\Phi_n)\|_{\Omega_T} \leqslant \|\!|\Phi - \Phi_n|\!\|_T$.

Since $\Phi \in L_2^\Sigma$ then for $p > 0$ $\Phi \in {}^{\mathsf{H}}M_G^p\big(0,T\big)$ and by **Th.5.5** it may be concluded that $\mathbb{E}\big[\|I\|_{\mathsf{H}}^p\big] \leqslant C_p \cdot \Big(\int\limits_0^T \|\Phi(t)\|_{L_2^\Sigma}^2\, dt \Big)^{\frac{p}{2}} < \infty$.

Lemma 5.1. *$\int\limits_0^T \Phi_n(t) Q \Phi_n^*(t) dt \xrightarrow[n\to\infty]{} \int\limits_0^T \Phi(t) Q \Phi^*(t) dt$ in the trace-class topology.*

Proof.

$\|\!|\Phi - \Phi_n|\!\|_T^2 = \int\limits_0^T \|\Phi(t) - \Phi_n(t)\|_{L_2^\Sigma}^2 dt$

$= \int\limits_0^T \sup\limits_{Q\in\Sigma} \mathrm{Tr}\Big[\big(\Phi(t) - \Phi_n(t)\big) Q \big(\Phi(t) - \Phi_n(t)\big)^*\Big] dt$

$$= \int_0^T \sup_{Q\in\Sigma} \big\|\big(\Phi(t) - \Phi_n(t)\big) Q^{1/2}\big\|_{L_2(\mathsf{H})}^2 dt \xrightarrow[n\to\infty]{} 0.$$

Denote $A_n := \Phi_n(t) \cdot Q^{1/2}$ and $A := \Phi(t) \cdot Q^{1/2}$.

We have that

$$\Big\| \int_0^T \Phi_n(t) Q \Phi_n^*(t) dt - \int_0^T \Phi(t) Q \Phi^*(t) dt \Big\|_{L_1(\mathsf{H})} = \Big\| \int_0^T \big(A_n A_n^* - A A^*\big) dt \Big\|_{L_1(\mathsf{H})}$$

$$\leqslant \int_0^T \|A_n A_n^* - A A^*\|_{L_1(\mathsf{H})} dt = \int_0^T \|A_n A_n^* - A A_n^* + A A_n^* - A A^*\|_{L_1(\mathsf{H})} dt$$

$$\leqslant \int_0^T \Big(\|A_n^*\|_{L_2(\mathsf{H})} \cdot \|A - A_n\|_{L_2(\mathsf{H})} + \|A\|_{L_2(\mathsf{H})} \cdot \|(A - A_n)^*\|_{L_2(\mathsf{H})} \Big) dt$$

$$\leqslant \Big(\int_0^T \|A_n\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2} \cdot \Big(\int_0^T \|A - A_n\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2}$$

$$+ \Big(\int_0^T \|A\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2} \cdot \Big(\int_0^T \|A - A_n\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2}$$

$$\leqslant \Bigg(\Big(\int_0^T \sup_{Q \in \Sigma} \big\| \big(\Phi_n(t)\big) Q^{1/2} \big\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2} + \Big(\int_0^T \sup_{Q \in \Sigma} \big\| \big(\Phi(t)\big) Q^{1/2} \big\|^2_{L_2(\mathsf{H})} dt \Big)^{1/2} \Bigg) \times$$

$$\times \int_0^T \sup_{Q \in \Sigma} \big\| \big(\Phi(t) - \Phi_n(t)\big) Q^{1/2} \big\|^2_{L_2(\mathsf{H})} dt$$

$$= \Big(|\!|\!| \Phi_n |\!|\!|_T + |\!|\!| \Phi |\!|\!|_T \Big) \cdot |\!|\!| \Phi - \Phi_n |\!|\!|_T \xrightarrow[n \to \infty]{} 0\,.$$

□

Theorem 5.7. *Stochastic integral* $I(\Phi) = \int_0^T \Phi(t) dB_t$ *with nonrandom integrand* $\Phi(t)$ *is* G*-normal distributed, where* $B_t \sim N_G\big(0, t \cdot \Sigma\big)$ *is a* G*-Brownian motion. I.e.,*

$$I(\Phi) \sim N_G\big(0, \Sigma_I\big)\ , \quad \text{where } \Sigma_I = \Big\{ \int_0^T \Phi(t) Q \Phi^*(t) dt \ \big| \ Q \in \Sigma \Big\}\,.$$

Proof.

1) From the first part of this section we have got that

$$I_n := I(\Phi_n) \sim N_G\big(0, \Sigma_{I_n}\big)\,, \ \text{ where } \ \Sigma_{I_n} = \Big\{ \int_0^T \Phi_n(t) Q \Phi_n^*(t) dt \ \big| \ Q \in \Sigma \Big\}\,.$$

2) $G_{I_n}(A) = \dfrac{1}{2} \mathbb{E}\big[\langle A I_n, I_n \rangle\big]\,;\ G_I(A) = \dfrac{1}{2} \mathbb{E}\big[\langle A I, I \rangle\big]\,.$

Then $2 \cdot \Big| G_{I_n}(A) - G_I(A) \Big| = \Big| \mathbb{E}\big[\langle A I_n, I_n \rangle\big] - \mathbb{E}\big[\langle A I, I \rangle\big] \Big|$

$$= \Big| \mathbb{E}\big[\langle A I_n, I_n \rangle\big] - \mathbb{E}\big[\langle A I_n, I \rangle\big] + \mathbb{E}\big[\langle A I_n, I \rangle\big] - \mathbb{E}\big[\langle A I, I \rangle\big] \Big|$$

$$\leqslant \Big|\mathbb{E}\big[\langle AI_n, I_n\rangle - \langle AI_n, I\rangle\big]\Big| + \Big|\mathbb{E}\big[\langle AI_n, I\rangle - \langle AI, I\rangle\big]\Big|$$
$$\leqslant \mathbb{E}\Big[\big|\langle AI_n, I_n - I\rangle\big|\Big] + \mathbb{E}\Big[\big|\langle A(I_n - I), I\rangle\big|\Big]$$
$$\leqslant \left(\mathbb{E}\Big[\|AI_n\|_{\mathsf{H}}^2\Big]\right)^{1/2} \cdot \left(\mathbb{E}\Big[\|I_n - I\|_{\mathsf{H}}^2\Big]\right)^{1/2} + \left(\mathbb{E}\Big[\|A(I_n - I)\|_{\mathsf{H}}^2\Big]\right)^{1/2} \cdot \left(\mathbb{E}\Big[\|I\|_{\mathsf{H}}^2\Big]\right)^{1/2}$$
$$\leqslant \|A\|_{L(\mathsf{H})} \cdot \left[\left(\mathbb{E}\Big[\|I_n\|_{\mathsf{H}}^2\Big]\right)^{1/2} + \left(\mathbb{E}\Big[\|I\|_{\mathsf{H}}^2\Big]\right)^{1/2}\right] \cdot \left(\mathbb{E}\Big[\|I_n - I\|_{\mathsf{H}}^2\Big]\right)^{1/2}$$
$$= \|A\|_{L(\mathsf{H})} \cdot \Big(\|I_n\|_{\Omega_T} + \|I\|_{\Omega_T}\Big) \cdot \|I_n - I\|_{\Omega_T} \xrightarrow[n\to\infty]{} 0\,.$$

So that $G_{I_n}(A) \xrightarrow[n\to\infty]{} G_I(A)$.

We know that G_I defines a covariation set Σ_I of the $I(\Phi)$.

3) Now we are going to proof that $I(\Phi) \sim N_G\big(0, \Sigma_I\big)$.

In order to show such a fact we consider $u_n(t,x) := \mathbb{E}\big[f(x + \sqrt{T-t}\, I_n)\big]$ and $u(t,x) := \mathbb{E}\big[f(x + \sqrt{T-t}\, I)\big]$, with a B-continuous function $f \in \mathbf{C}_{p.Lip}(\mathsf{H})$;

Since $I_n \sim N_G\big(0, \Sigma_{I_n}\big)$ then by **Th.4.1** we have that u_n is a unique viscosity solution to the equation
$$\begin{cases} \partial_t u + G_{I_n}(D_{xx}^2 u) = 0\,; \\ u(T,x) = f(x)\,. \end{cases} \qquad (\,*\,)$$

So, we need to show that u is a viscosity solution to the equation
$$\begin{cases} \partial_t u + G_I(D_{xx}^2 u) = 0\,; \\ u(T,x) = f(x)\,. \end{cases} \qquad (\,\#\,)$$

(i) We claim that for every fixed point (t,x) : $u_n(t,x) \xrightarrow[n\to 0]{} u(t,x)$.

In fact, $\big|u_n(t,x) - u(t,x)\big| = \Big|\mathbb{E}\big[f(x + \sqrt{T-t}\, I_n)\big] - \mathbb{E}\big[f(x + \sqrt{T-t}\, I)\big]\Big|$
$$\leqslant \mathbb{E}\Big[\big|f(x + \sqrt{T-t}\, I_n) - f(x + \sqrt{T-t}\, I)\big|\Big]$$
$$\leqslant \mathbb{E}\Big[C \cdot \big(1 + \|x + \sqrt{T-t}\, I_n\|_{\mathsf{H}}^m + \|x + \sqrt{T-t}\, I\|_{\mathsf{H}}^m\big) \cdot \big\|\sqrt{T-t}(I_n - I)\big\|_{\mathsf{H}}\Big]$$

$$\leqslant C \cdot \left(\mathbb{E}\Big[\Big(1 + \|x + \sqrt{T-t}\, I_n\|_{\mathsf{H}}^m + \|x + \sqrt{T-t}\, I\|_{\mathsf{H}}^m \Big)^2 \Big] \right)^{1/2} \times$$

$$\times \left((T-t) \cdot \mathbb{E}\Big[\|I_n - I\|_{\mathsf{H}}^2 \Big] \right)^{1/2}$$

$$\leqslant \widetilde{C} \cdot \|I_n - I\|_{\Omega_T} \xrightarrow[n\to\infty]{} 0 .$$

(ii) Also we claim that u, u_n are continuous at $(t,x) \in [0,T] \times \mathsf{H}$.
In fact, let us show that $\mathbb{E}\big[f(x+\sqrt{T-t-\delta}\, I)\big] \xrightarrow[\delta\to 0]{} \mathbb{E}\big[f(x+\sqrt{T-t}\, I)\big]$:

$$\Big| \mathbb{E}\big[f(x+\sqrt{T-t-\delta}\, I)\big] - \mathbb{E}\big[f(x+\sqrt{T-t}\, I)\big] \Big|$$

$$\leqslant \mathbb{E}\Big[\big| f(x+\sqrt{T-t-\delta}\, I) - f(x+\sqrt{T-t}\, I) \big| \Big]$$

$$\leqslant \mathbb{E}\Big[C \cdot \big(1 + \|x + \sqrt{T-t-\delta}\, I)\|_{\mathsf{H}}^m + \|x + \sqrt{T-t}\, I)\|_{\mathsf{H}}^m \big) \times$$

$$\times \big\| (\sqrt{T-t-\delta} - \sqrt{T-t}) I \big\|_{\mathsf{H}} \Big]$$

$$\leqslant C \cdot \left(\sqrt{T-t-\delta} - \sqrt{T-t} \right) \times$$

$$\times \left(\mathbb{E}\Big[\Big(1 + \|x + \sqrt{T-t-\delta}\, I)\|_{\mathsf{H}}^m + \|x + \sqrt{T-t}\, I)\|_{\mathsf{H}}^m \Big)^2 \Big] \right)^{1/2} \times \left(\mathbb{E}\Big[\|I\|_{\mathsf{H}}^2 \Big] \right)^{1/2}$$

$$\leqslant \widetilde{C} \cdot \left(\sqrt{T-t-\delta} - \sqrt{T-t} \right) \cdot \|I\|_{\Omega_T} \xrightarrow[\delta\to 0]{} 0 .$$

So that u is continuous. It is clear that u_n is continuous too.

(iii) Let ψ be a test function, such that: $u(t,x) \leqslant \psi(t,x)$;
$u(t_0,x_0) = \psi(t_0,x_0)$.
Since for every fixed point (t,x): $u_n(t,x) \xrightarrow[n\to 0]{} u(t,x)$ then there exists a sequence of test functions $\{\psi_n\}$, such that: $u_n(t,x) \leqslant \psi_n(t,x)$;
$u_n(t_0,x_0) = \psi_n(t_0,x_0)$;
$\psi_n(t,x) \xrightarrow[n\to 0]{} \psi(t,x)$.
In order to show it we can take $\widetilde{\psi}_n := \psi + u_n - u$ that satisfies above written required properties, and in the points where it is not enough smooth we need to alter it in the proper way to get the test function ψ_n.

We know that u_n is a viscosity sub- (and super-) solution to equation $(\,*\,)$.
So that $u_n(T,x) \leqslant f(x)$;

$$\Big[\partial_t\psi_n + G_{I_n}(D^2_{xx}\psi_n)\Big](t_0,x_0) \geqslant 0\,.$$

Hence $u(T,x) \leqslant f(x)$, since $u_n(t,x) \xrightarrow[n\to 0]{} u(t,x)$.

And $\Big[\partial_t\psi_n + G_{I_n}(D^2_{xx}\psi_n)\Big](t_0,x_0) \xrightarrow[n\to 0]{\mathbf{2)}} \Big[\partial_t\psi + G_I(D^2_{xx}\psi)\Big](t_0,x_0)$.

Therefore $\Big[\partial_t\psi + G_I(D^2_{xx}\psi)\Big](t_0,x_0) \geqslant 0$.

So, we have that u is a viscosity subsolution to equation $(\,\#\,)$.
And in the same way we can show that u is a viscosity supersolution.
So, we can conclude that u is a viscosity solution to equation $(\,\#\,)$.

3) Now we are going to describe the structure of the covariation set Σ_I.
Let us define a set $\Sigma := \overline{\mathrm{conv}(\Sigma'_I)}$ in the trace-class topology, where

$$\Sigma'_I := \Big\{B \in C_1(\mathsf{H})\,,\quad B = B^* \geqslant 0 \;|\; \forall\,\varepsilon > 0 \quad \frac{1}{2}\mathrm{Tr}\big[AB\big] \leqslant G_I(A) < \frac{1}{2}\mathrm{Tr}\big[AB\big] + \varepsilon\Big\}.$$

Therefore $G_I(A) = \dfrac{1}{2}\sup\limits_{B\in\Sigma_I}\mathrm{Tr}\big[AB\big] = \dfrac{1}{2}\sup\limits_{B\in\Sigma'_I}\mathrm{Tr}\big[AB\big]$.

Analogously, $G_{I_n}(A) = \dfrac{1}{2}\sup\limits_{B\in\Sigma_{I_n}}\mathrm{Tr}\big[AB\big] = \dfrac{1}{2}\sup\limits_{B\in\Sigma'_{I_n}}\mathrm{Tr}\big[AB\big]$.

Let us show that:

(a) If $\{B_n\} \subset \Sigma'_{I_n}$, such that $B_n \xrightarrow[n\to\infty]{} B$ then $B \in \Sigma'_I$.

(b) For every $B \in \Sigma'_I$ there exists a sequence $\{B_n\} \subset \Sigma'_{I_n}$, such that $B_n \xrightarrow[n\to\infty]{} B$.

In fact, let us fix an operator A then:
(a) If $\{B_n\} \subset \Sigma'_{I_n}$, such that $B_n \xrightarrow[n\to\infty]{} B$ then we have:

$$\frac{1}{2}\mathrm{Tr}\big[AB_n\big] \leqslant G_{I_n}(A) < \frac{1}{2}\mathrm{Tr}\big[AB_n\big] + \varepsilon\,;$$

Letting $n \to \infty$ yields $\dfrac{1}{2}\mathrm{Tr}\big[AB\big] \leqslant G_I(A) < \dfrac{1}{2}\mathrm{Tr}\big[AB\big] + \varepsilon$.

Hence $B \in \Sigma'_I$.

(b) Let there exists $B \in \Sigma'_I$, and a sequence $\{B_n\} \subset \Sigma'_{I_n}$ converges to the operator $C \neq B$: $B_n \xrightarrow[n\to\infty]{} C \in \Sigma'_I$. Then we have:

$\frac{1}{2}\mathrm{Tr}\big[AB\big] \leqslant G_I(A) < \frac{1}{2}\mathrm{Tr}\big[AB\big] + \varepsilon$;

$\frac{1}{2}\mathrm{Tr}\big[AC\big] \leqslant G_I(A) < \frac{1}{2}\mathrm{Tr}\big[AC\big] + \varepsilon$.

Therefore for every A : $\mathrm{Tr}\big[AB\big] = \mathrm{Tr}\big[AC\big]$.

So, it easy to check that $B = C$, a contradiction.

So, from **(a)** and **(b)** we have that

$\Sigma'_I = \Big\{B \mid \ B_n \xrightarrow[n\to\infty]{C_1(\mathsf{H})} B\,, \quad B_n \in \Sigma'_{I_n}\Big\}$, where $B_n = \int\limits_0^T \Phi_n(t)Q\Phi_n^*(t)dt$.

Hence $\Sigma_I = \Big\{B \mid \ B_n \xrightarrow[n\to\infty]{C_1(\mathsf{H})} B\,, \quad B_n \in \Sigma_{I_n}\Big\}$.

Applying **Lm.**5.1 we can conclude that $\Sigma_I = \Big\{\int\limits_0^T \Phi(t)Q\Phi^*(t)dt \mid Q \in \Sigma\Big\}$.

□

5.6 The continuity property of stochastic convolution

Define a **stochastic convolution** as the integral

$$I_t := \int\limits_0^t e^{(t-s)A}dB_s\,,$$

where $A : D(A) \to \mathsf{H}$ is the infinitesimal generator of C_0-semigroup $\big(e^{tA}\big)$.

Theorem 5.8. *The integral* $I_t := \int\limits_0^t e^{(t-s)A}dB_s$ *is continuous for quasi every* ω *if there exists* $\beta > 0$, *such that*

$$\int\limits_0^T \|e^{tA}\|^2_{L_2^\Sigma} \cdot t^{-\beta}dt < \infty\ .$$

Proof.
We shall use the factorization method (see [31]).
For this reason we will use the following elementary inequality:
Let $\alpha \in (0,1)$ then
$\int_0^1 (1-r)^{\alpha-1} \cdot r^{-\alpha}\, dr = B(\alpha, 1-\alpha) = \dfrac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = \dfrac{\pi}{\sin \pi\alpha}\,.$
It follows that

$$\int_\sigma^t (t-s)^{\alpha-1} \cdot (s-\sigma)^{-\alpha}\, ds = \frac{\pi}{\sin \pi\alpha} \quad , \tag{15}$$

$$0 \leqslant \sigma \leqslant s \leqslant t\,, \quad \text{where } s := r(t-\sigma) + \sigma\,,$$

because
$$\int_\sigma^t (t-s)^{\alpha-1} \cdot (s-\sigma)^{-\alpha}\, ds = \int_\sigma^t \big((1-r)(t-\sigma)\big)^{\alpha-1} \cdot \big(r(t-\sigma)\big)^{-\alpha}\, d[r(t-\sigma)+\sigma]$$
$$= \int_0^1 (1-r)^{\alpha-1} \cdot r^{-\alpha} \cdot (t-\sigma)^{\alpha-1} \cdot (t-\sigma)^{-\alpha} \cdot (t-\sigma)\, dr$$
$$= \int_0^1 (1-r)^{\alpha-1} \cdot r^{-\alpha}\, dr = \frac{\pi}{\sin \pi\alpha}\,.$$

Let $\alpha \in (0, \frac{1}{2})$ be fixed, and $m > \dfrac{1}{2\alpha}$ then we have
$I_t = \dfrac{\sin \pi\alpha}{\pi} \int_0^t e^{(t-s)A} \int_\sigma^t (t-s)^{\alpha-1} \cdot (s-\sigma)^{-\alpha}\, ds\; dB_s\,.$

From the Fubini theorem (**Th.5.6**) we get that
$I_t = \dfrac{\sin \pi\alpha}{\pi} \int_0^t e^{(t-s)A} \cdot (t-s)^{\alpha-1} Y(s)\, ds$ quasi surely,

$$\text{where } Y(s) = \int_0^s e^{(s-\sigma)A}\, (s-\sigma)^{-\alpha}\, dB_\sigma\,.$$

Then by **Th.5.7** we have for every s $\;Y(s) \sim N_G(0, \Sigma_{I_s})$

$$\text{where } \Sigma_{I_s} = \left\{ \int_0^s e^{(s-\sigma)A}\, Q\, e^{(s-\sigma)A^*}\, (s-\sigma)^{-2\alpha} d\sigma\,,\ Q \in \Sigma \right\}.$$

Therefore

$$\mathbb{E}\Big[\big\|Y(s)\big\|^2_{\mathsf{H}}\Big] \overset{\textbf{Th.5.3}}{\leqslant} \int_0^s \|Y(\sigma)\|^2_{L_2^\Sigma}\, d\sigma = \int_0^s \sup_{Q\in\Sigma} \mathrm{Tr}\Big[e^{(s-\sigma)A}\, Q\, e^{(s-\sigma)A^*}\,(s-\sigma)^{-2\alpha}\Big] d\sigma$$

$$= \int_0^s \|e^{(s-\sigma)A}\|^2_{L_2^\Sigma}\,(s-\sigma)^{-2\alpha}\, d\sigma = \Big\{ s-\sigma = t \in (s,0) \Big\} = \int_0^s \|e^{tA}\|^2_{L_2^\Sigma}\, t^{-2\alpha}\, dt < \infty\,.$$

Then by **Prop.2.4** it follows $\mathbb{E}\Big[\big\|Y(s)\big\|^{2m}_{\mathsf{H}}\Big] \leqslant C_m\,,\quad s\in[0,T]$.

Hence $\mathbb{E}\Big[\int_0^T \big\|Y(s)\big\|^{2m}_{\mathsf{H}}\, ds\Big] \leqslant C_m \cdot T\,,$ so that $Y \in L^{2m}(0,T;\mathsf{H})\,.$

Let us consider $z(t) = \int_0^t e^{(t-s)A}\,(t-s)^{\alpha-1} y(s)\, ds\,.$

Set $z_\varepsilon(t) := \int_0^{t-\varepsilon} e^{(t-s)A}\,(t-s)^{\alpha-1} y(s)\, ds\,,$ for a small enough $\varepsilon > 0\,.$

So, we have $\big|z(t) - z_\varepsilon(t)\big| = \int_{t-\varepsilon}^{t} e^{(t-s)A}\,(t-s)^{\alpha-1} y(s)\, ds$

$$\overset{\textbf{Hölder ineq.}}{\leqslant} \Big(\int_{t-\varepsilon}^{t} \big\|e^{(t-s)A}\big\|^{\frac{2m}{2m-1}}_{L(\mathsf{H})} \cdot (t-s)^{\frac{2m(\alpha-1)}{2m-1}}\, ds \Big)^{\frac{2m-1}{2m}} \cdot \Big(\int_{t-\varepsilon}^{t} \big\|y(s)\big\|^{2m}_{\mathsf{H}}\, ds \Big)^{\frac{1}{2m}}$$

$$\leqslant M \cdot e^{\frac{2m\varepsilon a}{2m-1}} \cdot \Big(\int_0^\varepsilon r^{\frac{2m(\alpha-1)}{2m-1}}\, dr \Big)^{\frac{2m-1}{2m}} \cdot \|y\|_{L^{2m}(0,\,t;\,\mathsf{H})} \leqslant K_\varepsilon \cdot \|y\|_{L^{2m}(0,\,t;\,\mathsf{H})}\,,$$

$$K_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0\,.$$

So that $z(\cdot)$ is continuous if $y(\cdot) \in L^{2m}(0,T;\mathsf{H})\,.$

And we have that I_t is continuous for quasi every $\omega\,.$

□

6 Viscosity solution for other parabolic PDEs

6.1 Ornstein-Uhlenbeck process

Consider the following SDE:

$$\begin{cases} dX_\tau = AX_\tau \, d\tau + dB_\tau \ , & \tau \in [t,T] \subset [0,T] \\ X_t = x \ . \end{cases} \tag{S}$$

where $X_t : [0,T] \times \Omega \to \mathsf{H}$;
B_t is a G-Brownian motion in H;
$A : D(A) \to \mathsf{H}$ is an infinitesimal generator of C_0-semigroup $\left(e^{tA}\right)$.

Definition 6.1. *A process*

$$X_\tau := X_\tau^{t,x} = e^{(\tau-t)A}x + \int\limits_t^\tau e^{(\tau-s)A} dB_s$$

will be called a mild solution to (S)*.*

Definition 6.2. *Stochastic process*

$$I_t = \int\limits_0^t e^{(t-s)A} dB_s \tag{16}$$

will be called Ornstein-Uhlenbeck process (or a stochastic convolution as we have already mentioned above).
Ornstein-Uhlenbeck process is well defined under the condition

$$\int\limits_0^t \sup_{Q \in \Sigma} \|e^{sA}Q^{1/2}\|^2_{L_2(\mathsf{H})} ds < \infty. \tag{17}$$

Remark 6.1. *The condition* (17) *holds true if* $\sup\limits_{Q \in \Sigma} \mathrm{Tr} Q < \infty$*.*

In other words the condition (17) (or the condition of **Rem.6.1**) implies that the map $s \mapsto e^{(t-s)A}$ belongs to the ${}^{\mathsf{H}}M_G^2\left(0,t\right)$, what shows us the following proposition.

Proposition 6.1. *If* $\sup\limits_{Q\in\Sigma} \mathrm{Tr} Q < \infty\,,\ \Phi(s) := e^{(t-s)A}$ *then*

$$\Phi(s) \in {}^{\mathsf{H}}M_G^2\big(0,t\big)\,.$$

Proof.
In order to prove the statement of proposition according to **Rem.5.4** we need to show that $|||\Phi|||_t < \infty$.
According to theory of C_0-semigroup it is known that
$e^{sA} \in L(\mathsf{H})\,,\quad \|e^{sA}\|^2_{L(\mathsf{H})} \leqslant M \cdot e^{as} \quad M \geqslant 1\,,\ a \in \mathbb{R}.$

Therefore $|||\Phi|||_t^2 = \mathbb{E}\Big[\int\limits_0^t \sup\limits_{Q\in\Sigma} \|\Phi(s)Q^{1/2}\|^2_{L_2(\mathsf{H})}\,ds\Big] = \int\limits_0^t \sup\limits_{Q\in\Sigma} \|e^{(t-s)A}Q^{1/2}\|^2_{L_2(\mathsf{H})}\,ds$

$$= \underbrace{\int\limits_0^t \sup_{Q\in\Sigma} \|e^{sA}Q^{1/2}\|^2_{L_2(\mathsf{H})}\,ds}_{\text{condition(17)}} \leqslant \int\limits_0^t \sup_{Q\in\Sigma}\Big[\|ne^{sA}\|^2_{L(\mathsf{H})} \cdot \|Q^{1/2}\|^2_{L_2(\mathsf{H})}\Big]\,ds$$

$$\leqslant \int\limits_0^t M \cdot e^{as} \cdot \sup_{Q\in\Sigma} \mathrm{Tr} Q\,ds \leqslant M \cdot \frac{e^{at}-1}{a} \cdot \sup_{Q\in\Sigma} \mathrm{Tr} Q < \infty\,.$$

□

Proposition 6.2. $X_\tau^{t,x} = X_\tau^{s,X_s^{t,x}},\quad 0 \leqslant t \leqslant s \leqslant \tau \leqslant T\,.$

Proof.

$$X_\tau^{t,x} = e^{(\tau-t)A}x + \int\limits_t^\tau e^{(\tau-\sigma)A}dB_\sigma$$

$$= e^{(\tau-s)A}\cdot e^{(s-t)A}x + \int\limits_t^s e^{(s-\sigma)A}\cdot e^{(\tau-s)A}dB_\sigma + \int\limits_s^\tau e^{(\tau-\sigma)A}dB_\sigma$$

$$= e^{(\tau-s)A}\Big(e^{(s-t)A}x + \int\limits_t^s e^{(s-\sigma)A}dB_\sigma\Big) + \int\limits_s^\tau e^{(\tau-\sigma)A}dB_\sigma$$

$$= e^{(\tau-s)A}\cdot X_s^{t,x} + \int\limits_s^\tau e^{(\tau-\sigma)A}dB_\sigma$$

$$= X_\tau^{s,X_s^{t,x}}.$$

□

6.2 Solving the fully nonlinear parabolic PDE with unbounded first order term

Lemma 6.1. *Let B_t be a G-Brownian motion and A be an infinitesimal generator of C_0 semigroup. A mapping $\psi : \mathbb{R}, \mathsf{H} \to \mathbb{R}$ is twice Fréchet differentiable by x.*
For the small $\delta > 0$ define the following random variable as:
$L_\delta := \frac{1}{\delta}\Big\langle D^2_{xx}\psi(t,x)[\int_0^\delta e^{(\delta-s)A}dB_s], \int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle$.
Then $\mathbb{E}[L_\delta] \xrightarrow[\delta\to 0]{} \mathbb{E}\Big[\big\langle D^2_{xx}\psi(t,x)B_1, B_1\big\rangle\Big] \equiv 2\,G\big(D^2_{xx}\psi(t,x)\big)$.

Proof.
Let $L_0 := \big\langle D^2_{xx}\psi(t,x)B_1, B_1\big\rangle$.

Note that

$$\mathbb{E}\Big[\langle AX, Y\rangle\Big] \leqslant \mathbb{E}\Big[\|AX\|_{\mathsf{H}} \cdot \|Y\|_{\mathsf{H}}\Big] \leqslant \mathbb{E}\Big[\|A\|_{L(\mathsf{H})} \cdot \|X\|_{\mathsf{H}} \cdot \|Y\|_{\mathsf{H}}\Big]$$
$$\overset{\textbf{Prop.2.3}}{\leqslant} \|A\|_{L(\mathsf{H})}\Big(\mathbb{E}\Big[\|X\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\|Y\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}. \quad (18)$$

Consider $K_\delta := \frac{1}{\delta}\big\langle D^2_{xx}\psi(t,x)\int_0^\delta(e^{(\delta-s)A} - I)dB_s, B_\delta\big\rangle$

and $M_\delta := \frac{1}{\delta}\big\langle D^2_{xx}\psi(t,x)\int_0^\delta e^{(\delta-s)A}dB_s, \int_0^\delta(e^{(\delta-s)A} - I)dB_s\big\rangle$.

Then we have:

$$0 \leqslant \mathbb{E}\big[|K_\delta|\big] \overset{(18)}{\leqslant} \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})}\Big(\mathbb{E}\Big[\|\int_0^\delta(e^{(\delta-s)A}-I)dB_s\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\|B_\delta\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}$$
$$\overset{\textbf{Th.5.5}}{\leqslant} \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})} \cdot C\Big(\mathbb{E}\Big[\int_0^\delta\|e^{(\delta-s)A} - I\|^2_{L_2^\Sigma}ds\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[\|B_\delta\|^2_{\mathsf{H}}\Big]\Big)^{\frac{1}{2}}$$
$$\overset{\textbf{Rem.4.2}}{=} \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})} \cdot C\Big(\int_0^\delta \sup_{Q\in\Sigma}\|e^{sA}Q^{1/2} - Q^{1/2}\|^2_{L_2(\mathsf{H})}ds\Big)^{\frac{1}{2}} \cdot \sqrt{\delta G(I)}$$
$$\leqslant \|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})} \cdot C \cdot \delta \cdot \max_{0\leqslant\theta\leqslant\delta}\sup_{Q\in\Sigma}\|e^{\theta A}Q^{1/2} - Q^{1/2}\|_{L_2(\mathsf{H})} \cdot \sqrt{G(I)} \xrightarrow[\delta\to 0]{} 0.$$

In much the same way:

$$0 \leqslant \mathbb{E}\Big[\big|M_\delta\big|\Big] \overset{(18)}{\leqslant} \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})}\bigg(\mathbb{E}\Big[\|\int_0^\delta e^{(\delta-s)A}dB_s\|^2_{\mathsf{H}}\Big]\bigg)^{\frac{1}{2}}\times$$

$$\times\bigg(\mathbb{E}\Big[\|\int_0^\delta (e^{(\delta-s)A}-I)dB_s\|^2_{\mathsf{H}}\Big]\bigg)^{\frac{1}{2}}$$

$$\overset{\textbf{Th.5.5}}{\leqslant} \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})}\cdot C\bigg(\mathbb{E}\Big[\int_0^\delta \|e^{(\delta-s)A}\|^2_{L_2^\Sigma}ds\Big]\bigg)^{\frac{1}{2}}\bigg(\mathbb{E}\Big[\int_0^\delta \|e^{(\delta-s)A}-I\|^2_{L_2^\Sigma}ds\Big]\bigg)^{\frac{1}{2}}$$

$$= \frac{1}{\delta}\|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})}\cdot C\Big(\int_0^\delta \sup_{Q_1\in\Sigma}\|e^{sA}Q_1^{\frac{1}{2}}\|^2_{L_2(\mathsf{H})}ds\Big)^{\frac{1}{2}}\Big(\int_0^\delta \sup_{Q_2\in\Sigma}\|e^{sA}Q_2^{\frac{1}{2}}-Q_2^{\frac{1}{2}}\|^2_{L_2(\mathsf{H})}ds\Big)^{\frac{1}{2}}$$

$$\leqslant \|D^2_{xx}\psi(t,x)\|_{L(\mathsf{H})}\cdot C\cdot\delta\cdot\max_{0\leqslant\theta_1\leqslant\delta}\sup_{Q_1\in\Sigma}\|e^{\theta_1 A}Q_1^{\frac{1}{2}}\|_{L_2(\mathsf{H})}\cdot\delta\times$$

$$\times\max_{0\leqslant\theta_2\leqslant\delta}\sup_{Q_2\in\Sigma}\|e^{\theta_2 A}Q_2^{\frac{1}{2}}-Q_2^{\frac{1}{2}}\|_{L_2(\mathsf{H})}\xrightarrow[\delta\to 0]{} 0\,.$$

Therefore

$$0 \leqslant \Big|\mathbb{E}[L_\delta]-\mathbb{E}[L_0]\Big| \overset{\textbf{Rem.4.2}}{=} \bigg|\frac{1}{\delta}\,\mathbb{E}\Big[\Big\langle D^2_{xx}\psi(t,x)\Big[\int_0^\delta e^{(\delta-s)A}dB_s\Big],\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle\Big]$$

$$-\frac{1}{\delta}\,\mathbb{E}\Big[\Big\langle D^2_{xx}\psi(t,x)B_\delta,B_\delta\Big\rangle\Big]\bigg|$$

$$\overset{\textbf{Prop.2.2, 2)}}{\leqslant} \frac{1}{\delta}\,\mathbb{E}\Bigg[\bigg|\Big\langle D^2_{xx}\psi(t,x)\Big[\int_0^\delta e^{(\delta-s)A}dB_s\Big],\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle$$

$$-\Big\langle D^2_{xx}\psi(t,x)B_\delta,B_\delta\Big\rangle\bigg|\Bigg]$$

$$\leqslant \frac{1}{\delta}\,\mathbb{E}\Bigg[\bigg|\Big\langle D^2_{xx}\psi(t,x)\Big[\int_0^\delta e^{(\delta-s)A}dB_s\Big],\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle$$

$$-\Big\langle D^2_{xx}\psi(t,x)\Big[\int_0^\delta e^{(\delta-s)A}dB_s\Big],B_\delta\Big\rangle\bigg|\Bigg]$$

$$+\frac{1}{\delta}\,\mathbb{E}\Bigg[\bigg|\Big\langle D^2_{xx}\psi(t,x)\Big[\int_0^\delta e^{(\delta-s)A}dB_s\Big],B_\delta\Big\rangle-\Big\langle D^2_{xx}\psi(t,x)B_\delta,B_\delta\Big\rangle\bigg|\Bigg]$$

$$= \mathbb{E}\Big[\big|K_\delta\big|\Big]+\mathbb{E}\Big[\big|M_\delta\big|\Big]\xrightarrow[\delta\to 0]{} 0\,.$$

□

Now let us turn back to equation (P):

$$\begin{cases} \partial_t u + \langle Ax, D_x u\rangle + G(D^2_{xx}u) = 0\,, & t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{P}$$

$u : [0,T] \times \mathsf{H} \to \mathbb{R}\,;$
$f \in \mathbf{C}_{p.Lip}(\mathsf{H})\,;$
$G : K_S(\mathsf{H}) \to \mathbb{R}$ is a G-functional;
$A : D(A) \to \mathsf{H}$ is a generator of C_0-semigroup $\left(e^{tA}\right)$.
B_t is a G-Brownian motion with corresponding G-functional $G(\,\cdot\,)$,

$$\text{i.e. } G(A) = \frac{1}{2t}\,\mathbb{E}\Big[\langle AB_t, B_t\rangle\Big]\,;$$

$X^{t,x}_\tau = e^{(\tau-t)A}x + \int\limits_t^\tau e^{(\tau-s)A}dB_s$ be a mild solution to equation (S):

$$\begin{cases} dX_\tau = AX_\tau\, d\tau + dB_\tau\,, & \tau \in [t,T] \subset [0,T] \\ X_t = x\,. \end{cases} \tag{S}$$

Theorem 6.1. *Let f is a B-continuous of $\mathbf{C}_{p.Lip}(\mathsf{H})$-class real function. Then $u(t,x) := \mathbb{E}\big[f(X^{t,x}_T)\big]$ is a unique viscosity solution to equation (P):*

$$\begin{cases} \partial_t u + \langle Ax, D_x u\rangle + G(D^2_{xx}u) = 0\,, & t \in [0,T)\,,\ x \in \mathsf{H}; \\ u(T,x) = f(x)\,. \end{cases} \tag{P}$$

Proof.
Let ψ be a test function, and for every fixed point $(t,x) \in [0,T] \times \mathsf{H}$ we have: $u \leqslant \psi\,;$

$$u(t,x) = \psi(t,x)\,.$$

Taking a small enough δ yields:

$$\psi(t,x) = u(t,x) = \mathbb{E}[f(X^{t,x}_T)] \overset{\textbf{Prop.6.2}}{=} \mathbb{E}[f(X^{s,X^{t,x}_s}_T)]$$
$$= \mathbb{E}\left[\mathbb{E}\Big[f(X^{s,y}_T)\Big]_{y=X^{t,x}_s}\right] = \mathbb{E}\Big[u(s,X^{t,x}_s)\Big] \leqslant \mathbb{E}\Big[\psi(s,X^{t,x}_s)\Big]\,.$$

Then putting $s := t+\delta,\ \delta > 0$ by the Taylor formula (**Lm.4.2**) we have:

$$\psi(s, X_s^{t,x}) = \psi(t+\delta, X_{t+\delta}^{t,x}) = \psi(t+\delta, e^{\delta A}x + \int_t^{t+\delta} e^{(t+\delta-s)A} dB_s)$$

$$\overset{\textbf{Rem.5.2}}{=} \psi\Big(t+\delta, x + (e^{\delta A}x - x) + \int_0^{\delta} e^{(\delta-s)A} dB_s\Big) = \psi(t,x) + \delta\, \partial_t \psi(t,x)$$

$$+ \Big\langle D_x\psi(t,x), e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s \Big\rangle + \frac{1}{2}\delta^2\, \partial_{tt}^2 \psi(t,x)$$

$$+ \delta \cdot \partial_t \Big\langle D_x\psi(t,x), e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s \Big\rangle$$

$$+ \frac{1}{2}\delta \Big\langle D_{xx}^2\psi(t,x)\Big[e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s\Big], e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s \Big\rangle$$

$$+ o\Big(\delta^2 + \big\|e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s\big\|_{\mathsf{H}}^2\Big).$$

We can say that $\mathbb{E}\Big[o\Big(\delta^2 + \big\|e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s\big\|_{\mathsf{H}}^2\Big)\Big] = o\Big(\delta^2\Big)$,

because
$$\mathbb{E}\Big[\big\|e^{\delta A}x - x + \int_0^{\delta} e^{(\delta-s)A} dB_s\big\|_{\mathsf{H}}^2\Big] \leqslant 2\,\mathbb{E}\Big[\big\|e^{\delta A}x - x\big\|_{\mathsf{H}}^2\Big] + 2\,\mathbb{E}\Big[\big\|\int_0^{\delta} e^{(\delta-s)A} dB_s\big\|_{\mathsf{H}}^2\Big]$$

$$\overset{\textbf{Th.5.5}}{\leqslant} 2\,\big\|e^{\delta A}x - x\big\|_{\mathsf{H}}^2 + 2\int_0^{\delta} \sup_{Q\in\Sigma} \big\|e^{(\delta-s)A} Q^{1/2}\big\|_{L_2(\mathsf{H})}^2 ds \xrightarrow[\delta\to 0]{} 0.$$

Then we have

$$0 \leqslant \frac{1}{\delta}\Big(\mathbb{E}\Big[\psi(t+\delta, X_{t+\delta}^{t,x})\Big] - \psi(t,x)\Big) \leqslant \frac{1}{\delta}\mathbb{E}\Big[\psi(t+\delta, X_{t+\delta}^{t,x}) - \psi(t,x)\Big]$$

$$= \mathbb{E}\Bigg[\partial_t\psi(t,x) + \Big\langle D_x\psi(t,x), \frac{e^{\delta A}x - x}{\delta}\Big\rangle + \frac{1}{\delta}\Big\langle D_x\psi(t,x), \int_0^{\delta} e^{(\delta-s)A} dB_s \Big\rangle$$

$$+ \frac{1}{2}\delta\, \partial_{tt}^2\psi(t,x) + \delta \cdot \partial_t \Big\langle D_x\psi(t,x), \frac{e^{\delta A}x - x}{\delta}\Big\rangle + \partial_t \Big\langle D_x\psi(t,x), \int_0^{\delta} e^{(\delta-s)A} dB_s \Big\rangle$$

$$+\frac{1}{2}\Big\langle D^2_{xx}\psi(t,x)\Big[\frac{e^{\delta A}x-x}{\delta}\Big],\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle$$

$$+\frac{1}{2}\Big\langle D^2_{xx}\psi(t,x)\int_0^\delta e^{(\delta-s)A}dB_s,\,\frac{e^{\delta A}x-x}{\delta}\Big\rangle$$

$$+\,\frac{\delta}{2}\Big\langle D^2_{xx}\psi(t,x)\Big[\frac{e^{\delta A}x-x}{\delta}\Big],\frac{e^{\delta A}x-x}{\delta}\Big\rangle$$

$$+\frac{1}{2\,\delta}\Big\langle D^2_{xx}\psi(t,x)\int_0^\delta e^{(\delta-s)A}dB_s,\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle\Bigg]+o\Big(\delta\Big)$$

$$\overset{\textbf{Prop.2.2.5).(a)-(b)}}{=}\;\partial_t\psi(t,x)+\Big\langle D_x\psi(t,x),\frac{e^{\delta A}x-x}{\delta}\Big\rangle+\frac{1}{2}\,\delta\,\partial^2_{tt}\psi(t,x)$$

$$+\,\delta\cdot\partial_t\Big\langle D_x\psi(t,x),\frac{e^{\delta A}x-x}{\delta}\Big\rangle+\frac{\delta}{2}\Big\langle D^2_{xx}\psi(t,x)\Big[\frac{e^{\delta A}x-x}{\delta}\Big],\frac{e^{\delta A}x-x}{\delta}\Big\rangle$$

$$+\,\frac{1}{2\,\delta}\,\mathbb{E}\Bigg[\Big\langle D^2_{xx}\psi(t,x)\int_0^\delta e^{(\delta-s)A}dB_s,\int_0^\delta e^{(\delta-s)A}dB_s\Big\rangle\Bigg]+o\Big(\delta\Big)$$

$$\xrightarrow[\delta\to 0]{\textbf{Lm.6.1}}\;\partial_t\psi(t,x)+\big\langle x,\;A^*D_x\psi(t,x)\big\rangle+G\big(D^2_{xx}\psi(t,x)\big)\,.$$

Letting $\delta\to 0$ yields $\Big[\partial_t\psi+\big\langle x,\;A^*D_x\psi(t,x)\big\rangle+G(D^2_{xx}\psi)\Big](t,x)\geqslant 0\,.$

Note, that u is continuous at $(t,x)\in[0,T]\times\mathsf{H}$.

In fact, let us show that $\mathbb{E}\big[f(x+B_s)\big]\xrightarrow[s\to t]{}\mathbb{E}\big[f(x+B_t)\big]$, $t\in[0,T]$:

$$0\leqslant\Big|\mathbb{E}[f(X_T^{t+\delta,x})]-\mathbb{E}[f(X_T^{t,x})]\Big|\leqslant\mathbb{E}\Big[\big|f(X_T^{t+\delta,x})-f(X_T^{t,x})\big|\Big]$$

$$\leqslant C\cdot\mathbb{E}\Big[\big(1+\|X_T^{t+\delta,x}\|^m_{\mathsf{H}}+\|X_T^{t,x}\|^m_{\mathsf{H}}\big)\cdot\|X_T^{t+\delta,x}-X_T^{t,x}\|_{\mathsf{H}}\Big]$$

$$\leqslant C\cdot\left(\mathbb{E}\Big[\big(1+\|X_T^{t+\delta,x}\|^m_{\mathsf{H}}+\|X_T^{t,x}\|^m_{\mathsf{H}}\big)^2\Big]\cdot\mathbb{E}\Big[\|X_T^{t+\delta,x}-X_T^{t,x}\|^2_{\mathsf{H}}\Big]\right)^{\frac{1}{2}}$$

$$\leqslant 2C\cdot\left(\Big(1+\mathbb{E}\Big[\|X_T^{t+\delta,x}\|^{2m}_{\mathsf{H}}\Big]+\mathbb{E}\Big[\|X_T^{t,x}\|^{2m}_{\mathsf{H}}\Big]\Big)\cdot\mathbb{E}\Big[\|X_T^{t+\delta,x}-X_T^{t,x}\|^2_{\mathsf{H}}\Big]\right)^{\frac{1}{2}}\xrightarrow[\delta\to 0]{}0,$$

this convergence is true because:

(a) $\mathbb{E}\Big[\|X_T^{t,x}\|_{\mathsf{H}}^{2m}\Big] = \mathbb{E}\Big[\|e^{(T-t)A}x + \int_t^T e^{(T-s)A}dB_s\|_{\mathsf{H}}^{2m}\Big]$

$$\leqslant 2m \cdot \left(\|e^{(T-t)A}x\|_{\mathsf{H}}^{2m} + \mathbb{E}\Big[\|\int_t^T e^{(T-s)A}dB_s\|_{\mathsf{H}}^{2m}\Big]\right)$$

$$\overset{\textbf{Th.5.5}}{\leqslant} 2m \cdot \left(\|e^{(T-t)A}x\|_{\mathsf{H}}^{2m} + C_m \cdot \Big(\int_t^T \sup_{Q\in\Sigma} \|e^{(T-s)A}\,Q^{1/2}\|^2_{L_2(\mathsf{H})} ds\Big)^m\right) < \infty\,.$$

(b) $\mathbb{E}\Big[\|X_T^{t+\delta,x} - X_T^{t,x}\|_{\mathsf{H}}^2\Big] = \mathbb{E}\Big[\|e^{(T-t)A}\big(e^{-\delta A}x - x\big) + \int_{t+\delta}^T e^{(T-s)A}dB_s - \int_t^T e^{(T-s)A}dB_s\|_{\mathsf{H}}^2\Big]$

$$\leqslant \Big(\|e^{(T-t)A}\big(e^{-\delta A}x - x\big)\|_{\mathsf{H}}^2 + \mathbb{E}\Big[\|\int_t^{t+\delta} e^{(T-s)A}dB_s\|_{\mathsf{H}}^2\Big]\Big)$$

$$\overset{\textbf{Th.5.5}}{\leqslant} \Big(\|e^{(T-t)A}\big(e^{-\delta A}x - x\big)\|_{\mathsf{H}}^2 + \|\int_t^{t+\delta} \sup_{Q\in\Sigma} \|e^{(T-s)A}\,Q^{1/2}\|^2_{L_2(\mathsf{H})} ds\Big) \xrightarrow[\delta\to 0]{} 0\,.$$

Also $u(T,x) = \mathbb{E}\big[f(x)\big] = f(x) \leqslant f(x)\,.$

So we see that u is a viscosity subsolution to equation (P).
In the same way one can prove that u is a viscosity supersolution, and the existence is proved.

It is clear that if f is B-continuous and has a polynomial growth that u is also B-continuous and has a polynomial growth, because a sublinear expectation $\mathbb{E}$ does not influence on it. So we can conclude that u is a unique viscosity solution by **Th.3.2**.

□

References

[1] Aggoun L., Elliott R. (2004). Measure Theory and Filtering. Cambridge University Press, Cambridge.

[2] Akhiezer N.I., Glazman I.M. (1993). Theory of Linear Operators in Hilbert Space. Ungar, New York.

[3] Ash R.B., Doléans-Dade C. (2000). Probability and Measure Theory (2nd ed.). Academic Press, San Diego.

[4] Barbu V., Da Prato G. (1983). Hamilton-Jacobi Equations in Hilbert Spaces. Research Notes in Math., 86. Pitman.

[5] Barbu V., Da Prato G. (1985). A note on a Hamilton-Jacobi Equation in Hilbert Space. Nonlinear Anal., 9, No. 12, pp. 1337-1345.

[6] Bardi, M., Capuzzo-Dolcetta, I. (1997). Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Birkhäuser, Boston.

[7] Bardi, M., Crandall, M.G., Evans L.C., Soner H.M., Souganidis P.E. (1995). Viscosity Solutions and Applications. Lecture notes in mathematics, 1660. Springer, Berlin.

[8] Berg, J., Löfström, J. (1976). Interpolation Spaces. Springer, Berlin-New York.

[9] Billingsley, P. (1999). Convergence of Probability Measures. Wiley, New York.

[10] Brezis, H. (2011). Functional Analysis, Sobolev Spaces and Partial Differential Equation. Springer, New York.

[11] Cannarsa P. (1989). Regularity Properties of Solutions to Hamilton-Jacobi Equations in Infinite Dimensions and Nonlinear Optimal Control. Differential Integral Equations, 2, No. 4, pp. 479-493.

[12] Cannarsa P., Da Prato G. (1991). Second Order Hamilton-Jacobi Equations in Infinite Dimensions. SIAM J. Control Optim., 29. pp. 474-492.

[13] Cannarsa P., Da Prato G. (1990). Some Results on Nonlinear Optimal Control Problems and Hamilton-Jacobi Equations in Infinite Dimensions. J. Funct. Anal., 90, No. 1, pp, 27-47.

[14] Cannarsa P., Da Prato G. (1991). A Semigroup Approach to Kolmogorov Equations in Hilbert Spaces. Appl. Math. Lett., 4, pp. 49-52.

[15] Cannarsa P., Da Prato G. (1991). On a Functional Analysis Approach to Parabolic Equations in Infinite Dimensions. J. Funct. Anal., 118, pp. 22-42.

[16] Cannarsa P., Da Prato G. (1991). Direct Solution of a Second Order Hamilton-Jacobi Equation in Hilbert Spaces, Stochastic Partial Differential Equations and Applications. Pitman Research Notes in Mathematics Series, 268, pp. 72-85.

[17] Cannarsa P., Da Prato G. (1992). Second Order Hamilton-Jacobi Equations in Infinite Dimensions and Stochastic Optimal Control Problems. Probabilistic and Stochastic Methods in Analysis, with Applications. NATO ASI Series C 372. Kluwer Academin Publishers, Boston, pp. 617-629.

[18] Cannarsa P., Gozzi F., Soner H.M. P. (1991). A boundary Value Problem for Hamilton-Jacobi Equations in Hilbert Spaces. Applied Math. Optim., 24 , pp. 197-220.

[19] Cannarsa P., Tessitore M.E. (1996) Infinite Dimensional Hamilton-Jacobi Equations and Dirichlet Boundary Control Problems of Parabolic Type. SIAM J. Control Optim., 34, No. 6, pp. 1831-1847.

[20] Cerrai S. (2001). Second Order PDEs in Finite and Infinite Dimensions. A Probabilistic Approach. Lecture Notes in Mathematics, 1762. Springer, Berlin.

[21] Choquet, G. (1953). Theory of Capacities. Annales de Institut Fourier, 5. pp. 131–295.

[22] Conway, J.B. (1990). A Course in Functional Analysis. Springer, New York.

[23] Crandall, M.G., Ishii H., Lions P.-L. (1992). User's Guide to Viscosity Solutions of Second Order PDEs. Bulletin (New Series). of the American Mathematical Society, Vol.27, No. 1, pp. 1-67.

[24] Crandall, M.G., Lions P.-L. (1983). Viscosity Solutions of Hamilton-Jacobi Equations. Transactions of the American Mathematical Society, Vol. 277, No. 1, pp. 1-42.

[25] Crandall, M.G., Lions P.-L. (1985). Hamilton-Jacobi Equations in Infinite Dimensions. I. Uniqueness of Viscosity Solutions. J. Funct. Anal., 62, No. 3, pp. 379–396.

[26] Crandall, M.G., Lions P.-L. (1986). Hamilton-Jacobi Equations in Infinite Dimensions. II. Existence of Viscosity Solutions. J. Funct. Anal., 65, No. 3, pp. 368–405.

[27] Crandall, M.G., Lions P.-L. (1986). Hamilton-Jacobi Equations in Infinite Dimensions. III. J. Func. Anal., 68, pp. 214-247.

[28] Crandall, M.G., Lions P.-L. (1990). Hamilton-Jacobi Equations in Infinite Dimensions. IV. Hamiltonians with Unbounded Linear Terms. J. Funct. Anal., 90, pp. 237-283.

[29] Crandall, M.G., Lions P.-L. (1991). Hamilton-Jacobi Equations in Infinite Dimensions. V. Unbounded Linear Terms and B-continuous Solutions. J. Funct. Anal., 97, pp. 417-465.

[30] Da Prato, G. (2006). An Introduction to Infinite Dimensional Analysis. Springer, Berlin.

[31] Da Prato, G., Kwapien S., Zabczyk J. (1987). Regularity of Solutions of Linear Stochastic Equations in Hilbert Spaces. Stoch., 23, pp. 1-23.

[32] Da Prato, G., Zabczyk, J. (1992). Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge.

[33] Da Prato, G., Zabczyk, J. (1996). Ergodicity for Infinite Dimensional Systems. Cambridge University Press, Cambridge.

[34] Da Prato, G., Zabczyk, J. (2002). Second Order Partial Differential Equations in Hilbert Spaces. London Mathematical Society Lecture Notes, 293. Cambridge University Press, Cambridge.

[35] Dellacherie, C. (1972). Capacité et Processus Stochastiques. Springer, Berlin.

[36] Denis, L., Hu, M., Peng, S. (2010). Function Spaces and Capacity Related to a Sublinear Expectation: Application to G-Brownian Motion Paths, in arXiv:0802.1240v2 [math.PR]

[37] Engelking R. (1989). General Topology. Heldermann, Berlin.

[38] Friedman A. (1982). Foundations of Modern Analysis. Dover, New York.

[39] Fuhrman, M., Tessitore, G. (2002). Nonlinear Kolmogorov Equations in Infinite Dimensional Spaces: The BSDEs Approach and Applications to Optimal Control. The Annals of Probability, Vol. 30, No. 3, pp. 1397–1465.

[40] Gao, F. (2009). Pathwise Properties and Homeomorphic Flows for Stochastic Differential Equations driven by G-Brownian Motion. Stochastic Processes and their Applications 119, pp. 3356–3382.

[41] Gawarecki, L., Mandrekar, V. (2011). Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations. Springer, Berlin.

[42] Gikhman, I.I., Skorohod, A.N. (1969). Introduction to the Theory of Random Processes. Saunders, Philadelphia.

[43] Gohberg I.C., Krein M.G. (1969). Introduction to the Theory of Linear Nonselfadjoint Operators, Trans. Mathem. Monographs, v. 18, Amer. Math. Soc., Providence, R. I.

[44] Gozzi F. (1996). Global Regular Solutions of Second Order Hamilton-Jacobi Equations in Hilbert Spaces with Locally Lipschitz Nonlinearities. Mathematical Analysis and Appl., Vol. 198, pp. 399-443.

[45] Gozzi F., Rouy E., Święch, A. (2000). Second Order Hamilton-Jacobi Equations in Hilbert Spaces and Stochastic Boundary Control. SIAM J. Control Optim., Vol. 38-2, pp. 400-430.

[46] Halmos P.R. (1974). Measure Theory. Springer, Berlin.

[47] Hu, M., Peng, S. (2009). On Representation Theorem of G-Expectations and Paths of G-Brownian Motion. Acta Mathematicae Applicatae Sinica, English Series, 25(3), pp. 539-546.

[48] Ikeda, N., Watanabe, S. (1989). Stochastic Differential Equations and Diffusion Processes (2nd ed.). North-Holland, Amsterdam.

[49] Ishii H. (1992). Viscosity Solutions for a Class of Hamilton-Jacobi Equations in Hilbert Spaces, J. Funct. Anal., 105, pp. 301-341.

[50] Ishii H. (1993). Viscosity Solutions of Nonlinear Second-Order Partial Differential Equations in Hilbert Spaces. Comm. Partial Differential Equations, 18, pp. 601-651.

[51] Jacod J., Protter P. (2000). Probability Essentials (2nd ed.). Springer, Berlin.

[52] Kadison R.V., Ringrose J.R. (1983). Fundamentals of the Theory of Operator Algebras. Vol.I. Elementary theory. Academic Press, New York.

[53] Kadison R.V., Ringrose J.R. (1986). Fundamentals of the Theory of Operator Algebras. Vol.II. Advanced theory. Academic Press, New York.

[54] Karatzas, I., Shreve, S. E. (1991). Brownian Motion and Stochastic Calculus (2nd ed.). Springer, New York.

[55] Kelome, D. (2002). Ph.D. thesis: Viscosity Solutions of Second Order Equations in a Separable Hilbert Space and Applications to Stochastic Optimal Control.

[56] Kelome, D., Święch, A. (2003). Viscosity Solutions of an Infinite-Dimensional Black–Scholes–Barenblatt Equation. Applied Mathematics & Optimization, 47, pp. 253–278.

[57] Kocan M., Soravia P. (1998). A Viscosity Approach to Infinite-Dimensional Hamilton-Jacobi Equations Arising in Optimal Control with State Constraints. SIAM J. Control Optim., 36, pp. 1348-1375.

[58] Kolmogorov A.N., Fomin S.V. (1999). Elements of the Theory of Functions and Functional Analysis. Dover, Mineola.

[59] Krylov, N.V., Röckner, M.,Zabczyk, J.,Da Prato, G. (1999). Stochastic PDEs and Kolmogorov Equations in Infinite Dimensions. Lecture Notes in Math., 1715. Cetraro, 1998. Springer, Berlin.

[60] Lions P.-L. (1988). Viscosity Solutions of Fully Nonlinear Second-Order Equations and Optimal Stochastic Control in Infinite Dimensions. I. The Case of Bounded Stochastic Evolutions, Acta Math., 161, No. 3-4, pp. 243–278.

[61] Lions P.-L. (1989). Viscosity Solutions of Fully Nonlinear Second-Order Equations and Optimal Stochastic Control in Infinite Dimensions. III. Uniqueness of Viscosity Solutions for General Second-Order Equations, J. Funct. Anal. 86 (1989), No. 1, pp. 1–18.

[62] Ljusternik. L.A, Sobolev. V.I. (1974). Elements of Functional Analysis. Hindustan Publishing Corp. Delhi; Halstadt Press, New York.

[63] Ma J., Yong J. (1999). Forward-Backward Stochastic Differential Equations and their Applications. Lecture Notes in Math., 1702. Springer, Berlin.

[64] van Mill, J. (2001). The Infinite-Dimensional Topology of Function Spaces. North-Holland, Amsterdam.

[65] Neveu J. (1970). Bases Mathématiques du Calcul des Probabilités (2nd ed.). Masson, Paris.

[66] Øksendal, B. (2010). Stochastic Differential Equations (6th ed.). Springer, Berlin.

[67] Pazy A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equation. Springer, New York.

[68] Peng, S. (2006)G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô Type, in arXiv:math.PR/0601035v2 [math.PR].

[69] Peng, S. (2007). Lecture Notes: G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty, in arXiv:0711.2834v1 [math.PR].

[70] Peng, S. (2008). Multi-dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation. Stochastic Processes and Their Applications, Vol. 118, pp. 2223–2253.

[71] Peng, S. (2008). A New Central Limit Theorem under Sublinear Expectations, in arXiv:0803.2656v1 [math.PR].

[72] Peng, S. (2010). Nonlinear Expectations and Stochastic Calculus under Uncertainty, in arXiv:1002.4546v1 [math.PR].

[73] Prévôt, C., Röckner, M. (2007). A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Mathematics, 1905. Springer, Berlin.

[74] Renardy M. (1995). Polar Decomposition of Positive Operators and Problem of Crandall and Lions, Appl. Anal., 57-3, pp. 383-385.

[75] Ringrose, J. R. (1971). Compact Non-self-adjoint Operators. Van Nostrand Reinhold, London.

[76] Shimano K. (2002). A Class of Hamilton-Jacobi Equations with Unbounded Coefficients in Hilbert Spaces, Appl. Math. Optim., 45, No. 1, pp. 75-98.

[77] Shiryaev A.N. (1995). Probability (2nd ed.). Springer, Berlin.

[78] Święch, A. (1994). "Unbounded" Second Order Partial Differential Equations in Infinite Dimensional Hilbert Spaces. Comm. Partial Differential Equations 19 , pp. 1999–2036.

[79] Tataru D. (1992). Viscosity Solutions of Hamilton-Jacobi Equations with Unbounded Linear Terms. J. Math. Anal. Appl., 163, pp. 345-392.

[80] Tataru D. (1994). Viscosity Solutions for Hamilton-Jacobi Equations with Unbounded Nonlinear Term: A Simplified Approach. J. Differential Equations, 111, pp. 123-146.

[81] Teschl, G. (2009). Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators, Graduate Studies in Mathematics, 99. Amer. Math. Soc., Providence.

[82] Vrabie, I. (2003). Co-Semigroups and Applications. North-Holland, Amsterdam.

[83] Yong, J., Zhou, X.Y. (1999). Stochastic Controls. Hamiltonian Systems and HJB Equations. Springer, New York.

[84] Yosida, K. (1980). Functional Analysis (6th ed.). Springer, Berlin.