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Harmonic Bergman spaces, Hardy-type spaces and harmonic analysis of a symmetric diffusion semigroup on $\mathbb{R}^n$

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Introduction

This thesis is divided into two parts, which deal with quite diverse subjects.

The first part is, in turn, divided into two chapters. The first focuses on the development of new function spaces in $\mathbb{R}^n$, called generalized Bergman spaces, and on their application to the Hardy space $H^1(\mathbb{R}^n)$. The second is devoted to the theory of Bergman spaces on noncompact Riemannian manifolds which possess the doubling property and to its relationships with spaces of Hardy type. The latter are tailored to produce endpoint estimates for interesting operators, mainly related to the Laplace–Beltrami operator.

The second part is devoted to the study of some interesting properties of the operator

$$\mathscr{A} f = -\frac{1}{2}\,\Delta f - x\cdot \nabla f \qquad \forall f \in C_c^\infty(\mathbb{R}^n),$$

which is essentially self-adjoint with respect to the measure

$$\mathrm{d}\gamma_{-1}(x) = \pi^{n/2}\,\mathrm{e}^{|x|^2}\,\mathrm{d}\lambda(x) \qquad \forall x \in \mathbb{R}^n,$$

where λ denotes the Lebesgue measure, and of the semigroup that $\mathscr{A}$ generates.

The generalized Bergman spaces in $\mathbb{R}^n$ and their applications to $H^1(\mathbb{R}^n)$ are discussed in Chapter 1, their extension to Riemannian manifolds is described in Chapter 2, and the analysis of the operator $\mathscr{A}$ occupies Chapter 3. Here we briefly describe the main results we have obtained, and illustrate the relationships with related results in the literature.

Bergman spaces and applications to Hardy spaces

The Euclidean case

Suppose that p is a number satisfying $1 \leq p < \infty$, and let Ω be an open bounded set in $\mathbb{R}^n$. The harmonic Bergman space $b^p(\Omega)$ is the space of all harmonic functions u on Ω such that

$$\|u\|_{b^p} := \Big(\int_\Omega |u|^p \,\mathrm{d}\lambda\Big)^{1/p} < \infty. \tag{0.0.1}$$

The analogous space of holomorphic functions was introduced and studied by S. Bergman [Be]. The Bergman spaces of holomorphic functions have been and still are the object of intensive studies in harmonic analysis of functions of several complex variables and in a variety of related contexts. The interest in harmonic Bergman spaces is comparatively recent (see, for instance, [ABR, KK, CKY] and the refences therein). One source of interest lies in the fact that the harmonic Bergman kernel, i.e., the reproducing kernel of $b^2(\Omega)$, is related to the Green function for the bi-harmonic Laplace equation on Ω (see, for instance, [Ma] and the references therein).

For $k \geq 1$ we define the generalized harmonic Bergman space $b^p_k(\Omega)$ to be the space of all functions satisfying (0.0.1) that are k-harmonic, i.e., such that

$$\Delta^k u = 0 \qquad \text{in } \Omega$$

in the sense of distributions (hence in the classical sense, by elliptic regularity). In Chapter 1 we establish some basic properties of $b^2_k(B)$, where B is a ball of $\mathbb{R}^n$. In particular, we find an orthonormal basis of $B^2_k(B)$ consisting of k-harmonic polynomials of $b^2_k(B)$ and compute the reproducing kernel (or Bergman kernel) of $b^2_k(B)$. The latter is the unique function R^k_B on $B \times B$ such that

$$u(x) = \int_B R^k_B(x,y)\, u(y)\,\mathrm{d}y \qquad \forall x \in B \quad \forall u \in b^2_k(B).$$

A closed formula for R^1_B is known (see [ABR, Theorem 8.13]), and estimates for derivatives of R^1_B may be easily deduced from it. In fact, a closed formula for R^k_B is also known for $k > n/2$ [Ma]. This formula is obtained by first computing explicitly the Green function for Δ^k and then deducing the exact formula for R^k_B. This procedure is interesting in itself, but gives a formula for R^k_B only for large values of k (compared to the dimension n) and it is a little bit involved. Our approach works for every n and k, and it is based on a simple trick, which relates R^k_B to the extended Poisson kernel on B (see (1.4.1) below), and on the behaviour of the L^2-norms of certain k-harmonic polynomials $\{\alpha^k_j : j \in \mathbb{N}\}$ as j tends to infinity. In fact, we first compute the exact values of these L^2-norms, which are solutions to certain linear systems, and then deduce their asymptotic behaviour as j tends to infinity. The major drawback of our approach is that we have not found a direct and elegant way to find these solutions. Anyway, apart from this, estimates for R^k_B and its derivatives follow quite easily from our formula.

The theory of k-harmonic Bergman spaces we developed has an interesting application to the theory of the Hardy space $H^1(\mathbb{R}^n)$. Recall that $H^1(\mathbb{R}^n)$ is a natural substitute for the Lebesgue space $L^1(\mathbb{R}^n)$ in many problems arising in real and harmonic analysis. The seminal works of C. Fefferman and E.M. Stein [FeS], R.R. Coifman [Co] and R.H. Latter [La] provide many different characterisations of $H^1(\mathbb{R}^n)$. In particular, the atomic characterisation, proved in one dimension by Coifman and in higher dimensions by Latter, opened the way to generalisations of the theory of Hardy spaces to more general settings, such as spaces of homogeneous type in the sense of Coifman and G. Weiss [CW].

These are metric measured spaces (X, ρ, μ), where ρ is a (pseudo-) distance on X and μ is a Borel measure satisfying some mild assumptions and possessing the *doubling* property. Recall that μ is doubling if there exists a constant D such that

$$\mu(2B) \leq D\, \mu(B) \qquad \text{for every metric ball } B \text{ in } X. \tag{0.0.2}$$

Here $2B$ is the ball with the same centre as B and twice the radius.

We briefly recall the definition of the Hardy space $H^1(X)$ [CW].

A function a in $L^2(X)$ with support contained in a metric ball B is said to be a $(1,2)$-atom if it satisfies the following size and cancellation conditions

(i) $\left(\int_B |a|^2 \,\mathrm{d}\mu\right)^{1/2} \leq \mu(B)^{-1/2}$,

(ii) $\int_X a \,\mathrm{d}\mu = 0$.

A function f in $L^1(X)$ belongs to $H^1(X)$ if it admits a decomposition of the form

$$f = \sum_j c_j\, a_j,$$

where $\{c_j\}$ is in ℓ^1 and the a_j are $(1,2)$-atoms. Coifman and Weiss proved that we obtain the same space if we consider $(1,p)$-atoms instead of $(1,2)$-atoms. Here p is in $(1,\infty]$, and a

$(1,p)$-atom is defined much as an $(1,2)$-atom, but with the size condition (i) above replaced by the following

(i)' $\left(\int_B |a|^p \,\mathrm{d}\mu\right)^{1/p} \leq \mu(B)^{-1/p'}$, where p' denotes the index conjugate to p, and the integral must be suitably interpreted when $p=\infty$.

There are interesting variants of the atomic decomposition of $H^1(\mathbb{R}^n)$ in the literature. Indeed, as a consequence of the maximal characterisation of $H^1(\mathbb{R}^n)$ (see, for instance, [St2, Chapter III]), it is known that we may consider $(1,p)$-atoms with more than one vanishing moment. More precisely, for every nonnegative integer N, denote by $\mathscr{P}_N$ the finite dimensional space of polynomials of degree at most N in $\mathbb{R}^n$. Then we still get $H^1(\mathbb{R}^n)$ if we consider functions in $L^1(\mathbb{R}^n)$ which admit an atomic decomposition in terms of $(1,p)$-atoms satisfying the following modified cancellation condition

(ii)' $\int_{B_j} a_j(x)\, q(x) \,\mathrm{d}x = 0 \qquad \forall q \in \mathscr{P}_N.$

An application of the theory of k-harmonic Bergman spaces developed in Chapter 1 is to prove that functions in $H^1(\mathbb{R}^n)$ admit an atomic decomposition in terms of atoms satisfying suitable *infinite dimensional cancellation conditions*. This turns out to have interesting links with the theory of k-harmonic functions defined on balls of $\mathbb{R}^n$.

Suppose that k is a positive integer, p is in $(1,\infty)$ and B is a ball. Define $\mathscr{A}_k^p(B)$ to be as the space of all functions A in $L^p(B)$ satisfying the following conditions

(i) $\left(\int_B |A(x)|^p \,\mathrm{d}x\right)^{1/p} \leq |B|^{-1/p'}$;

(ii) $\int_B A(x)\, q(x) \,\mathrm{d}x = 0$ for all k-harmonic polynomials q.

The collection of all functions belonging to $\mathscr{A}_k^p(B)$ for some ball B will be denoted by $\mathscr{A}_k^p$ and its elements will be called $X^{k,p}$-atoms. Note that the cancellation condition (ii) above may be equivalently expressed as "orthogonality" to the k-harmonic Bergman spaces $b_k^{p'}(B)$ (here p' denotes the index conjugate to p).

In Theorem 1.5.7 we shall give an elementary proof that every function in $H^1(\mathbb{R}^n)$ indeed admits an atomic decomposition in terms of $X^{k,p}$-atoms.

The case of Riemannian manifolds

In Chapter 2, we consider the case of connected complete noncompact Riemannian manifolds M, which possess the doubling property of balls (see (0.0.2)) and satisfy a relative

Faber–Krahn inequality (see (2.1.2)). Recall that these properties are equivalent to a on-diagonal upper estimate for the heat kernel of the type

$$\text{(DUE)} \qquad h_t(x,x) \leq \frac{C}{\mu\big(B(x,\sqrt{t})\big)} \qquad \forall x \in M \quad \forall t > 0.$$

Recently, Hardy-type spaces associated to operators have been introduced in various settings (see [R, AMR, MMV1, MMV2, HLMMY] and the references therein).

T. Coulhon and X.T. Duong [CD] proved that under the above assumptions on M, the Riesz transform $f \mapsto \big|\nabla \mathscr{L}^{-1/2}\big|$ is of weak type $(1,1)$. An interpolation argument with the trivial $L^2(M)$ boundedness of the Riesz transform gives the $L^p(M)$ boundedness for all p in $(1,2)$. P. Auscher, A. McIntosh and E. Russ [AMR] proved that the Riesz transform is bounded from a Hardy-type space $H^1_{\mathrm{d}^*}(\Lambda^0 T^* M)$ to $L^1(M)$. It is quite a difficult task to prove that this Hardy-type space agrees with ours. A proof of this equivalence is implicit in [HLMMY].

In the special case where M supports a scaled Poincaré inequality (see (2.6.1)) Russ [Ru] proved that the Riesz transform is bounded from $H^1(M)$ to $L^1(M)$. Moreover, $H^1(M) = H^1_{\mathrm{d}^*}(\Lambda^0 T^* M)$ (see [AMR]), hence the Riesz transform is bounded from $H^1_{\mathrm{d}^*}(\Lambda^0 T^* M)$ to $L^1(M)$. We give a different proof of the fact that these two Hardy spaces agree, based on De Giorgi's regularity theorem for elliptic equations, which, in turn, in the setting of Riemannian manifolds is a well-known consequence of an important result obtained independently by Grigor'yan and Saloff-Coste (see [Sa1]).

It is fair to say that, although some of the results contained in this chapter are already present in the literature, our point of view is, to the best of our knowledge, new. Furthermore, we believe that our approach is somewhat simpler, and that it may be adapted to other situations where the doubling condition fails (see, for instance, [MMV2, MMV4]).

The operator $\mathscr{A}$

This part of the thesis is dedicated to the analysis of the operator $\mathscr{A}$, defined by

$$\mathscr{A} f = -\frac{1}{2}\,\Delta f - x\cdot \nabla f \qquad \forall f \in C^\infty_c(\mathbb{R}^n).$$

For every real number β, denote by γ_β both the function

$$\gamma_\beta(x) = \pi^{-n\beta/2}\ \mathrm{e}^{-\beta|x|^2} \qquad \forall x \in \mathbb{R}^n, \tag{0.0.3}$$

and the measure on $\mathbb{R}^n$ whose density with respect to the Lebesgue measure λ is γ_β. In particular, γ_1 is the Gauss measure on $\mathbb{R}^n$. Denote by Q_{γ_β} the Dirichlet form, defined by

$$Q_\beta(f) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f(x)|^2 \,\mathrm{d}\gamma_\beta(x) \qquad \forall f \in \mathscr{C}_c^\infty(\mathbb{R}^n).$$

A simple integration by parts shows that $\mathscr{A}$ is the operator associated to Q_{-1}, and that the Ornstein–Uhlenbeck operator

$$\mathscr{L} f = -\frac{1}{2}\,\Delta f + x\cdot\nabla f \qquad \forall f \in C_c^\infty(\mathbb{R}^n)$$

is the operator associated to Q_1. The Ornstein–Uhlenbeck operator generates a diffusion semigroup, which has been the object of many investigations during the last two decades. In particular, efforts have been made to study operators related to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators (see [MPS, GMMST2] and the references therein), Riesz transforms [MUu, GMST1, MMS] and functional calculus [GMST2, GMMST1, HMM].

The purpose of this part of the thesis is to investigate the analogues for the operator $\mathscr{A}$ of some of the results contained in the aforementioned papers. In many cases, the methods developed for $\mathscr{L}$ go through for $\mathscr{A}$, and the proofs of the results for $\mathscr{A}$ are straightforward modifications of the proofs of the corresponding results for $\mathscr{A}$. There are exceptions, however, as in the determination of the region of holomorphy in $L^p(\gamma_{-1})$ of the semigroup generated by $\mathscr{A}$.

We note that the operators $\mathscr{A}$ and $\mathscr{L}$ are unitarily equivalent, so that the analysis of $\mathscr{A}$ on $L^2(\gamma_{-1})$ is equivalent to the analysis of $\mathscr{L}$ on $L^2(\gamma_1)$. There is no such equivalence as far as analysis on L^p is concerned. Therefore, although there are analogies between the analysis of $\mathscr{A}$ (and of related operators) on $L^p(\gamma_{-1})$ and that of $\mathscr{L}$ (and of related operators) on $L^p(\gamma_1)$, the results for $\mathscr{A}$ are not directly deducible from those for $\mathscr{L}$.

Note also that the measure γ_{-1} is infinite, whereas γ_1 is a probability measure. Furthermore, if $B(x,r)$ denotes the Euclidean ball with centre x and radius r, then $\gamma_{-1}\big(B(x,r)\big)$ grows more than exponentially with r, as r tends to infinity. An interesting line of research in analysis on Riemannian manifolds aims at understanding the behaviour of the Laplace–Beltrami operator on a Riemannian manifold and of certain related operators (spectral multipliers, Riesz transforms, ...) under certain geometric assumptions. Quite often these concern the volume growth and the curvature of the manifold. Many investigations have been made in the case where the volume growth of the manifold is polynomial, or at most exponential, but there are virtually no results in the case of superexponential growth. Though $\mathscr{A}$ is not the Laplace–Beltrami operator of any Riemannian metric on $\mathbb{R}^n$, it is the "radial part" of the Laplace–Beltrami operator of a Riemannian metric on a suitable warped product of $\mathbb{R}^n$ and $\mathbb{T}^n$. We hope that some of the results contained in Chapter 3 will give some indications for further investigations on manifolds with superexponential volume growth.

Altogether, we believe that the result presented here will be valuable for any further investigation of $\mathscr{A}$.

Here is a summary of the results contained in Chapter 3. In Section 3.1, we find an explicit formula for the semigroup $\{\mathscr{H}_t\}_{t\geq 0}$ generated by $\mathscr{A}$, and we shall prove that $\{\mathscr{H}_t\}_{t\geq 0}$ is a symmetric diffusion semigroup on $(\mathbb{R}^n, \gamma_{-1})$.

A well known result by V.A. Liskevich and M.A. Perelmuter [LP] states that each symmetric diffusion semigroup $\{\mathscr{T}_t\}_{t\geq 0}$ on (X, μ) extends to a bounded holomorphic semigroup on $L^p(\mu)$ with angle at least $\phi_p = \arccos(2/p-1)$, i.e., the map $t \to \mathscr{T}_t$ extends to an analytic L^p-operator-valued function $z \to \mathscr{T}_z$ defined on the sector $S_{\phi_p} = \{z \in \mathbb{C} : |\arg(z)| < \phi_p\}$, such that $\|\mathscr{T}_z\|_{L^p(\mu)} \leq 1$ for each $z \in S_{\phi_p}$. It is known that the angle ϕ_p in the theorem of Liskevich and Perelmuter is optimal. Indeed, a celebrated theorem of J.B. Epperson [E, Theorem 1.1] asserts that the semigroup $\mathscr{M}_z$ generated by the Ornstein–Uhlenbeck operator extends to a bounded operator on $L^p(\gamma)$ if and only if z belongs to the set

$$\mathbf{E}_p = \{x + iy \in \mathbb{C} \ : \ |\sin y| \leq \tan\phi_p \sinh x\},$$

which contains the sector S_{ϕ_p} and is tangent to the rays $e^{\pm i\phi_p}\mathbb{R}^+$ at the origin. We shall prove that the same result holds for the analytic continuation $\{\mathscr{H}_z\}$ of the semigroup $\{\mathscr{H}_t\}$.

In Section 3.2, we develop the theory of local Calderón–Zygmund operators.

In Section 3.3 we consider the maximal operator associated to the semigroup $\{\mathscr{H}_t\}$, defined by

$$\mathscr{H}^* f = \sup_{t>0} |\mathscr{H}_t f| \,. \tag{0.0.4}$$

By the Banach principle (see [G]), it is well known that the study of the boundedness of $\mathscr{H}^*$ on $L^p(\gamma_{-1})$ is related to the problem of the almost everywhere convergence of $\mathscr{H}_t f$ to f as t tends to 0, for $f \in L^p(\gamma_{-1})$. The operator $\mathscr{H}^*$ is known to be bounded on $L^p(\gamma_{-1})$ for $1 < p \leq \infty$, by standard Littlewood–Paley–Stein theory for symmetric diffusion semigroups

[St1, C]. We shall prove that $\mathscr{H}^*$ is also of weak type 1 with respect to the measure γ_{-1}. Our proof follows the same lines of the proof of the corresponding result for the Ornstein–Uhlenbeck semigroup [GMST1].

In Section 3.4 we state some results concerning a functional calculus for the operator $\mathscr{A}$ on $L^p(\gamma_{-1})$. We do not give proofs of these results, mainly because the proofs, though lengthy, may be obtained from the corresponding results for the Ornstein–Uhlenbeck operator with minor changes. Nevertheless, we believe that it is worth recording these results for future reference. It is straightforward to show that the spectral resolution of $\mathscr{A}$ is

$$\mathscr{A} = \sum_{k=0}^{\infty} (k+n)\, \mathscr{E}_k, \tag{0.0.5}$$

where $\mathscr{E}_k$ is the orthogonal projection of $L^2(\gamma_{-1})$ onto the linear span of the functions of the form $\gamma_1\, p$, and p is a Hermite polynomial of degree j in n variables. Given a bounded sequence $M : \{n, n+1, \dots\} \to \mathbb{C}$, we define the *spectral multiplier operator* associated to the *spectral multiplier* M by

$$M(\mathscr{A})f = \sum_{k=0}^{\infty} M(k+n)\mathscr{E}_k f \qquad \forall f \in L^2(\gamma_{-1}). \tag{0.0.6}$$

Clearly $M(\mathscr{A})$ is bounded on $L^2(\gamma_{-1})$; an interesting question is find necessary and/or sufficient conditions on M so that $M(\mathscr{A})$ extends to a bounded operator on $L^p(\gamma_{-1})$, for some $1 < p < \infty$, or of weak type 1 with respect to γ_{-1}. Our results may be stated as follows:

(i) if M is the restriction to the L^2-spectrum of $\mathscr{A}$ of a function $\widetilde{M}$ of Laplace transform type, then $M(\mathscr{A})$ is of weak type $(1,1)$;

(ii) if $1 < p < \infty$ and $u \in \mathbb{R}$, then

$$|||\mathscr{A}^{iu}|||_{L^p(\gamma_{-1})} \asymp \mathrm{e}^{\phi_p^*|u|} \qquad \text{as } u \text{ tends to } \infty :$$

here $\phi_p^* = \arcsin|2/p - 1|$;

(iii) if $1 < p < \infty$, $p \neq 2$, a is a positive number and M is the restriction of a bounded holomorphic function on $a + S_{\phi_p^*}$ and satisfies some mild conditions on the boundary, then $M(\mathscr{A})$ is bounded on $L^p(\gamma_{-1})$.

(iv) if M is the restriction of a continuous function on $[0,\infty)$ and

$$\sup_{t>0} \left\|\left|M(t\mathscr{A})\right|\right\|_{L^p(\gamma_{-1})} < \infty,$$

then M extends to a bounded holomorphic function on the sector $S_{\phi_p^*}$.

A few comments on these results are in order. By general semigroup theory [St1, C], if $1 < p < \infty$ and if M is the restriction to the L^2-spectrum of $\mathscr{A}$ of a function $\widetilde{M}$ of Laplace transform type, then $\mathscr{A}$ is bounded on $L^p(\gamma_{-1})$. Thus, (i) above complements this result by proving a limiting result for $p = 1$, which may be false for general symmetric diffusion semigroups.

A result similar to (ii) holds for the Ornstein–Uhlenbeck operator [MMS, HMM]. Our proof simplifies considerably the proof of the upper estimate given in [MMS]. The proof of the lower estimate is very similar to that of the corresponding result for the Ornstein–Uhlenbeck operator [HMM].

Recently, A. Carbonaro and O. Dragicevic [CD], improving previous result of Cowling [C] and using a general multiplier result of Meda [Me], proved that all submarkovian semigroups admit a functional calculus similar to that considered in (iii), but they need to assume that $a = 0$, and require a slightly stronger condition on the boundary of $S_{\phi_p^*}$.

Finally (iv) is the analogue for $\mathscr{A}$ of a result for the Ornstein–Uhlenbeck operator [HMM].

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

Part I

Generalized Bergman spaces and Hardy-type spaces

Chapter 1

Generalized Bergman spaces in $\mathbb{R}^n$

1.1 Mean value properties for polyharmonic functions

For each x in $\mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the Euclidean ball with centre x and radius r and by $\mathscr{B}$ the family of all (open) balls in $\mathbb{R}^n$. For each B in $\mathscr{B}$ we denote by c_B and r_B the centre and the radius of B, respectively. For $c > 0$, we denote by $c\,B$ the ball with centre c_B and radius $c\,r_B$. We shall denote by B_1 the unit ball centred at the origin and by $c(n)$ its Lebesgue measure. The Lebesgue measure of the measurable set E will be denoted by $|E|$.

It is well known that harmonic functions can be characterized by mean value properties. In particular, a twice continuously differentiable function u is harmonic on a domain $\Omega \subseteq \mathbb{R}^n$ if and only if for every ball $B(x,r) \subseteq \Omega$ the average of u over $B(x,r)$ is equal to $u(x)$. A similar result holds for polyharmonic functions. Recall that a function u is *polyharmonic* of degree k (or *k-harmonic*) on Ω if $u \in \mathscr{C}^{2k}(\Omega)$ and $\Delta^k u = 0$ on Ω (here Δ denotes the Laplace operator). Notice that, by elliptic regularity, if u is a distribution on Ω such that $\Delta^k u = 0$, then u is in $\mathscr{C}^{2k}(\Omega)$, hence it is k-harmonic in Ω.

In 1909 P. Pizzetti [Piz] proved, for $n = 2, 3$, a mean value property for polyharmonic functions. He showed that the spherical mean of a polyharmonic function u on $\partial B(x,r)$ may be expressed as a linear combination of $\Delta^j u(x)$, $j = 1, \ldots, k$, with coefficients depending on r. In 1936, M. Nicolesco [Nic] extended Pizzetti's formula to all dimensions and to the case of solid means. In particular, he proved the following.

Theorem 1.1.1. *Assume that k is a positive integer and that Ω is a domain in $\mathbb{R}^n$. If u is*

k-harmonic on Ω, then for every $x \in \Omega$ and $r < d(x, \partial\Omega)$

$$\fint_{B(x,r)} u = \sum_{j=0}^{k-1} d(j)\ \Delta^j u(x)\ r^{2j},$$

where $d(0) = 1$, $d(j) = 1/(2^j\ j!\ (n+2)\dots(n+2j))$ for $j \in \{1, \dots, k-1\}$ and $\fint_{B(x,r)} u = |B(x,r)|^{-1} \int_{B(x,r)} u(y)\, \mathrm{d}y$.

By using this result, in 1966, J.H. Bramble and L.E. Payne [BP] proved the following characterisation of k-harmonic functions.

Theorem 1.1.2. *Assume that k is a positive integer and that Ω is a domain in $\mathbb{R}^n$. The following hold:*

(i) if u is k-harmonic on Ω, then for every $x \in \Omega$ and $r < d(x, \partial\Omega)$, and for every choice of k distinct values $\beta_j \in (0,1)$, $j = 1, \dots, k$,

$$u(x) = \sum_{j=1}^{k} C_j \fint_{B_j} u, \tag{1.1.1}$$

where $C_j = \frac{\prod_{m \neq j} \beta_m}{\prod_{m \neq j (\beta_m - \beta_j)}}$ and $B_j = B(x, \sqrt{\beta_j} r)$;

(ii) if $u \in \mathscr{C}^{2k}(\Omega)$ satisfies (1.1.1) for every $x \in \Omega$, for every $r < d(x, \partial\Omega)$ and for every choice of k distinct values $\beta_j \in (0,1)$, $j = 1, \dots, k$, then u is k-harmonic in Ω.

1.2 Generalized Bergman spaces

Definition 1.2.1. Suppose that k is a positive integer, p is in $[1, \infty)$ and B is an open ball. The generalised *Bergman space* $b^p_k(B)$ is the space of all k-harmonic functions u which belong to $L^p(B)$, endowed with the $L^p(B)$-norm.

Observe that

$$b^p_1(B) \subset b^p_2(B) \subset \dots \subset L^p(B),$$

with proper inclusions. It is straightforward to check that for every integer k the Bergman space $b^p_k(B)$ is a closed subspace of $L^p(B)$. Hence $b^p_k(B)$ is a closed subspace of $b^p_{k+1}(B)$.

In particular, when $p = 2$ we denote by $M_{k-1}(B)$ the orthogonal complement of $b^2_{k-1}(B)$ in $b^2_k(B)$. Thus,

$$b^2_k(B) = b^2_{k-1}(B) \perp M_{k-1}(B).$$

Working recursively, we obtain the following orthogonal decomposition of $b^2_k(B)$

$$b^2_k(B) = b^2_1(B) \perp M_1 \perp \cdots \perp M_{k-1}(B). \tag{1.2.1}$$

Clearly Δ^k maps $b^2_{k+1}(B)$ into the vector space of harmonic functions on B, but not necessarily into $b^2_1(B)$, for functions in the range of Δ^k may not belong to $L^2(B)$. Note that the restriction of Δ^k to $M_k(B)$ is injective. Indeed, if u is in $M_k(B)$ and $\Delta^k u = 0$, then u is in $b^2_k(B)$, whence $u = 0$, because $b^2_k(B) \cap M_k(B) = \{0\}$.

Next we examine the structure of $b^2_k(B_1)$ more closely (recall that B_1 denotes the unit ball in $\mathbb{R}^n$ centred at 0). Preliminarily, we prove some facts about polynomials in $\mathbb{R}^n$. Denote by $\mathscr{P}_m(\mathbb{R}^n)$ the vector space of all homogeneous polynomials of degree m in $\mathbb{R}^n$ and by $\mathscr{H}_m(\mathbb{R}^n)$ the space of harmonic polynomials in $\mathscr{P}_m(\mathbb{R}^n)$. A well known result [ABR, Proposition 5.5] asserts that

$$\mathscr{P}_m(\mathbb{R}^n) = \mathscr{H}_m(\mathbb{R}^n) \oplus |x|^2 \; \mathscr{P}_{m-2}(\mathbb{R}^n).$$

For the sake of brevity, we set

$$\nu^k_m := 2^k\, k! \prod_{j=0}^{k-1} \big(n + 2m + 2j\big). \tag{1.2.2}$$

Lemma 1.2.2. *The operator Δ^k is an isomorphism between $|x|^{2k} \mathscr{H}_m(\mathbb{R}^n)$ and $\mathscr{H}_m(\mathbb{R}^n)$, with inverse*

$$\Delta^{-k} h(x) = \frac{1}{\nu^k_m} \; |x|^{2k} \; h(x) \qquad \forall h \in \mathscr{H}_m(\mathbb{R}^n).$$

Proof. We observe that if h is in $\mathscr{H}_m$, then $\Delta^k(|\cdot|^{2k} \; h)$ is a multiple of h, hence harmonic. Indeed, by Leibnitz's formula and Euler's formula,

$$\begin{aligned} \Delta(|\cdot|^{2k} \; h)(x) &= (\Delta\, |x|^{2k})\, h(x) + 2\, \nabla\, |x|^{2k} \cdot \nabla h(x) + |x|^{2k} \; \Delta h(x) \\ &= 2k(n + 2(k-1))\, |x|^{2(k-1)} \; h(x) + 4k\, |x|^{2(k-1)} \; x \cdot \nabla h(x) \\ &= 2k(n + 2m + 2(k-1))\, |x|^{2(k-1)} \; h(x). \end{aligned}$$

By arguing recursively, we see that

$$\Delta^k(|\cdot|^{2k} \; h) = \nu^k_m\, h. \tag{1.2.3}$$

Therefore Δ^k is a bijection between $|x|^{2k}\mathscr{H}_m(\mathbb{R}^n)$ and $\mathscr{H}_m(\mathbb{R}^n)$, and its inverse Δ^{-k} is given by

$$\Delta^{-k}h(x) = \frac{1}{\nu_m^k}\ |x|^{2k}\ h(x) \qquad \forall h \in \mathscr{H}_m(\mathbb{R}^n),$$

as required. $\square$

Lemma 1.2.3. *Suppose that p is a homogeneous polynomial of degree j in $b^2_k(B_1)$. Then there exist unique homogeneous harmonic polynomials $p_{j-2m} \in \mathscr{H}_{j-2m}(\mathbb{R}^n)$, $m = 0,\ldots,k-1$, such that*

$$p(x) = p_j(x) + |x|^2\ p_{j-2}(x) + \cdots + |x|^{2(k-1)}\ p_{j-2(k-1)}(x). \tag{1.2.4}$$

Proof. Set $J = [j/2]$. By a well known result [ABR, Theorem 5.7], there exist unique homogeneous harmonic polynomials $\{p_{j-2m}\}_{m=0}^J$, with p_{j-2m} of degree $j-2m$, such that

$$p(x) = p_j(x) + |x|^2\ p_{j-2}(x) + \cdots + |x|^{2J}\ p_{j-2J}(x).$$

We now impose the condition $\Delta^k p = 0$. By (1.2.3), if $k > m$, then

$$\Delta^k(|\cdot|^{2m}\, p_{j-2m}) = \Delta^{k-m}\Delta^m(|\cdot|^{2m}\, p_{j-2m}) = \nu_{j-2m}^m\ \Delta^{k-m}p_{j-2m} = 0,$$

because p_{j-2m} is harmonic. If, instead, $k \leq m$, then, by arguing as in the proof of Lemma 1.2.2, we see that

$$\Delta^k(|\cdot|^{2m}\, p_{j-2m}) = 2^k\frac{m!}{(m-k)!}\prod_{i=1}^k(n+2j-2m-2i)\ |\cdot|^{2(m-k)}\ p_{j-2m}.$$

Altogether, we see that $\Delta^k p$ is a linear combination with nonvanishing coefficients of the polynomials $|x|^{2(m-k)}\ p_{j-2m}$, where $m = k-1,\ldots,J$. Observe that $\Delta^k p$ is homogeneous of degree $j-2k$. Therefore, by the uniqueness in the decomposition of homogeneous polynomials [ABR, Theorem 5.7], $\Delta^k p = 0$ if and only if $p_{j-2m} = 0$ for all $m = k-1,\ldots,J$, as required. $\square$

Remark 1.2.4. For later purposes, it is desirable to decompose every homogeneous polynomial of degree j in $b^2_k(B_1)$ according to the orthogonal decomposition (1.2.1). Formula (1.2.4) expresses a homogeneous polynomial p of degree j in $b^2_k(B_1)$ as the sum of the homogeneous polynomials p_j, $|\cdot|^2 p_{j-2}$, ..., $|\cdot|^{2(k-1)}p_{j-2(k-1)}$, which belong to $b^2_1(B_1)$, $b^2_2(B_1)$, ..., $b^2_{k-1}(B_1)$, respectively. These are pairwise orthogonal in $L^2(B_1)$, because, for $m \neq l$,

$$\int_{B_1} |x|^{2m+2l}\ p_{j-2m}(x)\,\overline{p_{j-2l}(x)}\,\mathrm{d}x = \int_0^1\ s^{n+2j-1}\,\mathrm{d}s\int_{\mathbb{S}^{n-1}} p_{j-2m}(\omega)\,\overline{p_{j-2l}(\omega)}\,\mathrm{d}\omega,$$

and spherical harmonics of different degrees are orthogonal on the unit sphere. Here, $\overline{p_{j-2l}(\omega)}$ denotes the complex conjugate of $p_{j-2l}(\omega)$. However, this is not the desired orthogonal decomposition, because, for instance, p_{j-2m}, which belongs to $b^2_1(B_1)$, and $|x|^{2j}\, p_{j-2m}$ are obviously not orthogonal in $L^2(B_1)$.

We now explain how to perform the desired decomposition. It may be worth dealing first with the case $k = 2$. Given f in $b^2_k(B_1)$, denote by $\Pi_0 f$ the orthogonal projection of f onto $b^2_1(B_1)$, and, for $j = 1, \ldots, k-1$, denote by $\Pi_j f$ the orthogonal projection of f onto $M_j(B_1)$.

Lemma 1.2.5. *Let p be a homogeneous polynomial of degree j in $b^2_2(B_1)$ and let p_j and p_{j-2} be as in (1.2.4). Then*

$$\Pi_0 p = p_j + \frac{n+2j-4}{n+2j-2}\, p_{j-2} \qquad \textit{and} \qquad \Pi_1 p = \left(|\cdot|^2 - \frac{n+2j-4}{n+2j-2} \right) p_{j-2}.$$

Proof. It is clear that $p = \Pi_0 p + \Pi_1 p$ and that $p_j + (n+2j-4)/(n+2j-2)p_{j-2}$ is harmonic. Denote by p^1_{j-2} the polynomial defined by

$$p^1_{j-2}(x) = \left(|x|^2 - \frac{n+2j-4}{n+2j-2} \right) p_{j-2}(x) \tag{1.2.5}$$

By Lemma 1.2.2, p^1_{j-2} is in $b^2_2(B_1)$. Thus, it remains to show that p^1_{j-2} is orthogonal to all harmonic polynomials, for their restrictions to B_1 are dense in $b^2_1(B_1)$. Suppose that q_k is a homogeneous harmonic polynomial of degree k. Then

$$\begin{aligned} &\int_{B_1} q_k(x) \left(|x|^2 - \frac{n+2j-4}{n+2j-2} \right) \overline{p_{j-2}(x)}\, \mathrm{d}x \\ &= \int_0^1 s^{n+k+j-3} \left(s^2 - \frac{n+2j-4}{n+2j-2} \right) \mathrm{d}s \int_{\mathbb{S}^{n-1}} q_k(\omega)\, \overline{p_{j-2}(\omega)}\, \mathrm{d}\omega. \end{aligned}$$

If $j - 2 \neq k$, then the inner integral vanishes, for spherical harmonics of different degrees are orthogonal. If $j-2 = k$, then the outer integral vanishes, as a straighforward calculation shows, and the required result follows. $\square$

We aim at extending the decomposition above to $b^2_k(B_1)$. The extension hinges on the following technical lemma.

Lemma 1.2.6. *There exists a unique sequence of polynomials $\{\alpha^k_j : j, k = 0, 1, 2, \ldots\}$ on $\mathbb{R}$ with the following properties:*

(i) α^k_j is a monic even polynomial of degree $2k$ for every j in $\mathbb{N}$;

(ii) for every $k \geq 1$ the following orthogonality relations hold

$$\int_0^1 \alpha_j^k(s)\, \alpha_j^l(s)\, s^{n+2j-1}\, \mathrm{d}s = 0 \qquad l = 0, \ldots, k-1, \quad j = 0, 1, 2, \ldots. \tag{1.2.6}$$

Furthermore

$$\int_0^1 \alpha_j^k(s)\, s^{n+2j-1+2k}\, \mathrm{d}s \neq 0 \qquad j = 0, 1, 2, \ldots. \tag{1.2.7}$$

For $k \geq 1$, the polynomial α_j^k is given by

$$\alpha_j^k(s) = s^{2k} + \sum_{i=0}^{k-1} C_i^k(j,n)\, s^{2i}, \tag{1.2.8}$$

where

$$C_i^k(j,n) = (-1)^{k-i} \binom{k}{i} \frac{\prod_{l=i}^{k-1}(n+2j+2l)}{\prod_{l=k+i}^{2k-1}(n+2j+2l)}. \tag{1.2.9}$$

Hereafter, to simplify notation, we shall often omit the dependence of the coefficients $C_i^k(j,n)$ on j and n, and write simply C_i^k. We shall prove that the polynomials α_j^k are given by (1.2.8) and (1.2.9) by brute force and, although conceptually very simple, the proof requires long and tedious calculation. The precise form of the coefficients will be used only later to estimate the generalized Bergman projections. Therefore we have chosen to postpone this part of the proof of the lemma to the Appendix to this chapter.

Proof. (of the existence and uniqueness of α_j^k) For the duration of this proof, for each j in $\mathbb{N}$ we shall denote by $(\cdot,\cdot)_j$ the inner product in $L^2([0,1], s^{n+2j-1}\,\mathrm{d}s)$. Observe preliminarly that, for each $k \geq 1$, (1.2.6) is equivalent to the following:

$$\int_0^1 \alpha_j^k(s)\; s^{n+2j+2m-1}\; \mathrm{d}s = 0 \qquad m = 0, 1, \ldots, k-1 \quad j = 0, 1, 2, \ldots. \tag{1.2.10}$$

Furthermore, if α_j^k exists, then, by (i), it must be of the form:

$$\alpha_j^k(s) = s^{2k} + \sum_{i=0}^{k-1} C_i^k(j,n)\, s^{2i},$$

where the coefficients $C_i^k(j,n)$ must be determined.

We argue by induction on k. Suppose first that $k = 1$. Then a straightforward calculation shows that for every nonnegative integer j the system (1.2.10) has a unique solution, given by

$$\alpha_j^1(s) = s^2 - \frac{n+2j}{n+2j+2},$$

(see also Lemma 1.2.5). Furthermore, $(\alpha_j^1, s^2)_j \neq 0$.

Now suppose that for each $\ell \leq k-1$, there exists a unique sequence $\{\alpha_j^\ell : j \in \mathbb{N}\}$ of polynomials with the required properties (in particular $(\alpha_j^\ell, s^{2\ell})_j \neq 0$). Since (1.2.6) is equivalent to (1.2.10), (1.2.10), with ℓ in place of k, implies that $(\alpha_j^\ell, s^{2m})_j = 0$ for each $m \in \{0, \ldots, \ell-1\}$. Therefore,

$$(\alpha_j^k, \alpha_j^\ell)_j = \int_0^1 \Big(s^{2k} + \sum_{i=\ell}^{k-1} C_i^k s^{2i}\Big)\, \alpha_j^\ell(s)\, s^{n+2j-1}\, \mathrm{d}s = 0 \qquad \forall \ell = 0, \ldots, k-1. \tag{1.2.11}$$

Observe that (1.2.11) is an upper-triangular nonhomogeneous system with unknowns $C_0^k, \ldots, C_{k-1}^k$. Hence it has a unique solution if and only if the diagonal entries of the associated matrix do not vanish. These are

$$\int_0^1 \alpha_j^\ell(s)\, s^{n+2j-1+2\ell}\, \mathrm{d}s \qquad \ell = 0, \ldots, k-1,$$

which indeed do not vanish by the inductive hypothesis. This concludes the proof of the inductive step, and of the first part of the lemma. □

With a slight abuse of notation, we also denote by α_j^k the polynomial on $\mathbb{R}^n$, defined by

$$\alpha_j^k(x) = |x|^{2k} + C_{k-1}^k\, |x|^{2(k-1)} + \cdots + C_0^k, \tag{1.2.12}$$

where the coefficients $\{C_i^k\}_{i=0}^{k-1}$ are as in Lemma 1.2.6. Denote by $\mathscr{P}$ the linear space of all polynomials in $\mathbb{R}^n$. For each j in $\mathbb{N}$ and each $k \geq 1$, define the map $\mathscr{E}_j^k : \mathscr{H}_j \to \mathscr{P}$ by

$$\mathscr{E}_j^k p = \alpha_j^k\, p \qquad \forall p \in \mathscr{H}_j. \tag{1.2.13}$$

It will be convenient to denote the identity map by $\mathscr{E}_j^0 : \mathscr{H}_j \to \mathscr{H}_j$. For each $\ell \geq 1$, set

$$\mathscr{Q}_\ell := \operatorname{span}\big\{\operatorname{Ran}(\mathscr{E}_j^{m-1}) : j \in \mathbb{N},\ m = 1, \ldots, \ell\big\}.$$

For j in $\mathbb{N}$, let $\{p_{j,1}, \ldots, p_{j,d_j}\}$ be any orthonormal basis of $\mathscr{H}_j$.

Lemma 1.2.7. *Suppose that $k \geq 1$. The following hold:*

(i) $\operatorname{Ran}\big(\mathscr{E}_j^{k-1}\big)$ *is contained in* $b_k^2(B_1)$ *and*

$$\operatorname{Ran}\big(\mathscr{E}_j^{k-1}\big) \perp \operatorname{Ran}\big(\mathscr{E}_j^{m-1}\big) \qquad \forall m \in \{1, \ldots, k-1\};$$

(ii) $\mathscr{Q}_k$ *is dense in* $b^2_k(B_1)$*;*

(iii) $\operatorname{span}\{\operatorname{Ran}(\mathscr{E}_j^{k-1} : j \in \mathbb{N})\}$ *is dense in* $M_{k-1}(B_1)$*.*

(iv) the sequence

$$\mathscr{E}_j^{k-1} p_{j,1}, \ldots, \mathscr{E}_j^{k-1} p_{j,d_j} \qquad j = 0, 1, 2, \ldots$$

is an orthogonal basis of $M_{k-1}(B_1)$*. Hence*

$$\mathscr{E}_j^{h} p_{j,1}, \ldots, \mathscr{E}_j^{h} p_{j,d_j} \qquad h = 0, \ldots, k-1, \quad j = 0, 1, 2, \ldots$$

is an orthogonal basis of $b^2_k(B_1)$*.*

Proof. First we prove (i). Observe that

$$\mathscr{E}_j^{k-1} p = |\cdot|^{2(k-1)}\, p + C_{k-2}^{k-1}\, |\cdot|^{2(k-2)}\, p + \cdots + C_0^{k-1}\, p,$$

whence

$$\Delta^k \big(\mathscr{E}_j^{k-1} p\big) = \Delta \Delta^{k-1}\big(\, |\cdot|^{2(k-1)}\, p\big) + C_{k-2}^{k-1}\, \Delta^2 \Delta^{k-2}\big(\, |\cdot|^{2(k-2)}\, p\big) + \cdots + C_0^{k-1}\, \Delta^{k-1} \Delta p.$$

For $\ell \geq 1$, the polynomial $\Delta^\ell\big(\, |\cdot|^{2\ell}\, p\big)$ is harmonic by Lemma 1.2.2. Therefore all the summands on the right hand side vanish, whence $\mathscr{E}_j^{k-1} p$ is in $b^2_k(B_1)$.

Now, suppose that p and q are homogeneous harmonic polynomials of degree j and h, respectively. Then

$$\begin{aligned}(\mathscr{E}_j^{k-1} p, \mathscr{E}_h^{m-1} q) &= \int_{B_1} \alpha_j^{k-1}(x)\, p(x)\; \alpha_h^{m-1}(x)\, \overline{q(x)}\, \mathrm{d}x \\ &= \int_0^1 \alpha_j^{k-1}(s)\, \alpha_h^{m-1}(s)\, s^{n+j+h-1}\, \mathrm{d}s \int_{\mathbb{S}^{n-1}} p(\omega)\, \overline{q(\omega)}\, \mathrm{d}\omega.\end{aligned} \tag{1.2.14}$$

The inner integral vanishes if $j \neq h$ (spherical harmonics of different degrees are orthogonal on $\mathbb{S}^{n-1}$), and the outer integral vanishes if $j = h$, by (1.2.6). This concludes the proof of (i).

Next we prove (ii). It is well known (see [ABR, Corollary 5.34]) that the span of all harmonic homogeneous polynomials is dense in $b^2_1(B_1)$, hence the result holds for $k = 1$.

Henceforth we assume $k \geq 2$. We claim that the space of all u in $b^2_k(B_1)$ that extend to a bounded k-harmonic function in a neighbourhood of $\overline{B}_1$ is dense in $b^2_k(B_1)$. Indeed, suppose that u is in $b^2_k(B_1)$. Then for every $r \in (0, 1)$, the r-dilate u^r of u, defined by

$$u^r(x) = u(rx) \qquad \forall x \in (1/r)\, B_1,$$

is in $b^2_k((1/r)\, B_1)$, hence it is k-harmonic and bounded in a neighbourhood of $\overline{B}_1$. The claim then follows from the fact that u^r tends to u in $b^2_k(B_1)$.

Thus, to prove that $\mathscr{Q}_k$ is dense in $b^2_k(B_1)$ it suffices to prove that every function u in $b^k_2(B_1)$ that extends to a k-harmonic function in a neighbourhood of $\overline{B}_1$ may be approximated with arbitrary degree of precision by polynomials in $\mathscr{Q}_k$.

We argue by induction on k. As we have already said, the property holds for $k = 1$. Now suppose that $\mathscr{Q}_\ell$ is dense in $b^2_\ell(B_1)$ for $\ell = 1, \ldots, k-1$, and let u be a function in $b^2_k(B_1)$ that extends to a k-harmonic function in RB_1 for some $R > 1$. Then $\Delta^{k-1}u$ is harmonic in RB_1. Hence there exist homogeneous harmonic polynomials $\{p_j : j \in \mathbb{N}\}$ such that

$$\Delta^{k-1}u = \sum_{j=0}^{\infty} p_j, \tag{1.2.15}$$

where p_j is a homogeneous harmonic polynomial of degree j. It is known [ABR, Corollary 5.34] that the series (1.2.15) is absolutely and uniformly convergent in every compact subset of RB_1. In particular, for every r in $(1, R)$

$$\lim_{j\to\infty} \sup_{x\in rB_1} |p_j(x)| = 0.$$

Therefore, for every r' in $(1, r)$

$$\begin{aligned}
\sup_{x\in r'B_1} |p_j(x)| &= \sup_{0\le s\le r'} \sup_{x'\in\mathbb{S}^{n-1}} s^j\; |p_j(x')| \\
&= \sup_{0\le s\le r'} \sup_{x'\in\mathbb{S}^{n-1}} \left(\frac{s}{r}\right)^j r^j\; |p_j(x')| \\
&\le C\left(\frac{r'}{r}\right)^j \qquad \forall j \in \mathbb{N}.
\end{aligned}$$

Define U by

$$U = \sum_{j=0}^{\infty} \Delta^{1-k} p_j, \tag{1.2.16}$$

where

$$\Delta^{1-k} p_j(x) = \left(\nu_j^{k-1}\right)^{-1} |x|^{2(k-1)}\; p_j(x)$$

and ν_j^{k-1} is as in Lemma 1.2.2. Note that $\nu_j^{k-1} \asymp j^k$ as j tends to infinity. Hence the series (1.2.16) is absolutely and uniformly convergent in every compact subset of RB_1. Moreover the estimates above imply that $\Delta^{k-1}U = \sum_j p_j$, which is also equal to $\Delta^{k-1}u$. Therefore $\Delta^{k-1}(U - u) = 0$ in RB_1, i.e., $U - u$ is a $(k-1)$-harmonic function in RB_1. By the induction hypothesis, $U - u$ may be approximated to any degree of precision by $(k-1)$-harmonic polynomials. Thus, to prove that u may be well approximated by k-harmonic polynomials, it suffices to show that U does. This is straightforward, for the series (1.2.16) converges uniformly in a neighbourhood of $\overline{B}_1$, hence in $L^2(B_1)$. This concludes the proof of the inductive step, and of (ii).

Part (iii) is a direct consequence of (i) and (ii).

To prove (iv), observe that, by (iii), the span of

$$\mathscr{E}_j^{k-1} p_{j,1}, \ldots, \mathscr{E}_j^{k-1} p_{j,d_j} \qquad j = 0, 1, 2, \ldots \tag{1.2.17}$$

is dense in $M_{k-1}(B_1)$. Furthermore

$$\begin{aligned}(\mathscr{E}_j^{k-1} p_{j,\ell}, \mathscr{E}_h^{k-1} p_{j,m}) &= \int_0^1 \alpha_j^{k-1}(s)^2\, s^{n+2j-1}\,\mathrm{d}s \int_{\mathbb{S}^{n-1}} p_{j,\ell}(\omega)\, \overline{p_{j,m}(\omega)}\,\mathrm{d}\omega \\ &= 0\end{aligned} \tag{1.2.18}$$

because $p_{j,\ell}$ and $p_{j,m}$ are orthogonal in $L^2(B_1)$, whence so are their restrictions to $\mathbb{S}^{n-1}$. Therefore (1.2.17) is an orthogonal basis of $M_{k-1}(B_1)$. The last statement of (iv) follows from this and the orthogonal decomposition (1.2.1).

This concludes the proof of the lemma. □

1.3 The reproducing kernel of generalized Bergman spaces

Let B be an open ball in $\mathbb{R}^n$ and suppose that p is in $[1, \infty)$. For each x in B and each multiindex γ, denote by Λ_x^γ the *evaluation functional* at x on $b_k^p(B)$, defined by

$$\Lambda_x^\gamma u = D^\gamma u(x) \qquad \forall u \in b_k^p(B).$$

A noteworthy consequence of the mean value property for k-harmonic functions (1.1.1) is that Λ_x^γ is continuous on $b_k^p(B)$, as shown in the next proposition.

Proposition 1.3.1. *Suppose that p is in $[1, \infty)$, that x is in B and that γ is a multiindex. Then there exists a constant C, depending only on n, k and γ, such that*

$$|||\Lambda_x^\gamma|||_{b_k^p} \le \frac{C}{d(x, \partial B)^{|\gamma| + n/p}}. \tag{1.3.1}$$

Proof. The proof is by induction on $|\gamma|$. First we consider the case where $|\gamma| = 0$. Let $r < d(x, \partial B)$. Set $\beta_j = 1 - 2^{-j}$. It is straighforward to check that

$$\Big|\prod_{m\neq j} \frac{\beta_m}{\beta_m - \beta_j}\Big| \leq 2^{k(k-1)}.$$

Thus, $|C_j| \leq 2^{k(k-1)}$, for $j = 1, \dots, k$, where C_j is defined in (1.1.1). Hence, by the mean value property (1.1.1),

$$\begin{aligned}
|u(x)| &\leq C \sum_{j=1}^{k} \frac{1}{\big|B(x, \sqrt{\beta_j} r)\big|} \int_{B(x,\sqrt{\beta_j}r)} |u(y)| \, \mathrm{d}y \\
&\leq C \sum_{j=1}^{k} \frac{\|u\|_p}{\big|B(x, \sqrt{\beta_j} r)\big|^{1/p}} \\
&\leq C \; r^{-n/p} \, \|u\|_p.
\end{aligned}$$

We have used Hölder's inequality in the second inequality above. Then (1.3.1) (with $\gamma = 0$) follows by letting r tend to $d(x, \partial B)$.

Next, suppose that (1.3.1) holds for all multiindices γ such that $|\gamma| \leq m$, and suppose that $|\gamma'| = m + 1$. Then there exists $i \in \{1, \dots, n\}$ such that $D^{\gamma'} = \partial_i D^{\gamma}$, where $|\gamma| = m$. Choose $r < d(x, \partial B)/2$. Observe that, by the mean value property of k-harmonic function applied to $\partial_i D^{\gamma} u$ and by translation invariance of the Lebesgue measure,

$$\partial_i D^{\gamma} u(x) = \sum_{j=1}^{k} C_j \, \big|B(0, \sqrt{\beta_j} r)\big|^{-1} \int_{B(0,\sqrt{\beta_j}r)} \partial_i D^{\gamma} u(x + y) \, \mathrm{d}y.$$

By the divergence theorem and the inductive hypothesis

$$\begin{aligned}
|\partial_i D^{\gamma} u(x)| &\leq \frac{C}{r^n} \sum_{j=1}^{k} \int_{\partial B(0,\sqrt{\beta_i}r)} |D^{\gamma} u(x + \omega)| \, \mathrm{d}\omega \\
&\leq \frac{C}{r^n} \sum_{j=1}^{k} \|u\|_p \, \big[\inf_{|\omega|=r} d(x + \omega, \partial B)\big]^{-|\gamma|-n/p} \int_{\partial B(0,\sqrt{\beta_i}r)} \mathrm{d}\omega \\
&\leq C \, r^{-|\gamma|-1-n/p} \, \|u\|_p,
\end{aligned}$$

from which the required estimate for $\big\|\Lambda_x^{\gamma'}\big\|_{b_k^p}$ follows. $\square$

Definition 1.3.2. Denote by $R_B^k(x,\cdot)$ the unique function in $b_k^2(B)$ that represents the continuous linear functional Λ_x^0. Then

$$u(x) = \int_B \overline{R_B^k(x,y)}\, u(y)\,\mathrm{d}y \qquad \forall u \in b_k^2(B). \tag{1.3.2}$$

The function R_B^k on $B\times B$ will be called the *(generalized) Bergman kernel* or the *reproducing kernel* of $b_k^2(B)$.

In the following proposition we collect some elementary properties of R_B^k.

Proposition 1.3.3. *The following hold:*

(i) R_B^k *is real-valued;*

(ii) if $\{u_n\}$ *is an orthonormal basis of* $b_k^2(B)$*, then*

$$R_B^k(x,y) = \sum_{j=1}^{\infty} \overline{u_j(x)}\, u_j(y) \qquad \forall x,y \in B;$$

(iii) $R_B^k(x,y) = R_B^k(y,x)$ *for all* x *and* y *in* B*;*

(iv) $\|R_B^k(x,\cdot)\|_{b_k^2} = R_B^k(x,x)^{1/2}$ *for all* x *in* B*.*

Proof. These are standard properties of reproducing kernels. The proof is almost identical to the proof of [ABR, Proposition 8.4], and is omitted. □

Recall the orthogonal decomposition

$$b_k^2(B) = b_1^2(B) \perp M_1(B) \perp \cdots \perp M_{k-1}(B). \tag{1.3.3}$$

Each of the subspaces of $b_k^2(B)$ that appear on the right hand side is closed in $b_k^2(B)$. Therefore the restriction of Λ_x^0 to $M_j(B)$ is a continuous linear functional on $M_j(B)$, so that, by the Riesz representation theorem, there exists a unique function $R_B^{M_j}(x,\cdot)$ in $M_j(B)$ that represents $\Lambda_x^0\big|_{M_j}$.

Proposition 1.3.4. *Suppose that* B *is an open ball in* $\mathbb{R}^n$*. The following hold:*

(i) if B *has radius* r*, then*

$$R_B^k(x,y) = r^{-n}\, R_{B_1}^k\Big(\frac{x-c_B}{r}, \frac{y-c_B}{r}\Big) \qquad \forall (x,y) \in B\times B; \tag{1.3.4}$$

(ii) $R^k_B = R^1_B + R^{M_1}_B + \cdots + R^{M_{k-1}}_B$.

Proof. First we prove (i). Suppose that u is in $b^2_k(B)$. Then the function

$$\tau_{c_B} u^r(x) := u(rx + c_B) \qquad \forall x \in B_1$$

is in $b^2_k(B_1)$. Since $R^k_{B_1}$ is the reproducing kernel for $b^2_k(B_1)$,

$$\begin{aligned} u(rx + c_B) &= \int_{B_1} \tau_{c_B} u^r(y)\, R^k_{B_1}(x, y)\, \mathrm{d}y \\ &= r^{-n} \int_B u(y)\, R^k_{B_1}(x, (y - c_B)/r)\, \mathrm{d}y \qquad \forall x \in B_1, \end{aligned}$$

which may be rewritten in the form

$$u(x) = r^{-n} \int_B u(y)\, R^k_{B_1}((x - c_B)/r, (y - c_B)/r)\, \mathrm{d}y \qquad \forall x \in B,$$

and (i) is proved.

Next we prove (ii). Suppose that u is in $b^2_k(B)$. Then there exist u_0 in $b^2_1(B)$, u_h in $M_h(B)$, $h = 1, \ldots, k-1$, such that $u = u_0 + u_1 + \cdots + u_{k-1}$. Then

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + \cdots + u_{k-1}(x) \\ &= \int_B R^1_B(x, y)\, u_0(y)\, \mathrm{d}y + \sum_{h=1}^{k-1} \int_B R^{M_h}_B(x, y)\, u_h(y)\, \mathrm{d}y \end{aligned}$$

Since the decomposition (1.3.3) is orthogonal, the right hand side is equal to

$$\int_B R^1_B(x, y)\, u(y)\, \mathrm{d}y + \sum_{h=1}^{k-1} \int_B R^{M_h}_B(x, y)\, u(y)\, \mathrm{d}y.$$

Therefore $R^1_B(x, \cdot) + R^{M_1}_B(x, \cdot) + \cdots + R^{M_{k-1}}_B(x, \cdot)$ represents Λ^0_x on $b^2_k(B)$. By uniqueness, this must be $R^k_B(x, \cdot)$, as required. □

Our aim is to establish an explicit formula for R^k_B. By Proposition 1.3.4 (i) it suffices to determine $R^k_{B_1}$. It is known [ABR, Theorem 8.9] that

$$R^1_{B_1}(x, y) = \frac{1}{n\, c(n)} \sum_{j=0}^{\infty} (n + 2j)\, \widetilde{Z}_j(x, y), \tag{1.3.5}$$

where

$$\widetilde{Z}_j(x, y) := |x|^j\ |y|^j\ Z_j\Big(\frac{x}{|x|}, \frac{y}{|y|}\Big) \qquad \forall x, y \in \mathbb{R}^n$$

and Z_j denotes the j^{th} zonal harmonic on $\mathbb{S}^{n-1}$. Clearly $\widetilde{Z}_j(x,\cdot)$ is in $\mathscr{H}_j$. For any $j \in \mathbb{N}$, define σ_j^h by the rule

$$(\sigma_j^h)^{-1} = \int_0^1 \alpha_j^h(s)^2\, s^{n+2j-1}\,\mathrm{d}s,$$

where α_j^h is as in (1.2.8). Observe that

$$\big(\sigma_j^h\big)^{-1} = \int_0^1 \alpha_j^h(s)\, s^{n+2j+2h-1}\,\mathrm{d}s,$$

for $\alpha_j^h(s)$ is orthogonal to s^{2m} for each m in $\{0,\ldots,h-1\}$, by (1.2.10). Then formula (1.3.15), with k replaced by h and $m=h$, gives

$$\sigma_j^h = \frac{n+2j+4h}{2^{2h}\,(h!)^2} \prod_{l=h}^{2h-1} (n+2j+2l)^2. \tag{1.3.6}$$

Proposition 1.3.5. *The reproducing kernel of $M_h(B_1)$ is given by*

$$R_{B_1}^{M_h}(x,y) = \frac{1}{n\,c(n)} \sum_{j=0}^{\infty} \sigma_j^h\, \alpha_j^h(x)\, \widetilde{Z}_j(x,y)\, \alpha_j^h(y) \qquad \forall x,y \in B_1. \tag{1.3.7}$$

Proof. For each j in $\mathbb{N}$ let $\{p_{j,1},\ldots,p_{j,d_j}\}$ be an orthonormal basis of $\mathscr{H}_j$.

We *claim* that the function R_j^h, given by

$$R_j^h(x,y) = \frac{1}{n\,c(n)}\, \sigma_j^h\, \alpha_j^h(x)\, \widetilde{Z}_j(x,y)\, \alpha_j^h(y) \qquad \forall x,y \in B_1,$$

is the reproducing kernel of $\mathscr{E}_j^h \mathscr{H}_j$. Clearly, it suffices to show that

$$\big(\mathscr{E}_j^h p_{j,i}, R_j^h(x,\cdot)\big) = \mathscr{E}_j^h p_{j,i}(x) \qquad i = 1,\ldots,d_j.$$

To prove this, notice that

$$\begin{aligned}
\big(\mathscr{E}_j^h p_{j,i}, R_j^h(x,\cdot)\big) &= \frac{\sigma_j^h\, \alpha_j^h(x)}{n\,c(n)} \int_{B_1} \alpha_j^h(y)\, p_{j,i}(y)\, \widetilde{Z}_j(x,y)\, \alpha_j^h(y)\,\mathrm{d}y \\
&= \sigma_j^h\, \alpha_j^h(x)\ |x|^j \int_0^1 \alpha_j^h(s)^2\, s^{n+2j-1}\,\mathrm{d}s \int_{\mathbb{S}^{n-1}} Z_j(x/|x|,\omega)\, p_{j,i}(\omega)\,\mathrm{d}\sigma(\omega) \\
&= \alpha_j^h(x)\ |x|^j\ p_{j,i}(x/|x|) \\
&= \mathscr{E}_j^h p_{j,i}(x) :
\end{aligned}$$

we have used the fact that Z_j is the reproducing kernel of spherical harmonics of degree j in the third equality above.

Observe that if $j \neq \ell$, then

$$\begin{aligned}\big(R_\ell^h(x,\cdot), R_j^h(x,\cdot)\big) &= \frac{1}{n^2\, c(n)^2}\, \sigma_j^h\, \sigma_\ell^h\, \alpha_j^h(x)\, \alpha_\ell^h(x)\, \big(\mathscr{E}_\ell^h \widetilde{Z}_\ell(x,\cdot), \mathscr{E}_j^h \widetilde{Z}_j(x,\cdot)\big) \\ &= 0,\end{aligned}$$

because $\widetilde{Z}_\ell(x,\cdot)$ is in $\mathscr{H}_\ell$, $\widetilde{Z}_j(x,\cdot)$ is in $\mathscr{H}_j$, and the ranges of $\mathscr{E}_\ell^h$ and $\mathscr{E}_j^h$ are orthogonal by (1.2.14).

Next we show that for every x in B_1 the series $\sum_j R_j(x,\cdot)$ is convergent in $L^2(B_1)$. Indeed, observe that there exists a constant C such that

$$\sum_{i=0}^{h-1} \big|C_i^h(j,n)\big| \leq C \qquad \forall j \in \mathbb{N}.$$

This may be easily deduced from (1.2.9), with h in place of k, and the fact that

$$\frac{\prod_{l=i}^{h-1}(n+2j+2l)}{\prod_{l=h+i}^{2h-1}(n+2j+2l)} \leq 1.$$

Therefore

$$\big|\alpha_j^h(x)\big| \leq C \qquad \forall x \in B_1 \quad \forall j \in \mathbb{N},$$

and

$$\begin{aligned}\big\|R_j^h(x,\cdot)\big\|_{L^2(B_1)} &\leq C\, \sigma_j^h\, \|\widetilde{Z}_j(x,\cdot)\|_{L^\infty(B_1)}\, \|\alpha_j^h\|_{L^2(B_1)} \\ &\leq C\, (\sigma_j^h)^{1/2}\, |x|^j \dim(\mathscr{H}_j).\end{aligned}$$

We have used the well known estimate $\big|Z_j(x,\cdot)\big| \leq \dim(\mathscr{H}_j)$ and the fact that $(\sigma_j^h)^{-1/2} = \|\alpha_j^h\|_{L^2(B_1)}$ (see the definition of σ_j^h) in the second inequality. Now, from (1.3.6) we deduce that

$$\big(\sigma_j^h\big)^{1/2} \asymp j^{h+1/2} \qquad \text{as } j \text{ tends to infinity.} \tag{1.3.8}$$

Furthermore

$$\dim(\mathscr{H}_j) \asymp j^{n-2} \qquad \text{as } j \text{ tends to infinity.}$$

Altogether, we have proved that there exists a constant C, independent of j, such that

$$\big\|R_j^h(x,\cdot)\big\|_{L^2(B_1)} \leq C\, j^{n+h-3/2}\, |x|^j\,.$$

This implies that the series $\sum_j R_j^h(x,\cdot)$ is convergent in $L^2(B_1)$. Clearly $\sum_j R_j^h(x,\cdot)$ reproduces all polynomials in $\operatorname{span}\big(\bigcup_{j=0}^\infty \mathscr{E}_j^h \mathscr{H}_j\big)$, which is dense in $M_h(B_1)$ by Lemma (1.2.7) (iii). A density argument then shows that $\sum_j R_j^h(x,\cdot)$ is a reproducing kernel of $M_h(B_1)$. Hence it must be $R_{B_1}^{M_h}(x,\cdot)$, by uniqueness, as required. $\square$

1.3.1 Appendix

In this appendix, we present the proof of second part of Lemma 1.2.6. It would be highly desirable to find a simpler proof thereof.

Proof. (of the second part of Lemma 1.2.6) We prove that the polynomial (1.2.8) with coefficients (1.2.9) satifies condition (1.2.10). For notational convenience, set

$$\beta^{(0)}_{j,k}(s) := \alpha^k_j(s) \qquad \text{and} \qquad \beta^{(i)}_{j,k}(s) = \frac{(\beta^{(i-1)}_{j,k})'(s)}{2s} \qquad i = 1, 2, \dots .$$

We integrate repeatedly by parts in (1.2.10), and obtain, for every $m \in \mathbb{N}$,

$$\begin{aligned}
\int_0^1 \alpha^k_j(s)\, s^{n+2j+2m-1}\, \mathrm{d}s &= \Big[\frac{\beta^{(0)}_{j,k}(s)\, s^{n+2j+2m}}{n+2j+2m}\Big]_0^1 - \frac{2}{n+2j+2m}\int_0^1 \beta^{(1)}_{j,k}(s)\, s^{n+2j+2m+1}\, \mathrm{d}s \\
&= \frac{\beta^{(0)}_{j,k}(1)}{n+2j+2m} - \frac{2}{n+2j+2m}\int_0^1 \beta^{(1)}_{j,k}(s)\, s^{n+2j+2m+1}\, \mathrm{d}s \\
&= \dots \\
&= \sum_{i=0}^{k} \frac{(-1)^i\, 2^i\, \beta^{(i)}_{j,k}(1)}{\prod_{l=0}^{i}(n+2j+2(m+l))}.
\end{aligned}$$

We claim that

$$\beta^{(0)}_{j,k}(1) = \frac{2^k\, k!}{\prod_{l=k}^{2k-1}(n+2j+2l)}. \tag{1.3.9}$$

For the rest of this proof, we set $\eta_l := n + 2j + 2l$. Clearly, $\beta^{(0)}_{j,k}(1) = 1 + \sum_{i=0}^{k-1} C^k_i$. To prove the claim, we first show that

$$\frac{\beta^0_{j,k}(1)}{2^N\, k \dots (k-N+1)} = \sum_{i=0}^{k-N-1} (-1)^{k-N-i}\binom{k-N}{i}\frac{\prod_{l=i+N}^{k-1}\eta_l}{\prod_{l=k+i}^{2k-1}\eta_l} + \frac{1}{\prod_{l=2k-N}^{2k-1}\eta_l} \tag{1.3.10}$$

for each $N \in \{1, \dots, k-1\}$. We argue by finite induction on N. Consider $1+\sum_i C^k_i$. It is an elementary fact that $\binom{k}{i}$, which is one of the factors that appear in formula (1.2.9) for C^k_i, is equal to $\binom{k-1}{i}$ if $i = 0$, and decomposes as $\binom{k-1}{i} + \binom{k-1}{i-1}$ if $i \geq 1$. Observe that $\binom{k-1}{i}$ is also one of the two binomial coefficients which give the analogous decomposition of $\binom{k}{i+1}$, that appears in C^k_{i+1}. Moreover, C^k_i and C^k_{i+1} share $\prod_{l=i+1}^{k-1}\eta_l / \prod_{l=k+i+1}^{2k-1}\eta_l$. It is then convenient to sum up by collecting like terms, yielding

$$\sum_{i=0}^{k-1} C_i^k + 1 = \sum_{i=0}^{k-2} (-1)^{k-i-1} \binom{k-1}{i} \frac{\prod_{l=i+1}^{k-1} \eta_l}{\prod_{l=k+i+1}^{2k-1} \eta_l} \left[1 - \frac{n+2j+2i}{n+2j+2(k+i)}\right] + 1 - \frac{n+2j+2(k-1)}{n+2j+2(2k-1)}$$
$$= \sum_{i=0}^{k-2} (-1)^{k-i-1} \binom{k-1}{i} \frac{2k \prod_{l=i+1}^{k-1} \eta_l}{\prod_{l=k+i}^{2k-1} \eta_l} + \frac{2k}{n+2j+2(2k-1)},$$

which gives (1.3.10) for $N = 1$.

The inductive step may be completed by similar calculations. For, consider the right hand side of (1.3.10). We write $\binom{k-N-1}{0}$ instead of $\binom{k-N}{0}$ and $\binom{k-N-1}{i} + \binom{k-N-1}{i-1}$ instead of $\binom{k-N}{i}$. Now, for each $i = 0, \ldots, k-N-2$, the expression $\binom{k-N-1}{i}$ appears two times, the first multiplied by $(-1)^{k-N-i} \prod_{l=i+N}^{k-1} \eta_l / \prod_{i=k+i}^{2k-1} \eta_l$, the second by $(-1)^{k-N-i-1} \prod_{l=i+N+1}^{k-1} \eta_l / \prod_{i=k+i+1}^{2k-1} \eta_l$. The sum of these two terms equals

$$(-1)^{k-N-i-1} \binom{k-N-1}{i} \frac{\prod_{l=i+N+1}^{k-1} \eta_l}{\prod_{l=k+i+1}^{2k-1} \eta_l} \left[1 - \frac{n+2j+2(i+N)}{n+2j+2(k+i)}\right],$$

which is

$$(-1)^{k-N-i-1} \binom{k-N-1}{i} \frac{2(k-N) \prod_{l=i+N+1}^{k-1} \eta_l}{\prod_{l=k+i}^{2k-1} \eta_l}.$$

Similarly, we sum up $-(n+2j+2(k-1))/\prod_{l=2k-N-1}^{2k-1} \eta_l$, which comes from the decomposition of the term corresponding to $i = k-N-1$, and $1/\prod_{l=2k-N}^{2k-1} \eta_l$, obtaining

$$\frac{1}{\prod_{l=2k-N}^{2k-1} \eta_l} \left[1 - \frac{n+2j+2(k-1)}{n+2j+2(2k-N-1)}\right] = \frac{2(k-N)}{\prod_{l=2k-N-1}^{2k-1} \eta_l}.$$

It follows that

$$\frac{1 + \sum_{i=0}^{k-1} C_i^k}{2^N k \ldots (k-N+1)}$$
$$= \sum_{i=0}^{k-N-2} (-1)^{k-N-i-1} \binom{k-N-1}{i} \frac{2(k-N) \prod_{l=i+N+1}^{k-1} \eta_l}{\prod_{l=k+i}^{2k-1} \eta_l} + \frac{2(k-N)}{\prod_{l=2k-N-1}^{2k-1} \eta_l},$$

which is (1.3.10) for $N + 1$.

Now, the claim follows easily. Indeed, by (1.3.10) with $N = k-1$,

$$\begin{aligned}1+\sum_{i=0}^{k-1} C_i^k &= 2^{k-1}\, k!\Big(-\frac{n+2j+2(k-1)}{\prod_{l=k}^{2k-1}\eta_l}+\frac{1}{\prod_{l=k+1}^{2k-1}\eta_l}\Big)\\ &= \frac{2^{k-1}\, k!}{\prod_{l=k+1}^{2k-1}\eta_l}\Big(1-\frac{n+2j+2(k-1)}{n+2j+2k}\Big)\\ &= \frac{2^k\, k!}{\prod_{l=k}^{2k-1}\eta_l},\end{aligned}$$

and (1.3.9) is proved.

Similar arguments lead to

$$\beta_{j,k}^{(i)}(1) = \frac{2^{k-i}\, k!\,\binom{k}{i}}{\prod_{l=k+i}^{2k-1}\eta_l} \qquad \forall i\in\{1,\dots,k-1\}. \tag{1.3.11}$$

For the sake of completeness we give a sketch of the proof of this fact as well.
Similarly as before, we need to prove the auxiliary relation

$$\begin{aligned}\frac{\beta_{j,k}^{(i)}(1)}{k\dots(k-i+1)} &= 2^N\, k\dots(k-N+1)\\ &\times\left\{\sum_{h=i}^{k-N-1}(-1)^{k-N-h}\binom{k-i-N}{h-i}\frac{\prod_{l=h+N}^{k-1}\eta_l}{\prod_{l=k+h}^{2k-1}\eta_l}+\frac{1}{\prod_{l=2k-N}^{2k-1}\eta_l}\right\}\end{aligned} \tag{1.3.12}$$

for each $N\in\{1,\dots,k-i-1\}$.
Once again, we proceed by finite induction on N. To prove (1.3.12) for $N=1$, we observe that clearly

$$\beta_{j,k}^{(i)}(1) = k\dots(k-i+1)+\sum_{h=i}^{k-1} h\dots(h-i+1)\, C_h^k.$$

An easy computation shows that, for each $h\in\{i,\dots,k-1\}$, the coefficient $h\dots(h-i+1)$, multiplied by $\binom{k}{h}$ (which appears in C_h^k), gives $k\dots(k-i+1)\binom{k-i}{h-i}$. It follows that

$$\frac{\beta_{j,k}^{(i)}(1)}{k\dots(k-i+1)} = 1+\sum_{h=i}^{k-1}(-1)^{k-h}\binom{k-i}{h-i}\frac{\prod_{l=h}^{k-1}\eta_l}{\prod_{l=k+h}^{2k-1}\eta_l}.$$

We may now write $\binom{k-i-1}{0}$ instead of $\binom{k-i}{0}$, decompose $\binom{k-i}{h-i}$ as $\binom{k-i-1}{h-i}+\binom{k-i-1}{h-i-1}$ for h in $\{i+1,\dots,k-1\}$, and sum up by collecting like terms similarly as before. We obtain

$$\frac{\beta_{j,k}^{(i)}(1)}{k\dots(k-i+1)} = 2k\left\{\sum_{h=i}^{k-2}(-1)^{k-h-1}\binom{k-i-1}{h-i}\frac{\prod_{l=h+1}^{k-1}\eta_l}{\prod_{l=k+h}^{2k-1}\eta_l}+\frac{1}{n+2j+2(2k-1)}\right\},$$

which is (1.3.12) for $N = 1$.
The inductive step can be proved by a similar argument. We omit the details.
Now, by (1.3.12) with $N = k - i - 1$,

$$\begin{aligned}
\frac{\beta_{j,k}^{(i)}(1)}{k\ldots(k-i+1)} &= 2^{k-i-1}\,k\ldots(i+2)\left\{-\frac{n+2j+2(k-1)}{\prod_{l=k+i}^{2k-1}\eta_l}+\frac{1}{\prod_{l=k+i+1}^{2k-1}\eta_l}\right\}\\
&= \frac{2^{k-i-1}\,k\ldots(i+2)}{\prod_{l=k+i+1}^{2k-1}\eta_l}\left(1-\frac{n+2j+2(k-1)}{n+2j+2(k+i)}\right)\\
&= \frac{2^{k-i}\,k\ldots(i+1)}{\prod_{l=k+i}^{2k-1}\eta_l}.
\end{aligned}$$

To conclude the proof of (1.3.11), it suffices to observe that

$$k\ldots(k-i+1)\,k\ldots(i+1) = k!\,\frac{k!}{(k-i)!\,i!} = k!\binom{k}{i}.$$

Finally, it is easy to see that $\beta_{j,k}^{(k)}(1) = k!$. This, together with (1.3.9) and (1.3.11), gives

$$\begin{aligned}
&\int_0^1 \alpha_j^k(s)\,s^{n+2j+2m-1}\,\mathrm{d}s\\
&= \frac{2^k\,k!}{n+2j+2m}\left\{\frac{\binom{k}{0}}{\prod_{l=k}^{2k-1}\eta_l}\right.\\
&\qquad\left.+\sum_{i=1}^{k-1}\frac{(-1)^i\binom{k}{i}}{\prod_{l=k+i}^{2k-1}\eta_l\,\prod_{l=1}^{i}(n+2j+2(m+l))}+\frac{(-1)^k\binom{k}{k}}{\prod_{l=1}^{k}(n+2j+2(m+l))}\right\}.
\end{aligned}\tag{1.3.13}$$

We now use a procedure similar to that used to calculate (1.3.9) and (1.3.11) to reduce further this expression. We claim that

$$\begin{aligned}
&\int_0^1 \alpha_j^k(s)\,s^{n+2j+2m-1}\,\mathrm{d}s\\
&= \frac{2^k\,k!\,2^N\,(k-M-1)\ldots(k-M-N)}{n+2j+2m}\\
&\qquad\times\sum_{i=0}^{k-N}\frac{(-1)^{i+N}\binom{k-N}{i}}{\prod_{l=k+i}^{2k-1}\eta_l\,\prod_{l=1}^{i+N}(n+2j+2(m+l))}
\end{aligned}\tag{1.3.14}$$

for each $N \in \{1,\ldots,k\}$.
We first prove the claim for $N = 1$. Once more, for each $i \in \{1,\ldots,k-1\}$, we decompose the term corresponding to i in (1.3.13) into the sum of two terms, by writing $\binom{k-1}{i}+\binom{k-1}{i-1}$

instead of $\binom{k}{i}$. Moreover, we write $\binom{k-1}{0}$ instead of $\binom{k}{0}$ and $\binom{k-1}{k-1}$ instead of $\binom{k}{k}$. We then sum up $\binom{k-1}{0}/\prod_{l=k}^{2k-1}\eta_l$ and the second term obtained by the decomposition of the term associated to $i=1$, and obtain

$$\frac{-\binom{k-1}{0}}{\prod_{l=k+1}^{2k-1}\eta_l}\left(\frac{1}{n+2j+2k}-\frac{1}{n+2j+2(m+1)}\right),$$

which equals

$$\frac{-\binom{k-1}{0}\,2(k-m-1)}{\prod_{l=k}^{2k-1}\eta_l\,(n+2j+2(m+1))}.$$

Similarly, for any $i\in\{1,\dots,k-2\}$, we sum up the first term from the decomposition of the term corresponding to i and the second term from the decomposition of the one corresponding to $i+1$, yielding

$$\frac{(-1)^{i+1}\binom{k-1}{i}}{\prod_{l=k+i+1}^{2k-1}\eta_l\,\prod_{l=1}^{i}(n+2j+2(m+l))}\left(\frac{1}{n+2j+2(m+i+1)}-\frac{1}{n+2j+2(k+i)}\right),$$

which is equal to

$$\frac{(-1)^{i+1}\binom{k-1}{i}\,2(k-m-1)}{\prod_{l=k+i}^{2k-1}\eta_l\,\prod_{l=1}^{i+1}(n+2j+2(m+l))}.$$

Finally, the binomial coefficient $\binom{k-1}{k-1}$ appears in one of the two terms in which the term corresponding to $i=k-1$ is decomposed. We sum up such term with the last term in (1.3.13), and we obtain

$$\frac{(-1)^k\binom{k-1}{k-1}}{\prod_{l=1}^{k-1}(n+2j+2(m+l))}\left(\frac{1}{n+2j+2(m+k)}-\frac{1}{n+2j+2(2k-1)}\right),$$

which is

$$\frac{(-1)^k\binom{k-1}{k-1}\,2(k-m-1)}{(n+2j+2(2k-1))\,\prod_{l=1}^{k}(n+2j+2(m+l))}.$$

Eventually, we obtain

$$\int_0^1\alpha_j^k(s)\,s^{n+2j+2m-1}\,\mathrm{d}s=2(k-m-1)\left\{\sum_{i=0}^{k-1}\frac{(-1)^{i+1}\binom{k-1}{i}}{\prod_{l=k+i}^{2k-1}\eta_l\,\prod_{l=1}^{i+1}(n+2j+2(m+l))}\right\},$$

which is (1.3.14) for $N=1$.

The proof of the fact that, if (1.3.14) holds for a certain N, then it also holds for $N+1$, is similar and is omitted.

In particular, (1.3.14) with $N = k$ gives

$$\int_0^1 \alpha_j^k(s)\, s^{n+2j+2m-1}\, \mathrm{d}s = \frac{2^{2k}\, k!\, \prod_{l=0}^{k-1}(m-l)}{\prod_{l=k}^{2k-1}\eta_l \prod_{l=0}^{k}(n+2j+2(m+l))}, \tag{1.3.15}$$

which clearly vanishes if $m \in \{0, \ldots, k-1\}$. This concludes the proof of (1.2.10) and of the lemma. □

1.4 Estimates for generalized Bergman kernels

In this section we prove pointwise estimates for the reproducing kernel R_B^k of $b_k^2(B)$ that generalise those obtained by B.R. Choe, H. Koo and H. Yi [CKY, Theorem 2.1] for R_B^1. It is worth mentioning that H. Kang and Koo proved similar estimates for the harmonic Bergman kernel on any smooth bounded domain in $\mathbb{R}^n$ [KK, Theorem 1.1]. They adapted to this setting a method, developed by A. Nagel, J.P. Rosay, E.M. Stein and S. Wainger [NRSW] in the setting of several complex variables, based on some careful estimates for the Green operator associated to the Dirichlet problem for the biharmonic equation. It is an interesting and open question whether this method can be pushed to give sharp estimates for generalized Bergman kernels on smooth domains of $\mathbb{R}^n$.

Our estimates, proved in Theorem 1.4.3 below, will be the key to prove mapping properties of the generalised Bergman projections (see Theorem 1.4.7 below) and for later developements concerning Hardy spaces (see Section 1.5).

For the sake of brevity, it is convenient to set, for every x and y in B_1,

$$\rho(x,y) := \sqrt{1 - 2x\cdot y + |x|^2\,|y|^2}, \qquad \theta(x,y) := d(x,\partial B_1) + d(y,\partial B_1) + |x-y|$$

and

$$\xi(x,y) = 1 - |x|^2\,|y|^2\,.$$

Define the *extended Poisson kernel*, by

$$P(x,y) = \sum_{j=0}^{\infty} \widetilde{Z}_j(x,y) \qquad \forall x,y \in B_1. \tag{1.4.1}$$

It is known [ABR, Formula 8.11] that

$$P(x,y) = \frac{\xi(x,y)}{\rho(x,y)^n} \qquad \forall x,y \in B_1. \tag{1.4.2}$$

Furthermore [ABR, Theorem 8.13]

$$R_{B_1}(x,y) = \frac{\rho(x,y)^{-n}}{n\,c(n)}\,\Big(\frac{n\,\xi(x,y)^2}{\rho(x,y)^2} - 4\,|x|^2\,|y|^2\Big).$$

We shall need the following technical lemma.

Lemma 1.4.1. *The following hold:*

(i) for each pair of multiindices α and β there exists a constant C such that

$$\big|D_x^\alpha D_y^\beta R^1_{B_1}(x,y)\big| \le C\,\rho(x,y)^{-(n+|\alpha+\beta|)} \qquad \forall x,y \in B_1; \tag{1.4.3}$$

(ii) for every x and y in B_1

$$\frac{1}{\sqrt{6}}\,\theta(x,y) \le \rho(x,y) \le \sqrt{2}\,\theta(x,y);$$

(iii) the polynomial α_j^h may be written as

$$\alpha_j^h(x) = \sum_{i=0}^{h} (-1)^i\,A_i^h\,(1-|x|^2)^i, \tag{1.4.4}$$

where

$$A_i^h = A_i^h(j,n) = \binom{h}{i}\,\frac{2^{h-i}\,h\ldots(i+1)}{\prod_{l=h+i}^{2h-1}(n+2j+2l)} \qquad \forall i \in \{0,\ldots,h-1\}. \tag{1.4.5}$$

Proof. First we prove (i) in the case where $\alpha=\beta=0$. Observe that

$$1 - 2x\cdot y + |x|^2\,|y|^2 \ge 1 - 2\,|x|\,|y| + |x|^2\,|y|^2 = (1-|x|\,|y|)^2.$$

Hence

$$\frac{\xi(x,y)}{\rho(x,y)} \le 1 + |x|\,|y| \le 2 \tag{1.4.6}$$

and the required estimate for $R^1_{B_1}$ follows. We refer to [CKY, Theorem 2.1] for the case $|\alpha+\beta|>0$.

Next we prove (ii). On the one hand

$$\begin{aligned}
\rho(x,y)^2 &= (1-|x|^2)(1-|y|^2) + |x-y|^2 \\
&\le (1+|x|)\,(1+|y|)\,\frac{1}{2}\,\big[(1-|x|)^2 + (1-|y|)^2\big] + |x-y|^2 \\
&\le 2\,[(1-|x|)^2 + (1-|y|)^2 + |x-y|^2] \\
&\le 2\,\theta(x,y)^2,
\end{aligned}$$

from which the required right hand inequality follows by taking square roots of both sides.

On the other hand

$$|x-y|^2 \geq \big||x|-|y|\big|^2 = \big|(1-|y|)-(1-|x|)\big|^2.$$

Therefore

$$\begin{aligned}
\rho(x,y)^2 &= (1-|x|^2)(1-|y|^2) + |x-y|^2 \\
&= (1-|x|^2)(1-|y|^2) + \frac{1}{2}\,|x-y|^2 + \frac{1}{2}\,|x-y|^2 \\
&\geq (1-|x|)(1-|y|) + \frac{1}{2}\,\big[(1-|x|)^2 + (1-|y|)^2 - 2\,(1-|x|)(1-|y|)\big] + \frac{1}{2}\,|x-y|^2 \\
&= \frac{1}{2}\,\big[(1-|x|)^2 + (1-|y|)^2 + |x-y|^2\big] \\
&\geq \frac{1}{6}\,\theta(x,y)^2.
\end{aligned}$$

This concludes the proof of (ii).

To prove (iii), observe that

$$\begin{aligned}
\alpha_j^h(x) &= \sum_{m=0}^{h-1} C_m^h\,(|x|^2-1+1)^m + (|x|^2-1+1)^h \\
&= \sum_{m=0}^{h-1} C_m^h \sum_{i=0}^{m}(-1)^i\binom{m}{i}(1-|x|^2)^i + \sum_{i=0}^{h}(-1)^i\binom{h}{i}(1-|x|^2)^i \qquad (1.4.7)\\
&= \sum_{i=0}^{h-1}(-1)^i\Big[\sum_{m=i}^{h-1} C_m^h\binom{m}{i} + \binom{h}{i}\Big](1-|x|^2)^i + (-1)^h(1-|x|^2)^h.
\end{aligned}$$

For the rest of this proof, we set $\eta_l := n+2j+2l$. The computations are similar to those in the proof of Lemma 1.2.6. We *claim* that, for any $N \in \{1,\dots,h-i-1\}$,

$$\frac{A_i^h}{\binom{h}{i}\,2^N\,h\ldots(h-N+1)} = \sum_{m=i}^{h-N-1}(-1)^{h-N-m}\binom{h-N-i}{m-i}\frac{\prod_{l=m+N}^{h-1}\eta_l}{\prod_{l=h+m}^{2h-1}\eta_l} + \frac{1}{\prod_{l=2h-N}^{2h-1}\eta_l}. \qquad (1.4.8)$$

We first prove (1.4.8) for $N=1$. We recall that $C_m^h = (-1)^{h-m}\binom{h}{m}\prod_{l=m}^{h-1}\eta_l/\prod_{l=h+m}^{2h-1}\eta_l$, and

$$\binom{h}{m}\binom{m}{i} = \binom{h}{i}\frac{1}{(m-i)!\,(h-m)!} = \binom{h}{i}\binom{h-i}{m-i}.$$

Then,

$$A_i^h = \binom{h}{i}\left\{\sum_{m=i}^{h-1}(-1)^{h-m}\binom{h-i}{m-i}\frac{\prod_{l=m}^{h-1}\eta_l}{\prod_{l=h+m}^{2h-1}\eta_l} + 1\right\}. \qquad (1.4.9)$$

Since $\binom{h-i}{0} = \binom{h-i-1}{0}$, and $\binom{h-i}{m-i} = \binom{h-i-1}{m-i} + \binom{h-i-1}{m-i-1}$ for $m \in \{i+1, \ldots, h-1\}$,

$$A_i^h = \binom{h}{i} \Bigg\{ (-1)^{h-i} \binom{h-i-1}{0} \frac{\prod_{l=i}^{h-1} \eta_l}{\prod_{l=h+i}^{2h-1} \eta_l} + \sum_{m=i+1}^{h-1} (-1)^{h-m} \left[\binom{h-i-1}{m-1} + \binom{h-i-1}{m-i-1} \right] \frac{\prod_{l=m}^{h-1} \eta_l}{\prod_{l=h+m}^{2h-1} \eta_l} + 1 \Bigg\}.$$

Thus, each term in the sum in (1.4.9), except the one corresponding to $m = i$, is decomposed into the sum of two terms, the first one containing $\binom{h-i-1}{m-1}$, the second one containing $\binom{h-i-1}{m-i-1}$. It is easy to see that, for each $m \in \{i+1, \ldots, h-2\}$, the first term from the decomposition of the term corresponding to m and the second one from the decomposition of the term corresponding to $m+1$ contain the same binomial coefficient. Moreover, they share $\prod_{l=m+1}^{h-1} \eta_l / \prod_{l=h+m+1}^{2h-1} \eta_l$. Similar considerations holds for the term corresponding to $m = i$ and the second term corresponding to $m = i+1$. We may then sum up by collecting like terms, and obtain

$$\begin{aligned} A_i^h =& \binom{h}{i} \Bigg\{ \sum_{m=i}^{h-2} (-1)^{h-m-1} \binom{h-i-1}{m-i} \frac{\prod_{l=m+1}^{h-1} \eta_l}{\prod_{l=h+m+1}^{2h-1} \eta_l} \left[1 - \frac{n+2j+2m}{n+2j+2(h+m)} \right] \\ &+1 - \frac{n+2j+2(h-1)}{n+2j+2(2h-1)} \Bigg\} \\ =& \binom{h}{i} 2h \Bigg\{ \sum_{m=i}^{h-1} (-1)^{h-m-1} \binom{h-m-1}{m-i} \frac{\prod_{l=m+1}^{h-1} \eta_l}{\prod_{l=h+m}^{2h-1} \eta_l} + \frac{1}{n+2j+2(2h-1)} \Bigg\}, \end{aligned}$$

which is (1.4.8) with $N = 1$. Next, we assume that (1.4.8) holds for a certain N and we prove it for $N+1$. Similarly as before, we write $\binom{h-i-N-1}{0}$ instead of $\binom{h-i-N}{0}$, and we decompose $\binom{h-i-N}{m-i}$ as $\left[\binom{h-i-N-1}{m-i} + \binom{h-i-N-1}{m-i-1}\right]$. Then, for each $m \in \{i, \ldots, k-N-2\}$, there are two terms containing $\binom{h-i-N-1}{m-i}$, and they also share the common factor $\prod_{l=m+N+1}^{h-1} \eta_l / \prod_{l=h+m+1}^{2h-1} \eta_l$. We sum up, yielding

$$\begin{aligned} &\frac{A_i^h}{\binom{h}{i} 2^N h \ldots (h-N+1)} \\ &= \sum_{m=i}^{h-N-2} (-1)^{h-N-m-1} \binom{h-N-i-1}{m-i} \frac{\prod_{l=m+N+1}^{h-1} \eta_l}{\prod_{l=h+m+1}^{2h-1} \eta_l} \left[1 - \frac{n+2j+2(m+N)}{n+2j+2(h+m)} \right] \\ &\quad + \frac{1}{\prod_{l=2h-N}^{2h-1} \eta_l} \left[1 - \frac{n+2j+2(h-1)}{n+2j+2(2h-N-1)} \right] \\ &= 2(h-N) \Big\{ \sum_{m=i}^{h-N-2} (-1)^{h-N-m-1} \binom{h-N-i-1}{m-i} \frac{\prod_{l=m+N+1}^{h-1} \eta_l}{\prod_{l=h+m}^{2h-1} \eta_l} + \frac{1}{\prod_{l=2h-N-1}^{2h-1} \eta_l} \Big\}. \end{aligned}$$

This concludes the proof of (1.4.8). In particular, (1.4.8) with $N = h - i - 1$ gives

$$\begin{aligned} A_i^h =& \frac{\binom{h}{i} 2^{h-i-1} h \dots (i+2)}{\prod_{l=h+i+1}^{2h-1} \eta_l} \left[1 - \frac{n+2j+2(h-1)}{n+2j+2(h+i)}\right] \\ =& \frac{\binom{h}{i} 2^{h-i} h \dots (i+1)}{\prod_{l=h+i}^{2h-1} \eta_l}, \end{aligned}$$

as required. □

Lemma 1.4.2. *Suppose that (α, β) is a pair of multi-indices. The following hold:*

(i) there exists a positive constant C such that

$$\left|D_x^\alpha D_y^\beta P(x,y)\right| \le C\, \rho(x,y)^{-(n-1+|\alpha+\beta|)} \qquad \forall x, y \in B_1;$$

(ii) for every integer $n \ge 2$ and for every nonnegative integer ν there exists a constant C such that

$$\left|D_x^\alpha D_y^\beta \sum_{j=0}^{\infty} \pi_\nu(j)\, \widetilde{Z}_j(x,y)\right| \le C\, \rho(x,y)^{-(n+\nu-1+|\alpha+\beta|)} \qquad \forall x, y \in B_1 \tag{1.4.10}$$

where $\pi_\nu(j) = 2j\,(2j-1)\dots(2j-\nu+1)$.

Proof. First we prove (i). Recall formula (1.4.2) for the Poisson kernel. Since P is smooth in $B_1 \times B_1$, the required estimate is trivial when $\rho(x,y) \ge 1/2$, so that we may assume that $\rho(x,y) < 1/2$. In particular, x and y are both away from the origin. By (1.4.6),

$$\xi(x,y) \le 2\,\rho(x,y).$$

Since ξ is a polynomial in x and y, the trivial estimate

$$\left|D_x^\alpha D_y^\beta \xi(x,y)\right| \le C \qquad \forall x, y \in B_1$$

holds for every pair of multi-indices α and β such that $|\alpha+\beta| \ge 1$. Note also that there exists a constant C such that

$$\left|D_x^\alpha D_y^\beta \rho^{-n}(x,y)\right| \le C\, \rho(x,y)^{-(n+|\alpha+\beta|)}.$$

(see [CKY, Lemma 2.1]). Therefore, by Leibnitz's rule,

$$\begin{aligned} \left|D_x^\alpha D_y^\beta P(x,y)\right| &\le \sum_{\alpha'+\alpha''=\alpha} \sum_{\beta'+\beta''=\beta} \left|D_x^{\alpha'} D_y^{\beta'} \xi(x,y)\right| \left|D_x^{\alpha''} D_y^{\beta''} \rho^{-n}(x,y)\right| \\ &\le C \sum_{\alpha'+\alpha''=\alpha} \sum_{\beta'+\beta''=\beta} \rho(x,y)^{1-|\alpha'+\beta'|}\, \rho(x,y)^{-n-|\alpha''+\beta''|}, \end{aligned}$$

and (i) is proved.

To prove (ii) we argue as in [ABR, page 179] and write

$$\begin{aligned}
\Big|D_x^\alpha D_y^\beta \sum_{j=0}^{\infty} \pi_\nu(j)\, \widetilde{Z}_j^n(x,y)\Big| &= \Big|D_x^\alpha D_y^\beta \sum_{j=0}^{\infty} \partial_t^\nu \big[t^{2j} \widetilde{Z}_j^n(x,y)\big]_{|t=1}\Big| \\
&= \Big|D_x^\alpha D_y^\beta \partial_t^\nu \sum_{j=0}^{\infty} \widetilde{Z}_j^n(tx,ty)\Big._{|t=1}\Big| \\
&= \big|D_x^\alpha D_y^\beta \partial_t^\nu \big[P(tx,ty)\big]_{|t=1}\big| \\
&\le C \sum_{|\gamma+\delta|=\nu} \big|D_x^{\alpha+\gamma} D_y^{\beta+\delta} P(x,y)\big| \\
&\le C\, \rho(x,y)^{-(n+\nu-1+|\alpha+\beta|)},
\end{aligned}$$

where the last inequality follows from (i). This concludes the proof of (ii) and of the lemma. $\square$

We are now in position to prove the main theorem of this section.

Theorem 1.4.3. *Let B be an open ball in $\mathbb{R}^n$. For any pair (α,β) of multi-indices there exists a constant C such that*

$$\big|D_x^\alpha D_y^\beta R_B^k(x,y)\big| \le C\, \rho(x,y)^{-(n+|\alpha+\beta|)} \qquad \forall x,y \in B. \tag{1.4.11}$$

Proof. By Proposition 1.3.4 and Lemma 1.4.1, it remains to estimate the derivatives of $R_{B_1}^{M_h}$, $h=1,\ldots,k-1$. By (1.3.7) and (1.4.4)

$$R_{B_1}^{M_h}(x,y) = \frac{1}{n\,c(n)} \sum_{j=0}^{\infty} \sigma_j^h \sum_{i_1=0}^{h} A_{i_1}^h(j,n)\,(1-|x|^2)^{i_1}\, \widetilde{Z}_j(x,y) \sum_{i_2=0}^{h} A_{i_2}^h(j,n)\,(1-|y|^2)^{i_2}.$$

Thus, $R_{B_1}^{M_h}(x,y)$ is a finite linear combination of terms of the form

$$(1-|x|^2)^{i_1}\ (1-|y|)^{i_2}\ \sum_{j=0}^{\infty} \sigma_j^h\, A_{i_1}^h(j,n)\, A_{i_2}^h(j,n)\, \widetilde{Z}_j(x,y) \qquad i_1, i_2 \in \{0,\ldots,h\}.$$

Since $R_{B_1}^{M_h}$ is h-harmonic in each variable, it is smooth in $B_1 \times B_1$, hence (1.4.11) is trivially satisfied when x or y are far from ∂B_1. Therefore, we may assume that both x and y are close to ∂B_1. By (1.3.6) and (1.4.5), $\sigma_j^h\, A_{i_1}^h(j,n)\, A_{i_2}^h(j,n)$ is a polynomial of degree i_1+i_2+1. Since the polynomials

$$1, \pi_1(s), \ldots, \pi_\nu(s)$$

form a basis for the vector space of all polynomials of degree at most ν, there exist constants $d_0, \ldots, d_{i_1+i_1+1}$ such that

$$\sigma_j^h \, A_{i_1}^h(j,n) \, A_{i_2}^h(j,n) = \sum_{\ell=0}^{i_1+i_1+1} d_\ell \, \pi_\ell(j).$$

Thus the problem of estimating $R_{B_1}^{M_h}$ is reduced to that of estimating terms of the form

$$T_{i_1,i_2,\ell}(x,y) := (1-|x|^2)^{i_1} \; (1-|y|)^{i_2} \; \sum_{j=0}^{\infty} \pi_\ell(j) \, \widetilde{Z}_j(x,y), \tag{1.4.12}$$

where i_1 and i_2 are in $\{0, \ldots, h\}$ and $\ell \leq i_1 + i_2 + 1$. Note that

$$\left| D_x^\alpha (1-|x|^2)^{i_1} \right| \leq C \, \min\left(\rho(x,y)^{i_1-|\alpha|}, 1\right) \qquad \forall x, y \in B_1, \tag{1.4.13}$$

and that a similar estimate holds for the derivatives of $(1-|y|^2)^{i_2}$. Indeed, if $|\alpha| \geq i_1$, then $\left| D_x^\alpha (1-|x|^2)^{i_1} \right|$ is uniformly bounded in B_1, and if $|\alpha| < i_1$, then

$$\left| D_x^\alpha (1-|x|^2)^{i_1} \right| \leq C \, (1-|x|)^{i_1-|\alpha|} \leq C \, \theta(x,y)^{i_1-|\alpha|}$$

and the required estimate follows from Lemma 1.4.1 (ii).

Finally, by Leibnitz's rule and estimates (1.4.10) and (1.4.13),

$$\begin{aligned}
&\left| D_x^\alpha D_y^\beta T_{i_1,i_2,\ell}(x,y) \right| \\
&\leq \sum_{\alpha'+\alpha''=\alpha} \; \sum_{\beta'+\beta''=\beta} \left| D_x^{\alpha'} (1-|x|^2)^{i_1} \, D_y^{\beta'} (1-|y|)^{i_2-1} \right| \left| D_x^{\alpha''} D_y^{\beta''} \sum_{j=0}^{\infty} \pi_\ell(j) Z_j(x,y) \right| \\
&\leq C \, \rho(x,y)^{i_1-|\alpha'|} \, \rho(x,y)^{i_2-|\beta'|} \, \rho(x,y)^{-(n+\ell-1+|\alpha''+\beta''|)} \\
&\leq C \, \rho(x,y)^{-(n+|\alpha+\beta|)},
\end{aligned}$$

as required to complete the proof of the estimates for $R_{B_1}^k$.

The required estimates for a generic ball B follow from the estimates above and formula (1.3.4). $\square$

In the last part of this section we prove some interesting consequences of Theorem 1.4.3.

Definition 1.4.4. Let B be an open ball in $\mathbb{R}^n$. The orthogonal projection $\mathscr{P}_B^k$ of $L^2(B)$ onto $b_k^2(B)$ is called the *k-harmonic Bergman projection* on B.

Remark 1.4.5. The k-harmonic Bergman projection is given by

$$\mathscr{P}_B^k u(x) = \int_B R_B^k(x,y)\, u(y)\, \mathrm{d}y \qquad \forall x \in B. \tag{1.4.14}$$

Indeed, on the one hand, we already know that $\mathscr{P}_B^k$, restricted to $b_k^2(B)$ is the identity operator, for R_B^k is the reproducing kernel of $b_k^2(B)$.

On the other hand, suppose that v is in $L^2(B)$ and it is orthogonal to $b_k^2(B)$. Then, in particular, v is orthogonal to all k-harmonic polynomials, hence to $R_B^k(x,\cdot)$, for the reproducing kernel is a combination of k-harmonic polynomials (see Propositions 1.3.4 (ii) and 1.3.5).

Remark 1.4.6. Notice that the space of k-harmonic polynomials is dense in $b_k^p(B)$ for all p in $[1,\infty)$. First observe that it suffices to prove the result in the case where $B = B_1$. Next, let u be in $b_k^p(B_1)$. Then its r-dilate u^r, defined by

$$u^r(x) = u(rx) \qquad \forall x \in (1/r)\, B_1,$$

is in $b_k^p((1/r)\, B_1)$. Moreover, u^r is k-harmonic and bounded in a neighbourhood of $\overline{B}_1$. Since u^r tends to u in $b_k^p(B_1)$, it suffices to prove that k-harmonic functions which are bounded in a neighbourhood of B_1 may be approximated in the L^p norm by k-harmonic polynomials. Let v be any such function. By [ABR, Corollary 5.34], v may be approximated uniformly in a neighbourhood of $\overline{B}_1$, hence in the L^p norm.

It is natural to speculate whether $\mathscr{P}_B^k$ extends to a bounded operator on $L^p(B)$ for some $p \in (1,\infty)$. It is known that $\mathscr{P}_B^1$ possesses this property. It may be interesting to notice that the analogous property for general domains Ω (in place of B) in $\mathbb{R}^n$ is false in general. Indeed, there are starlike domains Ω in $\mathbb{R}^n$ with sharp intruding corners for which $\mathscr{P}_\Omega^1$ fails to extend to a bounded operator on $L^p(\Omega)$ for some $p \neq 2$ [CC]. Furthermore, for these starlike domains the Banach dual of $b_1^p(\Omega)$ fails to be $b_1^{p'}(\Omega)$. Here p' denotes the index conjugate to p. Set

$$b_k^{p'}(B)^\perp := \Big\{ v \in b_k^p(B) : \int_B g(x)\, v(x)\, \mathrm{d}x = 0 \ \text{ for all } g \in b_k^{p'}(B) \Big\} \tag{1.4.15}$$

The following result holds.

Theorem 1.4.7. *Let B denote an open ball in $\mathbb{R}^n$ and suppose that p is in $(1,\infty)$. Then*

(i) the k-harmonic Bergman projection $\mathscr{P}_B^k$ extends to a bounded operator on $L^p(B)$;

(ii) the Banach dual of $b^p_k(B)$ is $b^{p'}_k(B)$, where p' denotes the index conjugate to p;

(iii) for every f in $L^p(B)$ there exist unique functions u in $b^p_k(B)$ and v in $b^{p'}_k(B)^\perp$ such that $f = u + v$. Furthermore,

$$u = \mathscr{P}^k_B f \qquad \textit{and} \qquad v = (\mathscr{I} - \mathscr{P}^k_B) f.$$

Proof. Part (i) may be obtained by arguing much as in the proof of [KK, Theorem 4.2]. We omit the details.

Next we prove (ii). Clearly, any function in $b^{p'}_k(B)$ represents a continuous linear functional on $b^p_k(B)$.

Conversely, consider a continuous linear functional λ on $b^p_k(B)$. By the Hahn–Banach theorem, λ has an extension $\widetilde{\lambda}$ to a continuous linear functional on $L^p(B)$. Denote by $f_{\widetilde{\lambda}}$ the function in $L^{p'}(B)$ that represents $\widetilde{\lambda}$. From (i) we deduce that $\mathscr{P}^k_B f_{\widetilde{\lambda}}$ is in $b^{p'}_k(B)$. Clearly, for every k-harmonic polynomial v

$$\begin{aligned}\lambda(v) &= \int_B v(x)\, f_{\widetilde{\lambda}}(x)\,\mathrm{d}x \\ &= \int_B f_{\widetilde{\lambda}}(x)\,\mathrm{d}x \int_B R^k_B(x,y)\, v(y)\,\mathrm{d}y.\end{aligned}$$

By Theorem 1.4.3, for every y in B the function $R^k_B(\cdot, y)$ is in $L^p(B)$. Therefore, by Fubini's theorem, we may interchange the order of integration and obtain

$$\begin{aligned}\lambda(v) &= \int_B v(y)\,\mathrm{d}y \int_B f_{\widetilde{\lambda}}(x)\, R^k_B(x,y)\ \mathrm{d}x \\ &= \int_B v(y)\, \mathscr{P}^k_B f_{\widetilde{\lambda}}(y)\,\mathrm{d}y.\end{aligned}$$

Thus, the restriction of λ to the space of k-harmonic polynomials is represented by $\mathscr{P}^k_B f_{\widetilde{\lambda}}$. Since the space of k-harmonic polynomials is dense in $b^p_k(B)$ by Remark 1.4.6, the function $\mathscr{P}^k_B f_{\widetilde{\lambda}}$ represents λ on $b^p_k(B)$. This proves (ii).

Part (iii) is a routine consequence of (ii). We omit the details. □

Corollary 1.4.8. *Let p be in $(1,\infty)$, and denote by p' its conjugate index. There exists a constant C such that for every ball B in $\mathbb{R}^n$*

$$\|R^k_B(x,\cdot)\|_p \leq C\, d(x,\partial B)^{-n/p'} \qquad \forall x \in B \tag{1.4.16}$$

and

$$\|D^\alpha_x R^k_B(x,\cdot)\|_p \leq C\, d(x,\partial B)^{-|\alpha|-n/p'} \qquad \forall x \in B. \tag{1.4.17}$$

Proof. By the Hahn–Banach Theorem, the evaluation functional Λ_x on $b^p_k(B)$ extends to a continuous linear functional $\widetilde{\Lambda}_x$ on $L^p(B)$ such that $\|\widetilde{\Lambda}_x\|_{L^p(B)} = \|\Lambda_x\|_{b^p_k(B)}$. Denote by h the function in $L^{p'}(B)$ that represents $\widetilde{\Lambda}_x$. By Theorem 1.4.7 (iii) (with the role of p and p' interchanged) $h = \mathscr{P}^k_B h + (\mathscr{I} - \mathscr{P}^k_B)h$, with $\mathscr{P}^k_B h$ in $b^{p'}_k(B)$ and $(\mathscr{I} - \mathscr{P}^k_B)h$ in $b^p_k(B)^\perp$. It is straightforward to check that $\mathscr{P}^k_B h$ represents Λ_x, whence $\mathscr{P}^k_B h = R^k_B(x, \cdot)$. Since $\mathscr{P}^k_B$ is bounded on $L^{p'}(B)$ by Theorem 1.4.7 (i),

$$\begin{aligned}
\|R^k_B(x,\cdot)\|_{p'} &\leq \left\|\mathscr{P}^k_B\right\|_{p'} \|h\|_{p'} \\
&= \left\|\mathscr{P}^k_B\right\|_{p'} \left\|\widetilde{\Lambda}_x\right\|_{L^p(B)} \\
&= \left\|\mathscr{P}^k_B\right\|_{p'} \left\|\Lambda_x\right\|_{b^p_k(B)} \\
&\leq C\, d(x, \partial B)^{-n/p'},
\end{aligned}$$

as required.

The estimate (1.4.17) is proved similarly. Indeed, a straightforward consequence of the reproducing formula (1.3.2) is that

$$D^\alpha u(x) = \int_B D^\alpha_x R^k_B(x,y)\, u(y)\, \mathrm{d}y \qquad \forall u \in b^2_k(B).$$

Thus, the continuous linear functional $u \in b^{p'}_k(B) \mapsto D^\alpha u(x)$ is represented by $D^\alpha_x R^k_B(x, \cdot)$, and the required estimate follows from (1.3.1). □

1.5 Application to Hardy spaces

We recall the definition of the atomic Hardy space $H^1(\mathbb{R}^n)$.

Definition 1.5.1. Suppose that $1 < p \leq \infty$. An $H^{1,p}$-*atom* a is a function in $L^p(\mathbb{R}^n)$, supported in a ball $B \in \mathscr{B}$, with the following properties:

(i) $\int_B a(x)\, \mathrm{d}x = 0$;

(ii) $\|a\|_p \leq |B|^{-1/p'}$.

Definition 1.5.2. The *Hardy space* $H^{1,p}(\mathbb{R}^n)$ is the space of all functions f in $L^1(\mathbb{R}^n)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} c_j\, a_j, \tag{1.5.1}$$

where a_j are $H^{1,p}$-atoms, and $\sum_{j=1}^{\infty}|c_j| < \infty$. The norm $\|f\|_{H^1}$ of f is the infimum of $\sum_{j=1}^{\infty}|c_j|$ over all decompositions (2.3.1) of f.

It is a beautiful result of Coifman and Weiss [CW, Theorem A, p. 592] that all the spaces $H^{1,p}(\mathbb{R}^n)$ agree, when $1 < p \leq \infty$, and the corresponding norms are equivalent. We will simply denote them all by $H^1(\mathbb{R}^n)$, endowed with any of the equivalent norms above.

We now define *special atoms*, which satisfy, instead of (i) above, the much stronger cancellation condition of being orthogonal to the generalised harmonic Bergman space $b^p_k(B)$.

Definition 1.5.3. Suppose that k is a positive integer, p is in $(1,\infty)$ and denote by p' its conjugate index. A *special k-atom in L^p* (or *$X^{k,p}$-atom*) associated to the open ball B is a function A in $L^p(B)$ such that

(i) $\int_B A(x)\,q(x)\,\mathrm{d}x = 0$ for each k-harmonic polynomial q;

(ii) $\|A\|_p \leq |B|^{-1/p'}$.

Remark 1.5.4. Note that condition (i) implies that $\int_B A(x)\,\mathrm{d}x = 0$. Thus, an $X^{k,p}$-atom is an H^1-atom.

Definition 1.5.5. We define $X^{k,p}(\mathbb{R}^n)$ as the space of all functions in $L^1(\mathbb{R}^n)$ which admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} c_j\,A_j, \tag{1.5.2}$$

where A_j are $X^{k,p}$-atoms, and $\sum_{j=1}^{\infty}|c_j| < \infty$. We define

$$\|f\|_{X^{k,p}} = \inf \sum_j |\lambda_j|, \tag{1.5.3}$$

the infimum being taken over all representations of f of the form (1.5.2).

Lemma 1.5.6. *Let $p \in (1,\infty)$. There exists a constant C such that for every H^1-atom a there exist a summable sequence of complex numbers $\{c_j\}$ and a sequence $\{A_j\}$ of $X^{k,p}$-atoms such that*

$$a = \sum_j c_j\,A_j \qquad \textit{and} \qquad \sum_j |c_j| \leq C.$$

Proof. The translation $x \mapsto x_0 + x$ maps an atom associated to a ball with centre x_0 and radius t to an atom associated to a ball with centre 0 and radius t. Hence we may assume that B is a ball with centre 0.

First, we assume that that the support of the atom a is contained in the unit ball B_1.

We denote by B_j the ball with centre 0 and radius 2^{j-1} and by $\mathscr{P}_j^k$ the projection onto $b_k^2(B_j)$. Sometimes it will be convenient to write $\mathscr{P}_0^k$ for $\mathscr{I}$, the identity operator. We recall that, by Theorem 1.4.7, each $\mathscr{P}_j^k$ extends to a bounded operator on $L^p(B_1)$. Define

$$A_j = \frac{\mathscr{P}_{j-1}^k a - \mathscr{P}_j^k a}{|B_j|^{1/p'} \, \|\mathscr{P}_{j-1}^k a - \mathscr{P}_j^k a\|_p}.$$

Clearly the support of A_j is contained in B_j and

$$\|A_j\|_p \leq |B_j|^{-1/p'}.$$

Observe also that $\mathscr{P}_j^k a = \mathscr{P}_j^k(\mathscr{P}_{j-1}^k a)$. Now suppose that q is a k-harmonic polynomial in $\mathbb{R}^n$. Then

$$\int_{B_j} \left[\mathscr{P}_{j-1}^k a(x) - \mathscr{P}_j^k a(x)\right] q(x) \, \mathrm{d}x = \int_{B_j} (\mathscr{I} - \mathscr{P}_j^k)(\mathscr{P}_{j-1}^k a)(x) \, q(x) \, \mathrm{d}x$$

By Theorem 1.4.7 (iii), $(\mathscr{I} - \mathscr{P}_j^k)(\mathscr{P}_{j-1}^k a)$ is in $(b_k^{p'}(B_j))^\perp$, so that the last integral vanishes. Thus, A_j is an $X^{k,p}$-atom with support contained in B_j. At least formally, we may write

$$\begin{aligned} a &= a - \mathscr{P}_1^k a + \sum_{j=2}^\infty \left[\mathscr{P}_{j-1}^k a - \mathscr{P}_j^k a\right] \\ &= \sum_{j=1}^\infty c_j \, A_j, \end{aligned}$$

where $c_j = |B_j|^{1/p'} \, \|\mathscr{P}_{j-1} a - \mathscr{P}_j a\|_p$. To conclude the proof of the lemma, it suffices to show that $\sum_{j=1}^\infty |c_j| \leq C$, where C does not depend on the atom a. We denote by R_j^k the reproducing kernel of $b_k^2(B_j)$. Note that

$$\begin{aligned} \mathscr{P}_j^k a(x) &= \int_{B_1} a(y) \, R_j^k(x, y) \, \mathrm{d}y \\ &= \int_{B_1} a(y) \left[R_j^k(x, y) - R_j^k(x, 0)\right] \mathrm{d}y \\ &= \int_0^1 \mathrm{d}t \int_{B_1} a(y) \, \nabla_y R_j^k(x, ty) \cdot y \, \mathrm{d}y. \end{aligned}$$

Hence the generalised Minkowski inequality and (1.4.17) imply that

$$\begin{aligned}
\|\mathscr{P}_j^k a\|_p &\leq \int_0^1 \mathrm{d}t \int_{B_1} |a(y)| \, \|\nabla_y R_j(\cdot, ty)\|_p \, |y| \, \mathrm{d}y \\
&\leq C \int_{B_1} |a(y)| \, d(0, \partial B_j)^{-1-n/p'} \, \mathrm{d}y \qquad (1.5.4) \\
&\leq C \, 2^{-(1+n/p')j}.
\end{aligned}$$

Therefore

$$|c_j| \leq C \, 2^{-j},$$

with C independent of a, so that the sequence $\{c_j\}$ is summable, as required to conclude the proof of the lemma.

Next, suppose that t is positive and that a is an H^1-atom with support contained in $B(0,t)$. It is straightforward to check that the function $a_{1/t}$, defined by

$$a_{1/t}(x) = t^n a(tx),$$

is an H^1-atom with support contained in B_1. Now,

$$\begin{aligned}
\mathscr{P}_j^k(a_{1/t})(x) &= \int_{B_1} t^n \, a(ty) \, R_j^k(x,y) \, \mathrm{d}y \\
&= \int_{B(0,t)} a(y) \, R_j^k(x, y/t) \, \mathrm{d}y \\
&= t^n \int_{B(0,t)} a(y) \, t^{-n} \, R_j^k(tx/t, y/t) \, \mathrm{d}y \\
&= t^n \, \mathscr{P}_{tB_j}^k(a)(tx) \\
&= \left[\mathscr{P}_{tB_j}^k(a)\right]_{1/t}(x),
\end{aligned}$$

so that

$$\mathscr{P}_{tB_j}^k(a) = \left[\mathscr{P}_{B_j}^k(a_{1/t})\right]_t.$$

Then

$$\begin{aligned}
\left\|\mathscr{P}_{tB_j}^k(a)\right\|_p &= t^{-n/p} \left\|\mathscr{P}_{B_j}^k(a_{1/t})\right\|_p \\
&\leq C \, t^{-n/p} \, 2^{-(1+n/p')j},
\end{aligned}$$

where we have used the estimate (2.6.4) in the proof of the lemma. Therefore

$$\begin{aligned}
\left|tB_j\right|^{1/p'} \left\|\mathscr{P}_{tB_{j-1}}^k(a) - \mathscr{P}_{tB_j}^k(a)\right\|_p &\leq C \, (t2^j)^{n/p'} \, t^{-n/p} \, 2^{-(1+n/p')j} \\
&\leq C \, 2^{-j},
\end{aligned}$$

and we proceed as in the case of B_1. $\square$

By Lemma 1.5.6, $\|f\|_{X^{k,p}}$ is finite for every f in $H^1(\mathbb{R}^n)$. Moreover, observe that

$$\|f\|_{H^1} \leq \|f\|_{X^{k,p}}, \tag{1.5.5}$$

for every $X^{k,p}$-atom is also an H^1-atom. It is straightforward to check that $f \mapsto \|f\|_{X^{k,p}}$ is a norm on $H^1(\mathbb{R}^n)$.

Theorem 1.5.7. *Every function f in $H^1(\mathbb{R}^n)$ admits a decomposition of the form*

$$f = \sum_j \lambda_j A_j,$$

where $\{\lambda_j\}$ is a summable sequence and the A_j are $X^{k,p}$-atoms. Furthermore, there exists a positive constant c such that

$$c\,\|f\|_{X^{k,p}} \leq \|f\|_{H^1} \leq \|f\|_{X^{k,p}} \qquad \forall f \in H^1(\mathbb{R}^n).$$

Proof. The right hand inequality has already been proved in (1.5.5). Then the identity map ι is continuous from $H^1(\mathbb{R}^n)$, endowed with the topology induced by the norm $\|\cdot\|_{X^{k,p}}$, to $H^1(\mathbb{R}^n)$, endowed with the topology induced by the norm $\|\cdot\|_{H^1}$. Clearly ι is bijective, so that ι^{-1} is continuous, i.e., the left hand inequality holds. $\square$

Chapter 2

Bergman and Hardy spaces on Riemannian manifolds

2.1 Basic definitions and background material

We consider a connected noncompact Riemannian manifold M with Riemannian measure μ and Laplace–Beltrami operator $\mathscr{L}$. We denote by $\mathscr{B}$ the family of all balls in M. For each B in $\mathscr{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, we denote by $c\,B$ the ball with centre c_B and radius $c\,r_B$.
We shall assume throughout the following:

(i) M possesses the *volume doubling property*, i.e., there exists a positive constant D_0 such that
$$\mu(2B) \leq D_0\,\mu(B) \qquad \forall B \in \mathscr{B}; \tag{2.1.1}$$

(ii) there exist positive constants b and ν such that the *relative Faber–Krahn inequality*
$$\lambda_1(U) \geq \frac{b}{r_B^2}\left(\frac{\mu(B)}{\mu(U)}\right)^{2/\nu} \tag{2.1.2}$$
holds for any $B \in \mathscr{B}$ and for any relatively compact open set $U \subset B$. Here $\lambda_1(U)$ denotes the bottom of the spectrum of the Dirichlet Laplacian $\mathscr{L}_U$ on U.

In $\mathbb{R}^n$ and in many applications, ν is the dimension of M. It is known [Gr1] that every complete noncompact manifold with nonnegative Ricci curvature admits a relative Faber–Krahn inequality. An important result [Gr2, p. 410] states that, under assumption (i) above, (ii) is equivalent to the following diagonal upper estimate for the *heat kernel* $\{h_t\}_{t>0}$ associated to the Laplace-Beltrami operator on M

$$\text{(DUE)} \qquad h_t(x,x) \leq \frac{C}{\mu\big(B(x,\sqrt{t})\big)} \qquad \forall x \in M \quad \forall t > 0.$$

We now describe some noteworthy consequences of the relative Faber–Krahn inequality (2.1.2) concerning mean value inequalities for $\mathscr{L}$-harmonic functions. Recall that a $\mathscr{L}$-*harmonic function* (or simply a *harmonic function* for short) in an open set Ω is a smooth function u such that $\mathscr{L}u = 0$ in Ω. These inequalities will be used in the next section in the study of Bergman spaces and will be the key to obtain estimates for the corresponding Bergman projections.

Preliminarily, we recall that a *subsolution* of the heat equation in $I \times \Omega$, where I is an interval in the real line and Ω is an open subset of M, is a real function u in $\mathscr{C}^2(I \times \Omega)$ such that

$$\frac{\partial u}{\partial t} \leq \mathscr{L}u.$$

Theorem 2.1.1. *Suppose that B is a relatively compact ball in M which admits a Faber–Krahn inequality*

$$\lambda_1(U) \geq a\, \mu(U)^{-2/\nu} \tag{2.1.3}$$

for some positive constants a, ν and for any open subset U of B. Then there exists a constant C, which depends only on ν, such that the following hold:

(i) for any $T > 0$ and for any subsolution $u(t,x)$ of the heat equation in the cylinder $\mathscr{C} = (0,T] \times B$

$$u_+(T, c_B)^2 \leq \frac{C\, a^{-\nu/2}}{\min(\sqrt{T}, r_B)^{\nu+2}} \int_{\mathscr{C}} u_+^2(t,x)\, \mathrm{d}t\, \mathrm{d}\mu(x); \tag{2.1.4}$$

(ii) for every $\mathscr{L}$-harmonic function u on B

$$|u(c_B)|^2 \leq \frac{C\, a^{-\nu/2}}{r_B^{\nu}} \int_B |u|^2 \, \mathrm{d}\mu. \tag{2.1.5}$$

Proof. A proof of (i) may be found in [Gr1, Gr2].

To prove (ii) observe that since u is $\mathscr{L}$-harmonic, u_+ is $\mathscr{L}$-subharmonic. Similarly, since $-u$ is $\mathscr{L}$-harmonic, $(-u)_+$, which is equal to u_-, is $\mathscr{L}$-subharmonic. Then we may apply (i) to both u_+ and u_-, and the required estimate follows by setting $T = r_B^2$. $\square$

Theorem 2.1.2. *Suppose that M possesses the doubling property* (2.1.1) *and that the relative Faber–Krahn inequality* (2.1.2) *holds. Then there exists a constant C such that the following hold:*

(i) for every ball B and every $\mathscr{L}$-harmonic function u in B

$$|u(c_B)|^2 \leq \frac{C}{\mu(B)} \int_B |u|^2 \, \mathrm{d}\mu; \tag{2.1.6}$$

(ii) for every ball B and every $\mathscr{L}$-harmonic function u in B

$$|u(c_B)| \leq \frac{C}{\mu(B)} \int_B |u| \, \mathrm{d}\mu. \tag{2.1.7}$$

Proof. Observe that (i) follows simply by inserting $a = b\, r_B^{-2}\, \mu(B)^{2/\nu}$ in (2.1.5).

Part (ii) is essentially due to Li and Wang [LW]. For the sake of completeness we give full details of the proof. As in [LW], we use an inductive argument.

Set $Q := \mu(B)^{-1} \int_B |u| \, \mathrm{d}\mu$. By the L^2-mean value inequality applied to u on $(1/2)B$ and by the doubling property (2.1.1),

$$\begin{aligned} |u(c_B)|^2 &\leq \frac{C}{\mu(2^{-1}B)} \int_{2^{-1}B} |u|^2 \, \mathrm{d}\mu \\ &\leq \frac{C}{\mu(2^{-1}B)} \sup_{2^{-1}B} |u| \int_{2^{-1}B} |u| \, \mathrm{d}\mu \\ &\leq CQ\, \frac{\mu(B)}{\mu(2^{-1}B)} \sup_{2^{-1}B} |u| \\ &\leq CQ\, D_0 \sup_{2^{-1}B} |u| \,. \end{aligned} \tag{2.1.8}$$

For each positive integer k, set $R_k := \sum_{j=1}^k 2^{-j}$ and $S_k := \sup_{B_k} |u|$, where B_k denotes the ball $R_k\, B$. Note that $\{B_k\}$ is an increasing sequence of balls, which contain B_1 and approach asymptotically B.

We *claim* that

$$|u(c_B)| \leq (CQ)^{R_k}\, D_0^{\sum_{i=1}^k (i+1)2^{-i}}\, S_k^{2^{-k}}, \tag{2.1.9}$$

where C is the same constant as in (2.1.8).

By (2.1.8), the claim holds for $k = 1$.

Assume that (2.1.9) holds for k. Choose x in B_k such that

$$|u(x)| = \sup_{B_k} |u| \,.$$

Observe that

$$B(x, 2^{-k-1} r_B) \subset B_{k+1} \subset B \subset B(x, 2r_B).$$

By the L^2-mean value property (2.1.6) applied to u on $B(x, 2^{-k-1} r_B)$ and the doubling property,

$$\begin{aligned}
S_k^2 = |u(x)|^2 &\leq \frac{C}{\mu(B(x, 2^{-k-1} r_B))} \int_{B(x,2^{-k-1} r_B)} |u|^2 \, \mathrm{d}\mu \\
&\leq \frac{C}{\mu(B(x, 2^{-k-1} r_B))} \sup_{B(x,2^{-k-1} r_B)} |u| \int_{B(x,2^{-k-1} r_B)} |u| \, \mathrm{d}\mu \\
&\leq \frac{C}{\mu(B(x, 2^{-k-1} r_B))} \sup_{B_{k+1}} |u| \int_B |u| \, \mathrm{d}\mu \\
&\leq C D_0^{k+2} \frac{\mu(B)}{\mu(B(x, 2r_B))} Q \, S_{k+1} \\
&\leq C \, Q \, D_0^{k+2} \, S_{k+1}.
\end{aligned}$$

This, together with (2.1.9), gives

$$\begin{aligned}
|u(c_B)| &\leq (CQ)^{R_k} \, D_0^{\sum_{i=1}^{k} (i+1) 2^{-i}} \left[C \, Q \, D_0^{k+2} \, S_{k+1} \right]^{2^{-k-1}} \\
&= (CQ)^{R_{k+1}} \, D_0^{\sum_{i=1}^{k+1} (i+1) 2^{-i}} \, S_{k+1}^{2^{-k-1}},
\end{aligned}$$

which is (2.1.9) for $k+1$.

The required L^1-mean value inequality follows by taking the limit of both sides of (2.1.9) as k tends to ∞. □

2.2 Harmonic Bergman spaces

In this section we define the Hardy-type spaces we shall study in the rest of this chapter. Their definition involves the so-called harmonic Bergman spaces.

Definition 2.2.1. For every $p \in [1, \infty)$ and for every open subset Ω of M, the *Bergman space* $b^p(\Omega)$ is the space of all harmonic functions in $L^p(\Omega)$, i.e., the space of all functions H in $L^p(\Omega)$ such that $\mathscr{L} H = 0$ on Ω.

The Bergman space $b^p(\Omega)$, endowed with the $L^p(\Omega)$ norm, is a closed subspace of $L^p(\Omega)$, hence a Banach space. Indeed, given a Cauchy sequence $\{f_n\}$ in $b^p(\Omega)$, there exists a function

f in $L^p(\Omega)$ such that $\|f - f_n\|_p$ is convergent to 0. To prove that f is harmonic, observe that for every smooth function φ with compact support contained in Ω

$$\begin{aligned}\langle \varphi, \mathscr{L} f\rangle &= \langle \mathscr{L}\varphi, f\rangle \\ &= \lim_{n\to\infty} \langle \mathscr{L}\varphi, f_n\rangle \\ &= \lim_{n\to\infty} \langle \varphi, \mathscr{L} f_n\rangle \\ &= 0.\end{aligned}$$

Hence $\mathscr{L} f = 0$ in Ω in the sense of distributions. By elliptic regularity f is smooth, whence $\mathscr{L} f = 0$ pointwise and f is harmonic.

The spaces $b^p(\Omega)$ have been studied in various settings. For basic properties of $b^p(\Omega)$ when $\Omega \subset \mathbb{R}^n$, and in particular when Ω is an Euclidean ball, see [ABR, CKY] and the first chapter of this thesis. Other interesting results are contained in [KK], where estimates for the Bergman kernel on smooth domains in $\mathbb{R}^n$ are established.

A noteworthy consequence of the L^1-mean value inequality is that the *evaluation functional* λ_x at a point x of a domain Ω, i.e., the linear functional $\lambda_x(u) = u(x)$, is continuous on $b^p(\Omega)$ for any $p \in [1, \infty)$.

Proposition 2.2.2. *There exists a constant C, independent of x in Ω and p in $[1, \infty)$, such that*

$$|\!|\!|\lambda_x|\!|\!|_{b^p(\Omega)} \leq \frac{C^{1/p}}{\mu\big(B(x,R)\big)^{1/p}} \qquad \forall u \in b^p(\Omega),$$

where R denotes the distance of x from $\partial\Omega$.

Proof. Indeed, denote by R the distance of x from $\partial\Omega$. By Theorem 2.1.2 (ii) and Hölder's inequality,

$$\begin{aligned}|u(x)| &\leq \Big[\frac{C}{\mu\big(B(x,R)\big)} \int_{B(x,R)} |u|^p \, \mathrm{d}\mu\Big]^{1/p} \\ &\leq \frac{C^{1/p}}{\mu\big(B(x,R)\big)^{1/p}} \|u\|_{b^p(\Omega)} \qquad \forall u \in b^p(\Omega),\end{aligned} \tag{2.2.1}$$

where C is the constant appearing in (2.1.7), which is independent of u in $b^p(\Omega)$, x in Ω and p in $[1, \infty)$. The required estimate of $|\!|\!|\lambda_x|\!|\!|_{b^p(\Omega)}$ follows. $\square$

In the case where $p = 2$, by the Riesz representation theorem, there exists a unique function $R_\Omega(x, \cdot)$ in $b^2(\Omega)$ that represents the functional λ_x, i.e.,

$$u(x) = \int_\Omega R_\Omega(x,y)\, u(y) \, \mathrm{d}\mu(y) \qquad \forall u \in b^2(\Omega).$$

The function R_Ω is called the *Bergman kernel* for the domain Ω. Quite a few properties of R_Ω may be established by abstract nonsense. We refer the reader to the classical paper of N. Aronszajn [A] for a nice exposition of the theory of reproducing kernels. Recall that $b^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. The orthogonal projection of $L^2(\Omega)$ onto $b^2(\Omega)$ is called the *Bergman projection* and will be denoted by $\mathscr{P}_\Omega$. Various properties of $\mathscr{P}_\Omega$ will play an important role in the sequel. We collect them in the following proposition.

Proposition 2.2.3. *Suppose that M possesses the doubling property* (2.1.1) *and that the relative Faber–Krahn inequality* (2.1.2) *holds. Let Ω be a bounded domain in M. Then the following hold:*

(i) the function R_Ω is real-valued and $R_\Omega(x,y) = R_\Omega(y,x)$ for every x, y in Ω. Furthermore, $R_\Omega(x,\cdot)$ is in $b^2(\Omega)$;

(ii) the orthogonal projection $\mathscr{P}_\Omega$ of $L^2(\Omega)$ onto $b^2(\Omega)$ is given by

$$\mathscr{P}_\Omega f(x) = \int_\Omega R_\Omega(x,y)\, f(y)\,\mathrm{d}\mu(y) \qquad \forall f \in L^2(\Omega);$$

(iii) $\|R_\Omega(x,\cdot)\|_{b^2(\Omega)} = R_\Omega(x,x)^{1/2}$ for every x in Ω.

Proof. The proof of (i) and (ii) is classical and may be found in [A]. The proof of (iii) is straightforward and may be found in [ABR, Chapter 8]. □

Observe that we may write

$$\mathscr{P}_\Omega f(x) = \int_\Omega R_\Omega(x,y)\, f(y)\,\mathrm{d}\mu(y) \qquad \forall f \in L^p(\Omega) \cap L^2(\Omega).$$

It is natural to speculate whether the projection $\mathscr{P}_\Omega$ extends to a bounded operator on $b^p(\Omega)$ for p in $(1,\infty)$ and the Banach dual of $b^p(\Omega)$ is $b^{p'}(\Omega)$. Here p' denotes the index conjugate to p. In Section 1.4, we have already observed that this is not always the case even in $\mathbb{R}^n$ [CC]. We now prove that these pathologies disappear if we consider domains Ω with smooth boundary.

Theorem 2.2.4. *Suppose that M possesses the doubling property* (2.1.1) *and that the relative Faber–Krahn inequality* (2.1.2) *holds. Let Ω be a bounded domain in M with smooth boundary. Then the following hold:*

(i) the Bergman projection extends to a bounded operator on $L^p(\Omega)$ for all p in $(1,\infty)$;

(ii) *the Banach dual of* $b^p(\Omega)$ *is* $b^{p'}(\Omega)$, *where* p' *denotes the index conjugate to* p.

(iii) *for every* f *in* $L^p(\Omega)$ *there exist unique functions* u *in* $b^p_k(\Omega)$ *and* v *in* $b^{p'}_k(\Omega)^\perp$ *such that* $f = u + v$. *Furthermore,*

$$u = \mathscr{P}^k_\Omega f \qquad \textit{and} \qquad v = (\mathscr{I} - \mathscr{P}^k_\Omega)f.$$

Proof. The proof of (i) may be deduced from the biharmonic equation approach using standard results in elliptic theory (see also [CC, p. 700] and the references therein). The result is contained in [Me1]. The proofs of (ii) and (iii) follow the same lines as the proofs of the corresponfing results in the Euclidean case (see Theorem 1.4.7), and are omitted. □

If $R < \mathrm{Inj}_p(M)$, then $\partial B(p,R)$ is a smooth hypersurface in M. Unfortunately, larger open balls in M may not have smooth boundary (think of the case of a cylinder in $\mathbb{R}^3$), so that the theorem above may not be applicable to (some) large balls. For later developments, we shall need to work with open domains, which resemble open balls, but have smooth boundary. Here is the precise definition.

Definition 2.2.5. Suppose that R and ε are positive numbers and p is a point in M. A connected open subset Ω of M with smooth boundary is said to be an *approximate* (R,ε)-*ball with centre* p if there exist two balls B and B' with centre p and radii R and $(1+\varepsilon)R$, respectively, such that

$$B \subseteq \Omega \subseteq B'.$$

Clearly, any open ball with smooth boundary and radius R is an approximate (R,ε)-ball for every $\varepsilon > 0$. A basic question is whether approximate (R,ε)-balls with centre p exist for every positive R and ε and for every p in M. The following proposition answers the question in the positive.

Proposition 2.2.6. *Suppose that* R *and* ε *are positive numbers and* p *is a point in* M. *Then there exists infinitely many approximate* (R,ε)*-balls with centre* p.

Proof. It suffices to prove the result for ε small. Denote by B and $2B$ the balls with centre p and radius R and $2R$, respectively. Since M is assumed to be complete, $\overline{2B}$ is compact. Denote by $\rho_{\varepsilon/100}$ the Gaffney regularised distance with base point p such that

$$\left|\rho_{\varepsilon/100}(x) - d(x,p)\right| \leq 10^{-2}\,\varepsilon \qquad \forall x \in \overline{2B}.$$

Such a distance exists and it is smooth, as shown by M. Gaffney in [Ga]. By Sard's theorem (see, for instance, [Au, Theorem 1.30]), the set of critical values of $\rho_{\varepsilon/100}$ is of null measure in $\big((1+\varepsilon/3)R, (1+2\varepsilon/3)R\big)$. Suppose that c is a noncritical value in this interval, and set

$$\Omega := \{x \in 2B : \rho_{\varepsilon/100}(x) < c\}.$$

Then $\partial\Omega$ is a smooth hypersurface in M. It is straighforward to check that $B \subset \Omega \subset (1+\varepsilon)B$, so that Ω is an approximate (R, ε)-ball with centre p, as required. $\square$

Proposition 2.2.7. *Suppose that M possesses the doubling property* (2.1.1) *and that the relative Faber–Krahn inequality* (2.1.2) *holds. Suppose that c_1 and c_2 are numbers such that $1 < c_1 < c_2$. Then the following hold:*

(i) for each p in $[1, \infty)$ there exists a constant C, independent of B in $\mathscr{B}$ such that for every c in (c_1, c_2) and for every approximate $(cr_B, c_2 - c)$-ball Ω with centre c_B and for every x in B

$$|u(x)| \leq C\,\mu(B)^{-1/p}\,\|u\|_p \qquad \forall u \in b^p(\Omega);$$

(ii) for each p in $(1, \infty)$ there exists a constant C, independent of B in $\mathscr{B}$ such that for every c in (c_1, c_2) and for every approximate $(cr_B, c_2 - c)$-ball Ω with centre c_B

$$\Big[\int_\Omega \big|R_\Omega(x, y)\big|^{p'} \,\mathrm{d}\mu(y)\Big]^{1/p'} \leq C\,\mu(B)^{-1/p} \qquad \forall x \in B, \tag{2.2.2}$$

where p' denotes the index conjugate to p;

(iii) there exists a constant C, independent of B in $\mathscr{B}$ such that for every c in (c_1, c_2) and for every approximate $(cr_B, c_2 - c)$-ball Ω with centre c_B

$$\sup_{y\in\Omega}\big|R_\Omega(x, y)\big| \leq C\,\mu(B)^{-1} \qquad \forall x \in B; \tag{2.2.3}$$

(iv) the projection operator $\mathscr{P}_\Omega$, is bounded from $L^p(B)$ to $L^\infty(\Omega)$ and

$$\sup_{B\in\mathscr{B}} \mu(B)^{1/p}\, \big|\!\big|\!\big|\mathscr{P}_\Omega\big|\!\big|\!\big|_{L^p(B);L^\infty(\Omega)} < \infty.$$

Consequently, $\mathscr{P}_\Omega$ is bounded from $L^p(B)$ to $L^p(\Omega)$ and

$$\sup_{B\in\mathscr{B}} \big|\!\big|\!\big|\mathscr{P}_\Omega\big|\!\big|\!\big|_{L^p(B);L^p(\Omega)} < \infty.$$

Proof. We first prove (i). Since x is in B and Ω is an approximate $(cr_B, c-c_2)$-ball, the distance of x from $\partial\Omega$ is at least $(c-1)r_B$. By (2.2.1),

$$\begin{aligned}
|u(x)| &\leq \frac{C^{1/p}}{\mu\big(B(x,(c-1)r_B)\big)^{1/p}} \, \|u\|_{b^p(\Omega)} \\
&\leq \frac{C^{1/p}}{\mu\big(B(x,4r_B)\big)^{1/p}} \, \frac{\mu\big(B(x,4r_B)\big)^{1/p}}{\mu\big(B(x,(c-1)r_B)\big)^{1/p}} \, \|u\|_{b^p(\Omega)} \\
&\leq \frac{C^{1/p}\, D_0^{k/p}}{\mu\big(B(x,4r_B)\big)^{1/p}} \, \|u\|_{b^p(\Omega)} \qquad \forall u \in b^p(\Omega),
\end{aligned} \tag{2.2.4}$$

where C is the constant appearing in (2.1.7) and k is an integer such that $2^k(c-1) \geq 4$. We have used the doubling property and the inclusion $\Omega \subset B(x, 4r_B)$. This proves (i).

Next we prove (ii). Suppose that x is in Ω. Recall that the evaluation functional λ_x, defined just above formula (2.2.1), is continuous on $b^p(\Omega)$. Clearly,

$$\begin{aligned}
\lambda_x(u) &= u(x) \\
&= \int_\Omega R_\Omega(x,y)\, u(y)\, \mathrm{d}\mu(y) \qquad \forall u \in b^2(\Omega) \cap b^p(\Omega).
\end{aligned}$$

We claim that $b^2(\Omega) \cap b^p(\Omega)$ is dense in $b^p(\Omega)$. Since $b^p(\Omega) \subset b^2(\Omega)$ when $p > 2$, it suffices to consider the case where $1 < p < 2$. Then $b^2(\Omega) \cap b^p(\Omega)$ is just $b^2(\Omega)$. We argue by contradiction. Suppose that $b^2(\Omega)$ is not dense in $b^p(\Omega)$. Then there exists a nontrivial continuous linear functional λ on $b^p(\Omega)$ which vanishes on $b^2(\Omega)$. By Theorem 2.2.4 (ii), there exists a function φ in $b^{p'}(\Omega)$ such that

$$\int_\Omega f\, \varphi \, \mathrm{d}\mu = 0 \qquad \forall f \in b^2(\Omega).$$

Since $b^{p'}(\Omega)$ is contained in $b^2(\Omega)$, the formula above implies that $\varphi \mapsto \int_\Omega f\, \varphi\, \mathrm{d}\mu$ is the null functional on $b^2(\Omega)$. Hence $\varphi = 0$, which contradicts the fact that $\lambda \neq 0$. Note that here we use the fact that $\partial\Omega$ is smooth.

Thus,

$$\begin{aligned}
|||\lambda_x|||_{b^p(\Omega)} &= \sup |\lambda_x(u)| \\
&= \sup \Big|\int_\Omega R_\Omega(x,y)\, u(y)\, \mathrm{d}\mu(y)\Big|,
\end{aligned}$$

where the supremum is taken over all u in $b^2(\Omega) \cap b^p(\Omega)$ with $\|u\|_p \leq 1$. By arguing much as in the proof of Corollary 1.4.8, we may prove that there exists a constant, depending on

p, but not on x in Ω such that

$$\|R_\Omega(x.\cdot)\|_{p'} \leq C \,\big|\!\big|\!\big|\lambda_x\big|\!\big|\!\big|_{b^p(\Omega)},$$

By combining this with (2.2.1), we conclude that

$$\|R_\Omega(x,\cdot)\|_{p'} \leq \frac{C^{1/p}}{\mu\big(B(x,R)\big)^{1/p}},$$

where C is the constant appearing in (2.1.7), which is independent of x in B and p in $(1,\infty)$, and R is the distance between x and $\partial\Omega$. Since R is at least $(c-1)r_B$,

$$\mu\big(B(x,R)\big) \geq C\,\mu(B),$$

by the doubling property. Here C depends on c, but not on B. This concludes the proof of (ii).

Statement (iii) is contained in [Me1], and follows from careful estimates for the Green function associated to the biharmonic equation.

To prove (iv), observe that, by (iii),

$$\begin{aligned}
\big|\mathscr{P}_\Omega f(x)\big| &\leq \int_B \big\|R_\Omega(\cdot,y)\big\|_{L^\infty(\Omega)}\,|f(y)|\,\mathrm{d}\mu(y)\\
&\leq \frac{C}{\mu(B)}\int_B |f(y)|\,\mathrm{d}\mu(y)\\
&\leq C\,\mu(B)^{-1/p}\,\big\|f\big\|_p \qquad \forall f\in L^p(B),
\end{aligned}$$

where C is independent of x in Ω. The first of the two required estimate follows by taking the supremum of both sides with respect to x in Ω.

To prove the second estimate, we observe that

$$\big\|\mathscr{P}_\Omega f\big\|_{L^p(\Omega)} \leq \mu(\Omega)^{1/p}\,\big\|\mathscr{P}_\Omega f\big\|_{L^\infty(\Omega)},$$

and that there exists a constant C, which depends on c_1, c_2, but not on the ball B, such that

$$\mu(\Omega) \leq C\,\mu(B).$$

Therefore,

$$\big\|\mathscr{P}_\Omega f\big\|_{L^p(\Omega)} \leq C^{1/p}\,\mu(B)^{1/p}\,\|f\|_{L^p(B)} \qquad \forall f\in L^p(B),$$

where C is the same as above, as required. $\square$

2.3 Hardy-type spaces

Under the standing assumption that the Riemannian measure μ is doubling, M is a space of homogeneous type in the sense of Coifman and Weiss. We recall the definition of the atomic Hardy space $H^1(M)$.

Definition 2.3.1. Suppose that $1 < p \leq \infty$. An $H^{1,p}$-*atom* a is a function in $L^p(M)$, with support contained in a ball B, with the following properties

(i) $\int_B a \, \mathrm{d}\mu = 0$;

(ii) $\|a\|_p \leq \mu(B)^{-1/p'}$ (where p' denotes the index conjugate to p).

Definition 2.3.2. The *Hardy space* $H^{1,p}(M)$ is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} c_j \, a_j, \tag{2.3.1}$$

where a_j are $H^{1,p}$-atoms and $\sum_{j=1}^{\infty} |c_j| < \infty$. The norm $\|f\|_{H^{1,p}}$ of f is the infimum of $\sum_{j=1}^{\infty} |c_j|$ over all decompositions (2.3.1) of f.

It is a beautiful result of R.R. Coifman and G. Weiss [CW, Theorem A, p. 592] that the spaces $H^{1,p}(M)$ agree, when $1 < p \leq \infty$, and the corresponding norms are equivalent.

We now define *special atoms*, which satisfy, instead of (i) above, the much stronger cancellation condition of being *orthogonal* to the harmonic Bergman space $b^p(B)$. Then a Hardy-type space is defined as in the classical case of Coifman and Weiss, but with special atoms in place of classical atoms.

Definition 2.3.3. Suppose that p is in $(1, \infty)$ and denote by p' its conjugate index. A *special atom in* L^p (or $X^{1,p}$-*atom*) associated to the ball B is a function A in $L^p(M)$, with support contained in B and such that

(i) $\int_M A \, H \, \mathrm{d}\mu = 0$ for all H in $b^{p'}(B)$;

(ii) $\|A\|_p \leq \mu(B)^{-1/p'}$.

Note that condition (i) implies that $\int_M A \, \mathrm{d}\mu = 0$, because $\mathbf{1}_B$ is in $b^{p'}(B)$. Thus, a special atom is an $H^1(M)$-atom.

Definition 2.3.4. The *Hardy-type space* $X^1(M)$ associated to special atoms is the space of all functions f that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} c_j \, a_j, \tag{2.3.2}$$

where a_j are $X^{1,p}$-atoms and $\sum_{j=1}^{\infty} |c_j| < \infty$. The norm $\|f\|_{X^{1,p}}$ of f is the infimum of $\sum_{j=1}^{\infty} |c_j|$ over all decompositions (2.3.2) of f.

It is a nontrivial question to determine whether all the Hardy-type spaces $X^{1,p}(M)$, $1 < p < \infty$, agree. Here is our result, whose proof hinges on a variant of an idea of Coifman and Weiss [CW].

Theorem 2.3.5. *Suppose that M possesses the doubling property* (2.1.1) *and that the relative Faber–Krahn inequality* (2.1.2) *holds. For each p in $(1, \infty)$, the space $X^{1,p}(M)$ agrees with $X^{1,2}(M)$, and their norms are equivalent.*

Proof. Clearly, if $1 < p_1 < p_2 < \infty$, then $X^{1,p_2}(M) \subset X^{1,p_1}(M)$. Thus, it suffices to prove that the reverse containment holds. Clearly, it suffices to show that if A is a X^{1,p_1}-atom, then A admits a representation of the form

$$A = \sum_j \alpha_j a_j, \tag{2.3.3}$$

where each a_j is a X^{1,p_2}-atom and $\sum_j |\alpha_j| \leq D$, with D indipendent of A. The proof of this follows the same lines of the proof of the original result of Coifman and Weiss. However, there are also differences.

Suppose that A is a X^{1,p_1}-atom supported in a ball B and set $A_0 = \mu(B)A$. Observe that

$$\|A_0\|_{p_1}^{p_1} = \mu(B)^{p_1} \|A\|_{p_1}^{p_1} \leq \mu(B)^{p_1 - p_1/p_1'} = \mu(B). \tag{2.3.4}$$

Let α be a positive number and denote by O_α the set $\{x \in M : \mathscr{M} \, |A_0|^{p_1} (x) > \alpha^{p_1}\}$. Here $\mathscr{M}$ denotes the uncentred Hardy–Littlewood maximal operator. We refer the reader to [CW] for all basic properties of the Hardy–Littlewood maximal operator on spaces of homogeneous type. We shall use all the properties we need of $\mathscr{M}$ without further reference to [CW].

Assume that α is so large that O_α is contained in $2B$. Then O_α is a bounded open set, satisfying

$$\mu(O_\alpha) \leq C_0\big(\|A_0\|_{p_1}/\alpha\big)^{p_1} \leq C_0\alpha^{-p_1}\mu(B)$$

for a certain $C_0 > 0$. Hence, if α is large enough, $\mu(O_\alpha) < \mu(B) \leq \mu(M)$ and $M \setminus O_\alpha$ is nonempty. We may therefore perform a Whitney-type decomposition of O_α; there exists a collection of balls $\{B_j : j \in \mathbb{N}\}$, such that

(i) $\bigcup_j B_j = O_\alpha$;

(ii) $B_j^{**} \cap O_\alpha^c \neq \emptyset$;

(iii) each point in M belongs to, at most, N balls B_j^*, i.e., the collection $\{B_j^*\}$ has the finite overlapping property. It is important to note that N does not depend on the set O_α, but depends only on the geometry of M.

Here B_j^* and B_j^{**} are balls with the same centre as B_j and radii $2r_{B_j}$ and $6r_{B_j}$, respectively. Notice that, by (2.1.1), there exists $k_0 > 0$ such that

$$\frac{\mu(B^{**})}{\mu(B)} \leq k_0 \quad \text{for any ball } B \subseteq M.$$

Denote by A_j the function defined by

$$A_j = \frac{\mathbf{1}_{B_j}}{\sum_\ell \mathbf{1}_{B_\ell}} A_0.$$

Clearly the support of A_j is contained in B_j. Let Ω_j denote an approximate $(r_{B_j}, 1)$-ball with centre c_{B_j}. Thus, Ω_j has smooth boundary and satisfies $B_j \subseteq \Omega_j \subseteq B_j^*$. Now, define $b_j := A_j - \mathscr{P}_{\Omega_j}(A_j)$,

$$b := \sum_j b_j \qquad \text{and} \qquad g := A_0\, \mathbf{1}_{\Omega_\alpha^c} + \sum_j \mathscr{P}_{\Omega_j}(A_j).$$

We first show that these equalities are valid in $L^1(M)$. Observe that

$$\|b_j\|_{L^1(\Omega_j)} \leq \big(1 + \|\mathscr{P}_{\Omega_j}\|_{L^1(B_j);L^1(\Omega_j)}\big)\|A_j\|_{L^1(B_j)}.$$

Then, by Proposition 2.2.7(iv), there exists C such that

$$\begin{aligned}\sum_j \|b_j\|_{L^1(\Omega_j)} &\le C \sum_j \|A_0\|_{L^1(B_j)} \\ &\le CN \int_{O_\alpha} |A_0| \,\mathrm{d}\mu \\ &\le CN \|A_0\|_1 \\ &\le CN \|A_0\|_{p_1}\, \mu(B)^{1/p_1'} \\ &\le CN\mu(B).\end{aligned}$$

Note that the decomposition of A_0 into a good part and a bad part is different from the classical, for we use the regularising operators $\mathscr{P}_{\Omega_j}$ instead of the usual averaging operator. Some properties of b and g are the following.

(1) For any $x \in M$, $|g(x)| \le C k_0^{1/p_1} \alpha N$, where C is the constant appearing in Proposition 2.2.7(iii). Indeed, if $x \notin O_\alpha$, by Lebesgue differentiability theorem

$$|g(x)| = |b(x)| \le \left(\mathscr{M}\, |A_0|^{p_1}(x)\right)^{1/p_1} \le \alpha,$$

whereas, if $x \in O_\alpha$, $g(x) = \sum_j \mathscr{P}_{\Omega_j}(A_j)(x)$. Notice that at most N terms in this sum are non-zero. By Proposition 2.2.7(iv), we then have that

$$\begin{aligned}|g(x)| &\le \sum_j \left|\mathscr{P}_{\Omega_j}(A_j)(x)\right| \\ &\le C \sum_j \mu(B_j)^{-1/p_1} \,\|A_j\|_{p_1} \\ &\le C \sum_j \left(\frac{1}{\mu(B_j^{**})} \int_{B_j^{**}} |A_0|^{p_1} \,\mathrm{d}\mu\right)^{1/p_1} \left(\frac{\mu(B_j^{**})}{\mu(B_j)}\right)^{1/p_1} \\ &\le C k_0^{1/p_1} \alpha N.\end{aligned}$$

(2) $\mathrm{supp}(g) \subseteq B^*$. Indeed, we already observed that $O_\alpha \subseteq B^*$. On the other hand, $g = A_0$ on O_α^c, and $\mathrm{supp}(A_0) \subseteq B \subseteq B^*$.

(3) $\mathrm{supp}(b_j) \subseteq \Omega_j \subseteq B_j^*$.

(4) $\int_{B_j^*} b_j\, H \,\mathrm{d}\mu = 0$ for each $H \in b^{p_2'}(B_j^*)$. Indeed,

$$\int_{B_j^*} b_j\, H \,\mathrm{d}\mu = \int_{\Omega_j} b_j\, H \,\mathrm{d}\mu \qquad \forall H \in b^{p_2'}(B_j^*)$$

and every such H belongs to $L^\infty(\Omega_j)$, hence to $b^q(\Omega_j)$ for every q in $[1,\infty]$. Denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega_j)$. Observe that if φ is a smooth function with compact support contained in B_j, and $\psi := \varphi - \mathscr{P}_{\Omega_j}\varphi$, then

$$\begin{aligned}\int_{\Omega_j} \psi\, H \,\mathrm{d}\mu &= \int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)\varphi\, H \,\mathrm{d}\mu \\ &= \big(\varphi, H\big) - \big(\mathscr{P}_{\Omega_j}\varphi, H\big) \\ &= \big(\varphi, H\big) - \big(\varphi, \mathscr{P}_{\Omega_j} H\big) \\ &= 0,\end{aligned}$$

because φ is in $L^2(\Omega_j)$, H belongs to $b^2(\Omega_j)$, $\mathscr{P}_{\Omega_j}$ is self-adjoint in $L^2(\Omega_j)$ and is the identity operator on $b^2(\Omega_j)$.

Now, since A_j is in $L^{p_1}(B_j)$, there exists a sequence $\{\varphi_m^j : m \in \mathbb{N}\}$ of smooth functions with compact support contained in B_j such that $\lim_{m\to\infty} \big\|A_j - \varphi_m^j\big\|_{L^{p_1}(B_j)} = 0$. Then

$$\begin{aligned}&\Big|\int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)(A_j - \varphi_m^j)\, H \,\mathrm{d}\mu\Big| \\ &\leq \big\|\big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)(A_j - \varphi_m^j)\big\|_{L^{p_1}(\Omega_j)}\, \|H\|_{L^{p_1'}(\Omega_j)} \\ &\leq \big(1 + \|\mathscr{P}_{\Omega_j}\|_{L^{p_1}(B_j);L^{p_1}(\Omega_j)}\big)\, \big\|A_j - \varphi_m^j\big\|_{L^{p_1}(B_j)}\, \|H\|_{L^{p_1'}(\Omega_j)},\end{aligned}$$

which tends to 0 as m tends to ∞. Consequently

$$\int_{\Omega_j} b_j\, H \,\mathrm{d}\mu = \lim_{m\to\infty} \int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)\varphi_m^j\, H \,\mathrm{d}\mu = 0.$$

Observe that, by (3) and (4), b_j is a multiple of a X^{1,p_2}-atom supported in B_j^*. It follows that b is a multiple of a X^{1,p_2}-atom supported in B^{**}. Indeed, if H is any function in $b^{p_2'}(B^{**})$, then H is in $L^\infty(\cup_j\Omega_j)$, hence in $L^q(\cup_j\Omega_j)$ for each $q \in [1,\infty]$. We then have that

$$\int_{B^{**}} b\, H \,\mathrm{d}\mu = \sum_j \int_{\Omega_j} b_j\, H \,\mathrm{d}\mu = 0.$$

Since both A_0 and b satisfy the cancellation condition on the ball B^{**}, g satisfies such cancellation condition as well. In particular, by (1) and (2),

$$a_0 := \frac{g}{C k_0^{1/p_1} \alpha N \mu(B^{**})}$$

is a X^{1,p_2} atom, supported in B^{**}.

We now iterate this procedure, performing the same modified Calderón–Zygmund decomposition to each b_j that appears, at each step, in the bad part. Namely, we *claim* that there exists a collection of balls $\{B_{j_l} : j_l \in \mathbb{N}^l, l \in \mathbb{N}\}$ such that, for each $n \geq 1$,

$$A_0 = C k_0^{1/p_1} \alpha N \sum_{l=0}^{n-1} \alpha^l \sum_{j_l \in \mathbb{N}^l} \mu(B_{j_l}^{**}) a_{j_l} + \sum_{j_n \in \mathbb{N}^n} b_{j_n}, \tag{2.3.5}$$

where $\alpha = \alpha(p_1, D_0)$ is sufficiently large, C is the constant that appears in Proposition 2.2.7(iii) and

(I) a_{j_l} is a X^{1,p_2}-atom supported in $B_{j_l}^{**}$, for each $l \in \{0, \dots, n-1\}$;

(II) $\bigcup_{j_n \in \mathbb{N}^n} B_{j_n} \subseteq \{x \in M : \mathscr{M} |A_0|^{p_1}(x) > (\alpha^n/2)^{p_1}\}$;

(III) the collection $\{B_{j_l}^*\}$ has the finite overlapping property, with constant N^l;

(IV) b_{j_n} is supported in $B_{j_n}^*$;

(V) $\int_{B_{j_n}^*} b_{j_n} H \, \mathrm{d}\mu = 0$ for any $H \in b^{p_2'}(B_{j_n}^*)$;

(VI) $|b_{j_n}(x)| \leq |A_0(x)| + C k_0^{1/p_1} \alpha^n \mathbf{1}_{B_{j_n}^*}(x)$;

(VII) $\left(\frac{1}{\mu(B_{j_n}^*)} \int_{B_{j_n}^*} |b_{j_n}|^{p_1} \, \mathrm{d}\mu\right)^{1/p_1} \leq (1 + C) k_0^{1/p_1} \alpha^n$.

The proof of the claim is by induction on n. We start by proving that

$$A_0 = C k_0^{1/p_1} \alpha N \mu(B_0^{**}) a_0 + \sum_j b_j$$

is the required decomposition for $n = 1$. Properties (I), (III), (IV) and (V) have already been established. Since

$$\bigcup_j B_j \subseteq \{x : \mathscr{M} |A_0|^{p_1}(x) > \alpha^{p_1}\} \subseteq \{x : \mathscr{M} |A_0|^{p_1}(x) > (\alpha/2)^{p_1}\},$$

also property (II) holds. Property (VI) follows from Proposition 2.2.7(iv) and the fact that $A_j \leq A_0$. Indeed,

$$\begin{aligned}
|b_j(x)| &\leq |A_j(x)| + \left|\mathscr{P}_{\Omega_j}(A_j)(x)\right| \mathbf{1}_{B_j^*}(x) \\
&\leq |A_0(x)| + C \mu(B_j)^{-1/p_1} \|A_j\|_{L^{p_1}(B_j)} \mathbf{1}_{B_j^*}(x) \\
&\leq |A_0(x)| + C \left(\frac{\mu(B_j^{**})}{\mu(B_j)}\right)^{1/p_1} \left(\frac{1}{\mu(B_j^{**})} \int_{B_j^{**}} |A_0|^{p_1} \, \mathrm{d}\mu\right)^{1/p_1} \mathbf{1}_{B_j^*}(x) \\
&\leq |A_0(x)| + C k_0^{1/p_1} \alpha \mathbf{1}_{B_j^*}(x).
\end{aligned}$$

Finally, (VII) is a consequence of (VI):

$$\begin{aligned}\left(\frac{1}{\mu(B_j^*)}\int_{B_j^*}|b_j|^{p_1}\,\mathrm{d}\mu\right)^{1/p_1} &\leq \left(\frac{1}{\mu(B_j^*)}\int_{B_j^*}|A_0|^{p_1}\,\mathrm{d}\mu\right)^{1/p_1} + Ck_0^{1/p_1}\alpha\\ &\leq (1+C)k_0^{1/p_1}\alpha.\end{aligned}$$

We now assume that (2.3.5) holds for n and we prove it for $n+1$.

For each $j_n \in \mathbb{N}^n$, set

$$O_{j_n} = \left\{x \in M : \mathscr{M}\,|b_{j_n}|^{p_1}(x) > \alpha^{(n+1)p_1}\right\}.$$

Similarly as before, if α is sufficiently large, $O_{j_n} \subseteq 2B_{j_n}^*$ and $M \setminus O_{j_n}$ is non-empty. Then O_{j_n} admits a Whitney-type decomposition and there exists a collection of balls $\left\{B_{j_n,i} : i \in \mathbb{N}\right\}$ satisfying

(i) $\bigcup_i B_{j_n,i} = O_{j_n}$;

(ii) $B_{j_n,i}^{**} \cap O_{j_n}^c \neq \emptyset$;

(iii) for each $x \in M$, $\sum_i \mathbf{1}_{B_{j_n,i}^*}(x) \leq N$.

Let $\Omega_{j_n,i}$ be an approximate $(r_{B_{j_n,i}},1)$-ball with centre $c_{B_{j_n,i}}$ and denote by $\mathscr{P}_{\Omega_{j_n,i}}$ the Bergman projection onto $b^2(\Omega_{j_n,i})$. Let

$$A_{j_n,i} := \frac{\mathbf{1}_{B_{j_n,i}}}{\sum_l \mathbf{1}_{B_{j_n,l}}}\, b_{j_n}.$$

We write

$$b_{j_n} = g_{j_n} + \sum_i b_{j_n,i},$$

where $g_{j_n} := h_{j_n}\mathbf{1}_{O_{j_n}^c} + \sum_i \mathscr{P}_{\Omega_{j_n,i}}(A_{j_n,i})$ and $b_{j_n,i} := A_{j_n,i} - \mathscr{P}_{\Omega_{j_n,i}}(A_{j_n,i})$. Similarly as before, these equalities make sense in $L^1(M)$, and the following hold:

(1)' $|g_{j_n}(x)| \leq Ck_0^{1/p_1}\alpha^{n+1}N$;

(2)' $\mathrm{supp}(g_{j_n}) \subseteq 2B_{j_n}^* \subseteq B_{j_n}^{**}$;

(3)' $\mathrm{supp}(h_{j_n,i}) \subseteq \Omega_{j_n,i} \subseteq B_{j_n,i}^*$;

(4)' $\int_{B_{j_n,i}^*} h_{j_n,i}\, H\,\mathrm{d}\mu = 0$ for each $H \in b^{p_2'}(B_{j_n,i}^*)$.

The proof of these properties is almost identical to the proof of (1), (2), (3) and (4) above, and is omitted. Moreover, we deduce that

$$\int_{B^{**}_{j_n}} g_{j_n}\, H \,\mathrm{d}\mu = 0 \qquad \forall H \in b^{p_2'}(B^{**}_{j_n}).$$

Hence

$$a_{j_n} := \frac{g_{j_n}}{C k_0^{1/p_1} \alpha^{n+1} N \mu(B^{**}_{j_n})}$$

is a X^{1,p_2}-atom supported in $B^{**}_{j_n}$. We then have that

$$\begin{aligned} A_0 &= C k_0^{1/p_1} \alpha N \sum_{l=0}^{n-1} \alpha^l \sum_{j_l \in \mathbb{N}^l} \mu(B^{**}_{j_l}) a_{j_l} + \sum_{j_n \in \mathbb{N}^n} \big(g_{j_n} + \sum_{i \in \mathbb{N}} b_{j_n,i} \big) \\ &= C k_0^{1/p_1} \alpha N \sum_{l=0}^{n} \alpha^l \sum_{j_l \in \mathbb{N}^l} \mu(B^{**}_{j_l}) a_{j_l} + \sum_{j_n \in \mathbb{N}^n} \sum_{i \in \mathbb{N}} b_{j_n,i}. \end{aligned}$$

We now prove that this is (2.3.5) for $n+1$. Properties (I), (IV) and (V) have already been proved. Property (III) follows from the facts that the balls $\{B^*_{j_n} : j_n \in \mathbb{N}^n\}$ are M^n-disjoint and the balls $\{B^*_{j_n,i} : i \in \mathbb{N}\}$ are M-disjoint. To prove (VI), we use the same argument used to prove (VI) for $n = 1$ and the inductive hypothesis. Namely,

$$\begin{aligned} |b_{j_n,i}(x)| &\le |b_{j_n}(x)| + |\mathscr{P}_{j_n,i}(A_{j_n,i})(x)| \; \mathbf{1}_{B^*_{j_n,i}}(x) \\ &\le \big(|b_{j_n}(x)| + C\mu(B_{j_n,i})^{-1/p_1} \|A_{j_n,i}\|_{L^{p_1}(B_{j_n,i})} \big) \, \mathbf{1}_{B^*_{j_n,i}}(x) \\ &\le \big(|b_{j_n}(x)| + C k_0^{1/p_1} \alpha^{n+1} \big) \, \mathbf{1}_{B^*_{j_n,i}}(x) \\ &\le \big(|A_0(x)| + C k_0^{1/p_1} \alpha^n \mathbf{1}_{B^*_{j_n}}(x) + c k_0^{1/p_1} \alpha^{n+1} \big) \, \mathbf{1}_{B^*_{j_n,i}}(x) \\ &\le |A_0(x)| + C k_0^{1/p_1} \alpha^{n+1} \, \mathbf{1}_{B^*_{j_n,i}}(x). \end{aligned}$$

Similarly as before, (VII) follows easily from (VI). It only remains to prove (II). For, observe that if x is in O_{j_n}, then by (VI)

$$\begin{aligned} \alpha^{n+1} &< \big(\mathscr{M} \, |b_{j_n}|^{p_1} (x) \big)^{1/p_1} \\ &\le \big(\mathscr{M} \, |A_0|^{p_1} (x) \big)^{1/p_1} + C k_0^{1/p_1} \alpha^n. \end{aligned}$$

It follows that, if $\alpha > 2Ck_0^{1/p_1}$, $\big(\mathscr{M} \, |A_0|^{p_1} (x) \big)^{1/p_1} > \alpha^{n+1}/2$. Therefore

$$\begin{aligned} \bigcup_{j_n \in \mathbb{N}^n} \Big(\bigcup_{i \in \mathbb{N}} B_{j_n,i} \Big) &= \bigcup_{j_n \in \mathbb{N}^n} O_{j_n} \\ &\subseteq \big\{ x \in M : \mathscr{M} \, |A_0|^{p_1} (x) > \big(\alpha^{n+1}/2 \big)^{p_1} \big\}, \end{aligned}$$

and (II) is proved.

Then, (2.3.5) is valid for any $n \in \mathbb{N}$.

We now prove that (2.3.5) implies (2.3.3), with $\sum_j |\alpha_j| \le D$, D indipendent of A.

First, we prove that there exists $D = D(p_1, D_0, \alpha)$ such that

$$\sum_{n=0}^{\infty} \alpha^n \sum_{j_n \in \mathbb{N}^n} \mu(B^{**}_{j_n}) \le D. \tag{2.3.6}$$

Observe that

$$\begin{aligned}
\sum_{j_n \in \mathbb{N}^n} \mu(B^{**}_{j_n}) &\le k_0 \sum_{j_n \in \mathbb{N}^n} \mu(B_{j_n}) \\
&\le k_0 N^n \mu\Big(\bigcup_{j_n \in \mathbb{N}^n} B_{j_n} \Big) \\
&\le k_0 N^n \mu\Big(\big\{ x \in M : \mathscr{M}\big(|A_0|^{p_1} \big)(x) > \big(\alpha^{n+1}/2\big)^{p_1} \big\} \Big) \\
&\le C_0 k_0 N^n \big(2/\alpha^n\big)^{p_1} \|A_0\|_{p_1}^{p_1}.
\end{aligned}$$

Here we have used (II) and the fact that $\mathscr{M}$ is of weak type 1. It is then enough to choose $\alpha > N^{1-p_1}$ to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \alpha^n \sum_{j_n \in \mathbb{N}^n} \mu(B^{**}_{j_n}) &\le C k_0 2^{p_1} \|A_0\|_{p_1}^{p_1} \sum_{n=0}^{\infty} \big(N\alpha^{1-p_1}\big)^n \\
&\le C k_0 2^{p_1} \mu(B) \sum_{n=0}^{\infty} \big(N\alpha^{1-p_1}\big)^n,
\end{aligned}$$

and (2.3.6) follows.

Next, we prove that

$$A_0 = C k_0^{1/p_1} \alpha N \sum_{n=0}^{\infty} \alpha^n \sum_{j_n \in \mathbb{N}^n} \mu(B^{**}_{j_n}) a_{j_n}, \tag{2.3.7}$$

where the equality is to be interpreted in $L^1(M)$. For any n, set $H_n := \sum_{j_n \in \mathbb{N}^n} b_{j_n}$. We show that $\|H_n\|_1 \to 0$ as $n \to \infty$. We recall that b_{j_n} is supported in $B^*_{j_n}$ and we observe that, by (VII),

$$\begin{aligned}
\int_{B^*_{j_n}} |b_{j_n}| \, \mathrm{d}\mu &\le \Big(\int_{B^*_{j_n}} |b_{j_n}|^{p_1} \, \mathrm{d}\mu \Big)^{1/p_1'} \\
&\le k_0^{1/p_1} \alpha^n \mu(B^*_{j_n}).
\end{aligned}$$

Therefore, by the above estimate for $\sum \mu(B^{**}_{j_n})$,

$$\begin{aligned}\int |H_n| \,\mathrm{d}\mu &\leq \sum_{j_n \in \mathbb{N}^n} \int |b_{j_n}| \,\mathrm{d}\mu \\ &\leq k_0^{1/p_1} \alpha^n \sum_{j_n \in \mathbb{N}^n} \mu(B^*_{j_n}) \\ &\leq C 2^{p_1} k_0^{1+1/p_1} \big(\alpha^{1-p_1} N\big)^n \|A_0\|_{p_1}^{p_1},\end{aligned}$$

which tends to 0 as $n \to \infty$, because $\alpha^{1-p_1} N < 1$. This implies (2.3.7) and concludes the proof of the theorem. □

2.4 Calderón–Zygmund decomposition and interpolation

In this section we will prove that, under the assumptions that M is doubling and admits a relative Faber–Krahn inequality (2.1.2), $L^p(M)$ is an interpolation space between $X^1(M)$ and $L^\infty(M)$. We recall that M is then a space of homogeneous type in the sense of Coifman and Weiss. Our interpolation result will be a consequence of a variant of the classical Calderón–Zygmund decomposition and an argument of J.-L. Journé [J]. Note that this result is essentially known, though the proofs available in the literature are considerably involved (see, for instance, [AMR, HLMMY]). It is fair to say, however, that the proof in [AMR] applies also to Hardy spaces of differential forms, and we do not know whether our ideas can be pushed to give results also for Hardy spaces of differential forms.

Theorem 2.4.1. *Suppose that p is in $(1,\infty)$ and that f is in $L^p(M)$. For every positive number α there exist functions b in $X^1(M)$ and g in $L^\infty(M)$ and a constant C such that $f = b + g$ and*

(i) $|g| \leq C\,\alpha$;

(ii) $\|b\|_{X^1} \leq C\,\alpha\,|\Omega_\alpha|$, *where Ω_α denotes the set $\{\mathscr{M}(|f|^p) > \alpha^p\}$, and $\mathscr{M}$ is the uncentred Hardy–Littlewood maximal operator.*

Proof. We refer the reader to [CW] for all basic properties of the Hardy–Littlewood maximal operator on spaces of homogeneous type. We shall use all the properties we need of $\mathscr{M}$ without further reference to [CW].

Denote by $\{B_j\}$ a sequence of balls contained in Ω_α with the bounded overlapping property and such that

$$\frac{1}{\mu(B_j)}\int_{B_j}|f|^p\,\mathrm{d}\mu > \alpha^p,$$

but

$$\frac{1}{\mu(B_j^*)}\int_{B_j^*}|f|^p\,\mathrm{d}\mu \leq \alpha^p. \tag{2.4.1}$$

Here B_j^* denotes the ball with the same centre as B_j and twice the radius. Clearly

$$f\,\mathbf{1}_{\Omega_\alpha} = \sum_j \frac{\mathbf{1}_{B_j}}{\sum_\ell \mathbf{1}_{B_\ell}}\,f.$$

It will be convenient to set

$$f_j = \frac{\mathbf{1}_{B_j}}{\sum_\ell \mathbf{1}_{B_\ell}}\,f.$$

Note that, by the bounded overlapping property, there exists a positive constant c such that

$$c\,f\,\mathbf{1}_{B_j} \leq f_j \leq f\,\mathbf{1}_{B_j} \qquad j = 1, 2, \ldots.$$

For every positive integer j, denote by Ω_j an approximate $(2r_{B_j}, 10^{-2})$-ball with centre c_{B_j}. We write

$$f_j = f_j - \mathscr{P}_{\Omega_j}(f_j) + \mathscr{P}_{\Omega_j}(f_j),$$

where $\mathscr{P}_{\Omega_j}(f_j)$ denotes the Bergman projection of f_j onto $b^p(\Omega_j)$. Now, define $b_j := f_j - \mathscr{P}_{\Omega_j}(f_j)$,

$$b := \sum_j b_j \qquad \text{and} \qquad g := f\,\mathbf{1}_{\Omega_\alpha^c} + \sum_j \mathscr{P}_{\Omega_j}(f_j). \tag{2.4.2}$$

It is a standard consequence of the Lebesgue differentiability theorem that $|f|^p \leq \alpha^p$ in Ω_α^c. Thus, to prove (i) it suffices to show that there exists a constant C such that

$$\Big\| \sum_j \mathscr{P}_{\Omega_j}(f_j)\Big\|_\infty \leq C\,\alpha.$$

Since the sequence $\{\Omega_j\}$ has the finite overlapping property, it is enough to show that

$$\big\|\mathscr{P}_{\Omega_j}(f_j)\big\|_\infty \leq C\,\alpha,$$

for some constant C, independent of j. By Proposition 2.2.7 (iv),

$$\begin{aligned}\big\|\mathscr{P}_{\Omega_j}(f_j)\big\|_\infty &\leq \big|\!\big|\!\big|\mathscr{P}_{\Omega_j}\big|\!\big|\!\big|_{L^p(B_j);L^\infty(\Omega_j)}\,\big\|f_j\big\|_{L^p(B_j)}\\ &\leq C\,\mu(B_j)^{-1/p}\,\big\|f_j\big\|_{L^p(B_j)}\\ &\leq C\,\alpha,\end{aligned}$$

where we have used the doubling condition and (2.4.1) in the last inequality. This concludes the proof of (i).

Next we prove (ii). We *claim* that b_j is a multiple of a $X^{1,p}$-atom, and that there exists a constant C, independent of j, such that

$$\|b_j\|_{X^1} \leq C\,\mu(B_j^*)\,\alpha. \tag{2.4.3}$$

Indeed, denote by B_j^{**} the ball with centre c_{B_j} and radius $(2+10^{-1})r_{B_j}$. Clearly B_j^{**} contains Ω_j, and the distance between B_j^{**} and $\partial\Omega_j$ is positive. Furthermore

$$\mathrm{supp}(b_j) \subset \Omega_j \subset B_j^{**}.$$

First we prove that

$$\int_{B_j^{**}} b_j\,H\,\mathrm{d}\mu = 0 \qquad \forall H \in b^{p'}(B_j^{**}).$$

Obviously

$$\int_{B_j^{**}} b_j\,H\,\mathrm{d}\mu = \int_{\Omega_j} b_j\,H\,\mathrm{d}\mu \qquad \forall H \in b^{p'}(B_j^{**})$$

and every such H belongs to $L^\infty(\Omega_j)$, hence to $b^q(\Omega_j)$ for every q in $[1,\infty]$. Denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega_j)$. Observe that if φ is a smooth function with compact support contained in B_j, and $\psi := \varphi - \mathscr{P}_{\Omega_j}\varphi$, then

$$\begin{aligned}
\int_{\Omega_j} \psi\,H\,\mathrm{d}\mu &= \int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)\varphi\,H\,\mathrm{d}\mu \\
&= \big(\varphi, H\big) - \big(\mathscr{P}_{\Omega_j}\varphi, H\big) \\
&= \big(\varphi, H\big) - \big(\varphi, \mathscr{P}_{\Omega_j}H\big) \\
&= 0,
\end{aligned}$$

because φ is in $L^2(\Omega_j)$, H belong to $b^2(\Omega_j)$, $\mathscr{P}_{\Omega_j}$ is self-adjoint in $L^2(\Omega_j)$ and is the identity operator on $b^2(\Omega_j)$.

Now, if f_j is any function in $L^p(B_j)$, then there exists a sequence $\{\varphi_n\}$ (which of course depends on j) of smooth functions with compact support contained in B_j such that $\lim_{n\to\infty} \|f_j - \varphi_n\|_{L^p(B_j)} = 0$. Then

$$\begin{aligned}
\Big|\int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)(f_j - \varphi_n)\,H\,\mathrm{d}\mu\Big| &\leq \big\|\big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)(f_j - \varphi_n)\big\|_{L^p(\Omega_j)}\,\|H\|_{L^{p'}(\Omega_j)} \\
&\leq \big(1 + |||\mathscr{P}_{\Omega_j}|||_{L^p(B_j);L^p(\Omega_j)}\big)\,\big\|(f_j - \varphi_n)\big\|_{L^p(B_j)}\,\|H\|_{L^{p'}(\Omega_j)},
\end{aligned}$$

which tends to 0 as n tends to ∞. Consequently

$$\int_{\Omega_j} b_j\, H \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\Omega_j} \big(\mathscr{I} - \mathscr{P}_{\Omega_j}\big)(f_j - \varphi_n)\, H \,\mathrm{d}\mu = 0,$$

which proves that b_j satisfies the cancellation condition on the ball B_j^{**}. Hence b_j is a multiple of a $X^{1,p}$-atom. To prove that b_j satisfies (2.4.3), observe that

$$\begin{aligned}
\Big[\int_{B_j^{**}} |b_j|^p \,\mathrm{d}\mu\Big]^{1/p} &\leq \big(1 + \| \mathscr{P}_{\Omega_j} \|_{L^p(B_j);L^p(\Omega_j)}\big)\, \big\|f_j\big\|_{L^p(B_j)} \\
&\leq C\, \big\|f\big\|_{L^p(B_j)} \\
&\leq C\,\mu(B_j^*)^{1/p}\Big[\frac{1}{\mu(B_j^*)} \int_{B_j^*} |f|^p \,\mathrm{d}\mu\Big]^{1/p} \\
&\leq C\,\mu(B_j^*)^{1/p}\,\alpha;
\end{aligned}$$

the first inequality is a straighforward consequence of the definition of b_j, the second follows from the fact that $\| \mathscr{P}_{B_j^*} \|_p$ is uniformly bounded with respect to j, see Proposition 2.2.7 (iv), and the fourth from (2.4.1). The above estimate implies that $b_j/\big(C\mu(B_j^{**})\,\alpha\big)$ is a $X^{1,p}$-atom. Hence

$$\big\|b_j\big\|_{X^1} \leq C\,\mu(B_j^{**})\,\alpha \leq C\,\mu(B_j^{*})\,\alpha;$$

we have used the doubling condition in the last inequality above. This concludes the proof of the claim.

Now, the claim implies that

$$\begin{aligned}
\sum_j \big\|b_j\big\|_{X^1} &\leq C\,\alpha \sum_j \mu(B_j^*) \\
&\leq C\,\alpha\,\mu(\Omega_\alpha),
\end{aligned}$$

by the bounded overlapping property and the doubling condition.

This concludes the proof (ii), and of the theorem. □

Theorem 2.4.2. *Suppose that $\mathscr{T}$ is a linear operator that is bounded from $X^1(M)$ to $L^1(M)$ and on $L^q(M)$ for some q in $(1,\infty]$. Then $\mathscr{T}$ is bounded on $L^p(M)$ for all p in $(1,q)$.*

Proof. We shall prove that $\mathscr{T}$ is of weak type (p,p) for all p in $(1,q)$. Then the Marcinkiewicz interpolation theorem will imply that $\mathscr{T}$ is, in fact, bounded on $L^p(M)$ for all p in $(1,q)$.

First we assume that $q = \infty$. Fix $\alpha > 0$ and f in $L^p(M)$, and consider a Calderón–Zygmund decomposition $f = b + g$ at height $C'\alpha$ as in the previous theorem, where C' is chosen so small that $\{|\mathscr{T} g| > \alpha/2\}$ is empty. Then clearly

$$\begin{aligned}
\mu\big(\{|\mathscr{T} f| > \alpha\}\big) &\le \mu\big(\{|\mathscr{T} b| > \alpha/2\}\big) + \mu\big(\{|\mathscr{T} g| > \alpha/2\}\big) \\
&\le \frac{2}{\alpha} \int_M \big|\mathscr{T} b\big| \,\mathrm{d}\mu \\
&\le \frac{2}{\alpha} \,|\!|\!|\mathscr{T}|\!|\!|_{X^1;L^1} \,\|b\|_{X^1} \\
&\le \frac{C}{\alpha} \,|\!|\!|\mathscr{T}|\!|\!|_{X^1;L^1} \,\alpha\, \mu(\Omega_\alpha) \\
&\le C \,\frac{\|f\|_p{}^p}{\alpha^p} \,|\!|\!|\mathscr{T}|\!|\!|_{X^1;L^1},
\end{aligned}$$

thereby proving that $\mathscr{T}$ is of weak type (p,p), as required.

Next, assume that $q < \infty$. Fix $\alpha > 0$ and f in $L^p(M)$, and consider a Calderón–Zygmund decomposition $f = b + g$ at height α as in the previous theorem. The measure of the level set $\alpha/2$ of $\mathscr{T} b$ is estimated exactly as above. To estimate the measure of the level set $\alpha/2$ of $\mathscr{T} g$ we proceed as follows:

$$\begin{aligned}
\mu\big(\{|\mathscr{T} g| > \alpha/2\}\big) &\le \frac{2^q}{\alpha^q} \int_M \big|\mathscr{T} g\big|^q \,\mathrm{d}\mu \\
&\le \frac{2^q}{\alpha^q} \,|\!|\!|\mathscr{T}|\!|\!|_q^q \int_M \big|g\big|^p \,\big|g\big|^{q-p} \,\mathrm{d}\mu \\
&\le C \,\frac{\|f\|_p{}^p}{\alpha^p} \,|\!|\!|\mathscr{T}|\!|\!|_q^q;
\end{aligned}$$

in the last inequality we have used the fact that $|g| \le C\,\alpha$ and the estimate $\|g\|_p \le C\,\|f\|_p$. This last inequality follows from the definition of g (see (2.4.2)) and the fact that $\big|\!\big|\!\big|\mathscr{P}_{B_j}\big|\!\big|\!\big|_p$ is uniformly bounded with respect to j.

Thus $\mathscr{T}$ is of weak type (p,p), as required. □

2.5 Spectral multipliers

In this section we prove a multiplier result of Mihlin–Hörmander type for the Laplace–Beltrami operator. We emphasize the fact that its proof is fairly simple and avoids using the theory of singular integral operators. After we completed the proof of this result, X.T. Duong and L. Yan [DY1] proved a more general result concerning estimates for spectral multipliers for operators satisfying a Davies–Gaffney inequality. In fact, when restricted to spectral multipliers of the Laplace–Beltrami operator on manifolds with the doubling property and the relative Faber–Krahn inequality, their result is a strict relative of ours. However, we believe that our proof is much simpler than that of Duong and Yan, and emphasizes the role played by the geometric assumptions on M.

For an open bounded set Ω, define

$$\mathrm{Dom}_{\overline{\Omega}}(\mathscr{L}) := \{f \in \mathrm{Dom}(\mathscr{L}) : \mathrm{supp}(f) \subset \overline{\Omega}\}.$$

In addition to $\mathscr{L}$ acting on $\mathrm{Dom}_{\overline{\Omega}}(\mathscr{L})$, we consider also the Dirichlet Laplacian $\mathscr{L}_\Omega$ on the set Ω. We shall restrict our attention to bounded open sets Ω so that $\partial\Omega$ is smooth. Then

$$\mathrm{Dom}(\mathscr{L}_\Omega) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega),$$

and $\mathscr{L}_\Omega u$ is the distributional Laplacian of u for u in $\mathrm{Dom}(\mathscr{L}_\Omega)$. The following proposition will be useful in the proof of Theorem 2.5.4. A version thereof for balls of small radius in Riemannian manifolds with bounded geometry and spectral gap has been recently proved in [MMV4, Proposition 3.1]. The proof of Proposition 2.5.1 below follows almost *verbatim* the proof of [MMV4, Proposition 3.1], and is omitted.

Given a ball Ω, we denote by $\mathscr{T}_\Omega$ the restriction of $\mathscr{L}$ to $\mathrm{Dom}_{\overline{\Omega}}(\mathscr{L})$.

Proposition 2.5.1. *Assume that Ω is a bounded open subset of M such that $\partial\Omega$ is smooth. The following hold:*

(i) $W_0^{2,2}(\Omega) = \mathrm{Dom}_{\overline{\Omega}}(\mathscr{L}) \subset \mathrm{Dom}(\mathscr{L}_\Omega) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$*;*

(ii) $\mathscr{L}_\Omega$ *is an extension of* $\mathscr{T}_\Omega$*;*

(iii) the operator $\mathscr{T}_\Omega$ *is an isomorphism between* $W_0^{2,2}(\Omega)$ *and* $b^2(\Omega)^\perp$*, and the inverse operator* $\mathscr{T}_\Omega^{-1}$ *agrees with the restriction of* $\mathscr{L}_\Omega^{-1}$ *to* $b^2(\Omega)^\perp$*;*

(iv) there exists a constant C, independent of Ω, such that

$$\big\|\mathscr{T}_\Omega^{-1} f\big\|_2 \le \frac{C}{\lambda_1(\Omega)} \big\|f\big\|_{L^2(\Omega)} \qquad \forall f \in b^2(\Omega)^\perp,$$

where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian $\mathscr{L}_\Omega$.

The last ingredient of the proof of Theorem 2.5.4 is a lemma, whose statement requires more notation. For every ν in $(-1/2,\infty)$, denote by $\mathscr{J}_\nu$ the Bessel function of the first kind and of order ν. For notational convenience, we shall denote the operator $\sqrt{\mathscr{L}}$ by $\mathscr{D}$. Note that for every ν in $[-1/2,\infty)$ the function $\lambda \mapsto \mathscr{J}_\nu(t\lambda)$ is even and entire of exponential type t, so that the kernel $k_{\mathscr{J}_\nu(t\mathscr{D})}$ of the operator $\mathscr{J}_\nu(t\mathscr{D})$ is supported in the set $\{(x,y)\in M\times M : d(x,y)\leq t\}$ by the property of finite propagation speed.

Lemma 2.5.2. *For N large enough, the following estimate holds*

$$\sup_{t>0} \left\|\!\left| \mathscr{J}_{N-1/2}(t\mathscr{D})\right|\!\right\|_1 < \infty.$$

Proof. For notational convenience, in this proof we shall write $\mathscr{J}$ instead of $\mathscr{J}_{N-1/2}$. By Schwarz's inequality,

$$\begin{aligned}\left\|\!\left| \mathscr{J}(t\mathscr{D})\right|\!\right\|_1 &= \sup_{y\in M} \int_M \left|k_{\mathscr{J}(t\mathscr{D})}(x,y)\right| \mathrm{d}\mu(x) \\ &\leq \sup_{y\in M} \mu\big(B(y,t)\big)^{1/2} \Big[\int_M \left|k_{\mathscr{J}(t\mathscr{D})}(x,y)\right|^2 \mathrm{d}\mu(x)\Big]^{1/2}.\end{aligned} \tag{2.5.1}$$

We write

$$\mathscr{J}(t\mathscr{D}) = \big[\mathscr{J}(t\mathscr{D})\,\big(\mathscr{I}+t^2\mathscr{L}\big)^{N/2}\big]\,\big(\mathscr{I}+t^2\mathscr{L}\big)^{-N/2},$$

denote by B_t the operator $\mathscr{J}(t\mathscr{D})\,\big(\mathscr{I}+t^2\mathscr{L}\big)^{N/2}$, and observe that, by spectral theory and the asymptotic behaviour of Bessel functions,

$$\sup_{t>0} \left\|\!\left| B_t\right|\!\right\|_2 = \sup_{\lambda>0} \left|\mathscr{J}(t\lambda)\,(1+t^2\lambda)^{N/2}\right| < \infty. \tag{2.5.2}$$

Denote by k_t the kernel of the operator $\big(\mathscr{I}+t^2\mathscr{L}\big)^{-N/2}$. Notice that

$$k_{\mathscr{J}(t\mathscr{D})}(x,y) = B_t\big[k_t(\cdot,y)\big](x).$$

Since B_t is bounded on $L^2(M)$, uniformly in t by (2.5.2),

$$\begin{aligned}\Big[\int_M \left|k_{\mathscr{J}(t\mathscr{D})}(x,y)\right|^2 \mathrm{d}\mu(x)\Big]^{1/2} &\leq \left\|\!\left| B_t\right|\!\right\|_2 \Big[\int_M \left|k_t(x,y)\right|^2 \mathrm{d}\mu(x)\Big]^{1/2} \\ &\leq C\Big[\int_M \left|k_t(x,y)\right|^2 \mathrm{d}\mu(x)\Big]^{1/2}.\end{aligned}$$

Recall the classical subordination formula

$$k_t(x,y) = c_N \int_0^\infty s^N \mathrm{e}^{-s}\, h_{t^2 s}(x,y) \frac{\mathrm{d}s}{s} \qquad \forall x,y\in M : x\neq y,$$

and the following upper estimate for the heat kernel on M, which holds under the assumptions (2.1.1) and (2.1.2),

$$h_u(x,y) \leq \frac{C}{\mu\big(B(y,\sqrt{u})\big)}\,\mathrm{e}^{-cd(x,y)^2/u} \qquad \forall x,y\in M \quad \forall u>0.$$

By inserting this estimate in the subordination formula above, and using the generalised Minkowski inequality, we have that

$$\Big[\int_M \big|k_t(x,y)\big|^2\,\mathrm{d}\mu(x)\Big]^{1/2} \leq C\int_0^\infty \frac{\mathrm{d}s}{s}\,s^{N/2}\mathrm{e}^{-s}\,\Big[\int_M \frac{\mathrm{e}^{-2cd(x,y)^2/(t^2s)}}{\mu\big(B(y,t\sqrt{s})\big)^2}\,\,\mathrm{d}\mu(x)\Big]^{1/2}.$$

To estimate the inner integral on the right hand side, we write M as the union of the annuli A_ℓ, with $\ell=0,1,2,\ldots$, where

$$A_\ell := \big\{x\in M : \ell t\sqrt{s}\leq d(x,y)<(\ell+1)\,t\sqrt{s}\big\}.$$

Note that

$$\begin{aligned}\int_{A_\ell}\mathrm{e}^{-2cd(x,y)^2/(t^2s)}\,\mathrm{d}\mu(x) &\leq \mu\big(B(y,(\ell+1)\,t\sqrt{s})\big)\,\mathrm{e}^{-2c\ell^2}\\ &\leq \mu\big(B(y,t\sqrt{s})\big)\,D_0^{\ell+1}\,\mathrm{e}^{-2c\ell^2}.\end{aligned}$$

Therefore, by summing these estimates with respect to ℓ, we obtain that

$$\Big[\int_M \frac{\mathrm{e}^{-2cd(x,y)^2/(t^2s)}}{\mu\big(B(y,t\sqrt{s})\big)^2}\,\,\mathrm{d}\mu(x)\Big]^{1/2} \leq C\,\mu\big(B(y,t\sqrt{s})\big)^{-1/2},$$

where C does not depend on y and on t. Thus,

$$\begin{aligned}\big|\!\big|\!\big|\mathscr{J}(t\mathscr{D})\big|\!\big|\!\big|_1 &\leq \sup_{y\in M}\,\mu\big(B(y,t)\big)^{1/2}\,\Big[\int_M\big|k_t(x,y)\big|^2\,\mathrm{d}\mu(x)\Big]^{1/2}\\ &\leq C\,\sup_{y\in M}\int_0^\infty s^{N/2}\mathrm{e}^{-s}\,\Big[\frac{\mu\big(B(y,t)\big)}{\mu\big(B(y,t\sqrt{s})\big)}\Big]^{1/2}\frac{\mathrm{d}s}{s}.\end{aligned}$$

Clearly

$$\frac{\mu\big(B(y,t)\big)}{\mu\big(B(y,t\sqrt{s})\big)}\leq 1 \qquad \forall s\in[1,\infty).$$

Moreover, by the doubling property, there exist positive numbers C and ν such that

$$\frac{\mu\big(B(y,t)\big)}{\mu\big(B(y,t\sqrt{s})\big)}\leq C\,s^{-\nu/2} \qquad \forall s\in(0,1)\quad\forall y\in M.$$

By combining the last three estimates, we obtain

$$\big\vert\!\big\vert\!\big\vert \mathscr{J}(t\mathscr{D}) \big\vert\!\big\vert\!\big\vert_1 \leq C \Big[\int_0^1 s^{N-\nu/2}\, \mathrm{e}^{-s} \,\frac{\mathrm{d}s}{s} + \int_1^\infty s^N\, \mathrm{e}^{-s} \,\frac{\mathrm{d}s}{s} \Big]^{1/2},$$

where C does not depend on t. The right hand side is finite (and uniformly bounded with respect to t), as long as $N > \nu$, as required to conclude the proof of the lemma. □

Definition 2.5.3. Suppose that J is a positive integer. The space $\mathrm{Mih}(J)$ is the vector space of all *even* functions f on $\mathbb{R}$ for which there exists a positive constant C such that

$$\big| D^j f(\zeta) \big| \leq C \; |\zeta|^{-j} \qquad \forall \zeta \in \mathbf{R} \setminus \{0\} \quad \forall j \in \{0, 1, \ldots, J\}. \tag{2.5.3}$$

We denote by $\|f\|_{\mathrm{Mih}(J)}$ the infimum of all constants C for which (2.5.3) holds.

Theorem 2.5.4. *Suppose that J is a sufficiently large positive integer. Then there exists a constant C such that*

$$\vert\!\vert\!\vert m(\sqrt{\mathscr{L}}) \vert\!\vert\!\vert_{X^1;L^1} \leq C \, \|m\|_{\mathrm{Mih}(J)} \qquad \forall m \in \mathrm{Mih}(J).$$

Proof. In view of the theory developed in [MMV3] it suffices to prove that

$$\sup \big\{ \big\| m(\mathscr{D}) A \big\|_1 : A \text{ is an } X^1\text{-atom} \big\} < \infty.$$

Suppose that A is an X^1-atom, with support contained in B. Observe that

$$\big\| m(\mathscr{D}) A \big\|_1 = \big\| \mathbf{1}_{4B}\, m(\mathscr{D}) A \big\|_1 + \big\| \mathbf{1}_{(4B)^c}\, m(\mathscr{D}) A \big\|_1.$$

We estimate the two summands on the right hand side separately. To estimate the first, simply observe that, by Schwarz's inequality, the size condition for A, and the spectral theorem,

$$\begin{aligned} \big\| \mathbf{1}_{4B}\, m(\mathscr{D}) A \big\|_1 &\leq \mu(4B)^{1/2} \, \vert\!\vert\!\vert m(\mathscr{D}) \vert\!\vert\!\vert_2 \, \big\| A \big\|_2 \\ &\leq \Big(\frac{\mu(4B)}{\mu(B)} \Big)^{1/2}. \end{aligned}$$

The right hand side is bounded independently of B, because M is doubling.

Thus, to conclude the proof of the theorem it suffices to show that

$$\sup \big\| m(\mathscr{D}) A \big\|_{L^1((4B)^c)} < \infty, \tag{2.5.4}$$

where the supremum is taken over all X^1-atoms A. Suppose that A is a X^1-atom associated to the ball B. Choose $\varepsilon > 0$ and let Ω be an approximate (r_B, ε)-ball with centre c_B. Note that A is in $b^2(\Omega)^\perp$, for every harmonic function in Ω clearly belongs to $b^2(B)$. Then, by Proposition 2.5.1 (ii) and (iii),

$$A = \mathscr{T}_\Omega \mathscr{T}_\Omega^{-1} A = \mathscr{L} \mathscr{T}_\Omega^{-1} A.$$

Thus,

$$m(\mathscr{D})A = m(\mathscr{D})\mathscr{L}\big(\mathscr{T}_\Omega^{-1} A\big).$$

Recall that the support of $\mathscr{T}_\Omega^{-1} A$ is contained in Ω and that

$$\begin{aligned} \big\|\mathscr{T}_\Omega^{-1} A\big\|_{L^2(\Omega)} &\leq \big|\!\big|\!\big|\mathscr{T}_\Omega^{-1}\big|\!\big|\!\big|_{L^2(\Omega)} \big\|A\big\|_{L^2(B)} \\ &\leq \big|\!\big|\!\big|\mathscr{T}_\Omega^{-1}\big|\!\big|\!\big|_{L^2(\Omega)} \mu(B)^{-1/2}, \end{aligned}$$

so that

$$\begin{aligned} \big\|\mathscr{T}_\Omega^{-1} A\big\|_{L^1(\Omega)} &\leq \mu(\Omega)^{1/2} \big\|\mathscr{T}_\Omega^{-1} A\big\|_{L^2(\Omega)} \\ &\leq \Big(\frac{\mu(\Omega)}{\mu(B)}\Big)^{1/2} \big|\!\big|\!\big|\mathscr{T}_\Omega^{-1}\big|\!\big|\!\big|_{L^2(\Omega)} \\ &\leq \frac{C}{\lambda_1(\Omega)}. \end{aligned}$$

Here we have used the doubling property and Proposition 2.5.1 (iv).

We *claim* that there exists a constant C such that

$$\|m(\mathscr{D})f\|_{L^1((4B)^c)} \leq C\, r_B^{-2} \big\|f\big\|_{L^1(\Omega)} \qquad \forall f \in L^1(\Omega). \tag{2.5.5}$$

Given the claim, we conclude that

$$\begin{aligned} \|m(\mathscr{D})A\|_{L^1((4B)^c)} &= \|m(\mathscr{D})\mathscr{L}\big(\mathscr{T}_\Omega^{-1} A\big)\|_{L^1((4B)^c)} \\ &\leq C\, r_B^{-2} \big\|\mathscr{T}_\Omega^{-1} A\big\|_{L^1(B)} \\ &\leq C\, r_B^{-2}\, \lambda_1(B)^{-1} \\ &\leq C, \end{aligned} \tag{2.5.6}$$

thereby completing the proof of (2.5.4), and of the theorem. Note that the first inequality above follows from the claim, the second from Proposition 2.5.1 (iv), and the last from (2.1.2).

Thus, to conclude the proof of the theorem it remains to prove the claim. By Fourier analysis, the restriction of $m(\mathscr{D})\mathscr{L}f$ to $(4B)^c$ is given by

$$m(\mathscr{D})\mathscr{L}f = \frac{1}{2\pi}\int_{|t|\geq(3-\varepsilon)r_B} \widehat{m}(t)\,\cos(t\mathscr{D})\mathscr{L}f\,\mathrm{d}t.$$

Denote by ϕ a smooth, even function on $\mathbb{R}$ that vanishes in the complement of the set $\{t\in\mathbb{R} : 1/4\leq|t|\leq 4\}$. For a fixed R in $(0,1]$ and for each positive integer i, denote by E_i the set $\{t\in\mathbb{R} : 4^{i-1}r_B\leq|t|\leq 4^{i+1}r_B\}$. Clearly $\phi^{1/(4^i r_B)}$ is supported in E_i, and $\sum_{i=1}^{\infty}\phi^{1/(4^i r_B)} = 1$ in $\mathbb{R}\setminus(-r_B,r_B)$. Define $m_i(\mathscr{D})\mathscr{L}f$ as follows:

$$m_i(\mathscr{D})\mathscr{L}f = \frac{1}{2\pi}\int_{E_i}\phi^{1/(4^i r_B)}(t)\,\widehat{m}(t)\,\cos(t\mathscr{D})\mathscr{L}f\,\mathrm{d}t. \tag{2.5.7}$$

Recall that $\mathscr{L}=\mathscr{D}^2$, so that, at least formally,

$$\cos(t\mathscr{D})\mathscr{L} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(t\mathscr{D}).$$

Thus, by integrating by parts, we see that

$$m_i(\mathscr{D})\mathscr{L}f = -\frac{1}{2\pi}\int_{E_i}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[\phi^{1/(4^i r_B)}\,\widehat{m}\big](t)\,\cos(t\mathscr{D})f\,\mathrm{d}t. \tag{2.5.8}$$

Now, recall [MMV1, Lemma 5.1] that for every positive integer k there exists a polynomial P_{k+1} of degree $k+1$, without constant term, such that

$$\int_{-\infty}^{\infty} f(t)\,\cos(vt)\,\mathrm{d}t = \int_{-\infty}^{\infty} P_{k+1}(\mathscr{O})f(t)\,\mathscr{J}_{k+1/2}(tv)\,\mathrm{d}t \tag{2.5.9}$$

for all functions f such that $\mathscr{O}^\ell f\in L^1(\mathbb{R})\cap C_0(\mathbb{R})$ for all ℓ in $\{0,1,\ldots,k+1\}$. Here $\mathscr{O}^\ell$ denotes the differential operator $t^\ell\,(\mathrm{d}/\mathrm{d}t)^\ell$ on the real line. Thus,

$$m_i(\mathscr{D})\mathscr{L}f = -\frac{1}{2\pi}\int_{E_i}P_N(\mathscr{O})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[\phi^{1/(4^i r_B)}\,\widehat{m}\big](t)\,\mathscr{J}_{N-1/2}(t\mathscr{D})f\,\mathrm{d}t. \tag{2.5.10}$$

It is straightforward to check that there exists a constant C such that

$$\Big|P_N(\mathscr{O})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[\phi^{1/(4^i r_B)}\,\widehat{m}\big](t)\Big| \leq \frac{C}{t^3}\,\|m\|_{\mathrm{Mih}(J)} \qquad \forall t\in E_i. \tag{2.5.11}$$

Given (2.5.10) and (2.5.11), we obtain that

$$\begin{aligned}
\|m(\mathscr{D})f\|_{L^1((4B)^c)} &= \sum_{i=1}^{\infty} \|m_i(\mathscr{D})f\|_{L^1((4B)^c)} \\
&\leq \frac{1}{2\pi} \sum_{i=1}^{\infty} \int_{E_i} \Big| P_N(\mathscr{O}) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \big[\phi^{1/(4^i r_B)}\, \widehat{m}\big](t) \Big| \, \big\| \mathscr{J}(t\mathscr{D})f \big\|_1 \,\mathrm{d}t \\
&\leq C \sum_{i=1}^{\infty} \Big(\int_{E_i} t^{-3}\,\mathrm{d}t \Big) \, \|f\|_{L^1(\Omega)} \\
&\leq C\, r_B^{-2} \, \big\|f\big\|_{L^1(\Omega)},
\end{aligned}$$

thereby proving (2.5.5), and concluding the proof of the claim. We have used Lemma 2.5.2 in the second inequality above. □

2.6 Doubling property and scaled Poincaré inequality

In this section we prove that, under the additional assumption that M supports a scaled L^2 Poincaré inequality, the Hardy space $H^1(M)$ of Coifman and Weiss and the Hardy-type space $X^1(M)$ coincide. We also show that this may not be true if M does not support a scaled Poincaré inequality. These facts are already known. The equivalence of $H^1(M)$ and $X^1(M)$ appears, for instance, in [AMR, HLMMY]. Our proof of this result is quite elementary, and hinges on the fact that, by a result of A. Grigor'yan [Gr1] and of L. Saloff-Coste [Sa1], if a manifolds possesses the doubling property and supports a scaled Poincaré inequality, then it supports also a uniform Harnack inequality, hence harmonic functions on M are (uniformly) Hölder regular by the celebrated result of E. De Giorgi. We believe that, though the result is already known, the proof we give is considerably different from the original one and sheds more light on the role played by the scaled Poincaré inequality.

Furthermore, in [AMR] the authors attribute to A. Hassell the result (to the best of our knowledge still unpublished) that $H^1(M)$ and $X^1(M)$ are different when M is the connected product of two Euclidean spaces and claim that this fact may be proved as an application of methods developed in [CCH]. We give a direct proof of this result that may be applied also to manifolds which are not necessarily Euclidean at infinity.

Definition 2.6.1. We say that M supports a *scaled Poincaré inequality* if there exists a constant P such that

$$\int_B |f - f_B|^2 \, \mathrm{d}\mu \leq P \, r_B^2 \int_B |\nabla f|^2 \, \mathrm{d}\mu \tag{2.6.1}$$

for every f in $\mathscr{C}^\infty(B)$ and for all geodesic ball B. Here f_B denotes the average of f over B and r_B is the radius of B.

Before proving the main result of this section, we recall that A. Grigor'yan and L. Saloff-Coste proved that (2.1.1) and (2.6.1) are equivalent to a *uniform parabolic Harnack principle*. This uniform parabolic Harnack principle implies (but it is not equivalent to, see [HS]) a uniform elliptic Harnack principle, hence the uniform Hölder continuity of $\mathscr{L}$-harmonic functions.

Theorem 2.6.2. *Suppose that M possesses the volume doubling property* (2.1.1) *and admits the scaled Poincaré inequality* (2.6.1). *Then $H^1(M) = X^1(M)$, and their norms are equivalent.*

Proof. Clearly a X^1-atom A is also an H^1-atom. Hence a function F in $X^1(M)$ is also in $H^1(M)$ and

$$\|F\|_{H^1} \leq \|F\|_{X^1} \qquad \forall F \in X^1(M). \tag{2.6.2}$$

Therefore the identity map ι is a continuous injection from $X^1(M)$ to $H^1(M)$. Thus, to conclude the proof of the theorem it suffices to show that ι is surjective.

In fact, we shall prove the following *claim*: there exists a constant C such that for every H^1-atom a there exist a summable sequence of complex numbers $\{c_j\}$ and a sequence $\{A_j\}$ of special atoms such that

$$a = \sum_j c_j \, A_j \qquad \text{and} \qquad \sum_j |c_j| \leq C.$$

Suppose that the support of the atom a is contained in the ball B. Fix a positive number ε. For $j = 1, 2, 3, \ldots$, we denote by B_j and B_j' the balls with centre c_B and radii $2^{j-1} r_B$ and $(1+\varepsilon)2^{j-1} r_B$, respectively, and by $\mathscr{P}_j$ the orthogonal projection of $L^2(B_j)$ onto $b^2(\Omega_j)$, where Ω_j is an approximate $(2^{j-1} r_B, \varepsilon)$-ball with centre c_B (see Definition 2.2.5). Note that

$$B_j \subset \Omega_j \subset B_j' \subset B_{j+1}$$

for small ε. Sometimes it will be convenient to write $\mathscr{P}_0$ for $\mathscr{I}$, the identity operator from $L^2(B_j)$ to $L^2(\Omega_j)$, and B_1 instead of B. Define, for any j,

$$A_j = \frac{\mathscr{P}_{j-1} a - \mathscr{P}_j a}{\mu(B_j')^{1/2} \, \|\mathscr{P}_{j-1} a - \mathscr{P}_j a\|_2}.$$

Clearly the support of A_j is contained in B_j', and

$$\|A_j\|_2 \leq \mu(B_j')^{-1/2}.$$

Observe also that $\mathscr{P}_j a = \mathscr{P}_j(\mathscr{P}_{j-1}a)$, where we tacitly identify $L^2(\Omega_{j-1})$ with the corresponding closed subspace of $L^2(\Omega_j)$. Then we may write

$$\begin{aligned}\int_{B_j'} \big[\mathscr{P}_{j-1}a - \mathscr{P}_j a\big]\, H(x)\,\mathrm{d}\mu &= \int_{\Omega_j} \big[\mathscr{P}_{j-1}a - \mathscr{P}_j a\big]\, H(x)\,\mathrm{d}\mu \\ &= \int_{\Omega_j} (\mathscr{I} - \mathscr{P}_j)\big(\mathscr{P}_{j-1}a\big)\, H\,\mathrm{d}\mu\end{aligned}$$

for any H in $b^2(B_j')$. The operator $\mathscr{I} - \mathscr{P}_j$ is the orthogonal projection of $L^2(\Omega_j)$ onto the orthogonal complement of $b^2(\Omega_j)$ in $L^2(\Omega_j)$. Since $\mathscr{I} - \mathscr{P}_j$ is a self-adjoint operator in $L^2(\Omega_j)$, and the restriction of H to Ω_j is in the kernel of $\mathscr{I} - \mathscr{P}_j$, the last integral vanishes. Thus, A_j is a special atom with support contained in B_j'. At least formally, we may write

$$\begin{aligned}a &= a - \mathscr{P}_1 a + \sum_{j=2}^{\infty} \big[\mathscr{P}_{j-1}a - \mathscr{P}_j a\big] \\ &= \sum_{j=1}^{\infty} c_j\, A_j,\end{aligned}$$

where $c_j = \mu(B_j')^{1/2}\, \|\mathscr{P}_{j-1}a - \mathscr{P}_j a\|_2$. To conclude the proof of the claim, it suffices to show that

$$\sum_{j=1}^{\infty} |c_j| \leq C,$$

where C does not depend on the atom a. We denote by R_j the reproducing kernel of $b^2(\Omega_j)$. Since a has vanishing integral, we may write

$$\begin{aligned}\mathscr{P}_j a(x) &= \int_{B_1} a(y)\, R_j(x,y)\,\mathrm{d}\mu(y) \\ &= \int_{B_1} a(y)\, \big[R_j(x,y) - R_j(x,c_B)\big]\,\mathrm{d}\mu(y).\end{aligned}$$

Moreover, by the reproducing and the symmetry properties of R_j,

$$\begin{aligned}\|R_j(\cdot,y) - R_j(\cdot,c_B)\|^2_{L^2(\Omega_j)} &= \big(R_j(\cdot,y) - R_j(\cdot,c_B), R_j(\cdot,y) - R_j(\cdot,c_B)\big) \\ &= R_j(y,y) + R_j(c_B,c_B) - 2R_j(y,c_B) \\ &\leq \big|R_j(y,y) - R_j(y,c_B)\big| + \big|R_j(y,c_B) - R_j(c_B,c_B)\big|.\end{aligned}$$

The functions $R_j(y,\cdot)$ and $R_j(\cdot,c_B)$ are harmonic in Ω_j. Since c_B and y belong to $B_1 \subseteq 2^{-1}B_j$, and a uniform parabolic Harnack inequality holds on balls of M, by De Giorgi's theorem (see, for instance, [Sa1]) there exist α in $(0,1)$ and a constant C, independent of y and j, such that

$$\big|R_j(y,y) - R_j(y,c_B)\big| \leq C\,\Big(\frac{|y-c_B|}{2^{j-1}r_B}\Big)^{\alpha} \sup_{z\in 2B_1}\big|R_j(y,z)\big|. \tag{2.6.3}$$

For j large, both y and z are in $2^{-1}B_j$. Therefore, by Proposition 2.2.7 (iii), we can conclude that there exists a constant C, independent of j, such that

$$\sup_{z\in 2B_1}\big|R_j(y,z)\big| \leq C\,\mu(B_j)^{-1},$$

so that, by (2.6.3),

$$\big|R_j(y,y) - R_j(y,c_B)\big| \leq C\,2^{-\alpha j}\,\mu(B_j)^{-1}.$$

This, the generalised Minkowski inequality and the doubling property imply that

$$\begin{aligned}
\|\mathscr{P}_j a\|_2 &\leq \int_{B_1} |a(y)|\,\big\|R_j(\cdot,y) - R_j(\cdot,c_B)\big\|_2\,\mathrm{d}\mu(y)\\
&\leq C\,2^{-\alpha j}\,\mu(B_j)^{-1/2}\int_{B_1}|a(y)|\,\mathrm{d}\mu(y)\\
&\leq C\,2^{-\alpha j}\,\mu(B_j)^{-1/2}\\
&= C\,2^{-\alpha j}\,\mu(B_j')^{-1/2}\,\Big(\frac{\mu(B_j')}{\mu(B_j)}\Big)^{1/2}\\
&\leq C\,2^{-\alpha j}\,\mu(B_j')^{-1/2}.
\end{aligned} \tag{2.6.4}$$

Therefore

$$|c_j| \leq C\,2^{-\alpha j},$$

with C independent of a and j, so that the sequence $\{c_j\}$ is summable, as required to conclude the proof of the claim, and of the theorem. □

An example of a Riemannian manifold which does not support a uniform Harnack inequality is the following. Denote by M the connected product $\mathbb{R}^n \# \mathbb{R}^n$ of two copies of $\mathbb{R}^n$, when $n \geq 3$. The manifold M is obtained by "gluing" smoothly the two copies of $\mathbb{R}^n \setminus \{B(0,1)\}$, which we denote by C_1 and C_2. The metric on M agrees with the Euclidean metric on C_1 and C_2, and is globally smooth. It is straightforward to check that the scaled Poincaré inequality (2.6.1) fails on M (hence M does not support a uniform Harnack inequality, by a well known result of Grigor'yan and Saloff-Coste). Indeed, choose x on the "cylinder" that joins C_1 and C_2. Let $R > 0$ and let f_R be a function on M, with support contained in $\overline{B}(x, R)$, that is equal to -1 on C_1 and $+1$ on C_2. Clearly ∇f_R vanishes on C_1 and C_2, hence its L^2-norm does not depend on R, as R tends to infinity. Hence the right hand side of (2.6.1) grows like R^2 as R tends to $+\infty$. However, by carefully choosing the values of f_R on the "cylinder" joining C_1 and C_2, we may assume that $\int_{B(x,R)} f_R \, \mathrm{d}\mu = 0$. Then the left hand side of (2.6.1) grows as R^n, as R tends to infinity, whence the scaled Poincaré inequality cannot possibly hold for large values of R.

However, note that $\mathbb{R}^n \# \mathbb{R}^n$ satisfies a relative Faber–Krahn inequality (2.1.2). Indeed, by the Federer–Fleming theorem (see, for instance, [Ch]), the isoperimetric property of $\mathbb{R}^n \# \mathbb{R}^n$ implies the Sobolev inequality

$$\|f\|_{n/(n-1)} \leq C \, \|\nabla f\|_1 \qquad \forall f \in \mathscr{C}_c^\infty(\mathbb{R}^n \# \mathbb{R}^n). \tag{2.6.5}$$

This, in turn, implies the following estimate for the heat kernel:

$$h_t(x, x) \leq C \, t^{-n/2} \qquad \forall x \in \mathbb{R}^n \# \mathbb{R}^n, \quad \forall t > 0$$

(see [Gr2, Corollary 14.23, p. 383]). Finally, the relative Faber–Krahn inequality is a consequence of the last inequality and the doubling property.

Theorem 2.6.3. *Denote by M the connected product $\mathbb{R}^n \# \mathbb{R}^n$. Then $\nabla \mathscr{L}^{-1/2}$ is unbounded from $H^1(M)$ to $L^1(M)$.*

Proof. By (2.6.5), there exists a positive constant C such that

$$\|\nabla \mathscr{L}^{-1/2} f\|_1 \geq C \, \|\mathscr{L}^{-1/2} f\|_{n/(n-1)}.$$

If $\nabla \mathscr{L}^{-1/2}$ were bounded from $H^1(M)$ to $L^1(M)$, the following estimate would hold:

$$\|f\|_{H^1(M)} \geq C \, \|\mathscr{L}^{-1/2} f\|_{n/(n-1)} \qquad \forall f \in H^1(M).$$

In particular,

$$\sup \|\mathscr{L}^{-1/2} a\|_{n/(n-1)} < \infty,$$

where the supremum is taken over all H^1-atoms a. Denote by k the Schwartz kernel of $\mathscr{L}^{-1/2}$. We have

$$\mathscr{L}^{-1/2} a(x) = \int_M k(x, y) \, a(y) \, \mathrm{d}\mu(y) \tag{2.6.6}$$

Choose a with support contained in $B(p,R)$, where p is on the cylinder which joins C_1 and C_2, and R is large. Suppose that a is equal to -1 on C_1 and to $+1$ on C_2, and carefully choose the values of a on the "cylinder", so that $\int_M a \,\mathrm{d}\mu = 0$. Then a is a multiple of an H^1-atom. Let x be a point in C_2, far from the "cylinder". Then denote by y and $\widetilde{y}$ two points in $B(p,R)$, the first on C_2 and the second on C_1. By using the estimates for the heat kernel on M proved by Grygor'yan and Saloff-Coste [GS], it is not hard to prove that

$$k(x,y) \gg k(x,\widetilde{y}).$$

In fact, we can prove that there exists a positive constant C such that for R large enough

$$\int_B k(x,y)\, a(y)\, \mathrm{d}\mu(y) \geq c \int_{B_+} k(x,\widetilde{y})\, \mathrm{d}\mu(y),$$

where B_+ denotes $B(p,R) \cap C_2$. Similar estimates show that the function $x \mapsto \int_{B_+} k(x,\widetilde{y})\, \mathrm{d}\mu(y)$ does not belong to $L^{n/(n-1)}(M)$. Therefore $\mathscr{L}^{-1/2} a$ is not in $L^{n/(n-1)}(M)$. $\square$

Part II

A symmetric diffusion semigroup

Chapter 3

The semigroup generated by the operator $\mathscr{A}$

This chapter contains the analysis of the semigroup generated on $(\mathbb{R}^n, \gamma_{-1})$ by the operator $\mathscr{A}$. We recall that $\mathscr{A}$ is defined by

$$\mathscr{A} f = -\frac{1}{2}\,\Delta f - x\cdot \nabla f \qquad \forall f \in C_c^\infty(\mathbb{R}^n),$$

and that $\gamma_{-1}$is the measure whose density with respect to Lebesgue measure is $\pi^{n/2}\,\mathrm{e}^{|x|^2}$. In Section 3.1, we shall prove some preliminary properties of $\mathscr{A}$, produce a formula for the semigroup $\mathrm{e}^{-t\mathscr{A}}$, and investigate its region of holomorphy in $L^p(\gamma_{-1})$. In Section 3.2 we shall briefly study local singular integral operators in our setting. In Section 3.3 we shall prove that the maximal operator $\mathscr{H}^*$ associated to the semigroup generated by $\mathscr{A}$ is of weak type 1. Finally, Section 3.4 contains the statement of all results we have obtained concerning the functional calculus for $\mathscr{A}$.

3.1 Background material and preliminary results

Let $1 \le p < \infty$. Denote by $\mathscr{U}_p$ the operator defined by

$$\mathscr{U}_p f = \gamma_{-1/p}\; f \qquad \forall f \in \mathscr{C}_c^\infty(\mathbb{R}^n) \tag{3.1.1}$$

(see (0.0.3) for the definition of $\gamma_{-1/p}$). Clearly $\mathscr{U}_p$ extends to an isometry of $L^p(\gamma_{-1})$ onto $L^p(\lambda)$ and of $L^p(\lambda)$ onto $L^p(\gamma_1)$. We denote by $\mathscr{L}$ the Ornstein–Uhlenbeck operator. A straightforward computation shows that

$$\mathscr{U}_2 \mathscr{A} \mathscr{U}_2^{-1} = \frac{1}{2}\left(-\Delta + |\cdot|^2 + n\mathscr{I}\right)$$

and

$$\mathscr{U}_2^{-1} \mathscr{L} \mathscr{U}_2 = \frac{1}{2}\left(-\Delta + |\cdot|^2 - n\mathscr{I}\right).$$

Consequently,

$$\begin{aligned} \mathscr{A} f &= \frac{1}{2}\, \mathscr{U}_2^{-1}\left(-\Delta + |\cdot|^2 + n\mathscr{I}\right) \mathscr{U}_2 f \\ &= \frac{1}{2}\, \mathscr{U}_2^{-1}\left(-\Delta + |\cdot|^2 - n\mathscr{I} + 2n\mathscr{I}\right) \mathscr{U}_2 f \\ &= \mathscr{U}_2^{-1}\mathscr{U}_2^{-1}(\mathscr{L} + n\mathscr{I})\mathscr{U}_2\mathscr{U}_2 f \qquad \forall f \in \mathscr{C}_c^\infty(\mathbb{R}^n). \end{aligned}$$

It is well known that the Ornstein–Uhlenbeck operator $\mathscr{L}$ is essentially self-adjoint. We abuse the notation and still denote by $\mathscr{L}$ its self-adjoint extension, and by $\mathrm{Dom}(\mathscr{L})$ its domain. Clearly $\mathscr{L} + n\mathscr{I}$ is also self-adjoint. Since $\mathscr{U}_2^2 = \mathscr{U}_1$, which extends to an isometry of $L^2(\gamma_{-1})$ onto $L^2(\gamma_1)$, the operator

$$\mathscr{U}_1^{-1}\left(\mathscr{L} + n\mathscr{I}\right)\mathscr{U}_1 f \qquad \forall f \in L^2(\gamma_1) : \mathscr{U}_1 f \in \mathrm{Dom}(\mathscr{L})$$

is a self-adjoint extension of $\mathscr{A}$, which, with abuse of notation, we still denote by $\mathscr{A}$. Its domain is

$$\mathrm{Dom}(\mathscr{A}) = \{f \in L^2(\gamma_{-1}) : \mathscr{U}_{-1} f \in \mathrm{Dom}(\mathscr{L})\}.$$

Thus,

$$\mathscr{A} f = \mathscr{U}_1^{-1}\left(\mathscr{L} + n\mathscr{I}\right)\mathscr{U}_1 f \qquad \forall f \in \mathrm{Dom}(\mathscr{A}). \tag{3.1.2}$$

The spectral resolution of the identity of $\mathscr{L}$ is

$$\mathscr{L} = \sum_{j=0}^{\infty} j\, \mathscr{P}_j,$$

where $\mathscr{P}_j$ denotes the orthogonal projection onto the linear span of the Hermite polynomials of degree j in n variables. Therefore, the spectral resolution of $\mathscr{A}$ is

$$\mathscr{A} = \sum_{k=0}^{\infty} (k+n)\, \mathscr{E}_k, \tag{3.1.3}$$

where $\mathscr{E}_k = \mathscr{U}_1^{-1}\, \mathscr{P}_k \mathscr{U}_1$. Thus, $\mathscr{E}_k$ is the orthogonal projection of $L^2(\gamma_{-1})$ onto the linear span of the functions of the form $\gamma_1\, p$, where p is a Hermite polynomial of degree j in n variables.

By spectral theory,

$$\mathrm{e}^{-t\mathscr{A}} = \mathscr{U}_1^{-1}\, \mathrm{e}^{-t(\mathscr{L}+n\mathscr{I})}\, \mathscr{U}_1 f \qquad \forall f \in L^2(\gamma_{-1}).$$

We shall denote by $\{\mathscr{H}_t\}_{t\geq 0}$ the semigroup $\{\mathrm{e}^{-t\mathscr{A}}\}$ on $L^2(\gamma_{-1})$, and by $\{\mathscr{M}_t\}_{t\geq 0}$ the semigroup $\{\mathrm{e}^{-t\mathscr{L}}\}$ on $L^2(\gamma)$. We recall Mehler's formula for the Ornstein–Uhlenbeck semigroup. For $t > 0$,

$$\mathscr{M}_t f(x) = \int m_t(x,y) f(y)\, \mathrm{d}\gamma(y), \tag{3.1.4}$$

where

$$m_t(x,y) = \frac{1}{(1-\mathrm{e}^{-2t})^{n/2}}\, \exp\Big(\frac{|x+y|^2}{2(\mathrm{e}^t+1)} - \frac{|x-y|^2}{2(\mathrm{e}^t-1)}\Big) \qquad \forall x,y \in \mathbb{R}^n \quad \forall t > 0.$$

It is known that $\{\mathscr{M}_t\}_{t\geq 0}$ is a symmetric diffusion semigroup on $(\mathbb{R}^n, \gamma_1)$; we refer to [B] and the references therein for more on the Ornstein–Uhlenbeck semigroup.

Theorem 3.1.1. *For any test function f*

$$\mathscr{H}_t f(x) = \begin{cases} \int h_t(x,y) f(y)\, \mathrm{d}\gamma_{-1}(y) & \text{if } t > 0 \\ f(x) & \text{if } t = 0, \end{cases} \tag{3.1.5}$$

where

$$h_t(x,y) = \frac{\mathrm{e}^{-tn}}{\pi^n(1-\mathrm{e}^{-2t})^{n/2}} \exp\Big(-\frac{|x+y|^2}{2(1+\mathrm{e}^{-t})} - \frac{|x-y|^2}{2(1-\mathrm{e}^{-t})}\Big). \tag{3.1.6}$$

Moreover, $\{\mathscr{H}_t\}_{t\geq 0}$ is a symmetric diffusion semigroup on $(\mathbb{R}^n, \gamma_{-1})$.

Proof. To prove (3.1.5), notice that

$$\begin{aligned}
\mathscr{H}_t f(x) &= \mathscr{U}_1^{-1} \mathrm{e}^{-t(\mathscr{L}+n\mathscr{I})} \mathscr{U}_1 f(x) \\
&= \frac{\mathrm{e}^{-tn}}{(1-\mathrm{e}^{-2t})^{n/2}} \mathrm{e}^{-|x|^2} \int \exp\Big(\frac{|x+y|^2}{2(\mathrm{e}^t+1)} - \frac{|x-y|^2}{2(\mathrm{e}^t-1)}\Big) \mathrm{e}^{|y|^2} f(y)\, \mathrm{d}\gamma(y) \\
&= \frac{\mathrm{e}^{-tn}}{\pi^n(1-\mathrm{e}^{-2t})^{n/2}} \int \exp\Big(\frac{|x+y|^2}{2(\mathrm{e}^t+1)} - \frac{|x-y|^2}{2(\mathrm{e}^t-1)} - |x|^2 - |y|^2\Big) f(y)\, \mathrm{d}\gamma_{-1}(y) \\
&= \frac{\mathrm{e}^{-tn}}{\pi^n(1-\mathrm{e}^{-2t})^{n/2}} \int \exp\Big(-\frac{|x+y|^2}{2(1+\mathrm{e}^{-t})} - \frac{|x-y|^2}{2(1-\mathrm{e}^{-t})}\Big) f(y)\, \mathrm{d}\gamma_{-1}(y),
\end{aligned}$$

as required.

We already know that $\{\mathrm{e}^{-t\mathscr{A}}\}$ is symmetric. It is also positivity preserving, for its kernel is positive. We also note that, for $f \in L^1(\gamma_{-1}) \cap L^2(\gamma_{-1})$,

$$\begin{aligned}\int f(y)\, \mathrm{d}\gamma_{-1}(y) &= \frac{\mathrm{e}^{-tn}}{\pi^{n/2}(1-\mathrm{e}^{-2t})^{n/2}} \int f(y)\mathrm{e}^{|y|^2}\, \mathrm{d}y \\ &= \frac{\mathrm{e}^{-tn}}{\pi^{n/2}(1-\mathrm{e}^{-2t})^{n/2}} \int\int \exp\Big(-\frac{|x}{1} \\ &= \int \mathscr{H}_t f(x)\, \mathrm{d}\gamma_{-1}(x).\end{aligned}$$

Thus, $\mathscr{H}_t$ is contractive on $L^1(\gamma_{-1})$. Hence, by interpolation and duality, $\mathscr{H}_t$ is contractive on $L^p(\gamma_{-1})$ for $1 < p < \infty$. Finally,

$$\mathscr{H}_t 1 = 1 \qquad \forall t > 0,$$

i.e., $\{\mathscr{H}_t\}$ is markovian, and it is contractive also on $L^\infty(\gamma_{-1})$. □

Remark 3.1.2. A noteworthy consequence of (3.1.6) is that the semigroup $\{\mathscr{H}_t\}$ satisfies

$$\begin{aligned}\|\mathscr{H}_t\|_{1;\infty} &= \sup_{x,y\in\mathbb{R}^n} h_t(x,y) \\ &= \frac{\mathrm{e}^{-tn}}{\pi^n(1-\mathrm{e}^{-2t})^{n/2}} \qquad \forall t > 0,\end{aligned} \tag{3.1.7}$$

where $\|\mathscr{H}_t\|_{1;\infty}$ denotes the operator norm of $\mathscr{H}_t$ from $L^1(\gamma_{-1})$ to $L^\infty(\gamma_{-1})$. Thus, the semigroup $\{\mathscr{H}_t\}$ is *ultracontractive*. By interpolation, $\{\mathscr{H}_t\}$ is also *hypercontractive*, i.e., $\mathscr{H}_t$ is bounded from $L^p(\gamma_{-1})$ to $L^{p'}(\gamma_{-1})$, for $p \in (1,2)$. We recall that the Gaussian semigroup $\{\mathscr{M}_t\}$ is not ultracontractive, and that $\mathscr{M}_t$ is bounded from $L^p(\gamma_1)$ to $L^{p'}(\gamma_1)$ if and only if $t \geq \frac{1}{2}\log\left(\frac{p'-1}{p-1}\right)$, by a well known theorem of E. Nelson [Nel, Theorem 2].

By spectral theory, the map $t \mapsto \mathscr{H}_t$ from $[0,\infty)$ to the space of bounded operators on $L^2(\gamma_{-1})$ is continuous with respect to the strong operator topology of $L^2(\gamma_{-1})$, and it extends to a continuous map from the right half plane $\{z : \mathrm{Re}(z) \geq 0\}$ to the space of bounded operators on $L^2(\gamma_{-1})$, which is analytic in $\{z : \mathrm{Re}(z) > 0\}$. Correspondingly, the kernel $t \mapsto h_t$ has an analytic continuation to a distribution-valued function $z \mapsto h_z$, which is continuous in $\mathrm{Re}\, z \geq 0$ and holomorphic in $\mathrm{Re}\, z > 0$. Clearly h_z is the kernel of $\mathscr{H}_z$. An interesting question is to determine for which z in the right half plane the operator $\mathscr{H}_z$ extends to a bounded operator on $L^p(\gamma_{-1})$, for some $1 \leq p \leq \infty$. Since

$$\mathscr{H}_{\overline{z}} f(x) = \overline{\mathscr{H}_z \overline{f(x)}}, \tag{3.1.8}$$

$\mathscr{H}_z$ is bounded on $L^p(\gamma_{-1})$ if and only if $\mathscr{H}_{\overline{z}}$ is. Moreover, since

$$\int \mathscr{H}_z f\ \overline{g}\, \mathrm{d}\gamma_{-1} = \int f\ \overline{\mathscr{H}_{\overline{z}} g}\, \mathrm{d}\gamma_{-1},$$

we may conclude that $\mathscr{H}_z$ is bounded on $L^p(\gamma_{-1})$ if and only if it is bounded on $L^{p'}(\gamma_{-1})$, where $1/p + (1,1)/p' = 1$, and $\|\mathscr{H}_z\|_{L^p(\gamma_{-1})} = \|\mathscr{H}_z\|_{L^{p'}(\gamma_{-1})}$.

The final part of this section is dedicated to proving that $\mathscr{H}_z$ extends to a bounded operator on $L^p(\gamma_{-1})$ if and only if z belongs to the set $\mathbf{E}_p$, defined by

$$\mathbf{E}_p = \{x + iy \in \mathbb{C} \ : \ |\sin y| \le (\tan \phi_p) \sinh x\}, \tag{3.1.9}$$

where $\phi_p := \arccos(2/p - 1)$. We recall that the set $\mathbf{E}_p$ is the region of boundedness on $L^p(\gamma)$ for the analytic continuation of the Ornstein–Uhlenbeck semigroup $\{\mathscr{M}_t\}$ [E, Theorem 1.1]. By looking at the spectral resolution of the identity of $\mathscr{A}$, it is clear that

$$\mathscr{H}_{z+i\pi} f(x) = \mathscr{H}_z f(-x).$$

Thus, $\mathbf{E}_p$ is a closed $i\pi$–periodic subset of the half–plane $\operatorname{Re} z \ge 0$, symmetric with respect to the x-axis, and $\mathbf{E}_p = \mathbf{E}_{p'}$. So, to investigate the boundedness properties of the semigroup $\{\mathscr{H}_z\}$, we may restrict the parameter z to the set

$$\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \le \operatorname{Im} z \le \frac{\pi}{2}\}, \tag{3.1.10}$$

and we may consider only the case when $1 \le p \le 2$.

The proofs of most of the results below concerning $\mathscr{A}$ and $\{\mathscr{H}_z\}$ make use of the following change of variables, which was introduced in [GMMST1]. Define the $\mathbb{C}\cup\{\infty\}$-valued function τ on the set $\{\xi \in \mathbb{C} : |\xi| \le 1, |\arg \xi| \le \frac{\pi}{2}\}$ to be as

$$\tau(\xi) = \begin{cases} \log \frac{1+\xi}{1-\xi} & \text{if } \xi \ne 1, \\ \infty & \text{if } \xi = 1. \end{cases} \tag{3.1.11}$$

It is not difficult to see that τ maps its domain onto the halfstrip

$$\{z \in \mathbb{C} : \operatorname{Re} z \ge 0, |\operatorname{Im} z| \le \frac{\pi}{2}\} \cup \{\infty\},$$

the interval $(0,1)$ onto $(0,\infty)$ and the sector $S_{\phi_p} = \{\xi \in \mathbb{C} : |\xi| \le 1, 0 \le \arg(\xi) \le \phi_p\}$ onto the set $\mathbf{F}_p \cup \{\infty\}$. It is straightforward to check that, if $\xi \ne 1$, then

$$h_{\tau(\xi)}(x,y) = \frac{(1-\xi)^n}{\pi^n (4\xi)^{n/2}} \exp\Big(- \frac{|x|^2 + |y|^2}{2} - \frac{1}{4}\big(\xi\, |x+y|^2 + \frac{1}{\xi}\, |x-y|^2\big)\Big). \tag{3.1.12}$$

We also set $h_\infty(x,y) := 1$ for all x, y.

Theorem 3.1.3. *Let $1 \leq p \leq 2$. Then $\mathscr{H}_z$ is bounded on $L^p(\gamma_{-1})$ if and only if $z \in \mathbf{E}_p$, and in this case it is a contraction.*

Proof. The case where $p = 2$ follows directly from spectral theory. Thus, we may assume that $1 \leq p < 2$.

We observe that $\mathscr{H}_{\tau(\xi)}$ is bounded on $L^p(\gamma_{-1})$ if and only if $\mathscr{H}^{\lambda}_{\tau(\xi)} = \mathscr{U}_p \mathscr{H}_{\tau(\xi)} \mathscr{U}_p^{-1}$ is bounded on $L^p(\lambda)$. The kernel of the latter operator with respect to the Lebesgue measure is of Gaussian type and is given by

$$\begin{aligned} h^{\lambda}_{\tau(\xi)}(x,y) &= \frac{(1-\xi)^n}{\pi^{n/2}(4\xi)^{n/2}} \exp\Big(- \frac{|x|^2+|y|^2}{2} - \frac{1}{4}\big(\xi\,|x+y|^2 + \frac{1}{\xi}\,|x-y|^2\big) + \frac{|x|^2}{p} + \frac{|y|^2}{p'}\Big) \\ &= \frac{(1-\xi)^n}{\pi^{n/2}(4\xi)^{n/2}} \exp\Big(- \beta(\xi)\,|x|^2 - \epsilon(\xi)\,|y|^2 + 2\delta(\xi) x\cdot y\Big), \end{aligned}$$

where

$$\beta(\xi) = \frac{1}{2} + \frac{1}{4}\Big(\xi + \frac{1}{\xi}\Big) - \frac{1}{p}, \qquad \epsilon(\xi) = \frac{1}{2} + \frac{1}{4}\Big(\xi + \frac{1}{\xi}\Big) - \frac{1}{p'}, \qquad \delta(\xi) = \frac{1}{4}\Big(\frac{1}{\xi} - \xi\Big).$$

We rewrite the condition $\xi \in S_{\phi_p}$ in terms of the coefficients β, ϵ and δ. It is easy to see that, for $|\xi| \leq 1$, $|\arg \xi| \leq \phi_p$ if and only if

$$(\operatorname{Re}\delta(\xi))^2 \leq (\operatorname{Re}\beta(\xi))\;(\operatorname{Re}\epsilon(\xi)). \tag{3.1.13}$$

Indeed,

$$\begin{aligned} &(\operatorname{Re}\delta(\xi))^2 - (\operatorname{Re}\beta(\xi))(\operatorname{Re}\epsilon(\xi)) \\ &= \Big(\frac{\operatorname{Re}\xi}{4}\Big(\frac{1}{|\xi|^2} - 1\Big)\Big)^2 - \Big(\frac{\operatorname{Re}\xi}{4}\Big(1 + \frac{1}{|\xi|^2}\Big) + \frac{1}{2} - \frac{1}{p}\Big)\Big(\frac{\operatorname{Re}\xi}{4}\Big(1 + \frac{1}{|\xi|^2}\Big) - \frac{1}{2} + \frac{1}{p}\Big) \\ &= -\frac{(\operatorname{Re}\xi)^2}{4\,|\xi|^2} + \Big(\frac{1}{p} - \frac{1}{2}\Big)^2 \\ &= \frac{1}{4}\Big(- \frac{(\operatorname{Re}\xi)^2}{(\operatorname{Re}\xi)^2 + (\operatorname{Im}\xi)^2} + \frac{(2-p)^2}{p^2}\Big), \end{aligned}$$

so that $(\operatorname{Re}\delta(\xi))^2 \leq (\operatorname{Re}\beta(\xi))(\operatorname{Re}\epsilon(\xi))$ if and only if

$$\left|\frac{\operatorname{Im}\xi}{\operatorname{Re}\xi}\right| \leq \frac{2\sqrt{p-1}}{2-p} = \tan\phi_p.$$

We prove that, for $\operatorname{Re}\xi \geq 0$ and $\operatorname{Im}\xi \geq 0$, $\mathscr{H}^{\lambda}_{\tau(\xi)}$ is bounded on $L^p(\lambda)$ if and only if (3.1.13) holds.

First we prove that if (3.1.13) holds, then $\mathscr{H}^\lambda_{\tau(\xi)}$ is bounded on $L^p(\lambda)$. We observe that

$$\begin{aligned}\mathscr{H}^\lambda_{\tau(\xi)} &= \Big(\frac{\pi\gamma^*}{\delta(\xi)}\Big)^{n/2} \frac{(1-\xi)^n}{\pi^{n/2}(4\xi)^{n/2}} M_{\beta(\xi)-\gamma^*\delta(\xi)}\, T_{\gamma^*}\, \mathrm{e}^{-\gamma^*/(4\delta(\xi))\,\Delta}\, M_{\epsilon(\xi)-\delta(\xi)/\gamma^*} \\ &= \frac{(\gamma^*)^{n/2}(1-\xi)^{n/2}}{(1+\xi)^{n/2}} M_{\beta(\xi)-\gamma^*\delta(\xi)}\, T_{\gamma^*}\, \mathrm{e}^{-\gamma^*/(4\delta(\xi))\,\Delta}\, M_{\epsilon(\xi)-\delta(\xi)/\gamma^*},\end{aligned} \tag{3.1.14}$$

where $\gamma^* > 0$ is a constant which will be chosen later. Here M_α, for $\alpha \in \mathbb{C}$, denotes the multiplication operator given by

$$M_\alpha f(x) = \mathrm{e}^{-\alpha|x|^2} f(x),$$

T_{γ^*}, for $\gamma^* > 0$, the dilation operator

$$T_{\gamma^*} f(x) = f(\gamma^* x),$$

and $\{\mathrm{e}^{-z\Delta} : \operatorname{Re} z \geq 0\}$ the heat semigroup on $(\mathbb{R}^n, \lambda)$, defined by

$$\mathrm{e}^{-z\Delta} f(x) = \frac{1}{(4\pi z)^{n/2}} \int \exp\Big(\frac{-|x-y|^2}{4z}\Big) f(y)\, \mathrm{d}y.$$

A straightforward calculation shows that

$$\begin{aligned}&\exp\Big(-\beta(\xi)\,|x|^2 - \epsilon(\xi)\,|y|^2 + 2\delta(\xi)x\cdot y\Big) \\ &= \exp\Big((-\beta(\xi)+\gamma^*\delta(\xi))\,|x|^2\Big) \exp\Big(-\delta(\xi)\,|\gamma^* x - y|^2/\gamma^*\Big) \exp\Big((-\epsilon(\xi)+\delta(\xi)/\gamma^*)\,|y|^2\Big),\end{aligned}$$

so that (3.1.14) holds. Now, write $\xi = t\mathrm{e}^{i\phi}$, with $t \in (0,1]$ and $0 \leq \phi \leq \phi_p$. We shall treat the two cases $\operatorname{Re}\delta(\xi) = 0$ or $\operatorname{Re}\delta(\xi) > 0$ separately.

Suppose first that $\operatorname{Re}\delta(\xi) > 0$, i.e., $t \neq 1$. Then $\operatorname{Re}\big(\gamma^*/(4\delta(\xi))\big) > 0$, hence $\mathrm{e}^{-\gamma^*/(4\delta(\xi))\,\Delta}$ is bounded on $L^p(\lambda)$. The operator $M_{\beta(\xi)-\gamma^*\delta(\xi)}$ is bounded on $L^p(\lambda)$ if and only if

$$\operatorname{Re}\beta(\xi) - \gamma^* \operatorname{Re}\delta(\xi) \geq 0,$$

whereas $M_{\epsilon(\xi)-\delta(\xi)/\gamma^*}$ is bounded on $L^p(\lambda)$ if and only if

$$\operatorname{Re}\epsilon(\xi) - \operatorname{Re}\delta(\xi)/\gamma^* \geq 0.$$

Therefore the two operators are simultaneously bounded if and only if

$$\frac{\operatorname{Re}\delta(\xi)}{\operatorname{Re}\epsilon(\xi)} \leq \gamma^* \leq \frac{\operatorname{Re}\beta(\xi)}{\operatorname{Re}\delta(\xi)}; \tag{3.1.15}$$

since $\xi \in S_{\phi_p}$ implies (3.1.13), such a positive γ^* exists. Then $\mathscr{H}^\lambda_{\tau(\xi)}$ is composition of bounded operators, hence it is bounded on $L^p(\lambda)$, and

$$\left|\!\left|\!\left| \mathscr{H}^\lambda_{\tau(\xi)} \right|\!\right|\!\right|_{L^p(\lambda)} \leq \left|\frac{1-\xi}{1+\xi}\right|^{n/2} \frac{(\gamma^*)^{n/2}}{(\gamma^*)^{n/p}}. \tag{3.1.16}$$

Observe that for all ξ we are considering, we have $|(1-\xi)/(1+\xi)| \leq 1$. We need to show that for every such ξ we may choose γ^* such that the right hand side of the inequality above is less than or equal to 1. Note that

$$\frac{\operatorname{Re}\delta(\xi)}{\operatorname{Re}\epsilon(\xi)} = \frac{\frac{\cos\phi}{4}(1/t - t)}{\frac{\cos\phi}{4}(1/t+t) + \frac{\cos\phi_p}{2}} < 1 \qquad \forall t \in (0,1] \quad \forall \phi \in [0,\phi_p],$$

and that

$$\frac{\operatorname{Re}\beta(\xi)}{\operatorname{Re}\delta(\xi)} = \frac{\frac{\cos\phi}{4}(1/t+t) - \frac{\cos\phi_p}{2}}{\frac{\cos\phi}{4}(1/t-t)} \geq 1$$

if and only if $t \geq \cos\phi_p/\cos\phi$. Therefore, if ξ is such that $t \geq \cos\phi_p/\cos\phi$, then there exists $\gamma^* \geq 1$, satisfying (3.1.15), so that, by (3.1.16),

$$\left|\!\left|\!\left| \mathscr{H}^\lambda_{\tau(\xi)} \right|\!\right|\!\right|_{L^p(\lambda)} \leq (\gamma^*)^{\frac{n}{2}-\frac{n}{p}} \leq 1.$$

If, instead, ξ is such that $t < \cos\phi_p/\cos\phi$, then every γ^* which satisfies (3.1.15) is smaller than 1. In this case, we set

$$\gamma^* = \frac{\operatorname{Re}\beta(\xi)}{\operatorname{Re}\delta(\xi)},$$

so that

$$\begin{aligned}
\left|\!\left|\!\left| \mathscr{H}^\lambda_{\tau(\xi)} \right|\!\right|\!\right|_{L^p(\lambda)} &\leq \left|\frac{1-\xi}{1+\xi}\right|^{n/2} \Big(\frac{\operatorname{Re}\beta(\xi)}{\operatorname{Re}\delta(\xi)}\Big)^{n/2-n/p} \\
&\leq \Big(\frac{1+t^2-2t\cos\phi}{1+t^2+2t\cos\phi}\Big)^{n/4} \Big(\frac{\cos\phi(1-t^2)}{\cos\phi(1+t^2)-2t\cos\phi_p}\Big)^{n/2} \\
&= \Big(\frac{1+t^2-2t\cos\phi}{1+t^2+2t\cos\phi}\ \frac{(1-t^2)^2}{(1+t^2-2t\cos\phi_p/\cos\phi)^2}\Big)^{n/4}.
\end{aligned}$$

The right hand side is at most 1 if and only if

$$(1+t^2-2t\cos\phi)(1-t^2)^2 - (1+t^2+2t\cos\phi)(1+t^2-2t\cos\phi_p/\cos\phi)^2 \leq 0,$$

which is equivalent to

$$a(\phi)t^4 + b(\phi)t^3 + c(\phi)t^2 + b(\phi)t + a(\phi) \geq 0, \tag{3.1.17}$$

where

$$
\begin{aligned}
a(\phi) &= \cos\phi - \cos\phi_p/\cos\phi,\\
b(\phi) &= 1 - 2\cos\phi_p + \cos^2\phi_p/\cos^2\phi,\\
c(\phi) &= 2\cos\phi_p/\cos\phi\,(\cos\phi_p - 1).
\end{aligned}
$$

It is easy to see that (3.1.17) is satisfied for each $t < \cos\phi_p/\cos\phi$. Indeed,

$$
\begin{aligned}
&a(\phi)t^4 + b(\phi)t^3 + c(\phi)t^2 + b(\phi)t + a(\phi)\\
&= t^4\cos\phi + t^3\frac{\cos\phi_p}{\cos\phi}\big(\frac{\cos\phi_p}{\cos\phi} - t\big) + t^3(1-\cos\phi_p) + t^2\frac{\cos\phi_p}{\cos\phi}(2\cos\phi_p - 1)\\
&\quad + t(1 - t^2\cos\phi_p) + t\frac{\cos\phi_p}{\cos\phi}\big(\frac{\cos\phi_p}{\cos\phi} - t\big) + \cos\phi - t\cos\phi_p + \frac{\cos\phi_p}{\cos\phi}(1 - t\cos\phi).
\end{aligned}
$$

It is clear that all the terms here are nonnegative if $\cos\phi_p \geq 1/2$. Similarly, by writing

$$
\begin{aligned}
&a(\phi)t^4 + b(\phi)t^3 + c(\phi)t^2 + b(\phi)t + a(\phi)\\
&= t^4\cos\phi + t^3\frac{\cos\phi_p}{\cos\phi}\big(\frac{\cos\phi_p}{\cos\phi} - t\big) + t^3(1-2\cos\phi_p) + 2t^2\frac{\cos^2\phi_p}{\cos\phi}\\
&\quad + t\frac{\cos\phi_p}{\cos\phi}\big(\frac{\cos\phi_p}{\cos\phi} - t\big) + \frac{\cos\phi_p}{\cos\phi}(1-t^2) + t(1-2\cos\phi_p) + \big(\cos\phi + \frac{\cos\phi_p}{\cos\phi}\big),
\end{aligned}
$$

we see that (3.1.17) is satisfied also if $\cos\phi_p < 1/2$. This concludes the proof of the fact that, if $\operatorname{Re}\delta(\xi) > 0$, then condition (3.1.13) implies the contractivity of $\mathscr{H}^\lambda_{\tau(\xi)}$ on $L^p(\lambda)$.

We now turn to the case $\operatorname{Re}\delta(\xi) = 0$, i.e., $t = 1$. Write $\xi = \mathrm{e}^{i\phi}$, and assume momentarily that $\phi < \phi_p$. Then

$$
\begin{aligned}
&\left|\exp\left(-\beta(\xi)\,|x|^2 - \epsilon(\xi)\,|y|^2 + 2\delta(\xi)x\cdot y\right)\right|\\
&= \exp\left(-\operatorname{Re}\beta(\xi)\,|x|^2 - \operatorname{Re}\epsilon(\xi)\,|y|^2 + 2\operatorname{Re}\delta(\xi)x\cdot y\right)\\
&= \exp\left(-\frac{\cos\phi - \cos\phi_p}{2}\,|x|^2 - \frac{\cos\phi + \cos\phi_p}{2}\,|y|^2\right).
\end{aligned}
$$

Therefore

$$
\int \left|h^\lambda_{\tau(\xi)}(x,y)\right|\,\mathrm{d}x < \infty \qquad \text{and} \qquad \int \left|h^\lambda_{\tau(\xi)}(x,y)\right|\,\mathrm{d}y < \infty.
$$

Thus, $\mathscr{H}^\lambda_{\tau(\xi)}$ is bounded on $L^1(\lambda)$ and on $L^\infty(\lambda)$. By interpolation, $\mathscr{H}^\lambda_{\tau(\xi)}$ is bounded on $L^p(\lambda)$. Moreover, by Fatou's lemma, for each $0 < \phi < \phi_p$,

$$
\int \left|\mathscr{H}^\lambda_{\tau(\mathrm{e}^{i\phi})} f\right|^p\,\mathrm{d}\lambda \leq \liminf_{t\to 1} \int \left|\mathscr{H}^\lambda_{\tau(t\mathrm{e}^{i\phi})} f(x)\right|^p\,\mathrm{d}x.
$$

Since $\mathscr{H}^{\lambda}_{\tau(te^{i\phi})}$ is a contraction on $L^p(\lambda)$ for $t<1$, it follows that each $\mathscr{H}^{\lambda}_{\tau(e^{i\phi})}$ is a contraction as well. Similarly,

$$\int \left|\mathscr{H}^{\lambda}_{\tau(e^{i\phi_p})} f(x)\right|^p \, dx \leq \liminf_{\phi\to\phi_p} \int \left|\mathscr{H}^{\lambda}_{\tau(e^{i\phi})} f(x)\right|^p \, dx,$$

and so also $\mathscr{H}^{\lambda}_{\tau(e^{i\phi_p})}$ is a contraction on $L^p(\lambda)$.

Next we prove that if $\mathscr{H}^{\lambda}_{\tau(\xi)}$ is bounded on $L^p(\lambda)$, then (3.1.13) holds. Notice that the assumption $p<2$ implies that

$$\operatorname{Re}\epsilon(\xi) = \frac{\cos\phi}{4}(t+1/t) + \frac{\cos\phi_p}{2} > 0. \tag{3.1.18}$$

We consider the action on $\mathscr{H}^{\lambda}_{\tau(\xi)}$ on the family of Gaussian functions $g_s(x) = e^{s|x|^2}$, for $s\in\mathbb{C}$. Observe that if $\operatorname{Re} s \leq \operatorname{Re}\epsilon(\xi) - \operatorname{Re}\delta(\xi)$, then

$$\mathscr{H}^{\lambda}_{\tau(\xi)} g_s(x) = \frac{(1-\xi)^n}{(4\xi)^{n/2}(\epsilon(\xi)-s)^{n/2}} \exp\left(\frac{\delta^2(\xi)-\beta(\xi)\epsilon(\xi)+\beta(\xi)s}{\epsilon(\xi)-s}\,|x|^2\right) \qquad \forall x\in\mathbb{R}^n.$$

If $\operatorname{Re} s<0$, then $g_s \in L^p(\lambda)$. Since of $\mathscr{H}^{\lambda}_{\tau(\xi)}$ is bounded on $L^p(\lambda)$,

$$\operatorname{Re}\frac{\delta^2(\xi)-\beta(\xi)\epsilon(\xi)+\beta(\xi)s}{\epsilon(\xi)-s} < 0,$$

whence

$$\operatorname{Re}\frac{\delta^2(\xi)-\beta(\xi)\epsilon(\xi)+\beta(\xi)s}{\epsilon(\xi)-s} + \operatorname{Re}\beta(\xi) < \operatorname{Re}\beta(\xi). \tag{3.1.19}$$

Consider the complex map

$$M(s) = \frac{\delta^2(\xi)-\beta(\xi)\epsilon(\xi)+\beta(\xi)s}{\epsilon(\xi)-s} + \beta(\xi) = \frac{\delta^2(\xi)}{\epsilon(\xi)-s}.$$

Clearly M is a Möbius transformation, hence if maps generalized circles in the extended complex plane into generalized circles. In particular, M carries the line $\operatorname{Re} s = 0$ into a circle $\widetilde{C}$ passing through the origin.

We *claim* that there exists a point s_1 such that $\operatorname{Re} s_1 = 0$ and

$$\operatorname{Re} M(s_1) = \frac{(\operatorname{Re}\delta(\xi))^2}{\operatorname{Re}\epsilon(\xi)}.$$

Assuming the claim for the moment, we immediately conclude the proof of the theorem by observing that (3.1.19) implies that $\operatorname{Re} M(s_1) \leq \operatorname{Re}\beta(\xi)$, i.e., $(\operatorname{Re}\delta(\xi))^2 \leq (\operatorname{Re}\beta(\xi))(\operatorname{Re}\epsilon(\xi))$ holds.

To prove the claim, consider the point $s_2 = i \operatorname{Im} \epsilon(\xi)$, which minimizes $|\epsilon(\xi) - s|$ subject to the constraint $\operatorname{Re} s = 0$. Since

$$|M(s_2)| = \frac{|\delta^2(\xi)|}{|\epsilon(\xi) - s_2|} \geq \frac{|\delta^2(\xi)|}{|\epsilon(\xi) - s|}$$

for any s such that $\operatorname{Re} s = 0$, we see that the segment joining 0 and $M(s_2)$ is a diameter of $\widetilde{C}$, so that the point $M(s_2)/2$ is the centre of $\widetilde{C}$. The point $(M(s_2) + |M(s_2)|)/2$ lies on $\widetilde{C}$ and has the desired real part, indeed

$$\begin{aligned}
\frac{1}{2} \operatorname{Re}(M(s_2) + |M(s_2)|) &= \frac{1}{2} \Big(\operatorname{Re} \frac{\delta^2(\xi)}{\epsilon(\xi) - i \operatorname{Im} \epsilon(\xi)} + \Big| \frac{\delta^2(\xi)}{\epsilon(\xi) - i \operatorname{Im} \epsilon(\xi)} \Big| \Big) \\
&= \frac{1}{2} \Big(\frac{(\operatorname{Re} \delta(\xi))^2 - (\operatorname{Im} \delta(\xi))^2}{\operatorname{Re} \epsilon(\xi)} + \frac{(\operatorname{Re} \delta(\xi))^2 + (\operatorname{Im} \delta(\xi))^2}{\operatorname{Re} \epsilon(\xi)} \Big) \\
&= \frac{(\operatorname{Re} \delta(\xi))^2}{\operatorname{Re} \epsilon(\xi)}.
\end{aligned}$$

The claim is proved. □

3.2 Local Calderón–Zygmund theory

The kernels of many operators naturally associated to the operator $\mathscr{A}$ are singular integral operators in suitable neighbourhoods of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$, which we now define.

Definition 3.2.1. For every $\rho > 0$, let

$$N_\rho = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \ |x - y| \leq \frac{\rho}{1 + |x| + |y|}\}, \tag{3.2.1}$$

and denote by G_ρ its complementary set. We call N_ρ and G_ρ the *local region* and the *global region*, respectively.

Definition 3.2.2. A linear operator T, mapping the space of test functions into the space of measurable functions on $\mathbb{R}^n$, is a *local Calderón–Zygmund operator* if it satisfies the following assumptions:

(a) T extends to a bounded operator either on $L^q(\lambda)$ or on $L^q(\gamma_{-1})$ for some q, $1 < q < \infty$, or is of weak type 1 with respect to λ or to γ_{-1};

(b) there exists a measurable function K, defined off the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, such that for every test function f

$$Tf(x) = \int K(x,y) f(y) \, \mathrm{d}y$$

for all x outside the support of f;

(c) the function K satisfies

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \quad |\partial_x K(x,y)| + |\partial_y K(x,y)| \leq \frac{C}{|x-y|^{n+1}},$$

for all (x,y) in the local region N_2, $x \neq y$.

If φ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\varphi(x,y) = 1$ for $(x,y) \in N_1$, $\varphi(x,y) = 0$ for $(x,y) \notin N_2$ and

$$|\partial_x \varphi(x,y)| + |\partial_y \varphi(x,y)| \leq \frac{C}{|x-y|} \quad \text{for } x \neq y, \tag{3.2.2}$$

we define the *local part* and the *global part* of the local Calderón–Zygmund operator T by

$$\begin{aligned} T_{\mathrm{loc}} f(x) &= \int K(x,y) \; \varphi(x,y) \, \mathrm{d}y, \\ T_{\mathrm{glob}} f(x) &= Tf(x) - T_{\mathrm{loc}} f(x). \end{aligned} \tag{3.2.3}$$

Notice that the kernel of T_{loc} is supported in the local region N_2, so that, for all test functions f and g such that $f \otimes g$ is supported in the global region G_2,

$$\int T_{\mathrm{loc}} f(x) \; g(x) \, \mathrm{d}x = 0.$$

More generally, we say that a linear operator S, mapping $\mathscr{C}_c^\infty(\mathbb{R}^n)$ into $\mathscr{C}_c^\infty(\mathbb{R}^n)'$ is a *local operator* if

$$\int Sf(x) \; g(x) \, \mathrm{d}x = 0 \tag{3.2.4}$$

for all test functions f and g such that $f \otimes g$ is supported in a global region.

In this section we shall prove that if T is a local Calderón–Zygmund operator, then its local part is bounded both on $L^p(\lambda)$ and on $L^p(\gamma_{-1})$, whenever $1 < p < \infty$, and it is of weak type 1 with respect to both measures. Furthermore, we shall prove that, for local operators, strong and weak boundedness with respect to Lebesgue measure and the measure γ_{-1} are equivalent. These results are extensions to our setting of a well established technique, which has been developed in previous papers on the Ornstein–Uhlenbeck operator. In particular, we follow closely the treatment in [GMST1]. We shall make use of the following covering lemma, which appears in [GMST1] (except for (7), whose proof, however, is almost *verbatim* as that of the corresponding statement for the Gauss measure).

Lemma 3.2.3. *There exists a collection of balls*

$$B_j = B(x_j, \kappa/(1+|x_j|)),$$

where $\kappa = 1/20$, *such that*

1. *the collection* $\{B_j : j \in \mathbb{N}\}$ *covers* $\mathbb{R}^n$;
2. *the balls* $\{1/4B_j : j \in \mathbb{N}\}$ *are pairwise disjoint;*
3. *for any* $A > 0$, *the collection* $\{AB_j : j \in \mathbb{N}\}$ *has the bounded overlap property, i.e.,* $\sup \sum_j \chi_{AB_j} < \infty$;
4. $B_j \times 4B_j \subset N_1$ *for all* $j \in \mathbb{N}$;
5. *for each* $\rho > 0$ *there exists* $\delta > 0$ *such that* $B_j \times (\delta B_j)^c \subset G_\rho$;
6. $N_{1/7} \subseteq \cup_j (B_j \times 4B_j) \subseteq N_1$;
7. *there exist positive constants* C_1 *and* C_2 *such that for all* $j \in \mathbb{N}$,
$$C_1\, \mathrm{e}^{|x_j|^2} \lambda(E) \leq \gamma_{-1}(E) \leq C_2\, \mathrm{e}^{|x_j|^2} \lambda(E) \tag{3.2.5}$$
for each measurable subset E *of* δB_j (δ *is as in 5.).*

We shall also need the following lemma, whose proof can be found in [GMST2].

Lemma 3.2.4. *Let* μ *be a nonnegative Borel measure on* $\mathbb{R}^n$. *Given a sequence of nonnegative measurable functions* $\{f_j\}$, *let* $f = \sum_j \chi_{B_j} f_j$, *where* $\{B_j : j \in \mathbb{N}\}$ *is the collection of balls in Lemma 3.2.3. Then*

$$\mu\{x : f(x) > \alpha\} \leq \sum_j \mu\{x \in B_j : f_j(x) > \frac{\alpha}{M}\} \tag{3.2.6}$$

for all $\alpha > 0$, *where* $M = \sup \sum_j \chi_{B_j}$. *Moreover,*

$$\|f\|_q \leq M \left(\sum_j \int_{B_j} |f_j|^q \ \mathrm{d}\mu \right)^{1/q}, \tag{3.2.7}$$

for $1 \leq q < \infty$.

Proposition 3.2.5. *Let S be a local operator and p be in $(1,\infty)$. Then S is of weak type 1 with respect to λ if and only if it is of weak type 1 with respect to γ_{-1}. The same holds for the L^p boundedness. Moreover the norms of S on $L^p(\gamma_{-1})$ and $L^p(\lambda)$ are comparable.*

Proof. Assume that (3.2.4) holds for all $f, g \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $f \otimes g$ is supported in the global region G_ρ and that S is of weak type 1 with respect to Lebesgue measure. Let $\alpha > 0$. Then, by Lemma 3.2.3, there exists $\delta > 0$ such that $B_j \times (\delta B_j)^c \subset G_\rho$. Since S is local, the values of Sf on B_j depend only on the values of f on δB_j, i.e., $\chi_{B_j} Sf = \chi_{B_j} S(f\chi_{\delta B_j})$. Then, by Lemma 3.2.4 and (3.2.5),

$$
\begin{aligned}
\gamma_{-1}\{x : |Sf(x)| > \alpha\} &\leq \sum_j \gamma_{-1}\{x \in B_j : |Sf(x)| > \frac{\alpha}{M}\} \\
&\leq C_2 \sum_j \mathrm{e}^{|x_j|^2} \lambda\{x \in B_j : \left|S(f\chi_{\delta B_j})(x)\right| > \frac{\alpha}{M}\} \\
&\leq C_2 \sum_j \mathrm{e}^{|x_j|^2} \frac{C}{\alpha} \int_{\delta B_j} |f(x)| \, \mathrm{d}\lambda \\
&\leq \frac{C_2}{C_1} \sum_j \frac{C}{\alpha} \int_{\delta B_j} |f(x)| \, \mathrm{d}\gamma_{-1} \\
&\leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |\phi(x)| \, \mathrm{d}\gamma_{-1}.
\end{aligned}
$$

The proofs of the converse and of the second part of the theorem are similar. We omit the details. □

Similarly, one can prove the following result.

Lemma 3.2.6. *Assume that T is a linear operator mapping the space of L^∞ functions with compact support into the space of measurable functions on $\mathbb{R}^n$. Define, for any measurable and locally bounded function f,*

$$T^1 f := \sum_j \chi_{B_j} T(\chi_{4B_j} f),$$

where $\{B_j\}$ is the covering whose existence is established in Lemma 3.2.3. The following hold:

(i) if T is of weak type 1 with respect to λ or to γ_{-1}, then T^1 is of weak type 1 with respect to both measures;

(ii) if T is bounded on $L^q(\lambda)$ or on $L^q(\gamma_{-1})$ for some $q \in (1, \infty)$, then T^1 is bounded on $L^q(\lambda)$ and on $L^q(\gamma_{-1})$.

Proposition 3.2.7. *Suppose that T is a local Calderón–Zygmund operator. The following hold:*

(i) if $1 \leq q < \infty$ and T is bounded on $L^q(\gamma_{-1})$, then the operator T_{loc}, defined in (3.2.3), is bounded on $L^q(\gamma_{-1})$;

(ii) if T is of weak type 1 with respect to the measure γ_{-1}, then so is the operator T_{loc}.

A similar statement holds with the Lebesgue measure λ in place of γ_{-1}.

Proof. We shall prove (i). The proof of (ii) follows the same lines, and is omitted. The last statement of the proposition follows directly from (i), (ii) and Lemma 3.2.6.

Suppose that f is a test function and x is in B_j, where B_j is one of the balls introduced in Lemma 3.2.3. Then

$$\begin{aligned} T_{\mathrm{loc}} f(x) &= Tf(x) - T_{\mathrm{glob}} f(x) \\ &= T(f\chi_{4B_j})(x) + T(f(1-\chi_{4B_j}))(x) - \int K(x,y)(1-\varphi(x,y))f(y)\,\mathrm{d}y \\ &= T(f\chi_{4B_j})(x) + \int K(x,y)(\varphi(x,y) - \chi_{4B_j}(y))f(y)\,\mathrm{d}y. \end{aligned}$$

Since

$$T_{\mathrm{loc}} f(x) = \frac{\sum_j T_{\mathrm{loc}} f(x)\chi_{B_j}(x)}{\sum_j \chi_{B_j}(x)},$$

$$\begin{aligned} &|T_{\mathrm{loc}} f(x)| \\ &\leq \Big|\sum_j \chi_{B_j}(x) T(f\chi_{4B_j})(x)\Big| + \int \sum_j \chi_{B_j}(x)\,|K(x,y)|\,\big|\varphi(x,y) - \chi_{4B_j}(y)\big|\,|f(y)|\,\mathrm{d}y \qquad (3.2.8) \\ &= \big|T^{(1)} f(x)\big| + T^{(2)}\,|f|\,(x), \end{aligned}$$

where $T^{(1)}$ is as in Lemma 3.2.6 and $T^{(2)}$ is the operator with kernel

$$H(x,y) = \sum_j \chi_{B_j}(x)\,|K(x,y)|\,\big|\varphi(x,y) - \chi_{4B_j}(y)\big|\,.$$

By Lemma 3.2.6 and property (a) of Definition 3.2.2, $T^{(1)}$ is bounded on $L^q(\gamma_{-1})$. Next, we prove that $T^{(2)}$ is bounded on $L^p(\gamma_{-1})$ for all p in $[1, \infty)$. We observe that if $(x,y) \in N_{1/7}$, then

$(x, y) \in N_1$ by 6. of Lemma 3.2.3, whence $\varphi(x, y) - \chi_{4B_j}(y) = 0$. Moreover, if $(x, y) \notin N_2$ and $x \in B_j$, then $y \notin N_2$, and so again $\varphi(x, y) - \chi_{4B_j}(y) = 0$. Thus H is supported in $N_2 \setminus N_{1/7}$. Let (x, y) be such that

$$\frac{1}{7(1 + |x| + |y|)} \leq |x - y| \leq \frac{2}{1 + |x| + |y|}.$$

By (c) of Definition 3.2.2,

$$\begin{aligned} |K(x, y)| &\leq \frac{C}{|x - y|^n} \\ &\leq C\,(1 + |x| + |y|)^n \\ &\leq C\,(1 + |y|)^n, \end{aligned}$$

because $|x| \leq |x - y| + |y| \leq 1 + |y|$. Therefore,

$$\begin{aligned} |H(x, y)| &\leq C\,(1 + |y|)^n \sum_J \chi_{B_j}(x) \\ &\leq C\,(1 + |y|)^n. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} \int \left|T^{(2)} f(x)\right| \; \mathrm{d}\gamma_{-1}(x) &\leq C \int\!\!\int (1 + |y|)^n \chi_{N_2}(x, y)\, |f(y)| \; \mathrm{d}y \; \mathrm{d}\gamma_{-1}(x) \\ &= C \int (1 + |y|)^n\, |f(y)| \; \mathrm{d}y \int_{\{x:\, |x-y| \leq 2/(1+|x|+|y|)\}} \mathrm{d}\gamma_{-1}(x). \end{aligned}$$

Since

$$\lambda\big(B(y, 2/(1 + |y|))\big) \leq C\big(2/(1 + |y|)\big)^n$$

and

$$\gamma_{-1}\big(B(y, 2/(1 + |y|))\big) \leq C\, \mathrm{e}^{|y|^2} \lambda\big(B(y, 2/(1 + |y|))\big),$$

$T^{(2)}$ is bounded on $L^1(\gamma_{-1})$, that is

$$\int \left|T^{(2)} f(x)\right| \; \mathrm{d}\gamma_{-1}(x) \leq C \int |f(x)| \; \mathrm{d}\gamma_{-1}(x).$$

By a similar argument,

$$\begin{aligned} \left|T^{(2)} f(x)\right| &\leq C \int_{\{y:\, |x-y| \leq 2/(1+|x|+|y|)\}} (1 + |y|)^n\, |f(y)| \; \mathrm{d}\gamma_{-1}(y) \\ &\leq C \frac{1}{\gamma_{-1}(B(x, 2/(1 + |x|)))} \int_{B(x, 2/(1+|x|))} |f(y)| \; \mathrm{d}\gamma_{-1}(y). \end{aligned}$$

The boundedness of $T^{(2)}$ on $L^\infty(\gamma_{-1})$ then follows from the boundedness of the Hardy–Littlewood maximal operator on $L^\infty(\gamma_{-1})$. By interpolation, $T^{(2)}$ is bounded on $L^p(\gamma_{-1})$ for each $p \in (1, \infty)$. By (3.2.8), this concludes the proof of the theorem. □

Theorem 3.2.8. *Assume that T is a local Calderón–Zygmund operator. Then T_{loc} is bounded on $L^p(\lambda)$ and on $L^p(\gamma_{-1})$ for any $1 < p < \infty$, and it is of weak type 1 with respect to both measures.*

Proof. By assumption (a) of Definition 3.2.2, T is either bounded on $L^q(\lambda)$ or on $L^q(\gamma_{-1})$, or it is of weak type 1 with respect to one of those measures. Then, by Proposition 3.2.7, T_{loc} is of weak type 1 and of strong type q with respect to both measures. Moreover, T_{loc} is a Calderón–Zygmund operator. Indeed, by Definition 3.2.2 (c) and (3.2.2), its kernel $K_{\text{loc}} = K\varphi$ satisfies

$$|K_{\text{loc}}(x, y)| \leq \frac{C}{|x - y|^n}, \quad |\partial_x K_{\text{loc}}(x, y)| + |\partial_y K_{\text{loc}}(x, y)| \leq \frac{C}{|x - y|^{n+1}}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n,\ x \neq y$.

The conclusion then follows from standard Calderón–Zygmund theory and from Proposition 3.2.5. □

3.3 The maximal operator for the semigroup $\mathscr{H}_t$

The aim of this section is to prove weak type 1 estimates for the maximal operator $\mathscr{H}^*$ associated to the semigroup $\{\mathscr{H}_t\}_{t\geq 0}$, defined by

$$\mathscr{H}^* f(x) = \sup_{t>0} |\mathscr{H}_t f(x)| \,. \tag{3.3.1}$$

We recall that, by Littlewood–Paley–Stein theory, the maximal operator associated to any symmetric diffusion semigroup, on any positive measure space (X, μ), is bounded on $L^p(\mu)$, for each $p \in (1, \infty]$ [St1, Maximal Theorem, page 73], (see also [C]). In general, the maximal operator may fail to be of weak type 1. In the Gaussian context, the weak type $(1, 1)$ estimate for the maximal operator associated to the Ornstein–Uhlenbeck semigroup is due to B. Muckenhoupt [MUu] in the 1-dimensional case and to P. Sjögren [SJ] in higher dimensions. Later, T. Menárguez, S. Pérez and F. Soria [MPS] gave another different proof of the theorem in higher dimensions. Finally, a simpler proof was obtained by J. Garcia-Cuerva, G. Mauceri.

S. Meda, P. Sjögren and J.L. Torrea [GMMST2], as part of a more general theory concerning the maximal operator associated to the holomorphic Ornstein–Uhlenbeck semigroup. We now prove that the approach in [GMMST2] applies also in our context. The strategy is to decompose the maximal operator into the sum of a global and a local part, and analyze them separately. We define the *local* and the *global parts* of $\mathscr{H}^*$ by

$$\mathscr{H}^*_{\mathrm{loc}} f(x) = \sup_{t>0} \Big| \int h_t(x,y)\chi_N(x,y) f(y) \,\mathrm{d}\gamma_{-1}(y)\Big|,$$
$$\mathscr{H}^*_{\mathrm{glob}} f(x) = \sup_{t>0} \Big| \int h_t(x,y)\chi_G(x,y) f(y) \,\mathrm{d}\gamma_{-1}(y)\Big|,$$

where we write N and G instead of N_1 and G_1, and where χ_N and χ_G denote the characteristic functions of the sets N and G, respectively. We shall reduce the study of the local part on $L^p(\gamma_{-1})$ to the analogous problem on $L^p(\lambda)$, where standard Calderón–Zygmund theory applies, and we shall carefully estimate the kernel in the global region. Then the boundedness properties of the operator $\mathscr{H}^*$ will follow from the trivial inequality

$$\mathscr{H}^* f(x) \le \mathscr{H}^*_{\mathrm{loc}} f(x) + \mathscr{H}^*_{\mathrm{glob}} f(x).$$

We recall that, by (3.1.12), for any $s \in (0,1)$

$$h_{\tau(s)}(x,y) = \frac{(1-s)^n}{\pi^n (4s)^{n/2}} \exp\Big(-\frac{|x|^2+|y|^2}{2} - \frac{1}{4}\big(s\,|x+y|^2 + \frac{1}{s}\,|x-y|^2\big)\Big), \tag{3.3.2}$$

where τ is defined in (3.1.11).

We start by considering $\mathscr{H}^*_{\mathrm{loc}}$. We shall need the following preliminary result.

Lemma 3.3.1. *There exists a constant C such that, for every $s \in (0,1]$ and $(x,y) \in N$,*

$$\big|h_{\tau(s)}(x,y)\big| \le C \frac{\mathrm{e}^{-|y|^2}}{s^{n/2}} \exp\Big(-\frac{1}{4s}\,|x-y|^2\Big). \tag{3.3.3}$$

Proof. Clearly,

$$\big|h_{\tau(s)}(x,y)\big| \le \frac{C}{s^{n/2}} \exp\Big(-\frac{|x|^2+|y|^2}{2} - \frac{1}{4}\big(s\,|x+y|^2 + \frac{1}{s}\,|x-y|^2\big)\Big). \tag{3.3.4}$$

If $(x,y) \in N$, then $|x+y|\,|x-y| \le |x-y|\,(|x|+|y|+1) \le 1$, so that $|x|^2 \ge |y|^2 - 1$. Hence

$$\exp\Big(-\frac{|x|^2+|y|^2}{2}\Big) \le C\exp(-|y|^2).$$

Then (3.3.3) follows from this, (3.3.4), and the trivial estimate $\exp(-1/4\, s\,|x+y|^2) \le 1$. $\square$

Theorem 3.3.2. *The operator $\mathscr{H}^*_{\mathrm{loc}}$ is of weak type 1 and of strong type q for each $q \in (1, \infty]$.*

Proof. Let $f \geq 0$. By Lemma 3.3.1,

$$\begin{aligned}\mathscr{H}^*_{\mathrm{loc}} f(x) &\leq C \sup_{t>0} t^{-n/2} \int \exp\Big(\frac{|x-y|^2}{4t}\Big) \chi_L(x,y) f(y) \,\mathrm{d}y \\ &= C \mathscr{W}^* f(x),\end{aligned}$$

where $\mathscr{W}^*$ is the maximal operator associated to the Gauss–Weierstrass semigroup. Therefore $\mathscr{H}^*_{\mathrm{loc}}$ is bounded on $L^q(\lambda)$ for all $q \in (1, \infty]$ and of weak type 1 with respect to Lebesgue measure. Since $\mathscr{H}^*_{\mathrm{loc}}$ is local, the same is true with respect to the measure γ_{-1}, by Proposition 3.2.5. $\square$

Next, we turn to $\mathscr{H}^*_{\mathrm{glob}}$. We introduce the quadratic form

$$\widetilde{Q}_s(x,y) = |(1+s)y - (1-s)x|^2 . \tag{3.3.5}$$

It is straightforward to check that

$$s\,|x+y|^2 + \frac{1}{s}\,|x-y|^2 = \frac{1}{s}\widetilde{Q}_s(x,y) - 2\,|y|^2 + 2\,|x|^2 ,$$

whence

$$h_{\tau(s)}(x,y) = \frac{(1-s)^n}{\pi^n (4s)^{n/2}} \exp\Big(-|x|^2 - \frac{1}{4s}\widetilde{Q}_s(x,y)\Big). \tag{3.3.6}$$

Lemma 3.3.3. *Denote by $\theta = \theta(x,y)$ the angle between the two non–zero vectors x and y (understood to be 0 if $n = 1$). Let $\delta > 0$. Then there exists a constant C such that, for each $x, y \neq 0$ with $(x,y) \in G$,*

$$\sup_{s \in (0,1]} \frac{(1-s)^n}{s^{n/2}} \exp\Big(-\frac{\delta}{s}\widetilde{Q}_s(x,y)\Big) \leq C[(1+|x|)^n \wedge (|x| \sin\theta)^{-n}].$$

Proof. Let (x,y) be in G, with $x \neq y$. We start by proving the first estimate. We claim that

$$|x-y| \geq \frac{1}{2}(1+|x|)^{-1}. \tag{3.3.7}$$

Indeed, if $|y| \geq 1 + |x|$, then

$$|x-y| \geq |y| - |x| \geq 1 \geq \frac{1}{2(1+|x|)},$$

whereas $|y| < 1 + |x|$ implies $|x| + |y| \le 1 + 2\,|x|$, and so

$$|x-y| \ge \frac{1}{1+|x|+|y|} \ge \frac{1}{2(1+|x|)}.$$

Write

$$\begin{aligned}\widetilde{Q}_s(x,y) &= |(1+s)(y-x) - 2sx|^2 \\ &\ge |(1+s)\,|x-y| - 2s\,|x||^2\,,\end{aligned}$$

and note that, if $s \le 1/(8(1+|x|)^2)$, then

$$s\,|x| \le \frac{|x|}{4(1+|x|)^2} \le \frac{1}{4(1+|x|)},$$

whereas

$$(1+s)\,|x-y| \ge \frac{1}{2(1+|x|)}.$$

Hence

$$\widetilde{Q}_s(x,y) \ge C\frac{1}{(1+|x|)^2}. \tag{3.3.8}$$

It follows that

$$\begin{aligned}\sup_{0<s\le\frac{1}{8(1+|x|)^2}} \frac{(1-s)^n}{s^{n/2}} \exp\Big(-\frac{\delta}{s}\widetilde{Q}_s(x,y)\Big) &\le \sup_{0<s\le\frac{1}{8(1+|x|)^2}} \frac{C}{s^{n/2}} \exp\Big(-\frac{\delta}{s(1+|x|)^2}\Big) \\ &\le C(1+|x|)^n.\end{aligned}$$

Finally, it is enough to majorize the exponential by 1 to be able to deduce that

$$\begin{aligned}\sup_{\frac{1}{8(1+|x|)^2}<s\le 1} \frac{(1-s)^n}{s^{n/2}} \exp\Big(-\frac{\delta}{s}\widetilde{Q}_s(x,y)\Big) &\le C\big(8(1+|x|)^2\big)^{n/2} \\ &\le C(1+|x|)^n.\end{aligned}$$

To prove the second estimate, observe that $|(1+s)y-(1-s)x| \ge (1-s)\,|x|\sin\theta$, (the right hand side is the length of the projection of $(1-s)x$ on the hyperplane orthogonal to y). Then

$$\begin{aligned}\sup_{0<s\le 1} \frac{(1-s)^n}{s^{n/2}} \exp\left(-\frac{\delta}{s}\widetilde{Q}_s(x,y)\right) &\le \sup_{0<s\le 1} \frac{(1-s)^n}{s^{n/2}} \exp\left(-\frac{\delta(1-s)^2}{s}\,|x|^2\sin^2\theta\right) \\ &\le C\,(|x|\sin\theta)^{-n},\end{aligned}$$

as required. $\square$

We need one more preliminary result, which is contained in the following lemma. Its proof follows closely the lines of the proof of [GMST1, Theorem 1].

Lemma 3.3.4. *The operator $\widetilde{T}$, defined on test functions by*

$$\widetilde{T}f(x) = \mathrm{e}^{-|x|^2} \int (1+|x|)^n \wedge (|x|\sin\theta)^{-n} f(y) \,\mathrm{d}\gamma_{-1}(y), \tag{3.3.9}$$

extends to an operator of weak type 1 *with respect to the measure γ_{-1}.*

Proof. Assume that $\|f\|_{L^1(\gamma_{-1})} = 1$ and $f \geq 0$, and fix $\alpha > 0$. Let E_α and ξ_α denote the level set $\{x : \widetilde{T}f(x) > \alpha\}$ and the biggest amongst the solutions of the equation

$$\mathrm{e}^{-r^2}\,(1+r)^n = \alpha, \tag{3.3.10}$$

respectively. Then $E_\alpha \subseteq B(0, \xi_\alpha)$, because $|x| \geq \xi_\alpha$ implies

$$\widetilde{T}f(x) \leq \mathrm{e}^{-|x|^2}\,(1+|x|)^n \leq \alpha. \tag{3.3.11}$$

We need only to consider the intersection between E_α and the ring $R = \{\xi_\alpha/2 \leq |x| \leq \xi_\alpha\}$, because

$$\begin{aligned}
\gamma_{-1}\big(B\big(0,\xi_\alpha/2\big)\big) &\leq C\int_0^{\xi_\alpha/2} \mathrm{e}^{\rho^2}\ \rho^{n-1}\,\mathrm{d}\rho \\
&\leq C\,\mathrm{e}^{\xi_\alpha^2/4}\big(\xi_\alpha/2\big)^{n-2} \\
&\leq C\,\mathrm{e}^{\xi_\alpha^2}\,(1+\xi_\alpha)^{-n} \\
&\leq C\,\alpha^{-1}.
\end{aligned}$$

Set $E'_\alpha = \{x' \in \mathbb{S}^{n-1} : \exists \rho \in (\xi_\alpha/2, \xi_\alpha) \text{ such that } \rho x' \in E_\alpha\}$ and, for $x' \in E'_\alpha$, denote by $r(x')$ the biggest such ρ. Then $\widetilde{T}f(r(x')x') = \alpha$ by continuity, and this implies

$$\mathrm{e}^{r(x')^2} \sim \frac{1}{\alpha}\int \xi_\alpha^n \wedge (\xi_\alpha \sin\theta)^{-n} f(y)\,\mathrm{d}\gamma_{-1}(y). \tag{3.3.12}$$

Let $\mathrm{d}x'$ denote the surface measure on $\mathbb{S}^{n-1}$. We have

$$\begin{aligned}
\gamma_{-1}(E_\alpha \cap R) &= \int_{E'_\alpha} \mathrm{d}x' \int_{\xi_\alpha/2}^{r(x')} \mathrm{e}^{\rho^2}\ \rho^{n-1}\,\mathrm{d}\rho \\
&\leq C\int_{E'_\alpha} \mathrm{e}^{r(x')^2} r(x')^{n-2}\,\mathrm{d}x' \\
&\leq \frac{C}{\alpha\,\xi_\alpha^2}\int_{E'_\alpha} \mathrm{d}x' \int \xi_\alpha^{2n} \wedge (\sin\theta)^{-n} f(y)\,\mathrm{d}\gamma_{-1}(y).
\end{aligned}$$

Now, we change the order of integration and observe that

$$\int_{\mathbb{S}^{n-1}} \xi_\alpha^{2n} \wedge (\sin\theta)^{-n} \, \mathrm{d}x' \leq C\xi_\alpha^2, \tag{3.3.13}$$

as required to obtain the estimate for $\gamma_{-1}(E_\alpha)$. □

Theorem 3.3.5. *The operator $\mathscr{H}^*_{\mathrm{glob}}$ is of weak type* 1 *and of strong type q for each $q \in (1,\infty]$.*

Proof. The operator $\mathscr{H}^*_{\mathrm{glob}}$ is trivially bounded on $L^\infty(\gamma_{-1})$; moreover, it is controlled by the operator $\widetilde{T}$ defined in (3.3.9), whence it follows from formula (3.3.6), Lemma 3.3.3 and Lemma 3.3.4 that $\widetilde{T}$ is also of weak type 1. The boundedness of $\mathscr{H}^*_{\mathrm{glob}}$ on $L^p(\gamma_{-1})$ follows from these estimates by interpolation. □

Theorem 3.3.6. *The operator $\mathscr{H}^*$, defined in (3.3.1), is of weak type* 1 *and of strong type q for each $q \in (1,\infty]$.*

Proof. The result follows directly from Theorem 3.3.2 and Theorem 3.3.5. □

3.4 Multiplier operators for $\mathscr{A}$

In this section we summarize some of the results we have obtained concerning spectral multipliers associated to the operator $\mathscr{A}$. As explained in the introduction, we do not provide details of the proofs.

Recall formula (3.1.3), which we rewrite as

$$\mathscr{A} = \sum_{k=n}^{\infty} k \, \mathscr{E}_{k-n}, \tag{3.4.1}$$

where $\mathscr{E}_k = \mathscr{U}_1^{-1} \mathscr{P}_k \mathscr{U}_1$. Given a bounded sequence $M : \{n, n+1, \dots\} \to \mathbb{C}$, we define the *spectral multiplier operator* associated to the *spectral multiplier* M by

$$M(\mathscr{A})f = \sum_{j=n}^{\infty} M(j)\mathscr{E}_j f \qquad \forall f \in L^2(\gamma_{-1}). \tag{3.4.2}$$

Assume now that M is of *Laplace transform type*, i.e., it is of the form

$$M(j) = j \int_0^\infty \phi(t) \, \mathrm{e}^{-tj} \, \mathrm{d}t \qquad \forall j \geq n, \tag{3.4.3}$$

for some bounded measurable function ϕ defined on $(0,\infty)$. Then we say that the operator $M(\mathscr{A})$ associated to M is a *multiplier operator of Laplace transform type.* Since M is bounded on the spectrum of $\mathscr{A}$, $M(\mathscr{A})$ is bounded on $L^2(\gamma_{-1})$.

The following result is the counterpart for $\mathscr{A}$ of a well known theorem concerning the Ornstein–Uhlenbeck operator [GMST2].

Theorem 3.4.1. *If the function M is of Laplace transform type, then the multiplier operator $M(\mathscr{A})$ is of weak type 1 with respect to the measure γ_{-1}.*

Next, we state a result concerning bounded holomorphic functional calculus of order 1/2 in $L^p(\gamma_{-1})$, for $1<p<\infty$. We need some notation. For any $\psi\in(0,\pi)$, we denote by $H^\infty(S_\psi)$ the space of bounded holomorphic functions on the open sector $S_\psi=\{z\in\mathbb{C}:|\arg z|<\psi\}$. It is known that every function M in $H^\infty(S_\psi)$ admits a bounded extension to $\overline{S}_\psi$, also denoted by M.

Definition 3.4.2. Assume that J is a nonnegative integer and that $\psi\in(0,\pi/2)$. Define $H^\infty(S_\psi;J)$ as the Banach space of all the functions M in $H^\infty(S_\psi)$ for which there exists a constant C such that

$$\sup_{R>0}\int_R^{2R}\left|\lambda^j D^j M(\mathrm{e}^{\pm i\psi}\lambda)\right|^2\frac{\mathrm{d}\lambda}{\lambda}\leq C^2\qquad\forall j\in\{0,1,\dots,J\},\tag{3.4.4}$$

endowed with the norm

$$\|M\|_{\psi;J}=\inf\{C:(3.4.4)\text{ holds}\}.$$

Condition (3.4.4) is called a *Hörmander condition of order J.*

For any b in $\mathbb{R}$, denote by τ_b the translation operator

$$\tau_b(x)=x-b\qquad\forall x\in\mathbb{R}.$$

Our main results are the following theorems. Set $\phi_p^*=\arcsin|2/p-1|$.

Theorem 3.4.3. *If $1<p<\infty$ and $u\in\mathbb{R}$, then*

$$\left|\!\left|\!\left|\mathscr{A}^{iu}\right|\!\right|\!\right|_{L^p(\gamma_{-1})}\asymp\mathrm{e}^{\phi_p^*|u|}\qquad\textit{as } u \textit{ tends to } \infty.$$

.

Theorem 3.4.4. *Suppose that a is a positive number, and that $1 < p < \infty$, $p \neq 2$. Let $M : \{n, n+1, \dots\} \to \mathbb{C}$ be a bounded sequence and assume that there exists a bounded holomorphic function $\widetilde{M}$ in $a + S_{\phi_p^*}$ such that*

$$\widetilde{M}(k) = M(k) \qquad \forall k \geq n.$$

Then the following hold:

(i) if $\tau_{-a}\widetilde{M} \in H^\infty(S_{\phi_p^*}; 1/2)$, then $M(\mathscr{A})$ extends to a bounded operator on $L^p(\gamma_{-1})$, hence on $L^q(\gamma_{-1})$ for all q such that $|1/q - 1/2| \leq |1/p - 1/2|$. Furthermore,

$$|||M(\mathscr{A})|||_{L^p(\gamma_{-1})} \leq C\left(\|M\|_\infty + \|\widetilde{M}\|_{\phi_p^*;J}\right),$$

where C does not depend on M;

(ii) if $\tau_{-a}\widetilde{M} \in H^\infty(S_{\phi_p^*})$ and $|1/q - 1/2| < |1/p - 1/2|$, then $M(\mathscr{A})$ extends to a bounded operator on $L^q(\gamma_{-1})$.

We may assume that $a > n$. Observe that

$$\begin{aligned} M(\mathscr{A}) &= \sum_{k \leq a-n} M(k+n)\,\mathscr{E}_k + \sum_{k > a-n} \widetilde{M}(k+n)\,\mathscr{E}_k \\ &= \sum_{k \leq a-n} M(k+n)\,\mathscr{E}_k + \sum_{k > a-n} \widetilde{M}(k+n-a+a)\,\mathscr{E}_k \\ &= \sum_{k \leq a-n} M(k+n)\,\mathscr{E}_k + \sum_{j=1}^\infty \left(\tau_{-a}\widetilde{M}\right)(j)\,\mathscr{E}_{j+a-n}. \end{aligned} \tag{3.4.5}$$

Observe that $\tau_{-a}\widetilde{M}$ is in $H^\infty(S_{\phi_p^*}; J)$. Denote by $\mathscr{S}_a$ the operator, spectrally defined on $L^2(\gamma_{-1})$ by

$$\mathscr{S}_a f = \sum_{j=1}^\infty \mathscr{E}_{j+a-n} f.$$

Thus,

$$M(\mathscr{A}) = \sum_{k \leq a-n} M(k+n)\,\mathscr{E}_k + \left(\tau_a\widetilde{M}\right)(\mathscr{A})(\mathscr{S}_a f).$$

Since the projection $\mathscr{E}_k$ extend to operators bounded on $L^p(\gamma_{-1})$, $\mathscr{S}_a$ extends to a bounded operator on $L^p(\gamma_{-1})$.

Now we briefly discuss some necessary conditions for $M(\mathscr{A})$ to be an $L^p(\gamma_{-1})$ spectral multiplier. Our approach follows the same lines as [HMM].

Following Hebisch, Mauceri and Meda [HMM], we introduce the class of $L^p(\gamma_{-1})$ *uniform spectral multipliers* of $\mathscr{A}$, as the space of those multipliers M satisfying

$$\sup_{t>0} |||M(t\mathscr{L})|||_{L^p(\gamma_{-1})} < \infty.$$

We have proved the following.

Theorem 3.4.5. *Assume that $M : [0,\infty) \to \mathbb{C}$ is bounded, and continuous on $(0,\infty)$. The following hold:*

(i) if p is in $(1,\infty) \setminus \{2\}$ and M is a $L^p(\gamma_{-1})$ uniform spectral multiplier of $\mathscr{A}$, then M extends to a bounded holomorphic function in the sector $S_{\phi_p^*}$ and

$$\|M\|_{H^\infty(S_{\phi_p^*})} \leq \sup_{t>0} |||M(t\mathscr{A})|||_{L^p(\gamma_{-1})};$$

(ii) $\sup_{t>0} |||M(t\mathscr{A})|||_{L^1(\gamma_{-1})} < \infty$ if and only if M extends to a bounded holomorphic function in the half plane $S_{\pi/2}$, and $M(-2i\cdot)$ is the Fourier transform of a finite measure μ^M on $\mathbb{R}$, supported in $[0,\infty)$; furthermore,

$$\|\mu^M\|_{M(\mathbb{R})} = \sup_{t>0} |||M(t\mathscr{A})|||_{L^1(\gamma_{-1})}.$$

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