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D-Complex Structures on Manifolds: Cohomological Properties and Deformations

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Introduction

An *almost **D**-structure* on a $2n$-dimensional manifold M is an endomorphism K of the tangent bundle TM such that $K^2 = \mathrm{Id}_{TM}$ and the two eigendistributions $TM^{\pm}$ with eigenvalue ± 1 have the same rank n. An almost **D**-structure K is said to be *integrable* or to be a ***D**-structure* if $TM^{\pm}$ are involutive. It turns out that the integrability condition is equivalent to the vanishing of the *torsion tensor* N_K.

The basic idea of the **D**-geometry is to replace the field of complex numbers $\mathbb{C}$ with the algebra of *double numbers* $\mathbf{D}$ (namely $\mathbf{D} := \{x + \tau y \mid x, y \in \mathbb{R}\}$ where $\tau^2 = +1$), and consequently the model space of such a **D**-geometry is given by the set $\mathbf{D}^n$ of the n-tuples of double numbers. In some sense, the **D**-geometry is a counterpart of complex geometry, and while this one is elliptic, that one shows his hyperbolic side (e.g., the characteristic equation of **D**-holomorphic functions is the wave equation, while that one of holomorphic complex functions is elliptic).

Obviously there are a lot of parallelisms between **D**-geometry and complex geometry, but there is also a large number of links between **D**-structures and other geometric structures (e.g., *product structures* [5], *pseudo-Riemannian geometry*, *Bochner-Kähler metrics* [16], *bi-Lagrangian structures* [30], *Lorentzian surfaces*).

For example, **D**-structures naturally appear in the context of *bi-Lagrangian structures* on a symplectic manifold (M, ω) (namely a pair of Lagrangian transverse foliations on M) which is equivalent to assign a **D**-Kähler structure on M (i.e. a pair (K, g) of a **D**-structure K and a pseudo-Riemannian metric g of signature (n, n) such that $g(K\cdot, K\cdot) = -g(\cdot, \cdot)$ and whose fundamental form $\omega(\cdot, \cdot) = g(\cdot, K\cdot)$ is closed, see [16, § 5.2]).

The **D**-complex structures have been introduced more than fifty years ago and studied by many authors: e.g. P.K. Rashevskij [68], P. Libermann [57]. Because of the many connections to various branches of mathematics and physics as listed before, the **D**-structures are presented with many different names, e.g. *almost product structures* or *hyperbolic structures*. A very used name is the one introduced by P. Libermann in her Ph.D. Thesis [56], that is *para-complex structures* (that name is also used by many other authors in the literature, see e.g. [2, 20, 21], and the reference therein). Nevertheless, through this thesis we will follow the recent terminology introduced by F.R. Harvey and H.B. Lawson in [40].

Indeed, F.R. Harvey and H.B. Lawson [40], motivated by the study of calibrated submanifolds in semi-Riemannian geometry (see [60]) and by the optimal transport problem (see [48]), focused on the geometry of Special Lagrangian submanifolds in a **D**-Kähler manifold with trivial canonical bundle, that is the **D**-analogous of the Calabi-Yau case (this arguments will be covered in Chapter 4). Other recent results relate the **D**-complex structures to important analysis problem, e.g., C.D. Hill and P. Nurowsky in [42] gave an application of CR **D**-complex structures to the study of systems of partial differential equations and systems of ordinary differential equations, while M. Chursin, L. Schäfer and K. Smoczyk [18] studied the mean curvature flow in (almost) **D**-Kähler manifolds.

Nevertheless, there are also differences with the complex geometry. For example, there exist compact manifolds without any complex structure which admit a **D**-Kähler metric (see e.g. [61] where it is constructed a family of **D**-Kähler metrics on a nilmanifold which does not carry any complex structure). On the other hand, it is known that a sphere of even dimension does not have any **D**-structure by topological obstructions (see [29, Corollary 2.5] or [71]). Furthermore we find a **D**-Kähler Ricci flat metric on a manifold which does not admit any Kähler structure (see 4.2). As far as we know there is not a cohomological decomposition which could give topological obstructions to the existence of **D**-structures or **D**-Kähler structures as in the complex case.

In some sense, the **D**-geometry is settled between the complex geometry and a real pseudo-Riemannian geometry, depending on the choice of the coordinates, and both viewpoints have their strengths and weaknesses. If on a manifold we use the **D**-coordinates, we get a lot of parallelisms to complex geometry, but we have to be careful because we do not work with a field and we have divisors of zero (indeed, **D** is a ring and is not an integral domain so it is not a field). If we work with the null coordinates, we lose the complex viewpoint and a lot of good ideas coming from complex geometry. Through this work we will try to keep an eye on both these possibilities, to keep the best from each side.

We start this thesis recalling the standard definitions and the main properties of **D**-complex structures, trying to underline useful similarities and different behavior with respect to the complex structures. We also turn our attention on CR **D**-structures, which we will use through this work. At the end of the first chapter, we focus on automorphisms group of **D**-structures, which is infinite dimensional. However, under some restricting assumptions, we show that such a group is finite (see Proposition 1.8.20 and Corollary 1.8.21).

The first question we wonder relates to the study of small deformations of **D**-structures on a compact **D**-manifold. The starting point is the paper by C. Medori and A. Tomassini [61], in which are defined and constructed curves of (almost) **D**-complex structures.

In analogy to the classical theory of deformations of complex structures developed by K. Kodaira and D.C. Spencer (see [51, 52]), a *Differential Graded Lie Algebra* $(\mathcal{A}, [[\cdot,\cdot]], \overline{\partial}_K)$ (shortly *DGLA*) is introduced in [61] to characterize small deformations of a **D**-structure K on a compact manifold M. It turns out that these deformations are parametrized by 1-degree elements of $\mathcal{A}$ satisfying the *Maurer-Cartan equation*, where the space of 1-degree elements is $\mathcal{A}_1 = \Gamma(M, \wedge_K^{0,1}(M) \otimes T^{1,0}M)$, i.e. the space of endomorphisms of TM anticommuting with K (see 2.3.9).

We first construct a new DGLA $\widehat{\mathcal{A}}$ such that $\widehat{\mathcal{A}}_1 = \mathcal{A}_1$ (i.e. in the new DGLA we have the 1-degree elements of $\mathcal{A}$ that parametrized the deformations), then we construct a DGLA injective homomorphism which allows to embed $\widehat{\mathcal{A}}$ in the differential graded Lie algebra $\mathcal{F}$ of skew-symmetric derivations on $\wedge_K^{0,*}(M)$. More precisely we get:

Theorem 2.4.2 ([69, Theorem 2.3])**.** *The map $q : \widehat{\mathcal{A}} \to \mathcal{F}$ is a DGLA homomorphism, i.e. q is an injective map satisfying:*

$$[q(\varphi), q(\psi)] = q([[\varphi,\psi]]) \qquad \textit{(equivalently } [\urcorner_\varphi, \urcorner_\psi] = \urcorner_{[[\varphi,\psi]]}\textit{)} \tag{2.4.7}$$

and

$$\delta_K q(\varphi) = q(\overline{\partial}_K \varphi) \qquad \textit{(equivalently } \delta_K \urcorner_\varphi = \urcorner_{\overline{\partial}_K \varphi}\textit{).} \tag{2.4.8}$$

(Such a theorem, as well as other results of Chapter 2 of this thesis, is already published in our paper [69] *J. Geom. Phys.*, 2012. In this thesis we try to motivate and explain better these results).

As a consequence, we can describe the integrability condition of the deformations in terms of a suitable operator $\urcorner$ (see Section 2.4 and Corollary 2.4.5). Moreover, we are able to restate the Maurer-Cartan integrability condition of [61] in the **D**-holomorphic setting (see Remark 2.4.8).

Always in the paper [69], we study the analogous problem for *CR* $\boldsymbol{D}$*-structures*. An almost CR $\mathbf{D}$-structure of codimension $2k$ on a $(m+2k)$-manifold M is a pair $(\mathcal{H}, K)$ where $\mathcal{H}$ is a distribution and $K \in \mathrm{End}(\mathcal{H})$ is an almost $\mathbf{D}$-complex structure. Inspired by a work of P. de Bartolomeis and F. Meylan [22], we focus on *strictly CR* $\boldsymbol{D}$*-structures*, that is a contact manifold (M, ξ) with an integrable CR $\mathbf{D}$-complex structure K on the contact distribution. We construct a deformation theory for such structures and an appropriate DGLA and we show that the integrable deformations must satisfy a Maurer-Cartan condition, as in the previous case. Indeed, we get the following:

Theorem 2.5.11 ([69, Theorem 3.9])**.** *Let $K \in \mathcal{D}(\xi)$ be a strictly CR $\boldsymbol{D}$-structure on a compact contact manifold (M, ξ), and let $\hat{K} \in \mathcal{D}(\xi)$ be given by:*

$$\hat{K} = (\mathrm{Id} + \varphi) K (\mathrm{Id} + \varphi)^{-1}, \quad \textit{where} \ \ \varphi K + K \varphi = 0, \quad \varphi^t = \varphi. \tag{2.5.10}$$

Let $\widetilde{\varphi}$ be the operator associated to φ via the isomorphism m:

$$m : \xi \longrightarrow \xi^{0,1}, \qquad X \longmapsto \widetilde{X} = \frac{1}{2}(X + \tau K X).$$

Then

$$N_{\hat{K}} = 0 \Longleftrightarrow \overline{\partial}_K \widetilde{\varphi} + \frac{1}{2}[[\widetilde{\varphi}, \widetilde{\varphi}]] = 0.$$

Furthermore, we restate the integrability condition in the DGLA of skew-symmetric derivations of $\wedge_K^{0,*}(\xi)$ (see Theorem 2.5.9 and Remark 2.5.13), as done for $\mathbf{D}$-structures. We stress that the study of deformations using DGLAs does not involve the local ($\mathbf{D}$-holomorphic) coordinates, hence it describes intrinsically the deformations of the $\mathbf{D}$-structure K on a manifold, as well as it describes intrinsically the deformations of the CR $\mathbf{D}$-structure K.

We construct some examples of strictly CR $\mathbf{D}$-structures on some nilmanifolds, proving that on nilmanifold with structure equations isomorphic to $(0, 0, 12, 13, 14+23)$ (for notations and conventions, see Chapter 1, more precisely on nilmanifold see Section 1.6) there does not exist any strictly CR $\mathbf{D}$-structure (see Proposition 2.6.4).

A problem dealing with $\mathbf{D}$-complex structures is that one has to handle with semi-Riemannian metrics and not with Riemannian ones. In particular, one can try to reformulate a $\boldsymbol{D}$*-Dolbeault cohomological theory* for $\mathbf{D}$-complex structures, in the same vein as Dolbeault cohomology theory for complex manifolds: but one suddenly finds that such $\mathbf{D}$-Dolbeault groups are in general not finite-dimensional (for example, yet the space of $\mathbf{D}$-holomorphic functions on the product of two equi-dimensional manifolds is not finite-dimensional). In fact, one loses the ellipticity of the second-order differential operator associated to such $\mathbf{D}$-Dolbeault cohomology. Therefore, it would be interesting to find some other (well-behaved) counterpart to $\mathbf{D}$-Dolbeault cohomology groups.

It turns out that also the investigation of a $\mathbf{D}$-complex version of the complex $\partial\overline{\partial}$-Lemma is not a good way to look for such structures. In fact, we prove in this thesis that there are no $\mathbf{D}$-complex manifolds satisfying a $\mathbf{D}$-complex $\partial\overline{\partial}$-Lemma (see Proposition 3.1.11).

Recently, T.-J. Li and W. Zhang considered in [55] some subgroups, called $H^+_J(M;\mathbb{R})$ and $H^-_J(M;\mathbb{R})$, of the real second de Rham cohomology group $H^2_{dR}(M;\mathbb{R})$ of an almost complex manifold $(M,\ J)$, characterized by the type of their representatives with respect to the almost complex structure (more precisely, $H^+_J(M;\mathbb{R})$, respectively $H^-_J(M;\mathbb{R})$, contains the de Rham cohomology classes admitting a J-invariant, respectively J-anti-invariant, representative): in a sense, these subgroups behave as a "generalization" of the Dolbeault cohomology groups to the complex non-Kähler and non-integrable cases. In particular, these subgroups seem to be very interesting for 4-dimensional compact almost complex manifolds and in studying relations between cones of metric structures, see [55, 27, 11]. In fact, T. Drăghici, T.-J. Li and W. Zhang proved in [27, Theorem 2.3] that every almost complex structure J on a 4-dimensional compact manifold M induces the decomposition $H^2_{dR}(M;\mathbb{R}) = H^+_J(M;\mathbb{R}) \oplus H^-_J(M;\mathbb{R})$; the same decomposition holds true also for compact Kähler manifolds, thanks to Hodge decomposition (see, e.g., [55]), while examples of complex and almost complex structures in dimension greater than 4 for which it does not hold are known.

In our joint work with D. Angella [9] *Differ. Geom. and App.* (2012), we reformulate T.-J. Li and W. Zhang's theory in the almost **D**-complex case, constructing subgroups of the de Rham cohomology linked with the almost **D**-complex structure. In particular, we are interested in studying when an almost **D**-complex structure K on a manifold M induces the cohomological decomposition

$$H^2_{dR}(M;\mathbb{R}) \;=\; H^{2+}_K(M;\mathbb{R}) \oplus H^{2-}_K(M;\mathbb{R})$$

through the **D**-complex subgroups $H^{2+}_K(M;\mathbb{R})$, $H^{2-}_K(M;\mathbb{R})$ of $H^2_{dR}(M;\mathbb{R})$, made up of the classes admitting a K-invariant, respectively K-anti-invariant representative; such almost **D**-complex structures will be called *$\mathcal{C}^\infty$-pure-and-full (at the 2-nd stage)*, miming T.-J. Li and W. Zhang's notation in [55].

We prove some results and provide some examples showing that the situation, in the (almost) **D**-complex case, is very different from the (almost) complex case.

In particular, in Example 3.6.1 and in Example 3.6.2, we show that compact ***D**-Kähler* manifolds (that is, **D**-complex manifolds endowed with a symplectic form which is anti-invariant under the action of the **D**-complex structure) need not to satisfy the cohomological decomposition through the **D**-complex subgroups we have introduced. Furthermore, Example 3.6.5 shows a 4-dimensional almost **D**-complex nilmanifold that does not satisfy such a **D**-complex cohomological decomposition, providing a difference with [27, Theorem 2.3] by T. Drăghici, T.-J. Li and W. Zhang.

Nevertheless, with the aim to write a partial counterpart of [27, Theorem 2.3] in the **D**-complex case, we prove in Theorem 3.3.4 that a product $M \times N$ of two equidimensional manifolds equipped with the standard **D**-complex structure is always $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage (see Section 3.2 for the definition of *pure-and-full* property). Moreover, we get the following:

Theorem 3.5.14 ([9, Theorem 3.17]). *Every invariant **D**-complex structure on a 4-dimensional nilmanifold is $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage and hence also pure-and-full at the 2-nd stage.*

We also prove that such a theorem is optimal, because the hypothesis on integrability, nilpotency and dimension can not be dropped out (see Remark 3.5.15).

However, in this thesis we improve such a result, showing that on any Lie algebra the **D**-complex structures are always $\mathcal{C}^\infty$-pure at the 1-st stage (see Proposition 3.5.16), and proving that on a 4-dimensional nilmanifold, the dimensions of H^{2+}_K and of H^{2-}_K depend only on the underlying Lie algebra (see Theorem 3.5.18).

Lastly, in [9] we study explicit examples of deformations of **D**-complex structures. In particular, we provide examples showing that the dimensions of the **D**-complex subgroups of the cohomology can jump along a curve of **D**-complex structures (see Proposition 3.7.7). Next we show this following result:

Theorem 3.7.3 ([9, Theorem 4.2]). *The property of being **D**-Kähler is not stable under small deformations of the **D**-complex structure.*

Indeed, we construct on a nilmanifold which admits only a symplectic 2-form ω, a deformation of a **D**-complex structures K_t such that ω is a Kähler form only for $t = 0$ (see Example 3.7.2).

Such a theorem provides another strong difference with respect to the complex case (indeed, recall that the property of admitting a Kähler metric is stable under small deformations of the complex structure, as proved by K. Kodaira and D.C. Spencer in [52]).

For more information and results on deformations of complex structures, see e.g. [70, 8].

Finally, we study the Ricci-flat **D**-Kähler manifolds. As their name suggests, these manifolds are **D**-Kähler manifolds with Ricci-flat metric, and in the **D**-settings they are the **D**-analogous of the usual complex Calabi-Yau manifolds. Since there is a symplectic form ω, it makes sense to consider Lagrangian submanifolds. F.R. Harvey and H.B. Lawson in [40] studied a particular submanifold class of Ricci-flat **D**-Kähler manifolds, namely the *Split Special Lagrangian Submanifold*, proving that they are closed related to calibrated submanifolds in semi-Riemannian geometry (see the Ph.D. Thesis of J. Mealy [60]) and that they provide a natural setting for the optimal transport problem (see, e.g., Y.-H. Kim, R.J. McCann and M. Warren [48]). Moreover, they found some useful properties, e.g. a Lagrangian submanifold has constant phase if and only if it is a minimal submanifold.

With the aim to extend such a result to a larger class of manifolds, we drop the integrability of the **D**-structure (i.e. the Kähler hypothesis), and turn our attention to a more general class of manifolds, these endowed with a symplectic structure and having a parallel $(n, 0)$-form.

The first problem to deal with is to find a suitable connection which could give the required parallel condition. Since the **D**-structures are no more integrable, we need a connection different from the usual Levi-Civita connection. We focus on the set of **D**-Hermitian connections, i.e. metric connections such that $\nabla K = 0$, and we give some properties (see Section 4.4). Indeed, we developed a theory which is the **D**-complex analogous of the study of Hermitian connection made by P. Gauduchon [37] (see also [45]). Moreover, we found that the **D**-Chern connection is what we need, and we are able to state the following result (which can be thought as an extension of [40, Proposition 16.3]):

Theorem 4.5.7. *Let (M, g, K) be an almost **D**-Hermitian manifold such that the fundamental 2-form ω is closed and there exists a no-where vanishing $(n, 0)$-form ε that is parallel with respect to the **D**-Chern connection, i.e. $\nabla^1\varepsilon = 0$. Let $L \subset M$ be an oriented non-degenerate Lagrangian submanifold of M. Then for any vector $V \in TL$ tangent to the Lagrangian submanifold it holds:*

$$V(\theta) = -i_{\widetilde{H}_L}\omega = -i_{H^1_L}\omega + \sum_{i=1} g(V, T^1(e_i, e_i)). \tag{4.5.16}$$

In the previous theorem, we have used the following notation (see Chapter 4 for more informations): θ is the phase function, $\widetilde{H}_L$ is the **D**-complex mean curvature vector field, T^1 is the torsion of the **D**-Chern connection and H^1_L is the mean curvature vector of the **D**-Chern connection.

This thesis is organized as follows.
In Chapter 1 we introduce the basic notions of **D**-structures and of CR **D**-structures and other notations, results and properties useful through the further Chapters.
Chapter 2 is devoted to the deformation theory of **D**-structures and of strictly pseudoconvex CR structures. First we recall some definitions and properties of the DGLA, and we review the classical deformation theory of complex manifolds. Then, with the aid of [61], we construct the deformation theory for **D**-complex structures. In Section 2.4 we construct another DGLA and we build the DGLA injection q which allows to restate the Maurer-Cartan condition for integrable deformations. Finally, we study the deformation of CR **D**-complex manifolds, providing some examples.
The cohomological properties of **D**-manifolds is the topic discussed in Chapter 3. We

start recalling some properties of **D**-complex cohomology. In particular we investigate K-invariant and K-anti-invariant subgroups of the de Rham cohomology group, defining the notion of pure-and-full of a **D**-complex structure. Then we focus on solvmanifolds, and we construct some examples which show that **D**-complex case is very different from the complex case. In Section 3.7 we show that being a **D**-Kähler manifold is not stable for deformations of the **D**-complex structure.
Finally, in Chapter 4, we study Ricci-flat **D**-Kähler manifolds and **D**-Hermitian connections, to extend a result by F.R. Harvey and H.B. Lawson (see [40]) to non-integrable case, i.e. we restate their result for Lagrangian submanifolds of almost **D**-complex manifolds. We also construct some explicit examples of Ricci-flat **D**-Kähler manifolds over nilmanifolds (it is known that the only Calabi-Yau nilmanifold is the torus, hence again we note a strong difference with the complex case).

Chapter 1

Introduction to D-structures

D-complex geometry arises naturally as a counterpart of complex geometry. Indeed, an *almost **D**-complex structure* (also called *almost **D**-structure*) on a manifold M is an endomorphism K of the tangent bundle TM whose square K^2 is equal to the identity on TM and such that the rank of the two eigenbundles corresponding to the eigenvalues $\{-1, 1\}$ of K are equi-dimensional (for an *almost complex* structure $J \in \mathrm{End}(TM)$, one requires just that $J^2 = -\mathrm{Id}_{TM}$).

In this chapter we (briefly) recall some definitions and results on **D**-complex structures (for more details, results and motivations see [2], [20], [21] and [40] and the references therein). We start with the definition of the *Double numbers* **D** and then we turn our attention to **D**-structure on vector spaces and then on manifolds.

In Sections 1.4 and 1.5 we study the relations between the **D**-complex structure K and the pseudo-Riemannian metric g, recalling the definitions and properties of ***D**-Kähler manifolds*.

The last part of this chapter is devoted to *CR **D**-manifolds*. We focus on **D**-structures on *contact manifolds*, in particular on *strictly CR **D**-structures*, recalling some definitions and results. We end with a study of automorphism group of **D**-structures and CR **D**-structures.

Notation. Through the paper we will use the following conventions: *smooth* functions (also *differentiable* functions) means functions of class C^∞, and *differentiable maps* between manifolds are also assumed to be of class C^∞.
Manifolds are assumed to be C^∞-manifolds and satisfying the second axiom of countability. We denote by $\otimes^k V$ and $\wedge^k V$ the *k-th tensor power* and *k-th exterior power* of a vector bundle V over a manifold M. If the vector bundle is the cotangent bundle T^*M we shorten the notation for the exterior product by $\wedge^k M$. We denote by TM the *tangent bundle* to the manifold M, and we often will use the same symbol TM for the differentiable sections $\mathfrak{X}(M) = \Gamma(M, TM)$. This identification between the vector bundle and the space of differentiable sections of such a bundle will be done for most of the vector bundles (for example, $\Omega^k(M) = \Gamma(M, \wedge^k T^*M)$ will be also denoted by $\wedge^k M$).
All manifolds are assumed orientable, except in this first Chapter, where non-orientable manifolds are allowed when explicitly specified (this because here we introduce the main tools and describe the general aspects of Double manifolds). In the rest of the paper any non-orientable manifold is explicitly declared (see also Remark 1.4.6).

1.1 Preliminaries on Double numbers

1.1.1 Algebra of Double numbers

We denote by **D** the set of the *double numbers* (*para-complex, hyperbolic* or *Lorentzian numbers*), that is the 2-dimensional algebra over $\mathbb{R}$ endowed with a natural operation of sum, a multiplication by real number and a (distributive and associative) multiplication $*$ defined by:

$$(x, y) * (x', y') := (xx' + yy', xy' + x'y). \tag{1.1.1}$$

In analogy with the complex numbers, we set $\tau := (0, 1)$ (called *para-complex imaginary unit* or also ***D**-complex imaginary unit*), then $\tau^2 = 1$ and we can write

$$\mathbf{D} := \{z = x + \tau y \mid x, y \in \mathbb{R},\ \tau^2 = 1\}, \tag{1.1.2}$$

so $\mathbf{D}$ is the real algebra generated by 1 and τ. We call $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ the *real* and *(para-)imaginary part* of the double number $z = x + \tau y$.
However, choosing the basis $e := \frac{1}{2}(1 + \tau)$, $\bar{e} := \frac{1}{2}(1 - \tau)$, we get

$$\mathbf{D} = e\mathbb{R} \oplus \bar{e}\mathbb{R} = \{z = ue + v\bar{e} \mid u, v \in \mathbb{R}\} \tag{1.1.3}$$

where u, v are called the *adapted-coordinates* or *null-coordinates* of z, since $e^2 = e$, $\bar{e}^2 = \bar{e}$ and $e\bar{e} = 0$. We will call *standard coordinates* the real-imaginary coordinates x, y. Note also the relations:

$$\tau e = e, \qquad \tau \bar{e} = -\bar{e}. \tag{1.1.4}$$

Thus we see that $\mathbf{D}$ is just the algebra of 2×2 diagonal matrices $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ over $\mathbb{R}$. (Note that our notation for the adapted coordinates is slightly different from the one used by F.R. Harvey and H.B. Lawson in [40], in fact they switch e with $\bar{e}$ and u with v).

The *conjugate* of $z = x + \tau y$ is naturally defined by

$$\bar{z} := x - \tau y = ve + \bar{e}u \tag{1.1.5}$$

and moreover $z\bar{z}$ defines a quadratic form of signature $(1, 1)$:

$$\langle z, z \rangle := z\bar{z} = x^2 - y^2 = uv. \tag{1.1.6}$$

The algebra $\mathbf{D}$ is normed in the sense that $\langle z * w, z * w \rangle = \langle z, z \rangle \langle w, w \rangle$ and this is the only commutative normed algebra other than the real and complex numbers, in fact $\mathbf{D}$ is the Clifford algebra $Cl_{0,1}$ (see [20]).

The double number $z \in \mathbf{D}$ is invertible if and only if $z\bar{z} \neq 0$ (hence $\mathbf{D}$ is not an integral domain), and double numbers such that $z\bar{z} = 0$ are called *null elements*. We denote by $\mathbf{D}^*$ the *group of the invertible* elements and for $z \in \mathbf{D}^*$ we have

$$z^{-1} := \frac{\bar{z}}{\langle z, z \rangle}. \tag{1.1.7}$$

Obviously the null elements are of the form $z = x \pm \tau x$ (or $z = ue$, $z = v\bar{e}$) and $\mathbf{D}^*$ has four connected components. Often, especially in physics, elements with strictly negative norm are called *time-like numbers*, and elements with strictly positive norm are called *space-like numbers*. We denote by $\mathbf{D}^+$ the space-like component of $\mathbf{D}^*$ containing 1.

We can define the exponential function

$$\begin{aligned} \exp : \mathbf{D} &\xrightarrow{\simeq} \mathbf{D}^+ \\ z &\longmapsto \exp(z) := \exp(x)(\cosh y + \tau \sinh y) \\ &\qquad = e\exp(u) + \bar{e}\exp(v) \end{aligned} \tag{1.1.8}$$

which gives an isomorphism with inverse

$$\begin{aligned} \log : \mathbf{D}^+ &\to \mathbf{D} \\ z &\longmapsto \log(z) := \log\left(\sqrt{(x+y)(x-y)}\right) + \tau \log\left(\sqrt{\frac{x+y}{x-y}}\right) \\ &\qquad = e\log(u) + \bar{e}\log(v). \end{aligned} \tag{1.1.9}$$

The *unitary subgroup* $U_1(\mathbf{D}) := \{z \in \mathbf{D} \mid z\bar{z} = 1\}$ has two connected components, parameterized by $\pm\exp(\tau\theta)$ for $\theta \in \mathbb{R}$, and it is an hyperbola replacing the unit circle of complex numbers $\mathbb{C}$, for this it is also known as the *space-like unit sphere*.

1.1.2 D-holomorphic functions

Now treat $\mathbf{D}$ as $\mathbb{R}^2$ and consider a smooth $\mathbf{D}$-valued function F on an open set $U \in \mathbf{D}$. We can split F, using standard-coordinates (i.e. the real and imaginary) or the adapted-coordinates, in the following sense:

$$\begin{aligned} F : U &\longrightarrow \mathbf{D} \\ z = x + \tau y &\longmapsto F(z) &&= f(x,y) + \tau g(x,y) \\ = ue + v\bar{e} & &&= \hat{f}(u,v)e + \hat{g}(u,v)\bar{e} \end{aligned} \tag{1.1.10}$$

where $f, g, \hat{f}, \hat{g}$ are real-valued functions. We can define also the 1-forms:

$$dz := dx + \tau\, dy = e\, du + \bar{e}\, dv \quad \text{and} \quad d\bar{z} := dx - \tau\, dy = e\, dv + \bar{e}\, du \tag{1.1.11}$$

with duals

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y}\right) = e\frac{\partial}{\partial u} + \bar{e}\frac{\partial}{\partial v}, \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - \tau\frac{\partial}{\partial y}\right) = e\frac{\partial}{\partial v} + \bar{e}\frac{\partial}{\partial u}. \end{aligned} \tag{1.1.12}$$

A differentiable function $F : U \to \mathbf{D}$ is $\mathbf{D}$-*holomorphic* or *para-holomorphic* if

$$\frac{\partial}{\partial \bar{z}} F = 0, \tag{1.1.13}$$

that is, writing F as in (1.1.10) and $\frac{\partial}{\partial \bar{z}}$ as in (1.1.12), if it satisfies the *(para-)Cauchy-Riemann conditions*:

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial y} g(x,y) \qquad \frac{\partial}{\partial x} g(x,y) = \frac{\partial}{\partial y} f(x,y). \tag{1.1.14}$$

Writing F in the null coordinates, the $\mathbf{D}$-holomorphic condition becomes:

$$\frac{\partial}{\partial \bar{z}} F = e\frac{\partial}{\partial v}\hat{f}(u,v) + \bar{e}\frac{\partial}{\partial u}\hat{g}(u,v) = 0 \tag{1.1.15}$$

showing that a smooth function is $\mathbf{D}$-holomorphic if and only if $\hat{f} = \hat{f}(u)$ and $\hat{g} = \hat{g}(v)$. It follows that satisfying the para-Cauchy-Riemann conditions (1.1.14) (or (1.1.15)) does not assure the analyticity as in the complex case. In fact, it is possible to construct the following example:

Example 1.1.1. Let $F : \mathbf{D} \to \mathbf{D}$ defined as

$$\begin{aligned} F(x + \tau y) = &\left[\exp\left(-\frac{1}{(x+y)^2}\right) + \exp\left(-\frac{1}{(x-y)^2}\right)\right] \\ &+ \tau\left[\exp\left(-\frac{1}{(x+y)^2}\right) - \exp\left(-\frac{1}{(x-y)^2}\right)\right]. \end{aligned} \tag{1.1.16}$$

It easily follows that F is a $\mathbf{D}$-holomorphic function but it is not analytic. Moreover the conditions (1.1.14) (or (1.1.15)) alone do not imply the existence of further derivatives other than the first one: defining F as $F := \hat{f}(u)e + \hat{g}(v)\bar{e}$ with $\hat{f}, \hat{g} \in C^1(\mathbb{R})$ (that is real continuous function with continuous first derivative), we get such an example.

Note that:

$$\overline{\partial} F = 0 \text{ and } F \in \mathbb{R} \quad \text{implies} \quad F = c \in \mathbb{R} \text{ is constant.}$$

In fact, writing in null-coordinates, we get from $\overline{\partial} F = 0$:

$$F(u,v) = e f(u) + \bar{e} g(v)$$

since $F \in \mathbb{R}$, we have that $f(u) - g(v) = 0$, and by differentiating in $\dfrac{\partial}{\partial u}$ and in $\dfrac{\partial}{\partial v}$, we obtain that $f(u) = g(v) = c$. In conclusion $F(u,v) = ec + \bar{e}c = c$.

Remark 1.1.2. Indeed, we have:

$$\frac{\partial^2}{\partial z \, \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^2}{\partial u \, \partial v} \tag{1.1.17}$$

which is the wave equation and an hyperbolic operator, so we lack the regularity of the complex case (where $\dfrac{\partial^2}{\partial z \, \partial \bar{z}} = \dfrac{1}{4}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$ is an elliptic operator).

We introduce the following operators, in analogy with the complex case:

$$\begin{aligned} d &:= dx \wedge \frac{\partial}{\partial x} + dy \wedge \frac{\partial}{\partial y}, \qquad & d^{\mathbf{D}} &:= dx \wedge \frac{\partial}{\partial y} + dy \wedge \frac{\partial}{\partial x}, \\ \partial &:= dz \wedge \frac{\partial}{\partial z}, & \overline{\partial} &:= d\bar{z} \wedge \frac{\partial}{\partial \bar{z}}, \\ \partial_+ &:= du \wedge \frac{\partial}{\partial u}, & \partial_- &:= dv \wedge \frac{\partial}{\partial v}. \end{aligned} \tag{1.1.18}$$

These operators satisfy the following relations:

$$\begin{gathered} d = \partial + \overline{\partial} = \partial_+ + \partial_-, \qquad d^{\mathbf{D}} = \tau(\partial - \overline{\partial}) = \tau(\partial_+ - \partial_-), \\ \partial = \frac{1}{2}(d + \tau\, d^{\mathbf{D}}) = e\,\partial_+ + \bar{e}\,\partial_-, \qquad \overline{\partial} = \frac{1}{2}(d - \tau\, d^{\mathbf{D}}) = e\,\partial_- + \bar{e}\,\partial_+, \\ d\,d = d^{\mathbf{D}}\, d^{\mathbf{D}} = \partial^2 = \overline{\partial}^2 = \partial_+^2 = \partial_-^2 = \partial\,\overline{\partial} + \overline{\partial}\,\partial = \partial_+\,\partial_- + \partial_-\,\partial_+ = 0, \\ d\,d^{\mathbf{D}} = -2\tau\,\partial\,\overline{\partial} = 2\,\partial_+\,\partial_- \,. \end{gathered} \tag{1.1.19}$$

Remark 1.1.3. Some authors use the notations d_u and d_v for ∂_+ and ∂_- respectively (e.g. [40]).

1.1.3 Double n-Space

The set of n-ples of double numbers is denoted by $\mathbf{D}^n$ and is called the *double n-space*. Obviously as before

$$\mathbf{D}^n := \{(z^1, \ldots, z^n) \mid z^i \in \mathbf{D}\} \tag{1.1.20}$$

is isomorphic to $\mathbb{R}^n \oplus \mathbb{R}^n$ and we will call $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ (respectively $(u_+^1, \ldots, u_+^n, u_-^1, \ldots, u_-^n)$) the underlying real coordinates depending on the choice of the standard-coordinates (respectively the null-coordinates). We have replaced the previous notation (u,v) for the adapted-coordinates with the most glaring notation (u_+, u_-) (see e.g. [2]).

As before, we can introduce differential operators on **D**-valued forms by extending the previous ones (see (1.1.12) and (1.1.18)) in a natural way, for example:

$$\begin{aligned} \partial &:= \sum_{i=1}^n dz^i \wedge \frac{\partial}{\partial z^i}, \qquad & \overline{\partial} &:= \sum_{i=1}^n d\bar{z}^i \wedge \frac{\partial}{\partial \bar{z}^i}, \\ \partial_+ &:= \sum_{i=1}^n du_+^i \wedge \frac{\partial}{\partial u_+^i}, & \partial_- &:= \sum_{i=1}^n du_-^i \wedge \frac{\partial}{\partial u_-^i}. \end{aligned} \tag{1.1.21}$$

Again, the relations (1.1.19) are satisfied.

A $\mathbf{D}$-valued smooth function F over an open set $U \subset \mathbf{D}^n$ is ***D**-holomorphic* if $\overline{\partial} F = 0$. As before, it follows that F is $\mathbf{D}$-holomorphic if it satisfies the para-Cauchy-Riemann conditions if and only if in the null-coordinates $F = f(u_+^1, \dots, u_+^n)e + g(u_-^1, \dots, u_-^n)\bar{e}$ with $f,\ g$ real functions.

On $\mathbf{D}^n$ we have the *standard **D**-valued Hermitian inner product* $\langle \cdot, \cdot \rangle$ and a real inner product (a pseudo-metric) $(\cdot, \cdot)$ defined by:

$$\langle z, w \rangle := \sum_{i=1}^{n} z^i \bar{w}^i \qquad (z, w) := \mathrm{Re}\langle z, w \rangle := \mathrm{Re}\left(\sum_{i=1}^{n} z^i \bar{w}^i \right) \qquad z, w \in \mathbf{D}^n. \tag{1.1.22}$$

The associated quadratic form $\langle z, z \rangle = (z, z) = \sum_i^n (x^i)^2 - (y^i)^2 = \sum_i^n u_+^i u_-^i$ has (n, n)-signature (*neutral or split signature*). If we denote by $\omega = \sum_i^n dx^i \wedge dy^i = -\frac{1}{2} \sum_i^n du_+^i \wedge du_-^i$ the *standard symplectic form* on $\mathbb{R}^{2n}$, we get as in the complex case:

$$\langle z, w \rangle = (z, w) - \tau\omega(z, w) \qquad z, w \in \mathbf{D}^n. \tag{1.1.23}$$

The multiplication by τ induces an automorphism K in $\mathbf{D}^n$ $(Kz = \tau z)$. This isomorphism acts on standard-coordinates (resp. null-coordinates) in the following way:

$$\begin{aligned} K(x^i) &= y^i & K(y^i) &= x^i \\ K(u_+^i) &= +u_+^i & K(u_-^i) &= -u_-^i. \end{aligned} \tag{1.1.24}$$

Note that $K^2 = +\mathrm{Id}$ in contrast to the complex case where $J^2 = -\mathrm{Id}$. Such an endomorphism is what we use to generalize the Double numbers to manifold and to construct the $\mathbf{D}$-complex structure (see next sections 1.2 and 1.3).
We see also that K is an anti-isometry for the Hermitian inner product and for the symplectic form: $\langle Kz, Kw \rangle = -\langle z, w \rangle$, $\omega(Kz, Kw) = -\omega(z, w)$. Moreover these elements are related by the equation:

$$(z, Kw) = \omega(z, w). \tag{1.1.25}$$

The ***D**-linear maps* from $\mathbf{D}^n$ to $\mathbf{D}^m$ correspond to the set $M_{m,n}(\mathbf{D})$ of the $m \times n$-matrices with entries in $\mathbf{D}$. By standard algebra, square matrices $A, B \in M_n(\mathbf{D})$ have a ***D**-determinant* with the usual properties, e.g.:

$$\det_{\mathbf{D}} A^t = \det_{\mathbf{D}} A, \quad \det_{\mathbf{D}}(AB) = \det_{\mathbf{D}} A \det_{\mathbf{D}} B, \quad A\widetilde{A}^t = (\det_{\mathbf{D}} A)\, \mathrm{Id}_n \tag{1.1.26}$$

(where $\cdot^t$ denotes the transpose operator and $\widetilde{A}$ is the cofactor matrix of A). Hence A has *inverse* A^{-1} if and only if $\det_{\mathbf{D}} A$ has an inverse, that is if $\det_{\mathbf{D}} A \in \mathbf{D}^*$. We will denote by $\mathrm{GL}_n(\mathbf{D}) \subset M_n(\mathbf{D})$ the *group of invertible square matrices* of $M_n(\mathbf{D})$.

We have, as in the complex case $\det_{\mathbf{D}} A\, \overline{\det_{\mathbf{D}} A} = \det_{\mathbb{R}} A$, where in $\det_{\mathbb{R}} A$ the matrix is thought as an element of $M_{2n}(\mathbb{R})$. In the null-coordinates $A = eB + \bar{e}C \in M_n(\mathbf{D})$ with $B, C \in M_n(\mathbb{R})$ and for the $\mathbf{D}$-determinant the formula

$$\det_{\mathbf{D}} A = e\det_{\mathbb{R}} B + \bar{e}\det_{\mathbb{R}} C \tag{1.1.27}$$

holds. It follows that $\mathrm{GL}_n(\mathbf{D}) \cong \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ and that $\det_{\mathbf{D}} A \in \mathbf{D}^+$ if and only if $\det_{\mathbb{R}} B > 0$ and $\det_{\mathbb{R}} C > 0$.
Each linear map from $\mathbf{D}^n$ to $\mathbf{D}^m$ decomposes into the sum of a $\mathbf{D}$-linear map (the so called

(1,0)-*part*) and an anti-**D**-linear map ((0,1)-*part*), then the Jacobian of a smooth map $F : U \subset \mathbf{D}^n \to \mathbf{D}^m$ splits as:

$$\mathrm{Jac}_{\mathbb{R}}(F) = \mathrm{Jac}^{1,0}_{\mathbf{D}}(F) + \mathrm{Jac}^{0,1}_{\mathbf{D}}(F) = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \bar{z}} \tag{1.1.28}$$

and F is ***D**-holomorphic* if $J^{0,1}_{\mathbb{R}}(F) = \dfrac{\partial F}{\partial \bar{z}} = 0$, that is if $J_{\mathbb{R}}(F)$ is **D**-linear. A smooth function between open subsets of $\mathbf{D}^n$ is *bi-**D**-holomorphic* if F is **D**-holomorphic and $J^{1,0}_{\mathbf{D}}(F)$ is non-null (i.e. invertible).

Since **D** is not a field, $\mathbf{D}^n$ is not, strictly speaking, a vector space over **D**, but an algebra over **D**. By this reason a vector $Z = (z^i) \in \mathbf{D}^n$ will be called *regular* if one of the following equivalent conditions is satisfied:

1. Z and KZ are linearly independent on $\mathbb{R}$, where K is the natural extension of (1.1.24) to the component z^i of Z, i.e., it is the multiplication by τ of every component z^i of Z,

2. for $z \in \mathbf{D}$, it holds that if $zZ = 0$, then $z = 0$.

Proof of the equivalence. Setting $z = a + \tau b$ with $a, b \in \mathbb{R}$ we see that

$$zZ = (a + \tau b)(z^1, \ldots, z^n) = a(z^1, \ldots, z^n) + bK(z^1, \ldots, z^n), \tag{1.1.29}$$

this easily shows the equivalence between previous points 1 and 2. □

Given a matrix $A \in M_n(\mathbf{D})$, denote by A^* the conjugate transpose $\bar{A}^t$. Let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be the standard basis for $\mathbf{D}^n$. A set of regular vectors $v^1, \ldots, v^n \in \mathbf{D}^n$ is a *space-like **D**-unitary basis* for $\mathbf{D}^n$ if $v^1, \ldots, v^n, Kv^1, \ldots, Kv^n$ is a real orthonormal basis with $(v^i, v^i) = 1$ and $(Kv^i, Kv^i) = -1$ for all i. The *unitary group* $\mathrm{U}_n(\mathbf{D})$ is the set of matrices A satisfying one of the following equivalent conditions:

1. $(Az, Az) = (z, z)$ for all $z \in \mathbf{D}^n$,

2. $AA^* = \mathrm{Id}$ (or equivalently $A^*A = \mathrm{Id}$),

3. $Ax^1, \ldots, Ax^n$ is a space-like **D**-unitary basis.

Note that $U_n(\mathbf{D})$ has two components determined by $\det_{\mathbf{D}} A = \pm e^{\tau\theta} \in U_1(\mathbf{D})$ (see (1.1.8)). We call $U_1^+(\mathbf{D})$ the component containing the identity. We have also the *special linear group* $\mathrm{SL}_n(\mathbf{D}) := \{A \in GL_n(\mathbf{D}) \mid \det_{\mathbf{D}} A = 1\}$ and the *special unitary group* $\mathrm{SU}_n(\mathbf{D}) := \{A \in U_n(\mathbf{D}) \mid \det_{\mathbf{D}} A = 1\}$. Computing in the null-coordinate $A = eB + \bar{e}C$ we have:

$$\begin{aligned} A \in \mathrm{U}_n(\mathbf{D}) &\Leftrightarrow A = eB + \bar{e}(B^t)^{-1} \text{ for some } B \in \mathrm{GL}_n(\mathbb{R}), \\ A \in \mathrm{U}^+_n(\mathbf{D}) &\Leftrightarrow A = eB + \bar{e}(B^t)^{-1} \text{ for some } B \in \mathrm{GL}^+_n(\mathbb{R}), \\ A \in \mathrm{SL}^+_n(\mathbf{D}) &\Leftrightarrow A = eB + \bar{e}C \text{ for some } B, C \in \mathrm{SL}^+_n(\mathbb{R}), \\ A \in \mathrm{SU}_n(\mathbf{D}) &\Leftrightarrow A = eB + \bar{e}(B^t)^{-1} \text{ for some } B \in \mathrm{SL}_n(\mathbb{R}), \end{aligned} \tag{1.1.30}$$

thus $\mathrm{U}_n(\mathbf{D}) \cong \mathrm{GL}_n(\mathbb{R})$, $\mathrm{U}^+_n(\mathbf{D}) \cong \mathrm{GL}^+_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbf{D}) \cong \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SU}_n(\mathbf{D}) \cong \mathrm{SL}_n(\mathbb{R})$ and no one of these subgroups of $\mathrm{GL}_n(\mathbf{D}) \cong \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ is compact.

1.2 D-complex vector spaces

Our goal is to carry the structures of $\mathbf{D}$ and $\mathbf{D}^n$ on a manifold. The first step is to define a $\mathbf{D}$-complex structure over a vector space. We start with the following:

Definition 1.2.1. Let V be a finite dimensional real vector space. A ***D**-complex structure* (or, briefly ***D**-structure*, also called *para-complex structure*) on V is an endomorphism $K : V \to V$ such that:

1. K is an involution, that is $K^2 = \mathrm{Id}_V$;
2. the eigenspaces $V^\pm := \ker(\mathrm{Id}_V \mp K)$ of K with eigenvalues ± 1 respectively have the same dimension.

A vector space V endowed with a $\mathbf{D}$-complex structure K, denoted by (V, K), will be called ***D**-complex* (or *para-complex*) vector space.
A pseudo-Euclidean metric $(\cdot,\cdot)$ on V is said to be *compatible* with the $\mathbf{D}$-complex structure if $(K\cdot, K\cdot) = -(\cdot,\cdot)$, that is if K is an anti-isometry, and $(\cdot,\cdot)$ will be called a $\mathbf{D}$-Hermitian.
A *homomorphism* between $\mathbf{D}$-complex vector spaces (V, K), (V', K') is a linear map $L : V \to V'$ satisfying $L \circ K = K' \circ L$.

It follows from point 2 of the above Definition 1.2.1 that V must have even dimension $2n$, that $\dim V^+ = \dim V^- = n$ and that K has to be non-trivial (i.e. $K \neq \mathrm{Id}_V$). We note also that if $(\cdot,\cdot)$ is compatible with K, then it must have signature (n, n) and $V^\pm$ are null-subspaces for the pseudo-Euclidean metric, since

$$(X^\pm, X^\pm) = -(KX^\pm, KX^\pm) = -(X^\pm, X^\pm) \quad \text{for every } X^\pm \in V^\pm. \tag{1.2.1}$$

The double space $\mathbf{D}^n$ is a $\mathbf{D}$-complex vector space with the multiplication by τ as a $\mathbf{D}$-complex structure. Conversely, any $\mathbf{D}$-complex vector space (V, K) can be regarded as a $\mathbf{D}$-module via the isomorphism

$$(x + \tau y)v = xv + yKv \qquad v \in V,\ x, y \in \mathbb{R} \tag{1.2.2}$$

(we see that from this point of view, homomorphisms between $\mathbf{D}$-complex vector spaces correspond to $\mathbf{D}$-linear maps). This isomorphism can be set also in the null-coordinates as $(eu_+ + \bar{e}u_-)v = \frac{1}{2}u_+(v + Kv) + \frac{1}{2}u_-(v - Kv)$. Note that $v + Kv$ (resp. $v - Kv$) is the projection of v over the eigenspace V^+ (resp. V^-).

In fact, if (V, K) is a $\mathbf{D}$-complex vector space, then there exists a basis of eigenvectors $\{u_+^1, \ldots, u_+^n, u_-^1, \ldots, u_-^n\}$ such that $Ku_+^j = u_+^j$ and $Ku_-^j = -u_-^j$, and we can identify K with the diagonal matrix:

$$K = \begin{pmatrix} \mathrm{Id}_n & \\ & -\mathrm{Id}_n \end{pmatrix}. \tag{1.2.3}$$

However, there exists also a basis of vectors $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ (called standard-coordinates) such that $Kx^i = y^i$ and $Ky^i = x^i$. In such a basis K can be identified with the matrix

$$K = \begin{pmatrix} & \mathrm{Id}_n \\ \mathrm{Id}_n & \end{pmatrix}. \tag{1.2.4}$$

Summarizing up, we have the following possible identifications:

$$\begin{array}{rcccl} \mathbf{D}^n & \cong & (\mathbb{R}^n \times \mathbb{R}^n, \begin{pmatrix} \mathrm{Id}_n & \\ & -\mathrm{Id}_n \end{pmatrix}) & \cong & (\mathbb{R}^n \times \mathbb{R}^n, \begin{pmatrix} & \mathrm{Id}_n \\ \mathrm{Id}_n & \end{pmatrix}) \\ (z^i = x^i + \tau y^i) & \mapsto & (x^i + y^i, x^i - y^i) & \mapsto & (x^i, y^i) \\ (z^i = eu_+^i + \bar{e}u_-^i) & \mapsto & (u_+^i, u_-^i) & \mapsto & \left(\frac{1}{2}(u_+^i + u_-^i), \frac{1}{2}(u_+^i - u_-^i)\right). \end{array} \tag{1.2.5}$$

As one can see, the two basis $\{u_+^1, \ldots, u_+^n, u_-^1, \ldots, u_-^n\}$ and $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ are different. In particular the last one is composed by regular vectors, while no-one of the first one is so.

Definition 1.2.2. Let (V, K) be a $\mathbf{D}$-complex vector space. A vector $v \in V$ is called *regular* if it satisfies one of the following equivalent conditions:

1. $v \notin V^+$ and $v \notin V^-$,
2. $\mathbf{D} \cdot v$ is a one-dimensional $\mathbf{D}$-complex vector space over $\mathbf{D}$,
3. v and Kv are linearly independent over $\mathbb{R}$,
4. for $z \in \mathbf{D}$, it holds that $zv = 0 \Rightarrow z = 0$.

Proof of the equivalence. $1 \Leftrightarrow 2$: Writing $v = v^+ + v^- \in V^+ \oplus V^-$, we have $(\mathbf{D} \cdot v)^{\pm} = \mathbb{R} \cdot v^{\pm}$. Then $\mathbf{D} \cdot v$ is a one-dimensional $\mathbf{D}$-complex vector space if and only if $v^{\pm} \neq 0$.
$1 \Leftrightarrow 3$: Assuming $v \neq 0$, if $Kv = av$ for any $a \in \mathbb{R}$, then $v = K^2 v = a^2 v$ and $a = \pm 1$. The other implication is trivial.
$3 \Leftrightarrow 4$: It is a consequence from $(a + \tau b)v = av + bKv$. □

Let K be a $\mathbf{D}$-complex structure on V. We define the ***D**-complexification* (or *para-complexification*) of V as $V^{\mathbf{D}} := V \otimes_{\mathbb{R}} \mathbf{D}$, and we extend K to a $\mathbf{D}$-linear endomorphism of $V^{\mathbf{D}}$ also called K. Note that the $\mathbf{D}$-complexification of a vector space is different from the isomorphism (1.2.2): indeed, we have that $\dim_{\mathbb{R}} V = 2n$ and $\dim_{\mathbb{R}} V^{\mathbf{D}} = 4n$ while the dimensions as a free module over $\mathbf{D}$ are respectively $\dim_{\mathbf{D}} V = n$ and $\dim_{\mathbf{D}} V^{\mathbf{D}} = 2n$. If we set

$$\begin{aligned} V^{1,0} &:= \{v \in V^{\mathbf{D}} \mid Kv = \tau v\} = \{v + \tau K v \mid v \in V\}, \\ V^{0,1} &:= \{v \in V^{\mathbf{D}} \mid Kv = -\tau v\} = \{v - \tau K v \mid v \in V\} \end{aligned} \tag{1.2.6}$$

then we have $V^{\mathbf{D}} = V^{1,0} \oplus V^{0,1}$ and the subspace $V^{1,0}$ (resp. $V^{0,1}$) is called ***D**-holomorphic* subspace (resp. *anti-**D**-holomorphic*).

The $\mathbf{D}$-complex structure on V induces a $\mathbf{D}$-complex structure K^* on the dual space V^* by:

$$K^*(\alpha)(V) = \alpha(Kv) \tag{1.2.7}$$

for $\alpha \in V^*$, $v \in V$. If there is no confusion, we will denote by K the $\mathbf{D}$-complex structure both on V and on V^*. Consequently, V^* splits in the two eigespaces V_+ and V_- of K. As before, we have the following split of $V^{*\mathbf{D}} = V \otimes_{\mathbb{R}} \mathbf{D} = V_{1,0} \oplus V_{0,1}$, where

$$\begin{aligned} V_{1,0} &:= \{\alpha \in V^{*\mathbf{D}} \mid K\alpha = \tau v\} = \{\alpha + \tau K \alpha \mid \alpha \in V^*\}, \\ V_{0,1} &:= \{\alpha \in V^{*\mathbf{D}} \mid K\alpha = -\tau v\} = \{\alpha - \tau K \alpha \mid \alpha \in V^*\}. \end{aligned} \tag{1.2.8}$$

We denote by $\wedge^{p,q} V^{*\mathbf{D}}$ the subspace of $V^{*\mathbf{D}}$ spanned by the elements $\alpha \wedge \beta$ with $\alpha \in \wedge^p V_{1,0}$ and $\beta \in \wedge^q V_{0,1}$. Then:

$$\wedge^r V^{*\mathbf{D}} = \bigoplus_{p+q=r} \wedge^{p,q} V^{*\mathbf{D}}, \tag{1.2.9}$$

and if $\{\alpha^1, \ldots \alpha^n\}$ is a basis of $V_{1,0}$, then $\{\bar{\alpha}^1, \ldots \bar{\alpha}^n\}$ is a basis of $V_{0,1}$, and thus $\{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \wedge \bar{\alpha}^{i_1} \cdots \wedge \bar{\alpha}^{i_q}\}$ with $1 \leqslant i_1 < \cdots < i_p \leqslant n$, $1 \leqslant i_1 < \cdots < i_q \leqslant n$ is a basis for $\wedge^{p,q} V^{*\mathbf{D}}$.

Remark 1.2.3. Since $\mathbf{D}$ is a ring, we note some differences with the classical linear algebra:

1. In a $\mathbf{D}$-complex vector space (V, K), the $\mathbf{D}$-span of a vector v is not necessarily a $\mathbf{D}$-vector space. This is the reason for which we introduce the definition of regular vector (Definition 1.2.2).

2. If $L : V \to V'$ is $\mathbf{D}$-linear map, the image and kernel of L are K-invariant real subspace of V and V', but they may not be $\mathbf{D}$-complex subspaces, since $L(V^+)$ and $L(V^-)$ have not equal dimension. The same for $\ker(L)^{\pm}$.

3. It still holds that: L is injective if and only if $\ker L = \{0\}$.

4. Note that the spaces $V^{1,0}$ and $V^{0,1}$ are not the "eigenspaces" (and $\pm\tau$ are not "eigenvalues") of the extension K of the $\mathbf{D}$-complex structure K in a $\mathbf{D}$-complexificated space $V^{\mathbf{D}}$, because $V^{\mathbf{D}}$ is a $\mathbf{D}$-module, and not a vector space over $\mathbf{D}$.

On a $\mathbf{D}$-complexificated vector space, it is possible to define an Hermitian form in two ways. A ***D**-Hermitian form* on $V^{\mathbf{D}}$ is a map $h : V^{\mathbf{D}} \times V^{\mathbf{D}} \to \mathbf{D}$ such that

1. h is $\mathbf{D}$-linear in the first entry and $\mathbf{D}$-anti-linear in the second entry:

$$h(zZ, W) = zh(Z, W) \quad h(Z, zW) = \bar{z}h(Z, W) \qquad Z, W \in V^{\mathbf{D}},\ z \in \mathbf{D}; \tag{1.2.10}$$

2. $h(Z, W) = \overline{h(W, Z)}$.

Otherwise $\hat{h} : V^{\mathbf{D}} \times V^{\mathbf{D}} \to \mathbf{D}$ is a ***D**-Hermitian symmetric form* on $V^{\mathbf{D}}$ if it is a symmetric $\mathbf{D}$-linear form such that:

1. the spaces $V^{1,0}$ and $V^{0,1}$ are isotropy:

$$\hat{h}(V^{1,0}, V^{1,0}) = \hat{h}(V^{0,1}, V^{0,1}) = 0; \tag{1.2.11}$$

2. $\hat{h}(\bar{Z}, \bar{W}) = \overline{\hat{h}(Z, W)},\ Z, W \in V^{\mathbf{D}}$.

A $\mathbf{D}$-Hermitian (symmetric) form is said to be *non-degenerate* if it has a trivial kernel, i.e. $\ker h = \{Z \in V^{\mathbf{D}} \mid h(Z, V^{\mathbf{D}}) = 0\} = \{0\}$.

Obviously these definitions are related: if $\hat{h}$ is a $\mathbf{D}$-Hermitian symmetric form, then the form h defined by $h(Z, W) := \hat{h}(Z, \bar{W})$ is a $\mathbf{D}$-Hermitian form, and vice-versa. Moreover giving a pseudo-Euclidean metric $(\cdot, \cdot)$ compatible with K there exists a $\mathbf{D}$-Hermitian symmetric form $\hat{h} := (\cdot, \cdot)^{\mathbf{D}}$, where $(\cdot, \cdot)^{\mathbf{D}}$ is the natural $\mathbf{D}$-bilinear extension of $(\cdot, \cdot)$. Hence we have:

Proposition 1.2.4 ([2, Lemma 3.4])**.** *On a **D**-complex vector space (V, K) there exists a natural 1-1 correspondence between pseudo-Euclidean metric g compatible with K and the non degenerate **D**-Hermitian symmetric form $\hat{h}$, and hence there is also a 1-1 correspondence with the set of non-degenerate Hermitian form h.*

1.3 D-structures on manifolds

Now we are able to carry $\mathbf{D}$-structures over manifolds. Indeed, we will see that a $\mathbf{D}$-manifold is locally like $\mathbf{D}^n$, moreover it has a $\mathbf{D}$-complex structure on the tangent bundle.

Definition 1.3.1. An *almost **D**-complex structure* on a $2n$-manifold M is an endomorphism field $K \in \mathrm{End}(TM)$ such that:

1. $K^2 = +\mathrm{Id}_{TM}$,

2. the two eigendistributions $TM^{\pm} := \ker(\mathrm{Id} \mp K)$ have the same rank n.

An almost **D**-structure is said to be *integrable* if the eigendistribution $TM^{\pm}$ are integrable (remind that a distribution D is integrable if it is tangent to a foliation $\mathcal{F}$), in this case K is called a ***D**-complex structure*.
The pair (M, K) is called an *(almost) Double manifold*, or briefly an *(almost) **D**-complex manifold*.
A smooth map $f : (M, K) \to (M', K')$ between two (almost) **D**-complex manifolds is ***D**-holomorphic* if

$$d\,f \circ K = K' \circ d\,f. \tag{1.3.1}$$

If we focus on this definition, we note that the dimension of an almost **D**-complex manifold is necessarily even. Equivalently, an almost **D**-complex structure on M is a splitting of the tangent bundle TM in a direct sum of two subbundles $TM^{\pm}$ of the same fiber dimension, or it may be alternatively defined as a G-structure on M with structural group $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ (and the last one was the definition used by Libermann [57], [56]).

Remark 1.3.2. There is also a strong link with complex geometry. In fact, let M be a $2n$-dimensional compact manifold. Consider $K \in \mathrm{End}(TM)$ such that $K^2 = \lambda\, \mathrm{Id}_{TM}$ where $\lambda \in \{-1, 1\}$: if $\lambda = -1$, we call K an *almost complex structure*; if $\lambda = 1$, we call K an *almost **D**-complex structure.*

Example 1.3.3. It has been shown by many authors (for example [21]) that there are almost **D**-complex structures also on the tangent bundle of a manifold M. Let ∇ be a linear connection on M. For every $X \in TM$ we denote by X^v and X^h the vertical and horizontal lift with respect to the connection ∇. We set:

$$K(X^v) = X^v, \quad K(X^h) = -X^h; \qquad K'(X^v) = X^h, \quad K'(X^h) = X^v. \tag{1.3.2}$$

We see that (TM, K) and (TM, K') are almost **D**-complex manifolds. Moreover, it is known that K is integrable if and only if the connection ∇ has vanishing curvature, while K' is integrable if and only if ∇ has vanishing both curvature and torsion.

Example 1.3.4. Consider the Klein bottle $\mathcal{K}$ as a quotient of the square $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$, where we make the following identification:

$$\begin{aligned} (-1, y) &\sim (1, y) \qquad \text{for } -1 \le y \le 1 \\ (x, -1) &\sim (-x, 1) \qquad \text{for } -1 \le x \le 1. \end{aligned} \tag{1.3.3}$$

Now consider the foliations determinated by the vertical and horizontal lines of $\mathbb{R}^2$. It is easy to see that these foliations define a **D**-complex integrable structure on the quotient $\mathcal{K}$.

The *Nijenhuis tensor* N_K (also called *torsion tensor*) of an almost **D**-complex structure K is defined by:

$$N_K(X, Y) := [KX, KY] - K[KX, Y] - K[X, KY] + [X, Y] \qquad X, Y \in TM. \tag{1.3.4}$$

In the well known complex case, the integrable condition is related with the vanishing of Nijenhuis tensor (Newlander-Niremberg theorem [65]). In the **D**-complex case not only we have an analogous result by Frobenius theorem, but we can also obtain other equivalence to the integrability condition. In fact we have:

Proposition 1.3.5. *Given an almost **D**-complex manifold (M, K), the following conditions are equivalent:*

1. *(integrability) K is integrable (as in Definition 1.3.1);*
2. *(involutive property) the subbundles $TM^{\pm}$ are involutive, i.e. $[TM^+, TM^+] \subset TM^+$ and $[TM^-, TM^-] \subset TM^-$;*

3. *the Nijenhuis tensor of K vanishes, $N_K = 0$;*

4. *(existence of adapted-coordinates) for any point p there exist local real coordinates $u_+^1, \ldots, u_+^n, u_-^1, \ldots, u_-^n : U \to A_+ \times A_- \subset \mathbb{R}^n \times \mathbb{R}^n$ defined on an open neighbourhood U of p such that $d\,u_\pm^i \circ K = \pm\, d\,u_\pm^i$;*

5. *(existence of **D**-holomorphic coordinates) there exists on M an atlas $(U_\alpha, \varphi_\alpha)$ of local **D**-holomorphic chart, i.e. $\varphi_\alpha : M \supset U_\alpha \to \boldsymbol{D}^n$ are **D**-holomorphic in the sense of (1.3.1): $d\,\varphi_\alpha \circ K = \tau\, d\,\varphi_\alpha$ (or that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are **D**-holomorphic: $\overline{\partial}\,\varphi_\alpha \circ \varphi_\beta^{-1} = 0$);*

Proof. $1 \Leftrightarrow 2$: This is the Frobenius theorem.
$2 \Leftrightarrow 3$: Writing $X = X^+ + X^-$ and $Y = Y^+ + Y^-$ with respect to the decomposition $TM = TM^+ \oplus TM^-$, an easy computation gives:

$$N_K(X, Y) = 2(\mathrm{Id} - K)[X^+, Y^+] + 2(\mathrm{Id} + K)[X^-, Y^-], \tag{1.3.5}$$

and noting that $\mathrm{Id} \mp K$ are the projections over $TM^\pm$, we get the equivalence.
$1 \Leftrightarrow 4$: Fix a point $p \in M$. By Frobenius theorem applied to distribution TM^+, there exist functions u_-^i on an open neighborhood $p \in U$ which are constant on the leaves of TM^+ and such that $d\,u_-^i$ are linearly independent. Similarly, we can find functions u_+^i constant on the leaves of TM^- and such that $d\,u_+^i$ are linearly independent. From transversality of the two foliations we conclude that u_+^i, u_-^i is a system of local coordinates. The property $d\,u_\pm^i \circ K = \pm\, d\,u_\pm^i$ easily follows from the construction of $u_\pm^i$.
$5 \Rightarrow 3$: Given a **D**-complex atlas as in 5. we see that K on M is the pull-back of the **D**-complex structure from $\mathbf{D}^n$ (consider K to be the multiplication by τ), and obviously the associated Nijenhuis tensor $N_K = N_\tau$ is zero.
$4 \Rightarrow 5$: The adapted system of coordinates defines a system of **D**-holomorphic coordinates by $\mathrm{Re}\, z^i = \dfrac{u_+^i + u_-^i}{2}$ and $\mathrm{Im}\, z^i = \dfrac{u_+^i - u_-^i}{2}$, where K is the multiplication by τ. From $d\,u_\pm^i \circ K = \pm\, d\,u_\pm^i$ follows that

$$d\,z^i \circ K = (d\,\mathrm{Re}\, z^i + \tau\, d\,\mathrm{Im}\, z^i) \circ K = d\,\mathrm{Im}\, z^i + \tau\, d\,\mathrm{Re}\, z^i = \tau\, d\,z^i. \tag{1.3.6}$$

This shows that the functions z^i are indeed **D**-holomorphic, and form a coordinate system. Now it is sufficient to observe that we can cover M by coordinate domains U as above and that the coordinate changes are **D**-holomorphic. □

Example 1.3.6. Any **D**-complex vector space (V, K) is a **D**-manifold, as well as $\mathbf{D}^n$ is a **D**-manifold, with the multiplication by τ as a **D**-complex structure. In analogy with (1.2.5) we will call the *standard **D**-structure* K_n of $\mathbf{D}^n \cong \mathbb{R}^{2n} = \{(x^1, \ldots, x^n, y^1, \ldots, y^n) \mid x^i, y^j \in \mathbb{R}\}$ the following:

$$K_n\left(\frac{\partial}{\partial\, x^j}\right) = \frac{\partial}{\partial\, y^j} \qquad K_n\left(\frac{\partial}{\partial\, y^j}\right) = \frac{\partial}{\partial\, x^j}, \tag{1.3.7}$$

while we will called the *adapted **D**-structure* $\hat{K}_n$ of $\mathbf{D}^n \cong \mathbb{R}^{2n} = \{(u_+^1, \ldots, u_+^n, u_-^1, \ldots, u_-^n) \mid u_+^i, u_-^j \in \mathbb{R}\}$ the following one:

$$\hat{K}_n\left(\frac{\partial}{\partial\, u_+^j}\right) = \frac{\partial}{\partial\, u_+^j} \qquad \hat{K}_n\left(\frac{\partial}{\partial\, u_-^j}\right) = -\frac{\partial}{\partial\, u_-^j}. \tag{1.3.8}$$

Example 1.3.7. Any product $M = M_+ \times M_-$ of two smooth n-manifolds M_+, M_- is a **D**-manifold. In fact, on $TM = T(M_+ \times M_-) = T(M_+) \oplus T(M_-)$ we set a **D**-complex structure K on M by $K|_{TM_+} = \mathrm{Id}$ and $K|_{TM_-} = -\mathrm{Id}$. It is easy to see that K is integrable

and $T(M_\pm) = TM^\pm$. We will call such a K the *natural* ***D****-complex structure* or the *product structure* on $M_+ \times M_-$.
The $2n$-torus $\mathbb{T}^{2n} := S^n \times S^n$ has a natural **D**-complex structure K given by the product structure, and we will call $(\mathbb{T}^{2n}, K)$ the *standard* ***D****-complex torus* (obviously, here S^n denotes the product of n circumferences, and not the n-sphere).

Remark 1.3.8. It has to be noted that the notion of **D**-complex structures (Definition 1.3.1) can be generalized to the notion of (almost) product structures. Indeed, a pair of complementary distributions $\mathcal{B}$, $\mathcal{C}$ of constant rank p and q respectively, defines an *almost product structure* on a manifold M if $TM = \mathcal{B} \oplus \mathcal{C}$ and $\dim M = p+q$. The almost product structure is called integrable if both the distributions are integrable. Equivalently, there exists an endomorphism field $K \in \mathrm{End}(TM)$ such that $+1$ is an eigenvalue of multiplicity p and -1 is an eigenvalue of multiplicity q. Then we see that (almost) **D**-complex structures are (almost) product structures where the two distributions have the same constant rank (for more results on (almost) product structures see, e.g., [77, 78], [6] or [64]).

The previous Proposition 1.3.5 shows that any **D**-complex manifold is locally of the form as Example 1.3.7, that is locally there is a product structure (for this reason (almost) **D**-complex structures are often called (almost) product structures, despite the rank dimension of the distributions). It is also true that a **D**-manifold can be much more complicated as the following example shows.

Example 1.3.9. Let M_1, M_2 be two smooth 3-dimensional manifolds, and let S_1 (resp. S_2) be a 2-dimensional integrable foliation on M_1 (resp. M_2). We will denote by $L_1,\ L_2$ the 1-dimensional foliations transversal to $S_1,\ S_2$. Then we have the two transversal foliations $\mathcal{F}_1 = S_1 \times L_2$ and $\mathcal{F}_2 = L_1 \times S_2$ on $M = M_1 \times M_2$. By the integrability of S_k (since L_k is a 1-dimensional foliation it is always integrable), the pair $\mathcal{F}_1,\ \mathcal{F}_2$ defines an integrable **D**-complex structure K on $M = M_1 \times M_2$ which is different from that one of Example 1.3.7.

Let (M, K) be an almost **D**-complex manifold (not necessarily integrable). Then it is possible to extend the **D**-complex structure K on the cotangent bundle (as done for vector space in (1.2.7))

$$K^*(\alpha)(X) = \alpha(KX) \qquad \text{for } \alpha \in T^*M,\ X \in TM. \tag{1.3.9}$$

Note that this extension is so that K (and K^*) can commute with the usual duality between TM and T^*M. The decomposition $TM = TM^+ \oplus TM^-$ implies an analogous decomposition on $T^*M = T^*M_+ \oplus T^*M_-$, and hence induces a bigrading on $\wedge^\bullet T^*M$. Therefore:

$$\wedge^r T^*M = \bigoplus_{p+q=r} \wedge^{p,q}_{+,-}(M) \quad \text{where } \wedge^{p,q}_{+,-}(M) = \wedge^p T^*M_+ \otimes \wedge^q T^*M_-. \tag{1.3.10}$$

Sections of $\wedge^{p,q}_{+,-}T^*M$ are called *differential forms of degree* $(p+, q-)$ (or briefly $(p+, q-)$-*form*), and the space of these sections will be denoted by $\Omega^{p,q}_{+,-}(M)$ or by the same symbols $\wedge^{p,q}_{+,-}$ if no confusion is possible (read the Notation remark at the beginning of this Chapter 1). We will use something similar for the other sections of $\wedge^\bullet T^*M$.
Note that the space T^*M_+ (resp. T^*M_-) is the annihilator of TM^- (resp. TM^+).

We define in an almost **D**-complex manifold (M, K) the operator $d^{\mathbf{D}} := K^* \circ d \circ K^*$ and the following:

$$\begin{aligned} \partial_+ &:= \pi_{\wedge^{p+1,q}_{+-}} \circ d : \wedge^{p,q}_{+,-}(M) \longrightarrow \wedge^{p+1,q}_{+,-}(M), \\ \partial_- &:= \pi_{\wedge^{p,q+1}_{+-}} \circ d : \wedge^{p,q}_{+,-}(M) \longrightarrow \wedge^{p,q+1}_{+,-}(M). \end{aligned} \tag{1.3.11}$$

Similar considerations can be made on the ***D**-complex tangent bundle* $T^{\mathbf{D}}M = TM \otimes \mathbf{D}$. For any $p \in M$ we have the following decomposition of $T^{\mathbf{D}}_p M$:

$$T^{\mathbf{D}}_p M = T^{1,0}_p M \oplus T^{0,1}_p M \tag{1.3.12}$$

where:

$$\begin{aligned} T^{1,0}_p M &:= \{Z \in T^{\mathbf{D}}_p M \mid KZ = \tau Z\} = \{X + \tau KX \mid X \in T_p M\}, \\ T^{0,1}_p M &:= \{Z \in T^{\mathbf{D}}_p M \mid KZ = -\tau Z\} = \{X - \tau KX \mid X \in T_p M\}. \end{aligned} \tag{1.3.13}$$

Also the ***D**-complex cotangent bundle* $(T^{\mathbf{D}})^* M$ can be split into the "$\pm\tau$-eigenbundles" (see the point 4 of Remark 1.2.3):

$$\wedge^{1,0}_K(M) = \{\alpha + \tau K^* \alpha \mid \alpha \in (T^{\mathbf{D}})^* M\} \qquad \wedge^{0,1}_K(M) = \{\alpha - \tau K^* \alpha \mid \alpha \in (T^{\mathbf{D}})^* M\} \tag{1.3.14}$$

(we will drop the subscript K if it is clear from the context) and the bundle $\wedge^r (T^{\mathbf{D}})^* M$ of the **D**-complex r-forms divides in

$$\wedge^r (T^{\mathbf{D}})^* M = \bigoplus_{p+q=r} \wedge^{p,q}_K(M) \quad \text{where } \wedge^{p,q}_K(M) = \wedge^{p,0}_K(M) \otimes \wedge^{0,q}_K(M). \tag{1.3.15}$$

Again we can introduce the following operators:

$$\begin{aligned} \partial &:= \pi_{\wedge^{p+1,q}_K} \circ d : \wedge^{p,q}_K(M) \longrightarrow \wedge^{p+1,q}_K(M), \\ \overline{\partial} &:= \pi_{\wedge^{p,q+1}_K} \circ d : \wedge^{p,q}_K(M) \longrightarrow \wedge^{p,q+1}_K(M). \end{aligned} \tag{1.3.16}$$

If K is integrable and (M, K) is a **D**-complex manifold, then we can use null-coordinates. Hence

$$TM^+ = \operatorname{span}\left\{ \frac{\partial}{\partial u^i_+} \mid i = 1, \ldots, n \right\} \qquad TM^- = \operatorname{span}\left\{ \frac{\partial}{\partial u^i_-} \mid i = 1, \ldots, n \right\} \tag{1.3.17}$$

and setting:

$$\partial_+ := \sum_{i=1}^n d u^i_+ \wedge \frac{\partial}{\partial u^i_+}, \quad \partial_- := \sum_{i=1}^n d u^i_- \wedge \frac{\partial}{\partial u^i_-} \tag{1.3.18}$$

the exterior differential d can be decomposed as $d = \partial_+ + \partial_-$. In a similar way using **D**-holomorphic coordinates we get $T^{1,0}_p M = \operatorname{span}\{\partial / \partial z^i\}$, $\wedge^{1,0}_K(M) = \operatorname{span}\{d z^i\}$ and analogous for $T^{0,1}_p M$ and $\wedge^{0,1}_K(M)$. Moreover, the exterior differential can be written as $d = \partial + \overline{\partial}$ where:

$$\partial := \sum_{i=1}^n d z_i \wedge \frac{\partial}{\partial z_i}, \quad \overline{\partial} := \sum_{i=1}^n d \bar{z}_i \wedge \frac{\partial}{\partial \bar{z}_i}. \tag{1.3.19}$$

From $d^2 = 0$, these operators are related by the following equations:

$$\begin{gathered} d = \partial + \overline{\partial} = \partial_+ + \partial_-, \qquad d^{\mathbf{D}} = \tau(\partial - \overline{\partial}) = \tau(\partial_+ - \partial_-) = K^* \circ d \circ K^*, \\ \partial = \frac{1}{2}(d + \tau\, d^{\mathbf{D}}) = e\,\partial_+ + \bar{e}\,\partial_-, \qquad \overline{\partial} = \frac{1}{2}(d - \tau\, d^{\mathbf{D}}) = e\,\partial_- + \bar{e}\,\partial_+, \\ d\,d = d^{\mathbf{D}}\, d^{\mathbf{D}} = \partial^2 = \overline{\partial}^2 = \partial_+^2 = \partial_-^2 = \partial\,\overline{\partial} + \overline{\partial}\,\partial = \partial_+\,\partial_- + \partial_-\,\partial_+ = 0, \\ d\,d^{\mathbf{D}} = -2\tau\,\partial\,\overline{\partial} = 2\,\partial_+\,\partial_- . \end{gathered} \tag{1.3.20}$$

As in the complex case, we can read the integrability condition N_K as a splitting of the differential d.

Lemma 1.3.10. *The following conditions are equivalent:*

1. $d(\Lambda^{1,0}_{+,-}(M)) \subseteq \Lambda^{2,0}_{+,-}(M) \oplus \Lambda^{1,1}_{+,-}(M)$,
2. TM^+ *is closed under bracket of vectors,*
3. $d(\Lambda^{1,0}_{K}(M)) \subseteq \Lambda^{2,0}_{K}(M) \oplus \Lambda^{1,1}_{K}(M)$,
4. $TM^{1,0}$ *is closed under bracket of vectors,*
5. $N_K(X,Y) = 0$ *for all vector fields* X, Y.

Proof. To see the equivalence of 1 and 2, let $\alpha \in \Lambda^{0,1}_{+,-}(M)$, and use the formula

$$d\,\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha(X,Y) = -\alpha([X,Y]) \quad X, Y \in TM^+. \tag{1.3.21}$$

We proceed analogously to show the equivalence between 3 and 4.
Now from (1.3.5) we have:

$$N_K(X,Y) = 4\big([X^+,Y^+]^- + [X^-,Y^-]^+\big) \tag{1.3.22}$$

that shows the equivalence of 1 and 2 with the last point 5. We note also that the following formula:

$$\begin{aligned} N_K(X,Y) &= \mathrm{Re}\big([X-\tau KX, Y-\tau KY] + \tau K[X-\tau KX, Y-\tau KY]\big) \\ &= 8\,\mathrm{Re}\big([X^{1,0}, Y^{1,0}]^{0,1}\big) \end{aligned} \tag{1.3.23}$$

leads to the equivalence between 5 and the points 3 and 4. □

Corollary 1.3.11. *An almost **D**-complex structure K is integrable if and only if* d *splits as*

$$d = \partial_+ + \partial_- \tag{1.3.24}$$

or, equivalently, if d *splits as*

$$d = \partial + \overline{\partial}\,. \tag{1.3.25}$$

1.4 D-Hermitian metrics

Now we introduce a suitable notion of metric:

Definition 1.4.1. An *(almost) **D**-hermitian manifold* (M,K,g) is an (almost) **D**-complex manifold endowed with a pseudo-Riemannian metric g such that K is an anti-isometry for g, i.e. $g(K\cdot, K\cdot) = -g(\cdot,\cdot)$.

Note that g has signature (n,n) and the spaces TM^+ and TM^- are null spaces for g.
As explained before (see Proposition 1.2.4) we can extend g to a **D**-complexification form h such that:

$$h(X,Y) = \overline{h(Y,X)} \quad \text{and} \quad h(KX,Y) = \tau h(X,Y) = -h(X,KY) \quad X,Y \in TM. \tag{1.4.1}$$

(In the sequel, we identify g with its **D**-complexification if it makes no confusion.)
We can define the *fundamental 2-form* ω (or *Kähler form*) of the **D**-hermitian manifold (M,K,g) as:

$$\omega(\cdot,\cdot) := g(\cdot, K\cdot). \tag{1.4.2}$$

Moreover, if we express h in terms of its real and imaginary parts we get:

$$h(\cdot,\cdot) = g(\cdot,\cdot) - \tau\omega(\cdot,\cdot). \tag{1.4.3}$$

Notice that the fundamental 2-form satisfies $K^*\omega(\cdot,\cdot) = \omega(K\cdot, K\cdot) = -\omega(\cdot,\cdot)$ and is hence of type $(1+, 1-)$ (or, equivalently when considered as a **D**-valued form, ω is of type $(1,1)$).

Now let (M, K, g) be a **D**-Hermitian manifold (that is K integrable), then we can write h and ω in terms of local **D**-holomorphic coordinates. Setting $h_{i,\bar{j}} = h(\partial / \partial z^i, \partial / \partial \bar{z}^j)$ we get:

$$\begin{aligned} h &= \sum_{i,\bar{j}} \Big(h_{i,\bar{j}}\, d\, z^i \otimes d\, \bar{z}^j + h_{\bar{j},i}\, d\, \bar{z}^j \otimes d\, z^i \Big) \quad \text{where } h_{\bar{j},i} = \overline{h_{i,\bar{j}}}, \\ g &= \mathrm{Re}(h) = \frac{1}{2}(h + \bar{h}) = \sum_{i,\bar{j}} g_{i.\bar{j}} (d\, z^i \otimes d\, \bar{z}^j + d\, \bar{z}^i \otimes d\, z^j), \\ \omega &= -\,\mathrm{Im}(h) = -\frac{\tau}{2}(h - \bar{h}) = \sum_{i,\bar{j}} \omega_{i,\bar{j}}\, d\, z^i \wedge d\, \bar{z}^j, \end{aligned} \tag{1.4.4}$$

and from $\omega(\cdot,\cdot) = g(\cdot, K\cdot)$, we have $\omega_{i,\bar{j}} = -\tau g_{i,\bar{j}}$. Moreover, since TM^+ and TM^- are null for both ω and g, we have that for every $X \in TM$ $\omega(X,\cdot)$ and $g(X,\cdot)$ are 1-forms. Hence we can introduce a bundle isomorphism:

$$\begin{aligned} A : TM^+ &\xrightarrow{\simeq} T^*M_- \\ X^+ &\longmapsto A_{X^+} \end{aligned} \tag{1.4.5}$$

such that, writing $X = X^+ + X^-$, $Y = Y^+ + Y^-$ with respect to the decomposition $TM^+ \oplus TM^-$, we have:

$$g(X,Y) = A_{X^+}(Y^-) + A_{Y^-}(X^+) \qquad \omega(X,Y) = A_{X^+}(Y^-) - A_{Y^-}(X^+). \tag{1.4.6}$$

This leads us to the following:

Remark 1.4.2. As opposed to the complex Hermitian case, the isomorphism (1.4.5) shows that the existence of a **D**-Hermitian metric on M puts further topological restrictions on the bundle TM. For example, any product $M = M_+ \times M_-$ of two n-manifolds with the **D**-complex structure as in Example 1.3.7 is a **D** manifold, but (1.4.5) tells us that a Hermitian metric exists on M if and only if both M_+, M_- are parallelizable. In fact, for any (x, y) in $M_+ \times M_-$ we have an isomorphism

$$A : (T_xM)^+ = T_x(M_+) \longrightarrow T_y^*(M_-) = (T_y^*M)_- \tag{1.4.7}$$

given by (1.4.5). Picking a basis for $T_x(M_+)$ and moving $y \in M_-$ we get a parallelization of $T(M_-)$. Symmetry gives a parallelization of the other factor.

Furthermore, we have:

Proposition 1.4.3. *An almost **D**-Hermitian $2n$-manifold (M, K, g) is almost symplectic, and hence orientable.*

Proof. We use the fundamental 2-form $\omega(\cdot,\cdot) = g(\cdot, K\cdot)$. From the properties of g, we see that ω is bilinear, skew-symmetric and non-degenerate, then $\omega(\cdot,\cdot)$ is an almost symplectic form, $\omega^n \neq 0$ and it is a multiple of the volume form. Hence M is orientable. □

Now let us investigate a bit on the existence of such a metric on an almost complex manifold (M, K) with the following two examples.

Example 1.4.4. Take $M = N \times N$ to be the product of two copies of a non-orientable manifold N. We have seen (Example 1.3.7) that M has a natural almost **D**-complex structure K, but it follows from Proposition 1.4.3 that there exists no pseudo-Riemannian metric g compatible with the natural complex structure K.

On the other hand, it is possible to find pseudo-Riemannian metric g with signature (n,n) which does not admit any (almost) **D**-complex structure, as shown in the next Example 1.4.5.

Example 1.4.5. Let N be a non-orientable n-manifold as in the previous Example 1.4.4, and take on N a positive definite Riemannian metric g_+ and a negative definite Riemannian metric g_-. Then on $M = N \times N$ the product metric $g = g_+ \times g_-$ has signature (n,n), but using Proposition 1.4.3 we see that there is no (almost) **D**-structure K which is an anti-isometry for g. In fact, using the natural almost **D**-complex structure K of Example 1.3.7, we see that $g(KX, KY) = g(X, Y)$.

Remark 1.4.6. As Examples 1.3.4 and 1.4.4 show, the existence of a **D**-complex structure does not implies the orientability of the manifold, unlike the complex case. Also the existence of a neutral metric does not assure the existence of a compatible **D**-complex metric (see Example 1.4.5). To avoid such cases, and since we used to work with metric on manifold, we require for the rest of this paper to deal with *orientable manifold.* All the definitions made till now are still valid, with the obvious slight changes. E.g. on Proposition 1.3.5 we will require that the transition functions are orientation-preserving. In the **D**-complex atlas this means that the coordinate functions not only are **D**-holomorphic, but also that the Jacobian of the changes of coordinates has determinant in $\mathbf{D}^+$.

1.5 D-Kähler metrics

We now consider the following natural class of Hermitian double manifolds.

Definition 1.5.1. A Hermitian **D**-manifold (M, K, g) is said to be *Kähler* if the fundamental 2-form
$$\omega(\cdot,\cdot) = g(\cdot, K\cdot) \tag{1.5.1}$$
is closed, i.e. $d\omega = 0$.

Some basic properties of complex Kähler manifolds carry over to this context.

Remark 1.5.2. Even if the metric g is not a definite-metric, there still exists the Levi-Civita connection $\mathcal{D}^g$ of g (we will drop the upper index g if not necessary). This connection is the unique connection such that g is parallel along $\mathcal{D}$ (i.e. $\mathcal{D}g = 0$) and torsion-free (see e.g. [14]).

Proposition 1.5.3. *Let (M, K, g) be an almost **D**-Hermitian manifold. Then M is Kähler if and only if K is parallel in the Levi-Civita connection $\mathcal{D}$ (i.e. $\mathcal{D}K = 0$).*

Proof. If K is parallel with respect to $\mathcal{D}$ then:
$$\begin{aligned} N_K(X,Y) &= [KX,KY] - K[KX,Y] - K[X,KY] + [X,Y] \\ &= \mathcal{D}_{KX}KY - \mathcal{D}_{KY}KX - K\mathcal{D}_{KX}Y + K\mathcal{D}_Y KX \\ &\quad - K\mathcal{D}_X KY + K\mathcal{D}_{KY}X + \mathcal{D}_X Y - \mathcal{D}_Y X \\ &= \big(K(\mathcal{D}_Y K) - (\mathcal{D}_{KY}K)\big)X - \big(K(\mathcal{D}_X K) - (\mathcal{D}_{KX}K)\big)Y = 0 \end{aligned} \tag{1.5.2}$$

so K is integrable. Moreover, setting $\omega(\cdot,\cdot) = g(\cdot, K\cdot)$, we easily see from $\mathcal{D}K = \mathcal{D}g = 0$ that $\mathcal{D}\omega = 0$.
Vice-versa, a computation shows that for every $X, Y, Z \in TM$:

$$2g\big((\mathcal{D}_X K)Y, Z\big) = 2\big(g(\mathcal{D}_X(KY), Z) - g(K\mathcal{D}_X Y, Z)\big) = 2\big(g(\mathcal{D}_X(KY), Z) + g(\mathcal{D}_X Y, KZ)\big) \tag{1.5.3}$$

then expanding we get:

$$
\begin{aligned}
2g\big((\mathcal{D}_X K)Y, Z\big) &= Xg(KY,Z) + KYg(X,Z) - Zg(X,KY) \\
&\quad + g([X,KY],Z) - g([X,Z],KY) - g([KY,Z],X) \\
&\quad + Xg(Y,KZ) + Yg(X,KZ) - KZg(X,Y) \\
&\quad + g([X,Y],KZ) - g([X,KZ],Y) - g([Y,KZ],X) \\
&= -X\omega(Y,Z) - Y\omega(Z,X) - Z\omega(X,Y) \\
&\quad + \omega([X,Y],Z) + \omega([Z,X],Y) \pm \omega([Y,Z],X) \\
&\quad + KYg(X,Z) + Xg(Y,KZ) - KZg(X,Y) \\
&\quad + g([X,KY],Z) - g([KY,Z],X) - g([X,KZ],Y) - g([Y,KZ],X).
\end{aligned} \tag{1.5.4}
$$

Using the formula of differential of a 2-form we obtain:

$$
\begin{aligned}
2g\big((\mathcal{D}_X K)Y, Z\big) &= -\,d\omega(X,Y,Z) - \omega([Y,Z],X) \pm \omega([KY,KZ],X) \\
&\quad - KY\omega(KZ,X) - X\omega(KY,KZ) - KZ\omega(X,KY) + \omega([X,KY],KZ) \\
&\quad - \omega([X,KZ],KY) - g([KY,Z],X) - g([Y,KZ],X) \\
&= -\,d\omega(X,Y,Z) - \omega([Y,Z],X) - d\omega(X,KY,KZ) \\
&\quad - \omega([KY,KZ],X) - g([KY,Z],X) - g([Y,KZ],X)
\end{aligned} \tag{1.5.5}
$$

and finally:

$$
\begin{aligned}
2g\big((\mathcal{D}_X K)Y, Z\big) &= -\,d\omega(X,Y,Z) - d\omega(X,KY,KZ) \\
&\quad g\big(K[Y,Z] + K[KY,KZ] - [KY,Z] - [Y,KZ], X\big) \\
&= d\omega(X,Y,Z) + d\omega(X,KY,KZ) - g\big(N_K(Y,Z),KX\big).
\end{aligned} \tag{1.5.6}
$$

If $N_K = d\omega = 0$ it follows that $\mathcal{D}K = 0$. □

The **D**-Kähler manifolds are closely related to the existence of Lagrangian foliations. In fact we have:

Proposition 1.5.4. *A* ***D****-Hermitian manifold* (M,K,g) *is Kähler if and only if* (M,ω) *is symplectic and there is a pair of transversal Lagrangian foliations* $\mathcal{F}^{\pm}$ *(i.e. a bi-Lagrangian manifold).*

Proof. If (M,K,g) is **D**-Kähler, then the fundamental 2-form $\omega(\cdot,\cdot) := g(\cdot,K\cdot)$ is a symplectic form and $TM^{\pm}$ are involutive null-spaces for ω, hence they are a pair of transversal Lagrangian foliations.
Vice-versa, given a symplectic manifold (M,ω) with two transversal Lagrangian foliations $\mathcal{F}^{\pm}$, we define $K : TM \to TM$ such that $K|_{\mathcal{F}^{\pm}} = \pm\,\mathrm{Id}$. Since $\mathcal{F}^{\pm}$ are foliations, we see that K is an integrable **D**-complex structure. Moreover, it is known (see Chapter 4) that there exists an unique torsion-free connection ∇ such that $\nabla K = \nabla\omega = 0$. Setting $g(\cdot,\cdot) = \omega(\cdot,K\cdot)$ it follows that g is a **D**-Hermitian metric and $\nabla g = 0$, then the Levi-Civita connection of g is $\mathcal{D} = \nabla$. We conclude using the previous Proposition 1.5.3. □

Because of the previous result, **D**-Kähler manifolds are also referred to as *bi-Lagrangian manifolds* (e.g. [16] and [31]).

Proposition 1.5.5. *Let* (M,g) *be a connected pseudo-Riemannian* $2n$*-manifold. Then there exists a* ***D****-complex structure* K *such that* (M,K,g) *is* ***D****-Kähler if and only if the holonomy group of* (M,g) *is a subgroup of the* ***D****-unitary group, i.e. if and only if there exists a* $p \in M$ *and a linear isometry* $T_pM \cong \mathbb{R}^{2n}$ *which identifies the holonomy group* $\mathrm{Hol}_p(M,g)$ *with* $\mathrm{U}_n(\boldsymbol{D})$.

Proof. If (M, g, K) is a **D**-Kähler manifold, then follows that

$$\mathrm{Hol}_p(M, g) \subset \mathrm{U}_n(\mathbf{D}) \tag{1.5.7}$$

because of $\mathrm{U}_n(\mathbf{D})$ is isomorphic to $\mathrm{Aut}(T_pM, g_p, K_p)$.
Vice-versa, assume that $\psi \circ \mathrm{Hol}_p(M, g) \circ \psi^{-1} \subset \mathrm{U}_n(\mathbf{D})$ where $\psi : T_pM \to \mathbf{D}^n \cong \mathbb{R}^{2n}$ is a linear isometry. We want to define a **D**-complex structure on (M, g). Then we can do it on T_pM by pulling back the standard **D**-complex structure on $\mathbb{R}^{2n}$, i.e. $K_p := \psi \circ K_0 \circ \psi^{-1}$ (where K_0, g_0 denote the standard structure on $\mathbb{R}^{2n}$, i.e. when identified with $\mathbf{D}^n$). We see that K_p is an anti-isometry for g_p (because K_0 is an anti-isometry for g_0 and ψ is an isometry). Now K_p can be extended (by parallel transport) to a parallel anti-isometric para-complex structure K on (M, g), since K_p is invariant under the holonomy group $\mathrm{Hol}_p(M, g)$. Hence, by Proposition 1.5.3, (M, g, K) is a **D**-Kähler manifold. □

Example 1.5.6. Let M be a orientable surface $\dim_{\mathbb{R}} M = 2$ which admits a pseudo-Riemannian metric g, whose signature is $(1, 1)$, i.e. let M be a *Lorentzian surface.* Then, there exists a basis of vector fields $\{X, Y\}$ such that

$$g(X, X) = +1 \qquad g(Y, Y) = -1 \qquad g(X, Y) = 0. \tag{1.5.8}$$

We can introduce an almost **D**-complex structure K by $K(X) = Y$ and $K(Y) = X$. It follows that K and g are compatible, and since the distributions $T^{\pm}M = \{X \pm Y\}$ are 1-dimensional, they are involutive and integrable, and so the almost **D**-complex structure K is integrable. Hence every Lorentzian surface is a **D**-complex manifold. Moreover, it is also symplectic, since every 2-form is closed. Hence we see that the manifold is a **D**-Kähler surface. Then we have that:

Proposition 1.5.7. *Every orientable surface is Lorentzian if and only if it is a **D**-Kähler surface.*
*The only compact Lorentzian (equivalently **D**-Kähler) surface is the torus $\mathbb{T}^2$, which is the unique compact example of a **D**-Hermitian surface.*

The last part of the proposition follows from topological obstruction (a surface admits a pseudo-Riemannian metric of signature $(1, 1)$ only if has vanishing Euler characteristic, see e.g. [14]), and it is known that the only compact Lorentzian surfaces is the torus. As obvious, any **D**-Hermitian surface of dimension 2 is Lorentzian.

These **D**-complex manifolds, called *Ricci-flat **D**-Kähler manifolds*, that play a role similar to that one of Calabi-Yau manifolds in complex geometry, will be detailed in the Chapter 4.

1.6 Invariant D-complex structures on solvmanifolds

In this section we will recall some notions on solvmanifolds and nilmanifolds, which form a large class of examples of **D**-manifolds.

Let $M := \Gamma \backslash G$ be a $2n$-dimensional *solvmanifold* (resp. *nilmanifold*), that is, a compact quotient of a connected simply-connected solvable (resp. nilpotent) Lie group G by a co-compact discrete subgroup Γ. Set $(\mathfrak{g}, [\cdot, \cdot])$ the Lie algebra which is naturally associated to the Lie group G; given a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$, the Lie algebra structure of $\mathfrak{g}$ is characterized by the *structure constants* $\{c^k_{\ell m}\}_{\ell,m,k \in \{1,\ldots,2n\}} \subset \mathbb{R}$ such that, for any $k \in \{1, \ldots, 2n\}$,

$$d_{\mathfrak{g}}\, e^k := \sum_{\ell,m} c^k_{\ell m}\, e^\ell \wedge e^m \ , \tag{1.6.1}$$

where $\{e^1, \ldots, e^{2n}\}$ is the dual basis of $\mathfrak{g}^*$ of $\{e_1, \ldots, e_{2n}\}$ and $d_{\mathfrak{g}} := d\colon \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is defined by

$$\begin{aligned} \mathfrak{g}^* : &\longrightarrow \wedge^2\mathfrak{g}^* \\ \alpha &\longmapsto d_{\mathfrak{g}}\, \alpha(\cdot,\, \cdot) := -\alpha\left([\cdot,\, \cdot]\right). \end{aligned} \tag{1.6.2}$$

Notation. To shorten the notation, we will refer to a given solvmanifold $M := \Gamma\backslash\, G$ writing the structure equations of its Lie algebra: for example, writing

$$M \;:=\; \left(0^4,\; 12,\; 34\right) \;,$$

we mean that there exists a basis of the naturally associated Lie algebra $\mathfrak{g}$, let us say $\{e_1, \ldots, e_6\}$, whose dual will be denoted by $\{e^1, \ldots, e^6\}$ and with respect to which the structure equations are

$$\begin{cases} d\,e^1 &=\; d\,e^2 \;=\; d\,e^3 \;=\; d\,e^4 \;=\; 0 \\ d\,e^5 &=\; e^1 \wedge e^2 \;=:\; e^{12} \\ d\,e^6 &=\; e^1 \wedge e^3 \;=:\; e^{34} \;, \end{cases}$$

where we also shorten $e^{AB} := e^A \wedge e^B$. By identification (1.6.2) this also means that $[e_1, e_2] = -e_5$, $[e_1, e_3] = -e_6$ and all other brackets are zero.
Recall that, by Malcev theorem [59, Theorem 7], given a nilpotent Lie algebra $\mathfrak{g}$ with *rational* structure constants, then the connected simply-connected Lie group G naturally associated to $\mathfrak{g}$ admits a co-compact discrete subgroup Γ, and hence there exists a nilmanifold $M := \Gamma\backslash\, G$ whose Lie algebra is $\mathfrak{g}$ and such that the basis $\{e^1, \ldots, e^n\}$ of the dual algebra $\mathfrak{g}^*$ defines a basis of global 1-forms for $M := \Gamma\backslash\, G$. If the Lie algebra is not a nilmanifold, we will describe the existence of a compact quotient where again the basis of $\mathfrak{g}^*$ is a global frame of 1-forms.

A *linear almost **D**-complex* structure on $\mathfrak{g}$ is given by an endomorphism $K \in \mathrm{End}(\mathfrak{g})$ such that $K^2 = \mathrm{Id}_{\mathfrak{g}}$ and the eigenspaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ corresponding to the eigenvalues $+1$ and -1 respectively of K are equi-dimensional, i.e. $\dim_{\mathbb{R}} \mathfrak{g}^+ = \dim_{\mathbb{R}} \mathfrak{g}^- = \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{g}$.
Moreover, recall that a linear almost **D**-complex structure on $\mathfrak{g}$ is said to be *integrable* (and hence it is called a *linear **D**-complex* structure on $\mathfrak{g}$) if $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are Lie-subalgebras of $\mathfrak{g}$, i.e.

$$\left[\mathfrak{g}^+,\, \mathfrak{g}^+\right] \;\subseteq\; \mathfrak{g}^+ \qquad \text{and} \qquad \left[\mathfrak{g}^-,\, \mathfrak{g}^-\right] \;\subseteq\; \mathfrak{g}^- \;. \tag{1.6.3}$$

In fact, a Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_1, \mathfrak{g}_2$ satisfy the above equation (1.6.3) is also called a *double Lie algebra*, because of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras of $\mathfrak{g}$ and linear **D**-complex structure may be called in this way (see e.g. [58, 7, 6]).

A *G-invariant (almost) **D**-complex structure* K_{inv} on M is a **D**-complex structure on M induced by a **D**-complex structure on G which is invariant under the left-action of G on itself given by translations. Note that any G-invariant (almost) **D**-complex structure is determined by a linear almost **D**-complex structure on $\mathfrak{g}$, equivalently, it is defined by the datum of two subspaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ of $\mathfrak{g}$ such that

$$\mathfrak{g} \;=\; \mathfrak{g}^+ \oplus \mathfrak{g}^- \qquad \text{and} \qquad \dim_{\mathbb{R}} \mathfrak{g}^+ = \dim_{\mathbb{R}} \mathfrak{g}^- = \frac{1}{2}\, \dim_{\mathbb{R}} \mathfrak{g} \;; \tag{1.6.4}$$

indeed, one can define $K \in \mathrm{End}(\mathfrak{g})$ as $K\lfloor_{\mathfrak{g}^+} = \mathrm{Id}$ and $K\lfloor_{\mathfrak{g}^-} = -\mathrm{Id}$ and then $K \in \mathrm{End}(TM)$ by translations. Note that the almost **D**-complex structure K on M is integrable if and only if the linear almost **D**-complex structure K on $\mathfrak{g}$ is integrable.

Notation. On a solvmanifold $M := \Gamma\backslash\, G$, with respect to the given basis $\{e_j\}_j$, writing that the (almost) **D**-complex structure K is defined as

$$K \;:=\; (-\; +\; +\; -\; -\; +) \tag{1.6.5}$$

we mean that

$$\mathfrak{g}^{+} := \mathbb{R}\langle e_2,\ e_3,\ e_6\rangle \qquad \text{and} \qquad \mathfrak{g}^{-} := \mathbb{R}\langle e_1,\ e_4,\ e_5\rangle\ ,$$

or, equivalently, that

$$\begin{array}{lll} K(d\,e^1) = -\,d\,e^1 & K(d\,e^2) = +\,d\,e^2 & K(d\,e^3) = +\,d\,e^3 \\ K(d\,e^4) = -\,d\,e^4 & K(d\,e^5) = -\,d\,e^5 & K(d\,e^6) = +\,d\,e^6, \end{array}$$

this because of K^* commutes with the canonical isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ (see (1.3.9)). Dealing with *invariant* objects on M, we mean objects induced by objects on G which are invariant under the left-action of G on itself given by translations.

1.7 CR D-Manifolds

For the sake of completeness we start giving a general definition of *CR **D**-manifolds* (see [42]). Then we focus on those ones arising from a contact form and we recall some properties of contact manifolds. We also introduce the notion of *strictly CR **D**-structure*, which is analogue to the complex one.

Definition 1.7.1. An *almost CR **D**-structure* (also called *almost para-CR structure*) *of co-dimension* k on an $(2n+k)$-dimensional manifold M is a pair $(\mathcal{H}, K)$ where $\mathcal{H} \subset TM$ is a rank $2n$ distribution and $K \in \mathrm{End}(\mathcal{H})$ is a **D**-complex structure on $\mathcal{H}$, i.e. $K^2 = +\,\mathrm{Id}$ with $K \neq \pm\,\mathrm{Id}$ and the ± 1-eigendistributions $\mathcal{H}^{\pm} \subset \mathcal{H}$ of K have the same rank n. An *almost CR **D**-structure* is called *CR **D**-structure* if it is *integrable*, i.e. the eigendistributions $\mathcal{H}^{\pm}$ are involutive.

Remark 1.7.2. It has to be noted that this Definition 1.7.1 can be generalized in a similar way as the (almost) product structures generalized the (almost) **D**-complex structures (see Remark 1.3.8). Namely a *weak almost CR **D**-structure* (also called *weak almost para-CR structure*) *of co-dimension* k on an $(m+k)$-dimensional manifold M is a pair $(\mathcal{H}, K)$ where $\mathcal{H} \subset TM$ is a rank m distribution and $K \in \mathrm{End}(\mathcal{H})$ satisfies $K^2 = +\,\mathrm{Id}$ with $K \neq \pm\,\mathrm{Id}$. Again a *(weak) almost CR **D**-structure* is called *(weak) CR **D**-structure* if it is *integrable*, i.e. the eigendistributions $\mathcal{H}^{\pm}$ are involutive. We easily see that if the ± 1-eigendistributions $\mathcal{H}^{\pm}$ have the same rank n, then $(\mathcal{H}, K)$ is an *(almost) CR **D**-structure*, and some authors call them *strong* (almost) CR **D**-structure, to emphasize the difference with the weaker case (e.g. [3]).

Note that the *Nijenhuis tensor* N_K of an almost CR **D**-complex structure

$$N_K(X,Y) := [KX, KY] - K[KX, Y] - K[X, KY] + [X, Y] \qquad X, Y \in \mathcal{H}, \tag{1.7.1}$$

is not well defined in general, since $[X,Y]$ may not be in $\mathcal{H}$. So that the Nijenhuis tensor makes sense it is sufficient to require that

$$[KX, Y] + [X, KY] \in \mathcal{H} \quad \text{equivalently} \quad [X, Y] + [KX, KY] \in \mathcal{H} \quad X, Y \in \mathcal{H}. \tag{1.7.2}$$

Sometimes, the above condition is used by some authors as an integrability condition, but we stress that it is weaker than our request of involutive distributions. However, it is still true that if the distributions $\mathcal{H}^{\pm}$ are involutive, then N_K is well defined over $\mathcal{H}$ and $N_K = 0$. Vice versa, if K satisfies (1.7.2) and $N_K = 0$, then the distributions $\mathcal{H}^{\pm}$ are involutive and the CR **D**-structure is integrable.

In particular we have to deal with almost CR **D**-structures of co-dimension 1, and in this setting the contact structure plays a fundamental role, so we recall some notions.

Let M be a $(2n+1)$-dimensional manifold. A *contact form* is the datum of an $\alpha \in \wedge^1 M$ such that

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ everywhere on } M, \tag{1.7.3}$$

which is equivalent to say that:

1. α never vanishes on M, and

2. $d\alpha|_{\ker\alpha}$ is everywhere non degenerate (i.e. α restricts to a symplectic form on the $2n$-dimensional distribution $\xi = \ker\alpha$).

A tangent distribution ξ on M of co-dimension 1 is called a *contact structure* if it can be locally defined by the Pfaffian equation $\alpha = 0$ for some choice of the contact form α, and in this case (M, ξ) is called *contact manifold.* We denote the space of the sections of ξ by $\mathcal{H}(\xi)$, i.e. the space of ξ-valued vector fields on M.

Given a contact manifold (M, ξ) and a contact form α we denote with R_α the *Reeb vector field of* α, i.e. the unique vector field such that:

$$i_{R_\alpha} d\alpha = 0, \qquad \alpha(R_\alpha) = 1. \tag{1.7.4}$$

Remark 1.7.3. The Reeb vector field of a contact manifold (M, ξ) satisfies the following properties:

1. $TM = \xi \oplus \mathbb{R}R_\alpha$,

2. $[R_\alpha, X] \in \mathcal{H}(\xi)$ for every $X \in \mathcal{H}(\xi)$.

We give the following:

Definition 1.7.4. Let (M, ξ) be a contact manifold. We define $\mathfrak{D}(\xi)$ to be the set of the almost **D**-complex structures K on ξ which are *$d\alpha$-pseudo-calibrated*, namely $d\alpha(K\cdot, K\cdot) = -\,d\alpha(\cdot,\cdot)$ where α is a contact form.

Remark 1.7.5. We note that:

- $\mathfrak{D}(\xi)$ does not depend on the choice of α;

- Using that for a 2-form ω it holds:

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \tag{1.7.5}$$

  if $K \in \mathfrak{D}(\xi)$ then for $X, Y \in \mathcal{H}(\xi)$:

$$\begin{aligned} d\alpha(KX, KY) = -\,d\alpha(X, Y) &\Longleftrightarrow [KX, KY] + [X, Y] \in \mathcal{H}(\xi) \\ &\Longleftrightarrow [KX, Y] + [X, KY] \in \mathcal{H}(\xi) \end{aligned} \tag{1.7.6}$$

  and hence N_K is well defined on $\mathcal{H}(\xi)$. Moreover:

$$N_K \in (\wedge_K^{0,2}\xi^*) \otimes \xi. \tag{1.7.7}$$

  If $K \in \mathfrak{D}(\xi)$ then the condition of "weaker integrability" (1.7.2) is satisfied, and N_K is well defined.

Definition 1.7.6. A *strictly CR **D**-structure* on a contact manifold (M, ξ) is the datum of a $K \in \mathfrak{D}(\xi)$ satisfying $N_K(X, Y) = 0$ for every $X, Y \in \mathcal{H}(\xi)$.

Observe that:

- $K \in \mathfrak{D}(\xi)$ is a strictly CR $\mathbf{D}$-structure if and only if $[\xi^{0,1}, \xi^{0,1}] \subset \xi^{0,1}$, where $\xi^{0,1} = \{z \in \xi \otimes \mathbf{D} \mid KZ = -\tau Z\}$.
- Given $X \in T^{\mathbf{D}}M$, then:

$$X \in \xi^{\mathbf{D}} \Longleftrightarrow X^{1,0} \in \xi^{1,0} \Longleftrightarrow X^{0,1} \in \xi^{0,1}.$$

Given (M, ξ), set $N := M \times \mathbb{R}_s$ then (N, μ_α) is called the *symplectization of* (M, ξ) *with respect to* α, where the symplectic form μ_α is defined by

$$\mu_\alpha := d\left(\exp(s)\alpha\right). \tag{1.7.8}$$

Now starting with a given CR $\mathbf{D}$-structure on (M, ξ) we want to define a $\mathbf{D}$-structure on (N, μ_α). Let K be a $\mathbf{D}$-structure defined on $\ker \alpha$. We define the *extended* $\boldsymbol{D}$*-structure on* TN as follows: we set

$$KR_\alpha := S \qquad KS := R_\alpha \qquad \text{where } S := \frac{\partial}{\partial s}. \tag{1.7.9}$$

Remark 1.7.7. If $K \in \mathcal{D}(\xi)$ then $\mu_\alpha(\cdot, K\cdot)$ is a Hermitian metric of signature $(n+1, n+1)$ on TM.

1.8 Automorphisms of D-manifolds and CR D-manifolds

A $\mathbf{D}$-complex manifold M can be viewed as a manifold M together with a G-structure, and this view point is used in complex Kähler geometry to see that the automorphism group is finite. We recall briefly the G-structures (we refer to [72] or [49]).

Let M be a differentiable manifold of dimension n, and let $\mathcal{L}(M)$ be the bundle of linear frames over M. Then $\mathcal{L}(M)$ is a principal fiber bundle over M with structure group $\mathrm{GL}_n(\mathbb{R})$. Let G be a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$, by a *$G$-structure* on M we mean a differentiable subbundle P of $\mathcal{L}(M)$ with structure group G. A G-structure P on M is said *integrable* if every point of M has a coordinate neighbourhood U with local coordinate system $\{x^1, \ldots, x^n\}$ such that the cross section $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ of $\mathcal{L}(M)$ over U is a cross section of P over U (such a local coordinate system will be called *admissible*). If $\{y^1, \ldots, y^n\}$ is another admissible local coordinate system on V, then the Jacobian matrix $\left(\frac{\partial y^i}{\partial x^j}\right)_{i,j=1,\ldots,n}$ is in G at each point of $U \cup V$.

Let P and P' be two G-structures over M and M'. Let f be a diffeomorphism of M onto M' and $f_* : \mathcal{L}(M) \to \mathcal{L}(M')$ the induced isomorphism on bundles. If f_* maps P into P', then f is an *isomorphism* of the G-structure P onto G-structure P'. If $M = M'$ and $P = P'$, then an isomorphism f is called an *automorphism* of the G-structure P.

Now a vector field X on M is called a *infinitesimal automorphism* of a G-structure P if it generates a local 1-parameter group of automorphisms of P.

There is a correspondence between the G-structure and the linear transformation leaving some tensor K invariant. More precisely:

Proposition 1.8.1 ([49, Proposition 1.2])**.** *Let* $\mathtt{K}$ *be a tensor over the vector space* $\mathbb{R}^n$ *and* G *the group of linear transformations of* $\mathbb{R}^n$ *leaving* $\mathtt{K}$ *invariant. Let* P *be a* G*-structure on* M*, and let* K *the tensor field on* M *defined by both* P *and* $\mathtt{K}$ *in a natural manner.*

Then P *is integrable if and only if each point of* M *has coordinate neighbourhood with local coordinate system* $\{x^1, \ldots, x^n\}$ *with respect to which the components of* K *are constant functions on* U*.*

Now, from the above Proposition 1.8.1, we get:

Proposition 1.8.2 ([49, Proposition 1.3]). *Let* $\mathtt{K}$ *be a tensor over the vector space* $\mathbb{R}^n$ *and* G *the group of linear transformations of* $\mathbb{R}^n$ *leaving* $\mathtt{K}$ *invariant. Let* P *be a* G*-structure on* M *and let* K *be the tensor field on* M *defined by* $\mathtt{K}$ *and* P *as in Proposition 1.8.1. Then:*

1. *a diffeomorphism* $f : M \to M$ *is an automorphism of* P *if and only if* f *leaves* K *invariant;*
2. *a vector field* X *on* M *is an infinitesimal automorphism of* P *if and only if* $\mathcal{L}_X K = 0$*, where* $\mathcal{L}_X$ *denotes the Lie derivation with respect to* X*.*

This last proposition explains a possible link between G-structures and automorphisms.

Now we study infinitesimal automorphisms of an integrable G-structure by a local point of view. Without loss of generality, assume that $M = \mathbb{R}^n$ and $P = \mathbb{R}^n \times G$. Let $\{x^1, \ldots, x^n\}$ be the natural coordinate system in $\mathbb{R}^n$ and let X be a vector field in a neighbourhood of the origin 0 of $\mathbb{R}^n$. We expand its components into power series:

$$\begin{aligned} X &= \sum \xi^i \frac{\partial}{\partial x^i} \\ \xi^i &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1,\ldots,j_k=1}^{n} a^i_{j_1,\ldots,j_k} x^{j_1} \cdots x^{j_k} \end{aligned} \tag{1.8.1}$$

where $a^i_{j_1,\ldots,j_k} \in \mathbb{R}$ are symmetric in the subindex $j_1, \ldots, j_k$. Note that X is an infinitesimal automorphism of the G-structure P if and only if the matrix $\left(\dfrac{\partial \xi^i}{\partial x^j}\right)_{i,j}$ belongs to the Lie algebra $\mathfrak{g}$ of G, and we conclude the following remark.

Remark 1.8.3. X as before is an infinite automorphism of the G-structure P if and only if the matrix $\left(a^i_{j_1,\ldots,j_k}\right)_{i,j_\ell=1,\ldots,n} \in \mathfrak{g}$ for any fixed choice $j_1, \ldots, \hat{j}_\ell, \ldots, j_k$.

By the previous Remark 1.8.3, it makes sense to introduce the following definition: let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ and let $\mathfrak{g}_k$ be the space of symmetric multi-linear mappings:

$$S : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{1.8.2}$$

such that, for each fixed vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, the linear transformation

$$v \in \mathbb{R}^n \longmapsto S(v, v_1, \ldots, v_k) \in \mathbb{R}^n \tag{1.8.3}$$

is in $\mathfrak{g}$. In particular $\mathfrak{g}_0 = \mathfrak{g}$. We call $\mathfrak{g}_k$ the *k-th prolongation of* $\mathfrak{g}$. The first integer k such that $\mathfrak{g} = 0$ is called *the order of* $\mathfrak{g}$, and then $\mathfrak{g}_{k+1} = \mathfrak{g}_{k+2} = \cdots = 0$. If $\mathfrak{g}_k \neq 0$ for all k, then $\mathfrak{g}$ is of *infinite type.*

Proposition 1.8.4. *A Lie algebra* $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ *is of infinite type if it contains a matrix of rank* 1 *as an element.*

We will said that a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ is *elliptic* if it contains no matrix of rank 1. Hence, if $\mathfrak{g}$ is of finite order, then it is elliptic.

Example 1.8.5. Take $G = GL_p(\mathbb{R}) \times GL_q(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_p(\mathbb{R}) + \mathfrak{gl}_q(\mathbb{R})$ and $p + q = n$. Explicitly:

$$\begin{aligned} GL_p(\mathbb{R}) \times GL_q(\mathbb{R}) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in GL_p(\mathbb{R}), B \in GL_q(\mathbb{R}) \right\} \\ \mathfrak{gl}_p(\mathbb{R}) + \mathfrak{gl}_q(\mathbb{R}) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{gl}_p(\mathbb{R}), B \in \mathfrak{gl}_q(\mathbb{R}) \right\}. \end{aligned} \tag{1.8.4}$$

It is easy to see that $\mathfrak{g}$ contains elements of rank 1 and, hence, is of infinite type. The $GL_p(\mathbb{R}) \times GL_q(\mathbb{R})$-structure is in a natural one-to-one correspondence with the set of pairs (S, S'), where S and S' are complementary distributions of dimension p and q respectively, hence it defines an almost product structure (see Remark 1.3.8).

Example 1.8.6. Setting in the previous Example 1.8.5 $p = q = n$ we get that the $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$-structure defines a **D**-complex structure and it is of infinite type.

It should be recalled that the finiteness for the automorphism group of complex Kähler structures is based on the ellipticity of the geometric structure. But we have seen (Examples 1.8.5 and 1.8.6) that a product structure is of infinite type and not elliptic, and also the **D**-complex structure is not elliptic, hence the usual technique of complex Kähler manifolds does not work on **D**-complex manifolds. Till now, it is unknown for us if there is a general answer to this problem, while it is known that there are some results on homogeneous **D**-manifolds and on some class of CR **D**-manifold, and for the sake of completeness we will remind here.

1.8.1 Automorphisms of homogeneous D-manifolds

In this Section, we will remind a known results about automorphisms of **D**-complex manifolds. It is due to N. Tanaka [73], but it is far to be a general results: indeed, as we will see, this result concerns homogeneous **D**-Kähler manifolds (Remark 1.8.11).

We begin with two definitions.

Definition 1.8.7. A *simple graded Lie algebra of the first kind* is a datum of a Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that:

1. $\mathfrak{g}$ is finite dimensional simple Lie algebra;
2. $\mathfrak{g}$ is a *graded Lie algebra*, i.e. $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ (direct sum) and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for $i, j = -1, 0, 1$.

We will use (differential) graded Lie algebra in Chapter 2 (see 2.1).

Definition 1.8.8. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$, $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{m} \neq 0$ be a subspace of $\mathfrak{g}$. The system $(\mathfrak{g}; \mathfrak{h}; \mathfrak{m})$ is called an *affine symmetric triple* if it satisfies:

1. $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (as a direct sum);
2. $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Moreover $(\mathfrak{g}; \mathfrak{h}; \mathfrak{m})$ will be called *simple* if $\mathfrak{g}$ is simple, and will also be called *of reducible type* if it is reducible the linear isotropy representation ρ of $\mathfrak{h}$ on $\mathfrak{m}$ defined by:

$$\begin{aligned} \rho &: \mathfrak{h} \longrightarrow \mathfrak{m} \\ \rho(X)Y &:= [X, Y] \qquad X \in \mathfrak{h},\ Y \in \mathfrak{m}. \end{aligned} \tag{1.8.5}$$

Given a simple graded Lie algebra of the first kind $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ we can construct an affine symmetric triple $(\mathfrak{g}; \mathfrak{h}; \mathfrak{m})$ by:

$$\mathfrak{h} = \mathfrak{g}_0 \qquad \mathfrak{m} = \mathfrak{g}_{-1} + \mathfrak{g}_1. \tag{1.8.6}$$

Also the converse is true:

Lemma 1.8.9 ([73, Lemma 2.7] or [13])**.** *Any symmetric triple of simple and reducible type is associated with a simple graded Lie algebra of the first kind. Furthermore, such an association is unique in a suitable sense.*

The affine symmetric triples are strictly connected with the product structures, as shown by S. Kaneyuki and M. Kozai [47] (see also [36]).

The symmetric triple gives rise to a "standard" non-compact affine symmetric space G/H which is endowed with a product structure. In fact, let G be a connected simply-connected Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ be a subgroup with Lie algebra $\mathfrak{g}_0$, then $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are naturally associated to (invariant) subbundles E_+ and E_- of the tangent bundle $T(G/H)$ of G/H.

Theorem 1.8.10 ([73, Theorem 2.8]). *Let $\mathfrak{g}$ be simple graded Lie algebra of the first kind and assume that $\mathfrak{g}$ is of the classical type. If $\mathfrak{g}$ is isomorphic with a definite Möebius (graded) Lie algebra, then there is an isomorphism of* $\mathrm{Aut}(G/H)$ *onto the diffeomorphism group of a sphere. Otherwise, there is an isomorphism of G onto* $\mathrm{Aut}(G/H)$.

Remark 1.8.11. It has to be noted that such a result is far from being optimal. In fact, it treats of homogeneous manifolds, and moreover, such a manifolds are naturally endowed with a pseudo-Riemannian metric compatible with the **D**-complex structure (see [73, Section 2.5]).

There is another result concerning the homogeneous **D**-Kähler manifold, due to S. Kaneyuki. A triple $(M = G/H, F^{\pm}, \omega)$ is a ***D**-Kähler symmetric space* if $M = G/H$ is an (homogeneous) **D**-Hermitian manifold with a symplectic form ω and admitting a pair of two H-invariant transversal Lagrangian foliations $F^{\pm}$ (we recall that by Theorem 1.5.4 this is equivalent to the **D**-Kählerness).

Theorem 1.8.12 ([46, Theorem 8.1 and Theorem 8.4]). *Let $(M = G/H, F^{\pm}, \omega)$ be a **D**-Kähler symmetric space. If G is of type BC_r or if G has rank $r \geq 2$, then the automorphism group* $\mathrm{Aut}(M, F^{\pm}, \omega)$ *is equal to G.*

Some other results on homogeneous **D**-Kähler manifolds can be found in [1]. We also note that this last result can be seen in a broader framework questions: namely, given an homogeneous manifold $M = G/H$, an interesting problem is to wonder when the automorphism group (or other groups acting on the manifold M) is isomorphic to the Lie group G (for more about this subject see [67]).

1.8.2 Automorphisms of CR D-manifolds

Now we turn our attention to the CR **D**-structures. Giving an (almost) CR **D**-complex manifold $(M, \mathcal{H}, K)$, we study when the automorphism group $\mathrm{Aut}(\mathcal{H}, K)$ is finite dimensional. To answer to this problem, we will use a construction made by N. Tanaka about "tower" of canonical principal bundles obtained by extending a given Lie algebra (see [74, 75], see also [4]), jointly with a theorem by S. Kobayashi (see [49]) which allows to bound the dimension of $\mathrm{Aut}(\mathcal{H}, K)$ with the dimension of the Tanaka prolongation.

We will proceed as follows: we first associate a Lie algebra to the distribution $\mathcal{H}$, then we prolong that Lie algebra using the N. Tanaka construction, then we see when such a prolongation is finite, and finally we use the Kobayashi's Theorem to deduce the dimension of $\mathrm{Aut}(\mathcal{H}, K)$. We will mainly refer to [3].

Let $(M, \mathcal{H}, K)$ be a CR **D**-manifold. It is possible to associate a Lie algebra $\mathfrak{m}(x)$ to any point $x \in M$ in the following way.

We consider the filtration of the Lie algebra $\mathfrak{X}(M)$ of vector fields defined inductively by:

$$\begin{aligned} &\Gamma(\mathcal{H})_{-1} = \Gamma(\mathcal{H}), \\ &\Gamma(\mathcal{H})_{-i} = \Gamma(\mathcal{H})_{-i+1} + [\Gamma(H), \Gamma(H)_{-i+1}], \qquad \text{for } -i < -1. \end{aligned} \tag{1.8.7}$$

Then at any point $x \in M$ we get:

$$\mathcal{H}_x = \mathcal{H}_{-1}(x) \subset \mathcal{H}_{-2}(x) \subset \ldots \subset \mathcal{H}_{-d+1}(x) \subset \mathcal{H}_{-d}(x) \subset \ldots \subset T_xM \tag{1.8.8}$$

where we have defined

$$\mathcal{H}_{-i}(x) := \{X_x \mid X \in \Gamma(\mathcal{H}_{-i})\}. \tag{1.8.9}$$

Set

$$\mathfrak{m}_{-i}(x) = \frac{\mathcal{H}_{-i}(x)}{\mathcal{H}_{-i+1}(x)}. \tag{1.8.10}$$

We recall the following definitions.

Definition 1.8.13. A Lie algebra $\mathfrak{g}$ has a *gradation of depth k* if $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \sum_{-k\le i} \mathfrak{g}_i = \mathfrak{g}_{-k} + \mathfrak{g}_{-k+1} + \dots + \mathfrak{g}_0 + \dots \tag{1.8.11}$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset [\mathfrak{g}_{i+j}]$ for $i, j \ge -k$ and $\mathfrak{g}_{-k} \ne 0$. Such a Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ is also called *graded Lie algebra (of depth k).*
A graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ is called:

1. *fundamental* if the negative part $\mathfrak{m} = \sum_{i<0} \mathfrak{g}_i$ is generated by $\mathfrak{g}_{-1}$;
2. *non degenerate* if $X \in \mathfrak{g}_{-1}$ and $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$ (equivalently, if $0 \ne X \in \mathfrak{g}_{-1}$, then exists $Y \in \mathfrak{g}_{-1}$ such that $[X, Y] \ne 0$);
3. *effective* (or *transitive*) if the non-negative part $\mathfrak{g}_0 + \mathfrak{g}_1 + \dots$ contains no non-trivial ideal of $\mathfrak{g}$.

Assuming $\mathcal{H}_{-d}(x) = T_xM$, we easily see that $\mathfrak{m}(x) = \sum_i \mathfrak{m}_{-i}(x)$ defines a fundamental negatively graded Lie algebra of depth d.

A distribution $\mathcal{H}$ is called a *regular distribution of depth d and type* $\mathfrak{m}$ if all the graded fundamental Lie algebras $\mathfrak{m}(x)$ are isomorphic to a given graded Lie algebra $\mathfrak{m} = \sum_i \mathfrak{m}_{-i}$.

Definition 1.8.14. A pair $(\mathfrak{m} = \sum_i \mathfrak{m}_{-i}, K_0)$, where $\mathfrak{m}$ is a fundamental negatively graded Lie algebra, and K_0 is an involutive endomorphism of $\mathfrak{m}_{-1}$ such that $K_0^2 = +\mathrm{Id}$ and the ± 1-eigenspaces $\mathfrak{m}_{-1}^{\pm}$ are commutative subalgebras of $\mathfrak{m}(x)$, is called an *integrable CR **D**-algebra.*

An (almost) CR **D**-structure $(\mathcal{H}, K)$ on a manifold M is *regular of type* $(\mathfrak{m}, K_0)$ *and depth d* if, for any $x \in M$, the pair $(\mathfrak{m}(x), K_x)$ is isomorphic to $(\mathfrak{m}, K_0)$, and will be called *non-degenerate* if the corresponding graded Lie algebra is non-degenerate.

Remark 1.8.15. A regular almost CR **D**-structure of type $(\mathfrak{m}, K_0)$ is integrable (in the sense of Definition 1.7.1) if and only if the Lie algebra $(\mathfrak{m}, K_0)$ is integrable (in the sense of Definition 1.8.14).

Now we turn our attention to the *full prolongation* $\mathfrak{g}(\mathfrak{m})$ of a negatively graded fundamental Lie algebra $\mathfrak{m}$. Such a prolongation, introduced by N. Tanaka (see e.g. [74, 75]), is the maximal graded Lie algebra

$$\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}_{-d}(\mathfrak{m}) + \dots + \mathfrak{g}_{-1}(\mathfrak{m}) + \mathfrak{g}_0(\mathfrak{m}) + \mathfrak{g}_1(\mathfrak{m}) + \dots \tag{1.8.12}$$

whose negative part is $\mathfrak{g}_{-d}(\mathfrak{m}) + \dots + \mathfrak{g}_{-1}(\mathfrak{m}) = \mathfrak{m}$ and such that the following condition holds:

$$X \in \mathfrak{g}_k(\mathfrak{m}),\ k \ge 0 \text{ and } [X, \mathfrak{g}_{-1}(\mathfrak{m})] = 0 \quad \text{implies } X = 0. \tag{1.8.13}$$

This full prolongation can be defined inductively, and it was proved by N. Tanaka [74] that always exists and that it is unique up to automorphisms and it is related with a tower of canonical principal bundles.

Indeed, since the Tanaka prolongation is defined inductively, each positive element is constructed starting from $\mathfrak{g}_0(\mathfrak{m})$. However, it is possible to choose a subalgebra $\mathfrak{g}_0'$ of $\mathfrak{g}_0(\mathfrak{m})$

and repeat the inductively construction. This is useful, since such a construction gives a tower of canonical principal bundles even if it begins with $\mathfrak{g}_0'$ instead of $\mathfrak{g}_0(\mathfrak{m})$. Tanaka has also shown that when this prolongation is finite, then the tower ends with an absolute parallelism. Given a negatively graded fundamental Lie algebra $\mathfrak{m}$ and a subalgebra $\mathfrak{g}_0'$ of $\mathfrak{g}_0(\mathfrak{m})$, we will denote the prolongation of $\mathfrak{m}+\mathfrak{g}_0'$ constructed inductively from $\mathfrak{g}_0'$ by $(\mathfrak{m}+\mathfrak{g}_0')^\infty := \mathfrak{m}+\mathfrak{g}_0'+\mathfrak{g}_1'+\dots$.

Definition 1.8.16. A graded Lie algebra $\mathfrak{m}+\mathfrak{g}_0'$ is called of *finite type* if its prolongation $(\mathfrak{m}+\mathfrak{g}_0')^\infty$ is a finite dimensional Lie algebra, and is called of *semisimple type* if $(\mathfrak{m}+\mathfrak{g}_0')^\infty$ is a finite dimensional semisimple Lie algebra.

Provided that such a prolongation is finite, Tanaka has proved that in this tower the automorphisms of the "lower" levels can be pull back to automorphisms of "upper" levels, and that at the last level we obtain an $\{e\}$-structure. Moreover, by a result of Kobayashi, the dimension of automorphism group is less than or equal to the dimension of the maximal prolongation. We recall the Kobayashi Theorem.

Theorem 1.8.17 ([49, Theorem 3.2])**.** *Let M be a n-dimensional manifold with an $\{e\}$-structure (i.e. an absolute parallelism). Let $\mathfrak{U}$ be the group of automorphism of the $\{e\}$-structure. Then $\mathfrak{U}$ is a Lie transformation group such that* $\dim\mathfrak{U}\leq\dim M$.

For our purposes this theorem can be restate as:

Theorem 1.8.18 (see [4, Theorem 5.5])**.** *Let M be a manifold and let $\pi : P\to M$ be a G_0'-structure with $\mathfrak{g}_0'=\mathrm{Lie}(G_0')$ of finite k and $G_0'\subset \mathrm{GL}_n(\mathbb{R})$. Then, the automorphism group $\mathrm{Aut}(\pi)$ is a Lie group of dimension less then or equal to*

$$\dim(\mathfrak{m}+\mathfrak{g}_0')^\infty = \dim(\mathbb{R}^n+\mathfrak{g}_0'+\dots+\mathfrak{g}_{k-1}'). \tag{1.8.14}$$

From this last theorem we see that, if we want to study the automorphism group of CR **D**-manifolds, it is important that the first step of the prolongation, namely $\mathfrak{g}_0'$, is "compatible" with the **D**-structure K. Hence to construct our prolongation, we will look for $\mathfrak{g}_0'$ with this property.

Let $(\mathcal{H},K)$ be a regular CR **D**-structure of type $(\mathfrak{m},K_0)$. We define the following Lie algebra $\mathfrak{g}_0(\mathfrak{m},K_0)$ to be the subalgebra of $\mathfrak{g}_0(\mathfrak{m})$ consisting of any $A\in\mathfrak{g}_0(\mathfrak{m})$ such that $A|_{\mathfrak{m}_{-1}}$ commutes with K_0, more explicitly:

$$\begin{aligned}\mathfrak{g}_0(\mathfrak{m},K_0) :&= \mathrm{Der}(\mathfrak{m},K_0)\\ &=\{A\in\mathrm{Der}(\mathfrak{m}) \mid A(\mathfrak{m}_j)\subset\mathfrak{m}_j\ \forall j<0 \text{ and } A|_{\mathfrak{m}_{-1}}\circ K_0 = K_0\circ A|_{\mathfrak{m}_{-1}}\}.\end{aligned} \tag{1.8.15}$$

We see that $\mathfrak{g}_0(\mathfrak{m},K_0)$ is the Lie algebra of the Lie group $\mathrm{Aut}(\mathfrak{m},K_0)$, and we define the prolongation of $\mathfrak{g}(\mathfrak{m},K_0):=\mathfrak{m}+\mathfrak{g}_0(\mathfrak{m},K_0)$ as the Lie algebra:

$$(\mathfrak{g}(\mathfrak{m},K_0))^\infty = \mathfrak{m}_{-d}+\dots+\mathfrak{m}_{-1}+\mathfrak{g}_0(\mathfrak{m},K_0)+\mathfrak{g}_1+\dots \tag{1.8.16}$$

where inductively $\mathfrak{g}_i=\{X\in\mathfrak{g}(\mathfrak{m}) \mid [X,\mathfrak{m}_{-1}]\subset\mathfrak{g}_{i-1}\}$ for any $i\geq 1$. By construction, $(\mathfrak{g}(\mathfrak{m},K_0))^\infty$ is a subalgebra of the full prolongation $\mathfrak{g}(\mathfrak{m})$.

Now the problem turns to wonder when such a prolongation is finite, and the following lemma gives us a condition to understand the finiteness of the prolongation $(\mathfrak{g}(\mathfrak{m},K_0))^\infty$.

Lemma 1.8.19 ([3, Lemma 3.2])**.** *Let $(\mathfrak{m}=\sum_{i<0}\mathfrak{m}_i,K_0)$ be an integrable CR **D**-algebra and let $\mathfrak{g}_0(\mathfrak{m},K_0)$ to be as in* (1.8.15)*. Then the graded Lie algebra $\mathfrak{g}(\mathfrak{m},K_0)$ is of finite type if and only if $\mathfrak{m}$ is non-degenerate.*

Finally, we conclude with the following proposition, which is a consequence of Theorem 1.8.18 and of Lemma 1.8.19.

Proposition 1.8.20 (see [3, Section 4.3]). *Let $(M, \mathcal{H}, K)$ be a regular non-degenerate CR **D**-manifold of type $(\mathfrak{m}, K_0)$ and depth d. Then the automorphism group $\mathrm{Aut}(M, \mathcal{H}, K)$ has finite dimension.*

Now let K be a strictly CR **D**-structure on a contact manifold (M, ξ), with contact form α. Since $d\alpha$ is non-degenerate on ξ, we get that for any fixed $X \in \xi$, there exists a $Y \in \xi$ such that $0 \neq d\alpha(X, Y) = -\alpha([X, Y])$, i.e. $[X, Y] \in \mathbb{R}R_\alpha$ (here R_α is the Reeb vector field, see the previous Section 1.7). The corresponding filtration (1.8.8) is of depth 2:

$$\begin{aligned} &\mathcal{H}_{-1} = \xi, \\ &\mathcal{H}_{-2} = [\xi, \xi] = \mathbb{R}R_\alpha + \xi = TM \end{aligned} \tag{1.8.17}$$

which is the same in any point $x \in M$, and

$$\mathfrak{m}_{-1} = \xi \qquad \mathfrak{m}_{-2} = TM/\xi = \mathbb{R}R_\alpha. \tag{1.8.18}$$

Then a strictly CR **D**-structure on a contact manifold (M, ξ) is fundamental, regular, non-degenerate and of depth 2. We have proved the following:

Corollary 1.8.21. *Let let K be a strictly CR **D**-structure on a contact manifold (M, ξ). Then the automorphism group $\mathrm{Aut}(M, \mathcal{H}, K)$ has finite dimension.*

Chapter 2

Deformations of D-structures

In this chapter we are interested in the study of small deformations of **D**-structures on a compact **D**-manifold. In particular, we focus on the algebraic aspects of the theory.

We start this chapter by recalling the definition of *Differential Graded Lie Algebra* (shortly *DGLA*) $(C,\ [\cdot,\cdot],\ d)$ and some preliminary results and facts on this topic (see Section 2.1).

In Section 2.2 we review the classical theory of deformations of complex structures developed by K. Kodaira and D.C. Spencer (see [51] [52]), which involves the local holomorphic coordinates, into the DGLA setting. It has to be noted that the study of deformations of complex structure J (resp. **D**-complex structures K) using DGLA does not involve the local holomorphic (resp. **D**-holomorphic) coordinates, hence it describes intrinsically the deformations of the complex structure J (resp. of the **D**-complex structure K).

A *Differential Graded Lie Algebra* $(\mathcal{A}, [[\cdot,\cdot]], \overline{\partial}_K)$ (shortly *DGLA*) is introduced by C. Medori and A. Tomassini in [61] to characterize small deformations of a **D**-structure K on a compact manifold M. Such deformations are parametrized by 1-degree element of $\mathcal{A}$. In Section 2.3 we remind their construction, in particular we define the bracket $[[\cdot,\cdot]]$ and the operator $\overline{\partial}_K$, which will be useful in the following sections, and we describe the theory of curves of **D**-complex structures on a fixed manifold.

Section 2.4 is devoted to prove that the integrability condition of a small deformation can be viewed as a Maurer-Cartan equation in the space $\mathcal{F}$ of skew-symmetric derivations on $\wedge^{0,*}_K(M)$ (see Corollary 2.4.5). To do this, we first construct a new differential graded Lie algebra $\widehat{\mathcal{A}}$ and then we prove that there is a DGLA injective homomorphism $q : \widehat{\mathcal{A}} \to \mathcal{F}$ (see Theorem 2.4.2). The new DGLA $\widehat{\mathcal{A}}$ is constructed such that $\widehat{\mathcal{A}}_i = \mathcal{A}_i$ for $i \geq 1$, so the 1-degree elements that parametrize the deformations can be read as elements in the new DGLA $\widehat{\mathcal{A}}$. Moreover, we are able to show that the condition of integrability found in [61, Theorem 4.2] (see Theorem 2.3.9 below) can be tested both on the real setting or in the **D**-complexificated setting (see Remark 2.4.8).

In the second part of the chapter we study the analogous problem for *CR **D**-structures.* We focus on **D**-structures on *contact manifolds*, in particular on *strictly CR **D**-structures* (see Section 1.7). Recently C.D. Hill and P. Nurowsky in [42] gave an application of these structures in a context of ODE's and PDE's systems.

In 2.5 we investigate deformations of CR **D**-structures on contact manifolds (M, ξ) as done in the complex CR case by P. de Bartolomeis and F. Meylan (see [22]). We construct the DGLA $\mathcal{B}_K(\xi)$ of such deformations and we prove that the integrability condition is related to a Maurer-Cartan equation (see Theorem 2.5.11). Furthermore, we restate the integrability condition in the DGLA of skew-symmetric derivations of $\wedge^{0,*}_K(\xi)$ (see Theorem 2.5.9 and Remark 2.5.13).

Finally in 2.6, we construct some examples of families of CR **D**-structures on the generalized Heisenberg group and on another compact quotient of nilpotent Lie group studying their deformations, proving that there exists a 5-dimensional nilpotent Lie algebra which does not admit a CR **D**-structure (see Proposition 2.6.4).

The main results of this chapter have been published by the author in the paper [69].

2.1 Preliminaries on DGLA (differential graded Lie algebras)

In order to develop the deformation theory of **D**-manifold analogue to the theory of K. Kodaira and D.C. Spencer [51] for complex manifolds, we need to recall some algebraic facts.

Following K. Fukaya [34] we recall some notions on DGLA. By results in classical theory, these structures are related to the deformation theory (see [51], [52], [22] and [61]).

Let R be a commutative ring with unit.

Definition 2.1.1. A *differential graded Lie algebra* $(C,\ [\cdot,\cdot],\ d)$ (*DGLA* for short) is the datum of:

1 a graded R-module $C = \bigoplus_{p\in\mathbb{Z}} C_p$;

2 a bilinear map $[\cdot,\cdot] : C\times C \to C$ such that:

- **(a)** $[C_r, C_s] \subseteq C_{r+s}$,
- **(b)** for homogeneous elements a, b, c we have:
  - $[a,b] = -(-1)^{|a||b|}[b,a]$ (where $|a|$ denotes the degree of a, e.g. if $a \in C_s$ then $|a| = s$),
  - the graded Jacobi identity:

$$[[a,b],c] + (-1)^{(|a|+|b|)|c|}[[c,a],b] + (-1)^{(|b|+|c|)|a|}[[b,c],a] = 0; \tag{2.1.1}$$

3 an operator $d : C \to C$ of 1-degree such that:

- **(a)** $d\circ d = 0$,
- **(b)** for homogeneous elements $a, b \in C$ we have

$$d[a,b] = [d\,a, b] + (-1)^{|a|}[a, d\,b]. \tag{2.1.2}$$

Let $(C,\ [\cdot,\cdot],\ d)$ be a DGLA. For $\gamma \in C_1$ and $a \in C$ we set

$$d_\gamma\, a := d\,a + [\gamma, a], \tag{2.1.3}$$

and we have that:

$$\begin{aligned} d_\gamma[a,b] =&\ d[a,b] + [\gamma,[a,b]] \\ &= [d\,a,b] + (-1)^{|a|}[a,d\,b] \\ &\quad + (-1)^{(|a|+|b|)|\gamma|}\left((-1)^{(|a|+|b|)|\gamma|}[[\gamma,a],b] + (-1)^{(|b|+|\gamma|)|a|}[[b,\gamma],a]\right) \\ &= [d_\gamma\, a, b] + (-1)^{|a|}\left([a,d\,b] + [a,[\gamma,b]]\right) \\ &= [d_\gamma\, a, b] + (-1)^{|a|}[a, d_\gamma\, b]. \end{aligned}$$

We recall the *Maurer-Cartan equation*:

$$d\,\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \tag{2.1.4}$$

and we set

$$\mathfrak{MC}(C) := \{\gamma \in C_1 \mid \gamma \text{ satisfies } (2.1.4)\}. \tag{2.1.5}$$

In general d_γ fails to be a derivation, since $d_\gamma^2 \neq 0$, but it is known that (see [34]):

Proposition 2.1.2. *Let $(C,\ [\cdot,\cdot],\ d)$ be a DGLA. If $\gamma \in \mathfrak{MC}(C)$, then $d_\gamma\ \cdot := d\cdot + [\gamma, \cdot]$ is a derivation of 1-degree in the DGLA $(C,\ [\cdot,\cdot],\ d)$ (i.e. $d_\gamma^2 = 0$).*

Proof. In fact, we have:

$$\begin{aligned} d_\gamma^2\, a &= d_\gamma(d\, a + [\gamma, a]) \\ &= d^2\, a + [\gamma, d\, a] + d[\gamma, a] + [\gamma, [\gamma, a]] \\ &= [\gamma, d\, a] + [d\,\gamma, a] - [\gamma, d\, a] + \frac{1}{2}[[\gamma, \gamma], a] \\ &= [d\,\gamma + \frac{1}{2}[\gamma, \gamma], a] = 0. \end{aligned}$$

□

2.2 Review of deformations of complex structures

Here we briefly recall the Kodaira theory of deformations of complex structures. For further results on complex deformations, look at [51, 52] or [54], while we refer mainly to [63, 44] and [50].

Let B be a complex (respectively, differentiable) manifold. A family $\{M_t\}_{t\in B}$ of compact complex manifolds is said to be a *complex-analytic* (respectively, *differentiable*) *family of compact complex manifolds* if there exists a complex (respectively, differentiable) manifold $\mathfrak{X}$ and a surjective holomorphic (respectively, smooth) map $\pi \colon \mathfrak{X} \to B$ such that:

1. $\pi^{-1}(t) = M_t$ for any $t \in B$, and
2. π is a proper holomorphic (respectively, smooth) submersion.

A compact complex manifold M is said to be a *deformation* of a compact complex manifold N if there exists a complex-analytic family $\{M_t\}_{t\in B}$ of compact complex manifolds, and $b_0,\, b_1 \in B$ such that $M_{b_0} = M$ and $M_{b_1} = N$.

A complex-analytic family $\pi : \mathfrak{X} \to B$ of compact complex manifolds is said to be *trivial* if $\mathfrak{X}$ is bi-holomorphic to $\pi_B : B \times M_b \to B$ for some $b \in B$ (where $\pi_B : B \times M_b \to B$ denotes the natural projection onto B); it is said to be *locally trivial* if, for any $b \in B$, there exists an open neighbourhood U of b in B such that $\pi : \mathfrak{X}|_{\pi^{-1}(U)} \to U$ is trivial. The following theorem by C. Ehresmann [28] states the local triviality of a differentiable family of compact complex manifolds (see, e.g., [50, Theorem 2.3, Theorem 2.5], [63, Theorem 1.4.1]).

Theorem 2.2.1 (see [28])**.** *Let $\{M_t\}_{t\in B}$ be a differentiable family of compact complex manifolds. For any $s,\, t \in B$, the manifolds M_s and M_t are diffeomorphic.*

As a consequence of Ehresmann's theorem above, a complex-analytic family $\{M_t\}_{t\in B}$ of compact complex manifolds with B contractible can be viewed as a family of complex structures on a compact differentiable manifold.

Given a compact manifold M let J be a complex structure on M, let J' be an (almost) complex structure on M. We want to compare these structures. It is known that J induces a decomposition $TM^{\mathbb{C}} = T^{1,0}_J M \oplus T^{0,1}_J M$, and similarly J' induces $TM^{\mathbb{C}} = T^{1,0}_{J'} M \oplus T^{0,1}_{J'} M$. Suppose that J' is "sufficiently close" to J, then the projection $\pi_{T^{0,1}_J M}$ gives an isomorphism between $T^{0,1}_{J'} M$ and $T^{0,1}_J M$. Then we get a map:

$$T^{0,1}_J M \xrightarrow{(\pi_{T^{0,1}_J M})^{-1}} T^{0,1}_{J'} M \xrightarrow{(\pi_{T^{1,0}_J M})^{-1}} T^{1,0}_J M. \tag{2.2.1}$$

This map can be view as a $(0,1)$-form with values in the bundle $T^{1,0}_J M$, i.e. as an element $\xi \in \wedge^{0,1} M \otimes T^{0,1} M$. Conversely, this element determines the subspace $T^{1,0}_{J'} M$ and thus J'. In local coordinates, if $\{z^1, \dots, z^n\}$ are a local holomorphic coordinates on M for the complex structure J, then we can write $\xi \in \wedge^{0,1} M \otimes T^{0,1} M$ as:

$$\xi = \sum_{i,j} h_i^j(z)\, d\,\bar{z}^i \otimes \frac{\partial}{\partial\, z^j} \tag{2.2.2}$$

and the corresponding space TM is spanned by the images of $\dfrac{\partial}{\partial\, \bar{z}^i}$, in other words, by the n vector fields:

$$\frac{\partial}{\partial\, \bar{z}^i} + \sum_j h_i^j(z)\, d\,\bar{z}^i \otimes \frac{\partial}{\partial\, z^j} \qquad \text{for } i = 1, \dots, n. \tag{2.2.3}$$

From now on for this entire section, let M be a fixed compact n-dimensional manifold with a complex structure J, and let B be a small ball centered at the origin of $\mathbb{C}^m$. We will see B as the space of parameters, or the *basis* for the deformation. Let $\pi : \mathfrak{X} \to B$ be a *family of deformations* of M over B such that $M = \pi^{-1}(0)$. We will denote by $\mathcal{J}$ the complex structure of $\mathfrak{X}$. By Ehresmann Theorem, we see that all the fibers of the map π are compact complex manifolds, and they are all diffeomorphic to M. One can think the fibers $\pi^{-1}(z)$ as a small deformations of the *central fiber* M.

Since we have $\mathfrak{X} \simeq M \times B$ as differentiable manifold, we have that all vertical and horizontal slices are complex submanifolds: each horizontal slice $\{z\} \times B$ carries the same complex structure of B, while the complex structure J_t on the vertical slice $M \times \{t\}$ varies with $t \in B$ and, in general, agrees with that of M only for $t = 0$.

Now, we describe in local coordinates. Let $\{z^1, \dots, z^n\}$ be a local holomorphic coordinates on M and let $\{t^1, \dots, t^m\}$ be the holomorphic coordinates on the disk B. Of course, these coordinates are not holomorphic for the complex structure $\mathcal{J}$, but together with their conjugates $\{\bar{z}^1, \dots, \bar{z}^n, \bar{t}^1, \dots, \bar{t}^m\}$ we have a smooth coordinates system on $M \times B$. We may write a generic element $\xi(t) \in \wedge^{0,1} M \otimes T^{0,1} M$ describing the changing of holomorphic coordinates as:

$$\xi(t) = \sum_{i,j} h_i^j(z,t)\, d\,\bar{z}^i \otimes \frac{\partial}{\partial\, z^j}. \tag{2.2.4}$$

We see that $h_i^u(z, 0) = 0$ and at any point $(z, t) \in M \times B$ we have:

$$\wedge^{0,1} \mathfrak{X} = \operatorname{span} \left. \left\{ \frac{\partial}{\partial\, \bar{z}^i} + \sum_j h_i^j(z,t) \frac{\partial}{\partial\, z^j},\ \frac{\partial}{\partial\, \bar{t}^k} \right\} \right|_{1 \le i \le n,\ 1 \le k \le m}. \tag{2.2.5}$$

We turn our attention to the differential manifold $M \times B$, leaving the complex manifold $(\mathfrak{X}, \mathcal{J})$. In general a section $\xi(t) \in \wedge^{0,1} M \otimes T^{0,1} M$ defines an *almost complex structure* on $M \times B$ and it might not come from a complex structure. We are interested in when the deformation corresponding to $\xi(t)$ is integrable, i.e. we are looking for family of complex manifold. In order to do this, it is useful to introduce the following bracket on $\wedge^{0,1} M \otimes T^{1,0} M$. Given two sections $\psi, \varphi \in \wedge^{0,1} M \otimes T^{1,0} M$ such that

$$\psi = \sum_{i,j} a_i^j\, d\,\bar{z}^i \otimes \frac{\partial}{\partial\, z^j} = \alpha \otimes X \qquad \varphi = \sum_{\ell,k} b_k^j\, d\,\bar{z}^\ell \otimes \frac{\partial}{\partial\, z^k} = \beta \otimes Y \tag{2.2.6}$$

then we define the bracket $[\cdot,\cdot] : \wedge^{0,1} M \otimes T^{1,0} M \to \wedge^{0,2} M \otimes T^{1,0} M$ as:

$$[\psi, \varphi] := \sum_{i,j,k,\ell} d\,\bar{z}^i \wedge d\,\bar{z}^\ell \otimes \left[a_i^j \frac{\partial}{\partial\, z^j}, b_\ell^k \frac{\partial}{\partial\, z^k} \right] \in \wedge^{0,2} M \otimes T^{1,0} M. \tag{2.2.7}$$

We also define the del-bar operator $\overline{\partial} : \wedge^{0,1}M \otimes T^{1,0}M \to \wedge^{0,2}M \otimes T^{1,0}M$ by:

$$\overline{\partial}\,\psi = \sum_{i,j,k} \frac{\partial\, a_i^j}{\partial\, \bar{z}^k}\, d\,\bar{z}^k \wedge d\,\bar{z}^i \otimes \frac{\partial}{\partial\, z^j} \in \wedge^{0,2}M \otimes T^{1,0}M. \tag{2.2.8}$$

These definitions are independent of the choice of local holomorphic coordinates, and can be extended in a similar way to any bundles $\wedge^{0,p}M \otimes T^{1,0}M$ for $p > 1$.

Then it holds:

Theorem 2.2.2. *Let M be a compact manifold endowed with an integrable almost complex structure J. A smooth section $\xi(t) \in \wedge^{0,1}M \otimes T^{1,0}M$ defines an almost complex structure J_t on $M \times B$. The almost complex structure J_t is integrable if and only if $\xi(t)$ satisfies the Maurer-Cartan equation:*

$$\overline{\partial}\,\xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0 \qquad \forall\ t \in B. \tag{2.2.9}$$

It is easy to see that, with the definition of (2.2.7) and (2.2.8) suitably extended, $(\wedge^{0,p} \otimes T^{1,0}M,\ \overline{\partial},\ [\cdot,\cdot])$ defines a DGLA and that the condition (2.2.9) is the classical Maurer-Cartan obstruction. In fact, one has:

- The bracket is defined as:

$$\begin{aligned} &[\cdot,\cdot] : (\wedge^{0,p}M \otimes T^{1,0}M) \times (\wedge^{0,q}M \otimes T^{1,0}M) \longrightarrow \wedge^{0,p+q}M \otimes T^{1,0}M \\ &\quad \big[\bar{\alpha} \otimes X,\, \bar{\beta} \otimes Y\big] \ :=\ \big(\bar{\beta} \wedge \mathcal{L}_Y \bar{\alpha}\big) \otimes X + \big(\bar{\alpha} \wedge \mathcal{L}_X \bar{\beta}\big) \otimes Y + \big(\bar{\alpha} \wedge \bar{\beta}\big) \otimes [X,\, Y] \end{aligned} \tag{2.2.10}$$

  where $\mathcal{L}_W\varphi := \iota_W\, d\,\varphi + d\,(\iota_W \varphi)$ is the Lie derivative of φ along W;

- The del-bar operator is defined as:

$$\begin{aligned} &\overline{\partial} : \wedge^{0,1}M \otimes T^{1,0}M \longrightarrow \wedge^{0,2}M \otimes T^{1,0}M \\ &\quad \overline{\partial}\,\varphi\,(\bar{Z},\, \bar{W}) \ :=\ \big[\bar{Z},\, \varphi\,(\bar{W})\big]^{1,0} - \big[\bar{W},\, \varphi\,(\bar{Z})\big]^{1,0} - \varphi\,\big([\bar{Z},\, \bar{W}]\big)\ , \end{aligned} \tag{2.2.11}$$

  where $X^{1,0} := X - i\, J\, X$ is the $(1,0)$-component of X with respect to the complex structure J.

2.3 Deformations of D-manifolds

In this section, we first introduce some key tools to understand the space of deformations of **D**-structures, then we recall some results on the DGLA $(\mathcal{A}, [[\cdot,\cdot]], \overline{\partial}_K)$ governing such deformations. Finally we describe curves of **D**-structures. The main reference for this section is [61].

Given an almost **D**-manifold (M, K), we set:

$$\begin{aligned} (\wedge_K^{0,p}T^*M)^{\mathbb{R}} \otimes TM := \{&\varphi \in \Gamma(M, \wedge^p(M) \otimes TM) \mid \\ &\varphi(X_1, \ldots, KX_j, \ldots, X_p) = -K\varphi(X_1, \ldots, X_p)\ \forall j = 1, \ldots, p\}. \end{aligned} \tag{2.3.1}$$

Remark 2.3.1. We observe that

- $(\wedge_K^{0,0}T^*M)^{\mathbb{R}} \otimes TM$ is the set of smooth vector fields.
- $(\wedge_K^{0,1}T^*M)^{\mathbb{R}} \otimes TM = \{\varphi \in \mathrm{End}(TM) \mid \varphi K + K\varphi = 0\}$.

- with the notation of Definition 4.3.1 $(\wedge_K^{0,2}T^*M)^{\mathbb{R}} \otimes TM$ is a subspace of Ω^{III} (see Section 4.3).

For $X, Y \in \Gamma(M)$, we set

$$[[X,Y]] := \frac{1}{2}\left([X,Y] + [KX,KY] - \frac{1}{2}N_K(X,Y)\right). \tag{2.3.2}$$

Observe that if $\theta \in (\wedge_K^{0,p}T^*M)^{\mathbb{R}} \otimes TM$, then the map

$$(X_0,\ldots,X_p) \mapsto \sum_{0\le j\le k\le p} (-1)^{j+k}\theta([[X_j,X_k]],X_0,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_p)$$

defines an element of $(\wedge_K^{0,p+1}T^*M)^{\mathbb{R}} \otimes TM$.

Definition 2.3.2. Let

$$\overline{\partial}_K : (\wedge_K^{0,p}T^*M)^{\mathbb{R}} \otimes TM \longrightarrow (\wedge_K^{0,p+1}T^*M)^{\mathbb{R}} \otimes TM$$

be the operator defined as follows:

1. For $X \in \Gamma(M)$ set
$$(\overline{\partial}_K\, X)Y := \frac{1}{2}\left([Y,X] - K[KY,X] + \frac{1}{2}N_K(X,Y)\right).$$

2. For $\theta \in (\wedge_K^{0,p}T^*M)^{\mathbb{R}} \otimes TM$ set
$$\overline{\partial}_K\theta(X_0,\ldots,X_p) := \sum_{j=0}^{p}(-1)^j\left(\overline{\partial}_K\,\theta(X_0,\ldots,\hat{X}_j,\ldots,X_p)\right)(X_j) +$$
$$\sum_{0\le j\le k\le p}(-1)^{j+k}\theta([[X_j,X_k]],X_0,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_p).$$

Following [61], we construct the DGLA of deformations of **D**-structures over a fixed compact **D**-manifold (M,K).

Let $\Omega_K^{0,p}(M)$ be the space of the sections of the bundle of $(0,p)$-forms on (M,K). Denote by $\Gamma(M,\wedge_K^{0,p}(M)\otimes T^{1,0}M)$ the space of sections of the vector bundle $\wedge_K^{0,p}(M)\otimes T^{1,0}(M)$. Set

$$\mathcal{A}_p := \begin{cases} \Gamma(M,\wedge_K^{0,p}(M)\otimes T^{1,0}(M)) & 0\le p\le n,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathcal{A} := \bigoplus_{p\in\mathbb{Z}} \mathcal{A}_p.$$

Note that the real vector space $\mathcal{A}_p$ is a module over **D**.

In the sequel, we shall consider the following isomorphism m:

$$\begin{aligned} m : TM &\longrightarrow T^{1,0}M\\ X &\longmapsto \frac{1}{2}(X+\tau KX)\end{aligned} \tag{2.3.3}$$

and the corresponding isomorphism, also denoted by m:

$$\begin{aligned} m : (\wedge_K^{0,p}T^*M)^{\mathbb{R}}\otimes TM &\longrightarrow \mathcal{A}_p\\ \varphi &\longmapsto \frac{1}{2}(\varphi+\tau K\varphi).\end{aligned} \tag{2.3.4}$$

Note that $m^{-1}(\psi) = (\psi + \bar{\psi})$.

Remark 2.3.3. For Z, W sections of $T^{1,0}M$, we can define a bracket on $T^{1,0}M$ using the isomorphism m (see (2.3.3)):

$$[[Z, W]] := m[[m^{-1}(Z), m^{-1}(W)]] = [Z, W]$$

and identifying $\overline{\partial}_k$ with $m \circ \overline{\partial}_k \circ m^{-1}$ we can also define an operator $\overline{\partial}_K$ on $T^{1,0}M$:

$$(\overline{\partial}_K W)(\bar{Z}) := \frac{1}{2}([\bar{Z}, W] + \tau K[\bar{Z}, W]).$$

Moreover for $\varphi \in (\wedge_K^{0,1} T^*M)^{\mathbb{R}} \otimes TM$ we obtain that:

$$\overline{\partial}_K \varphi(X, Y) = (\overline{\partial}_K(\varphi Y))(X) - (\overline{\partial}_K(\varphi X))(Y) - \varphi([[X, Y]]).$$

We recall here the definitions of the bracket $[[\cdot,\cdot]]$ and of the operator $\overline{\partial}_K$ which make $(\mathcal{A},\ [[\cdot,\cdot]],\ \overline{\partial}_K)$ a DGLA (see [61]).

Definition 2.3.4. The bracket $[[\cdot,\cdot]] : \mathcal{A}_p \times \mathcal{A}_q \longrightarrow \mathcal{A}_{p+q}$ is defined in the following way:

1. For every $Z, W \in \mathcal{A}_0$ set:
$$[[Z, W]] = [Z, W]$$
where $[\cdot,\cdot]$ is the usual bracket on vector fields.

2. For every $Z \in \mathcal{A}_0$ and $\varphi \in \mathcal{A}_1$, then $[[\varphi, Z]] = -[[Z, \varphi]] \in \mathcal{A}_1$ is defined by:
$$[[\varphi, Z]](\bar{W}) = [\varphi\bar{W}, Z] + \frac{1}{2}\varphi([Z, \bar{W}] - \tau K[Z, \bar{W}]).$$

3. For every $\varphi \in \mathcal{A}_1$, then $[[\varphi, \varphi]] \in \mathcal{A}_2$ is defined by:
$$[[\varphi, \varphi]](\bar{Z}, \bar{W}) = 2[\varphi\bar{Z}, \varphi\bar{W}] - 2\varphi([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}]). \tag{2.3.5}$$

4. For $\varphi, \psi \in \mathcal{A}_1$, then $[[\varphi, \psi]] \in \mathcal{A}_2$ is defined by:
$$[[\varphi, \psi]] = \frac{1}{2}\left([[\varphi + \psi, \varphi + \psi]] - [[\varphi, \varphi]] - [[\psi, \psi]]\right).$$

5. For $\alpha \in \Omega_K^{0,p}(M)$ and $\beta \in \Omega_K^{0,q}(M)$, $\varphi, \psi \in \mathcal{A}_1$, set:
$$[[\alpha \wedge \varphi, \beta \wedge \psi]] = (-1)^q \alpha \wedge \beta \wedge [[\varphi, \psi]] + (-1)^q (\urcorner_\psi \alpha) \wedge \beta \wedge \varphi + \alpha \wedge (\urcorner_\varphi \beta) \wedge \psi,$$
where $\urcorner_\varphi$ is a skew-symmetric derivation of 1-degree of $\Omega_K^{0,*}(M)$ defined as follows: for smooth function f
$$(\urcorner_\varphi f)(\bar{Z}) = \partial f(\varphi(\bar{Z})) = \varphi(\bar{Z})f = (\varphi\bar{Z}f)$$
and for $\alpha \in \Omega_K^{0,1}(M)$
$$(\urcorner_\varphi \alpha)(\bar{Z}, \bar{W}) = \varphi\bar{Z}\alpha(\bar{W}) - \varphi\bar{W}\alpha(\bar{Z}) - \alpha([\varphi\bar{Z}, \bar{W}] - [\bar{Z}, \varphi\bar{W}]).$$

6. In the general case, we extend $[[\cdot,\cdot]]$ by bilinearity to any pair of elements of $\mathcal{A}$.

Definition 2.3.5. Define the $\overline{\partial}_K$ operator $\overline{\partial}_K : \mathcal{A}_p \to \mathcal{A}_{p+1}$ as follows:

1. For $Z \in \mathcal{A}_0$ and $W \in T^{0,1}M$, set
$$(\overline{\partial}_K Z)(\bar{W}) = \frac{1}{2}([\bar{W}, Z] + \tau K[\bar{W}, Z]).$$

2. For $\varphi \in \mathcal{A}_1$ and $Z, W \in T^{1,0}M$ set:

$$(\overline{\partial}_K \varphi)(\bar{Z}, \bar{W}) = (\overline{\partial}_K \varphi(\bar{W}))\bar{Z} - (\overline{\partial}_K \varphi(\bar{Z}))\bar{W} - \varphi([\bar{Z}, \bar{W}]).$$

3. Extend in the general case $\overline{\partial}_K$ by Leibniz rule, i.e.

$$(\overline{\partial}_K \alpha \wedge \varphi) = (\overline{\partial}_K \alpha) \wedge \varphi + (-1)^{|\alpha|}(\alpha \wedge (\overline{\partial}_K \varphi)).$$

Remark 2.3.6. Since the **D**-structure is integrable and using the isomorphism m between $(\wedge_K^{0,p} T^*M)^{\mathbb{R}} \otimes TM$ and $\mathcal{A}_p$ (see (2.3.4)), it follows that the bracket $[[\cdot, \cdot]]$ and the differential $\overline{\partial}_k$ in Definition 2.3.4 and in Definition 2.3.5 are related with the ones in the equation (2.3.2) and Definition 2.3.2. Namely, we have used the following identification (see also Remark 2.3.3):

$$[[X, Y]] = m^{-1}[[m(X), m(Y)]] \text{ and } \overline{\partial}_K = m^{-1} \circ \overline{\partial}_K \circ m. \tag{2.3.6}$$

Now we recall some facts and results on curves of **D**-structures over a compact almost **D**-manifold (M, K).

Let K_n denote the standard **D**-structure on $\mathbb{R}^{2n}$ (see Example 1.3.6) and consider the space of the linear **D**-structure on $\mathbb{R}^{2n}$:

$$\mathcal{X}(n) := \{P \in \mathrm{GL}(2n, \mathbb{R}) \mid P^2 = \mathrm{Id},\ \mathrm{tr}(P) = 0\}.$$

Then we have the following:

Proposition 2.3.7 ([61, Proposition 3.1])**.** *There exists a neighborhood U of K_n in $\mathcal{X}(n)$ such that every $P \in U$ can be written in a unique way as*

$$P = (\mathrm{Id} + L)K_n(\mathrm{Id} + L)^{-1},$$

where $K_nL + LK_n = 0$ and $\det(\mathrm{Id} + L) \neq 0$.

Proof. For the sake of completeness, we recall the proof. Observe that $\mathrm{GL}_{2n}(\mathbb{R})$ acts transitively on the space $\mathcal{X}(n)$ of the **D**-complex structures on $\mathbb{R}^{2n}$ by:

$$P \mapsto APA^{-1} \qquad A \in \mathrm{GL}_{2n}(\mathbb{R})\ P \in \mathcal{X}(n). \tag{2.3.7}$$

The isotropy group at K_n is:

$$\mathcal{H}(n) := \{A \in \mathrm{GL}_{2n}(\mathbb{R}) \mid AK_n - K_nA = 0\}. \tag{2.3.8}$$

Then $\mathcal{X}(n) = \mathrm{GL}_{2n}(\mathbb{R})/\mathcal{H}(n)$, the projection $\pi : \mathrm{GL}_{2n} \to \mathcal{X}(n)$ is a $\mathcal{H}(n)$-principal bundle and $\pi(A) = AK_nA^{-1}$. Since we have local triviality of this bundle, there exists a local section σ and an open set U such that:

$$\begin{aligned} \sigma : U &\longrightarrow \mathrm{GL}_{2n}(\mathbb{R}) \\ K_n &\mapsto \sigma(K_n) = \mathrm{Id} \end{aligned} \tag{2.3.9}$$

and for any $P \in U$ it holds that $\sigma(P)K_n\sigma^{-1}(P) = P$.
Now, we split the Lie algebra $\mathfrak{gl}_{2n}(\mathbb{R})$ into a sum $\mathfrak{s}(n) \oplus \mathfrak{h}(n)$ defined by:

$$\begin{aligned} \mathfrak{s}(n) &:= \{X \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid XK_n + K_nX = 0\} \\ \mathfrak{h}(n) &:= \{X \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid XK_n - K_nX = 0\}, \end{aligned} \tag{2.3.10}$$

and we define the projection over $\mathfrak{s}(n)$ (resp. $\mathfrak{h}(n)$) by:

$$\begin{aligned} S : \mathfrak{gl}_{2n}(\mathbb{R}) &\longrightarrow \mathfrak{s}(n) \qquad X \stackrel{S}{\longmapsto} \frac{1}{2}\big(X - K_nXK_n\big) \\ H : \mathfrak{gl}_{2n}(\mathbb{R}) &\longrightarrow \mathfrak{h}(n) \qquad X \stackrel{H}{\longmapsto} \frac{1}{2}\big(X + K_nXK_n\big). \end{aligned} \tag{2.3.11}$$

Now, by definition of H we get immediately $H(\sigma(K_n)) = \mathrm{Id}$, then, choosing U small enough, $H(\sigma(P)) \in \mathcal{H}(n)$ for all $P \in U$. Then we define a new section over U by:

$$\hat{\sigma}(P) = \sigma(P)\big(H(\sigma(P))\big)^{-1}. \tag{2.3.12}$$

Note that, since if $Y \in \mathfrak{h}(n)$ then also $Y^{-1} \in \mathfrak{h}(n)$, we have

$$\begin{aligned} H(\hat{\sigma}(P)) &= H\left(\sigma(P)\big(H(\sigma(P))\big)^{-1}\right) \\ &= \frac{1}{2}\left(\sigma(P)\big(H(\sigma(P))\big)^{-1} + K_n\sigma(P)\big(H(\sigma(P))\big)^{-1}K_n\right) \\ &= \frac{1}{2}\left(\sigma(P)\big(H(\sigma(P))\big)^{-1} + K_n\sigma(P)K_n\big(H(\sigma(P))\big)^{-1}\right) \\ &= \frac{1}{2}\big(\sigma(P) + K_n\sigma(P)K_n\big)\big(H(\sigma(P))\big)^{-1} \\ &= \big(H(\sigma(P)\big)\big(H(\sigma(P))\big)^{-1} = \mathrm{Id}, \end{aligned} \tag{2.3.13}$$

and also:

$$\hat{\sigma}(K_n) = \sigma(K_n)\big(H(\sigma(K_n))\big)^{-1} = \mathrm{Id}\big(H(\mathrm{Id})\big)^{-1} = \mathrm{Id}\,. \tag{2.3.14}$$

We see that the section $\hat{\sigma}$ is uniquely determined by the previous conditions (2.3.13) and (2.3.14), namely $\hat{\sigma}(K_n) = \mathrm{Id}$ and $H(\hat{\sigma}(P)) = \mathrm{Id}$. Then, using the split (2.3.10), for every **D**-structure P we have:

$$\begin{aligned} P &= \sigma(P)K_n\sigma^{-1}(P) = \sigma(P)\big(H(\sigma(P))\big)^{-1}\big(H(\sigma(P))\big)K_n\sigma^{-1}(P) \\ &= \sigma(P)\big(H(\sigma(P))\big)^{-1}K_n\big(H(\sigma(P))\big)\sigma^{-1}(P) = \hat{\sigma}(P)K_n\hat{\sigma}(P)^{-1} \\ &= \big(H(\hat{\sigma}(P)) + S(\hat{\sigma}(P))\big)K_n\big(H(\hat{\sigma}(P)) + S(\hat{\sigma}(P))\big)^{-1} \\ &= \big(\mathrm{Id} + L\big)K_n\big(\mathrm{Id} + L\big)^{-1}, \end{aligned} \tag{2.3.15}$$

where we have set $L := S(\hat{\sigma}(P))$, thence $LK_n + K_nL = 0$. This concludes the proof. □

Let K_t (for t small, $-\varepsilon < t < \varepsilon$) be a curve of almost **D**-structures on a compact M such that $K_0 = K$. Then, as a consequence of the previous proposition, K_t can be written as

$$K_t = (\mathrm{Id} + \varphi_t)K(\mathrm{Id} + \varphi_t)^{-1} \quad \text{where } \varphi_t = t\varphi + o(t),\ \varphi_t K + K\varphi_t = 0.$$

We note that the space of such φ_t is nothing else but $(\wedge_K^{0,1}T^*M)^{\mathbb{R}} \otimes TM$ (see Remark 2.3.1).

Moreover we have that

$$\frac{d}{dt}K_t\bigg|_{t=0} = 2\varphi K \tag{2.3.16}$$

and the following result:

Proposition 2.3.8 ([61, Proposition 3.2])**.** *Let K_t be a curve of almost para-complex structures, defined for $-\varepsilon < t < \varepsilon$, such that $K_0 = K$. Then*

$$\frac{d}{dt}N_{K_t}(X,Y)\bigg|_{t=0} = 4(\overline{\partial}_K\,\varphi)(X,Y) - N_K(\varphi X, Y) - N_K(X, \varphi Y) - \varphi\big(N_K(X,Y)\big). \tag{2.3.17}$$

Proof. By previous Proposition 2.3.7, we can write:

$$K_t = (\mathrm{Id} + \varphi_t)K(\mathrm{Id} + \varphi_t)^{-1} \quad \text{where } \varphi_t = t\varphi + o(t),\ \varphi_t K + K\varphi_t = 0. \tag{2.3.18}$$

Now, recalling (2.3.16), we get:

$$\begin{aligned}\frac{d}{dt}N_{K_t}(X,Y)\bigg|_{t=0} &= [2\varphi KX, KY] + [KX, 2\varphi KY] - 2\varphi K\left([KX,Y]+[X,KY]\right)\\ &\quad - K\left([2\varphi KX, Y] + [X, 2\varphi KY]\right)\\ &= 2\Big(-[K\varphi X, KY] - [KX, K\varphi Y] + K\varphi\left([KX,Y]+[X,KY]\right)\\ &\quad + K\left([K\varphi X, Y] + [X, K\varphi Y]\right) \pm K\left([\varphi X, KY] + [KX, \varphi Y]\right)\\ &\quad \pm [\varphi X, Y] \pm [X, \varphi Y]\Big)\\ &= 2\Big(-N_K(\varphi X, Y) - N_K(X, \varphi Y) + K\varphi\left([KX,Y]+[X,KY]\right)\\ &\quad - K\left([\varphi X, KY] + [KX, \varphi Y]\right) + [\varphi X, Y] + [X, \varphi Y]\Big).\end{aligned} \tag{2.3.19}$$

On the other hand, using Definition 2.3.2 we have:

$$\begin{aligned}4(\overline{\partial}_K\varphi)(X,Y) &= 4\big(\overline{\partial}_K(\varphi Y)(X) - \overline{\partial}_K(\varphi X)(Y) - \varphi([[X,Y]])\big)\\ &= 2\Big([X,\varphi Y] - K[KX,\varphi Y] + \frac{1}{2}N_K(\varphi Y, X)\\ &\quad - [Y, \varphi X] + K[KY, \varphi X] - \frac{1}{2}N_K(\varphi X, Y)\\ &\quad - \varphi\big([X,Y] + [KX,KY] - \frac{1}{2}N_K(X,Y)\big)\Big).\end{aligned} \tag{2.3.20}$$

Note that

$$-\varphi\big(2[X,Y] + 2[KX,KY] - N_K(X,Y)\big) - \varphi\big(N_K(X,Y)\big) = -2\varphi(K[KX,Y] + K[X,KY]), \tag{2.3.21}$$

hence the proof of the proposition follows from the above equality. □

Now we study the relations between a curve of **D**-structures and a fixed compatible 2-form.

Let (M,K) be a **D**-Hermitian manifold and let ω be a 2-form. K is ω-*calibrated* if

$$\omega(K\cdot, K\cdot) = -\omega(\cdot,\cdot)$$

that is, if $\omega(K\cdot,\cdot) = -\omega(\cdot, K\cdot)$. Let K_t be a curve of **D**-structures such that K_t is ω-calibrated for any $|t| < \varepsilon$. Then we have (using 2.3.16):

$$\begin{aligned}\omega(K_t\cdot,\cdot) &= -\omega(\cdot, K_t\cdot)\\ \frac{d}{dt}\omega(K_t\cdot,\cdot)\bigg|_{t=0} &= -\frac{d}{dt}\omega(\cdot, K_t\cdot)\bigg|_{t=0}\\ \omega(\varphi K\cdot,\cdot) &= -\omega(\cdot,\varphi K\cdot)\\ \omega(\varphi K\cdot,\cdot) &= -\omega(K\cdot,\varphi\cdot).\end{aligned}$$

Hence, if we set the pseudo-Riemannian metric $g(\cdot,\cdot) := \omega(\cdot, K\cdot)$, we have

$$g(\varphi K\cdot, K\cdot) = -g(K\cdot, K\varphi\cdot) = g(K\cdot, \varphi K\cdot), \quad \text{i.e. } \varphi = \varphi^t \tag{2.3.22}$$

where the transposition is taken with respect to the pseudo-metric g.

Finally, we recall the following theorem which characterizes the integrable deformations of **D**-structures.

Theorem 2.3.9 ([61, Theorem 4.2])**.** *Let* (M, K) *be a compact* **D***-manifold. Then the map between*

$$\{\varphi \in \Gamma(M, \mathrm{End}(TM)) \mid \varphi K + K\varphi = 0,\ \overline{\partial}_k\,\varphi + \frac{1}{2}[[\varphi,\varphi]] = 0,\ \det(\mathrm{Id}+\varphi) \neq 0\}$$

and

$$\{\hat{K} \in \Gamma(M, \mathrm{End}(TM)) \mid \hat{K}^2 = \mathrm{Id},\ N_{\hat{K}} = 0\},$$

given by

$$\varphi \longmapsto \hat{K} = (\mathrm{Id}+\varphi)K(\mathrm{Id}+\varphi)^{-1}$$

is a bijection between a neighborhood of $0 \in \Gamma(M, \mathrm{End}(TM))$ *and a neighborhood of* K*.*

The theorem is an easy consequence of the following lemma:

Lemma 2.3.10 ([61, Proposition 4.1])**.** *Let* K *be a* **D***-structure on a manifold* M*, and let* $\hat{K} = (\mathrm{Id}+\varphi)K(\mathrm{Id}+\varphi)^{-1}$ *be an almost* **D***-complex structure with* $\varphi K + K\varphi = 0$ *and* $\det(\mathrm{Id}+\varphi) \neq 0$*. Then*

$$(\mathrm{Id}+\varphi)^{-1}N_{\hat{K}}((\mathrm{Id}+\varphi)(X), (\mathrm{Id}+\varphi)(Y)) = 4(\mathrm{Id}-\varphi^2)^{-1}\left(\overline{\partial}_K\,\varphi + \frac{1}{2}[[\varphi,\varphi]]\right)(X, Y) \tag{2.3.23}$$

for $X, Y \in TM$*.*

Proof of Lemma 2.3.10. First note that the formula is equivalent to:

$$(\mathrm{Id}+\varphi)N_{\hat{K}}((\mathrm{Id}+\varphi)(X), (\mathrm{Id}+\varphi)(Y)) = 4\left(\overline{\partial}_K\,\varphi + \frac{1}{2}[[\varphi,\varphi]]\right)(X, Y). \tag{2.3.24}$$

This formula is proven by a straightforward computation, using that $N_K = 0$, $\varphi K + K\varphi = 0$ and the Definition 2.3.2 and equation (2.3.2). Because the techniques are similar to those ones of Lemma 2.4.7 the whole proof is omitted. □

Proof of Theorem 2.3.9. Now the theorem follows from the previous Proposition 2.3.7 and the Lemma 2.3.10. This concludes the proof of the theorem. □

2.4 Deformations of D-structures as derivations

Let (M, K) be a **D**-complex compact manifold. In this section, we restate the condition $\overline{\partial}_k\,\varphi + \frac{1}{2}[[\varphi,\varphi]] = 0$ of the previous theorem in terms of skew-symmetric derivations on $\wedge_K^{0,*}(M)$, giving a different (but equivalent) condition to verify if a deformation is integrable (Corollary 2.4.5). To do this, the key tool is the Theorem 2.4.2. We proceed as done by P. de Bartolomeis and F. Meylan in [22] for the CR-complex case.

Before stating the theorem, we need some preliminaries. First of all, we introduce a new graded algebra:

$$\widehat{\mathcal{A}}_p := \begin{cases} \Gamma(M, \wedge_K^{0,p}(M) \otimes T^{1,0}M) & 1 \leq p \leq n, \\ 0 & \text{otherwise} \end{cases} \quad \text{and set} \quad \widehat{\mathcal{A}} := \bigoplus_{p\in\mathbb{Z}} \widehat{\mathcal{A}}_p. \tag{2.4.1}$$

Note that $\widehat{\mathcal{A}}_p = \mathcal{A}_p$ for $p \neq 0$.

Now for any $\varphi \in \widehat{\mathcal{A}}_p$ we want to define a p-degree skew derivation $\urcorner_\varphi : \wedge_K^{0,*}(M) \to \wedge_K^{0,*}(M)$ (i.e. a p-degree skew derivation $\urcorner_\varphi : \Omega_K^{0,q}(M) \to \Omega_K^{0,q+p}(M)$ for any q). We proceed as follows:

For $(p = 1)$ take $\varphi \in \widehat{\mathcal{A}}_1$.

- If $q = 0$ we define $\urcorner_\varphi$ as follows (set f a smooth function and $Z \in T^{1,0}M$):

$$(\urcorner_\varphi f)(\bar{Z}) := \partial f(\varphi(\bar{Z})) = \varphi(\bar{Z})f = (\varphi\bar{Z}f). \tag{2.4.2}$$

If $q = 1$ we define $\urcorner_\varphi$ as follows (set $\gamma \in \Omega_K^{0,1}(M)$ and $Z, W \in T^{1,0}M$):

$$(\urcorner_\varphi\gamma)(\bar{Z}, \bar{W}) := \varphi\bar{Z}\gamma(\bar{W}) - \varphi\bar{W}\gamma(\bar{Z}) - \gamma([\varphi\bar{Z}, \bar{W}] - [\bar{Z}, \varphi\bar{W}]). \tag{2.4.3}$$

- Since

$$\urcorner_\varphi(f\gamma) = (\urcorner_\varphi f) \wedge \gamma + f\urcorner_\varphi\gamma,$$

we can extend $\urcorner_\varphi$ as a 1-degree skew derivation, that is if $\beta = \sum \beta_1 \wedge \cdots \wedge \beta_q \in \Omega_K^{0,q}(M)$, $q > 1$ we have:

$$\urcorner_\varphi\beta := \sum_{j=1}^{q}(-1)^{(1+j)}\beta_1 \wedge \cdots \wedge \urcorner_\varphi\beta_j \wedge \cdots \wedge \beta_q. \tag{2.4.4}$$

For $(p > 1)$ we write $\psi \in \widehat{\mathcal{A}}_p$ as a sum of elements of the form $\alpha_i \wedge \varphi_i$ with $\alpha_i \in \Omega_K^{0,p-1}(M)$ and $\varphi_i \in \widehat{\mathcal{A}}_1$ then we set

$$\urcorner_{\alpha_i \wedge \varphi_i} := \alpha_i \wedge \urcorner_{\varphi_i} \tag{2.4.5}$$

and $\urcorner_\psi$ is just the sum of the expression above.

Now we consider in $\widehat{\mathcal{A}}_p$ the same operators $[[\cdot, \cdot]]$ and $\overline{\partial}_K$ as defined in Definitions 2.3.4 and 2.3.5.

Let $\mathcal{F}$ be the space of skew-symmetric derivations on $\wedge_K^{0,*}(M)$ and consider the usual bracket defined on homogeneous elements as

$$[F, G] := F \circ G - (-1)^{|F||G|}G \circ F \quad \text{and set} \quad \delta_K F := [\overline{\partial}_K, F],$$

then an easy computation proves the following:

Proposition 2.4.1. $(\mathcal{F},\ [\cdot, \cdot],\ \delta_K)$ *is a DGLA.*

Define q to be a map between $(\widehat{\mathcal{A}},\ [[\cdot, \cdot]],\ \overline{\partial}_K)$ and $(\mathcal{F},\ [\cdot, \cdot],\ \delta_K)$:

$$q : \widehat{\mathcal{A}} \longrightarrow \mathcal{F}, \qquad \varphi \longmapsto \urcorner_\varphi. \tag{2.4.6}$$

We are ready to state the following:

Theorem 2.4.2. *The map $q : \widehat{\mathcal{A}} \to \mathcal{F}$ is a DGLA homomorphism, i.e. q is an injective map satisfying:*

$$[q(\varphi), q(\psi)] = q([[\varphi, \psi]]) \qquad \textit{(equivalently } [\urcorner_\varphi, \urcorner_\psi] = \urcorner_{[[\varphi,\psi]]}\textit{)} \tag{2.4.7}$$

and

$$\delta_K q(\varphi) = q(\overline{\partial}_K\, \varphi) \qquad \textit{(equivalently } \delta_K \urcorner_\varphi = \urcorner_{\overline{\partial}_K\, \varphi}\textit{)}. \tag{2.4.8}$$

To prove that q is an injective DGLA homomorphism we need the following two lemmata.

Lemma 2.4.3. *Let $\varphi \in \widehat{\mathcal{A}}_1$ and $\gamma \in \Omega_K^{0,2}(M)$. Then $\urcorner_\varphi$ satisfies the following:*

$$\urcorner_\varphi\gamma(\bar{Z}, \bar{W}, \bar{U}) = \underset{Z,W,U}{\mathfrak{S}}\varphi(\bar{Z})\gamma(\bar{W}, \bar{U}) - \underset{Z,W,U}{\mathfrak{S}}\gamma([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}], \bar{U}) \tag{2.4.9}$$

where $\underset{Z,W,U}{\mathfrak{S}}$ denotes the cyclic sum over $Z, W, U \in T^{1,0}M$.

Proof of Lemma 2.4.3. We apply the definition of $\urcorner_\varphi$ for $\gamma = \alpha \wedge \beta$, where $\alpha, \beta \in \Omega^{0,1}_K(M)$, i.e. using (2.4.4) we get:

$$\urcorner_\varphi \gamma = \urcorner_\varphi \alpha \wedge \beta = (\urcorner_\varphi \alpha) \wedge \beta - \alpha \wedge (\urcorner_\varphi \beta).$$

Now taking $Z, W, U \in T^{1,0}M$ we get:

$$\begin{aligned}
\urcorner_\varphi \gamma(\bar Z, \bar W, \bar U) &= (\urcorner_\varphi \alpha) \wedge \beta(\bar Z, \bar W, \bar U) - \alpha \wedge (\urcorner_\varphi \beta)(\bar Z, \bar W, \bar U) \\
&= \frac{1}{2}\Big\{ \urcorner_\varphi \alpha(\bar Z, \bar W)\beta(\bar U) - \urcorner_\varphi \alpha(\bar W, \bar Z)\beta(\bar U) \\
&\quad + \urcorner_\varphi \alpha(\bar U, \bar Z)\beta(\bar W) - \urcorner_\varphi \alpha(\bar Z, \bar U)\beta(\bar W) \\
&\quad + \urcorner_\varphi \alpha(\bar W, \bar U)\beta(\bar Z) - \urcorner_\varphi \alpha(\bar U, \bar W)\beta(\bar Z)\Big\} \\
&\quad + \frac{1}{2}\Big\{ \alpha(\bar Z)\urcorner_\varphi \beta(\bar W, \bar U) - \alpha(\bar Z)\urcorner_\varphi \beta(\bar U, \bar W) \\
&\quad + \alpha(\bar W)\urcorner_\varphi \beta(\bar U, \bar Z) - \alpha(\bar W)\urcorner_\varphi \beta(\bar Z, \bar U) \\
&\quad + \alpha(\bar U)\urcorner_\varphi \beta(\bar Z, \bar W) - \alpha(\bar U)\urcorner_\varphi \beta(\bar W, \bar Z)\Big\}.
\end{aligned}$$

Then we expand using (2.4.3):

$$\begin{aligned}
&\urcorner_\varphi \gamma(\bar Z, \bar W, \bar U) = \\
&= \frac{1}{2}\Big\{ \{\varphi \bar Z \alpha(\bar W) - \varphi \bar W \alpha(\bar Z) - \alpha([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W])\}\beta(\bar U) \\
&\quad -\{\varphi \bar W \alpha(\bar Z) - \varphi \bar Z \alpha(\bar W) - \alpha([\varphi \bar W, \bar Z] + [\bar W, \varphi \bar Z])\}\beta(\bar U) \\
&\quad +\{\varphi \bar U \alpha(\bar Z) - \varphi \bar Z \alpha(\bar U) - \alpha([\varphi \bar U, \bar Z] + [\bar U, \varphi \bar Z])\}\beta(\bar W) \\
&\quad -\{\varphi \bar Z \alpha(\bar U) - \varphi \bar U \alpha(\bar Z) - \alpha([\varphi \bar Z, \bar U] + [\bar Z, \varphi \bar U])\}\beta(\bar W) \\
&\quad +\{\varphi \bar W \alpha(\bar U) - \varphi \bar U \alpha(\bar W) - \alpha([\varphi \bar W, \bar U] + [\bar W, \varphi \bar U])\}\beta(\bar Z) \\
&\quad -\{\varphi \bar U \alpha(\bar W) - \varphi \bar W \alpha(\bar U) - \alpha([\varphi \bar U, \bar W] + [\bar U, \varphi \bar W])\}\beta(\bar Z)\Big\} \\
&- \frac{1}{2}\Big\{ \alpha(\bar Z)\{\varphi \bar W \beta(\bar U) - \varphi \bar U \beta(\bar W) - \beta([\varphi \bar W, \bar U] + [\bar W, \varphi \bar U])\} \\
&\quad -\alpha(\bar Z)\{\varphi \bar U \beta(\bar W) - \varphi \bar W \beta(\bar U) - \beta([\varphi \bar U, \bar W] + [\bar U, \varphi \bar W])\} \\
&\quad +\alpha(\bar W)\{\varphi \bar U \beta(\bar Z) - \varphi \bar Z \beta(\bar U) - \beta([\varphi \bar U, \bar Z] + [\bar U, \varphi \bar Z])\} \\
&\quad -\alpha(\bar W)\{\varphi \bar Z \beta(\bar U) - \varphi \bar U \beta(\bar Z) - \beta([\varphi \bar Z, \bar U] + [\bar Z, \varphi \bar U])\} \\
&\quad +\alpha(\bar U)\{\varphi \bar Z \beta(\bar W) - \varphi \bar W \beta(\bar Z) - \beta([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W])\} \\
&\quad -\alpha(\bar U)\{\varphi \bar W \beta(\bar Z) - \varphi \bar Z \beta(\bar W) - \beta([\varphi \bar W, \bar Z] + [\bar W, \varphi \bar Z])\}\Big\}.
\end{aligned} \tag{2.4.10}$$

Since $\varphi \bar Z \in T^{1,0}M$, we can see it as derivation, then:

$$(\varphi \bar Z \alpha(\bar W))\beta(\bar U) + \alpha(\bar W)(\varphi \bar Z \beta(\bar U)) = \varphi \bar Z(\alpha(\bar W)\beta(\bar U)).$$

Using this and summing up all the terms, we get from (2.4.10) the first term of the right hand side of formula (2.4.9):

$$\varphi \bar Z \alpha \wedge \beta(\bar W, \bar U) + \varphi \bar W \alpha \wedge \beta(\bar U, \bar Z) + \varphi \bar U \alpha \wedge \beta(\bar Z, \bar W).$$

In the same way, we can see that:

$$\begin{aligned}
-\alpha([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W])\beta(\bar U) + \alpha(\bar U)\beta([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W]) &= \\
= -\alpha \wedge \beta([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W], \bar U)& \\
= -\gamma([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W], \bar U).&
\end{aligned}$$

We sum again and we get the second term of (2.4.9):

$$-\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}],\bar{U})-\gamma([\varphi\bar{W},\bar{U}]+[\bar{W},\varphi\bar{U}],\bar{Z})-\gamma([\varphi\bar{U},\bar{Z}]+[\bar{U},\varphi\bar{Z}],\bar{U})$$

and the lemma is proved. □

Lemma 2.4.4. *Let $\psi \in \widehat{\mathcal{A}}_2$ and $\beta \in \Omega^{0,1}_K(M)$. Then $\daleth_\psi$ satisfies the following:*

$$\daleth_\psi\beta(\bar{Z},\bar{W},\bar{U}) = \underset{Z,W,U}{\mathfrak{S}}\psi(\bar{Z},\bar{W})\beta(\bar{U}) - \underset{Z,W,U}{\mathfrak{S}}\beta([\psi(\bar{Z},\bar{W}),\bar{U}]) \tag{2.4.11}$$

where $Z, W, U \in T^{1,0}M$.

Proof of Lemma 2.4.4. Applying the definition of $\daleth_\psi$ for $\psi = \alpha\wedge\varphi$ (see (2.4.5)), we have:

$$\begin{aligned}\daleth_\psi\beta(\bar{Z},\bar{W},\bar{U}) &= \alpha\wedge\daleth_\varphi\beta(\bar{Z},\bar{W},\bar{U})\\ &= \frac{1}{2}\Big\{\alpha(\bar{Z})\daleth_\varphi\beta(\bar{W},\bar{U})-\alpha(\bar{Z})\daleth_\varphi\beta(\bar{U},\bar{W})\\ &\quad+\alpha(\bar{W})\daleth_\varphi\beta(\bar{U},\bar{Z})-\alpha(\bar{W})\daleth_\varphi\beta(\bar{Z},\bar{U})\\ &\quad+\alpha(\bar{U})\daleth_\varphi\beta(\bar{Z},\bar{W})-\alpha(\bar{U})\daleth_\varphi\beta(\bar{W},\bar{Z})\Big\}\end{aligned}$$

and then we get the second part of (2.4.10). First, notice that we have

$$\alpha(\bar{Z})\varphi\bar{W}\beta(\bar{U})-\alpha(\bar{W})\varphi\bar{Z}\beta(\bar{U}) = \alpha\wedge\varphi(\bar{Z},\bar{W})\beta(\bar{U}) = \psi(\bar{Z},\bar{W})\beta(\bar{U})$$

and since there are two terms like this, we can sum and simplify the $\frac{1}{2}$, and we get the first term of the right hand of (2.4.11):

$$\psi(\bar{Z},\bar{W})\beta(\bar{U})+\psi(\bar{W},\bar{U})\beta(\bar{Z})+\psi(\bar{U},\bar{Z})\beta(\bar{W}).$$

Now observe that, since $\beta\in\Omega^{0,1}_K(M)$, we have that $\beta(X)=\beta(X^{0,1})=\beta(X+Y^{1,0})$ for all vector fields X, Y (here $X^{1,0}$ denotes the projection of X over $T^{1,0}M$). Hence, since $\varphi\bar{W}\in T^{1,0}M$, we have

$$\begin{aligned}-\alpha(\bar{Z})\beta([\varphi\bar{W},\bar{U}]) &= -\beta\big(\alpha(\bar{Z})(\varphi\bar{W}(\bar{U})-\bar{U}(\varphi\bar{W}))\big)\\ &= -\beta\big(\alpha(\bar{Z})\varphi\bar{W}(\bar{U})-\alpha(\bar{Z})\bar{U}(\varphi\bar{W})\big)\\ &= -\beta\big(\alpha(\bar{Z})\varphi\bar{W}(\bar{U})-\bar{U}(\alpha(\bar{Z})\varphi\bar{W})+\bar{U}(\alpha(\bar{Z}))\varphi\bar{W}\big)\\ &= -\beta\big([\alpha(\bar{Z})\varphi\bar{W},\bar{U}]\big)-\bar{U}(\alpha(\bar{Z}))\beta(\varphi\bar{W})\\ &= -\beta\big([\alpha(\bar{Z})\varphi\bar{W},\bar{U}]\big).\end{aligned}$$

Now we have

$$\begin{aligned}-\beta([\alpha(\bar{Z})\varphi\bar{W},\bar{U}])+\beta([\alpha(\bar{W})\varphi\bar{Z},\bar{U}]) &= -\beta([\alpha\wedge\varphi(\bar{Z},\bar{W}),\bar{U}])\\ &= -\beta([\psi(\bar{Z},\bar{W}),\bar{U}])\end{aligned}$$

and finally we get the last term of (2.4.11). □

Now we are ready to prove the Theorem 2.4.2.

Proof of Theorem 2.4.2. **Step 1:** Injectivity of the map q.
If $\daleth_\varphi = \daleth_\psi$ then, in particular, we have

$$\varphi(\bar{Z})(f) = \daleth_\varphi f(\bar{Z}) = \daleth_\psi f(\bar{Z}) = \psi(\bar{Z})(f)$$

for every smooth function f and for every vector field $Z\in T^{1,0}M$, and we get $\psi=\varphi$.
Step 2: Proof of (2.4.7).

We have to prove that $\daleth_\varphi \circ \daleth_\varphi = \frac{1}{2}\daleth_{[[\varphi,\varphi]]}$. First of all, take $\varphi \in \widehat{\mathcal{A}}_1$, a smooth function f and $Z, W \in T^{1,0}M$. By (2.4.4), (2.4.2) and (2.3.5) we get:

$$\begin{aligned}
\daleth_\varphi \circ \daleth_\varphi f(\bar{Z},\bar{W}) &= \varphi\bar{Z}\daleth_\varphi f(\bar{W}) - \varphi\bar{W}\daleth_\varphi f(\bar{Z}) - \daleth_\varphi f([\varphi\bar{Z},\bar{W}] + [\bar{Z},\varphi\bar{W}]) \\
&= \varphi\bar{Z}\varphi\bar{W}f - \varphi\bar{W}\varphi\bar{Z}f - \varphi([\varphi\bar{Z},\bar{W}] + [\bar{Z},\varphi\bar{W}])f \\
&= [\varphi\bar{Z},\varphi\bar{W}]f - \varphi([\varphi\bar{Z},\bar{W}] + [\bar{Z},\varphi\bar{W}])f \\
&= \frac{1}{2}[[\varphi,\varphi]](\bar{Z},\bar{W})f = \frac{1}{2}\daleth_{[[\varphi,\varphi]]}f(\bar{Z},\bar{W})
\end{aligned}$$

where the last equivalence follows from equation (2.4.5). Since $\sum \alpha_i \wedge \psi_i = [[\varphi,\varphi]] \in \widehat{\mathcal{A}}_2$ with $\alpha_i \in \Omega^{0,1}_K(M)$ and $\varphi \in \widehat{\mathcal{A}}_1$, by linearity we can write:

$$\begin{aligned}
\daleth_{[[\varphi,\varphi]]}f(\bar{Z},\bar{W}) &= \daleth_{\sum \alpha_i\wedge\psi_i}f(\bar{Z},\bar{W}) = \sum \alpha_i \wedge \daleth_{\psi_i}f(\bar{Z},\bar{W}) \\
&= \sum \Big(\alpha_i\bar{Z}\daleth_{\psi_i}f(\bar{W}) - \alpha_i\bar{W}\daleth_{\psi_i}f(\bar{Z})\Big) \\
&= \sum \Big(\alpha_i\bar{Z}\psi_i\bar{W}f - \alpha_i\bar{W}\psi_i\bar{Z}f\Big) \\
&= \sum \alpha_i \wedge \psi_i(\bar{Z},\bar{W})f = [[\varphi,\varphi]](\bar{Z},\bar{W})f.
\end{aligned} \tag{2.4.12}$$

Now take $\gamma \in \Omega^{0,1}_K(M)$. Using Lemma 2.4.3 we get (we drop the index Z, W, U on the cyclic sum, as it is clear from the context):

$$\begin{aligned}
\daleth_\varphi \circ \daleth_\varphi \gamma(\bar{Z},\bar{W},\bar{U}) &= (\mathfrak{S}\varphi\bar{Z}\daleth_\varphi\gamma(\bar{W},\bar{U})) - \mathfrak{S}\daleth_\varphi\gamma([\varphi\bar{Z},\bar{W}] + [\bar{Z},\bar{W}],\bar{U}) = \\
&= \mathfrak{S}(\varphi\bar{Z}\{\varphi\bar{W}\gamma(\bar{U}) - \varphi\bar{U}\gamma(\bar{W}) - \gamma([\varphi\bar{W},\bar{U}] + [\bar{W},\varphi\bar{U}])\}) \\
&\quad - \mathfrak{S}\Big(\varphi([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}])\gamma(\bar{U}) - \varphi\bar{U}\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]) \\
&\quad - \gamma\left([\varphi([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]),\bar{U}] + [[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}],\varphi\bar{U}]\right)\Big) \\
&= \mathfrak{S}(\varphi\bar{Z}\varphi\bar{W}\gamma(\bar{U})) - \mathfrak{S}(\varphi\bar{Z}\varphi\bar{U}\gamma(\bar{W})) \\
&\quad - \mathfrak{S}\varphi\bar{Z}\gamma([\varphi\bar{W},\bar{U}]+[\bar{W},\varphi\bar{U}]) \\
&\quad - \mathfrak{S}\varphi([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]\gamma(\bar{U})) \\
&\quad + \mathfrak{S}\varphi\bar{U}\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]) \\
&\quad - \mathfrak{S}\gamma([\varphi([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]),\bar{U}]) \\
&\quad + \mathfrak{S}\gamma([[\varphi\bar{Z},\bar{W}],\varphi\bar{U}]) + \mathfrak{S}\gamma([[\bar{Z},\varphi\bar{W}],\varphi\bar{U}]).
\end{aligned}$$

If we rewrite the terms, we can observe that:

$$\begin{aligned}
\mathfrak{S}(\varphi\bar{Z}\varphi\bar{W}\gamma(\bar{U})) - \mathfrak{S}(\varphi\bar{Z}\varphi\bar{U}\gamma(\bar{W})) &= \mathfrak{S}(\varphi\bar{Z}\varphi\bar{W}\gamma(\bar{U})) - \mathfrak{S}(\varphi\bar{W}\varphi\bar{Z}\gamma(\bar{U})) \\
&= \mathfrak{S}([\varphi\bar{Z},\varphi\bar{W}]\gamma(\bar{U}));
\end{aligned}$$

and that:

$$\begin{aligned}
&-\mathfrak{S}\varphi\bar{Z}\gamma([\varphi\bar{W},\bar{U}] + [\bar{W},\varphi\bar{U}]) + \mathfrak{S}\varphi\bar{U}\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]) = \\
&= -\mathfrak{S}\varphi\bar{Z}\gamma([\varphi\bar{W},\bar{U}] + [\bar{W},\varphi\bar{U}]) + \mathfrak{S}\varphi\bar{Z}\gamma([\varphi\bar{W},\bar{U}]+[\bar{W},\varphi\bar{U}]) = 0,
\end{aligned}$$

and using the Jacobi identity (2.1.1)

$$\begin{aligned}
&\mathfrak{S}\gamma([[\varphi\bar{Z},\bar{W}],\varphi\bar{U}]) + \mathfrak{S}\gamma([[\bar{Z},\varphi\bar{W}],\varphi\bar{U}]) \\
&= \mathfrak{S}\gamma([[\varphi\bar{Z},\bar{W}],\varphi\bar{U}]) + \mathfrak{S}\gamma([[\bar{W},\varphi\bar{U}],\varphi\bar{Z}]) \\
&= -\mathfrak{S}\gamma([[\varphi\bar{U},\varphi\bar{Z}],\bar{W}]).
\end{aligned}$$

Now using Lemma 2.4.4 we have:

$$\begin{aligned}
\urcorner_\varphi \circ \urcorner_\varphi \gamma(\bar Z, \bar W, \bar U) &= \mathfrak{S}(\varphi \bar Z \varphi \bar W \gamma(\bar U)) - \mathfrak{S}(\varphi \bar W \varphi \bar Z \gamma(\bar U)) \\
&\quad - \mathfrak{S}\varphi([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W]\gamma(\bar U)) \\
&\quad - \mathfrak{S}\gamma([[\varphi \bar U, \varphi \bar Z], \bar W]) \\
&\quad - \mathfrak{S}\gamma([\varphi([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W]), \bar U]) \\
&= \frac{1}{2}\mathfrak{S}\left\{(2[\varphi \bar Z, \varphi \bar W] - 2\varphi([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W]))\gamma(\bar U)\right\} \\
&\quad - \frac{1}{2}\mathfrak{S}\gamma([2[\varphi \bar Z, \varphi \bar W] - 2\varphi([\varphi \bar Z, \bar W] + [\bar Z, \varphi \bar W]), \bar U]) \\
&= \frac{1}{2}\mathfrak{S}[[\varphi, \varphi]](\bar Z, \bar W)\gamma(\bar U) - \frac{1}{2}\mathfrak{S}\gamma\big([[[\varphi, \varphi]](\bar Z, \bar W), \bar U]\big) \\
&= \frac{1}{2}\urcorner_{[[\varphi,\varphi]]}\gamma(\bar Z, \bar W, \bar U).
\end{aligned}$$

For the general case we can argue by induction: assume that, for all $\alpha \in \wedge_K^{0,q}(M)$:

$$\urcorner_\varphi \urcorner_\varphi \alpha = \frac{1}{2}\urcorner_{[[\varphi,\varphi]]}\alpha.$$

Let $\beta \in \wedge_K^{0,q+1}(M)$ such that $\beta = \gamma \wedge \alpha$ with $\alpha \in \wedge_K^{0,q}(M)$ and $\gamma \in \wedge_K^{0,1}(M)$. Then:

$$\begin{aligned}
\urcorner_\varphi \urcorner_\varphi \beta &= \urcorner_\varphi(\urcorner_\varphi \gamma \wedge \alpha + \gamma \wedge \urcorner_\varphi \alpha) \\
&= (\urcorner_\varphi \urcorner_\varphi \gamma) \wedge \alpha + \urcorner_\varphi \gamma \wedge \urcorner_\varphi \alpha - \urcorner_\varphi \gamma \wedge \urcorner_\varphi \alpha + \gamma(\urcorner_\varphi \urcorner_\varphi \alpha) \\
&= (\frac{1}{2}\urcorner_{[[\varphi,\varphi]]}\gamma) \wedge \alpha + \gamma \wedge (\frac{1}{2}\urcorner_{[[\varphi,\varphi]]}\alpha) \\
&= \frac{1}{2}\urcorner_{[[\varphi,\varphi]]}(\gamma \wedge \alpha) = \frac{1}{2}\urcorner_{[[\varphi,\varphi]]}\beta.
\end{aligned}$$

Moreover, take $\psi, \varphi \in \widehat{\mathcal{A}}_1$. Then:

$$\begin{aligned}
\urcorner_{[[\psi,\varphi]]} &= \frac{1}{2}(\urcorner_{[[\psi+\varphi,\psi+\varphi]]} - \urcorner_{[[\psi,\psi]]} - \urcorner_{[[\varphi,\varphi]]}) \\
&= \urcorner_{\psi+\varphi} \circ \urcorner_{\psi+\varphi} - \urcorner_\psi \circ \urcorner_\psi - \urcorner_\varphi \circ \urcorner_\varphi \\
&= \urcorner_\psi \urcorner_\psi + \urcorner_\psi \urcorner_\varphi + \urcorner_\varphi \urcorner_\psi + \urcorner_\varphi \urcorner_\varphi - \urcorner_\psi \urcorner_\psi - \urcorner_\varphi \urcorner_\varphi \\
&= [\urcorner_\psi, \urcorner_\varphi].
\end{aligned}$$

Finally, again by induction we have

$$\urcorner_{[[\alpha\wedge\psi,\beta\wedge\varphi]]} = [\urcorner_{\alpha\wedge\psi}, \urcorner_{\beta\wedge\varphi}],$$

and by the definition of the bracket $[\cdot,\cdot]$ in $\mathcal{F}$ we get:

$$[\urcorner_\varphi, \urcorner_\varphi] = \urcorner_\varphi \circ \urcorner_\varphi - (-1)^{|\urcorner_\varphi|}\urcorner_\varphi \circ \urcorner_\varphi = 2\urcorner_\varphi \circ \urcorner_\varphi.$$

By linearity we can extend this demonstration to elements $\beta \in \wedge_K^{0,q+1}(M)$ not of the form $\beta = \gamma \wedge \alpha$, and this achieves the proof of (2.4.7).

Step 3: Proof of (2.4.8).

Let f be a function, $Z, W \in T^{1,0}M$ and $\varphi \in \widehat{\mathcal{A}}_1$. Since $\overline{\partial}_K \varphi \in \widehat{\mathcal{A}}_2$ and using the same argument as in (2.4.12), we observe that the right side of (2.4.8) is:

$$\begin{aligned}
\urcorner_{\overline{\partial}_K \varphi}(f)(\bar Z, \bar W) &= (\overline{\partial}_K \varphi)(\bar Z, \bar W)(f) \\
&= \big((\overline{\partial}_K \varphi \bar W)(\bar Z) - (\overline{\partial}_K \varphi \bar Z)(\bar W) - \varphi([\bar Z, \bar W])\big)(f) = \\
= \Big(\frac{1}{2}([\bar Z,\varphi \bar W] + \tau K[\bar Z,\varphi \bar W] &- [\bar W,\varphi \bar Z] - \tau K[\bar W,\varphi \bar Z]) - \varphi[\bar Z,\bar W]\Big)(f).
\end{aligned} \tag{2.4.13}$$

On the left side of (2.4.8) we have:

$$\delta_K \lrcorner_\varphi(f) = [\overline{\partial}_K, \lrcorner_\varphi](f) = (\overline{\partial}_K \lrcorner_\varphi + \lrcorner_\varphi \overline{\partial}_K)(f).$$

Consequently:

$$\begin{aligned}\overline{\partial}_K \lrcorner_\varphi(f)(\bar{Z}, \bar{W}) &= \overline{\partial}_K(\lrcorner_\varphi f\bar{W}(\bar{Z})) - \overline{\partial}_K(\lrcorner_\varphi f\bar{Z})(\bar{W}) - \lrcorner_\varphi f([\bar{Z}, \bar{W}])\\ &= \bar{Z}(\lrcorner_\varphi f\bar{W}) - \bar{W}(\lrcorner_\varphi f\bar{Z}) - \lrcorner_\varphi f([\bar{Z}, \bar{W}])\end{aligned}$$

and

$$\lrcorner_\varphi \overline{\partial}_K(f)(\bar{Z}, \bar{W}) = \varphi\bar{Z}\,\overline{\partial}_K\, f(\bar{W}) - \varphi\bar{W}\,\overline{\partial}_K\, f(\bar{Z}) - \overline{\partial}_K\, f([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}]).$$

Hence we have:

$$\begin{aligned}(\overline{\partial}_K \lrcorner_\varphi + \lrcorner_\varphi \overline{\partial}_K)(f)(\bar{Z}, \bar{W}) &=\\ &= \bar{Z}(\lrcorner_\varphi f\bar{W}) - \bar{W}(\lrcorner_\varphi f\bar{Z}) - \lrcorner_\varphi f([\bar{Z}, \bar{W}])\\ &\quad + \varphi\bar{Z}\,\overline{\partial}_K\, f(\bar{W}) - \varphi\bar{W}\,\overline{\partial}_K\, f(\bar{Z})\\ &\quad - \overline{\partial}_K\, f([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}])\\ &= \bar{Z}(\varphi\bar{W})(f) - \bar{W}(\varphi\bar{Z})(f) - \varphi([\bar{Z}, \bar{W}])(f)\\ &\quad + \varphi\bar{Z}(\bar{W})(f) - \varphi\bar{W}(\bar{Z})(f)\\ &\quad - \frac{1}{2}([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}] - \tau K([\varphi\bar{Z}, \bar{W}] + [\bar{Z}, \varphi\bar{W}]))(f)\end{aligned}$$

where in the last equality we use that $\overline{\partial}_K\, f(X) = \overline{\partial}\, f(X) = \frac{1}{2}(X - \tau KX)(f)$ if $X \in TM$. We can see that this last expression coincides with (2.4.13), since $\bar{Z}(\varphi\bar{W})(f) - \varphi\bar{W}(\bar{Z})(f) = [\bar{Z}, \varphi\bar{W}](f)$. Then we have:

$$\lrcorner_{\overline{\partial}_K \varphi}(f) = \delta_K \lrcorner_\varphi(f).$$

Now let $\gamma \in \Omega_K^{0,1}(M)$ and $Z, W, U \in T^{1,0}M$. We have, using Lemma 2.4.4 and the definition of $\overline{\partial}_K\, \varphi$ (Definition 2.3.5):

$$\begin{aligned}\lrcorner_{\overline{\partial}_K \varphi}(\bar{Z}, \bar{W}, \bar{U}) &= \mathfrak{S}\,\overline{\partial}_K\, \varphi(\bar{Z}, \bar{W})\gamma(\bar{U}) - \mathfrak{S}\gamma([\overline{\partial}_K\, \varphi(\bar{Z}, \bar{W}), \bar{U}])\\ &= \mathfrak{S}(\overline{\partial}_K(\varphi\bar{W})(\bar{Z}) - \overline{\partial}_K(\varphi\bar{Z})(\bar{W}) - \varphi([\bar{Z}, \bar{W}]))\gamma(\bar{U})\\ &\quad - \mathfrak{S}\gamma([\overline{\partial}_K(\varphi\bar{W})(\bar{Z}) - \overline{\partial}_K(\varphi\bar{Z})(\bar{W}) - \varphi([\bar{Z}, \bar{W}]), \bar{U}]).\end{aligned}$$

Expanding the previous (using the definition of $\overline{\partial}_K\, \varphi\bar{W}$):

$$\begin{aligned}\lrcorner_{\overline{\partial}_K \varphi}(\bar{Z}, \bar{W}, \bar{U}) &= \mathfrak{S}\frac{1}{2}([\bar{Z}, \varphi\bar{W}] + \tau K[\bar{Z}, \varphi\bar{W}])\gamma(\bar{U})\\ &- \mathfrak{S}\frac{1}{2}([\bar{W}, \varphi\bar{Z}] + \tau K[\bar{W}, \varphi\bar{Z}])\gamma(\bar{U}) - \mathfrak{S}\varphi([\bar{Z}, \bar{W}])\gamma(\bar{U})\\ &- \mathfrak{S}\gamma(\frac{1}{2}[[\bar{Z}, \varphi\bar{W}] + \tau K[\bar{Z}, \varphi\bar{W}], \bar{U}])\\ &+ \mathfrak{S}\gamma(\frac{1}{2}[[\bar{W}, \varphi\bar{Z}] + \tau K[\bar{W}, \varphi\bar{Z}], \bar{U}]) + \mathfrak{S}\gamma([\varphi([\bar{Z}, \bar{W}]), \bar{U}]).\end{aligned} \tag{2.4.14}$$

The left hand of (2.4.8) is:

$$\delta_K \lrcorner_\varphi = [\overline{\partial}_K, \lrcorner_\varphi] = \overline{\partial}_K \lrcorner_\varphi + \lrcorner_\varphi \overline{\partial}_K\,.$$

Hence, using the definition of $\overline{\partial}_K$ and of $\lrcorner_\varphi$:

$$\begin{aligned}\overline{\partial}_K \lrcorner_\varphi\gamma(\bar{Z}, \bar{W}, \bar{U}) &= \mathfrak{S}\,\overline{\partial}_K \lrcorner_\varphi\gamma(\bar{W}, \bar{U})(\bar{Z}) - \mathfrak{S}\lrcorner_\varphi\gamma([\bar{Z}, \bar{W}], \bar{U})\\ &= \mathfrak{S}\bar{Z}(\lrcorner_\varphi\gamma)(\bar{W}, \bar{U}) - \mathfrak{S}\lrcorner_\varphi\gamma([\bar{Z}, \bar{W}], \bar{U})\\ &= \mathfrak{S}\bar{Z}\{\varphi\bar{W}\gamma(\bar{U}) - \varphi\bar{U}\gamma(\bar{W}) - \gamma([\varphi\bar{W}, \bar{U}] + [\bar{W}, \varphi\bar{U}])\}\\ &\quad - \mathfrak{S}\varphi([\bar{Z}, \bar{W}])\gamma(\bar{U}) + \mathfrak{S}\varphi\bar{U}\gamma([\bar{Z}, \bar{W}])\\ &\quad + \mathfrak{S}\gamma([\varphi([\bar{Z}, \bar{W}]), \bar{U}] + [[\bar{Z}, \bar{W}], \varphi\bar{U}]).\end{aligned}$$

We also have, using Lemma 2.4.3:

$$\begin{aligned}\daleth_\varphi \overline{\partial}_K \gamma(\bar{Z},\bar{W},\bar{U}) &= \mathfrak{S}\varphi\bar{Z}(\overline{\partial}_K \gamma)(\bar{W},\bar{U}) - \mathfrak{S}\,\overline{\partial}_K(\gamma)([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}],\bar{U})\\ &= \mathfrak{S}\varphi\bar{Z}\{\bar{W}\gamma(\bar{U}) - \bar{U}\gamma(\bar{W}) - \gamma([\bar{W},\bar{U}])\}\\ &\quad - \mathfrak{S}\,\overline{\partial}_K(\gamma)([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}],\bar{U}).\end{aligned}$$

Now, using the Jacobi identity (2.1.1), we observe that:

$$\gamma([[\bar{Z},\bar{W}],\varphi\bar{U}]) = \gamma([\bar{Z},[\bar{W},\varphi\bar{U}]]) - \gamma([\bar{W},[\bar{Z},\varphi\bar{U}]])$$

and using the fact that $\overline{\partial}_K \gamma \in \Omega^{0,2}_K(M)$ we can take the $(0,1)$-part, then we get:

$$\begin{aligned}&\overline{\partial}_K(\gamma)([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}],\bar{U}) =\\ &= \overline{\partial}_K \gamma(\frac{1}{2}\{[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}])\},\bar{U})\\ &= \frac{1}{2}\{[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}])\}\gamma(\bar{U})\\ &\quad - \frac{1}{2}\bar{U}\{\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]))\}\\ &\quad - \frac{1}{2}\gamma([[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]),\bar{U}]).\end{aligned}$$

Finally we have:

$$\begin{aligned}(\overline{\partial}_K \daleth_\varphi + \daleth_\varphi \overline{\partial}_K)\gamma(\bar{Z},\bar{W},\bar{U}) &= \mathfrak{S}\bar{Z}\{\varphi\bar{W}\gamma(\bar{U}) - \varphi\bar{U}\gamma(\bar{W})\}\\ &- \mathfrak{S}\bar{Z}\{\gamma([\varphi\bar{W},\bar{U}]+[\bar{W},\varphi\bar{U}])\}\\ &- \mathfrak{S}\varphi([\bar{Z},\bar{W}])\gamma(\bar{U}) + \mathfrak{S}\varphi\bar{U}\gamma([\bar{Z},\bar{W}])\\ &+ \mathfrak{S}\gamma([\varphi([\bar{Z},\bar{W}]),\bar{U}])\\ &+ \mathfrak{S}\gamma([\bar{Z},[\bar{W},\varphi\bar{U}]]) - \gamma([\bar{W},[\bar{Z},\varphi\bar{U}]])\\ &+ \mathfrak{S}\varphi\bar{Z}\{\bar{W}\gamma(\bar{U}) - \bar{U}\gamma(\bar{W}) - \gamma([\bar{W},\bar{U}])\}\\ &- \mathfrak{S}\frac{1}{2}\{[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}])\}\gamma(\bar{U})\\ &+ \mathfrak{S}\frac{1}{2}\bar{U}\{\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]))\}\\ &+ \mathfrak{S}\frac{1}{2}\gamma([[\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}] - \tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]),\bar{U}]).\end{aligned} \tag{2.4.15}$$

Using that $\frac{1}{2}\bar{U}\{\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]-\tau K([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}]))\} = \bar{U}\gamma([\varphi\bar{Z},\bar{W}]+[\bar{Z},\varphi\bar{W}])$ and possibly using the cycling sum, we see that (2.4.14) and (2.4.15) are equal.
As before, the general case follows by induction and this concludes the proof of the theorem.
□

From Theorem 2.4.2 we are able to rewrite the integrability condition $\overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] = 0$ as follows:

Corollary 2.4.5. *Let q given by*

$$q : \widehat{\mathcal{A}} \longrightarrow \mathcal{F} \qquad \varphi \longmapsto \daleth_\varphi,$$

then:

1. $(\mathrm{im}\ q,\ [\cdot,\cdot],\ \delta_K)$ *is a DGLA;*
2. $(\widehat{\mathcal{A}},\ [[\cdot,\cdot]],\ \overline{\partial}_K)$ *is a DGLA isomorphic to the previous one;*

3. *we have that*
$$\varphi \in \mathfrak{MC}(\widehat{\mathcal{A}}) \Longleftrightarrow \daleth_\varphi \in \mathfrak{MC}(\mathcal{F}),$$
i.e. $\overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] = 0$ *if and only if* $\delta_K \daleth_\varphi + \frac{1}{2}[\daleth_\varphi, \daleth_\varphi] = 0$.

Proof. The statement follows from Theorem 2.4.2 and from the Definition of $\mathfrak{MC}$ (equation (2.1.5)). $\square$

Remark 2.4.6. We observe the following:

- Let $C(\mathfrak{g})$ be the center of a Lie algebra $\mathfrak{g}$ (i.e. $C(\mathfrak{g}) := \{a \in \mathfrak{g} \mid [a,b] = 0\ \forall b \in \mathfrak{g}\}$). It is easy to see that $C(\widehat{\mathcal{A}}) = 0$. Indeed, if $\varphi, \psi \in \widehat{\mathcal{A}}$, and f is a smooth function, then
$$[[\varphi, f\psi]] = f[[\varphi,\psi]] + \daleth_\varphi f \wedge \psi.$$
Take now $\psi \in C(\widehat{\mathcal{A}})$, hence $0 = [[\varphi, f\psi]] = \daleth_\varphi f \wedge \psi$ for all f and for all ψ. If we choose $\psi \neq 0$ such that $\psi(\bar{Z}) = Z$, then $\daleth_\varphi f = 0$ for all functions f and hence $\daleth_\varphi = 0$. Now by the injectivity of the map q we have that $\varphi = 0$.

- Moreover setting $\overline{\partial}_\varphi := \overline{\partial}_K + [[\varphi, \cdot]]$ (see (2.1.3)) we have that:
$$\overline{\partial}_\varphi^2 = 0 \Longleftrightarrow \overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] = 0 \Longleftrightarrow$$
$$\Longleftrightarrow \delta_K \daleth_\varphi + \frac{1}{2}[\daleth_\varphi, \daleth_\varphi] = 0 \Longleftrightarrow (\overline{\partial}_K + \daleth_\varphi)^2 = 0,$$
where, on the first equivalence, we use that $\overline{\partial}_K^2 = 0$ and that the center of $\widehat{\mathcal{A}}$ is zero, the equivalence in the middle is given by Corollary 2.4.5, the last equivalence is a direct consequence of the definitions.

We end this section with the following lemma.

Lemma 2.4.7. *Let (M, K) be a compact **D**-manifold, and let $\hat{K}$ defined by:*
$$\hat{K} = (\mathrm{Id} + \varphi)K(\mathrm{Id} + \varphi)^{-1}, \quad \textit{where} \ \ K\varphi + \varphi K = 0,\ \det(\mathrm{Id} + \varphi) \neq 0. \tag{2.4.16}$$
Then:
$$(\mathrm{Id} + \varphi)^{-1} N_{\hat{K}}((\mathrm{Id} + \varphi)(\bar{Z}), (\mathrm{Id} + \varphi)(\bar{W})) = 4(\mathrm{Id} - \varphi^2)^{-1}\left(\overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]]\right)(\bar{Z}, \bar{W})$$
for $Z, W \in T^{1,0}M$.

Proof. If $Z, W \in T^{1,0}M$, then:
$$\begin{aligned} N_K(\bar{Z}, \bar{W}) &= [\bar{Z}, \bar{W}] + [K\bar{Z}, K\bar{W}] - K[K\bar{Z}, \bar{W}] - K[\bar{Z}, K\bar{W}] \\ &= 2[\bar{Z}, \bar{W}] + 2\tau K[\bar{Z}, \bar{W}]. \end{aligned}$$
Moreover we have:
$$\begin{aligned} (\mathrm{Id} + \varphi)\frac{1}{2}(X - \tau K X) &= \frac{1}{2}((\mathrm{Id} + \varphi)X - \tau(\mathrm{Id} + \varphi)KX) \\ &= \frac{1}{2}((\mathrm{Id} + \varphi)X - \tau \hat{K}(\mathrm{Id} + \varphi)X). \end{aligned}$$
Using this fact together with $(\mathrm{Id} - \varphi^2) = (\mathrm{Id} - \varphi)(\mathrm{Id} + \varphi)$ we have:
$$\begin{aligned} &(\mathrm{Id} - \varphi^2)(\mathrm{Id} + \varphi)^{-1} N_{\hat{K}}((\mathrm{Id} + \varphi)(\bar{Z}), (\mathrm{Id} + \varphi)(\bar{W})) \\ &\quad = (\mathrm{Id} - \varphi) N_{\hat{K}}((\mathrm{Id} + \varphi)(\bar{Z}), (\mathrm{Id} + \varphi)(\bar{W})) \\ &\quad = 2(\mathrm{Id} - \varphi)\Big\{[(\mathrm{Id} + \varphi)(\bar{Z}), (\mathrm{Id} + \varphi)(\bar{W})] \\ &\qquad + \tau\hat{K}[(\mathrm{Id} + \varphi)(\bar{Z}), (\mathrm{Id} + \varphi)(\bar{W})]\Big\}. \end{aligned}$$

By the definition of $\hat{K}$ (see equation (2.4.16)) we have $\hat{K} = (\mathrm{Id}+\varphi)(\mathrm{Id}-\varphi)^{-1}K$, and combining this with the previous equation we get

$$\begin{aligned}(\mathrm{Id}-\varphi^2)(\mathrm{Id}+\varphi)^{-1}N_{\hat{K}}((\mathrm{Id}+\varphi)(\bar{Z}),(\mathrm{Id}+\varphi)(\bar{W}))\\ = 2\Big\{(\mathrm{Id}-\varphi)[(\mathrm{Id}+\varphi)(\bar{Z}),(\mathrm{Id}+\varphi)(\bar{W})]\\ +\tau(\mathrm{Id}+\varphi)K[(\mathrm{Id}+\varphi)(\bar{Z}),(\mathrm{Id}+\varphi)(\bar{W})]\Big\}.\end{aligned}$$

Now applying the facts that $\widetilde{\varphi}(X) = \frac{1}{2}(\varphi X - \tau\varphi K X)$ and that $\widetilde{\varphi}(\bar{Z}) = \varphi(\bar{Z})$, a straightforward computation shows that:

$$\begin{aligned}&(\mathrm{Id}-\varphi^2)(\mathrm{Id}+\varphi)^{-1}N_{\hat{K}}((\mathrm{Id}+\varphi)(\bar{Z}),(\mathrm{Id}+\varphi)(\bar{W}))\\ &= 4\,\overline{\partial}_K\,\widetilde{\varphi}(\bar{Z},\bar{W}) + 4[\widetilde{\varphi}\bar{Z},\widetilde{\varphi}\bar{W}]\\ &\quad - 2\varphi\left([\bar{Z},\varphi\bar{W}] + [\varphi\bar{Z},\bar{W}] - \tau K[\bar{Z},\varphi\bar{W}] - \tau K[\varphi\bar{Z},\bar{W}]\right)\\ &= 4\,\overline{\partial}_K\,\widetilde{\varphi}(\bar{Z},\bar{W}) + 4[\widetilde{\varphi}\bar{Z},\widetilde{\varphi}\bar{W}] - 4\widetilde{\varphi}([\bar{Z},\varphi\bar{W}] + [\varphi\bar{Z},\bar{W}])\\ &= 4\left(\overline{\partial}_K\,\widetilde{\varphi} + \frac{1}{2}[[\widetilde{\varphi},\widetilde{\varphi}]]\right)(\bar{Z},\bar{W})\end{aligned}$$

where, in the last equivalence, we use the equation (2.3.5). This ends the proof of the lemma. □

Note that this Lemma 2.4.7 is the analogous of 2.3.10 (see [61]) written in the **D**-setting. We will need it in Section 2.5, but we put it here since it holds for a general **D**-manifold. We can also merge the Theorem 2.3.9 and Lemmas 2.3.10 and 2.4.7 to get the following:

Remark 2.4.8. Let (M,K) be a compact **D**-manifold, and let $\hat{K}$ defined by:

$$\hat{K} = (\mathrm{Id}+\varphi)K(\mathrm{Id}+\varphi)^{-1}, \quad \text{where } K\varphi + \varphi K = 0,\ \det(\mathrm{Id}+\varphi) \neq 0. \tag{2.4.17}$$

Then the integrability condition $N_{\hat{K}} = 0$ is equivalent to the vanishing of $\overline{\partial}_K\,\varphi + \frac{1}{2}[[\varphi,\varphi]]$, which can be tested either on real vectors $X, Y \in TM$ or on **D**-complexificated vectors $\bar{Z}, \bar{W} \in T^{0,1}$.

2.5 Deformations of CR D-structures

In this section, we are interested in studying the deformations of a **D**-structure on a fixed compact contact manifold (M,ξ). The deformations of strictly pseudo-convex CR-structures have been studied in [22].

As done in the **D**-structures case, we first construct the DGLA $\mathcal{B}_K(\xi)$ of deformations of strictly CR **D**-structures and discuss the integrability condition (Maurer-Cartan equation (2.1.4)). Then we explicit the injection of the DGLA $\mathcal{B}_K(\xi)$ in the space of skew derivations $\mathcal{E}$ and we write an equivalent condition to the integrability.

Let (M,ξ,K) be a strictly CR **D**-structure with $\dim M = 2n+1$ (see Section 1.7 for basic definitions). Fix a contact form α and extend K on the α-symplectization of (M,ξ) (see equations (1.7.8) and (1.7.9)). We can show the following proposition similar to the complex case.

Proposition 2.5.1. *Let (M,ξ) be a contact manifold, α be a contact form and $K \in \mathcal{D}(\xi)$. Then:*

$$\begin{aligned}N_K(X,Y) \in \xi \qquad \forall X,Y \in TN\\ \overline{\partial}_K\,R_\alpha = -\frac{1}{4}N_K(R_\alpha,\cdot).\end{aligned} \tag{2.5.1}$$

Proof. Since $K \in \mathcal{D}(\xi)$, we have that $N_K(X,Y) \in \xi$ for $X, Y \in \xi$ (see (1.7.7)). Moreover, since $i_{R_\alpha}\, d\alpha = 0$ and $\alpha(R_\alpha) = 1$, for every $X \in \xi$ we have

$$0 = d\alpha(R_\alpha, X) = R_\alpha(\alpha(X)) - X(\alpha(R_\alpha)) - \alpha([R_\alpha, X]) = -\alpha([R_\alpha, X])$$

namely $[R_\alpha, X] \in \xi$ for every $X \in \xi$. Furthermore $[R_\alpha, \frac{\partial}{\partial s}] = 0$, i.e. $[R_\alpha, X] = 0$ for every $X \in TN$. Now, since $\alpha(S) = 0$ and $[S, X] = 0$ for all $X \in TN$, we get the first part of the thesis from the definition of $N_K(X,Y)$.
To prove the second part, we calculate (using Definition 2.3.2):

$$\overline{\partial}_K\, R_\alpha(X) = \frac{1}{2}\left([X, R_\alpha] - K[KX, R_\alpha]\right) + \frac{1}{4} N_K(R_\alpha, X).$$

Since $[S, X] = 0$, we get:

$$\begin{aligned} N_K(R_\alpha, X) &= [KR_\alpha, KX] + [R_\alpha, X] - K[KR_\alpha, X] - K[R_\alpha, KX] \\ &= [S, KX] + [R_\alpha, X] - K[S, X] - K[R_\alpha, KX] \\ &= [R_\alpha, X] - K[R_\alpha, KX]. \end{aligned}$$

Hence:

$$\begin{aligned} \overline{\partial}_K\, R_\alpha(X) &= \frac{1}{2}\left([X, R_\alpha] - K[KX, R_\alpha]\right) + \frac{1}{4}\left([R_\alpha, X] - K[R_\alpha, KX]\right) \\ &= \frac{1}{4}\left([X, R_\alpha] - K[KX, R_\alpha]\right) = -\frac{1}{4} N_K(R_\alpha, X). \end{aligned}$$ □

Now on a strictly CR **D**-structure (M, ξ, K) we fix a contact form α and as before we extend K on the α-symplectization of (M, ξ). Define

$$\mathcal{B}^p_K(\xi) := \left\{\gamma \in (\wedge^{0,p}_K(\xi))^{\mathbb{R}} \otimes \xi \mid \overline{\partial}_K\, \gamma \in (\wedge^{0,p+1}_K(\xi))^{\mathbb{R}} \otimes \xi\right\}. \tag{2.5.2}$$

We can prove the following:

Lemma 2.5.2. *Let $\gamma \in (\wedge^{0,p}_K)^{\mathbb{R}} \otimes \xi$. Then $\gamma \in \mathcal{B}^p_K(\xi)$ if and only if for any $X_0, \dots, X_p \in \xi$ we have:*

$$\sum_{j=0}^{p} (-1)^j\, d\alpha(X_j, \gamma(X_0, \dots, \hat{X}_j, \dots, X_p)) = 0, \tag{2.5.3}$$

where α is a contact form for ξ. Consequently, the previous definition of $\mathcal{B}^p_K$ does not depend on α.

Proof. By Proposition 2.5.1 we have that $N_K(X,Y) \in \mathcal{H}(\xi)$. By Remark 1.7.5, we have that $[X,Y] + [KX, KY] \in \mathcal{H}(\xi)$, then by equation (2.3.2) $[[X,Y]] \in \mathcal{H}(\xi)$ for $X, Y \in \mathcal{H}(\xi)$ and $\gamma([[X_j, X_k]], X_0, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_p) \in \xi$. So, given $\gamma \in (\wedge^{0,p}_K(\xi))^{\mathbb{R}} \otimes \xi$ we have that:

$$\begin{gathered} \overline{\partial}_K\, \gamma \in (\wedge^{0,p+1}_K(\xi))^{\mathbb{R}} \otimes \xi \iff \\ \sum (-1)^j (\overline{\partial}_K\, \gamma(X_0, \dots, \hat{X}_j, \dots, X_p))(X_j) \in \xi \iff \\ \sum (-1)^j \frac{1}{2}\Big([X_j, \gamma(X_0, \dots, \hat{X}_j, \dots, X_p)] - K[KX_j, \gamma(X_0, \dots, \hat{X}_j, \dots, X_p)] \\ -\frac{1}{2} N_K(\gamma(X_0, \dots, \hat{X}_j, \dots, X_p), X_j)\Big) \in \xi \end{gathered} \tag{2.5.4}$$

for any $X_0, \dots, X_p \in \xi$. Since $N_K \in \xi$, we see that it is enough to compute the tangential component $[X,Y]_\xi$ of $[X,Y]$ with respect to the decomposition $TM = \xi \oplus \mathbb{R}R_\alpha$ (see Remark 1.7.5). Indeed, we have $d\alpha(X,Y) = -\alpha([X,Y])$ on ξ, then using $\alpha(R_\alpha) = 1$, it follows that

$$[X,Y]_\xi = [X,Y] + d\alpha(X,Y) R_\alpha. \tag{2.5.5}$$

Finally, using (2.5.4) and (2.5.5) we have that $\gamma \in \mathcal{B}^p_K(\xi)$ if and only if the condition (2.5.3) holds.
The last part of the lemma is a consequence of the properties of the Reeb vector field R_α (see Remark 1.7.5). □

Remark 2.5.3. It has to be noted that:

- $\mathcal{B}^0_K(\xi) = 0$.
- $\mathcal{B}^1_K(\xi) = \{\varphi \in \mathrm{End}(\xi) \mid \varphi K + K\varphi = 0,\ \varphi = \varphi^t\}$ where the transposition $\cdot^t$ is taken with respect to the pseudo-metric $g_K(\cdot,\cdot) = d\,\alpha(\cdot, K\cdot)$. The condition $\varphi = \varphi^t$ is due to the compatibility between the contact form α and the curve of **D**-structures for small t (see equation (2.3.22)).

We have the following:

Lemma 2.5.4.

$$\dim \mathcal{B}^p_K(\xi) = 2n\binom{n}{p} - 2\binom{n}{p+1}.$$

Proof. We fix a local basis of ξ. The lemma follows using (2.5.3) and that $d\,\alpha$ is everywhere non degenerate at ξ. □

Via the isomorphism m (see (2.3.3)):

$$\begin{aligned} m : \xi &\longrightarrow \xi^{1,0} \\ X &\longmapsto \frac{1}{2}(X + \tau K X) \end{aligned} \tag{2.5.6}$$

we define $S_p \subset (\wedge^{0,p}_K(\xi))^{\mathbb{R}} \otimes \xi$ as the space

$$S_p := \{\gamma \in (\wedge^{0,p}_K(\xi))^{\mathbb{R}} \otimes \xi \mid \gamma = \sum_{r=1}^{l} \beta_r \wedge \varphi_r\} \tag{2.5.7}$$

where $\beta_1, \ldots, \beta_l \in \wedge^{0,p-1}_K(\xi)$ and $\varphi_1, \ldots, \varphi_l \in \mathcal{B}^1_K(\xi)$. We have:

Lemma 2.5.5. *$S_p \subset \mathcal{B}^p_K(\xi)$ and*

$$\dim S_p = 2\sum_{k=0}^{n-1}\left(\binom{n}{p} - \binom{k}{p}\right)$$

where we use the convention that $\binom{k}{p} = 0$ if $k < p$.

Proof. Take for simplicity $\gamma = \beta \wedge \varphi \in (\wedge^{0,p}_K(\xi))^{\mathbb{R}} \otimes \xi$ (the general case follows by linearity). By Lemma 2.5.2, $\gamma \in \mathcal{B}^p_K(\xi)$ if and only if (2.5.3) holds. By definition we have that:

$$\gamma(X_1, \ldots, X_p) = \frac{1}{p!}\sum_\sigma \mathrm{sgn}(\sigma)\beta(X_{\sigma_1}, \ldots, X_{\sigma_{p-1}})\varphi(X_{\sigma_p})$$

and hence:

$$\begin{aligned} &\sum_j d\,\alpha(X_j, \gamma(X_0, \ldots, \hat{X}_j, \ldots, X_p)) = \\ =&\frac{1}{p!}\sum_\sigma \mathrm{sgn}(\sigma)\beta(X_{\sigma_0}, \ldots, X_{\sigma_{p-1}})\sum_j d\,\alpha(X_j, \varphi(X_{\sigma_p})) = 0, \end{aligned}$$

since $\varphi \in \mathcal{B}^1_K(\xi)$. Hence if $\gamma \in S_p$ then $\gamma \in \mathcal{B}^p_K(\xi)$.
The computation of $\dim S_p$ is easy and it is left to the reader. □

Proposition 2.5.6. *Let $\mathcal{B}^p_K(\xi)$ given by (2.5.2) and S_p given by (2.5.7). Then:*

$$\mathcal{B}^p_K(\xi) = S_p.$$

Proof. We already know that $S_p \subset \mathcal{B}^p_K(\xi)$, then we only need to show that the spaces have the same dimension. Using the fundamental relation

$$\binom{m+1}{n+1} = \binom{m}{n} + \binom{m}{n+1}$$

on the dimension of $\mathcal{B}^p_K(\xi)$ in Lemma 2.5.4 we have:

$$\dim \mathcal{B}^p_K(\xi) = 2n\binom{n}{p} - 2\binom{n}{p+1} = 2n\binom{n}{p} - 2\left(\binom{n-1}{p} + \binom{n-1}{p+1}\right).$$

Iterating, we get:

$$\dim \mathcal{B}^p_K(\xi) = 2n\binom{n}{p} - 2\sum_{k=0}^{n-1}\binom{k}{p} = 2\sum_{k=0}^{n-1}\left(\binom{n}{p} - \binom{k}{p}\right)$$

and by Lemma 2.5.5 we obtain that $\dim \mathcal{B}^p_K(\xi) = \dim S_p$. □

Now we are able to construct the DGLA of deformations of a strictly CR **D**-structure. Set

$$\mathcal{B}_K(\xi) := \bigoplus_{p\in\mathbb{Z}} \mathcal{B}^p_K(\xi).$$

Then we can show the following:

Theorem 2.5.7. *Let (M,ξ) be a compact contact manifold endowed with a strictly CR **D**-structure K and let $\overline{\partial}_K$ be as in Definition 2.3.2. Then:*

$$\overline{\partial}^2_K = 0 \ \text{on}\ \mathcal{B}_K(\xi). \tag{2.5.8}$$

Proof. By Proposition 2.5.6 it is enough to prove (2.5.8) on $\mathcal{B}^1_K(\xi)$. Recall (see Remark 2.5.3) that $\varphi \in \mathcal{B}^1_K(\xi)$ if and only if $K\varphi + \varphi K = 0$ and $\varphi = \varphi^t$ with respect to g_K. We have that:

$$d\,\alpha(\varphi(X), KY) = g_K(\varphi(X), Y) = g_K(X, \varphi(Y)).$$

Hence we have:

$$g_K(X, \varphi(Y)) = d\,\alpha(X, K\varphi(Y)) = -\,d\,\alpha(KX, \varphi(Y)) = d\,\alpha(\varphi(Y), KX).$$

By remark 1.7.5 and since $d\,\alpha(X,Y) = -\alpha([X,Y])$ for $X, Y \in \xi$ it follows that:

$$[KX, \varphi(Y)] + [\varphi(X), KY] \in \mathcal{H}(\xi) \tag{2.5.9}$$

for $X, Y \in \mathcal{H}(\xi)$. A straightforward computation, taking into account (2.5.9) and the fact that N_K vanishes on $\mathcal{H}(\xi)$, shows that $\overline{\partial}^2_K\,\varphi(X,Y,Z) = 0$ for all $X, Y, Z \in \mathcal{H}(\xi)$. Hence $\overline{\partial}^2_K = 0$. □

Summing up, in view of the isomorphism m (see (2.5.6)), $\mathcal{B}^p_K(\xi)$ can be viewed as the space:

$$\{\gamma \in \wedge^{0,p}_K(\xi) \otimes \xi^{0,1} \mid \overline{\partial}_K\,\gamma \in \wedge^{0,p+1}_K(\xi) \otimes \xi^{0,1}\}$$

or, equivalently (by Lemma 2.5.2), as:

$$\{\gamma \in \wedge_K^{0,p}(\xi) \otimes \xi^{0,1} \mid \sum_{j=0}^{p} (-1)^j \, d\alpha(\bar{Z}_j, \gamma(\bar{Z}_1, \ldots, \widehat{\bar{Z}_j} \ldots, \bar{Z}_p)) = 0,$$
$$\forall\, Z_0, \ldots, Z_p \in \xi^{0,1}\},$$

and hence we have that $\varphi \in \mathcal{B}_K^1(\xi)$ if and only if

$$0 = d\alpha(\varphi(\bar{Z}), \bar{W}) + d\alpha(\bar{Z}, \varphi(\bar{W}))$$

for all $Z, W \in \xi^{1,0}$. Thus, by Remark 1.7.5 we have:

$$[\varphi(\bar{Z}), \bar{W}] + [\bar{Z}, \varphi(\bar{W})] \in \xi^{1,0} \oplus \xi^{0,1}.$$

Now for any $\varphi \in \mathcal{B}_K^p(\xi)$ we want to define a p-degree skew derivation $\daleth_\varphi : \wedge_K^{0,*}(\xi) \to \wedge_K^{0,*}(\xi)$. We proceed in the same way as we did in Section 2.4, namely:

For $(p = 1)$ take $\varphi \in \mathcal{B}_K^1(\xi)$.

- If $q = 0$ we define $\daleth_\varphi$ as follows (set f a smooth function and $Z \in \xi^{1,0}$):
$$(\daleth_\varphi f)(\bar{Z}) := \partial f(\varphi(\bar{Z})) = \varphi(\bar{Z})f = (\varphi\bar{Z}f).$$
If $q = 1$ we define $\daleth_\varphi$ as follows (set $\gamma \in \wedge_K^{0,1}(\xi)$ and $Z, W \in \xi^{1,0}$):
$$(\daleth_\varphi \gamma)(\bar{Z}, \bar{W}) := \varphi\bar{Z}\gamma(\bar{W}) - \varphi\bar{W}\gamma(\bar{Z}) - \gamma([\varphi\bar{Z}, \bar{W}] - [\bar{Z}, \varphi\bar{W}]).$$
- Since
$$\daleth_\varphi(f\gamma) = (\daleth_\varphi f) \wedge \gamma + f \daleth_\varphi \gamma,$$
we can extend $\daleth_\varphi$ as a 1-degree skew derivation, i.e. if $\beta = \sum \beta_1 \wedge \cdots \wedge \beta_q \in \wedge_K^{0,q}(\xi)$, $q > 1$ we have:
$$\daleth_\varphi \beta := \sum_{j=1}^{q} (-1)^{(1+j)} \beta_1 \wedge \cdots \wedge \daleth_\varphi \beta_j \wedge \cdots \wedge \beta_q.$$

For $(p > 1)$ we write $\psi \in \mathcal{B}_K^p(\xi)$ as a sum of elements of the form $\alpha_i \wedge \varphi_i$ with $\alpha_i \in \wedge_K^{0,p-1}(\xi)$ and $\varphi_i \in \mathcal{B}_1(\xi)$ then we set
$$\daleth_{\alpha_i \wedge \varphi_i} := \alpha_i \wedge \daleth_{\varphi_i}$$
and $\daleth_\psi$ is just the sum of the expression above.

Now we consider the same operators $[[\cdot, \cdot]]$ and $\overline{\partial}_K$ in $\mathcal{B}_K(\xi)$ as defined before (see Definition 2.3.4 and Definition 2.3.5). Observe that the definitions are well posed, in fact we have the following:

Lemma 2.5.8. *Let $\varphi, \psi \in \mathcal{B}_K^1(\xi)$. Then $[[\varphi, \psi]] \in \mathcal{B}_K^2(\xi)$.*

Proof. A straightforward computation together with the Jacobi identity shows that $[[\varphi, \psi]]$ verifies the conditions of Lemma 2.5.2, and hence we get the proof of lemma. □

Again, let us introduce the space $\mathcal{E}$ of skew-symmetric derivations on $\wedge_K^{0,*}(\xi)$, with the bracket defined on homogeneous elements by

$$[F, G] := F \circ G - (-1)^{|F||G|} G \circ F$$

and set

$$\delta_K F := [\overline{\partial}_K, F],$$

then we have that $(\mathcal{E},\ [\cdot, \cdot],\ \delta_K)$ is a DGLA.

Now we define a map q between $(\mathcal{B}_K(\xi),\ [[\cdot, \cdot]],\ \overline{\partial}_K)$ and $(\mathcal{E},\ [\cdot, \cdot],\ \delta_K)$ as done in (2.4.6) and we get the analogous to the Theorem 2.4.2.

Theorem 2.5.9. *The map q defined by:*

$$q : \mathcal{B}_K(\xi) \longrightarrow \mathcal{E}, \qquad \varphi \longmapsto \urcorner_\varphi,$$

is a DGLA homomorphism, i.e. q is an injective map satisfying:

$$[q(\varphi), q(\psi)] = q([[\varphi, \psi]]) \qquad (\textit{equivalently } [\urcorner_\varphi, \urcorner_\psi] = \urcorner_{[[\varphi,\psi]]})$$

and

$$\delta_K q(\varphi) = q(\overline{\partial}_K \varphi) \qquad (\textit{equivalently } \delta_K \urcorner_\varphi = \urcorner_{\overline{\partial}_K \varphi}).$$

Proof. The proof proceeds as the one of Theorem 2.4.2. □

Corollary 2.5.10. *Let $q : \mathcal{B}_K(\xi) \to \mathcal{E}$ given as in Theorem 2.5.9. Then:*

1. $(\operatorname{im} q,\ [\cdot,\cdot],\ \delta_K)$ *is a DGLA;*
2. $(\mathcal{B}_K(\xi),\ [[\cdot,\cdot]],\ \overline{\partial}_K)$ *is a DGLA isomorphic to the previous one;*
3. *we have that*
$$\varphi \in \mathfrak{MC}(\mathcal{B}_K(\xi)) \Longleftrightarrow \urcorner_\varphi \in \mathfrak{MC}(\mathcal{E}),$$
i.e. $\overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] = 0$ *if and only if* $\delta_K \urcorner_\varphi + \frac{1}{2}[\urcorner_\varphi, \urcorner_\varphi] = 0$.

Now we can state the CR version of the Theorem 2.3.9:

Theorem 2.5.11. *Let $K \in \mathcal{D}(\xi)$ be a strictly CR **D**-structure on a compact contact manifold (M,ξ), and let $\hat{K} \in \mathcal{D}(\xi)$ be given by:*

$$\hat{K} = (\operatorname{Id} + \varphi) K (\operatorname{Id} + \varphi)^{-1}, \quad \textit{where} \ \ \varphi K + K \varphi = 0, \quad \varphi^t = \varphi. \tag{2.5.10}$$

Let $\widetilde{\varphi}$ be the operator associated to φ via the isomorphism m:

$$m : \xi \longrightarrow \xi^{0,1}, \qquad X \longmapsto \widetilde{X} = \frac{1}{2}(X + \tau K X).$$

Then

$$N_{\hat{K}} = 0 \Longleftrightarrow \overline{\partial}_K \widetilde{\varphi} + \frac{1}{2}[[\widetilde{\varphi}, \widetilde{\varphi}]] = 0.$$

Proof. The theorem is a consequence of the following Lemma 2.5.12, which is the CR version of Lemma 2.4.7 and whose proof is similar to that one of Lemma 2.4.7 and therefore it is omitted. □

Lemma 2.5.12. *Let $\hat{K} \in \mathcal{D}(\xi)$ given by (2.5.10) and such that* $\det(\operatorname{Id} + \varphi) \neq 0$. *Then:*

$$(\operatorname{Id} + \varphi)^{-1} N_{\hat{K}} \left((\operatorname{Id} + \varphi)(\bar{Z}), (\operatorname{Id} + \varphi)(\bar{W}) \right) = 4(\operatorname{Id} - \varphi^2)^{-1} \left(\overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] \right) (\bar{Z}, \bar{W})$$

for $Z, W \in \xi^{1,0}$.

Remark 2.5.13. Since the center of $\mathcal{B}_K(\xi)$ is zero, setting $\overline{\partial}_\varphi := \overline{\partial}_K + [[\varphi, \cdot]]$ (see equation (2.1.3)) and arguing as in Remark 2.4.6 it follows that:

$$\overline{\partial}_\varphi^2 = 0 \Longleftrightarrow \overline{\partial}_K \varphi + \frac{1}{2}[[\varphi,\varphi]] = 0 \Longleftrightarrow$$
$$\Longleftrightarrow \delta_K \urcorner_\varphi + \frac{1}{2}[\urcorner_\varphi, \urcorner_\varphi] = 0 \Longleftrightarrow (\overline{\partial}_K + \urcorner_\varphi)^2 = 0.$$

2.6 Examples of deformations of CR D-structures

We end this chapter with some examples of CR **D**-structures and their deformations. We focus on 5-dimensional nilmanifolds, proving that there exists a 5-dimensional nilpotent Lie algebra that does not admit any CR **D**-structure (see Proposition 2.6.4).

Example 2.6.1. Let $\mathfrak{h}(3)$ be the 3-dimensional real Heisenberg Lie algebra. Then we can find a basis $\{e^1, e^2, e^3\}$ of $\mathfrak{h}^*$ such that:

$$d e^1 = d e^3 = 0 \qquad d e^2 = e^1 \wedge e^3.$$

Therefore, denoting by $\{X_1, X_2, X_3\}$ the dual basis of $\{e^1, e^2, e^3\}$, we obtain that $[X_1, X_3] = -X_2$ and the other brackets are zero. Let $H(3)$ be the Heisenberg group:

$$H(3) = \left\{ A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and let $M^3 = \Gamma \backslash H(3)$ be any compact quotient of $H(3)$, then $\alpha := e^2$ is a contact form and $\xi = \ker\alpha = \mathrm{Span}\{X_1, X_3\}$, which makes (M^3, ξ) a contact manifold.
Let $K \in \mathrm{End}(\xi)$ be the **D**-structure defined by:

$$K(X_1) = X_3 \qquad K(X_3) = X_1.$$

It is easy to verify that K is a strictly CR **D**-structure on the contact manifold (M^3, ξ).

Now, setting $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, every invariant endomorphism φ of ξ anti-commuting with K and such that $\varphi^t = \varphi$ has the following form:

$$\varphi_{\mathbf{t}} = \begin{pmatrix} t_1 & t_2 \\ -t_2 & -t_1 \end{pmatrix}$$

with respect to the basis of ξ. A computation yields to

$$\overline{\partial}_k \, \widetilde{\varphi_{\mathbf{t}}} = 0 \qquad [[\widetilde{\varphi_{\mathbf{t}}}, \widetilde{\varphi_{\mathbf{t}}}]] = 0$$

then for $|\mathbf{t}| < \varepsilon$ it follows that $K_{\mathbf{t}} = (\mathrm{Id} + \varphi_{\mathbf{t}}) K (\mathrm{Id} + \varphi_{\mathbf{t}})^{-1}$ is a 2-parameter family of strictly CR **D**-structures on (M^3, ξ) with $K_0 = K$.

Example 2.6.2 (Generalized Heisenberg group)**.** Let $H(2n+1)$ denote the $(2n+1)$-dimensional real Heisenberg group, i.e.

$$H(2n+1) = \left\{ A = \begin{pmatrix} 1 & X & z \\ 0 & \mathrm{Id}_n & Y^t \\ 0 & 0 & 1 \end{pmatrix} \mid X, Y \in \mathbb{R}^n, z \in \mathbb{R} \right\}$$

(where ^{t} denote the transposition).

This is a connected, simply-connected nilpotent Lie group. Let $\mathfrak{h}_{2n+1}$ be the Lie algebra of $H(2n+1)$, then we can find a basis $\{e^1, \ldots, e^n, f^1, \ldots, f^n, \alpha\}$ of $\mathfrak{h}^*_{2n+1}$ such that:

$$d e^i = d f^j = 0 \qquad d \alpha = \sum_{i=1}^{n} e^i \wedge f^i.$$

Therefore, denoting by $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ the dual basis of $\{e^1, \ldots, e^n, f^1, \ldots, f^n, \alpha\}$, we obtain that $[X_i, Y_j] = -\delta_{ij} Z$ and the other brackets are zero. In view of the nilpotency of $H(2n+1)$, by Malcev theorem there exists uniform discrete Γ subgroup of H such that $M^{2n+1} = \Gamma \backslash H(2n+1)$ will be a compact quotient and hence a nilmanifold.

Then we get that α is a contact form and $\xi = \ker\alpha = \mathrm{Span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$, namely (M^{2n+1}, ξ) is a contact manifold.

Let $K \in \mathrm{End}(\xi)$ be the **D**-structure defined by:

$$K(X_i) = Y_i \qquad K(Y_i) = X_i.$$

It is easy to verify that K is a strictly CR **D**-structure on the contact manifold (M^{2n+1}, ξ). Moreover, with respect to the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ of ξ, every invariant endomorphism φ of ξ, anti-commuting with K, and such that $\varphi^t = \varphi$ has the following form:

$$\varphi_{\mathbf{t}} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

where A, B are real $n \times n$ symmetric matrices which depend on a $n(n+1)$-parameter family $\mathbf{t} = \{t_1, \ldots, t_{n(n+1)}\}$. A direct computation yields to

$$\overline{\partial}_K \widetilde{\varphi_{\mathbf{t}}} + \frac{1}{2}[[\widetilde{\varphi_{\mathbf{t}}}, \widetilde{\varphi_{\mathbf{t}}}]] = 0.$$

Therefore for $|\mathbf{t}| < \varepsilon$ it follows that $K_{\mathbf{t}} = (\mathrm{Id} + \varphi_{\mathbf{t}})K(\mathrm{Id} + \varphi_{\mathbf{t}})^{-1}$ is a $n(n+1)$-parameter family of strictly CR **D**-structures on (M, ξ) with $K_0 = K$.

Example 2.6.3. Let $\mathfrak{n}$ be a Lie algebra with basis $\{X_1, \ldots, X_5\}$ and let be the dual $\mathfrak{n}^*$ with basis $\{e^1, \ldots, e^5\}$ of invariant 1-forms.

Suppose that the structure equations are:

$$de^1 = de^2 = de^3 = 0,\ de^4 = e^{12},\ de^5 = e^{13} + e^{24},$$

i.e.

$$[X_1, X_2] = -X_4,\ [X_1, X_3] = -X_5,\ [X_2, X_4] = -X_5$$

and all the other brackets vanish. Let N be the simply-connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ and let $M = \Gamma\backslash N$ be any compact quotient of N (by Malcev theorem [59] there exists such a quotient). Hence, using the notation of Section 1.6, we get

$$M := (0, 0, 0, 12, 13 + 24). \tag{2.6.1}$$

We have that $\alpha = e^5$ is a contact form and $\xi = \ker\alpha = \mathrm{Span}\{X_1, \ldots, X_4\}$. Let $K \in \mathrm{End}(\xi)$ be the **D**-structure defined by

$$K(X_1) = X_2,\ K(X_2) = X_1,\ K(X_3) = -X_4,\ K(X_4) = -X_3.$$

It turns out that K is a strictly CR **D**-structure. Setting $\mathbf{t} = (t_1, \ldots, t_6)$, every invariant endomorphism φ anti-commuting with K and such that $\varphi^t = \varphi$ has the following form:

$$\varphi_{\mathbf{t}} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ -t_2 & -t_1 & t_4 & t_3 \\ t_5 & t_6 & -t_1 & t_2 \\ t_6 & t_5 & -t_2 & t_1 \end{pmatrix},$$

with respect to the basis $\{X_1, \ldots, X_4\}$ of ξ. A straightforward computation shows that:

$$\overline{\partial}_K \widetilde{\varphi_{\mathbf{t}}} = 0, \qquad [[\widetilde{\varphi_{\mathbf{t}}}, \widetilde{\varphi_{\mathbf{t}}}]] = 0.$$

Consequently, for $|\mathbf{t}| < \varepsilon$, $K_{\mathbf{t}} = (\mathrm{Id} + \varphi_{\mathbf{t}})K(\mathrm{Id} + \varphi_{\mathbf{t}})^{-1}$ gives rise to a 6-parameter family of strictly CR **D**-structures on (M, ξ), with $K_0 = K$.

We have the property below:

Proposition 2.6.4. *If $M = \Gamma\backslash G$ is a 5-dimensional nilmanifold endowed with an invariant contact form and with an invariant strictly CR* **D***-structure, then the Lie algebra of G can be isomorphic to*

$$(0,0,0,12,13+24) \qquad \text{or to} \qquad (0,0,0,0,12+34), \tag{2.6.2}$$

but it is not isomorphic to $(0,0,12,13,14+23)$.

Proof. We recall (see [19] or [76]) that if a nilmanifold M of dimension 5 admits an invariant contact form, then the Lie algebra of G is isomorphic to one of the following

$$(0,0,12,13,14+23), \quad (0,0,0,12,13+24), \quad (0,0,0,0,12+34). \tag{2.6.3}$$

By the previous examples, we have proved that if a 5-dimensional nilmanifold has Lie algebra isomorphic to $(0,0,0,12,13+24)$ (Example 2.6.3) or to $(0,0,0,0,12+34)$ (Example 2.6.2 with $n=2$), then M admits a strictly CR **D**-structures.
Let $\mathfrak{n}$ be the Lie algebra $(0,0,12,13,14+23)$ and let $\{e^1,\dots,e^5\}$ be a basis of $\mathfrak{n}^*$ with dual basis $\{X_1,\dots,X_5\}$. Now we prove that $\mathfrak{n}$ does not admit an invariant strictly CR **D**-structure. First of all, any generic contact form is proportional to $\alpha = e^5 + a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4$, with $a_i \in \mathbb{R}$. Then, since $X_5 \notin \ker\alpha$, we get $\ker\alpha = \ker e^5 = \xi = \operatorname{span}\{X_1,X_2,X_3,X_4\}$. Note that the structure equations imply that $[X_1,X_j] \neq 0$ for $j = 1,\dots,4$, furthermore, if $\{U_1,\dots,U_4\}$ is an other basis of ξ, then there is an element of such a basis, say U_1, such that

$$[U_1,U_j] \neq 0 \qquad \text{for } j = 2,3,4. \tag{2.6.4}$$

Now suppose that there exists an invariant strictly CR **D**-structure and, without losing of generality, let U_1,U_2 be a basis for $\xi^{1,0}$ and U_3,U_4 be a basis for $\xi^{0,1}$. By the integrability condition (see Remark 1.7.5), we have $[\xi^{1,0},\xi^{1,0}] \subset \xi^{1,0}$ but since $\xi^{1,0}$ is generated by two elements, for the nilpotent condition it must be a nilpotent 1-step group, hence $[\xi^{1,0},\xi^{1,0}] = 0$, and the same holds for $\xi^{0,1}$, and by (2.6.4) we get an absurd. $\square$

Chapter 3

Cohomological properties of D-manifolds

We start this chapter by recalling some cohomological properties of the **D**-complex manifolds, focusing on the definitions of $\partial_\pm$-Dolbeault groups and showing that these groups are not finite-dimensional. As a consequence, a **D**-complex version of the $\partial\overline{\partial}$-Lemma can not hold (see Section 3.1).

In Section 3.2, we recall what a **D**-complex structure is, we introduce the problem of studying **D**-complex subgroups of cohomology and we introduce the concept of $\mathcal{C}^\infty$-pure-and-full **D**-complex structure to mean a structure inducing a **D**-complex decomposition in cohomology.

The relations between $\mathcal{C}^\infty$-pureness, $\mathcal{C}^\infty$-fullness, pureness and fullness is the argument of Section 3.3. We also use these relations to study the **D**-complex decompositions in (co)-homology for product manifolds (we prove that every manifold given by the product of two equi-dimensional differentiable manifolds is $\mathcal{C}^\infty$-pure-and-full with respect to the natural **D**-complex structure, see Theorem 3.3.4).

In Section 3.4, we introduce analogous definitions at the linear level of the Lie algebra associated to a (quotient of a) Lie group. In particular, we prove that, for a completely-solvable solvmanifold with an invariant **D**-complex structure, the problem of the existence of a **D**-complex cohomological decomposition reduces to such a (linear) decomposition at the level of its Lie algebra (see Proposition 3.4.4).

In Section 3.5, we prove Theorem 3.5.14, saying that every invariant **D**-complex structure on a 4-dimensional nilmanifold is $\mathcal{C}^\infty$-pure-and-full at the 2nd stage. Moreover we show that the dimensions of H^{2+}_K and of H^{2-}_K depend only on the underlying Lie algebra (see Theorem 3.5.18).

In the next Section 3.6, we give some examples to show that the hypotheses we assume on Theorem 3.5.14 can not be dropped out. Moreover, we provide examples showing that admitting **D**-Kähler structures does not imply being $\mathcal{C}^\infty$-pure-and-full (see Proposition 3.6.4).

In Section 3.7, we study deformations of **D**-complex structures, providing an example to prove Theorem 3.7.3 and showing that, in general, jumping for the dimensions of the **D**-complex subgroups of cohomology can occur.

The main results of this chapter have been published by the author and D. Angella in [9]. The Section 3.1.2 and Theorem 3.5.18 are original results by the author.

Notation. We follow the notation of Chapter 1. In particular, for solvmanifold and nilmanifold, we refer to Section 1.6. Moreover, we add that in writing the cohomology of M (which is isomorphic to the cohomology of the complex $(\wedge^\bullet\mathfrak{g}^*,\, d_\mathfrak{g})$ if M is completely-solvable, see [41] and Section 3.4 below), we list the harmonic representatives with respect to the invariant metric $g := \sum_\ell e^\ell \odot e^\ell$ instead of their classes.

3.1 Preliminaries on D-complex cohomology

In this section we first briefly recall some definitions and results on the cohomology of **D**-complex manifold (e.g. a **D**-version of the Dolbeault Lemma), then we show that it is not possible to introduce a $\partial\overline{\partial}$-Lemma for **D**-complex manifolds.

3.1.1 Some simple remarks on D-complex Dolbeault cohomology

Let M be a $2n$-dimensional manifold with an almost **D**-complex structure K. In Chapter 1 we have seen that there is a natural decomposition $T^*M = T^*M_+ \oplus T^*M_-$ into the corresponding eigenbundles (see Section 1.3). Therefore, for any $\ell \in \mathbb{N}$, on the space of ℓ-forms on M, we have the decomposition:

$$\begin{aligned} \wedge^\ell M := \wedge^\ell (T^*M) \;&=\; \wedge^\ell (T^*M_+ \oplus T^*M_-) \\ &= \bigoplus_{p+q=\ell} \wedge^p (T^*M_+) \otimes \wedge^q (T^*M_-) \;=:\; \bigoplus_{p+q=\ell} \wedge^{p,q}_{+\,-} M \end{aligned} \tag{3.1.1}$$

where, for any $p, q \in \mathbb{N}$, the natural extension of K on $\wedge^\bullet M$ acts on $\wedge^{p,q}_{+\,-} M := \wedge^p (T^*M_+) \otimes \wedge^q (T^*M_-)$ as $(+1)^p (-1)^q \,\mathrm{Id}$. In particular, for any $\ell \in \mathbb{N}$,

$$\wedge^\ell M \;=\; \underbrace{\bigoplus_{p+q=\ell,\; q \text{ even}} \wedge^{p,q}_{+\,-} M}_{=:\, \wedge^{\ell\,+}_K M} \oplus \underbrace{\bigoplus_{p+q=\ell,\; q \text{ odd}} \wedge^{p,q}_{+\,-} M}_{=:\, \wedge^{\ell\,-}_K M} \tag{3.1.2}$$

where

$$K\lfloor_{\wedge^{\ell\,+}_K M} \;=\; \mathrm{Id} \qquad \text{and} \qquad K\lfloor_{\wedge^{\ell\,-}_K M} \;=\; -\,\mathrm{Id}\;. \tag{3.1.3}$$

If an (integrable) **D**-complex structure K is given, then the exterior differential splits as $d \;=\; \partial_+ + \partial_-$ (see Section 1.3).

In particular, the condition $d^2 = 0$ gives

$$\begin{cases} \partial_+^2 &=\; 0 \\ \partial_+ \partial_- + \partial_- \partial_+ &=\; 0 \\ \partial_-^2 &=\; 0 \end{cases} \tag{3.1.4}$$

and hence one could define the ∂_+-***D****-Dolbeault cohomology* as

$$H^{\bullet,\bullet}_{\partial_+}(M;\mathbb{R}) \;:=\; \frac{\ker \partial_+}{\mathrm{im}\, \partial_+}\;, \tag{3.1.5}$$

see [53]. Unfortunately, one can not hope to adjust the Hodge theory of the complex case to this non-elliptic context, as we show with the following example (see also Proposition 3.1.11 below).

Example 3.1.1. Take M_1 and M_2 two differentiable manifolds having the same dimension: then, $M_1 \times M_2$ has a natural **D**-complex structure whose eigenbundles decomposition corresponds to the decomposition $T(M_1 \times M_2) = TM_1 \oplus TM_2$; it is straightforward to compute that the space $H^{0,0}_{\partial_+}(M_1 \times M_2)$ of ∂_+-closed functions on $M_1 \times M_2$ is not finite-dimensional, being

$$H^{0,0}_{\partial_+}(M_1 \times M_2) \;\simeq\; \mathcal{C}^\infty (M_2)\;.$$

Obviously, on an (integrable) **D**-complex manifold (M, K), the exterior differential splits also as $d = \partial + \overline{\partial}$ (see Section 1.3), and the condition $d^2 = 0$ gives $\partial^2 = 0$ and $\overline{\partial}^2 = 0$, thus we could also define the *$\overline{\partial}$-**D**-Dolbeault cohomology* as

$$H^{\bullet,\bullet}_{\overline{\partial}}(M;\mathbb{R}) := \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}\,, \tag{3.1.6}$$

(see [53]). These operators, and these cohomological groups, are related to each others (see Theorem 3.1.4 below), and also with this cohomology one can not hope to adjust the Hodge theory of the complex case to this context.

Indeed, all these operators, namely ∂_+, ∂_-, ∂ and $\overline{\partial}$ verify a Dolbeault Lemma, as shown in [53] and in [20].

Lemma 3.1.2 (**D**-Dolbeault Lemma for ∂_+ and ∂_- [20, Lemma 1]). *Let $U \cong U^+ \times U^-$ be an open set such that $U^\pm$ are simply-connected. Then the equation $\partial_\pm\, \theta^\pm = 0$ implies the existence of a real-valued function $F^\pm$ such that*

$$\theta^\pm = \partial_\pm\, F^\pm\,. \tag{3.1.7}$$

The function $F^\pm$ is unique up to addition of a real-valued function $f_\mp$ which satisfies $\partial_\pm\, f_\mp = 0$.

Proof. The uniqueness statement is obvious. To prove the existence, suppose e.g. that $\partial_+\, \theta^+ = 0$ on $U \cong U^+ \times U^-$. Then we can define a function F^+ on $U^+ \times U^- \cong U$ by

$$F^+(z_+, z_-) := \int_\gamma \theta^+\, d\, s\,, \tag{3.1.8}$$

where $z_\pm := (z^1_\pm, \ldots, z^n_\pm)$ and the integration is over any path γ from $(0, z_-)$ to (z_+, z_-), (where s is its arc length) contained in $U^+ \times \{z_-\}$. The condition $\partial_+\theta^+ = 0$ ensures that the integral is path independent. In fact, it implies that the one-form $\theta^+|_{U^+\times\{z_-\}}$ is closed and hence exact, since U^+ is simply connected. □

By Proposition 1.3.5, it follows that each point in an arbitrary **D**-complex manifold (M, K) has a neighbourhood $U \cong U^+ \times U^- \subset M$. This leads us to the following theorem, which state that, locally, a ∂_+-closed $(p+, q-)$-form (with $p \geq 1$) is ∂_+ -exact (and analogously for ∂_-).

Theorem 3.1.3 ([53, Theorem 1.2.8]). *Let (M, K) be a **D**-complex manifold, and let $U \cong U^+ \times U^- \subset M$ be an open set such that $U^\pm$ are simply-connected.*

Then any ∂_+-closed form $\varphi \in \wedge^{p,q}_{+,-}(U)$, $p \geq 1$ is ∂_+-exact. Likewise, any ∂_--closed form $\psi \in \wedge^{p,q}_{+,-}(U)$, $q \geq 1$ is ∂_--exact.

It has to be noted that such a result holds for (**D**-complexificated) (p, q)-forms too. Indeed, using the isomorphism described in equation (1.2.5), it is possible to relate the $\partial_\pm$-Dolbeault Lemma with the one for the operators ∂, $\overline{\partial}$ in the following way:

Proposition 3.1.4 ([53, Proposition 1.2.10]). *For a **D**-complex manifold (M, K), there is an $\mathbb{R}$-linear isomorphism:*

$$\begin{aligned} j : \wedge^{p,q}_{+,-}(M) \times \wedge^{q,p}_{+,-}(M) &\longrightarrow \wedge^{p,q}_{K}(M) \\ (\varphi, \psi) &\longmapsto \frac{1}{2}(1+\tau)\varphi + \frac{1}{2}(1-\tau)\psi = e\varphi + \bar{e}\psi, \end{aligned} \tag{3.1.9}$$

such that the following diagram commutes:

$$\begin{tikzcd} \wedge^{p,q}_{+,-}(M) \times \wedge^{q,p}_{+,-}(M) \arrow[r, "j"] \arrow[d, "\partial_- \times \partial_+"'] & \wedge^{p,q}_{K}(M) \arrow[d, "\overline{\partial}"] \\ \wedge^{p,q+1}_{+,-}(M) \times \wedge^{q+1,p}_{+,-}(M) \arrow[r, "j"] & \wedge^{p,q+1}_{K}(M). \end{tikzcd} \tag{3.1.10}$$

This gives us the following:

Corollary 3.1.5 ([53, Corollary 1.2.11])**.** *For $U \cong U^+ \times U^- \subset M$ such that $U^\pm$ are simply-connected, any $\overline{\partial}$-closed form $\varphi \in \wedge^{p,q}_K(M)$, $q \geq 1$, is $\overline{\partial}$-exact.*

With suitable modifications an analogous statement says that any ∂-closed form $\varphi \in \wedge^{p,q}_K(M)$, $p \geq 1$, is ∂-exact.

Remark 3.1.6. Note that the isomorphism constructed in (1.2.5) (see also Proposition 3.1.4) holds for all $p, q \geq 0$, then cohomological properties can be read on the **D**-complexificated (p, q)-forms or on the $(p+, q-)$-forms, and the isomorphism j makes a correspondence between $\partial_\pm$-closed forms and $\overline{\partial}$-closed forms, and between $\partial_\pm$-exact forms and $\overline{\partial}$-exact forms (and similarly between $\partial_\pm$ and ∂).
For example, on a product manifold $M_1 \times M_2$ the Example 3.1.1 and Proposition 3.1.4 tell us that

$$H^{0,0}_{\overline{\partial}}(M_1 \times M_2) \simeq \mathcal{C}^\infty(M_2) , \tag{3.1.11}$$

and also $H^{0,0}_{\overline{\partial}}(M_1 \times M_2)$ is infinite dimensional.

3.1.2 $\partial\,\overline{\partial}$-Lemma for D-structures

It is natural to ask if there are relations between the **D**-complex Dolbeault cohomologies and the classical de-Rham cohomology, and also if there is a **D**-complex version of the $\partial\,\overline{\partial}$-Lemma for complex manifold (also called $d\,d^c$-*Lemma*). This Lemma says that on a compact Kähler manifolds it holds that:

$$\ker \partial \cap \ker \overline{\partial} \cap \operatorname{im} d = \operatorname{im}(\partial\,\overline{\partial}), \tag{3.1.12}$$

(equivalently, it holds $\ker d \cap \ker d^c \cap \operatorname{im} d = \operatorname{im}(d\,d^c)$). A complex structure J on a manifold M (not necessarily Kähler) which satisfies the $\partial\,\overline{\partial}$-Lemma is a complex structure J that verifies (3.1.12) (i.e. every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\,\overline{\partial}$-exact). It turns out that such equation (3.1.12) has interesting consequence on the cohomology groups of the complex manifolds (e.g. for compact complex manifolds satisfying the $\partial\,\overline{\partial}$-*Lemma*, the Bott-Chern cohomology coincides with the Dolbeault cohomology). See [25] or the recent [12] for more results on manifold satisfying the $\partial\,\overline{\partial}$-Lemma.

Coming back to the **D**-setting, the question is if it is possible to have a similar $\partial\,\overline{\partial}$-Lemma by using the operator ∂ and $\overline{\partial}$ (or ∂_+ and ∂_-). Unfortunately, the answer is no, as we shall prove (see Corollary 3.1.12 below). Indeed, we see that a **D**-complex manifold is provided with two differential forms ∂_+ and ∂_-, and there is a classical results that says that if a manifold with a double complex (∂_+, ∂_-) such that $d = \partial_+ + \partial_-$ satisfies the $\partial_+\,\partial_-$-Lemma, then the cohomology groups of de-Rham and of ∂_+-Dolbeault must have the same dimension. By Remark 3.1.6, also a $\partial\,\overline{\partial}$-Lemma can not hold for any compact **D**-complex manifold.

We briefly recall such a result, then we investigate the **D**-complex case. We start with the following Lemma.

Lemma 3.1.7 ([25, Lemma 5.15])**.** *Let* $(K^{\bullet,\bullet}, d', d'')$ *be a double complex of vector spaces (or objects of any abelian category), and let* $(K^{\bullet}, d)$ *be the associated simple complex (i.e.* $d = d' + d''$ *and* $K^n = \sum_{p+q=n} K^{p,q}$*). For each integer* n *the following conditions are equivalent:*

$(a)_n$ *in* K^n *it holds:*

$$\ker d' \cap \ker d'' \cap \operatorname{im} d = \operatorname{im}(d'\, d''); \tag{3.1.13}$$

$(b)_n$ *in* K^n *it holds:*

$$\ker d'' \cap \operatorname{im} d' = \operatorname{im}(d'\, d'') \quad \text{and} \quad \ker d' \cap \operatorname{im} d'' = \operatorname{im}(d'\, d''); \tag{3.1.14}$$

$(c)_n$ *in* K^n *it holds:*

$$\ker d' \cap \ker d'' \cap \left(\operatorname{im} d' + \operatorname{im} d''\right) = \operatorname{im}(d'\, d''); \tag{3.1.15}$$

$(a)^*_{n-1}$ *in* K^{n-1} *it holds:*

$$\operatorname{im} d' \cap \operatorname{im} d'' \cap \ker d = \ker(d'\, d''); \tag{3.1.16}$$

$(b)^*_{n-1}$ *in* K^{n-1} *it holds:*

$$\operatorname{im} d'' + \ker d' = \ker(d'\, d'') \quad \text{and} \quad \operatorname{im} d' + \ker d'' = \ker(d'\, d''); \tag{3.1.17}$$

$(c)^*_{n-1}$ *in* K^{n-1} *it holds:*

$$\operatorname{im} d' + \operatorname{im} d'' + \left(\ker d' \cap \ker d''\right) = \ker(d'\, d''). \tag{3.1.18}$$

Remark 3.1.8. Note that if the above equivalent conditions of Lemma 3.1.7 hold for every n, then the double complex vector space $(K^{\bullet,\bullet}, d', d'')$ satisfies a $d'\, d''$-Lemma, as a consequence of $(a)_n$. Moreover, if it happens, the natural maps in the following commutative diagram are all isomorphisms (see [25, Remark 5.16]):

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC} := \dfrac{\ker d' \cap \ker d''}{\operatorname{im}(d'\, d'')} \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{d'} := \dfrac{\ker d'}{\operatorname{im} d'} \arrow[dr] & H^{\bullet}_{d} := \dfrac{\ker d}{\operatorname{im} d} \arrow[d] & H^{\bullet,\bullet}_{d''} := \dfrac{\ker d''}{\operatorname{im} d''} \arrow[dl] \\
 & H^{\bullet,\bullet}_{A} := \dfrac{\ker(d'\, d'')}{\operatorname{im} d' + \operatorname{im} d''} &
\end{tikzcd} \tag{3.1.19}$$

Remark 3.1.9. Every integrable complex structure J on a manifold gives a double complex, since $d = \partial + \overline{\partial}$, and if (M, J) satisfies any of the equivalent conditions above for all n, then it is said to satisfies the $\partial\,\overline{\partial}$-Lemma, as specified before. In the complex setting the upper cohomology group is known as the *Bott-Chern cohomology group*, the lower one is the *Aeppli cohomology group*, the right (resp. left) ones are the classical Dolbeault cohomology groups and the central cohomology group is the classical de-Rham cohomology group, and we have isomorphism between all of this groups:

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC}(M) \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\overline{\partial}}(M) \arrow[dr] & H^{\bullet}_{dR}(M;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\partial}(M) \arrow[dl] \\
 & H^{\bullet,\bullet}_{A}(M) &
\end{tikzcd} \tag{3.1.20}$$

(Bott-Chern cohomology group has been recently studied by the author jointly with D. Angella and M.G. Franzini in the paper [8]).

Now coming back to **D**-complex structure, we know (see Section 1.3 equation (1.3.20)) that on a **D**-complex manifold we can split the differential in two ways, namely:

$$d = \partial + \overline{\partial} = \partial_+ + \partial_- \,. \tag{3.1.21}$$

Let (M, K) be a **D**-complex manifold, then $(\wedge^{\bullet,\bullet}_{+,-}(M), \partial_+, \partial_-)$ and $(\wedge^{\bullet,\bullet}(M), \partial, \overline{\partial})$ are double complexes and $(\wedge^{\bullet}(M), d)$ is the associated simple complex. We say that M satisfies the *$\partial_+\partial_-$-Lemma* if it satisfies any of the equivalent conditions of Lemma 3.1.7 for all n, with $d' = \partial_+$ and $d'' = \partial_-$ i.e. if it holds:

$$\ker \partial_+ \cap \ker \partial_- \cap \operatorname{im} d = \operatorname{im}(\partial_+ \partial_-). \tag{3.1.22}$$

Other than Dolbeault cohomology, it is possible to introduce the *$\partial_+\partial_-$-Bott-Chern* and *$\partial_+\partial_-$-Aeppli cohomologies*:

$$H^{\bullet,\bullet}_{BC}(M) \;:=\; \frac{\ker \partial_+ \cap \ker \partial_-}{\operatorname{im} \partial_+ \partial_-}\,, \qquad H^{\bullet,\bullet}_{A}(M) \;:=\; \frac{\ker \partial_+ \partial_-}{\operatorname{im} \partial_+ + \operatorname{im} \partial_-}\,, \tag{3.1.23}$$

and similarly we can introduce the ***D**-Bott-Chern* and ***D**-Aeppli cohomologies* using ∂ and $\overline{\partial}$.

Likewise we say that M satisfies the ***D**-complex $\partial\overline{\partial}$-Lemma* if it satisfies any of the equivalent conditions of Lemma 3.1.7 for all n with $d' = \partial$ and $d'' = \overline{\partial}$

Remark 3.1.10. It is possible to introduce the analogous notions of ***D**-complex $\partial\overline{\partial}$-Lemma* if holds:

$$\ker \partial \cap \ker \overline{\partial} \cap \operatorname{im} d = \operatorname{im}(\partial\overline{\partial}), \tag{3.1.24}$$

and of $d\,d^{\mathbf{D}}$-Lemma if holds:

$$\ker d \cap \ker d^{\mathbf{D}} \cap \operatorname{im} d = \operatorname{im}(d\,d^{\mathbf{D}}). \tag{3.1.25}$$

We observe that all these conditions are equivalent, indeed for **D**-complex forms being ∂_+-closed and ∂_--closed is the same as being ∂-closed and $\overline{\partial}$-closed (and it is also the same as being d-closed and $d^{\mathbf{D}}$-closed), while being $\partial_+\partial_-$-exact is equivalent to being $\partial\overline{\partial}$-exact (and equivalent to $d\,d^{\mathbf{D}}$-exact), because of (1.3.20).

If M satisfies the **D**-complex $\partial_+\partial_-$-Lemma then, by Remark 3.1.8, we have that the following maps are all isomorphisms:

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(M) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\partial_+}(M) \arrow[dr] & H^{\bullet}_{dR}(M;\mathbb{R}) \arrow[d] & H^{\bullet,\bullet}_{\partial_-}(M) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(M) & \end{tikzcd} \tag{3.1.26}$$

Note that by Remark 3.1.6 the same will happen using ∂ and $\overline{\partial}$ instead of $\partial_\pm$.

In particular, a compact **D**-complex manifold satisfying the $\partial_+\partial_-$-Lemma must have ∂_+-**D**-Dolbeault cohomology isomorphic to the de-Rham cohomology, and hence finite dimensional, but there are not examples of such manifolds. In fact we have:

Proposition 3.1.11. *In a $2n$-dimensional $\boldsymbol{D}$-manifold (M, K), the $\boldsymbol{D}$-complex Dolbeault cohomology groups $H^{p,0}_{\partial_-}(M, \mathbb{R})$ are infinite dimensional for $0 \le p \le n$.*
Analogously, the cohomological groups $H^{0,q}_{\partial_+}(M, \mathbb{R})$ (for $0 \le q \le n$), $H^{0,q}_{\partial}(M, \boldsymbol{D})$ (for $0 \le q \le n$) and $H^{p,0}_{\overline{\partial}}(M, \boldsymbol{D})$ (for $0 \le p \le n$) are not finite-dimensional.

Proof. The idea is basically to construct an example of form which is ∂_--closed but which can not be ∂_--exact (and the same for the other operators). We start by showing that the space of ∂_--closed functions is infinite dimensional, by arguing as in Example 3.1.1.

Let $U \cong U^+ \times U^- \subset M$ be an open set such that $U^\pm$ are simply-connected, and let (z^i_+, z^i_-) be the adapted-coordinates (as shown in the Proposition 1.3.5 every **D**-complex manifold has locally this product structure). It is easy to see that every $\mathcal{C}^\infty_c(U)$-function $f : U \to \mathbb{R}$ such that $f = f(z_+)$ is a ∂_--closed function (by a $\mathcal{C}^\infty_c(U)$-function we mean the set of smooth functions with compact support $K := \mathrm{supp}(f) \subset U$). However, since functions are 0-forms, there may not exist any form α such that $\partial_-\,\alpha = f$. Now we extend f to a global function $\widetilde{f}$ on M by setting $\widetilde{f} = 0$ out of the support $K := \mathrm{supp}(f)$ and $\widetilde{f} = f$ on K.
Using the isomorphism (1.2.5) we easily see that $e\widetilde{f}$ is a **D**-holomorphic function (and it is not $\overline{\partial}$-exact). Analogously, every smooth function $g \in \mathcal{C}^\infty_c(U)$ such that $g = g(z_-)$ can be extended to a global function $\widetilde{g}$ on M such that $\partial_+\,\widetilde{g} = 0$, and again $\bar{e}\widetilde{g}$ is a ∂-closed function.

Now it is easy to extend such a construction to p-forms. In the local coordinates (z^i_+, z^i_-) on $U \cong U^+ \times U^- \subset M$, let $\alpha = d\,z^+_{i_1} \wedge \cdots \wedge d\,z^+_{i_p}$ be a $(p+, 0-)$-form on U, where suppose $1 \le i_1 \le \ldots \le i_p \le n$, and let $\widetilde{f} = f(z_+)$ be as before. Then we easily have that $\widetilde{f}\alpha$ is a ∂_--closed $(p+, 0-)$-form, but it is not exact. Similarly, $e\widetilde{f}\alpha$ is a $\overline{\partial}$-closed non-$\overline{\partial}$-exact $(p, 0)$-form, and setting $\beta = d\,z^-_{i_1} \wedge \cdots \wedge d\,z^-_{i_p}$, we have that $\widetilde{g}\beta$ is a ∂_+-closed $(0+, p-)$-form, but it is not exact, as well as $\bar{e}\widetilde{g}\beta$ is a ∂-closed $(0, p)$-form but not ∂-exact. □

We summarize on the following:

Corollary 3.1.12. *Any compact $\boldsymbol{D}$-complex manifold does not satisfy the $\boldsymbol{D}$-complex $\partial\overline{\partial}$-Lemma.*

The fact that some groups $H^{\bullet,\bullet}_{\partial_+}(M, \mathbb{R})$ are infinite dimensional takes us to look for cohomological properties related to the **D**-complex structure in some finite subgroup of H^ℓ_{dR}. For this reason we will study subgroups of de-Rham cohomology in a compact almost **D**-manifold.

3.2 D-complex subgroups of cohomology and of homology

Let (M, K) be a compact almost **D**-complex manifold and let $2n := \dim M$.

The problem we are considering is when the decomposition

$$\wedge^\bullet M \;=\; \bigoplus_{p,q} \wedge^{p,\,q}_{+\,-}\, M \;=\; \wedge^{\bullet\,+}_K M \oplus \wedge^{\bullet\,-}_K M \tag{3.2.1}$$

moves to cohomology.

The same problem has been studied for the complex case by T.-J. Li and W. Zhang (see, e.g., [55]). While in the complex case this study is motivated by the analysis of symplectic cones (the *tamed one* and the *calibrated one*), in the **D**-complex case we are interested in understand the cohomology of **D**-manifolds, since we have to deal with a **D**-Dolbeault cohomology which is infinite dimensional, as seen in the previous Section 3.1.2 (we refer, e.g., to [55, 26] and the references therein for precise definitions, motivations and results concerning this problem, related ones and the notion of $\mathcal{C}^\infty$-pure-and-fullness in almost complex geometry).

Here and later, we mime T.-J. Li and W. Zhang (see e.g. [55]). For any $p, q, \ell \in \mathbb{N}$, we define

$$H_K^{(p,q)}(M;\mathbb{R}) := \left\{[\alpha] \in H_{dR}^{p+q}(M;\mathbb{R}) \mid \alpha \in \wedge_{+\,-}^{p,\,q} M\right\} \tag{3.2.2}$$

and

$$\begin{aligned} H_K^{\ell+}(M;\mathbb{R}) &:= \left\{[\alpha] \in H_{dR}^{\ell}(M;\mathbb{R}) \mid K\alpha = \alpha\right\} \\ &= \left\{[\alpha] \in H_{dR}^{\ell}(M;\mathbb{R}) \mid \alpha \in \wedge_K^{\ell+} M\right\} , \\ H_K^{\ell-}(M;\mathbb{R}) &:= \left\{[\alpha] \in H_{dR}^{\ell}(M;\mathbb{R}) \mid K\alpha = -\alpha\right\} \\ &= \left\{[\alpha] \in H_{dR}^{\ell}(M;\mathbb{R}) \mid \alpha \in \wedge_K^{\ell-} M\right\} . \end{aligned} \tag{3.2.3}$$

Remark 3.2.1. Note that, if K is integrable, then, for any $\ell \in \mathbb{N}$,

$$H_K^{\ell+} = \bigoplus_{p+q=\ell,\ q \text{ even}} H_K^{(p,q)}(M;\mathbb{R}) \tag{3.2.4}$$

and

$$H_K^{\ell-} = \bigoplus_{p+q=\ell,\ q \text{ odd}} H_K^{(p,q)}(M;\mathbb{R}) . \tag{3.2.5}$$

We introduce the following definitions.

Definition 3.2.2. For $\ell \in \mathbb{N}$, an almost **D**-complex structure K on the manifold M is said to be

- $\mathcal{C}^\infty$*-pure at the ℓ-th stage* if
$$H_K^{\ell+}(M;\mathbb{R}) \cap H_K^{\ell-}(M;\mathbb{R}) = \{0\} ; \tag{3.2.6}$$
- $\mathcal{C}^\infty$*-full at the ℓ-th stage* if
$$H_K^{\ell+}(M;\mathbb{R}) + H_K^{\ell-}(M;\mathbb{R}) = H_{dR}^{\ell}(M;\mathbb{R}) ; \tag{3.2.7}$$
- $\mathcal{C}^\infty$*-pure-and-full at the ℓ-th stage* if it is both $\mathcal{C}^\infty$-pure at the ℓ-th stage and $\mathcal{C}^\infty$-full at the ℓ-th stage, or, in other words, if it satisfies the cohomological decomposition
$$H_{dR}^{\ell}(M;\mathbb{R}) = H_K^{\ell+}(M;\mathbb{R}) \oplus H_K^{\ell-}(M;\mathbb{R}) . \tag{3.2.8}$$

Consider $(M,\, K)$ a compact almost **D**-complex manifold and let $2n := \dim M$. Denote by $D_\bullet M := D^{2n-\bullet} M$ the space of currents on M, that is, the topological dual space of $\wedge^\bullet M$. Define the de Rham homology $H_\bullet(M;\mathbb{R})$ of M as the homology of the complex $(D_\bullet M,\, d)$, where $d\colon D_\bullet M \to D_{\bullet-1} M$ is the dual operator of $d\colon \wedge^{\bullet-1} M \to \wedge^\bullet M$. Note that there is a natural inclusion $T_\cdot\colon \wedge^\bullet M \hookrightarrow D^\bullet M = D_{2n-\bullet} M$ given by

$$\eta \mapsto T_\eta := \int_M \cdot \wedge \eta , \tag{3.2.9}$$

and in particular, one has that $dT_\eta = T_{d\eta}$. Moreover, one can prove that $H_{dR}^\bullet(M;\mathbb{R}) \simeq H_{2n-\bullet}(M;\mathbb{R})$ (see, e.g., [24]).

The action of K on $\wedge^\bullet M$ induces, by duality, an action on $D_\bullet M$ (again denoted by K) and hence a decomposition

$$D_\ell M \;=\; \bigoplus_{p+q=\ell} D^{+\,-}_{p,\,q}\, M \;; \tag{3.2.10}$$

note that, for any $p, q \in \mathbb{N}$, the space $D^{+\,-}_{p,\,q}\, M = D^{n-p,\,n-q}_{+\,-}$ is the dual space of $\wedge^{p,\,q}_{+\,-}\, M$ and that $T.\colon\ \wedge^{p,\,q}_{+\,-}\ M \hookrightarrow D^{p,\,q}_{+\,-}\, M$. As in the smooth case, we set

$$D^K_{\bullet +} M \;:=\; \bigoplus_{q \text{ even}} D^{+\,-}_{\bullet,\,q}\, M \qquad \text{and} \qquad D^K_{\bullet -} M \;:=\; \bigoplus_{q \text{ odd}} D^{+\,-}_{\bullet,\,q}\, M \;, \tag{3.2.11}$$

so $K|_{D^K_{\bullet \pm} M} = \pm \operatorname{Id}$ for $\pm \in \{+, -\}$ and

$$D_\bullet M \;=\; D^K_{\bullet +} M \,\oplus\, D^K_{\bullet -} M \;. \tag{3.2.12}$$

For any $p, q, \ell \in \mathbb{N}$, we define

$$H^K_{(p,q)}(M;\mathbb{R}) \;:=\; \left\{ [\alpha] \in H_{p+q}(M;\mathbb{R}) \;\mid\; \alpha \in D^{+\,-}_{p,\,q}\, M \right\} \tag{3.2.13}$$

and

$$\begin{aligned} H^K_{\ell +}(M;\mathbb{R}) &:= \left\{ [\alpha] \in H_\ell(M;\mathbb{R}) \;\mid\; K\alpha = \alpha \right\} \;, \\ H^K_{\ell -}(M;\mathbb{R}) &:= \left\{ [\alpha] \in H_\ell(M;\mathbb{R}) \;\mid\; K\alpha = -\alpha \right\} \;. \end{aligned} \tag{3.2.14}$$

We introduce the following definitions.

Definition 3.2.3. For $\ell \in \mathbb{N}$, an almost **D**-complex structure K on the manifold M is said to be

- *pure at the ℓ-th stage* if

$$H^K_{\ell +}(M;\mathbb{R}) \cap H^K_{\ell -}(M;\mathbb{R}) \;=\; \{0\} \;; \tag{3.2.15}$$

- *full at the ℓ-th stage* if

$$H^K_{\ell +}(M;\mathbb{R}) \,+\, H^K_{\ell -}(M;\mathbb{R}) \;=\; H_\ell(M;\mathbb{R}) \;; \tag{3.2.16}$$

- *pure-and-full at the ℓ-th stage* if it is both pure at the ℓ-th stage and full at the ℓ-th stage, or, in other words, if it satisfies the homological decomposition

$$H_\ell(M;\mathbb{R}) \;=\; H^K_{\ell +}(M;\mathbb{R}) \,\oplus\, H^K_{\ell -}(M;\mathbb{R}) \;. \tag{3.2.17}$$

3.3 D-complex decompositions in homology and cohomology

As in the complex case (see [55, Proposition 2.30] and also [10, Theorem 2.1]), there are links between $\mathcal{C}^\infty$-pure-and-full and pure-and-full concepts. In fact, using the same arguments as in [55], we get:

Proposition 3.3.1 ([9, Proposition 1.4])**.** *Let $(M,\, K)$ be a $2n$-dimensional compact almost **D**-complex manifold. Then, for every $\ell \in \mathbb{N}$, the following implications hold:*

$$\begin{array}{ccc} \mathcal{C}^\infty\text{-}\textit{full at the } \ell\text{-}\textit{th stage} & \Longrightarrow & \textit{pure at the } \ell\text{-}\textit{th stage} \\ \Downarrow & & \Downarrow \\ \textit{full at the } (2n-\ell)\text{-}\textit{th stage} & \Longrightarrow & \mathcal{C}^\infty\text{-}\textit{pure at the } (2n-\ell)\text{-}\textit{th stage}\,. \end{array} \tag{3.3.1}$$

Proof. Since, for every $p, q \in \mathbb{N}$, we have $H_K^{(p,q)}(M;\mathbb{R}) \overset{T\cdot}{\hookrightarrow} H^K_{(n-p,n-q)}(M;\mathbb{R})$ and $H^\ell_{dR}(M;\mathbb{R}) \simeq H_{2n-\ell}(M;\mathbb{R})$, we get that the two vertical arrows are obvious.
To prove the horizontal arrows, consider $\langle \cdot, \cdot \rangle$ the duality pairing $D_\ell M \times \wedge^\ell M \to \mathbb{R}$ or the induced non-degenerate pairing $H^\ell_{dR}(M;\mathbb{R}) \times H_\ell(M;\mathbb{R}) \to \mathbb{R}$. Let K is $\mathcal{C}^\infty$-full at the ℓ-th stage; suppose that there exists $\mathfrak{c} = [\gamma_+] = [\gamma_-] \in H^K_{\ell+}(M;\mathbb{R}) \cap H^K_{\ell-}(M;\mathbb{R})$ with $\gamma_+ \in D^K_{\ell+}M$ and $\gamma_- \in D^K_{\ell-}M$, then

$$\left\langle H^\ell(M;\mathbb{R}),\, \mathfrak{c} \right\rangle \;=\; \left\langle H_K^{\ell+}(M;\mathbb{R}),\, [\gamma_-] \right\rangle + \left\langle H_K^{\ell+}(M;\mathbb{R}),\, [\gamma_-] \right\rangle \;=\; 0$$

and therefore $\mathfrak{c} = 0$ in $H_\ell(M;\mathbb{R})$; hence K is pure at the ℓ-th stage.
A similar argument proves the bottom arrow. □

Corollary 3.3.2 ([9, Corollary 1.5])**.** *If the almost* ***D****-complex structure K on the manifold M is $\mathcal{C}^\infty$-full at every stage, then it is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

Remark 3.3.3. It follows, from the Corollary above 3.3.2, that on a compact 4-dimensional manifold, being $\mathcal{C}^\infty$-full at the 2-nd stage implies being $\mathcal{C}^\infty$-pure at the 2-nd stage.

Recall (see Example 1.3.7) that, given M_1 and M_2 two differentiable compact manifolds with $\dim M_1 = \dim M_2 = n$, the product $M_1 \times M_2$ inherits a natural **D**-complex structure K, given by the decomposition

$$T\,(M_1 \times M_2) \;=\; TM_1 \,\oplus\, TM_2\;; \tag{3.3.2}$$

in other words, K acts as Id on M_1 and $-\mathrm{Id}$ on M_2. For any $\ell \in \mathbb{N}$, using the Künneth formula (see e.g. [17]), one gets

$$\begin{aligned}
H^\ell_{dR}\,(M_1 \times M_2;\mathbb{R}) \;&\simeq\; \bigoplus_{p+q=\ell} H^p\,(M_1;\mathbb{R}) \,\otimes\, H^q\,(M_2;\mathbb{R}) \\
&=\; \underbrace{\left(\bigoplus_{p+q=\ell,\; q \text{ even}} H^p\,(M_1;\mathbb{R}) \,\otimes\, H^q\,(M_2;\mathbb{R})\right)}_{\subseteq H_K^{\ell+}(M_1 \times M_2;\mathbb{R})} \\
&\quad \oplus \underbrace{\left(\bigoplus_{p+q=\ell,\; q \text{ odd}} H^p\,(M_1;\mathbb{R}) \,\otimes\, H^q\,(M_2;\mathbb{R})\right)}_{\subseteq H_K^{\ell-}(M_1 \times M_2;\mathbb{R})} \\
&\subseteq\; H_K^{\ell+}\,(M_1 \times M_2;\mathbb{R}) \,+\, H_K^{\ell-}\,(M_1 \times M_2;\mathbb{R})\;.
\end{aligned}$$

Therefore, using also Corollary 3.3.2, one gets the following result (compare it with [26, Proposition 2.6]).

Theorem 3.3.4 ([9, Corollary 1.6])**.** *Let M_1 and M_2 be two equi-dimensional compact manifolds. Then the natural* ***D****-complex structure on the product $M_1 \times M_2$ is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

Remark 3.3.5. Note that in the complex case things go different, indeed T. Drăghici, T.-J. Li and W. Zhang show in [26, Proposition 2.6], that the product $(M_1 \times M_2, J_1 + J_2)$, where (M_1, J_1) and (M_2, J_2) are two compact almost complex $\mathcal{C}^\infty$-pure-and-full manifolds such that $b_1(M_1) = 0$ or $b_1(M_2) = 0$, is a $\mathcal{C}^\infty$-pure-and-full manifold. As the authors say, in the complex setting it is not known if the statement holds without the assumption on b_1.

3.4 The cohomology of completely-solvable solvmanifolds and its D-complex subgroups

Now let us focus on the quotient manifolds, in particular on nilmanifolds and solvmanifolds. We will use the same notation as in Section 1.6.

Recall that the translation induces an isomorphism of differential algebras between the space of forms on $\mathfrak{g}^*$ and the space $\wedge^\bullet_{\mathrm{inv}} M$ of invariant differential forms on M:

$$\left(\wedge^\bullet \mathfrak{g}^*,\, d_{\mathfrak{g}}\right) \xrightarrow{\simeq} \left(\wedge^\bullet_{\mathrm{inv}} M,\, d\lfloor_{\wedge^\bullet_{\mathrm{inv}} M}\right) \;; \tag{3.4.1}$$

moreover, by K. Nomizu's and A. Hattori's theorems (see resp. [66] and [41]), if M is a nilmanifold or, more in general, a completely-solvable solvmanifold, then the natural inclusion

$$\left(\wedge^\bullet_{\mathrm{inv}} M,\, d\lfloor_{\wedge^\bullet_{\mathrm{inv}} M}\right) \hookrightarrow \left(\wedge^\bullet M,\, d\right) \tag{3.4.2}$$

is a quasi-isomorphism, hence

$$H^\bullet\left(\wedge^\bullet \mathfrak{g}^*,\, d_{\mathfrak{g}}\right) \;\simeq\; H^\bullet\left(\wedge^\bullet_{\mathrm{inv}} M,\, d\lfloor_{\wedge^\bullet_{\mathrm{inv}} M}\right) \;=:\; H^\bullet_{\mathrm{inv}}(M;\mathbb{R}) \;\xrightarrow{\simeq}\; H^\bullet_{dR}(M;\mathbb{R})\,. \tag{3.4.3}$$

In this section, we study **D**-complex decomposition in cohomology at the level of $H^\bullet(\mathfrak{g};\mathbb{R}) := H^\bullet\left(\wedge^\bullet \mathfrak{g}^*,\, d_{\mathfrak{g}}\right)$.

Recall that the linear almost **D**-complex structure K on $\mathfrak{g}$ defines a splitting $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ into eigenspaces and hence, for every $\ell \in \mathbb{N}$, one gets also the splitting

$$\wedge^\ell \mathfrak{g}^* \;=\; \bigoplus_{p+q=\ell} \wedge^p\left(\mathfrak{g}^+\right)^* \otimes \wedge^q\left(\mathfrak{g}^-\right)^* \;=:\; \bigoplus_{p+q=\ell} \wedge^{p,q}_{+\,-}\, \mathfrak{g}^*\,, \tag{3.4.4}$$

where, for any $p, q \in \mathbb{N}$, one has $K\lfloor_{\wedge^{p,q}_{+\,-}\mathfrak{g}^*} = (+1)^p\,(-1)^q\,\mathrm{Id}$.

As already done for manifolds, we introduce also the splitting of the differential forms into their K-invariant and K-anti-invariant components:

$$\wedge^\bullet \mathfrak{g}^* \;=\; \wedge^{\bullet\,+}_K \mathfrak{g}^* \,\oplus\, \wedge^{\bullet\,-}_K \mathfrak{g}^* \tag{3.4.5}$$

where

$$\wedge^{\bullet\,+}_K \mathfrak{g}^* \;:=\; \bigoplus_{q \text{ even}} \wedge^{\bullet,\,q}_{+\,-}\, \mathfrak{g}^* \qquad \text{and} \qquad \wedge^{\bullet\,-}_K \mathfrak{g}^* \;:=\; \bigoplus_{q \text{ odd}} \wedge^{\bullet,\,q}_{+\,-}\, \mathfrak{g}^*\,. \tag{3.4.6}$$

We define, for any $p, q, \ell \in \mathbb{N}$, the following subspace of the cohomology group $H^{p+q}(\mathfrak{g};\mathbb{R})$:

$$\begin{aligned}
H^{(p,q)}_K(\mathfrak{g};\mathbb{R}) \;&:=\; \left\{[\alpha] \in H^{p+q}(\mathfrak{g};\mathbb{R}) \mid \alpha \in \wedge^{p,q}_{+\,-}\,\mathfrak{g}^*\right\}\,,\\
H^{\ell\,+}_K(\mathfrak{g};\mathbb{R}) \;&:=\; \left\{[\alpha] \in H^{\ell}(\mathfrak{g};\mathbb{R}) \mid K\alpha = \alpha\right\}\,,\\
H^{\ell\,-}_K(\mathfrak{g};\mathbb{R}) \;&:=\; \left\{[\alpha] \in H^{\ell}(\mathfrak{g};\mathbb{R}) \mid K\alpha = -\alpha\right\}\,.
\end{aligned}$$

We have the following definitions.

Definition 3.4.1. For $\ell \in \mathbb{N}$, a linear almost **D**-complex structure on the Lie algebra $\mathfrak{g}$ is said to be

- *linear $\mathcal{C}^\infty$-pure at the ℓ-th stage* if

$$H^{\ell\,+}_K(\mathfrak{g};\mathbb{R}) \cap H^{\ell\,-}_K(\mathfrak{g};\mathbb{R}) \;=\; \{0\}\;;$$

- *linear $\mathcal{C}^\infty$-full at the ℓ-th stage* if

$$H^{\ell\,+}_K(\mathfrak{g};\mathbb{R}) + H^{\ell\,-}_K(\mathfrak{g};\mathbb{R}) = H^\ell(\mathfrak{g};\mathbb{R}) \;;$$

- *linear $\mathcal{C}^\infty$-pure-and-full at the ℓ-th stage* if it is both $\mathcal{C}^\infty$-pure at the ℓ-th stage and $\mathcal{C}^\infty$-full at the ℓ-th stage, or, in other words, if it satisfies the cohomological decomposition

$$H^\ell(\mathfrak{g};\mathbb{R}) = H^{\ell\,+}_K(\mathfrak{g};\mathbb{R}) \oplus H^{\ell\,-}_K(\mathfrak{g};\mathbb{R}) \;.$$

Given a completely-solvable solvmanifold, we want to study the connection between the $\mathcal{C}^\infty$-pure-and-fullness of an *invariant* almost **D**-complex structure and the linear $\mathcal{C}^\infty$-pure-and-fullness of the corresponding *linear* almost **D**-complex structure on the associated Lie algebra.
We need the following result by J. Milnor.

Lemma 3.4.2 ([62, Lemma 6.2])**.** *Any connected Lie group that admits a discrete subgroup with compact quotient is unimodular and in particular admits a bi-invariant volume form η.*

The previous Lemma is used to prove the following result, for which we refer to [32, Theorem 2.1] by A. Fino and G. Grantcharov.

Lemma 3.4.3. *Let $M := \Gamma\backslash G$ be a solvmanifold and call $\mathfrak{g}$ the Lie algebra that is naturally associated to the connected simply-connected Lie group G. Denote by K an invariant almost $\boldsymbol{D}$-complex structure on M or equivalently the associated linear almost $\boldsymbol{D}$-complex structure on $\mathfrak{g}$. Let η be the bi-invariant volume form on G given by Lemma 3.4.2 and suppose that $\int_M \eta = 1$. Define the map*

$$\mu\colon \wedge^\bullet M \to \wedge^\bullet_{inv} M \;, \qquad \mu(\alpha) := \int_M \alpha\lfloor_m \eta(m) \;.$$

One has that

$$\mu\lfloor_{\wedge^\bullet_{inv} M} = \mathrm{Id}\lfloor_{\wedge^\bullet_{inv} M}$$

and that

$$d\,(\mu(\cdot)) = \mu\,(d\,\cdot) \qquad \text{and} \qquad K\,(\mu(\cdot)) = \mu\,(K\cdot) \;.$$

Proof. The proof is similar to that one of [32, Theorem 2.1] and therefore it is omitted. □

Then we can prove the following result (a similar result for almost complex structures has been obtained also by A. Tomassini and A. Fino in [33, Theorem 3.4]).

Proposition 3.4.4 ([9, Proposition 2.4])**.** *Let $M := \Gamma\backslash G$ be a completely-solvable solvmanifold and call $\mathfrak{g}$ the Lie algebra that is naturally associated to the connected simply-connected Lie group G. Denote by K an invariant almost $\boldsymbol{D}$-complex structure on M or equivalently the associated linear almost $\boldsymbol{D}$-complex structure on $\mathfrak{g}$. Then, for every $\ell \in \mathbb{N}$ and for $\pm \in \{+, -\}$, the injective map*

$$H^{\ell\,\pm}_K(\mathfrak{g};\mathbb{R}) \to H^{\ell\,\pm}_K(M;\mathbb{R})$$

induced by translations is an isomorphism.
Furthermore, for every $\ell \in \mathbb{N}$, the linear $\boldsymbol{D}$-complex structure $K \in \mathrm{End}(\mathfrak{g})$ is linear $\mathcal{C}^\infty$-pure (respectively, linear $\mathcal{C}^\infty$-full) at the ℓ-th stage if and only if the $\boldsymbol{D}$-complex structure $K \in \mathrm{End}(TM)$ is $\mathcal{C}^\infty$-pure (respectively, $\mathcal{C}^\infty$-full) at the ℓ-th stage.

Proof. Consider the map $\mu\colon \wedge^\bullet M \to \wedge^\bullet_{\text{inv}} M$ defined in Lemma 3.4.3. The thesis follows from the following three observations.
Since $d\,(\mu(\cdot)) = \mu\,(d\,\cdot)$, one has that μ sends d-closed (respectively, d-exact) forms to d-closed (respectively, d-exact) invariant forms and so it induces a map

$$\mu\colon H^\bullet_{dR}(M;\mathbb{R}) \to H^\bullet_{\text{inv}}(M;\mathbb{R}) \;\simeq\; H^\bullet\left(\mathfrak{g};\mathbb{R}\right) \;.$$

Since $K\left(\mu(\cdot)\right) = \mu\left(K\cdot\right)$, for $\pm \in \{+,\,-\}$, one has

$$\mu\left(\wedge^{\bullet\,\pm}_K M\right) \;\subseteq\; \wedge^{\bullet\,\pm}_{K\,\text{inv}} M \;,$$

where $\wedge^{\bullet\,\pm}_{K\,\text{inv}} M := \wedge^{\bullet\,\pm}_K M \cap \wedge^\bullet_{\text{inv}} M \simeq \wedge^{\bullet\,\pm}_K \mathfrak{g}^*$, hence

$$\mu\left(H^{\bullet\,\pm}_K(M;\mathbb{R})\right) \;\subseteq\; H^{\bullet\,\pm}_K\left(\mathfrak{g};\mathbb{R}\right) \;.$$

Lastly, since M is a completely-solvable solvmanifold, its cohomology is isomorphic to the invariant one (see [41]) and hence the condition $\mu\lfloor_{\wedge^\bullet_{\text{inv}} M} = \mathrm{Id}\lfloor_{\wedge^\bullet_{\text{inv}} M}$ gives that μ is the identity in cohomology. □

3.5 $\mathcal{C}^\infty$-pure-and-fullness of low-dimensional D-complex solvmanifolds

We turn our attention on solvmanifold (see Section 1.6 for notations and basic properties).

Let $(\mathfrak{a},\, [\cdot,\cdot])$ be a Lie algebra and consider the *lower central series* $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ defined, by induction on $n \in \mathbb{N}$, as

$$\begin{cases} \mathfrak{a}^0 & := \quad \mathfrak{a} \\ \mathfrak{a}^{n+1} & := \quad [\mathfrak{a}^n,\, \mathfrak{a}] \quad \text{for } n \in \mathbb{N} \end{cases} \;;$$

note that $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is a descending sequence of Lie algebras:

$$\mathfrak{a} \;=\; \mathfrak{a}^0 \;\supseteq \mathfrak{a}^1 \;\supseteq\; \cdots \;\supseteq\; \mathfrak{a}^{j-1} \;\supseteq\; \mathfrak{a}^j \;\supseteq\; \cdots \;.$$

Recall that the *nilpotent step* of $\mathfrak{a}$ is defined as

$$s\left(\mathfrak{a}\right) \;:=\; \inf\left\{n \in \mathbb{N} \mid \mathfrak{a}^n = 0\right\} \;,$$

so $s\left(\mathfrak{a}\right) < +\infty$ means that $\mathfrak{a}$ is nilpotent.

In particular, if the linear **D**-complex structure K on the Lie algebra $\mathfrak{g}$ induces the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, we consider

$$s^+ \;:=\; s\left(\mathfrak{g}^+\right) \qquad \text{and} \qquad s^- \;:=\; s\left(\mathfrak{g}^-\right) \;;$$

since $\mathfrak{g}^+ \subset \mathfrak{g}$ and $\mathfrak{g}^- \subset \mathfrak{g}$, we have obviously that

$$s^+ \;\leq\; s\left(\mathfrak{g}\right) \qquad \text{and} \qquad s^- \;\leq\; s\left(\mathfrak{g}\right) \;.$$

We start with the following easy lemma.

Lemma 3.5.1 ([9, Lemma 3.5])**.** *Let $\mathfrak{g}$ be a $2n$-dimensional nilpotent Lie algebra, that is, $s\left(\mathfrak{g}\right) < +\infty$. Let K be a linear **D**-complex structure on $\mathfrak{g}$, inducing the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. Then, setting $s^\pm := s\left(\mathfrak{g}^\pm\right)$ for $\pm \in \{+,\,-\}$, we have*

$$1 \;\leq\; s^+ \;\leq\; n-1 \qquad \textit{and} \qquad 1 \;\leq\; s^- \;\leq\; n-1 \;.$$

Proof. The proof follows easily observing that, for $\pm \in \{+,\, -\}$, we have

$$\begin{cases} \dim_{\mathbb{R}} \left(\mathfrak{g}^{\pm}\right)^0 & = & n \\ \dim_{\mathbb{R}} \left(\mathfrak{g}^{\pm}\right)^k & \leq & \max\left\{n-k-1,\, 0\right\} \quad \text{for } k \geq 1 \end{cases},$$

as a consequence of the nilpotent condition and of the integrability property. □

We have the following result, to be compared with previous Theorem 3.3.4.

Proposition 3.5.2 ([9, Proposition 3.6]). *Let $\mathfrak{g}$ be a Lie algebra. If K is a linear **D**-complex structure on $\mathfrak{g}$ with eigenspaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ such that $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$, then K is linear $\mathcal{C}^\infty$-pure-and-full at every stage.*

Proof. Since $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, and since $\mathfrak{g}^+$ commutes with $\mathfrak{g}^-$ (i.e $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$), one can write $\mathfrak{g} = \mathfrak{g}^+ \times \mathfrak{g}^-$. Now using Künneth formula as in Theorem 3.3.4 one gets the thesis. □

Therefore, from Proposition 3.4.4, one gets the following corollary.

Corollary 3.5.3 ([9, Corollary 3.7]). *Let $M :=: \Gamma \backslash G$ be a completely-solvable solvmanifold endowed with an invariant **D**-complex structure K. Call $\mathfrak{g}$ the Lie algebra naturally associated to the Lie group G and consider the linear **D**-complex structure $K \in \mathrm{End}(\mathfrak{g})$ induced by $K \in \mathrm{End}(TM)$. Suppose that the eigenspaces $\mathfrak{g}^+$ and $\mathfrak{g}^-$ of $K \in \mathrm{End}(\mathfrak{g})$ satisfy $[\mathfrak{g}^+,\, \mathfrak{g}^-] = \{0\}$. Then K is $\mathcal{C}^\infty$-pure-and-full at every stage and pure-and-full at every stage.*

Recall the following definition.

Definition 3.5.4. A linear **D**-complex structure on a Lie algebra $\mathfrak{g}$ is said to be *Abelian* if the induced decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ satisfies $[\mathfrak{g}^+,\, \mathfrak{g}^+] = \{0\} = [\mathfrak{g}^-,\, \mathfrak{g}^-]$, namely, $s^+ = 1 = s^-$.
A **D**-complex structure on a solvmanifold is *Abelian*, if its associated linear **D**-complex structure on the corresponding Lie algebra is Abelian.

Remark 3.5.5. Note that every linear **D**-complex structure on a 4-dimensional nilpotent Lie algebra is Abelian, as a consequence of Lemma 3.5.1.

Theorem 3.5.6 ([9, Corollary 3.10]). *Let $\mathfrak{g}$ be a Lie algebra and K be a linear Abelian **D**-complex structure on $\mathfrak{g}$. Then K is linear $\mathcal{C}^\infty$-pure at the 2-nd stage.*

Proof. Denote by $\pi^+ \colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet +}_K \mathfrak{g}^*$ the map that gives the K-invariant component of a given form. Recall that $d\eta := -\eta\left([\cdot,\, \cdot]\right)$ for every $\eta \in \wedge^1 \mathfrak{g}^*$; therefore, since $[\mathfrak{g}^+,\, \mathfrak{g}^+] = 0$ and $[\mathfrak{g}^-,\, \mathfrak{g}^-] = 0$, we have that

$$\pi^+_K \left(\mathrm{im}\left(d\colon\ \wedge^1 \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*\right)\right) \;=\; \{0\}\;.$$

Suppose that there exists $[\gamma^+] = [\gamma^-] \in H^{2\,+}_K(\mathfrak{g};\,\mathbb{R}) \cap H^{2\,-}_K(\mathfrak{g};\,\mathbb{R})$, where $\gamma^+ \in \wedge^{2\,+}_K \mathfrak{g}^*$ and $\gamma^- \in \wedge^{2\,-}_K \mathfrak{g}^*$; let $\alpha \in \wedge^1 \mathfrak{g}^*$ be such that $\gamma^+ = \gamma^- + d\alpha$. Since $\pi^+_K(d\alpha) = 0$, we have that $\gamma^+ = 0$ and hence $[\gamma^+] = 0$, so K is linear $\mathcal{C}^\infty$-pure at the 2-nd stage. □

Remark 3.5.7. We note that the condition of K being Abelian in Theorem 3.5.6 can not be dropped, not even partially. In fact, Example 3.7.1 shows that the Abelian assumption just on $\mathfrak{g}^-$ is not sufficient to have $\mathcal{C}^\infty$-pureness at the 2-nd stage. Another example on a (non-unimodular) solvable Lie algebra is given below.

Example 3.5.8 (*There exists a 4-dimensional (non-unimodular) solvable Lie algebra with a non-Abelian* **D***-complex structure that is not linear $\mathcal{C}^\infty$-pure at the 2-nd stage*)**.** Consider the 4-dimensional solvable Lie algebra defined by

$$\mathfrak{g} := (0,\ 0,\ 0,\ 13+34)\ ;$$

note that $\mathfrak{g}$ is not unimodular, since $d\,e^{124} = e^{1234}$, see Lemma 3.5.12.
Set the linear **D**-complex structure

$$K := (+\ +\ -\ -)\ ;$$

note that K is not Abelian, since $[\mathfrak{g}^+,\, \mathfrak{g}^+] = 0$ but $[\mathfrak{g}^-,\, \mathfrak{g}^-] = \mathbb{R}\langle e_3\rangle \neq \{0\}$.
A straightforward computation yields that $\mathfrak{g}$ is linear $\mathcal{C}^\infty$-full (in fact we have $H^2(\mathfrak{g};\mathbb{R}) = \mathbb{R}\left\langle e^{12},\, e^{34}\right\rangle \oplus \left\langle e^{23},\, e^{13}\right\rangle$ and $H^+_K(\mathfrak{g};\mathbb{R}) = \mathbb{R}\left\langle e^{12},\, e^{34}\right\rangle$, $H^-_K(\mathfrak{g};\mathbb{R}) = \mathbb{R}\left\langle e^{23},\, e^{13}\right\rangle$) but linear non-$\mathcal{C}^\infty$-pure, since

$$H^{2\,+}_K(\mathfrak{g};\mathbb{R}) \ni \left[e^{34}\right] = \left[e^{34} - d\,e^4\right] = -\left[e^{13}\right] \in H^{2\,-}_K(\mathfrak{g};\mathbb{R})$$

and $\left[e^{34}\right] \neq 0$.

As a corollary of Theorem 3.5.6 and using Proposition 3.4.4, we get the following result.

Corollary 3.5.9 ([9, Corollary 3.13])**.** *Let $M :=: \Gamma\backslash G$ be a completely-solvable solvmanifold endowed with an invariant Abelian* **D***-complex structure K. Then K is $\mathcal{C}^\infty$-pure at the 2-nd stage.*

Remark 3.5.10. For a **D**-complex structure on a compact manifold, being Abelian or being $\mathcal{C}^\infty$-pure at the 2-nd stage is not a sufficient condition to have $\mathcal{C}^\infty$-fullness at the 2-nd stage. Indeed, Example 3.6.1 provides a **D**-complex structure on a 6-dimensional solvmanifold that is Abelian, $\mathcal{C}^\infty$-pure at the 2-nd stage and non-$\mathcal{C}^\infty$-full at the 2-nd stage.

Remark 3.5.11. In particular, recalling Remark 3.5.5, invariant **D**-complex structures on 4-dimensional nilmanifolds are $\mathcal{C}^\infty$-pure at the 2-nd stage.

Now we will focus on the $\mathcal{C}^\infty$-fullness property, but while for invariant Abelian **D**-complex structures on higher-dimensional nilmanifolds we can not hope to have, in general, $\mathcal{C}^\infty$-fullness at the 2-nd stage (see Example 3.6.1), for 4-dimensional nilmanifolds we can prove that every invariant **D**-complex structure is in fact also $\mathcal{C}^\infty$-full at the 2-nd stage, see Theorem 3.5.14: to prove this fact, we need the following lemmata.
The first one is a classical result (see, e.g., [39]).

Lemma 3.5.12. *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension n. Then*

$$d\lfloor_{\wedge^{n-1}\mathfrak{g}^*} = 0\,.$$

Lemma 3.5.13. *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension $2n$ endowed with an Abelian linear* **D***-complex structure K. Then*

$$d\lfloor_{\wedge^{n,\,0}_{+\,-}\mathfrak{g}^* \oplus \wedge^{0,\,n}_{+\,-}\mathfrak{g}^*} = 0\,.$$

Proof. Consider the basis

$$\left(\mathfrak{g}^+\right)^* = \mathbb{R}\left\langle e^1,\, \ldots,\, e^n\right\rangle \qquad \text{and} \qquad \left(\mathfrak{g}^-\right)^* = \mathbb{R}\left\langle f^1,\, \ldots,\, f^n\right\rangle$$

where $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ is the decomposition into eigenspaces induced by K. Being K Abelian, the general structure equations are of the form

$$\begin{cases} d\,e^j &= \sum_{h,\,k=1}^n a^j_{hk}\, e^h \wedge f^k \\ d\,f^j &= \sum_{h,\,k=1}^n b^j_{hk}\, e^h \wedge f^k \end{cases}$$

for some $a^j_{hk}, b^j_{hk} \in \mathbb{R}$ and $j \in \{1, \dots, n\}$.
A straightforward computation yields

$$\begin{aligned} d\left(e^1 \wedge \cdots \wedge e^n\right) &= \sum_{\ell=1}^{n} (-1)^{\ell+1} e^1 \wedge \cdots \wedge d e^\ell \wedge \cdots \wedge e^n \\ &= \sum_{\ell=1}^{n} (-1)^{\ell+1} e^1 \wedge \cdots \wedge \sum_{h,k=1}^{n} a^\ell_{hk}\, e^h \wedge f^k \wedge \cdots \wedge e^n \\ &= \sum_{\ell=1}^{n} (-1)^{\ell+1} e^1 \wedge \cdots \wedge \sum_{k=1}^{n} a^\ell_{\ell k}\, e^\ell \wedge f^k \wedge \cdots \wedge e^n \\ &= \sum_{\ell=1}^{n} (-1)^{\ell+1} (-1)^{n-\ell} \left(\sum_{k=1}^{n} a^\ell_{\ell k} \right) e^1 \wedge \cdots \wedge e^n \wedge f^k \\ &= (-1)^{n+1} \sum_{k=1}^{n} \left(\sum_{\ell=1}^{n} a^\ell_{\ell k} \right) e^1 \wedge \cdots \wedge e^n \wedge f^k \end{aligned} \tag{3.5.1}$$

where, for any $k \in \{1, \dots, n\}$,

$$\sum_{\ell=1}^{n} a^\ell_{\ell k} \; = \; 0 \,,$$

since it is the coefficient of

$$\begin{aligned} 0 &= d\left(e^1 \wedge \cdots \wedge e^n \wedge f^1 \wedge \cdots \wedge f^{k-1} \wedge f^{k+1} \wedge \cdots \wedge f^n\right) \\ &= d\left(e^1 \wedge \cdots \wedge e^n\right) \wedge f^1 \wedge \cdots \wedge f^{k-1} \wedge f^{k+1} \wedge \cdots \wedge f^n \\ &\quad + (-1)^n e^1 \wedge \cdots \wedge e^n \wedge d\left(f^1 \wedge \cdots \wedge f^{k-1} \wedge f^{k+1} \wedge \cdots \wedge f^n\right) \\ &= d\left(e^1 \wedge \cdots \wedge e^n\right) \wedge f^1 \wedge \cdots \wedge f^{k-1} \wedge f^{k+1} \wedge \cdots \wedge f^n + (-1)^n \\ &= (-1)^{n+1} \sum_{k=1}^{n} \left(\sum_{\ell=1}^{n} a^\ell_{\ell k} \right) e^1 \wedge \cdots \wedge e^n \wedge f^k \wedge f^1 \wedge \cdots \wedge f^{k-1} \wedge f^{k+1} \wedge \cdots \wedge f^n \end{aligned} \tag{3.5.2}$$

which is zero by Lemma 3.5.12. Analogously, we get the same for f^i and $\sum_h b^h_{hk} = 0$. □

We can now prove the following result.

Theorem 3.5.14 ([9, Theorem 3.17]). *Every invariant **D**-complex structure on a 4-dimensional nilmanifold is $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage and hence also pure-and-full at the 2-nd stage.*

Proof. (We see that the $\mathcal{C}^\infty$-pureness at the 2-nd stage follows from Remark 3.5.5 and Corollary 3.5.9.)
From Lemma 3.5.13 one gets that, on every 4-dimensional **D**-complex nilmanifold, the **D**-complex invariant component of an invariant 2-form is closed and hence also the **D**-complex anti-invariant component of a closed invariant 2-form is closed. In fact, we can say that the decomposition $\wedge^+\mathfrak{g}^* \oplus \wedge^-\mathfrak{g}^*$ moves to cohomology: let α be a closed 2-form, then:

$$0 = d\,\alpha = d(\alpha^{2,0} + \alpha^{0,2}) + d\,\alpha^{1,1} \tag{3.5.3}$$

and we see, by the Lemma 3.5.13, that both the invariant and anti-invariant components are closed. Then the linear **D**-complex structure is linear $\mathcal{C}^\infty$-full at the 2-nd stage; by Proposition 3.4.4, the **D**-complex structure is hence $\mathcal{C}^\infty$-full at the 2-nd stage (note that at this point we can deduce the $\mathcal{C}^\infty$-pure at the 2-nd stage from 3.3.1). □

Remark 3.5.15. We note that Theorem 3.5.14 is optimal. Indeed, we can not grow dimension (see Example 3.6.1 and Example 3.6.2), nor change the nilpotent hypothesis with solvable condition (see Example 3.7.1), nor drop the integrability condition on the **D**-complex structure (see Example 3.6.5).

It is possible to wonder what happens at other stages on nilmanifolds: in general, it is not possible to give a general behavior. However, for invariant structure and for the 1-stage, it is possible to give the following:

Proposition 3.5.16. *Let K be a linear **D**-complex structure on a Lie algebra $\mathfrak{g}$. Then it is $\mathcal{C}^\infty$-pure at the 1-st stage.*

Proof. We note that on $\mathfrak{g}^*$ there are not exact 1-forms, so, given a closed invariant (resp. anti-invariant) form $\alpha \in H^{1+}_K(\mathfrak{g},\mathbb{R})$ (resp. $\alpha \in H^{1-}_K(\mathfrak{g},\mathbb{R})$) we have that it is impossible to find an other representative of the class $[\alpha]$ which belongs to $H^{1-}_K(\mathfrak{g},\mathbb{R})$ (resp. $H^{1+}_K(\mathfrak{g},\mathbb{R})$). □

Remark 3.5.17. It follows that nilmanifolds and solvmanifolds endowed with invariant **D**-complex structure are all $\mathcal{C}^\infty$-pure at the 1-st stage.

There is another interesting consequence from Theorem 3.5.14 above (we will see that things go different on higher dimensions, see Examples 3.7.1, 3.7.5 and 3.7.6).

Theorem 3.5.18. *For every invariant **D**-complex structure on a 4-dimensional nilmanifold the dimension of H^{2+}_K and of H^{2-}_K are invariant of the **D**-complex structure, i.e. they are constant and depending only on the nilpotent Lie algebra.*
More precisely we have $\dim H^{2+}_K(M,\mathbb{R}) = 2$ *and* $\dim H^{2-}_K(M,\mathbb{R}) = \dim H^2_{\mathrm{dR}}(M,\mathbb{R}) - 2$.

Proof. Let $M = G/\Gamma$ be a compact quotient of a 4-dimensional nilpotent Lie group G whose Lie algebra is $\mathfrak{g}$. From Lemma 3.5.13, it follows that elements of $\wedge^{2,0}_{+-}\mathfrak{g}$ and of $\wedge^{0,2}_{+-}\mathfrak{g}$ are closed. It is easy to see that both $\wedge^{2,0}_{+-}\mathfrak{g}$ and of $\wedge^{0,2}_{+-}\mathfrak{g}$ are non-null and have only one element. Focus on $\alpha \in \wedge^{2,0}_{+-}\mathfrak{g}$, we will prove that α is not an exact form. Suppose

$$\alpha = d\beta \in \wedge^{2,0}_{+-}M \tag{3.5.4}$$

for some 1-form β. Since α is not zero and there exist 2 elements, said $X_1, X_2 \in \mathfrak{g}^+$ such that:

$$0 \neq \alpha(X_1,X_2) = d\beta(X_1,X_2) = -\beta([X_1,X_2]). \tag{3.5.5}$$

By Remark 3.5.5 the invariant **D**-complex structure need to be Abelian, hence $[X_1,X_2]=0$, which is a contradiction. The same happens for $\wedge^{0,2}_{+-}\mathfrak{g}$.
Now, using Proposition 3.4.4 we get the statement for $H^{2+}_K(M,\mathbb{R})$.
We see that $\wedge^{2,0}_{+-} \oplus \wedge^{0,2}_{+-} = H^{2+}_K(M,\mathbb{R})$ and $\dim H^{2+}_K(M,\mathbb{R}) = \dim\big(\wedge^{2,0}_{+-} \oplus \wedge^{0,2}_{+-}\big) = 2$. Now, we know that the Betti numbers are invariant of the nilmanifold, and by Theorem 3.5.14 we can write:

$$\dim H^{2-}_K(M,\mathbb{R}) = \dim H^2_{\mathrm{dR}}(M,\mathbb{R}) - \dim H^{2+}_K(M,\mathbb{R}) = \dim H^2_{\mathrm{dR}}(M,\mathbb{R}) - 2 \tag{3.5.6}$$

which shows that such a dimension does not depend on the **D**-complex structure, but only on the nilmanifold. □

3.6 Some examples of non-$\mathcal{C}^\infty$-pure-and-full (almost) D-complex nilmanifolds

Now, using the notation of 1.6 and the results in the previous sections, we provide examples of invariant (almost) **D**-complex structures on nilmanifolds.

Firstly, we give two examples of non-$\mathcal{C}^\infty$-pure or non-$\mathcal{C}^\infty$-full nilmanifolds admitting **D**-Kähler structures.

Example 3.6.1 (*There exists a* 6*-dimensional* ***D****-complex nilmanifold that is* $\mathcal{C}^\infty$*-pure at the* 2*-nd stage and non-*$\mathcal{C}^\infty$*-full at the* 2*-nd stage and admits a* ***D****-Kähler structure*)**.** Indeed, take the nilmanifold

$$M \;:=\; \left(0^4,\, 12,\, 13\right)$$

(as in Section 1.6 we refer to a nilpotent Lie group G, a compact quotient $M = \Gamma\backslash G$, whose Lie algebra $\mathfrak{g}$ has dual $\mathfrak{g}^*$ with structure equations define above) and define the invariant **D**-complex structure K setting

$$K \;:=\; (-\,+\,+\,-\,-\,+) \;.$$

By Nomizu's theorem (see [66]), the de Rham cohomology of M is given by

$$H^2_{dR}(M;\mathbb{R}) \;\simeq\; H^2_{dR}(\mathfrak{g};\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{16},\, e^{23},\, e^{24},\, e^{25},\, e^{34},\, e^{36},\, e^{26}+e^{35}\right\rangle \;.$$

Note that

$$H^{2+}_K(\mathfrak{g};\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{36}\right\rangle$$

and

$$H^{2-}_K(\mathfrak{g};\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{16},\, e^{24},\, e^{25},\, e^{34}\right\rangle \;,$$

hence $H^{2+}_K \cap H^{2-}_K = \{0\}$, since no invariant representative in the class $\left[e^{26}+e^{35}\right]$ is of pure type with respect to K (indeed, the space of invariant d-exact 2-forms is $\mathbb{R}\left\langle e^{12},\, e^{13}\right\rangle$). It follows that $K \in \mathrm{End}\,(\mathfrak{g})$ is linear non-$\mathcal{C}^\infty$-full at the 2-nd stage and linear $\mathcal{C}^\infty$-pure at the 2-nd stage and hence, by Proposition 3.4.4, $K \in \mathrm{End}(TM)$ is $\mathcal{C}^\infty$-pure at the 2-nd stage (being K Abelian, see Definition 3.5.4, one can also argue using Corollary 3.5.9) and non-$\mathcal{C}^\infty$-full at the 2-nd stage.
Moreover, we observe that

$$\omega \;:=\; e^{16}+e^{25}+e^{34}$$

is a symplectic form compatible with K, hence $(M,\, K,\, \omega)$ is a **D**-Kähler manifold.

Example 3.6.2 (*There exists a* 6*-dimensional* ***D****-complex nilmanifold that is non-*$\mathcal{C}^\infty$*-pure at the* 2*-nd stage (and hence non-*$\mathcal{C}^\infty$*-full at the* 4*-th stage) and admitting a* ***D****-Kähler structure*)**.** Take the nilmanifold M defined by

$$M \;:=\; \left(0^3,\; 12,\; 13+14,\; 24\right)$$

and the invariant **D**-complex structure K defined as

$$K \;:=\; (+\;-\;+\;-\;+\,-) \;.$$

Note that, since $[e_2, e_4] = -e_6$, one has that $[\mathfrak{g}^-,\, \mathfrak{g}^-] \neq \{0\}$ and hence K is not Abelian (see Definition 3.5.4).
We have

$$H^{2+}_K(\mathfrak{g};\mathbb{R}) \;\ni\; \left[e^{13}\right] \;=\; \left[e^{13}-d\,e^5\right] \;=\; -\left[e^{14}\right] \;\in\; H^{2-}_K(\mathfrak{g};\mathbb{R})$$

and therefore we get that

$$0 \;\neq\; \left[e^{13}\right] \;\in\; H^{2+}_K(\mathfrak{g};\mathbb{R}) \cap H^{2-}_K(\mathfrak{g};\mathbb{R}) \;,$$

namely, $K \in \mathrm{End}(\mathfrak{g})$ is not linear $\mathcal{C}^\infty$-pure at the 2-nd stage, hence $K \in \mathrm{End}(TM)$ is not $\mathcal{C}^\infty$-pure at the 2-nd stage; furthermore, by Proposition 3.3.1, we have also that K is not $\mathcal{C}^\infty$-full at the 4-th stage.
Moreover, we observe that

$$\omega \;:=\; e^{16}+e^{25}+e^{34}$$

is a symplectic form compatible with K, hence $(M,\, K,\, \omega)$ is a **D**-Kähler manifold.

Remark 3.6.3. Note that higher-dimensional examples of **D**-Kähler non-$\mathcal{C}^\infty$-full, respectively non-$\mathcal{C}^\infty$-pure, at the 2-nd stage structures can be obtained taking products with standard **D**-complex tori (see Example 1.3.7).

In contrast with the complex case (see, e.g., [55]), the previous two examples prove the following result:

Proposition 3.6.4 ([9, Proposition 3.3])**.** *To have a* **D**-*Kähler structure is not a sufficient condition to be* $\mathcal{C}^\infty$-*pure at the* 2-*nd stage nor being* $\mathcal{C}^\infty$-*full at the* 2-*nd stage.*

Moreover, also on 4 dimensional compact manifolds things go different between complex and the **D**-complex structures. In fact, it was proved by T. Drăghici, T.-J. Li and W. Zhang that every almost complex structure on a 4-dimensional compact manifold induces an almost complex decomposition at the level of the real second de Rham cohomology group, (see [27, Theorem 2.3]), while our following Example 3.6.5 shows that an analogous result does not hold in general in the almost **D**-complex case.

Example 3.6.5 (*There exists a* 4-*dimensional almost* **D**-*complex nilmanifold which is non-*$\mathcal{C}^\infty$*pure-and-full at the* 2-*nd stage*)**.** Indeed, take the nilmanifold M defined by

$$M \;:=\; (0,0,0,12)$$

(namely, the product of $\mathbb{R}$ and the Heisenberg group) and define the invariant almost **D**-complex structure by the eigenspaces

$$\mathfrak{g}^+ \;:=\; \mathbb{R}\left\langle e_1,\, e_3-e_2\right\rangle \qquad\text{and}\qquad \mathfrak{g}^- \;:=\; \mathbb{R}\left\langle e_2,\, e_4\right\rangle .$$

Note that K is not integrable, since $[e_1, e_3-e_2]=e_4$, $[e_2,e_4]=0$.
Note that we have

$$H^{2+}_K(\mathfrak{g};\mathbb{R}) \ni \left[e^{13}\right] = \left[e^{13}+d\,e^4\right] = \left[e^{13}+e^{12}\right] = \left[e^1\wedge\left(e^3+e^2\right)\right] \in H^{2-}_K(\mathfrak{g};\mathbb{R})$$

and therefore we get that

$$0 \;\neq\; \left[e^{13}\right] \;\in\; H^{2+}_K(\mathfrak{g};\mathbb{R}) \cap H^{2-}_K(\mathfrak{g};\mathbb{R}) \;.$$

Then, K is not $\mathcal{C}^\infty$-pure at the 2-nd stage and, by Proposition 3.3.1, is not $\mathcal{C}^\infty$-full at the 2-nd stage. Indeed, the closed 2-form e^{24} does not admit any representative in $H^{2+}_K(\mathfrak{g};\mathbb{R})$ nor in $H^{2-}_K(\mathfrak{g};\mathbb{R})$, since:

$$H^{2+}_K(\mathfrak{g};\mathbb{R}) = \{e^{13}\} \qquad H^{2-}_K(\mathfrak{g};\mathbb{R}) = \{e^1\wedge(e^3+e^2),\, e^{14},\, e^3\wedge(e^2+e^3)\}. \tag{3.6.1}$$

3.7 Behavior of H_K^{p+} and H_K^{p-} under small deformations

In this section, we study explicit examples of deformations of **D**-complex structures on nilmanifolds and solvmanifolds. We refer to Chapter 2 (see also [61, 69]) for more results about deformations of **D**-complex structures.

The following examples show a curve $\{K_t\}_{t\in\mathbb{R}}$ of **D**-complex structures on a 4-dimensional solvmanifold; while K_0 is linear $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage for $t\neq 0$ one proves that K_t is neither $\mathcal{C}^\infty$-pure at the 2-nd stage nor $\mathcal{C}^\infty$-full at the 2-nd stage (Example 3.7.1). Moreover, K_0 admits a **D**-Kähler structure and K_t does not admit a **D**-Kähler structure (Example 3.7.2). In particular, this curve provides an example of the instability of **D**-Kählerness under small deformations of the **D**-complex structure and it proves also that the nilpotency condition in Theorem 3.5.14 can not be dropped out.

Example 3.7.1 (*There exists a 4-dimensional solvmanifold endowed with an invariant* $\mathbf{D}$*-complex structure such that it is* $\mathcal{C}^\infty$*-pure-and-full at the 2-nd stage and it has small deformations that are not* $\mathcal{C}^\infty$*-pure-and-full at the 2-nd stage*)**.** Consider the 4-dimensional solvmanifold defined by

$$M \;:=\; (0,\, 0,\, 23,\, -24)$$

(see, e.g., [15]).
By Hattori's theorem (see [41]), it is straightforward to compute

$$H^2_{dR}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{12},\, e^{34}\right\rangle \;.$$

For every $t \in \mathbb{R}$, consider the invariant $\mathbf{D}$-complex structure with respect to the basis $\{e^1, \ldots, e^4\}$

$$K_t \;:=\; \left(\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) \;.$$

In particular, for $t=0$, we have

$$K_0 \;=\; (-\,+\,+-) \;.$$

It is straightforward to check that K_0 is $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage (note however that K_0 is not Abelian): in fact,

$$H^{2+}_{K_0}(M;\mathbb{R}) \;=\; \{0\} \qquad \text{and} \qquad H^{2-}_{K_0}(M;\mathbb{R}) \;=\; H^2_{dR}(M;\mathbb{R}) \;;$$

in particular, we have

$$\dim_{\mathbb{R}} H^{2+}_{K_0}(M;\mathbb{R}) \;=\; 0 \;, \qquad \dim_{\mathbb{R}} H^{2-}_{K_0}(M;\mathbb{R}) \;=\; 2.$$

For every $t \in \mathbb{R}$, we have that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_2,\, e_3\right\rangle \qquad \text{and} \qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_1,\, e_4 + t\, e_2\right\rangle \;:$$

in particular, $\left[\mathfrak{g}^+_{K_t},\, \mathfrak{g}^+_{K_t}\right] = \mathbb{R}\left\langle e_3\right\rangle \subseteq \mathfrak{g}^+_{K_t}$ and $\left[\mathfrak{g}^-_{K_t},\, \mathfrak{g}^-_{K_t}\right] = \{0\}$, which proves the integrability of K_t, for every $t \in \mathbb{R}$.
Furthermore, for $t \neq 0$, we get

$$\begin{aligned} H^{2-}_{K_t}(M;\mathbb{R}) \ni \left[e^{34}\right] \;&=\; \left[e^{34} + \frac{1}{t}\, d\, e^3\right] \;=\; \left[e^{34} + \frac{1}{t}\left(e^{23} + t\, e^{43} - t\, e^{43}\right)\right] \\ &=\; \left[\frac{1}{t}\left(e^2 - t\, e^4\right) \wedge e^3\right] \in H^{2+}_{K_t}(M;\mathbb{R}) \end{aligned}$$

and therefore we have that

$$0 \;\neq\; \left[e^{34}\right] \;\in\; H^{2-}_{K_t}(M;\mathbb{R}) \cap H^{2+}_{K_t}(M;\mathbb{R}) \;.$$

In particular, for $t \neq 0$, it follows that K_t is not $\mathcal{C}^\infty$-pure at the 2-nd stage and hence it is not $\mathcal{C}^\infty$-full at the 2-nd stage, as a consequence of Proposition 3.3.1 (in fact, no invariant representative in the class $\left[e^{12}\right] = \left[e^1 \wedge (e^2 - te^4) + te^{14}\right]$ is of pure type with respect to K_t, the space of invariant d-exact 2-forms being $\mathbb{R}\left\langle (e^2 - t\, e^4) \wedge e^3 - t\, e^{34},\, (e^2 - t\, e^4) \wedge e^4 \right\rangle$).
Therefore, for $t \neq 0$, we have

$$\dim_{\mathbb{R}} H^{2+}_{K_t}(M;\mathbb{R}) \;=\; 1 \;, \qquad \dim_{\mathbb{R}} H^{2-}_{K_t}(M;\mathbb{R}) \;=\; 1.$$

Note that, in this example, for every $t \in \mathbb{R}$ one has $s\left(\mathfrak{g}_{K_t}^{-}\right) = 0$ and $s\left(\mathfrak{g}_{K_t}^{+}\right) = 1$ but for $t \neq 0$ the **D**-complex structure K_t is not $\mathcal{C}^\infty$-pure at the 2-nd stage, therefore the Abelian condition on just $\mathfrak{g}^-$ in Theorem 3.5.6 is not sufficient to have $\mathcal{C}^\infty$-pureness at the 2-nd stage, as announced in Remark 3.5.7.
Note moreover that, in this example, the functions

$$\mathbb{R} \ni t \mapsto \dim_{\mathbb{R}} H^{2+}_{K_t}(M;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad \mathbb{R} \ni t \mapsto \dim_{\mathbb{R}} H^{2-}_{K_t}(M;\mathbb{R}) \in \mathbb{N}$$

are, respectively, lower-semi-continuous and upper-semi-continuous.

Example 3.7.2 (*There exists a* 4*-dimensional solvmanifold endowed with an invariant* ***D****-complex structure such that it admits a* ***D****-Kähler structure and it has small deformations that are not* ***D****-Kähler*)**.** Consider the solvmanifold M with K_0 and the deformations K_t as in the previous Example 3.7.1. We see that M admits a symplectic form $\omega := e^{12} + e^{34}$ which is compatible with the **D**-complex structure K_0: therefore, $(M,\, K_0,\, \omega)$ is a **D**-Kähler manifold. Instead, for $t \neq 0$, one has $H^{-}_{K_t}(M;\mathbb{R}) = \mathbb{R}\left\langle e^{34}\right\rangle$ and therefore, if a K_t-compatible symplectic form ω_t exists, it should be in the same cohomology class as e^{34} and then it should satisfy

$$\mathrm{Vol}(M) \;=\; \int_M \omega_t \wedge \omega_t \;=\; \int_M e^{34} \wedge e^{34} \;=\; 0 \,,$$

which is not possible; therefore, for $t \neq 0$, there is no symplectic structure compatible with the **D**-complex structure K_t: in particular, $(M,\, K_t)$ for $t \neq 0$ admits no **D**-Kähler structure.

Indeed, the previous example proves the following result, giving a strong difference between the **D**-complex and the complex cases (compare with the stability result of Kählerness proved by K. Kodaira and D. C. Spencer in [52], saying that on a Kähler manifold the deformations of complex structure J_t are all Kähler for t small).

Theorem 3.7.3 ([9, Theorem 4.2])**.** *The property of being* ***D****-Kähler is not stable under small deformations of the* ***D****-complex structure.*

Example 3.7.1 proves also the following instability result (a similar result holds also in the complex case, see [10, Theorem 3.2]).

Proposition 3.7.4 ([9, Proposition 4.3])**.** *The property of being* $\mathcal{C}^\infty$*-pure at the* 2*-nd stage or* $\mathcal{C}^\infty$*-full at the* 2*-nd stage is not stable under small deformations of the* ***D****-complex structure.*

We recall that T. Drăghici, T.-J. Li and W. Zhang proved in [27, Theorem 5.4] that, given a curve of almost complex structures on a 4-dimensional compact manifold, the dimension of the almost complex anti-invariant subgroup of the real second de Rham cohomology group is upper-semi-continuous and hence (as a consequence of [27, Theorem 2.3]) the dimension of the almost complex invariant subgroup of the real second de Rham cohomology group is lower-semi-continuous. This result holds no more true in dimension greater than 4 (see [11]).
We provide two examples showing that the dimensions of the **D**-complex invariant and anti-invariant subgroups of the cohomology can jump along a curve of **D**-complex structures, proving that the dimensions are neither upper- nor lower- semi-continuous.

Example 3.7.5 (*There exists a curve of* ***D****-complex structures on a* 6*-dimensional nilmanifold such that the dimensions of the* ***D****-complex invariant and anti-invariant subgroups* *of the real second de Rham cohomology group jump (lower-semi-continuously) along the curve)*. Consider the 6-dimensional nilmanifold

$$M \;:=\; (0,\,0,\,0,\,12,\,13,\,24)\;.$$

By Nomizu's theorem (see [66]), it is straightforward to compute

$$H^2_{dR}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26},\, e^{35},\, e^{25}+e^{34}\right\rangle\;.$$

For every $t\in[0,\,1]$, consider the invariant **D**-complex structure with respect to the basis $\{e^1,\ldots,e^6\}$

$$K_t \;:=\; \left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & -1 & & & & \\ \hline & & \frac{(1-t)^2-t^2}{(1-t)^2+t^2} & \frac{2t(1-t)}{(1-t)^2+t^2} & & \\ & & \frac{2t(1-t)}{(1-t)^2+t^2} & -\frac{(1-t)^2-t^2}{(1-t)^2+t^2} & & \\ \hline & & & & 1 & \\ & & & & & -1 \end{array}\right)\;. \tag{3.7.1}$$

For $0\le t\le 1$, one checks that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_1,\,(1-t)\,e_3+t\,e_4,\,e_5\right\rangle \qquad\text{and}\qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_2,\,t\,e_3-(1-t)\,e_4,\,e_6\right\rangle\;;$$

one can easily verify that the integrability condition of K_t is satisfied for every $t\in[0,\,1]$. Indeed:

$$\begin{array}{rr} [e_1,\,(1-t)e_3+te_4]=-(1-t)e_5\in\mathfrak{g}^+_{K_t} & [e_1,\,e_5]=[(1-t)e_3+te_4,\,e_5]=0\in\mathfrak{g}^+_{K_t} \\ [e_2,\,te_3-(1-t)e_4]=(1-t)e_6\in\mathfrak{g}^-_{K_t} & [e_2,\,e_6]=[te_3-(1-t)e_4,\,e_6]=0\in\mathfrak{g}^-_{K_t}. \end{array} \tag{3.7.2}$$

In particular, for $t\in\{0,\,1\}$, one has

$$K_0 \;=\; (+\,-\,+\,-\,+\,-) \qquad\text{and}\qquad K_1 \;=\; (+\,-\,-\,+\,+\,-)\;.$$

It is straightforward to check that K_0 and K_1 are $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage and

$$\begin{aligned} H^2_{dR}(M;\mathbb{R}) \;&=\; \underbrace{\mathbb{R}\left\langle e^{15},\,e^{26},\,e^{35}\right\rangle}_{=H^{2\,+}_{K_0}(M;\mathbb{R})}\oplus\underbrace{\mathbb{R}\left\langle e^{14},\,e^{23},\,e^{25}+e^{34}\right\rangle}_{=H^{2\,-}_{K_0}(M;\mathbb{R})} \\ &=\; \underbrace{\mathbb{R}\left\langle e^{14},\,e^{15},\,e^{23},\,e^{26}\right\rangle}_{=H^{2\,+}_{K_1}(M;\mathbb{R})}\oplus\underbrace{\mathbb{R}\left\langle e^{35},\,e^{25}+e^{34}\right\rangle}_{=H^{2\,-}_{K_1}(M;\mathbb{R})}\;; \end{aligned}$$

therefore

$$\dim_{\mathbb{R}} H^{2\,+}_{K_0}(M;\mathbb{R}) \;=\; 3 \qquad\text{and}\qquad \dim H^{2\,-}_{K_0}(M;\mathbb{R}) \;=\; 3$$

and

$$\dim_{\mathbb{R}} H^{2\,+}_{K_1}(M;\mathbb{R}) \;=\; 4 \qquad\text{and}\qquad \dim H^{2\,-}_{K_1}(M;\mathbb{R}) \;=\; 2\;.$$

Instead, for $0<t<1$, one has

$$H^{2\,+}_{K_t}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\,e^{15},\,e^{23},\,e^{26}\right\rangle$$

and

$$H^{2\,-}_{K_t}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\,e^{23},\,e^{25}+e^{34}\right\rangle\;;$$

it follows that, for $0 < t < 1$, the $\mathbf{D}$-complex structure K_t is neither $\mathcal{C}^\infty$-pure at the 2-nd stage nor $\mathcal{C}^\infty$-full at the 2-nd stage (in fact $e^{35} \notin H^{2+}_{K_t}(M;\mathbb{R})$ and $e^{35} \notin H^{2-}_{K_t}(M;\mathbb{R})$ while $e^{14} \in H^{2+}_{K_t}(M;\mathbb{R}) \cap H^{2-}_{K_t}(M;\mathbb{R})$); moreover, for $0 < t < 1$, one gets

$$\dim_\mathbb{R} H^{2+}_{K_t}(M;\mathbb{R}) \;=\; 4 \qquad \text{and} \qquad \dim H^{2-}_{K_t}(M;\mathbb{R}) \;=\; 3\;;$$

in particular, the functions

$$[0,\,1] \ni t \mapsto \dim_\mathbb{R} H^{2+}_{K_t}(M;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad [0,\,1] \ni t \mapsto \dim_\mathbb{R} H^{2-}_{K_t}(M;\mathbb{R}) \in \mathbb{N}$$

are non-constant and both lower-semi-continuous.

The previous examples show that the dimension of the $\mathbf{D}$-complex anti-invariant subgroup of the de Rham cohomology $\dim_\mathbb{R} H^-$ in general is not upper-semi-continuous (it is such in Example 3.7.1) nor lower-semi-continuous (it is such in Example 3.7.5). We end this section with an example showing that also the dimension of the $\mathbf{D}$-complex invariant subgroup of the de Rham cohomology (i.e. $\dim_\mathbb{R} H^+$) in general is not lower-semi-continuous (it is such in Example 3.7.1 and in Example 3.7.5).

Example 3.7.6 (*There exists a curve of* $\boldsymbol{D}$*-complex structures on a* 6*-dimensional nilmanifold such that the dimensions of the* $\boldsymbol{D}$*-complex invariant and anti-invariant subgroups of the real second de Rham cohomology group jump (upper-semi-continuously) along the curve*)**.** Consider the 6-dimensional nilmanifold

$$M \;:=\; (0,\,0,\,0,\,12,\,13,\,24)\;.$$

By Nomizu's theorem (see [66]), it is straightforward to compute

$$H^2_{dR}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26},\, e^{35},\, e^{25}+e^{34}\right\rangle\;.$$

For every $t \in [0,\,1]$, consider the invariant $\mathbf{D}$-complex structure

$$K_t \;:=\; \left(\begin{array}{cccc|cc} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ \hline & & & & \frac{(1-t)^2-t^2}{(1-t)^2+t^2} & \frac{2t(1-t)}{(1-t)^2+t^2} \\ & & & & \frac{2t(1-t)}{(1-t)^2+t^2} & -\frac{(1-t)^2-t^2}{(1-t)^2+t^2} \end{array}\right)\;.$$

For $0 \le t \le 1$, one verifies that

$$\mathfrak{g}^+_{K_t} \;=\; \mathbb{R}\left\langle e_1,\, e_4,\, (1-t)\,e_5 + t\,e_6\right\rangle \qquad \text{and} \qquad \mathfrak{g}^-_{K_t} \;=\; \mathbb{R}\left\langle e_2,\, e_3,\, t\,e_5 - (1-t)\,e_6\right\rangle\;;$$

one can straightforwardly check that the integrability condition of K_t is satisfied for every $t \in [0,\,1]$. Furthermore, one can prove that K_t is Abelian for every $t \in [0,\,1]$, hence it is in particular $\mathcal{C}^\infty$-pure at the 2-nd stage by Corollary 3.5.9.
In particular, for $t \in \{0,\,1\}$, one has

$$K_0 \;=\; (+--++-) \qquad \text{and} \qquad K_1 \;=\; (+--+-+)\;.$$

An easy computation shows that

$$H^2_{dR}(M;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{14},\, e^{15},\, e^{23},\, e^{26}\right\rangle}_{=H^{2+}_{K_0}(M;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{35},\, e^{25}+e^{34}\right\rangle}_{=H^{2-}_{K_0}(M;\mathbb{R})}$$

and K_0 is $\mathcal{C}^\infty$-pure-and-full at the 2-nd stage, while K_1 is $\mathcal{C}^\infty$-pure at the 2-nd stage, non-$\mathcal{C}^\infty$-full at the 2-nd stage and

$$H^2_{dR}(M;\mathbb{R}) \;=\; \underbrace{\mathbb{R}\left\langle e^{14},\, e^{23},\, e^{35}\right\rangle}_{=\,H^{2\,+}_{K_1}(M;\mathbb{R})} \oplus \underbrace{\mathbb{R}\left\langle e^{15},\, e^{26}\right\rangle}_{=\,H^{2\,-}_{K_1}(M;\mathbb{R})} \oplus \mathbb{R}\left\langle e^{25}+e^{34}\right\rangle \;,$$

where

$$\mathbb{R}\left\langle e^{25}+e^{34}\right\rangle \cap \left(H^{2\,+}_{K_1}(M;\mathbb{R}) \oplus H^{2\,-}_{K_1}(M;\mathbb{R})\right) \;=\; \{0\} \;;$$

therefore

$$\dim_{\mathbb{R}} H^{2\,+}_{K_0}(M;\mathbb{R}) \;=\; 4 \qquad \text{and} \qquad \dim H^{2\,-}_{K_0}(M;\mathbb{R}) \;=\; 2$$

and

$$\dim_{\mathbb{R}} H^{2\,+}_{K_1}(M;\mathbb{R}) \;=\; 3 \qquad \text{and} \qquad \dim H^{2\,-}_{K_1}(M;\mathbb{R}) \;=\; 2 \,.$$

Instead, for $0<t<1$, one has

$$H^{2\,+}_{K_t}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle e^{14},\, e^{23}\right\rangle$$

and

$$H^{2\,-}_{K_t}(M;\mathbb{R}) \;=\; \mathbb{R}\left\langle t\, e^{26} + (1-t)\, e^{25} + (1-t)\, e^{34}\right\rangle \;,$$

while

$$\mathbb{R}\left\langle e^{15},\, e^{35},\, e^{26}\right\rangle \cap \left(H^{2\,+}_{K_t}(M;\mathbb{R}) \oplus H^{2\,-}_{K_t}(M;\mathbb{R})\right) \;=\; \{0\} \;;$$

it follows that, for $0<t<1$, the **D**-complex structure K_t is $\mathcal{C}^\infty$-pure at the 2-nd stage and non-$\mathcal{C}^\infty$-full at the 2-nd stage; moreover, for $0<t<1$, one gets

$$\dim_{\mathbb{R}} H^{2\,+}_{K_t}(M;\mathbb{R}) \;=\; 2 \qquad \text{and} \qquad \dim H^{2\,-}_{K_t}(M;\mathbb{R}) \;=\; 1 \;:$$

in particular, the functions

$$[0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(M;\mathbb{R}) \in \mathbb{N} \qquad \text{and} \qquad [0,\,1] \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(M;\mathbb{R}) \in \mathbb{N}$$

are non-constant and both upper-semi-continuous.

We resume the contents of Examples 3.7.1, 3.7.5 and 3.7.6 in the following proposition.

Proposition 3.7.7. *Let M be a compact manifold and let $\{K_t\}_{t\in I}$ be a curve of **D**-complex structures on M, where $I\subseteq\mathbb{R}$. Then, in general, the functions*

$$I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,+}_{K_t}(M;\mathbb{R}) \in \mathbb{N} \qquad \textit{and} \qquad I \ni t \mapsto \dim_{\mathbb{R}} H^{2\,-}_{K_t}(M;\mathbb{R}) \in \mathbb{N}$$

are not upper-semi-continuous or lower-semi-continuous.

Chapter 4

D-Kähler Ricci-flat metrics

In this chapter, we will address some results concerning the Ricci-flat **D**-Kähler manifolds and their special Lagrangian submanifolds. These manifolds are **D**-Kähler manifolds with trivial canonical bundle, that is the **D**-analogous of the Calabi-Yau manifolds in the **D**-settings. Since there is a symplectic form ω, it makes sense to consider Lagrangian submanifolds. Recently, F.R. Harvey and H.B. Lawson [40] show that this setting is closed related to calibrated submanifolds in semi-Riemannian geometry (see [60]) and to the optimal transport problem (see [48]).

After presenting some properties of Ricci-flat **D**-Kähler manifolds, we try to extend such properties to a larger class of almost **D**-Hermitian manifolds.

We begin this chapter recalling some results and properties about Ricci-flat **D**-Kähler metrics (see Section 4.1) and giving some examples of such manifolds, remarking the difference of the **D**-setting with the usual complex case (see Section 4.2).

Then in Section 4.3 we introduce the space $\Omega^2(TM)$, that is the natural space where to study the **D**-Hermitian connections, which will be introduced in the next Section 4.4.

Finally in Section 4.5 we give a generalization of a result of F.R. Harvey and H.B. Lawson (see [40, Proposition 16.3]) concerning Lagrangian submanifolds of symplectic almost **D**-complex manifolds.

4.1 Properties of D-Kähler Ricci flat metrics

Definition 4.1.1. A ***D**-line bundle* on an arbitrary manifold M (not necessarily **D**-manifold) is a family of free **D**-modules over M which is locally isomorphic to $U \times \mathbf{D}$ on an open set U of M, and whose transition functions are smooth $\mathbf{D}^+$-valued functions.
A *(**D**-)holomorphic line bundle* over a **D**-manifold M is a **D**-line bundle such that the transition functions are **D**-holomorphic.

We note that the transition functions are chosen on $\mathbf{D}^+$ because we want the orientability of the line bundle.

The bundle $\kappa := \wedge_K^{n,0} M$ of the holomorphic n-forms on a **D**-complex manifold M is called *canonical bundle*. We see that it has transition functions given by $\det_{\mathbf{D}}\left(\dfrac{\partial z^i}{\partial z^j}\right)$, which are holomorphic with values in $\mathbf{D}^+$. In local standard-coordinates or null-coordinates we can write a holomorphic n-form Φ as:

$$\begin{aligned} \Phi &= F(z)\, d z^1 \wedge \cdots \wedge d z^n \\ &= e f(u_+^1, \ldots, u_+^n)\, d u_+^- \wedge \cdots \wedge d u_+^n + \bar{e} g(u_-^1, \ldots, u_-^n)\, d u_-^1 \wedge \cdots \wedge d u_-^n. \end{aligned} \tag{4.1.1}$$

We make the following definition, in analogy with the complex case.

Definition 4.1.2. A ***D**-Calabi-Yau manifold* is a **D**-Hermitian manifold M which has a nowhere vanishing holomorphic section ε of the canonical bundle $\wedge_K^{n,0}(M)$ (we do not require M to be Kähler). We call the *curvature of the canonical line bundle* on $\wedge_K^{n,0}$ the 2-form Ω defined by:

$$\Omega = \tau\,\partial\,\overline{\partial}\log||\varepsilon||. \tag{4.1.2}$$

We have:

Proposition 4.1.3. *On a **D**-Calabi-Yau manifold the curvature Ω of the canonical bundle is independent of the section ε.*

Proof. In local holomorphic coordinates $z = (z^1, \dots, z^n)$ we have $\varepsilon = f(z)\,d\,z^1 \wedge \cdots \wedge d\,z^n$, where $f(z)$ is a **D**-holomorphic function such that $\|\varepsilon\|^2 = cf(z)\overline{f(z)}$ with $c = c(z) > 0$. In particular $f(z)$ has positive norm:

$$\tau\,\partial\,\overline{\partial}\log||\varepsilon|| = \tau\,\partial\,\overline{\partial}\log\|c\| + \tau\,\partial\,\overline{\partial}\log(f(z)\overline{f(z)}) = \tau\,\partial\,\overline{\partial}\log||c||. \tag{4.1.3}$$

If we change coordinates to z', the determinant of $\partial\,z/\,\partial\,z'$ gives a new coefficient $c' = c(\partial\,z/\,\partial\,z')(\overline{\partial\,z/\,\partial\,z'})$ and again $\tau\,\partial\,\overline{\partial}\log\|c\| = \tau\,\partial\,\overline{\partial}\log\|c'\|$ since the coordinates change is holomorphic. □

Proposition 4.1.4. *Let (M, K, g) be a **D**-Kähler manifold, and Ω the curvature of its canonical bundle as above and set* Ric *as the Ricci curvature of the Levi-Civita connection on TM. Then we have:*

$$\Omega(X, Y) = -\operatorname{Ric}(X, KY). \tag{4.1.4}$$

Furthermore, the canonical bundle is flat if and only if the Ricci curvature of M is zero.

Proof. The proof is similar to the complex case and therefore it is omitted (see also [2, Proposition 5.6] to compare Ω and Ric in local coordinates). □

Definition 4.1.5. A *Ricci-flat Kähler **D**-manifold* $(M, g, \omega, \varepsilon)$ is a **D**-Kähler manifold with $\operatorname{Ric} = 0$.

Remark 4.1.6. We see that a Ricci-flat Kähler **D**-manifold has holonomy group contained in the group $\mathrm{SU}_n(\mathbf{D})$, exactly as, in the complex case, it happens to the Calabi-Yau manifolds to have holonomy group contained in $\mathrm{SU}_n(\mathbb{C})$.

We can characterize these manifolds with a property similar to that one which happens for complex case (see e.g. [43]). We describe this property in the **D**-complex setting or in the adapted-setting, as we done in the following two propositions.

Proposition 4.1.7 ([40, Proposition 11.2])**.** *A Ricci-flat Kähler **D**-manifold is equivalent to the data of a symplectic $2n$-dimensional manifold (M, ω) together with two* d*-closed real n-forms ψ, φ such that:*

1. *$\Phi := \psi + \tau\varphi$ is a decomposable (i.e. simple) **D**-valued n-form,*

2. *$\Phi \wedge \omega = 0$, i.e. $\psi \wedge \omega = \varphi \wedge \omega = 0$,*

3. *it happens that*

$$\Phi \wedge \bar{\Phi} = \begin{cases} \omega^n & \text{if } n \text{ even,} \\ -\tau\omega^n & \text{if } n \text{ odd.} \end{cases} \tag{4.1.5}$$

Proof. First of all, note that given a Ricci-flat Kähler **D**-manifold, then we have that $\Phi \in \kappa$ satisfies the previous conditions, provided that ψ and φ are its real and imaginary parts.
Conversely, let (M, ω) be a symplectic manifold satisfying point 1 to 3. Because of 1, we can write locally $\Phi := \psi + \tau\varphi = \theta^1 \wedge \cdots \wedge \theta^n$ for some 1-forms $\theta^1, \ldots, \theta^n$. Let Σ be the subbundle of $T^*M \otimes \mathbf{D}$ spanned by $\theta^1, \ldots, \theta^n$. Note that if θ^i is a null-element in $T^*M \otimes \mathbf{D}$ (say $\tau\theta^i = \theta^i$), then locally:

$$\theta^i \wedge \overline{\theta^i} = (d\,e^i + \tau\, d\,e^i) \wedge (d\,e^i - \tau\, d\,e^i) = 0 \tag{4.1.6}$$

which is in contrast with 3 and the fact that $\omega^n \neq 0$ (since the manifold is symplectic). Then by $0 \neq \Phi \wedge \bar{\Phi} = \omega^n$ (resp. $-\tau\omega^n$) we have $T^*M = \Sigma + \bar{\Sigma}$ and this defines an almost **D**-complex structure on M: a 1-form α is of type $(1,0)$ if and only if $\alpha \wedge \Phi = 0$. Obviously $\theta^i \in \wedge^{1,0}$ and they form a basis of $(1,0)$-forms.
Now let us show that such a **D**-structure is integrable: take $\alpha \in \wedge^{1,0}$ and write:

$$d\,\alpha = \sum_{i,j} a_{ij}\theta^i \wedge \theta^j + \sum_{i,j} b_{ij}\theta^i \wedge \bar{\theta}^j + \sum_{i,j} c_{ij}\bar{\theta}^i \wedge \bar{\theta}^j. \tag{4.1.7}$$

But since $\alpha \wedge \Phi = 0$ we get by the closeness of Φ:

$$0 = d(\alpha \wedge \Phi) = d(\alpha) \wedge \Phi - \alpha \wedge d(\Phi) = d(\alpha) \wedge \Phi; \tag{4.1.8}$$

this implies $c_{ij} = 0$ and hence $d\wedge^{1,0} \subset \wedge^{2,0} + \wedge^{1,1}$, which shows that such a **D**-structure is integrable. In fact, this implies that given $X, Y \in T^{1,0}M$ then $[X, Y] \in T^{1,0}M$ and then we apply the Frobenius theorem.
Finally we see that ω, as a 2-form, can be split in a similar way as in (4.1.7). Condition 2 implies that ω has no $(0,2)$-part, and since it is real we have $\omega \in \wedge^{1,1}M$. Setting $g(\cdot,\cdot) := \omega(\cdot, K\cdot)$ we have that (M, ω, K) is a **D**-Kähler manifold.
Finally, since Ψ is closed we see it is an holomorphic n-form, and by condition 3 we get that it is a non-vanishing holomorphic section of κ with constant length, hence $\partial\bar{\partial}\log||\Psi|| = 0$ and by Proposition 4.1.4 it is a Ricci-flat manifold. □

Proposition 4.1.8 ([40, Proposition 11.3])**.** *A Ricci-flat Kähler **D**-manifold is equivalent to the data of a symplectic $2n$-dimensional manifold (M, ω) together with two d-closed real n-forms α, β such that:*

1. *$\Phi := e\alpha + \bar{e}\beta$ is **D**-valued n-form and the real forms α, β are decomposable,*
2. *$\Phi \wedge \omega = 0$, i.e. $\alpha \wedge \omega = \beta \wedge \omega = 0$,*
3. *$\alpha \wedge \beta = \omega^n$.*

Proof. Set $\alpha = \psi + \varphi$ and $\beta = \psi - \varphi$ and note that

$$(\alpha^1 \wedge \cdots \wedge \alpha^n)e + (\beta^1 \wedge \cdots \wedge \beta^n)\bar{e} = (\alpha^1 e + \beta^1 \bar{e}) \wedge \cdots \wedge (\beta^n e + \alpha^n \bar{e}). \tag{4.1.9}$$

Then it easily follows the equivalence between Propositions 4.1.7 and 4.1.8. □

4.2 Examples of D-Kähler Ricci flat manifolds

In the following examples, we construct a Ricci-flat Kähler **D**-structure on nilmanifolds, showing how different are the complex and the double geometry. In fact, it is well known that the only Kähler nilmanifold is the torus, and hence it is the only example of Calabi-Yau structure over nilmanifolds.

Example 4.2.1. Let $\mathfrak{g}$ be the nilpotent Lie algebra which dual space $\mathfrak{g}^*$ is defined by the structure equations:

$$\begin{cases} d e^1 = d e^2 = 0 \\ d e^3 = e^1 \wedge e^2 \\ d e^4 = e^1 \wedge e^3 \\ d e^5 = e^2 \wedge e^3 \\ d e^6 = e^1 \wedge e^4. \end{cases}$$

Let G be a simply-connected nilpotent Lie group whose Lie algebra is $\mathfrak{g}$. By Malcev theorem, there exists a discrete subgroup $\Gamma \subset G$ and hence a quotient $M = \Gamma \backslash G$, and $\{e^i\}$ is a basis of global 1-forms. With the notation of Section 1.6 we have:

$$\mathfrak{g} := (0, 0, 12, 13, 23, 14). \tag{4.2.1}$$

We can define an almost **D**-complex structure K given by:

$$\begin{array}{lll} K(e^1) = e^1 & K(e^2) = -e^2 & K(e^3) = -e^3 \\ K(e^4) = e^4 & K(e^5) = -e^5 & K(e^6) = e^6. \end{array}$$

We set

$$g := -\frac{1}{6} e^1 \odot e^5 + \frac{1}{6} e^2 \odot e^4 - e^3 \odot e^4 + e^2 \odot e^6.$$

It is easy to verify that:

1. K is a **D**-complex integrable structure, since $N_K = 0$ where N_K is the torsion tensor of the **D**-structure K;

2. the pseudo-Riemannian metric g is compatible with K (i.e. $g(KX, KY) = -g(X, Y)$);

3. the fundamental Kähler form $\omega(\cdot, \cdot) = g(\cdot, K\cdot)$ is closed, where

$$\omega = +\frac{1}{6} e^1 \wedge e^5 + \frac{1}{6} e^2 \wedge e^4 - e^3 \wedge e^4 + e^2 \wedge e^6.$$

Then, (M, K, ω) defines a Kähler **D**-manifold. Now we want to prove that it is also a Ricci-Flat Kähler **D**-manifold. Now we set:

$$\begin{aligned} \varphi &= \frac{1}{2} \left(e^1 \wedge e^4 \wedge e^6 + e^2 \wedge e^3 \wedge e^5 \right) \\ \psi &= \frac{1}{2} \left(e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^5 \right) \\ \Phi &= \varphi + \tau \psi. \end{aligned}$$

We can see that Φ is a global $(3, 0)$-form no-where vanishing, indeed

$$\begin{aligned} \alpha^1 &:= \frac{(e^1 + e^2)}{2} + \frac{\tau(e^1 - e^2)}{2}, \\ \alpha^2 &:= \frac{(e^4 + e^3)}{2} + \frac{\tau(e^4 - e^3)}{2}, \\ \alpha^3 &:= \frac{(e^6 + e^5)}{2} + \frac{\tau(e^6 - e^5)}{2} \end{aligned}$$

is a basis of $(1, 0)$-forms, and is easy to see that $\Phi \wedge \alpha^i$ is zero for $i = 1, 2, 3$ (in fact we have $\alpha^1 \wedge \alpha^2 \wedge \alpha^3 = \Phi$).
Moreover we have:

1. φ, ψ are closed real 3-forms,
2. Φ is a simple (decomposable) **D**-valued 3-form,
3. $\Phi \wedge \omega = 0$,
4. $\Phi \wedge \overline{\Phi} = -\tau\omega^3$

then (M, g, K) is a **D**-Kähler Ricci flat manifold (as stated in Proposition 4.1.7), and the global $(n, 0)$-form (a $(3, 0)$-form in this case) is Φ.

Example 4.2.2. We construct a similar example for the nilpotent Lie algebra $\mathfrak{n}$ defined by the structure equations:

$$\begin{cases} d\,e^1 = d\,e^2 = d\,e^3 = 0 \\ d\,e^4 = e^1 \wedge e^2 \\ d\,e^5 = e^1 \wedge e^3 + e^1 \wedge e^4 \\ d\,e^6 = e^2 \wedge e^4. \end{cases}$$

Let G' be a simply-connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}$. By Malcev theorem, there exists a discrete subgroup $\Gamma \subset G'$ and hence a discrete quotient $N = \Gamma\backslash G'$, and $\{e^i\}$ is a basis of global 1-forms.
We define a **D**-complex structure K by:

$$\begin{array}{ccc} K(e^1) = e^1 & K(e^2) = -e^2 & K(e^3) = e^3 \\ K(e^4) = -e^4 & K(e^5) = e^5 & K(e^6) = -e^6, \end{array}$$

and set

$$\omega = +\frac{1}{\sqrt[3]{6}}e^1 \wedge e^6 + \frac{1}{\sqrt[3]{6}}e^3 \wedge e^4 - \frac{1}{\sqrt[3]{6}}e^5 \wedge e^2$$

which is a $(1, 1)$-closed form. Moreover we have the $(1, 0)$-forms:

$$\begin{aligned} \varphi^1 &:= \frac{(e^1 + e^2)}{2} + \frac{\tau(e^1 - e^2)}{2}, \\ \varphi^2 &:= \frac{(e^3 + e^4)}{2} + \frac{\tau(e^3 - e^4)}{2}, \\ \varphi^3 &:= \frac{(e^5 + e^6)}{2} + \frac{\tau(e^5 - e^6)}{2} \end{aligned}$$

and we define a $(3, 0)$ form $\Phi = \varphi^1 \wedge \varphi^2 \wedge \varphi^3$ closed and no-where vanishing and we have that:

$$\Phi \wedge \overline{\Phi} = -\tau\omega^3.$$

Again, by Proposition 4.1.7, we get an other example of Ricci-flat Kähler **D**-manifold.

4.3 The space $\Omega^2(TM)$

Our goal is to extend some results from the theory of Ricci-flat **D**-Kähler manifolds to other manifolds, dropping out the **D**-Kähler condition. But to obtain this, we have that K is no more parallel with respect to Levi-Civita connection (if not, by Proposition 1.5.3 it follows that the manifold is **D**-Kähler). As in the complex case, there are two ways to work with non-Kähler manifolds. The first one is working with Levi-Civita connection despite the fact that $\mathcal{D}K \neq 0$. A result that can be obtained in this way is a classification of almost **D**-Hermitian structures: by studying the behavior of $\mathcal{D}\omega$ (where ω is the fundamental 2-form), P. Gadea and J.M. Masque [35] obtained a classification of such structures similar to the classification of almost Hermitian structures made by A. Gray and L.M. Hervella [38].

The second way is to find some other connections ∇ with torsion, preserving both K and g (in fact, we use metric connections).

However, in this section we follow the second way, so we will study the set of ***D**-Hermitian canonical connections*:

$$\{\nabla g = \nabla K = 0\} \tag{4.3.1}$$

in a similar way as done by P. Gouduchon on almost complex Hermitian manifolds [37]. The study of this set has been done in the **D**-complex case by S. Ivanov and S. Zamkovoy [45]. We will recall some properties of such connection and try to complete the picture of these connections.

We begin with the study of the space $\Omega^2(TM)$.

Let (M, g, K) be an almost **D**-Hermitian manifold. The integrability and the closeness of the fundamental 2-form are not necessary conditions through this section.

First of all, consider the space of TM-valued 2-forms, that we will denoted by $\Omega^2(TM)$, and the same notation will be used for the space of sections. As obvious, such a space is isomorphic to $TM \otimes \wedge^2 TM$, hence a $B \in \Omega^2(TM)$ can be identified with a tri-linear form which is anti-symmetric on the last two entries, i.e.:

$$B(X, Y, Z) = g(X, B(Y, Z)). \tag{4.3.2}$$

In particular, $\wedge^3 M \subset \Omega^2(TM)$, and the *Bianchi projector*:

$$\begin{aligned} \mathfrak{b} : \Omega^2(TM) &\longrightarrow \wedge^3 M \\ \mathfrak{b}(B)(X, Y, Z) &= \frac{1}{3}\left(B(X, Y, Z) + B(Y, Z, X) + B(Z, X, Y)\right) \end{aligned} \tag{4.3.3}$$

is surjective, and $\ker \mathfrak{b}$ is the set of all the B satisfying the Bianchi identity.

Moreover, fixed an orthonormal basis $\{X_1, \ldots, X_{2n}\}$ for g such that

$$g(X_i, X_i) = \begin{cases} +1 & \text{if } 0 \leqslant i \leqslant n \\ -1 & \text{if } n+1 \leqslant i \leqslant 2n \end{cases} \quad \text{and } X_{n+i} = KX_i \text{ for } 0 \leqslant i \leqslant n, \tag{4.3.4}$$

we define the *trace projector* by:

$$\begin{aligned} \mathrm{tr} : \Omega^2(TM) &\longrightarrow \wedge^1 M \\ \mathrm{tr}B(X) &= \sum_{i=1}^{2n} B(X_i, X_i, X). \end{aligned} \tag{4.3.5}$$

Because of the trace is onto, we want to define a function, denoted by $\widetilde{\cdot}$, from the space of 1-forms into the space $\Omega^2(TM)$ such that for a 1-form α it happens that $\mathrm{tr}(\widetilde{\alpha}) = \alpha$. This function is defined by:

$$\begin{aligned} \widetilde{\cdot} : \wedge^1 M &\longrightarrow \Omega^2(TM) \\ \widetilde{\alpha}(X, Y, Z) &= \frac{1}{2n-1}\big(\alpha(Z)g(X, Y) - \alpha(Y)g(X, Z)\big). \end{aligned} \tag{4.3.6}$$

It easily follows from (4.3.5) that if $\beta \in \wedge^3 M$ then $\mathrm{tr}\beta = 0$ and if $\alpha \in \wedge^1 M$ then $\mathfrak{b}\widetilde{\alpha} = 0$, and we get the following decomposition as in the complex case (see [37, Equations (1.2.4), (1.2.5)]):

$$\Omega^2(TM) = \wedge^1 M \oplus (\Omega^2(TM))^0 \oplus \wedge^3 M, \tag{4.3.7}$$

this means that any $B \in \Omega^2(TM)$ can be written as:

$$B = \widetilde{\mathrm{tr}B} + B^0 + \mathfrak{b}B, \tag{4.3.8}$$

where $\wedge^1 M$ and $\wedge^3 M$ are the images of (4.3.6) and of (4.3.3) respectively, and $(\Omega^2(TM))^0$ is the set of elements satisfying Bianchi identity and trace-free.

It is possible to divide the space $\Omega^2(TM)$ into 3 other sub-classes as follows:

Definition 4.3.1. We say that element $B \in \Omega^2(TM)$ is:

1. of type I if $B(KX, KY) = -B(X, Y)$,
2. of type II if $B(KX, Y) = KB(X, Y)$,
3. of type III if $B(KX, Y) = -KB(X, Y)$,

for any $X, Y \in TM$. The subspaces of Ω^2 will be denoted respectively by Ω^{I}, Ω^{II} and Ω^{III}. All elements $B \in \Omega^2(TM)$ can be written, accordingly with this decomposition, in the following way $B = B^{\mathrm{I}} + B^{\mathrm{II}} + B^{\mathrm{III}}$.

It follows that the projections over Ω^{I}, Ω^{II} and Ω^{III} are given respectively by:

$$\begin{aligned} B^{\mathrm{I}}(X,Y) &= \frac{1}{2}\big(B(X,Y) - B(KX,KY)\big), \\ B^{\mathrm{II}}(X,Y) &= \frac{1}{4}\big(B(X,Y) + KB(KX,Y) + KB(X,KY) + B(KX,KY)\big), \\ B^{\mathrm{III}}(X,Y) &= \frac{1}{4}\big(B(X,Y) - KB(KX,Y) - KB(X,KY) + B(KX,KY)\big). \end{aligned} \tag{4.3.9}$$

Remark 4.3.2. Some authors use the notation type $(1,1)$, $(2,0)$, $(0,2)$ for type I, II, III respectively (see e.g. S. Ivanov and S. Zamkovoy [45]). We stress that this type decomposition is different from the decomposition in (1.3.10) (i.e. the $(p+, q-)$-type decomposition) or that one in (1.3.15) (i.e. the $(p.q)$-type decomposition). In particular, these elements are real (we have not **D**-complexificated the tangent bundle) and the type II $= (2,0)$ and III $= (0,2)$ are not conjugate each other, as (4.3.9) shows. To avoid this confusion we prefer the notation as in Definition 4.3.1. However, these type decompositions are related, as the next Proposition 4.3.3 shows.

We introduce the *involution* `In` of $\Omega^2(TM)$ defined by:

$$\mathtt{In}B(X,Y,Z) = B(X,KY,KZ) \qquad X,Y,Z \in TM. \tag{4.3.10}$$

Note that $\mathtt{In}^2 = +\mathrm{Id}$ and the eigenspace with respect to $+1$ (resp. -1) is $\Omega^{\mathrm{II}} \oplus \Omega^{\mathrm{III}}$ (resp. Ω^{I}).

Any $\psi \in \wedge^3 M$ has hence two decompositions: one as a 3-form, and one as a section of $\Omega^2(TM)$. In other words, in addition to the type decomposition $\psi = \psi^{\mathrm{I}} + \psi^{\mathrm{II}} + \psi^{\mathrm{III}}$, as a 3-form we can write $\psi = \psi^+ + \psi^-$ where $\psi^+ = \psi^{2,1}_{+-} + \psi^{1,2}_{+-}$ and $\psi^- = \psi^{3,0}_{+-} + \psi^{0,3}_{+-}$ with respect to $\wedge^3 M = \wedge^{3,0}_{+-} M \oplus \wedge^{2,1}_{+-} M \oplus \wedge^{1,2}_{+-} M \oplus \wedge^{0,3}_{+-} M$. Explicitly we have:

$$\begin{aligned} \psi^+(X,Y,Z) &= \frac{1}{4}\big(3\psi(X,Y,Z) - \psi(X,KY,KZ) - \psi(KX,Y,KZ) - \psi(KX,KY,Z)\big), \\ \psi^-(X,Y,Z) &= \frac{1}{4}\big(\psi(X,Y,Z) + \psi(X,KY,KZ) + \psi(KX,Y,KZ) + \psi(KX,KY,Z)\big), \end{aligned} \tag{4.3.11}$$

and we will denote these spaces by $\wedge^{3,+}M$ and $\wedge^{3,-}M$.

These two decompositions are related by the following:

Proposition 4.3.3. *For any $\psi \in \wedge^3 M \subset \Omega^2(TM)$ it holds:*

$$\psi^- = \psi^{\mathrm{III}} \qquad \text{and} \qquad \psi^+ = \psi^{\mathrm{II}} + \psi^{\mathrm{I}}. \tag{4.3.12}$$

Proof. The first equality is a consequence of (4.3.9) and of (4.3.11) when we read the III part as a 3-form through the identification (4.3.2). The second one now follows easily by subtraction and by the splitting of $\Omega^2(TM)$ and of $\wedge^3 M$ as explained before (or it follows by a similar argument to the previous one). □

Moreover, from this proposition and the fact that $\mathtt{In}\psi^{\mathrm{I}} = -\psi^{\mathrm{I}}$ and that $\mathtt{In}\psi^{\mathrm{II}} = \psi^{\mathrm{II}}$ (and the same for the (III)-type), we can write:

$$\psi^{\mathrm{I}} = \frac{1}{2}\big(\psi^+ - \mathtt{In}(\psi^+)\big), \tag{4.3.13}$$

and hence

$$\psi^{\mathrm{II}} = \frac{1}{2}\big(\psi^+ + \mathtt{In}(\psi^+)\big). \tag{4.3.14}$$

Now we investigate the relationship between the decomposition of type I, II and III and the decomposition (4.3.8). For any of the subspaces Ω^{I}, Ω^{II}, Ω^{III} the following properties are valid:

Proposition 4.3.4. *The space Ω^{I} can be split in $\Omega^{\mathrm{I}}_s \oplus \Omega^{\mathrm{I}}_a$, where $\Omega^{\mathrm{I}}_s := \ker(\mathfrak{b}|_{\Omega^{\mathrm{I}}})$ and Ω^{I}_a is orthogonal to Ω^{I}_s.*
Moreover, the Bianchi projector $\mathfrak{b} : \Omega^{\mathrm{I}}_a \overset{\cong}{\to} \wedge^{3,+}M$ is an isomorphism, in particular, for any $A \in \Omega^{\mathrm{I}}_a$ we have that $\mathfrak{b}A \in \wedge^{3,+}M$ and

$$A = \frac{3}{4}(\mathfrak{b}A - \mathtt{In}(\mathfrak{b}A)), \tag{4.3.15}$$

i.e. on $\alpha \in \wedge^{3,+}M$ the inverse of Bianchi projector is $\frac{3}{4}(\alpha - \mathtt{In}\alpha) =: \mathfrak{b}^{-1}(\alpha) \in \Omega^{\mathrm{I}}_a$.

Proof. The first part of the proposition is obvious.
Let $A \in \Omega^{\mathrm{I}}_a$, and consider $\mathfrak{b}A \in \wedge^3$ (this because of (4.3.8)). We see that if $X, Y, Z \in TM^+$ then $A(X,Y,Z) = A(X,KY,KZ) = -A(X,Y,Z)$ hence $\mathfrak{b}A$ has no $(3+,0-)$-part. Analogously we have the vanishing of the $(0+,3-)$-part, and we get that $\mathfrak{b}A \in \wedge^{3,+}M$.
The Bianchi projector is injective, since if $\mathfrak{b}A = 0$, then $A \in \Omega^{\mathrm{I}}_s \cap \Omega^{\mathrm{I}}_a = \{0\}$ and $A = 0$.
To prove the surjectiveness, let α^+ be in $\wedge^{3,+}M$. By Proposition 4.3.3 $\alpha^+ = \alpha^{\mathrm{I}} + \alpha^{\mathrm{II}}$ and by (4.3.13) $\alpha^{\mathrm{I}} = \frac{1}{2}\big(\alpha^+ - \mathtt{In}(\alpha^+)\big)$. By using (4.3.13) we have:

$$\begin{aligned}\mathfrak{b}(\alpha^{\mathrm{I}}) &= \frac{1}{2}\big(\mathfrak{b}(\alpha^+) - \mathfrak{b}(\mathtt{In}\alpha^+)\big)\\ &= \frac{1}{2}\Big(\alpha^+ + \frac{1}{3}\alpha^+\Big) = \frac{2}{3}\alpha^+\end{aligned} \tag{4.3.16}$$

i.e. $\alpha^+ = \frac{3}{2}\mathfrak{b}(\alpha^{\mathrm{I}})$. Again using (4.3.13) we obtain (4.3.15). □

Proposition 4.3.5. *The Bianchi projector $\mathfrak{b} : \Omega^{\mathrm{II}} \overset{\cong}{\to} \wedge^{3,+}M$ is an isomorphism. More precisely, for any $B \in \Omega^{\mathrm{II}}$, we have that $\mathfrak{b}B \in \wedge^{3,+}M$ and*

$$B = \frac{3}{2}(\mathfrak{b}B + \mathtt{In}(\mathfrak{b}B)) \tag{4.3.17}$$

i.e. on $\alpha \in \wedge^{3,+}M$ the inverse of Bianchi projector is $\frac{3}{2}(\alpha + \mathtt{In}\alpha) =: \mathfrak{b}^{-1}(\alpha) \in \Omega^{\mathrm{II}}$.
Moreover, if $B \in \Omega^{\mathrm{II}}$ then $\mathrm{tr}B = 0$, that is B is trace-free.

Proof. Let $B \in \Omega^{\mathrm{II}}$, and consider $\mathfrak{b}B \in \wedge^3 M$ (this because of (4.3.8)). If $X, Y, Z \in TM^+$ then $B(X,Y,Z) = B(X,KY,Z) = -B(KX,Y,Z) = -B(X,Y,Z)$ hence $\mathfrak{b}B$ has no $(3+,0-)$-part. An analogous computation for the $(0+,3-)$-part gets that $\mathfrak{b}B \in \wedge^{3,+}M$.
To prove injectiveness, let $\mathfrak{b}B = 0$, and computing on (X, KY, KZ) we obtain:

$$\begin{aligned} 0 &= B(X,KY,KZ) + B(KY,KZ,X) + B(KZ,X,KY) + 0 \\ &= B(X,Y,Z) - B(Y,Z,X) - B(Z,X,Y) + 3\mathfrak{b}B(X,Y,Z) \\ &= 2B(X,Y,Z) \end{aligned} \tag{4.3.18}$$

i.e. $B = 0$.
Now take $\beta^+ \in \wedge^{3,+}M$, then Proposition 4.3.3 yields $\beta^+ = \beta^{\mathrm{I}} + \beta^{\mathrm{II}}$ and equation (4.3.14) gives $\beta^{\mathrm{I}} = \dfrac{1}{2}\big(\beta^+ + \mathtt{In}(\beta^+)\big)$. Using (4.3.14) we have:

$$\begin{aligned} \mathfrak{b}(\beta^{\mathrm{II}}) &= \frac{1}{2}\big(\mathfrak{b}(\beta^+) + \mathfrak{b}(\mathtt{In}\beta^+)\big) \\ &= \frac{1}{2}\Big(\beta^+ - \frac{1}{3}\beta^+\Big) = \frac{1}{3}\beta^+, \end{aligned} \tag{4.3.19}$$

i.e. $\beta^+ = 3\mathfrak{b}(\beta^{\mathrm{II}})$. From this last equality, and using (4.3.14), we obtain (4.3.17).
Finally, an easy computation shows that, if $B \in \Omega^{\mathrm{II}}$, then:

$$\begin{aligned} \mathrm{tr}B(X) &= \sum_{i=1}^{2n} B(e_i,e_i,X) = \sum_{i=1}^{n} B(e_i,e_i,X) + B(e_{n+i},e_{n+i},X) \\ &= \sum_{i=1}^{n} B(e_i,e_i,X) + B(Ke_i,Ke_i,X) = \sum_{i=1}^{n} B(e_i,e_i,X) - B(e_i,e_i,X) = 0. \end{aligned} \tag{4.3.20}$$

This ends the proof of the proposition. □

Proposition 4.3.6. *For any $C \in \Omega^{\mathrm{III}}$, we have that $\mathfrak{b}C \in \Omega^{\mathrm{III}}$. In particular as a 3-form, $\mathfrak{b}C \in \wedge^{3,-}M$.*

Proof. By (4.3.8), given $C \in \Omega^{\mathrm{III}}$ it holds $\mathfrak{b}C \in \wedge^3 M$. If $X, Y \in TM^+,\ Z \in TM^-$ then:

$$\begin{aligned} \mathfrak{b}C(X,Y,Z) &= \frac{1}{3}\big(C(X,Y,Z) + C(Y,Z,X) + C(Z,X,Y)\big) \\ &= \frac{1}{3}\big(-C(X,Y,KZ) - C(Y,KZ,X) + C(KZ,X,Y)\big) \\ &= \frac{1}{3}\big(-C(KX,Y,Z) - C(KY,Z,X) - C(Z,KX,Y)\big) \\ &= \frac{1}{3}\big(-C(X,Y,Z) - C(Y,Z,X) - C(Z,X,Y)\big) = -\mathfrak{b}C(X,Y,Z). \end{aligned} \tag{4.3.21}$$

Hence $\mathfrak{b}C$ has no $(2+,1-)$-part. Analogously for the $(1+,2-)$-part, and we get that $\mathfrak{b}C \in \wedge^{3,+}M$. By Proposition 4.3.3, we have also $\mathfrak{b}C \in \Omega^{\mathrm{III}}$. □

Remark 4.3.7. From Propositions 4.3.4 and 4.3.5 we see that there is an isomorphism Ψ between Ω^{II} and Ω^{I}_a:

$$\begin{aligned} \Psi : \Omega^{\mathrm{II}} &\xrightarrow{\cong} \Omega^{\mathrm{I}}_a \\ \Psi(B) &= \frac{3}{4}(\mathfrak{b}B - \mathtt{In}(\mathfrak{b}B)) \quad \text{for } B \in \Omega^{\mathrm{II}}, \\ \Psi^{-1}(A) &= \frac{3}{2}(\mathfrak{b}A + \mathtt{In}(\mathfrak{b}A)) \quad \text{for } A \in \Omega^{\mathrm{I}}_a. \end{aligned} \tag{4.3.22}$$

4.4 Special Hermitian connection on D-manifolds

We now have all the necessary tools to introduce the **D**-Hermitian connections.

Definition 4.4.1. A connection ∇ on an almost **D**-Hermitian manifold is said to be ***D**-Hermitian* if ∇ is both metric:

$$\nabla g = 0 \tag{4.4.1}$$

and the almost **D**-complex structure K is parallel for ∇ (or ∇ is *compatible* with the almost **D**-complex structure, briefly ∇ is ***D**-compatible*):

$$\nabla K = 0, \quad \text{i.e.} \quad \nabla_X KY = K\nabla_X Y \qquad X, Y \in TM. \tag{4.4.2}$$

We define the *potential* A for a **D**-Hermitian connection ∇ as:

$$A(X, Y, Z) := g(\nabla_X Y, Z) - g(\mathcal{D}_X Y, Z) \qquad X, Y, Z \in TM, \tag{4.4.3}$$

and the *torsion* of a connection as:

$$T^{\nabla}(X, Y) := \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]} \qquad X, Y \in TM, \tag{4.4.4}$$

(we will drop the upper index in T^{∇} when there is no confusion about the connection with respect to the torsion is considered).

Recall the following properties for the Levi-Civita connection $\mathcal{D}$:

$$\mathcal{D}g = 0 \quad \text{i.e.} \quad \mathcal{D}_X g(Y, Z) = Xg(Y, Z) = g(\mathcal{D}_X Y, Z) + g(Y, \mathcal{D}_X Z) \tag{4.4.5}$$

$$T^{\mathcal{D}} = 0 \quad \text{i.e.} \quad [X, Y] = \mathcal{D}_X Y - \mathcal{D}_Y X. \tag{4.4.6}$$

Remark 4.4.2. Note that while the torsion T^{∇} of any connection is an element of $\Omega^2(TM)$ (even if the connection is not **D**-compatible or is not metric), the same is not true for a generic potential. However, the potential of a **D**-Hermitian connection is in $\Omega^2(TM)$, since $\nabla g = 0$, in fact (using the identification (4.3.2)):

$$\begin{aligned} A(X, Y, Z) &= g(\nabla_X Y, Z) - g(\mathcal{D}_X Y, Z) \\ &= -g(Y, \nabla_X Z) + \nabla_X g(Y, Z) + g(Y, \mathcal{D}_X Z) - \mathcal{D}_X g(Y, Z) \\ &= -g(\nabla_X Z, Y) + Xg(Y, Z) + g(\mathcal{D}_X Z, Y) - Xg(Y, Z) \\ &= -g(\nabla_X Z, Y) + g(\mathcal{D}_X Z, Y) = -A(X, Z, Y). \end{aligned} \tag{4.4.7}$$

Remark 4.4.3. Setting $T(X, Y, Z) = g(X, T(X, Y))$, the potential and the torsion of a **D**-Hermitian connection are related by:

$$T = -A + 3\mathfrak{b}A \qquad A = -T + \frac{3}{2}\mathfrak{b}T. \tag{4.4.8}$$

In particular it follows from (4.4.8):

$$\mathfrak{b}A = \frac{1}{2}\mathfrak{b}T \qquad \mathrm{tr}A = -\mathrm{tr}T. \tag{4.4.9}$$

Moreover, the decomposition (4.3.8) yields $A^0 = -T^0$.

We conclude that any **D**-Hermitian connection is completely determined by its torsion (or, equivalently, by its potential).

Proof of (4.4.8) *and* (4.4.9). In fact:

$$\begin{aligned} T(X,Y,Z) &= g(X, \nabla_Y Z - \nabla_Z Y - [Y,Z]) \\ &= g(X, \nabla_Y Z) + g(\nabla_Z X, Y) - Zg(X,Y) + g(X, -\mathcal{D}_Y Z + \mathcal{D}_Z Y) \\ &= A(Y,Z,X) + g(\nabla_Z X, Y) - Zg(X,Y) - g(\mathcal{D}_Z X, Y) + Zg(X,Y) \\ &= A(Y,Z,X) + A(Z,X,Y) \pm A(X,Y,Z) = -A(X,Y,Z) + 3\mathfrak{b}A(X,Y,Z), \end{aligned} \tag{4.4.10}$$

while for the second one of (4.4.8), using that both ∇ and $\mathcal{D}$ are metrics and that Levi-Civita is torsion-free, we get:

$$\begin{aligned} 2A(X,Y,Z) &= 2g(\nabla_X Y, Z) \pm g(\nabla_Y X, Z) \pm g([X,Y], Z) - 2g(\mathcal{D}_X Y, Z) \\ &= T(Z,X,Y) + g(\nabla_X Y, Z) + g(\nabla_Y X, Z) + g([X,Y], Z) - 2g(\mathcal{D}_X Y, Z) \\ &= T(Z,X,Y) \pm g([Z,X], Y) \pm g(\nabla_Z X, Y) - g(Y, \nabla_X Z) + Xg(Y,Z) \\ &\quad - g(X, \nabla_Y Z) + Yg(X,Z) + g(\mathcal{D}_X Y - \mathcal{D}_Y X, Z) - 2g(\mathcal{D}_X Y, Z) \\ &= T(Z,X,Y) + T(Y,Z,X) + g([Z,X], Y) - g(\nabla_Z X, Y) + g(\mathcal{D}_X Y, Z) \\ &\quad + g(Y, \mathcal{D}_X Z) - g(X, \nabla_Y Z) + g(\mathcal{D}_Y X, Z) + g(X, \mathcal{D}_Y Z) - g(\mathcal{D}_X Y + \mathcal{D}_Y X, Z) \\ &= T(Z,X,Y) + T(Y,Z,X) + g(\mathcal{D}_Z X - \mathcal{D}_X Z, Y) + g(X, \nabla_Z Y) \\ &\quad - Zg(X,Y) + g(Y, \mathcal{D}_X Z) - g(X, \nabla_Y Z) + g(X, \mathcal{D}_Y Z) \\ &= T(Z,X,Y) + T(Y,Z,X) + g(\mathcal{D}_Z X, Y) + g(X, \nabla_Z Y) - g(\mathcal{D}_Z X, Y) \\ &\quad - g(X, \mathcal{D}_Z Y) - g(X, \nabla_Y Z) + g(X, \mathcal{D}_Y Z) \\ &= T(Z,X,Y) + T(Y,Z,X) + g(X, \nabla_Z Y - \nabla_Y Z) + g(X, \mathcal{D}_Y Z - \mathcal{D}_Z Y) \\ &= T(Z,X,Y) + T(Y,Z,X) + g(X, \nabla_Z Y - \nabla_Y Z - [Z,Y]). \end{aligned} \tag{4.4.11}$$

Hence by definitions of Bianchi projector (4.3.3) and of torsion (4.4.4) we have from the previous equation:

$$\begin{aligned} A(X,Y,Z) &= \frac{1}{2}\left(T(Z,X,Y) + T(Y,Z,X) + T(X,Y,Z) - 2T(X,Y,Z)\right) \\ &= -T(X,Y,Z) + \frac{3}{2}\mathfrak{b}T(X,Y,Z). \end{aligned} \tag{4.4.12}$$

Now the first one of (4.4.9) follows from (4.4.8):

$$3\mathfrak{b}A = T + A = \frac{3}{2}\mathfrak{b}T \tag{4.4.13}$$

and the second one from the fact that $\mathrm{tr}(\mathfrak{b}) = 0$ (see (4.3.5)):

$$\mathrm{tr}A = -\mathrm{tr}T + \frac{3}{2}\mathrm{tr}(\mathfrak{b}T) = -\mathrm{tr}T. \tag{4.4.14}$$

This concludes the proof. □

Remark 4.4.4. On an almost **D**-Hermitian manifold, also the following tensors can be considered as elements of $\Omega^2(TM)$: the Nijenhuis tensor

$$N_K(X;Y,Z) = g(X, N_K(Y,Z)), \tag{4.4.15}$$

and the covariant derivative of the **D**-Kähler form with respect to the Levi-Civita connection

$$\mathcal{D}\omega(X,Y,Z) = \big(\mathcal{D}_X\omega\big)(Y,Z). \tag{4.4.16}$$

Indeed, while the first one is an immediate consequence of the definition of Nijenhuis tensor, for the second one we have:

$$\begin{aligned}\big(\mathcal{D}_X\omega\big)(Y,Z) &= X\omega(Y,Z) - \omega(\mathcal{D}_X Y, Z) - \omega(Y, \mathcal{D}_X Z)\\ &= -X\omega(Z,Y) + \omega(Z,\mathcal{D}_X Y) + \omega(\mathcal{D}_X Z, Y) = -\big(\mathcal{D}_X\omega\big)(Y,Z)\end{aligned} \tag{4.4.17}$$

which justifies that $\mathcal{D}\omega(X,Y,Z) \in \Omega^2(TM)$.

We have the following proposition, which explains the relations between N_K, $d\omega$, $\mathcal{D}\omega(X,Y,Z)$ and the decomposition type exposed before.

Proposition 4.4.5 ([45, Proposition 3.1])**.** *We have, on an almost **D**-Hermitian manifold, the following properties:*

1. *the Nijenhuis tensor N_K is of type* III. *Moreover we have:*

$$3\mathfrak{b}N_K = 4(d^{\mathbf{D}}\omega)^-; \tag{4.4.18}$$

2. *the component $(\mathcal{D}\omega)^{\mathrm{I}}$ vanishes identically, i.e. we have:*

$$\mathcal{D}\omega = (\mathcal{D}\omega)^{\mathrm{II}} + (\mathcal{D}\omega)^{\mathrm{III}}; \tag{4.4.19}$$

3. *more precisely, we have that the component $(\mathcal{D}\omega)^{\mathrm{III}}$ is determined by N_K as follows:*

$$\begin{aligned}(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) &= \frac{1}{4}\big(N_K(KX,Y,Z) - N_K(KY,Z,X) - N_K(KZ,X,Y)\big)\\ &= -\frac{1}{2}\left(g(\mathcal{D}_X(K)Y + \mathcal{D}_{KX}(K)KY, Z)\right)\end{aligned} \tag{4.4.20}$$

or, equivalently:

$$(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) = (d\omega)^-(X,Y,Z) + \frac{1}{2}N_K(KX,Y,Z); \tag{4.4.21}$$

4. *the component $(\mathcal{D}\omega)^{\mathrm{II}}$ is determinated by $(d\omega)^+$, more precisely:*

$$(\mathcal{D}\omega)^{\mathrm{II}}(X,Y,Z) = \frac{1}{2}\big(d\omega^+(X,Y,Z) + d\omega^+(X,KY,KZ)\big). \tag{4.4.22}$$

Proof. We separate the proof into four points.

Proof of 1. From (1.3.4) we have:

$$N_K(KX,Y) = -KN_K(X,Y) \tag{4.4.23}$$

then $N_K \in \Omega^{\mathrm{III}}$. Using (4.3.11) we have:

$$\begin{aligned}(d^{\mathbf{D}}\omega)^-(X,Y,Z) &= \frac{1}{4}\big(d^{\mathbf{D}}\omega(X,Y,Z) + d^{\mathbf{D}}\omega(X,KY,KZ) +\\ &\quad d^{\mathbf{D}}\omega(KX,Y,KZ) + d^{\mathbf{D}}\omega(KX,KY,Z)\big)\\ &= -\frac{1}{4}\big(+d\omega(KX,KY,KZ) + d\omega(KX,Y,Z)\\ &\quad + d\omega(X,KY,Z) + d\omega(X,Y,KZ)\big).\end{aligned} \tag{4.4.24}$$

We expand using the following relation for the differential $d\alpha$ of a 2-form α

$$\begin{aligned}d\alpha(X,Y,Z) &= X\alpha(Y,Z) + Y\alpha(Z,X) + Z\alpha(X,Y)\\ &\quad - \alpha([X,Y],Z) - \alpha([Y,Z],X) - \alpha([Z,X],Y)\end{aligned} \tag{4.4.25}$$

we get:

$$
\begin{aligned}
(d^{\mathbf{D}}\omega)^-(X,Y,Z) = -\frac{1}{4}\big(&KXg(KY,Z)+KYg(KZ,X)+KZg(KZX,Y)\\
&-g([KX,KY],Z)-g([KY,KZ],X)-g([KZ,KX],Y)\\
&+KXg(Y,KZ)+Yg(Z,X)+Zg(KX,KY)\\
&-g([KX,Y],KZ)-g([Y,Z],X)-g([Z,KX],KY)\\
&+Xg(KY,KZ)+KYg(Z,KX)+Zg(X,Y)\\
&-g([X,KY],KZ)-g([KY,Z],KX)-g([Z,X],Y)\\
&+Xg(Y,Z)+Yg(KZ,KX)+KZg(X,KY)\\
&-g([X,Y],Z)-g([Y,KZ],KX)-g([KZ,X],KY)\big).
\end{aligned}
\tag{4.4.26}
$$

We simplify using (1.4.2) and (4.3.3) to obtain

$$
\begin{aligned}
(d^{\mathbf{D}}\omega)^-(X,Y,Z) &= \frac{1}{4}\big(g(Z,[KX,KY]+[X,Y]-K[KX,Y]-K[X,KY])\\
&\quad+g(X,[KY,KZ]+[Y,Z]-K[KY,Z]-K[Y,KZ])\\
&\quad+g(Y,[KZ,KX]+[Z,X]-K[KZ,X]-K[Z,KX])\big)\\
&= \frac{1}{4}\big(g(Z,N_K(X,Y))+g(X,N_K(Y,Z)+g(Y,N_K(Z,X))\big)\\
&= \frac{3}{4}\mathfrak{b}N_K(X,Y,Z),
\end{aligned}
\tag{4.4.27}
$$

which concludes the proof of 1.

Proof of 2. An easy computation shows:

$$
\begin{aligned}
\mathcal{D}\omega(X,Y,Z) &= (\mathcal{D}_X\omega)(Y,Z)\\
&= \mathcal{D}_X\omega(Y,Z)-\omega(\mathcal{D}_XY,Z)-\omega(Y,\mathcal{D}_XZ)\\
&= Xg(Y,KZ)-g(\mathcal{D}_XY,KZ)-g(Y,K\mathcal{D}_XZ)\\
&= Xg(Y,KZ)-Xg(Y,KZ)+g(Y,\mathcal{D}_XKZ)+g(KY,\mathcal{D}_XZ)\\
&= g(Y,\mathcal{D}_XKZ)+Xg(KY,Z)-g(\mathcal{D}_XKY,Z)\\
&= -g(KY,K\mathcal{D}_XKZ)+Xg(KY,Z)-g(\mathcal{D}_XKY,Z)\\
&= -\omega(KY,\mathcal{D}_XKZ)+\mathcal{D}_X\omega(KY,KZ)-\omega(\mathcal{D}_XKY,KZ)\\
&= (\mathcal{D}_X\omega)(KY,KZ)=\mathcal{D}\omega(X,KY,KZ),
\end{aligned}
\tag{4.4.28}
$$

and by (4.3.9) we have:

$$
\mathcal{D}\omega^{\mathrm{I}}(X,Y,Z)=\frac{1}{2}\big(\mathcal{D}_X\omega(Y,Z)-\mathcal{D}_X\omega(KY,KZ)\big)=0. \tag{4.4.29}
$$

Proof of 3. We first prove the (4.4.20). Using (4.3.3) we have:

$$
\begin{aligned}
(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) &= \frac{1}{4}\big((\mathcal{D}\omega)(X,Y,Z)+(\mathcal{D}\omega)(KX,KY,Z)\\
&\quad+(\mathcal{D}\omega)(KX,Y,KZ)+(\mathcal{D}\omega)(X,KY,KZ)\big)\\
&= \frac{1}{4}\big((\mathcal{D}_X\omega)(Y,Z)+(\mathcal{D}_{KX}\omega)(KY,Z)\\
&\quad+(\mathcal{D}_{KX}\omega)(Y,KZ)+(\mathcal{D}_X\omega)(KY,KZ)\big).
\end{aligned}
\tag{4.4.30}
$$

Hence:

$$
\begin{aligned}
(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) = \frac{1}{4}\big(&Xg(Y,KZ) - g(\mathcal{D}_X Y, KZ) - g(Y, K\mathcal{D}_X Z) \\
&+ KXg(KY,KZ) - g(\mathcal{D}_{KX}KY,KZ) - g(KY,K\mathcal{D}_{KX}Z) \\
&+ KXg(Y,Z) - g(\mathcal{D}_{KX}Y,Z) - g(Y,K\mathcal{D}_{KX}KZ) \\
&+ Xg(KY,Z) - g(\mathcal{D}_X KY, Z) - g(KY, K\mathcal{D}_X KZ)\big) \\
= \frac{1}{4}\big(&-g(\mathcal{D}_X Y, KZ) + g(KY,\mathcal{D}_X Z) - g(\mathcal{D}_{KX}KY,KZ) + g(Y,\mathcal{D}_{KX}Z) \\
&- g(\mathcal{D}_{KX}Y,Z) + g(KY,\mathcal{D}_{KX}KZ) - g(\mathcal{D}_X KY,Z) + g(Y,\mathcal{D}_X KZ)\big).
\end{aligned}
\tag{4.4.31}
$$

Finally, since Levi-Civita is a metric connection:

$$
\begin{aligned}
(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) &= \frac{1}{4}\big(-g(\mathcal{D}_X Y,KZ) - g(\mathcal{D}_X KY,Z) + Xg(KY,Z) \\
&\quad - g(\mathcal{D}_{KX}KY,KZ) - g(\mathcal{D}_{KX}Y,Z) + KXg(Y,Z) \\
&\quad - g(\mathcal{D}_{KX}Y,Z) - g(\mathcal{D}_{KX}KY,KZ) + KXg(KY,KZ) \\
&\quad - g(\mathcal{D}_X KY,Z) - g(\mathcal{D}_X Y,KZ) + Xg(Y,KZ)\big) \\
&= -\frac{1}{2}\big(g(\mathcal{D}_X Y + \mathcal{D}_{KX}KY, KZ) + g(\mathcal{D}_{KX}Y + \mathcal{D}_X KY, Z)\big) \\
&= -\frac{1}{2}g(-K\mathcal{D}_X Y - K\mathcal{D}_{KX}KY + \mathcal{D}_{KX}Y + \mathcal{D}_X KY, Z) \\
&= -\frac{1}{2}g(\mathcal{D}_X(K)Y + \mathcal{D}_{KX}(K)Y, Z).
\end{aligned}
\tag{4.4.32}
$$

On the other hand we have:

$$
\begin{aligned}
N_K(KX,Y,Z) &- N_K(KY,Z,X) - N_K(KZ,X,Y) = \\
&= g(KX,[KY,KZ] - K[KY,Z] - K[Y,KZ] + [Y,Z]) \\
&\quad - g(KY,[KZ,KX] - K[KZ,X] - K[Z,KX] + [Z,X]) \\
&\quad - g(KZ,[KX,KY] - K[KX,Y] - K[X,KY] + [X,Y])
\end{aligned}
\tag{4.4.33}
$$

now, using that $\mathcal{D}$ is torsion-free:

$$
\begin{aligned}
N_K(KX,Y,Z) &- N_K(KY,Z,X) - N_K(KZ,X,Y) = \\
&= g(KX,\mathcal{D}_{KY}KZ - \mathcal{D}_{KZ}KY + \mathcal{D}_Y Z - \mathcal{D}_Z Y) \\
&\quad + g(X,\mathcal{D}_{KY}Z - \mathcal{D}_Z KY + \mathcal{D}_Y KZ - \mathcal{D}_{KZ}Y) \\
&\quad - g(KY,\mathcal{D}_{KZ}KX - \mathcal{D}_{KX}KZ + \mathcal{D}_Z X - \mathcal{D}_X Z) \\
&\quad - g(Y,\mathcal{D}_{KZ}X - \mathcal{D}_X KZ + \mathcal{D}_Z KX - \mathcal{D}_{KX}Z) \\
&\quad - g(KZ,\mathcal{D}_{KX}KY - \mathcal{D}_{KY}KX + \mathcal{D}_X Y - \mathcal{D}_Y X) \\
&\quad - g(Z,\mathcal{D}_{KX}Y - \mathcal{D}_Y KX + \mathcal{D}_X KY - \mathcal{D}_{KY}X).
\end{aligned}
\tag{4.4.34}
$$

Finally, using that $\mathcal{D}$ is metric we obtain:

$$\begin{aligned}
N_K(KX,Y,Z) &- N_K(KY,Z,X) - N_K(KZ,X,Y) = \\
&= -g(\mathcal{D}_{KY}KX,KZ) + KYg(KX,KZ) - g(\mathcal{D}_Y KX,Z) + Yg(KX,Z) \\
&\quad - g(KX,\mathcal{D}_{KZ}KY + \mathcal{D}_Z Y) - g(X,\mathcal{D}_Z KY + \mathcal{D}_{KZ}Y) \\
&\quad - g(\mathcal{D}_{KY}X,Z) + KYg(X,Z) - g(\mathcal{D}_Y X,KZ) + Yg(X,KZ) \\
&\quad + g(\mathcal{D}_{KZ}KY,KX) - KZg(KY,KX) \\
&\quad - g(\mathcal{D}_{KX}KY,KZ) + KXg(KY,KZ) \\
&\quad + g(\mathcal{D}_Z KY,X) - Zg(KY,X) - g(\mathcal{D}_X KY,Z) + Xg(KY,Z) \\
&\quad + g(\mathcal{D}_{KZ}Y,X) - KZg(Y,X) - g(\mathcal{D}_X Y,KZ) + Xg(Y,KZ) \\
&\quad + g(\mathcal{D}_Z Y,KX) - Zg(Y,KX) - g(\mathcal{D}_{KX}Y,Z) + KXg(Y,Z) \\
&\quad - g(KZ,\mathcal{D}_{KX}KY - \mathcal{D}_{KY}KX + \mathcal{D}_X Y - \mathcal{D}_Y X) \\
&\quad - g(Z,\mathcal{D}_{KX}Y - \mathcal{D}_Y KX + \mathcal{D}_X KY - \mathcal{D}_{KY}X) \\
&= -2\big(g(KZ,\mathcal{D}_{KX}KY + \mathcal{D}_X Y) + g(Z,\mathcal{D}_{KX}Y + \mathcal{D}_X KY)\big) \\
&= -2g(Z,-K\mathcal{D}_{KX}KY - K\mathcal{D}_X Y + \mathcal{D}_{KX}Y + \mathcal{D}_X KY) \\
&= -2g(KZ,\mathcal{D}_{KX}(K)Y + \mathcal{D}_X(K)Y).
\end{aligned} \tag{4.4.35}$$

Comparing this last equation with (4.4.32) we get the (4.4.20). Now the equation (4.4.21) follows from equations (4.4.18) and the fact that $N_K \in \Omega^{\mathrm{III}}$:

$$\begin{aligned}
(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) &= \frac{1}{4}\big(N_K(KX,Y,Z) - N_K(KY,Z,X) - N_K(KZ,X,Y)\big) \\
&= \frac{1}{4}\big(N_K(KX,KY,KZ) - N_K(KY,KZ,KX) \\
&\quad - N_K(KZ,KX,KY) \pm N_K(KX,KY,KZ)\big) \\
&= \frac{1}{2}N_K(KX,Y,Z) - \frac{3}{4}\mathfrak{b}N_K(KX,KY,KZ) \\
&= \frac{1}{2}N_K(KX,Y,Z) - (d^{\mathbf{D}}\omega)^-(KX,KY,KZ) \\
&= \frac{1}{2}N_K(KX,Y,Z) + (d\omega)^-(X,Y,Z).
\end{aligned} \tag{4.4.36}$$

Proof of 4. We have that

$$\mathfrak{b}(\mathcal{D}\omega) = \frac{1}{3}\,d\omega, \tag{4.4.37}$$

indeed:

$$\begin{aligned}
\mathfrak{b}(\mathcal{D}\omega)(X,Y,Z) &= \frac{1}{3}\left(\mathcal{D}\omega(X,Y,Z) + \mathcal{D}\omega(Y,Z,X) + \mathcal{D}\omega(Z,X,Y)\right) \\
&= \frac{1}{3}\big(X\omega(Y,Z) - \omega(\mathcal{D}_X Y,Z) - \omega(Y,\mathcal{D}_X Z) \\
&\quad + Y\omega(Z,X) - \omega(\mathcal{D}_Y Z,X) - \omega(Z,\mathcal{D}_Y X) \\
&\quad + Z\omega(X,Y) - \omega(\mathcal{D}_Z X,Y) - \omega(X,\mathcal{D}_Z Y)\big) \\
&= \frac{1}{3}\big(X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) \\
&\quad - \omega(\mathcal{D}_X Y - \mathcal{D}_Y X,Z) - \omega(\mathcal{D}_Z X - \mathcal{D}_X Z,Y) - \omega(\mathcal{D}_Y Z - \mathcal{D}_Z Y,X)\big)
\end{aligned} \tag{4.4.38}$$

and since $\mathcal{D}$ has no torsion:

$$\begin{aligned}\mathfrak{b}(\mathcal{D}\omega)(X,Y,Z) &= \frac{1}{3}\big(X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y)\\ &\quad - \omega([X,Y],Z) - \omega([Z,X],Y) - \omega([Y,Z],X)\big) = \frac{1}{3}\,d\omega(X,Y,Z).\end{aligned} \tag{4.4.39}$$

From (4.4.19), we split $\mathcal{D} = \mathcal{D}^{\mathrm{II}} + \mathcal{D}^{\mathrm{III}}$ getting:

$$\mathfrak{b}(\mathcal{D}\omega) = \mathfrak{b}(\mathcal{D}\omega^{\mathrm{II}}) + \mathfrak{b}(\mathcal{D}\omega^{\mathrm{III}}) = \frac{1}{3}\,d\omega^{+} + \frac{1}{3}\,d\omega^{-} \tag{4.4.40}$$

and by Propositions 4.3.5 and 4.3.6 we have:

$$\mathfrak{b}(\mathcal{D}\omega^{\mathrm{II}}) = \frac{1}{3}\,d\omega^{+}. \tag{4.4.41}$$

Finally, again using Proposition 4.3.5 we have:

$$\begin{aligned}\mathcal{D}\omega^{\mathrm{II}}(X,Y,Z) &= \frac{3}{2}\big(\mathfrak{b}(\mathcal{D}\omega^{\mathrm{II}})(X,Y,Z) + \mathtt{In}\big(\mathfrak{b}(\mathcal{D}\omega^{\mathrm{II}})\big)(X,Y,Z)\big)\\ &= \frac{3}{2}\big(\frac{1}{3}\,d\omega^{+}(X,Y,Z) + \mathtt{In}\frac{1}{3}\,d\omega^{+}(X,Y,Z)\big)\\ &= \frac{1}{2}\big(d\omega^{+}(X,Y,Z) + d\omega^{+}(X,KY,KZ)\big)\end{aligned} \tag{4.4.42}$$

which ends the proof. □

Proposition 4.4.6 ([45, Theorem 3.2])**.** *For any $\boldsymbol{D}$-Hermitian connection ∇, let T be its torsion, view as element of $\Omega^2(TM)$. Then:*

1. *The component T^{III} of T is independent of ∇ and verifies:*

$$T^{\mathrm{III}} = -4N_K; \tag{4.4.43}$$

where N_K is the Nijenhuis tensor of K.

2. *The skew symmetric part of $(T^{\mathrm{II}} - T^{\mathrm{I}}_a)$ is independent of ∇ and we have:*

$$\mathfrak{b}(T^{\mathrm{II}} - T^{\mathrm{I}}_a) = -\frac{1}{3}(d^{\boldsymbol{D}}\omega)^{+}. \tag{4.4.44}$$

Equivalently, the sum $(T^{\mathrm{II}} + \Psi^{-1}T^{\mathrm{I}}_a) \in \Omega^{\mathrm{II}}$ is independent of ∇ and satisfies:

$$(T^{\mathrm{II}} + \Psi^{-1}T^{\mathrm{I}}_a) = (\mathcal{D}\omega)^{\mathrm{II}}(K\cdot,\cdot,\cdot) = -\frac{1}{2}\big((d^{\boldsymbol{D}}\omega)^{+} + \mathtt{In}(d^{\boldsymbol{D}}\omega)^{+}\big). \tag{4.4.45}$$

3. *T is entirely determined by its component T^{I}_s and its component $(\mathfrak{b}T)^{+}$, which can be chosen arbitrarily.*

Proof. A straightforward computation shows that the linear connection $\nabla = \mathcal{D} + A$ is Hermitian if and only if the potential A satisfies

$$A(X,KY,Z) + A(X,Y,KZ) = (\mathcal{D}\omega)(X,Y,Z). \tag{4.4.46}$$

From equation (4.4.8), we get that ∇ is Hermitian if and only if

$$T(X,KY,Z) + T(X,Y,KZ) - \frac{3}{2}\big(\mathfrak{b}T(X,KY,Z) + \mathfrak{b}T(X,Y,KZ)\big) = -(\mathcal{D}\omega)(X,Y,Z). \tag{4.4.47}$$

Now T^{I} satisfies, by definition, $T^{\mathrm{I}}(KY,KZ) = -T^{\mathrm{I}}(Y,Z)$, i.e. as an element of $TM \otimes \wedge^2 M$ it holds: $T^{\mathrm{I}}(X,KY,Z) = -T^{\mathrm{I}}(X,Y,KZ)$. Hence the previous equation (4.4.47) can be split in two equations taking the II-part and III-part, since its I-part is zero, getting the following:

$$2T^{\mathrm{III}}(KX,Y,Z) - 3\big((\mathfrak{b}T)^{\mathrm{III}}(KX,Y,Z)\big) = -(\mathcal{D}\omega)^{\mathrm{III}}(X,Y,Z) \tag{4.4.48}$$

$$-2T^{\mathrm{II}}(KX,Y,Z) - \frac{3}{2}\big((\mathfrak{b}T)^{\mathrm{II}}(X,KY,Z) + (\mathfrak{b}T)^{\mathrm{II}}(X,Y,KZ)\big) = -(\mathcal{D}\omega)^{\mathrm{II}}(X,Y,Z) \tag{4.4.49}$$

Then the assertion 1. follows from equations (4.4.48) and (4.4.21), since an easy computation shows that:

$$(d\omega)^{-}(X,Y,Z) = 3(\mathfrak{b}T)^{\mathrm{III}}(KX,Y,Z). \tag{4.4.50}$$

From (4.4.49), using Proposition 4.3.5 and Remark 4.3.7 we obtain:

$$\begin{aligned} -(\mathcal{D}\omega)^{\mathrm{II}}(KX,Y,Z) &= -2T^{\mathrm{II}}(X,Y,Z) - \frac{3}{2}\big((\mathfrak{b}T)^{\mathrm{II}}(KX,KY,Z) + (\mathfrak{b}T)^{\mathrm{II}}(KX,Y,KZ)\big) \\ &= -2T^{\mathrm{II}}(X,Y,Z) - 3(\mathfrak{b}T)^{\mathrm{II}}(KX,KY,Z) \\ &= T^{\mathrm{II}}(X,Y,Z) + \frac{3}{2}\mathfrak{b}(T)^{\mathrm{I}}(X,Y,Z) + \frac{3}{2}\mathfrak{b}(T)^{\mathrm{I}}(X,KY,KZ) \\ &= T^{\mathrm{II}}(X,Y,Z) + \Psi^{-1}(T)^{\mathrm{I}}(X,Y,Z) \end{aligned} \tag{4.4.51}$$

because of a long computation shows

$$-3(\mathfrak{b}T)^{\mathrm{II}}(KX,KY,Z) = \frac{3}{2}\mathfrak{b}(T)^{\mathrm{I}}(X,Y,Z) + \frac{3}{2}\mathfrak{b}(T)^{\mathrm{I}}(X,KY,KZ) + 3T^{\mathrm{II}}(X,Y,Z). \tag{4.4.52}$$

The equation (4.4.45) follows now from the point 4 of the previous Proposition 4.4.5.
Equation (4.4.44) now is obtained by a direct computation, or by (4.4.18) together with the following identities:

$$T^{\mathrm{II}} - T^{\mathrm{I}} = \frac{1}{4}N_K + \mathtt{In}(T) \tag{4.4.53}$$

$$3\mathfrak{b}\big(\mathtt{In}(T)\big) = -\,d^{\mathbf{D}}\,\omega. \tag{4.4.54}$$

Finally, (4.4.52) follows from the previous points 1. and 2. □

We get the following corollary:

Corollary 4.4.7. *More precisely, for any real* 3*-form* ψ^+ *of type* $(1,2)+(2,1)$ *and any section* B_s *of* Ω^{I_s}*, there exists a unique* ***D****-Hermitian connection whose torsion* T *satisfies the following two conditions:*

$$T^{\mathrm{I}}_s = B_s \qquad (\mathfrak{b}T)^+ = \psi^+. \tag{4.4.55}$$

The other parts of the torsion are determined by

$$\begin{aligned} \mathfrak{b}(T^{\mathrm{I}}_a) &= \frac{1}{2}\big(\psi^+ + \frac{1}{3}(d^{\boldsymbol{D}}\omega)^+\big) \\ \mathfrak{b}(T^{\mathrm{II}}) &= \frac{1}{2}\big(\psi^+ - \frac{1}{3}(d^{\boldsymbol{D}}\omega)^+\big) \\ T^{\mathrm{III}} &= N_K. \end{aligned} \tag{4.4.56}$$

Explicitly, the torsion is given by:

$$T = -\frac{1}{4}N_K - \frac{1}{8}(d^{\boldsymbol{D}}\omega)^+ - \frac{3}{8}\mathtt{In}(d^{\boldsymbol{D}}\omega)^+ + \frac{9}{8}\psi^+ + \frac{3}{8}\mathtt{In}(\psi^+) + B_s. \tag{4.4.57}$$

Proof. The equation (4.4.56) easy follows from equations (4.4.43) and (4.4.44) and the fact that $(\mathfrak{b}T)^+ = \mathfrak{b}(T)^{\mathrm{II}} + \mathfrak{b}(T)^{\mathrm{I}} = \psi^+$. The second part of the corollary is a consequence of the split $T = T_a^{\mathrm{I}} + T_s^{\mathrm{I}} + T^{\mathrm{II}} + T^{\mathrm{III}}$ and the Propositions 4.3.5 and 4.4.6 $\square$

We are ready to introduce the *set of canonical* ***D****-connection*:

Definition 4.4.8. A **D**-Hermitian connection is called *canonical* if its torsion T satisfies the following conditions:

$$T_s^{\mathrm{I}} = 0 \qquad (\mathfrak{b}T)^+ = -\frac{2t-1}{3}(d^{\mathbf{D}}\,\omega)^+ \tag{4.4.58}$$

for some real number t. We will denote by ∇^t the **D**-Hermitian canonical connection corresponding to a parameter $t \in \mathbb{R}$.

Combining with (4.4.57), we see that the torsion T^t of a **D**-Hermitian canonical connection ∇^t is given by:

$$T^t = -N_K - \frac{3t-1}{4}(d^{\mathbf{D}}\,\omega)^+ - \frac{t+1}{4}\mathtt{In}(d^{\mathbf{D}}\,\omega)^+. \tag{4.4.59}$$

Moreover, by (4.4.8), the connection ∇^t itself is related to the Levi-Civita connection $\mathcal{D}$ by:

$$g(\nabla^t_X Y, Z) = g(\mathcal{D}_X Y, Z) - T^t(X,Y,Z) + \frac{3}{2}\mathfrak{b}T^t(X,Y,Z) \tag{4.4.60}$$

and by Proposition 4.4.5 we have also:

$$g(\nabla^t_X Y, Z) = g(\mathcal{D}_X Y, Z) - \frac{1}{2}g(\mathcal{D}_X K)(KY,Z) - \frac{t}{4}\big((d^{\mathbf{D}}\,\omega)^+(X,Y,Z) - (d^{\mathbf{D}}\,\omega)^+(X,KY,KZ)\big). \tag{4.4.61}$$

In the set of canonical **D**-Hermitian connections $\{\nabla^t \mid t \in \mathbb{R}\}$ we distinguish the following ones:

$t = 0$**.** The canonical connection ∇^0, called the *first canonical connection*, is the orthogonal projection of the Levi-Civita connection $\mathcal{D}$ into the affine space of the **D**-Hermitian connections. This connection is characterized by the conditions:

$$T_s^{\mathrm{I}} = 0 \qquad T^{\mathrm{II}} = 0. \tag{4.4.62}$$

In particular if K is integrable, the torsion of ∇^0 is of type I. Moreover, if $(d\,\omega)^+ = 0$, all the canonical connections degenerate to the first canonical connection ∇^0.

$t = -1$**.** The connection ∇^{-1} is characterized by the condition:

$$T + N_K \qquad \textit{is totally skew-symmetric} \tag{4.4.63}$$

i.e. $T + N_K$ is a 3-form. In particular, if K is integrable, ∇^{-1} is characterized by its torsion being totally skew-symmetric. We shall call this connection *the Bismut connection*, since in the complex case such a connection has been considered by J.M. Bismut.

$t = 1$**.** The connection ∇^1 is called *the second canonical connection* or *the Chern connection*. It is characterized by the condition:

$$T^{\mathrm{I}} = 0. \tag{4.4.64}$$

We focus on the Chern connection for which we have another characterization by using the relation with the intrinsic operator $\overline{\partial}_K$ defined before (see Chapter 2).

Remark 4.4.9. We see that any **D**-Hermitian connection determines a "Cauchy-Riemann operator", denoted by $\overline{\partial}^{\nabla}$ and defined as the $(0,1)$-part of ∇:

$$\overline{\partial}^{\nabla}_X Y = \frac{1}{2}\big(\nabla_X Y + K\nabla_{KX}Y\big). \tag{4.4.65}$$

In this manner a similar operator, denoted by $\overline{\partial}^t$, is attached to each canonical **D**-Hermitian connection ∇^t.

Proposition 4.4.10. *For any almost $\boldsymbol{D}$-Hermitian structure and any canonical connection ∇^t, the corresponding $\boldsymbol{D}$-Cauchy-Riemann operator $\overline{\partial}^t$ is related to the intrinsic Chauchy-Riemann operator $\overline{\partial}$ of K by:*

$$g(\overline{\partial}^t_X Y, Z) = g(\overline{\partial}_X Y, Z) + \frac{t-1}{4}\big((d^{\boldsymbol{D}}\omega)^+(X,Y,Z) - (d^{\boldsymbol{D}}\omega)^+(X,KY,KZ)\big). \tag{4.4.66}$$

Remark 4.4.11. In particular, for the Chern connection we have the identification, on the space of $(0,p)$-forms, of $\overline{\partial}^1 := (\nabla^1)^{0,1}$ with the usual operator $\overline{\partial}\,|\wedge^{0,p}_K := \pi_{\wedge^{0,p+1}_K} \circ d$.

Then, we have recovered on a (M,K) **D**-manifold with K integrable the following:

Corollary 4.4.12 ([45, Theorem 3.5])**.** *Let (M,K) be a $2n$-dimensional $\boldsymbol{D}$-Hermitian manifold, with integrable $\boldsymbol{D}$-structure K. Then:*

1. *there exists a unique $\boldsymbol{D}$-Hermitian connection ∇^1 on M with torsion $T^1 \in \Omega^2(TM)$ such that:*
$$T^1(KX,Y) = KT(X,Y). \tag{4.4.67}$$
This connection is the canonical connection obtained by $t = 1$ and given by:
$$g(\nabla^1_X Y, Z) = g(\mathcal{D}_X Y, Z) - \frac{1}{2}\, d\omega(KX,Y,Z). \tag{4.4.68}$$

2. *The curvature $R^1 := [\nabla^1,\nabla^1] - \nabla^1_{[\cdot,\cdot]}$ is of "type* I*" in the sense that*
$$R^1(KX,KY) = -R^1(X,Y). \tag{4.4.69}$$

Remark 4.4.13. If $N_K = 0$ and the **D**-structure is integrable, then the identification $\overline{\partial}^1 := (\nabla^1)^{0,1}$ with $\overline{\partial}$ (see Remark 4.4.11) can be extended on space of p-form. Hence the Chern connection is the unique connection such that:

$$\nabla^1 g = 0, \qquad \nabla^1 K = 0, \qquad \nabla^{0,1} = \overline{\partial}\,. \tag{4.4.70}$$

4.5 Some generalizations on minimal Lagrangian submanifolds

We recall some properties of Lagrangian submanifold of $\mathbf{D}^n$. Let LAG denote the set of oriented non-degenerate Lagrangian n-planes in $\mathbf{D}^n$.

The set LAG decomposes into $2n+2$ connected components

$$\mathrm{LAG} = \bigcup_{p+q=n} \mathrm{LAG}^{\pm}_{p,q} \tag{4.5.1}$$

where $\mathrm{LAG}^{\pm}_{p,q}$ consists of planes for which the induced metric has signature (p,q) and orientation $+$ or $-$ when compared to a fixed chosen model. In general such a positive model is the standard definite positive Lagrangian plane $\mathbb{R}^n$ in $\mathbf{D}^n$. Each $\mathrm{LAG}^{+}_{p,q}$ and

$\mathrm{LAG}^-_{p,q}$ is an orbit of the unitary group $\mathrm{U}^+_n(\mathbf{D})$, and the pair $\mathrm{LAG}^+_{p,q} \cup \mathrm{LAG}^-_{p,q}$ is an orbit of $\mathrm{U}_n(\mathbf{D})$.

Note that if we choose $\mathbb{R}^n$ as oriented model planes we have that

$$\pm\tau^q\, dz|_P = \exp(\tau\theta_P)\, d\,\mathrm{vol}_P \qquad \text{for } P \in \mathrm{LAG}^\pm_{p,q}, \tag{4.5.2}$$

where $\theta_P \in \mathbb{R}$ and $d\,\mathrm{vol}_P$ is the unit (positive) volume form on P.

If $L \subset \mathbf{D}^n$ is an oriented connected non-degenerate Lagrangian submanifold of signature p, q, then all its tangent planes lie either in $\mathrm{LAG}^+_{p,q}$ or in $\mathrm{LAG}^-_{p,q}$ depending on the orientation of L and we say that L is of *type* $\mathrm{LAG}^+_{p,q}$ or of *type* $\mathrm{LAG}^-_{p,q}$ respectively. Therefore by (4.5.2) we get that

$$\pm\tau^q\, dz|_L = \exp(\tau\theta)\, d\,\mathrm{vol}_L. \tag{4.5.3}$$

The smooth real-valued function θ on L is the *phase function* on L, and L has *constant phase* if θ is constant.

We have the following important result.

Proposition 4.5.1 ([40, Proposition 16.3])**.** *Let $L \subset \boldsymbol{D}^n$ be an oriented non-degenerate Lagrangian submanifold which is connected. Then L is a minimal (mean curvature zero) submanifold if and only if L has constant phase.*

This proposition is a consequence of the following Lemma.

Lemma 4.5.2 ([40, Formula (16.3)])**.** *Let $L \subset \boldsymbol{D}^n$ be an oriented non-degenerate Lagrangian submanifold which is connected. Then, for any tangent vector field V on L we have:*

$$V(\theta) = g(KV, H_L) = i_{H_L}\omega, \tag{4.5.4}$$

where H_L is the mean curvature flow of L.

Remark 4.5.3. It has been proved by F.R. Harvey and H.B. Lawson [40] that the previous relations (Proposition 4.5.1 and Lemma 4.5.2) holds also for a Lagrangian submanifold L of a **D**-Kähler Ricci-flat manifold M (see [40, Remark 16.6]).

We will extend these results to a class of symplectic almost **D**-Hermitian manifolds which admit a no-where vanishing **D**-holomorphic n-form. To do this, the necessary tool will be the **D**-Chern connection (or first canonical connection) defined in the previous Section 4.4.

In fact in this section we will consider (M, g, K, ε), where K is an almost **D**-complex structure, ω a symplectic form such that $\omega(\cdot,\cdot) = g(\cdot, K\cdot)$ is closed, and $\varepsilon \in \wedge^{n,0}_K M$ is a $(n,0)$-form satisfying:

1. $\varepsilon \wedge \bar\varepsilon = \dfrac{(-1)^{\frac{n(n+1)}{2}}\tau^n}{n!}\omega^n$, i.e. $\varepsilon \wedge \bar\varepsilon$ is a multiple of the volume form of M, and hence is a no-where vanishing $(n,0)$-form,
2. ε is parallel with respect to the **D**-Chern connection, i.e. $\nabla^1\varepsilon = 0$.

We remind also some definitions. Let $\{e_1, \ldots, e_n\}$ be a local g-orthonormal frame for a submanifold N of M. Denote by $(\cdot)^\perp$ the normal component in TM with respect to N. The *mean curvature vector* of the submanifold N, with respect to a connection ∇, is $H_N = \sum_{i=1}^n (\nabla_{e_i} e_i)^\perp$. In particular, we are interested in the mean curvature vector of the Chern connection, denoted by

$$H^1_N = \sum_{i=1}^n (\nabla^1_{e_i} e_i)^\perp. \tag{4.5.5}$$

We also define the ***D**-complex mean curvature vector* of the submanifold N in a similar way to the definition of the complex mean curvature vector (see e.g. [23]) as:

$$\widetilde{H}_N = \sum_{i=1}^{n} (K\mathcal{D}_{e_i} Ke_i)^{\perp}. \tag{4.5.6}$$

We now state two lemmata.

Lemma 4.5.4. *Let N be a Lagrangian submanifold of an almost **D**-Hermitian manifold M, and fix an orthonormal frame $\{e_1, \dots, e_d\}$ of TN where $d = \dim N$. The Levi-Civita and the Chern connections satisfy the following equation:*

$$\sum_{j}^{d} g(\nabla^1 e_j, Ke_j) = \sum_{j}^{d} g(\mathcal{D}e_j, Ke_j). \tag{4.5.7}$$

Proof. The equality follows from a simple calculation, and the fact that $\nabla^1 = \mathcal{D} + \frac{1}{2}K\mathcal{D}(K)$ $= \frac{1}{2}(\mathcal{D} + K\mathcal{D}K)$ (see Section 4.4). Indeed, we have:

$$\begin{aligned}
\sum_j g(\nabla^1 e_j, Ke_j) &= \sum_j g\big(\tfrac{1}{2}(\mathcal{D} + K\mathcal{D}K)e_k, Ke_k\big) \\
&= \sum_j \frac{1}{2}\big(g(\mathcal{D}e_j, Ke_j) + g(K\mathcal{D}Ke_j, Ke_j)\big) \\
&= \sum_j \frac{1}{2}\big(g(\mathcal{D}e_j, Ke_j) - g(\mathcal{D}Ke_j, e_j)\big) \\
&= \sum_j \frac{1}{2}\big(g(\mathcal{D}e_j, Ke_j) + g(Ke_j, \mathcal{D}e_j) + \mathcal{D}g(e_j, Ke_j)\big).
\end{aligned} \tag{4.5.8}$$

We conclude because of $g(e_j, Ke_j) = \omega(e_j, e_j) = 0$ since N is Lagrangian. □

Remark 4.5.5. It has to be noted that the previous Lemma 4.5.4 does not require that the Lagrangian submanifold is non-degenerate nor of dimension $n = \frac{1}{2}\dim M$.

Lemma 4.5.6. *It holds:*

$$V(\theta) = \sum_k g(\nabla^1_V e_k, Ke_k). \tag{4.5.9}$$

Proof. Let V be a vector field tangent to the Lagrangian submanifold L, and fix $\{e_1, \dots, e_n\}$ an orthonormal frame of L. Then, since V is also a derivation, we get:

$$V(\theta) = \tau \exp(-\tau\theta) V(\exp(\tau\theta)) = \tau \exp(-\tau\theta) V(\varepsilon(e_1, \dots, e_n)). \tag{4.5.10}$$

Using that $\nabla^1 \varepsilon = 0$ we have:

$$V(\varepsilon(e_1, \dots, e_n)) = \sum_k \varepsilon(e_1, \dots, \nabla^1_V e_k, \dots, e_n). \tag{4.5.11}$$

We can write $\nabla^1_V e_k$ in the orthonormal components. In fact, since L is non-degenerate, $\{e_1, \dots, e_n, Ke_1, \dots, Ke_n\}$ defines a local orthonormal frame of TM, hence:

$$\nabla^1_V e_k = \sum_i g(\nabla^1_V e_k, e_i)e_i + g(\nabla^1_V e_k, Ke_i)Ke_i, \tag{4.5.12}$$

and the fact that ∇^1 is metric implies that:

$$0 = (\nabla^1 g)(e_i, e_i) = \nabla^1(g(e_i.e_i)) - g(\nabla^1 e_i, e_i) - g(e_i, \nabla^1 e_i) = -2g(\nabla^1 e_i, e_i). \tag{4.5.13}$$

In a local frame we have that $\varepsilon(e_i, Ke_i, , W_1, \dots, W_n) = \varepsilon(e_i, e_i, W_1, \dots, W_n) = 0$ for any $W_i \in TL$, then:

$$\begin{aligned} V(\varepsilon(e^1, \dots, e^n)) &= \sum_k \varepsilon(e_1, \dots, \nabla^1_V e_k, \dots, e_n) \\ &= \sum_k \varepsilon(e_1, \dots, g(\nabla^1_V e_k, Ke_k)Ke_k, \dots, e_n) \\ &= \sum_k g(\nabla^1_V e_k, Ke_k)\varepsilon(e_1, \dots, Ke_k, \dots, e_n) \\ &= \sum_k g(\nabla^1_V e_k, Ke_k)\tau\exp(\tau\theta). \end{aligned} \tag{4.5.14}$$

Then, from (4.5.11) and (4.5.14), we have:

$$V(\theta) = \sum_k g(\nabla^1_V e_k, Ke_k), \tag{4.5.15}$$

which concludes the proof. □

We can now state the main result:

Theorem 4.5.7. *Let (M, g, K) be an almost $\boldsymbol{D}$-Hermitian manifold such that the fundamental 2-form ω is closed and there exists a no-where vanishing $(n, 0)$-form ε that is parallel with respect to the $\boldsymbol{D}$-Chern connection, i.e. $\nabla^1\varepsilon = 0$. Let $L \subset M$ be an oriented non-degenerate Lagrangian submanifold of M. Then for any vector $V \in TL$ tangent to the Lagrangian submanifold it holds:*

$$V(\theta) = -i_{\widetilde{H}_L}\omega = -i_{H^1_L}\omega + \sum_{i=1} g(V, T^1(e_i, e_i)). \tag{4.5.16}$$

Proof. From the definition of torsion (4.4.4) we obtain:

$$\begin{aligned} \sum_k g(\nabla^1_V e_k, Ke_k) &= \sum_k g(\nabla^1_{e_k} V, Ke_k) + \sum_k g([V, e_k], Ke_k) + g(T^1(V, e_k), Ke_k) \\ &= -\sum_k g(K\nabla^1_{e_k} V, e_k) + g(T^1(V, e_k), Ke_k) \end{aligned} \tag{4.5.17}$$

because of $\omega|_L = 0$ and $[V, e_k] \in TL$. Recalling that $\nabla^1 K = 0$ gives us:

$$-\sum_k g(K\nabla^1_{e_k} V, e_k) + g(T^1(V, e_k), Ke_k) = -\sum_k g(\nabla^1_{e_k} KV, e_k) + g(T^1(V, e_k), Ke_k), \tag{4.5.18}$$

and finally, since ∇^1 is metric, we have:

$$\begin{aligned} -\sum_k g(\nabla^1_{e_k} KV, e_k) + g(T^1(V, e_k), Ke_k) &= \sum_k g(KV, \nabla^1_{e_k} e_k) \\ &\quad + e_k g(KV, e_k) + g(T^1(V, e_k), Ke_k) \\ &= \sum_k g(KV, \nabla^1_{e_k} e_k) + g(T^1(V, e_k), Ke_k). \end{aligned} \tag{4.5.19}$$

Hence from this last equation and (4.5.5) and (4.5.9) we have the first part of theorem:

$$V(\theta) = -i_{H_L^1}\omega + \sum_k g(T^1(V, e_k), Ke_k). \tag{4.5.20}$$

Now, coming back to (4.5.9), we can change ∇^1 with the Levi-Civita connection $\mathcal{D}$, as done in (4.5.8):

$$\begin{aligned}
\sum_k g(\nabla^1_V e_k, Ke_k) &= \sum_k g(\mathcal{D}_V e_k, Ke_k) \\
&= \sum_k g(\mathcal{D}_{e_k} V, Ke_k) + g([V, e_k], Ke_k) \\
&= -\sum_k g(V, \mathcal{D}_{e_k} Ke_k) + \mathcal{D}_{e_k} g(V, Ke_k) \\
&= \sum_k g(KV, K\mathcal{D}_{e_k} Ke_k) \\
&= -i_{\widetilde{H}_L}\omega(V),
\end{aligned} \tag{4.5.21}$$

where in the first equality we use that $\mathcal{D}$ is torsion-free, in the second one that $\mathcal{D}$ is metric, and $g([V, e_k], Ke_k) = g(V, Ke_k)$ because L is Lagrangian. This concludes the proof. □

The following corollary is an easy consequence of this theorem, and it underlines the difference with the complex case, where the phase of a Lagrangian plane is related with the Maslov class, and hence it defines a closed form.

Corollary 4.5.8. *In the same hypothesis of Theorem 4.5.7, $V(\theta)$ defines a class in H^1 if and only if H_L is an Hamiltonian field for ω.*

Corollary 4.5.9. *In the same hypothesis of Theorem 4.5.7, a Lagrangian submanifold is minimal for the **D**-complex mean curvature if and only if it has constant phase.*

Proof. The proof is an easy consequence of the main Theorem 4.5.7. □

Bibliography

[1] Alekseevsky D.V., Medori C.: Bi-isotropic decompositions of semisimple Lie algebras and homogeneous bi-Lagrangian manifolds, *J. Algebra*, **313** (2007), No. 1, 8–27.

[2] Alekseevsky D.V., Medori C., Tomassini A.: Homogeneous para-Kähler Einstein manifolds, (English. Russian original) *Russ. Math. Surv.*, **64** (2009), No. 1, 1–43; translation from *Usp. Mat. Nauk*, **64** (2009), No. 1(385), 3–50.

[3] Alekseevsky D.V., Medori C., Tomassini A.: Maximally homogeneous para-CR manifolds of semisimple type, in *Handbook of pseudo-Riemannian geometry and supersymmetry*, 559–577, IRMA Lect. Math. Theor. Phys., **16**, Eur. Math. Soc., Zürich, (2010). DOI: `http://dx.doi.org/10.4171/079`.

[4] Alekseevsky D.V., Spiro A.F.: Prolongations of Tanaka structures and regular CR strctures, in *Selected topics in Cauchy-Riemann geometry*, 1–37, Quad. Mat., **9**, Department of Mathematics Seconda Università di Napoli, Caserta, (2001).

[5] Andrada A.: Complex product structures on 6-dimensional nilpotent Lie algebras, *Forum Math.*, **20** (2008), No. 2, 285–315.

[6] Andrada A., Barberis M.L., Dotti I.G., Ovando G.P., Product structures on four dimensional solvable Lie algebras, *Homology Homotopy Appl.* **7** (2005), No. 1, 9–37.

[7] Andrada A., Salamon S.: Complex product structures on Lie algebras, *Forum Math.*, **17** (2005), No. 2, 261–295.

[8] Angella D., Franzini M.G., Rossi F.A.: Degree of non-Kählerianity for 6-dimensional nilmanifolds, available Preprint on-line `arXiv: 1210.0406 [math.DG]`

[9] Angella D., Rossi F.A.: Cohomology of **D**-complex manifolds, *Differ. Geom. and App.*, **30** (2012), 530–547. DOI: `10.1016/j.difgeo.2012.07.003`. Available preprint on-line `arXiv: 1201.2503 [math.DG]`.

[10] Angella D., Tomassini A.: On cohomological decomposition of almost-complex manifolds and deformations, *J. Symplectic Geom.* **9** (2011), No. 3, 403–428.

[11] Angella D., Tomassini A.: On the cohomology of almost-complex manifolds, *Internat. J. Math.* **23**, 1250019 (2012), 25 pp., DOI: `10.1142/S0129167X11007604`.

[12] Angella D., Tomassini A.: On the $\partial\overline{\partial}$-Lemma and Bott-Chern cohomology, to appear in *Invent. Math.*, DOI: `10.1007/s00222-012-0406-3`.

[13] Berger M.: Les espaces symétriques noncompacts, *Ann. Sci. École Norm. Sup. (3)*, **74** (1957), 85–177.

[14] Besse, A.L.: *Einstein manifolds* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **10**, Springer-Verlag, Berlin, (1987). Reprint in Classics in Mathematics, Springer-Verlag, Berlin, (2008).

[15] Bock C.: *On Low-Dimensional Solvmanifolds*, PhD Thesis, `arXiv: 0903.2926v4 [math.DG]`.

[16] Bryant R.L.: Bochner-Kähler metrics, *J. Amer. Math. Soc.*, **14** (2001), No. 3, 623–715.

[17] Cartan H., Eilenberg S.: *Homological algebra*, Princeton Mathematical Series, **19**, Princeton University Press XV Princeton, New Jersey, (1956). Reprint in Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, (1999).

[18] Chursin M., Schäfer L., Smoczyk K.: Mean curvature flow of space-like Lagrangian submanifolds in almost para-Kähler manifolds., *Calc. Var. Partial Differential Equations* **41** (2011), No. 1-2, 111–125.

[19] Conti D., Salamon S.: Generalized Killing spinors in dimension 5, *Trans. Amer. Math. Soc.*, **359** (2007), No. 11, 5319–5343.

[20] Cortés V., Mayer C., Mohaupt T., Saueressig F.: Special Geometry of Euclidean Supersymmetry I: Vector Multiplets, *J. High Energy Phys.*, **No. 3** (2004), 028, 73 pp.

[21] Cruceanu V., Fortuny P., Gadea P.M.: A survey on paracomplex geometry, *Rocky Mountain J. Math.*, **26** (1996), 83–115.

[22] de Bartolomeis P., Meylan F.: Intrinsic deformation theory of CR structures, *Ann. Sc. Norm. Super. Pisa Cl. Sc.*, Vol. IX **3**, (2010), 459–494.

[23] de Bartolomeis P., Tomassini A.: On the Maslov index of Lagrangian submanifolds of generalized Calabi-Yau manifolds, *Internat. J. Math.*, **17** (2006), No. 8 921–947.

[24] de Rham G.: *Variétés différentiables. Formes, courants, formes harmoniques*, Publ. Inst. Math. Univ. Nancago III. Actualités Sci. Ind., no. 1222, Hermann et Cie, Paris, (1955). Translated from the French in *Differentiable manifolds. Forms, currents, harmonic forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 266, Springer-Verlag, Berlin, (1984).

[25] Deligne P., Griffiths P., Morgan J., Sullivan D.: Real homotopy theory of Kähler manifolds, *Invent. Math.*, **29** (1975), No. 3, 245–274.

[26] Drăghici T., Li T.-J., Zhang W.: Geometry of tamed almost complex structures in dimension 4, in *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, 233–251, AMS/IP Stud. Adv. Math., **51**, pt. 1, 2, Amer. Math. Soc., Providence, RI, (2012).

[27] Drăghici T., Li T.-J., Zhang W.: Symplectic forms and cohomology decomposition of almost complex four-manifolds *Int. Math. Res. Not. IMRN*, **2010** (2010), No. 1, 1–17.

[28] Ehresmann C.: Sur les espaces fibrés différentiables, *C. R. Acad. Sci. Paris*, **224** (1947), 1611–1612.

[29] Erdem S.: Some para-Hermitian related complex structures and non-existence of semi-Riemannian metric on some speheres, *Serdica Math. J.*, **25** (1999), 83–90.

[30] Etayo F., Santamaría R.: The canonical connection of a bi-Lagrangian manifold, *J. Phys. A.*, **34** (2001), No. 5, 981–987.

[31] Etayo F., Santamaría R., Trías U.R.: The geometry of a bi-Lagrangian manifold, *Differential Geom. Appl.*, **24** (2006), 33–59.

[32] Fino A., Grantcharov G.: Properties of manifolds with skew-symmetric torsion and special holonomy, *Adv. Math.*, **189** (2004), No. 2, 439–450.

[33] Fino A., Tomassini A.: On some cohomological properties of almost complex manifolds, *J. Geom. Anal.*, **20** (2010), 107–131.

[34] Fukaya, K.: Deformation theory, homological algebra and mirror symmetry, in: *Geometry and physics of branes (Como, 2001)*, 121–209, Ser. High Energy Phys. Cosmol. Gravit., IOP Bristol, (2003).

[35] Gadea P.M., Masque J.M.: Classification of almost para-Hermitian manifolds, *Rend. Mat. Appl. (7)*, **11** (1991), No. 2 377–396.

[36] Gadea P.M., Montesinos Amibilia A.: The paracomplex projective spaces as symmetric and natural spaces, *Indian J. Pure Appl. Math.* **23** (1992), No. 4, 261–275.

[37] Gauduchon P.: Hermitian connections and Dirac operators, *Boll. Un. Mat. Ital. B (7)* **11** (1997), No. 2 suppl., 257–288.

[38] Gray, A., Hervella, L.M.: The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl. (4)*, **123** (1980), 35–58.

[39] Greub W., Halperin S., Vanstone R.: *Connections, curvature, and cohomology. Volume III: Cohomology of principal bundles and homogeneous spaces*, Pure and Applied Mathematics, Vol. 47-III, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, (1976).

[40] Harvey F.R., Lawson H.B.Jr.: Split Special Lagrangian Geometry, (2010), *e-print* `arXiv: 1007.0450 [math.DG]`.

[41] Hattori A.: *Spectral sequence in the de Rham cohomology of fibre bundles*, *J. Fac. Sci. Univ. Tokyo Sect. I*, **8** (1960), No. 1960, 289–331.

[42] Hill C.D., Nurowski P.: Differential equations and para-CR structures, *Boll. Unione Mat. Ital. (9)*, **3** (2010), No. 1 25–91.

[43] Hitchin N.: The moduli space of special Lagrangian submanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25** (1997), No. 3-4, 503–515.

[44] Huybrechts D.: *Complex geometry. An introduction*, Universitext, Springer-Verlag, Berlin, (2005).

[45] Ivanov S., Zamkovoy S.: Parahermitian and paraquaternionic manifolds, *Differential Geom. Appl. 23*, (2005), No. 2, 205–234.

[46] Kaneyuki S.: Compactification of parahermitian symmetric spaces and its applications. II. Stratifications and automorphism groups, *J. Lie Theory* **13** (2003), No. 2, 535–563.

[47] Kaneyuki S., Kozai M.: Paracomplex structures and affine symmetric spaces, *Tokyo J. Math.* **8** (1985), No. 1, 81–98.

[48] Kim Y.-H., McCann R.J., Warren M.: Pseudo-Riemannian geometry calibrates optimal transportation, *Math. Res. Lett.* **17** (2010), No. 6, 1183–1197.

[49] Kobayashi S.: *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70, Springer-Verlag, New York-Heidelberg, (1972). Reprinted in Classics in Mathematics, Springer-Verlag, Berlin, (1995).

[50] Kodaira K.: *Complex manifolds and deformation of complex structures*, Grundlehren der Mathematischen Wissenschaften, **283**, Springer-Verlag, New York, (1986). Reprinted in Classics in Mathematics, Springer-Verlag, Berlin, (2005). Translated from the Japanese original by Kazuo Akao (1981).

[51] Kodaira K., Spencer D.C.: On deformations of complex analytic structures. I, II, *Ann. of Math. (2)*, **67** (1958), 328–466.

[52] Kodaira K., Spencer D.C.: On deformations of complex analytic structures. III. Stability theorems for complex structures, *Ann. of Math. (2)*, **71** (1960), 43–76.

[53] Krahe M., Para-pluriharmonic maps and twistor spaces, in *Handbook of pseudo-Riemannian geometry and supersymmetry*, 497–557, IRMA Lect. Math. Theor. Phys., **16**, Eur. Math. Soc., Zürich, 2010.

[54] Kuranishi, M.: On the locally complete families of complex analytic structures, *Ann. of Math. (2)*, **75** (1962), 536–577.

[55] Li T.-J., Zhang W., Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, *Comm. Anal. Geom.* **17** (2009), No. 4, 651–684.

[56] Libermann P.: Sur le problème d'équivalence de certaines structures infinitésimales, (French) *Ann. Mat. Pura Appl. (4)*, **36** (1954), 27–120.

[57] Libermann P.: Sur les structures presque paracomplexes, *C. R. Acad. Sci. Paris*, **234** (1952), 2517–2519.

[58] Lu J.-H., Weinstein A.: Poisson Lie groups, dressing transformations and Bruhat decompositions, *J. Diff. Geom.*, **31** (1990), 501–526.

[59] Malcev A.I.: On a class of homogeneous spaces, *Izvestiya Akad. Nauk SSSR Ser. Mat.*, **13** (1949) pp. 9–32; English translation, *Amer. Math. Soc. Translation Ser. 1*, **9** (1962), 276–307.

[60] Mealy J.: Calibrations on semi-Riemannian manifolds, Ph.D. Thesis, Rice University, (1989).

[61] Medori C., Tomassini A.: On small deformations of para-complex manifolds, *J. Noncommut. Geom.*, **5** (2010), No. 3, 507–522.

[62] Milnor J.: Curvature of left-invariant metrics on Lie groups, *Adv. in Math.*, **21** (1976), No. 3, 293–329.

[63] Morrow J., Kodaira, K,: *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, (1971). Reprinted in edition with errata, AMS Chelsea Publishing, Providence, RI, (2006).

[64] Naveira A.M.: A classification of Riemannian almost-product manifolds, *Rend. Mat. (7)*, **3** (1983), No. 3, 577–592.

[65] Newlander A., Nirenberg L.: Complex analytic coordinates in almost complex manifolds, *Ann. of Math. (2)*, **65** (1957), 391–404.

[66] Nomizu K.: On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* (2), **59** (1954), No. 3, 531–538.

[67] Onishchik A.L.: *Topology of transitive transformation groups*, Leipzig: Johann Ambrosius Barth, (1994).

[68] Rasevskij, P.K.: The scalar field in a stratified space, (Russian) *Trudy Sem. Vektor. Tenzor. Analizu* **6** (1948), 225–248.

[69] Rossi F.A.: On deformations of **D**-manifolds and CR **D**-manifolds, *J. Geom. Phys.*, **62** (2012), No. 2, 464–478. DOI: `10.1016/j.geomphys.2011.11.007`.

[70] Rossi F.A., Tomassini A.: On Strong Kähler and Astheno-Kähler metrics on nilmanifolds, *Adv. Geom.*, **12** (2012), No. 3, 431–446. DOI: `10.1515/advgeom-2011-057`.

[71] Steenrod, N.: *The Topology of Fibre Bundles*, Princeton Mathematical Series **14**, Princeton University Press, Princeton N.J., (1951).

[72] Sternberg S.: *Lectures on differential geometry*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. Second edition, Chelsea Publishing Co., New York, 1983.

[73] Tanaka N.: On affine symmetric spaces and the automorphisms groups of product manifolds, *Hokkaido math. J.*, **14** (1985), No. 3, 277–351.

[74] Tanaka N.: On differential systems, graded Lie algebras and pseudogroups, *J. Math. Kyoto Univ.*, **10** (1970), 1–82.

[75] Tanaka N.: On the equivalence problems associated with simple graded Lie algebras, *Hokkaido Math. J.*, **8** (1979), No. 1, 23–84.

[76] Tomassini A., Vezzoni L.: Contact Calabi-Yau manifolds and special Legendrian submanifolds, *Osaka J. Math.*, **45** (2008), No. 1, 127–147.

[77] Yano K.: Affine connexions in an almost product space, *Kodai Math. Sem. Rep.*, **11** (1959), 1–24.

[78] Yano K.: *Differential geometry on complex and almost complex spaces*, International Series of Monographs in Pure and Applied Mathematics, Vol. 49, A Pergamon Press Book, The Macmillan Co., New York, (1965).

[79] Yatsui T.: On pseudo-product graded Lie algebras, *Hokkaido Math. J.* **17** (1988), No. 3, 333–343