A note on local asymptotics of solutions to singular elliptic equations via monotonicity methods

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Abstract. This paper concerns the asymptotic behavior of solutions and their gradients to linear and nonlinear elliptic equations with singular coefficients of fuchsian type.

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1. Introduction and main results

Regularity properties of solutions to linear elliptic partial differential equations have been widely studied in the literature, both in the case of singular coefficients in the elliptic operator and in the case of domains with non smooth boundary. Asymptotic expansions near the singularity of the coefficients or near a non regular point of the boundary were derived e.g. in [2, 3, 5, 12, 13, 14, 15, 16], see also the references therein.

The present paper is concerned with the asymptotic behavior near the singularity of solutions to equations associated to the following class of Schrödinger operators with singular homogeneous electromagnetic potentials:

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\,\nabla + \frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|}\right)^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}. \tag{1.1}$$

We study both linear and nonlinear equations obtained as perturbations of the operator $\mathcal{L}_{\mathbf{A},a}$ in a domain $\Omega \subset \mathbb{R}^N$ containing the origin. In [5] asymptotic expansions at the singularity of solutions to linear equations of the

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type

$$\mathcal{L}_{\mathbf{A},a}u(x) = h(x)\,u(x), \quad \text{in } \Omega, \tag{1.2}$$

and semilinear equations with at most critical growth of the type

$$\mathcal{L}_{\mathbf{A},a}u(x) = f(x,u(x)), \quad \text{in } \Omega, \tag{1.3}$$

were derived. By solutions of (1.2) or (1.3) we mean functions which belong to a suitable Sobolev space depending on the magnetic potential $\mathbf{A}$ and solve the corresponding equations in a distributional sense. In [5], via a monotonicity approach, we proved a quantitative asymptotic formula under the following assumption on h, i.e.

$$h \in L^\infty_{\rm loc}(\Omega\setminus\{0\},\mathbb{C}),\ |h(x)| = O(|x|^{-2+\varepsilon}) \text{ as } |x|\to 0 \text{ for some } \varepsilon>0. \tag{1.4}$$

The validity of a Cauchy type formula, allowed to exclude the presence of logarithmic terms in the leading part of the asymptotic expansion. We emphasize that, in general, weaker negligibility type assumptions than (1.4) could not imply the absence of logarithms in the first expansion term; e.g. the condition $h \in L^{N/2}$ is not sufficient to that purpose, as the following example shows.

Example. The equation

$$-\Delta u = \frac{2-N}{|x|^2\log|x|}u \quad \text{in } B_{1/2} = \{x\in\mathbb{R}^N : |x|<1/2\},$$

admits as a weak $H^1(B_{1/2})$-solution the function $u(x) = \log|x|$, whose leading expansion term is of course of logarithmic type, although the perturbing potential $h(x) = (2-N)|x|^{-2}(\log|x|)^{-1}$ belongs to $L^{N/2}(B_{1/2})$.

In light of these considerations, we wonder about the legitimate question of what is the actual threshold for the perturbing potential that still gives the validity of a good asymptotics. A first result in this direction is the remark that the monotonicity method developed in [5] can be actually adapted to treat a class of perturbing potentials h satisfying pointwise upper estimates at the singularity of the type

$$\begin{aligned} &h\in L^\infty_{\rm loc}(\Omega\setminus\{0\},\mathbb{C}), \quad \lim_{r\to 0^+}\xi(r)=0, \quad \frac{\xi(r)}{r}\in L^1(0,\overline{R}),\\ &\text{and} \quad \frac{1}{r}\int_0^r \frac{\xi(s)}{s}\,ds \in L^1(0,\overline{R}) \end{aligned} \tag{1.5}$$

with $B_{\overline{R}}(0)\subset\Omega$ and $\xi(r) := \frac{1}{r}\sup_{x\in B_r}|x|^2|h(x)|$, including e.g. potentials satisfying

$$|h(x)| \leq \text{const}\,\frac{1}{|x|^2\,|\log|x||^\alpha}, \quad \alpha>2. \tag{1.6}$$

In the present paper, we show that the same asymptotic expansion as in [5] can be derived under integrability conditions on the best Hardy type constant on balls of h and its gradient, see (1.12–1.16). Such assumptions are alternative to (1.4) and (1.6) and require nontrivial adaptations of the techniques developed in [5]. Conditions (1.12–1.16) are also satisfied for example if

$$h \in L^s(B_{\overline{R}}, \mathbb{C}), \quad |x \cdot \nabla h| \in L^s(B_{\overline{R}}), \quad \text{for some } s > N/2$$

or

$$h \in K^{\mathrm{loc}}_{N,\delta}(B_{\overline{R}}) \qquad \text{and} \qquad \Re(x \cdot \nabla h(x)) \in K^{\mathrm{loc}}_{N,\delta}(B_{\overline{R}})$$

for some $\delta > 0$. Here $K^{\mathrm{loc}}_{N,\delta}(B_{\overline{R}})$ denotes a modified version of the usual Kato class $K^{\mathrm{loc}}_{N}(B_{\overline{R}})$ (see [8] for the definition of $K^{\mathrm{loc}}_{N}(B_{\overline{R}})$ and [9] for the definition of $K^{\mathrm{loc}}_{N,\delta}(B_{\overline{R}})$).

In order to relate the results of [5] and of the present paper with the previous literature on asymptotic analysis at singularities, let us consider the second order elliptic operator $\mathcal{L}$ on a domain $\Omega \ni 0$ written in polar coordinates (r, θ) as

$$\mathcal{L} = \sum_{0 \le j + |\beta| \le 2} a_{j,\beta}(r,\theta)(r\partial_r)^j \partial_\theta^\beta \tag{1.7}$$

where j is an integer, $\beta = (\beta_1, \dots, \beta_{N-1}) \in \mathbb{N}^{N-1}$ is a multi-index and $|\beta| = \sum_{j=1}^{N-1} \beta_j$. By [15, Theorem (7.3)], if u is a distributional solution of the equation $\mathcal{L}u = 0$ and $r^{-\delta}u(r,\theta) \in L^2(drd\theta)$, then u admits the following distributional asymptotic expansion

$$u \sim \sum_{\Re s_j > \delta - \frac{1}{2}} \sum_{\ell=0}^{\infty} \sum_{p=0}^{p_j} r^{s_j + \ell} (\log r)^p u_{j,\ell,p}(\theta) \tag{1.8}$$

where $\{s_j : j \in \mathbb{Z}\backslash\{0\}\}$ are the *indicial roots* defined in [15, Definition (2.21)]. Let us concentrate our attention on the first term of the expansion (1.8), i.e.

$$r^{s_{j_\delta}} \sum_{p=0}^{p_{j_\delta}} (\log r)^p u_{j_\delta,0,p}(\theta) \tag{1.9}$$

where j_δ is the smallest value of $j \in \mathbb{Z}$ for which $s_{j_\delta} > \delta - \frac{1}{2}$, see [15, Theorem (7.3)]. This term could be identically zero if δ is not optimal, whereas a finer choice of δ allows selecting the first nontrivial term in (1.8). In the case of operator (1.1), the indicial roots s_j can be written explicitly in terms of the eigenvalues of the operator on the sphere $L_{\mathbf{A},a} := (-i\nabla_{\mathbb{S}^{N-1}} + \mathbf{A})^2 - a$, i.e.

$$s_j = -\frac{N-2}{2} + \mathrm{sign}(j)\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{|j|}(\mathbf{A}, a)} \qquad \text{for all } j \in \mathbb{Z} \setminus \{0\}$$

where $\mu_1(\mathbf{A}, a) \le \mu_2(\mathbf{A}, a) \le \mu_3(\mathbf{A}, a) \le \cdots \le \mu_k(\mathbf{A}, a) \le \dots$ denote the eigenvalues of $L_{\mathbf{A},a}$. For more details on the meaning of the asymptotic expansion (1.8) see [15, Section 7].

According to (1.7), the elliptic operator $\mathcal{L}_{\mathbf{A},a} - h$ can be rewritten as

$$\mathcal{L}_{\mathbf{A},a,h} := -r^2\partial_r^2 - (N-1)r\partial_r + L_{\mathbf{A},a} - r^2 h(r,\theta),$$

which, after the change of variable $t = \log r$, takes the form

$$\mathcal{L}_{\mathbf{A},a,h} = -\partial_t^2 - (N-2)\partial_t + L_{\mathbf{A},a} - e^{2t}h(e^t,\theta),$$

which is asymptotically translation-invariant as $t \to -\infty$, since the potential $h(r,\theta)$ is negligible with respect to the inverse square potential r^{-2} as $r \to 0^+$.

As far as the linear equation (1.2) is concerned, the main result of [5] provides the leading term in the asymptotic expansion near the singularity of the coefficients. Similar asymptotic expansions were proved by Mazzeo [15], [16], with a completely different approach, in the setting of elliptic equations on compact manifolds with boundary (see also [10], [11], and [17]).

The main novelty of our approach in [5] is the use of Almgren's monotonicity formula [1]. The use of monotonicity methods to study unique continuation properties of solutions to elliptic partial differential equations dates back to the pioneering paper by Garofalo and Lin [7], see also [9] for the case of Schrödinger operators with magnetic potentials.

In the present paper we illustrate the strengths of the monotonicity formula approach, by completing some of the results in [5]. The main purposes of this note are essentially the following:

- to deduce from the monotonicity formula more precise informations on the first term in the asymptotic expansion of [15], [16] under some alternative assumptions on the perturbation h which require some integrability type conditions instead of pointwise decay as in [5],
- to provide a general method with the perspective of unifying the approach to linear and nonlinear equations with singular coefficients,
- to improve in the nonlinear case the results that in [5] were obtained by using a-priori pointwise estimates on solutions.

In the remaining part of the introduction we will examine these three goals with more detail.

Here and in [5], the indicial root of the leading term in the asymptotic expansion of finite energy solutions (namely H^1-weak solutions) to (1.2), is determined by introducing the following Almgren-type monotonicity function

$$\mathcal{N}_{u,h}(r) = \frac{r\int_{B_r}\left[\left|\nabla u + i\frac{A(x/|x|)}{|x|}u\right|^2 - \frac{a(x/|x|)}{|x|^2}|u|^2 - (\Re h)|u|^2\right]dx}{\int_{\partial B_r}|u|^2\,dS}, \tag{1.10}$$

for any $r \in (0,\overline{r})$, with $\overline{r} \in (0,R)$ sufficiently small. By a blow up argument, we are able to characterize the indicial root γ corresponding to the leading term in the asymptotic expansion as

$$\gamma = \lim_{r\to 0^+}\mathcal{N}_{u,h}(r). \tag{1.11}$$

We point out that the monotonicity argument does not need vanishing of solutions of (1.2) outside a small neighborhood of $r = 0$ which is instead required in the Mellin transform approach used in [15, Section 7]. Moreover, here and in [5], a characterization of the coefficient of the leading power is given by means of a Cauchy's integral type formula for u, see (1.27).

Let us now describe the integrability type assumptions on the perturbation h which are required by the forthcoming analysis. Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a domain containing the origin. Let $\overline{R} > 0$ be such that $B_{\overline{R}} \subset \Omega$ and let h satisfy

$$h \in L^\infty_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C}), \qquad \nabla h \in L^1_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C}^N). \tag{1.12}$$

Define, for any $r \in (0, \overline{R})$, the two functions

$$\eta_0(r) = \sup_{\substack{u \in H^1(B_r) \\ u \not\equiv 0}} \frac{\int_{B_r} h|u|^2 dx}{\int_{B_r} \Big[\big|\nabla u + i\frac{\mathbf{A}(\frac{x}{|x|})}{|x|}u\big|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u|^2\Big]dx + \frac{N-2}{2r}\int_{\partial B_r}|u|^2 dS} \tag{1.13}$$

$$\eta_1(r) = \sup_{\substack{u \in H^1(B_r) \\ u \not\equiv 0}} \frac{\int_{B_r} |\Re(x \cdot \nabla h)||u|^2 dx}{\int_{B_r} \Big[\big|\nabla u + i\frac{\mathbf{A}(\frac{x}{|x|})}{|x|}u\big|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u|^2\Big]dx + \frac{N-2}{2r}\int_{\partial B_r}|u|^2 dS}. \tag{1.14}$$

We observe that, under the assumption

$$\mu_1(\mathbf{A}, a) > -\left(\frac{N-2}{2}\right)^2,$$

the quadratic form appearing at the denominators of the two quotients in (1.13) and (1.14) is positive for any $u \in H^1(B_r) \setminus \{0\}$ and for any $r > 0$, and its square root is a norm equivalent to the $H^1(B_r)$-norm (see [5, Lemma 3.1]).

Let us assume that

$$\lim_{r \to 0^+} \eta_0(r) = 0, \qquad \frac{\eta_0(r)}{r} \in L^1(0, \overline{R}), \qquad \frac{1}{r}\int_0^r \frac{\eta_0(s)}{s}\, ds \in L^1(0, \overline{R}) \tag{1.15}$$

and that

$$\frac{\eta_1(r)}{r} \in L^1(0, \overline{R}), \qquad \frac{1}{r}\int_0^r \frac{\eta_1(s)}{s}\, ds \in L^1(0, \overline{R}). \tag{1.16}$$

A further aim of the present paper is to point out how the combination of monotonicity and blow-up techniques provides a powerful tool in the study of nonlinear problems of the type (1.3), where f is a nonlinearity with at most critical growth. In [5], the study of (1.3) was carried out as follows: a-priori upper bounds of solutions to (1.3) were first deduced by a classical iteration scheme, allowing treating the nonlinear term as a linear one of the type $h(x)u$ with a potential h depending nonlinearly on u but satisfying a suitable pointwise estimate. The linear result [5, Theorem 1.3] was thus invoked to prove its nonlinear version [5, Theorem 1.6]. In particular, in [5] a nonlinear version of the monotonicity formula was not needed being the asymptotics for the nonlinear problem deducible from the linear case. On the other hand, the a-priori pointwise estimate on solutions of (1.3) needed to reduce the nonlinear problem to a linear one required the further assumption

$$\mu_1(0, a) > -\left(\frac{N-2}{2}\right)^2, \tag{1.17}$$

see the statement of [5, Theorem 1.6] and [5, Theorem 9.4].

In the present paper, we remove condition (1.17) and prove Theorem 1.1 below under the less restrictive positive definiteness condition (**A.4**). Such improved result is obtained through a unified approach which allows treating simultaneously linear and nonlinear equations. A similar unified approach was previously introduced in the paper [6] dealing with elliptic equations with cylindrical and many-particle potentials, for which a-priori pointwise estimates seem to be quite more difficult to be proved, thus requiring a purely nonlinear approach based on a nonlinear monotonicity formula.

Let us consider a unified version of (1.2) and (1.3), i.e. an equation of the form

$$\mathcal{L}_{\mathbf{A},a}u = h(x)\, u + f(x,u), \quad \text{in } \Omega, \tag{1.18}$$

where h satisfies (1.12), (1.15), (1.16), f is of the type

$$f(x,z) = g(x,|z|^2)z, \qquad \text{for a.e. } x\in\Omega, \text{ for all } z\in\mathbb{C}, \tag{1.19}$$

$g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} g\in C^0(\Omega\times[0,+\infty)), \quad G\in C^1(\Omega\times[0,+\infty)), \\ |g(x,s)s| + |\nabla_x G(x,s)\cdot x| \leq C_g(|s|+|s|^{\frac{2^*}{2}}), \\ \qquad\qquad\qquad\qquad \text{for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R}, \end{cases} \tag{1.20}$$

$G(x,s) = \frac{1}{2}\int_0^s g(x,t)\,dt$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $C_g > 0$ is a constant independent of $x\in\Omega$ and $s\in\mathbb{R}$, and $\nabla_x G$ denotes the gradient of G with respect to the x variable.

The special form (1.19) chosen for the function f is invariant by gauge transformations and hence very natural in the study of nonlinear Schrödinger equations with magnetic fields, see for example [4]. We stress that our approach works for very general nonlinearities and also for perturbations of the homogeneous magnetic potential.

Let us recall the assumptions $(\mathbf{A.1}), (\mathbf{A.2}), (\mathbf{A.3}), (\mathbf{A.4})$ already introduced in [5]:

$$\boldsymbol{\mathcal{A}}(x) = \frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|}, \quad V(x) = -\frac{a\big(\frac{x}{|x|}\big)}{|x|^2} \qquad \textbf{(homogeneity)} \tag{A.1}$$

$$\begin{cases} \mathbf{A}\in C^1(\mathbb{S}^{N-1},\mathbb{R}^N) \\ a\in L^\infty(\mathbb{S}^{N-1},\mathbb{R}) \end{cases} \qquad \textbf{(regularity of angular coefficients)} \tag{A.2}$$

$$\mathbf{A}(\theta)\cdot\theta = 0 \quad \text{for all } \theta\in\mathbb{S}^{N-1}. \qquad \textbf{(transversality)} \tag{A.3}$$

$$\mu_1(\mathbf{A},a) > -\left(\frac{N-2}{2}\right)^2. \qquad \textbf{(positive definiteness)} \tag{A.4}$$

An equivalent version of (**A.4**) can be given by introducing the quantity

$$\Lambda(\mathbf{A},a) := \sup_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N}|x|^{-2}a(x/|x|)\,|u(x)|^2\,dx}{\int_{\mathbb{R}^N}\left|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 dx} \tag{1.21}$$

and by taking into account that

$$\mu_1(\mathbf{A},a) > -\left(\frac{N-2}{2}\right)^2 \quad \text{if and only if} \quad \Lambda(\mathbf{A},a) < 1 \ , \tag{1.22}$$

see [5, Lemma 1.1] and [6, Lemma 2.3]. It is easy to verity that $\Lambda(\mathbf{A},a) \geq 0$ and it is zero if and only if $a \leq 0$ a.e. in $\mathbb{S}^{N-1}$.

The following theorem characterizes the leading term of the asymptotic expansion of solutions to (1.18) by means of the limit of the associated Almgren-type function

$$\begin{aligned}\mathcal{N}_{u,h,f}(r) = &\frac{r\int_{B_r}\Big[\big|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\big|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2\Big]\,dx}{\int_{\partial B_r}|u(x)|^2\,dS} \\ &-\frac{r\int_{B_r}\big[(\Re h(x))|u(x)|^2 + g(x,|u(x)|^2)|u(x)|^2\big]\,dx}{\int_{\partial B_r}|u(x)|^2\,dS}.\end{aligned} \tag{1.23}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set containing* 0, (**A.1**), (**A.2**), (**A.3**), (**A.4**) *hold, and u be a weak $H^1(\Omega,\mathbb{C})$-solution to* (1.18), *$u \not\equiv 0$, with f satisfying* (1.19)–(1.20), *and h satisfying either* (1.5) *or* (1.12)–(1.16). *Then, letting $\mathcal{N}_{u,h,f}(r)$ as in* (1.23), *there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that*

$$\gamma := \lim_{r\to 0^+}\mathcal{N}_{u,h,f}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A},a)}. \tag{1.24}$$

Furthermore, if $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(\mathbf{A},a)$, and $\{\psi_i : j_0 \leq i \leq j_0+m-1\}$ ($j_0 \leq k_0 \leq j_0+m-1$) is an $L^2(\mathbb{S}^{N-1},\mathbb{C})$-orthonormal basis for the eigenspace of $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A},a)$, then

$$\lambda^{-\gamma}u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1}\beta_i\psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1},\mathbb{C}) \quad \text{as } \lambda \to 0^+, \tag{1.25}$$

and

$$\lambda^{1-\gamma}\nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1}\beta_i\big(\gamma\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)\big) \text{ in } C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C}^N), \tag{1.26}$$

as $\lambda \to 0^+$ for any $\tau \in (0,1)$, where $(\beta_{j_0},\beta_{j_0+1},\dots,\beta_{j_0+m-1}) \neq (0,0,\dots,0)$ and

$$\begin{aligned}\beta_i = &\int_{\mathbb{S}^{N-1}}\bigg[\frac{u(R\theta)}{R^\gamma} + \\ &+\int_0^R\frac{(h(s\,\theta)+g(s\theta,|u(s\theta)|^2))u(s\theta)}{2\gamma+N-2}\Big(s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}}\Big)ds\bigg]\overline{\psi_i(\theta)}\,dS(\theta),\end{aligned} \tag{1.27}$$

for all $R > 0$ such that $\overline{B}_R = \{x \in \mathbb{R}^N : |x| \leq R\} \subset \Omega$.

It is worth pointing out how convergence (1.25) excludes the presence of logarithmic factors in the leading term of the expansion (1.8).

Although the proof of Theorem 1.1 follows essentially the scheme of Theorem 1.3 in [5], the addition of the nonlinear term in the Almgren-type function (1.23) and the replacement of pointwise assumptions on h with the integral type ones (1.15–1.16), require some significant adaptations which are emphasized in Section 2. As a relevant byproduct of Theorem 1.1 we also obtain the following pointwise estimate on solutions to (1.18).

Corollary 1.2. *Let u be a weak $H^1(\Omega,\mathbb{C})$-solution to (1.18) and all the assumptions of Theorem 1.1 hold. Then for any $\Omega' \Subset \Omega$, there exists a constant $C = C(\Omega', u)$ such that*

$$|u(x)| \leq C|x|^\gamma \quad \text{for a.e. every } x \in \Omega' \tag{1.28}$$

where γ is the number defined (1.24).

We point out that Corollary 1.2 is a direct consequence of Theorem 1.1 which is proved by monotonicity and blow-up methods, and hence does not require any iterative Brezis-Kato scheme; in particular, here we can drop the strongest positivity condition (1.17), which was instead needed in [5] to start the iteration procedure.

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2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 with h under conditions (1.15–1.16). The case of h satisfying (1.5) is omitted since in that case the proof can be performed by following the scheme of [5] to estimate the linear perturbation involving h and the approach developed in [6] to treat the nonlinear term.

Solutions to (1.18) satisfy the following Pohozaev-type identity.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set such that $0 \in \Omega$. Let $a, \mathbf{A}$ satisfy $(\mathbf{A.2})$, and u be a weak $H^1(\Omega,\mathbb{C})$-solution to (1.18) in Ω, with h satisfying (1.12), (1.15)–(1.16), and f as in (1.19)–(1.20). Then*

$$\begin{aligned}
&-\frac{N-2}{2}\int_{B_r}\left[\left|\left(\nabla + i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\right)u\right|^2 - \frac{a\big(x/|x|\big)}{|x|^2}|u|^2\right]dx \\
&\quad+\frac{r}{2}\int_{\partial B_r}\left[\left|\left(\nabla + i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\right)u\right|^2 - \frac{a\big(x/|x|\big)}{|x|^2}|u|^2\right]dS \\
&\quad= r\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS - \frac{1}{2}\int_{B_r}\Re\big(\nabla h\cdot x\big)|u|^2\,dx \\
&\quad-\frac{N}{2}\int_{B_r}\Re(h)|u|^2\,dx + \frac{r}{2}\int_{\partial B_r}\Re(h)|u|^2\,dS + r\int_{\partial B_r}G(x,|u|^2)\,dS \\
&\quad-\int_{B_r}\big(\nabla_x G(x,|u|^2)\cdot x + NG(x,|u|^2)\big)\,dx
\end{aligned} \tag{2.1}$$

for all $r > 0$ *such that* $\overline{B_r} = \{x \in \mathbb{R}^N : |x| \le r\} \subset \Omega$, *where* $\nu = \nu(x)$ *is the unit outer normal vector* $\nu(x) = \frac{x}{|x|}$.

Proof. One can proceed similarly to the proof Theorem 4.1 in [5] by fixing $r \in (0, \overline{R})$ and finding a sequence $\{\delta_n\} \subset (0, r)$ such that $\lim_{n\to+\infty} \delta_n = 0$ and

$$\delta_n \int_{\partial B_{\delta_n}} \bigg[\bigg| \bigg(\nabla + i\, \frac{\mathbf{A}\big(x/|x|\big)}{|x|} \bigg) u \bigg|^2 + \frac{|u|^2}{|x|^2} + \bigg| \frac{\partial u}{\partial \nu} \bigg|^2 + \Re\big(h(x)\big)|u(x)|^2 + |G(x, |u(x)|^2)| \bigg]\, dS \to 0 \tag{2.2}$$

as $n \to +\infty$. This is possible by the fact that $\Re\big(h\big)|u|^2, G(x, |u|^2) \in L^1(B_r)$ in view of (1.13), (1.15) and (1.20).

By ($\mathbf{A.2}$) and (1.12) we deduce that $u \in C^{1,\tau}_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C})$ for every $\tau \in (0, 1)$ and $h \in W^{1,1}_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C})$ and hence, integrating by parts, we obtain

$$\begin{aligned} &\int_{B_r \setminus B_{\delta_n}} \Re\big(h(x)u(x)(x \cdot \overline{\nabla u(x)})\big)\, dx \\ &= -\frac{1}{2} \int_{B_r \setminus B_{\delta_n}} \Re\big(\nabla h(x) \cdot x\big)|u(x)|^2\, dx - \frac{N}{2} \int_{B_r \setminus B_{\delta_n}} \Re\big(h(x)\big)|u(x)|^2\, dx \\ &\quad + \frac{r}{2} \int_{\partial B_r} \Re\big(h(x)\big)|u(x)|^2\, dS - \frac{\delta_n}{2} \int_{\partial B_{\delta_n}} \Re\big(h(x)\big)|u(x)|^2\, dS. \end{aligned}$$

Passing to the limit as $n \to +\infty$, by (1.15), (1.16), and (2.2) we obtain

$$\begin{aligned} \lim_{n\to+\infty} &\int_{B_r \setminus B_{\delta_n}} \Re\big(h(x)u(x \cdot \overline{\nabla u(x)})\big)\, dx \\ &= -\frac{1}{2} \int_{B_r} \Re\big(\nabla h(x) \cdot x\big)|u(x)|^2\, dx - \frac{N}{2} \int_{B_r} \Re\big(h(x)\big)|u(x)|^2\, dx \\ &\quad + \frac{r}{2} \int_{\partial B_r} \Re\big(h(x)\big)|u(x)|^2\, dS. \end{aligned}$$

The proof of the proposition then follows proceeding as in the proof of Theorem 4.1 in [5] and Proposition A.1 in [6]. □

Proceeding as in [5], one can show that, under assumptions ($\mathbf{A.2}$), ($\mathbf{A.3}$), ($\mathbf{A.4}$), and (1.15), there exists $\overline{r} \in (0, \overline{R})$ such that $H(r) = r^{1-N} \int_{\partial B_r} |u|^2 dS$ is strictly positive for any $r \in (0, \overline{r})$ and $\sup_{r \in (0,\overline{r})} \eta_0(r) < +\infty$. In this way, if D is the function defined by

$$\begin{aligned} D(r) = \frac{1}{r^{N-2}} \int_{B_r} &\bigg[\bigg| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \bigg|^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \bigg]\, dx \\ &- \frac{1}{r^{N-2}} \int_{B_r} \big[(\Re h(x))|u(x)|^2 + g(x, |u(x)|^2)|u(x)|^2 \big]\, dx, \end{aligned}$$

then the quotient

$$\mathcal{N}(r) := \mathcal{N}_{u,h,f}(r) = \frac{D(r)}{H(r)}, \quad \text{for a.e. } r \in (0, \overline{r}), \tag{2.3}$$

is well defined. Arguing as in [5, (52)], it is easy to verify that

$$D(r) = \frac{r}{2} H'(r) \quad \text{for a.e. } r \in (0, \overline{r}). \tag{2.4}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set such that $0 \in \Omega$, a, $\mathbf{A}$ satisfy $(\mathbf{A.2}), (\mathbf{A.3})$, $(\mathbf{A.4})$, and $u \not\equiv 0$ be a weak $H^1(\Omega, \mathbb{C})$-solution to* (1.18) *in Ω, with h satisfying* (1.12), (1.15)–(1.16), *and f satisfying* (1.19)–(1.20). *Then, letting $\mathcal{N}$ as in* (2.3), *there holds $\mathcal{N} \in W^{1,1}_{\rm loc}(0, \overline{r})$ and*

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \tag{2.5}$$

in a distributional sense and for a.e. $r \in (0, \overline{r})$, where

$$\nu_1(r) = \frac{2r \left[\left(\int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right) \cdot \left(\int_{\partial B_r} |u|^2 dS \right) - \left(\int_{\partial B_r} \Re\left(u \frac{\partial \overline{u}}{\partial \nu} \right) dS \right)^2 \right]}{\left(\int_{\partial B_r} |u|^2 dS \right)^2} \tag{2.6}$$

and

$$\begin{aligned} \nu_2(r) = & -\frac{\int_{B_r} \Re(2h + \nabla h \cdot x)|u|^2 \, dx}{\int_{\partial B_r} |u|^2 dS} \\ & + \frac{r \int_{\partial B_r} \left(2G(x,|u|^2) - g(x,|u|^2)|u|^2 \right) dS}{\int_{\partial B_r} |u|^2 \, dS} \\ & + \frac{\int_{B_r} \left((N-2) g(x,|u|^2)|u|^2 - 2NG(x,|u|^2) - 2\nabla_x G(x,|u|^2) \cdot x \right) dx}{\int_{\partial B_r} |u|^2 \, dS}. \end{aligned} \tag{2.7}$$

Proof. One can proceed exactly as in the proof of Lemma 5.4 in [5] by using the Pohozaev-type identity (2.1) in place of (32) in [5]. $\square$

The following proposition provides an a-priori super-critical summability of solutions to (1.18) which will allow including the critical growth case in the Almgren type monotonicity formula.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set such that $0 \in \Omega$, a, $\mathbf{A}$ satisfy $(\mathbf{A.2}), (\mathbf{A.3}), (\mathbf{A.4})$, and u be a $H^1(\Omega, \mathbb{C})$-weak solution to*

$$\mathcal{L}_{\mathbf{A},a} u(x) = h(x)u(x) + V(x)u(x), \quad \text{in } \Omega, \tag{2.8}$$

with h satisfying (1.12), (1.15)–(1.16) *and $V \in L^{N/2}(\Omega, \mathbb{C})$. Letting*

$$q_{\rm lim} := \begin{cases} \frac{2^*}{2} \min\left\{ \frac{4}{\Lambda(\mathbf{A},a)} - 2, 2^* \right\}, & \text{if } \Lambda(\mathbf{A},a) > 0, \\ \frac{(2^*)^2}{2}, & \text{if } \Lambda(\mathbf{A},a) = 0, \end{cases}$$

then for any $1 \leq q < q_{\rm lim}$ there exists $r_q > 0$, depending only on N, $\mathbf{A}$, a, q, h such that $B_{r_q} \subset \Omega$ and $u \in L^q(B_{r_q}, \mathbb{C})$.

Proof. By $(\mathbf{A}.\mathbf{4})$ and (1.22) we have that $\frac{2}{2^*}q_{\lim} > 2$. For any $2 < \tau < \frac{2}{2^*}q_{\lim}$, define $C(\tau) := \frac{4}{\tau+2}$ and let $\ell_\tau > 0$ be so large that

$$\left(\int\limits_{|V(x)|\geq \ell_\tau} |V(x)|^{\frac{N}{2}}\, dx \right)^{\frac{2}{N}} < \frac{S(\mathbf{A})(C(\tau) - \Lambda(\mathbf{A},a))}{2} \tag{2.9}$$

where

$$S(\mathbf{A}) := \inf_{\substack{v \in \mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C}) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left| \nabla v(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} v(x) \right|^2 dx}{\left(\int_{\mathbb{R}^N} |v(x)|^{2^*}\, dx \right)^{\frac{2}{2^*}}} > 0.$$

Let $r > 0$ be such that $B_r \subset \Omega$. For any $w \in H^1_0(B_r,\mathbb{C})$, by Hölder and Sobolev inequalities and (2.9), we have

$$\begin{aligned}
\int_{B_r} |V(x)||w(x)|^2\, dx &= \int\limits_{B_r \cap \{|V(x)|\leq \ell_\tau\}} |V(x)||w(x)|^2\, dx \\
&\qquad + \int\limits_{B_r \cap \{|V(x)|\geq \ell_\tau\}} |V(x)||w(x)|^2\, dx \\
&\leq \ell_\tau \int_{B_r} |w(x)|^2\, dx + \left(\int\limits_{|V(x)|\geq \ell_\tau} |V(x)|^{\frac{N}{2}}\, dx \right)^{\frac{2}{N}} \left(\int_{B_r} |w(x)|^{2^*}\, dx \right)^{\frac{2}{2^*}} \\
&\leq \ell_\tau \int_{B_r} |w(x)|^2\, dx \\
&\quad + \frac{C(\tau) - \Lambda(\mathbf{A},a)}{2} \int_{B_r} \left| \nabla w(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} w(x) \right|^2 dx.
\end{aligned} \tag{2.10}$$

Let $\rho \in C^\infty_c(B_r,\mathbb{R})$ such that $\rho \equiv 1$ in $B_{r/2}$ and define $v := \rho u \in H^1_0(B_r,\mathbb{C})$. Then v is a $H^1(\Omega,\mathbb{C})$-weak solution of the equation

$$\mathcal{L}_{\mathbf{A},a} v(x) = h(x)v(x) + V(x)v(x) + g(x) \qquad \text{in } \Omega \tag{2.11}$$

where $g = -u\Delta\rho - 2\nabla u \cdot \nabla\rho - 2iu\frac{\mathbf{A}(x/|x|)}{|x|} \cdot \nabla\rho \in L^2(B_r,\mathbb{C})$. For any $n \in \mathbb{N}$, $n \geq 1$, let us define the function $v^n := \min\{|v|, n\}$. Testing (2.11) with

$(v^n)^{\tau-2}\overline{v} \in H^1_0(B_r,\mathbb{C})$ we obtain

$$\begin{aligned}
&\int_{B_r}(v^n(x))^{\tau-2}\left|\nabla v(x)+i\tfrac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx\\
&\qquad+(\tau-2)\int_{B_r}(v^n(x))^{\tau-2}|\nabla|v(x)||^2\chi_{\{|v(x)|<n\}}(x)\,dx\\
&\qquad-\int_{B_r}\frac{a(\frac{x}{|x|})}{|x|^2}(v^n(x))^{\tau-2}|v(x)|^2\,dx \qquad (2.12)\\
&=\int_{B_r}\Re(h(x))(v^n(x))^{\tau-2}|v(x)|^2\,dx+\int_{B_r}\Re(g(x)(v^n(x))^{\tau-2}\overline{v}(x))\,dx\\
&\qquad+\int_{B_r}\Re(V(x))(v^n(x))^{\tau-2}|v(x)|^2\,dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2\\
&=(v^n)^{\tau-2}\left|\nabla v+i\frac{\mathbf{A}(x/|x|)}{|x|}v\right|^2+\frac{(\tau-2)(\tau+2)}{4}(v^n)^{\tau-2}|\nabla|v||^2\chi_{\{|v(x)|<n\}},
\end{aligned}$$

then by (2.12), (1.21), (1.13), and (2.10) with $w=(v^n)^{\frac{\tau}{2}-1}v$, we obtain for any $r>0$ small enough such that $\eta_0(r)<1$,

$$\begin{aligned}
&C(\tau)\int_{B_r}\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2 dx\\
&\quad\le\int_{B_r}\frac{a(x/|x|)}{|x|^2}|(v^n)^{\frac{\tau}{2}-1}v|^2\,dx+\int_{B_r}\Re(h)|(v^n)^{\frac{\tau}{2}-1}v|^2\,dx\\
&\qquad+\int_{B_r}\Re(V)|(v^n)^{\frac{\tau}{2}-1}v|^2\,dx+\int_{B_r}\Re\big(g(v^n)^{\tau-2}\overline{v}\big)\,dx\\
&\quad\le\left[\Lambda(\mathbf{A},a)(1-\eta_0(r))+\eta_0(r)+\frac{C(\tau)-\Lambda(\mathbf{A},a)}{2}\right]\times \qquad (2.13)\\
&\qquad\qquad\times\int_{B_r}\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2 dx\\
&\qquad+\ell_\tau\int_{B_r}(v^n)^{\tau-2}|v|^2\,dx+\int_{B_r}|g|(v^n)^{\tau-2}|v|\,dx.
\end{aligned}$$

Since $g \in L^2(B_r, \mathbb{C})$, by Hölder inequality the last term in the right hand side of (2.13) can be estimated as

$$\begin{aligned}
\int_{B_r} |g(x)|(v^n(x))^{\tau-2}|v(x)|\,dx &\leq \|g\|_{L^2(\Omega,\mathbb{C})}\left(\int_{B_r}(v^n(x))^{2\tau-4}|v(x)|^2\,dx\right)^{\frac12}\\
&= \|g\|_{L^2(\Omega,\mathbb{C})}\left(\int_{B_r}(v^n(x))^{\frac{2(\tau-1)(\tau-2)}{\tau}}(v^n(x))^{\frac{2(\tau-2)}{\tau}}|v(x)|^2\,dx\right)^{\frac12}\\
&\leq \|g\|_{L^2(\Omega,\mathbb{C})}\left(\int_{B_r}|(v^n(x))^{\frac{\tau}{2}-1}v(x)|^{\frac{4(\tau-1)}{\tau}}\,dx\right)^{\frac12}
\end{aligned}$$

and, since $\frac{4(\tau-1)}{\tau} < 2^*$ for any $\tau < \frac{2}{2^*}q_{\lim}$, by Hölder inequality, Sobolev embedding, and Young inequality, we obtain

$$\begin{aligned}
&\int_{B_r}|g|(v^n)^{\tau-2}|v|\,dx\\
&\leq \|g\|_{L^2(\Omega,\mathbb{C})}\left(\frac{\omega_{N-1}}{N}\right)^{\frac12-\frac{2(\tau-1)}{2^*\tau}} r^{\frac{N}{2}-\frac{2N(\tau-1)}{2^*\tau}}\left(\int_{B_r}|(v^n)^{\frac{\tau}{2}-1}v|^{2^*}dx\right)^{\frac{2(\tau-1)}{2^*\tau}}\\
&\leq \|g\|_{L^2(\Omega,\mathbb{C})}\left(\frac{\omega_{N-1}}{N}\right)^{\frac12-\frac{2(\tau-1)}{2^*\tau}} r^{\frac{N}{2}-\frac{(N-2)(\tau-1)}{\tau}} S(\mathbf{A})^{-\frac{\tau-1}{\tau}}\times \qquad (2.14)\\
&\qquad\times\left(\int_{B_r}\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2 dx\right)^{\frac{\tau-1}{\tau}}\\
&\leq \frac{\frac{\tau-1}{\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}}{S(\mathbf{A})}\int_{B_r}\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(\frac{x}{|x|})}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2 dx\\
&\qquad+\frac{1}{\tau}\|g\|^{\tau}_{L^2(\Omega,\mathbb{C})},
\end{aligned}$$

where $\omega_{N-1} := \int_{\mathbb{S}^{N-1}} dS(\theta)$ denotes the volume of the unit sphere $\mathbb{S}^{N-1}$. Inserting (2.14) into (2.13) we obtain

$$\begin{aligned}
&\left[\frac{C(\tau)-\Lambda(\mathbf{A},a)}{2}-\eta_0(r)-\frac{\tau-1}{\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}S(\mathbf{A})^{-1}\right]\times\\
&\qquad\qquad\times\int_{B_r}\left|\nabla((v^n)^{\frac{\tau}{2}-1}v)+i\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{\frac{\tau}{2}-1}v\right|^2 dx\\
&\quad\leq \frac{1}{\tau}\|g\|^{\tau}_{L^2(\Omega,\mathbb{C})}+\ell_\tau\int_{B_r}(v^n(x))^{\tau-2}|v(x)|^2\,dx
\end{aligned}$$

and, by Sobolev embedding,

$$\begin{aligned}
&S(\mathbf{A})\left[\frac{C(\tau)-\Lambda(\mathbf{A},a)}{2}-\eta_0(r)-\frac{\tau-1}{S(\mathbf{A})\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}\right]\times\\
&\qquad\qquad\times\left(\int_{B_r}(v^n(x))^{\frac{2^*}{2}\tau-2^*}|v(x)|^{2^*}\,dx\right)^{2/2^*} \qquad (2.15)\\
&\quad\leq \frac{1}{\tau}\|g\|^{\tau}_{L^2(\Omega,\mathbb{C})}+\ell_\tau\int_{B_r}(v^n(x))^{\tau-2}|v(x)|^2\,dx.
\end{aligned}$$

Since $\tau < \frac{2}{2^*} q_{\lim}$ then $C(\tau) - \Lambda(\mathbf{A}, a)$ is positive and $\frac{N\tau}{2(\tau-1)} - N + 2$ is also positive. Moreover by (1.15), $\lim_{r\to 0^+} \eta_0(r) = 0$. Hence we may fix r small enough in such a way that the left hand side of (2.15) becomes positive. Since $v \in L^\tau(B_r, \mathbb{C})$, letting $n \to +\infty$, the right hand side of (2.15) remains bounded and hence, by Fatou Lemma, we infer that $v \in L^{\frac{2^*}{2}\tau}(B_r, \mathbb{C})$. Since $\rho \equiv 1$ in $B_{r/2}$, we may conclude that $u \in L^{\frac{2^*}{2}\tau}(B_{r/2}, \mathbb{C})$. This completes the proof of the lemma. □

According to the previous proposition, we may fix from now on a weak H^1-solution u to (1.18),

$$2^* < q < q_{\lim},$$

and r_q in such a way that $u \in L^q(B_{r_q})$. We omit the proof of the following lemma which can be deduced in a quite standard way by combining Hardy-Sobolev inequalities with boundary terms (see [5, §3]) with assumptions (1.15) and (1.20).

Lemma 2.4. *Under the same assumptions as in Lemma 2.2, there exist $\tilde r \in (0, \min\{\overline{r}, r_q\})$ and a positive constant $\overline{C} = \overline{C}(N, \mathbf{A}, a, h, f, u) > 0$ depending on N, $\mathbf{A}$, a, h, f, u but independent of r such that*

$$\int_{B_r} \left[\left| \nabla u + i \frac{\mathbf{A}(\frac{x}{|x|})}{|x|} u \right|^2 - \frac{a(x|x|)}{|x|^2} |u|^2 \right] dx - \int_{B_r} \left[(\Re h)|u|^2 + g(x, |u|^2)|u|^2 \right] dx$$

$$\geq -\frac{N-2}{2r} \int_{\partial B_r} |u|^2 dS + \overline{C} \left(\int_{B_r} |u|^{2^*} dx \right)^{\frac{2}{2^*}} \tag{2.16}$$

$$+ \overline{C} \left(\int_{B_r} \left[\left| \nabla u + i \frac{\mathbf{A}(\frac{x}{|x|})}{|x|} u \right|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} |u|^2 \right] dx + \frac{N-2}{2r} \int_{\partial B_r} |u|^2 dS \right)$$

and

$$\mathcal{N}(r) > -\frac{N-2}{2} \tag{2.17}$$

for every $r \in (0, \tilde r)$.

The term ν_2 introduced in Lemma 2.2 can be estimated as follows.

Lemma 2.5. *Under the same assumptions as in Lemma 2.2, let $\tilde r$ be as in Lemma 2.4 and ν_2 as in (2.7). Then there exist a positive constant $C_1 > 0$ depending on $N, q, C_g, \overline{C}, \tilde r, \|u\|_{L^q(B_{\tilde r}, \mathbb{C})}$ and a function $\omega \in L^1(0, \tilde r)$, $\omega \geq 0$ a.e. in $(0, \tilde r)$, such that*

$$|\nu_2(r)| \leq C_1 \left[\mathcal{N}(r) + \frac{N}{2} \right] \left[r^{-1}(\eta_0(r) + \eta_1(r)) + r^{-1+\frac{2(q-2^*)}{q}} + \omega(r) \right]$$

for a.e. $r \in (0, \tilde r)$ and

$$\int_0^r \omega(s)\, ds \leq \frac{\|u\|^{2^*(1-\alpha)}_{L^{2^*}(\Omega)}}{1-\alpha}\, r^{\frac{N(q-2^*)}{q}(\alpha - \frac{2}{2^*})}$$

for all $r \in (0, \tilde r)$ and for some α satisfying $\frac{2}{2^*} < \alpha < 1$.

Proof. The estimates on the terms in (2.7) involving g and G can be obtained by using Proposition 2.3, Lemma 2.4 and proceeding as in [6, Lemma 5.6].

Here we only estimate the term in (2.7) which involves the function h and its gradient. From (1.13), (1.14) and (2.4) we deduce that

$$\left|\int_{B_r}(2h(x)+\nabla h(x)\cdot x)|u(x)|^2\,dx\right| \leq (2\eta_0(r)+\eta_1(r))\overline{C}^{-1}r^{N-2}\left[D(r)+\frac{N-2}{2}H(r)\right]$$

and, therefore,

$$\left|\frac{\int_{B_r}(2h(x)+\nabla h(x)\cdot x)|u(x)|^2\,dx}{\int_{\partial B_r}|u|^2\,dS}\right| \leq \overline{C}^{-1}r^{-1}(2\eta_0(r)+\eta_1(r))\left[\mathcal{N}(r)+\frac{N-2}{2}\right] \tag{2.18}$$

for all $r\in(0,\tilde{r})$. □

Lemma 2.6. *Under the same assumptions as in Lemma* 2.2*, the limit*

$$\gamma := \lim_{r\to 0^+}\mathcal{N}(r)$$

exists and is finite.

Proof. From Schwarz's inequality, the function ν_1 defined in (2.6) is nonnegative. Furthermore, by Lemma 2.5 and assumptions (1.15) and (1.16), $\frac{\nu_2}{\mathcal{N}+N/2}\in L^1(0,\tilde{r})$. Hence, from (2.5) and integration we deduce that $\mathcal{N}$ is bounded in $(0,\tilde{r})$, thus implying, in view of Lemma 2.5, that $\nu_2\in L^1(0,\tilde{r})$. Therefore $\mathcal{N}'$ turns out to be an integrable perturbation of a nonnegative function and hence $\mathcal{N}(r)$ admits a finite limit as $r\to 0^+$. For more details, we refer the reader to Lemmas 5.7 and 5.8 in [6]. □

A first consequence of the convergence of $\mathcal{N}$ at 0 is the following estimate of H from above.

Lemma 2.7. *Under the same assumptions as in Lemma* 2.2*, there exists a constant $K_1>0$ such that*

$$H(r)\leq K_1 r^{2\gamma}\quad \textit{for all } r\in(0,\bar{r}), \tag{2.19}$$

with $\gamma:=\lim_{r\to 0^+}\mathcal{N}(r)$ being as in Lemma 2.6.

Proof. From (2.4), (2.5), and Schwarz's inequality, it follows that

$$\frac{H'(r)}{H(r)}=\frac{2}{r}\mathcal{N}(r)\geq\frac{2\gamma}{r}+\frac{2}{r}\int_0^r\nu_2(s)\,ds.$$

By Lemma 2.5, assumptions (1.15–1.16), and boundedness of $\mathcal{N}$, we have that $r\mapsto\frac{1}{r}\int_0^r\nu_2\in L^1(0,\tilde{r})$. Hence the conclusion follows from integration. □

We omit the proof of the following lemma which follows closely the blow up scheme developed in [5, Lemma 6.1].

Lemma 2.8. *Under the same assumptions as in Lemma* 2.2*, the following holds true:*

(i) *there exists $k_0 \in \mathbb{N}$ such that $\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}$;*
(ii) *for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ and an eigenfunction ψ of the operator $L_{\mathbf{A},a}$ associated to the eigenvalue $\mu_{k_0}(\mathbf{A}, a)$ such that $\|\psi\|_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = 1$ and*

$$\frac{u(\lambda_{n_k} x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^\gamma \psi\Big(\frac{x}{|x|}\Big)$$

weakly in $H^1(B_1, \mathbb{C})$, strongly in $H^1(B_r, \mathbb{C})$ for every $0 < r < 1$, and in $C^{1,\tau}_{\rm loc}(B_1 \setminus \{0\}, \mathbb{C})$ for any $\tau \in (0, 1)$.

A first step towards the description of the behavior of H as $r \to 0^+$ is the following lemma, whose proof is similar to [6, Lemma 6.6].

Lemma 2.9. *Let the assumptions of Lemma* 2.2 *hold and $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma* 2.6*. Then the limit*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r)$$

exists and it is finite.

Under the integral type assumptions (1.15–1.16), the proof that

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0$$

is more delicate than it was under the pointwise conditions required in [5] and a new argument is needed to prove it.

Lemma 2.10. *Suppose that all the assumptions of Lemma* 2.2 *hold true. Let k_0 be as in Lemma* 2.8 *and let $j_0, m \in \mathbb{N}$, $j_0, m \geq 1$ such that m is the multiplicity of $\mu_{k_0}(\mathbf{A}, a)$, $j_0 \leq k_0 \leq j_0 + m - 1$ and*

$$\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a) = \mu_{k_0}(\mathbf{A}, a).$$

Let $\{\psi_i : j_0 \leq i \leq j_0 + m - 1\}$ be an $L^2(\mathbb{S}^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of the operator $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A}, a)$. Then for any sequence $\lambda_n \to 0^+$ there exists $i \in \{j_0, \dots, j_0 + m - 1\}$ such that

$$\liminf_{n\to+\infty} \frac{\left|\int_{\mathbb{S}^{N-1}} u(\lambda_n\theta)\overline{\psi_i(\theta)}\, dS(\theta)\right|}{\sqrt{H(\lambda_n)}} > 0.$$

Proof. Suppose by contradiction that there exists a sequence $\lambda_n \to 0^+$ such that

$$\liminf_{n\to+\infty} \frac{\left|\int_{\mathbb{S}^{N-1}} u(\lambda_n\theta)\overline{\psi_i(\theta)}\, dS(\theta)\right|}{\sqrt{H(\lambda_n)}} = 0$$

for all $i \in \{j_0, \dots, j_0 + m - 1\}$. By Lemma 2.8 we deduce that there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction ψ of the operator $L_{\mathbf{A},a}$ corresponding to the eigenvalue $\mu_{k_0}(\mathbf{A}, a)$ with $\|\psi\|_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = 1$, such that

$$\frac{u(\lambda_{n_k}\theta)}{\sqrt{H(\lambda_{n_k})}} \to \psi(\theta)$$

strongly in $L^2(\mathbb{S}^{N-1})$ and

$$\lim_{k\to+\infty} \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_k}\theta)}{\sqrt{H(\lambda_{n_k})}} \overline{\psi_i(\theta)}\, dS(\theta) = 0.$$

Therefore

$$\int_{\mathbb{S}^{N-1}} \psi(\theta)\overline{\psi_i(\theta)}\, dS(\theta) = \lim_{k\to+\infty} \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_k}\theta)}{\sqrt{H(\lambda_{n_k})}} \overline{\psi_i(\theta)}\, dS(\theta) = 0 \tag{2.20}$$

for any $i \in \{j_0, \dots, j_0 + m - 1\}$. Hence $\psi \equiv 0$, thus giving rise to a contradiction. □

Lemma 2.11. *Let the assumptions of Lemma* 2.2 *hold and* $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ *be as in Lemma* 2.6. *Then there holds*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0.$$

Proof. For the sake of completeness, we report here part of the proof of Lemma 6.5 in [5]. Let $0 < R < \frac{\tilde{r}}{2}$, $\tilde{r}$ as in Lemma 2.4, and, for any $k \in \mathbb{N}\setminus\{0\}$, let ψ_k be a L^2-normalized eigenfunction of the operator $L_{\mathbf{A},a}$ on the sphere associated to the k-th eigenvalue $\mu_k(\mathbf{A}, a)$, i.e. satisfying

$$\begin{cases} L_{\mathbf{A},a}\psi_k(\theta) = \mu_k(\mathbf{A}, a)\, \psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\, dS(\theta) = 1. \end{cases} \tag{2.21}$$

We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1}, \mathbb{C})$, hence u and $hu + g(x, |u|^2)u$ can be expanded as

$$u(x) = u(\lambda\,\theta) = \sum_{k=1}^{\infty} \varphi_k(\lambda)\psi_k(\theta), \tag{2.22}$$

$$\begin{aligned} h(x)u(x) + g(x, |u(x)|^2)u(x) &= h(\lambda\,\theta)u(\lambda\,\theta) + g(\lambda\,\theta, |u(\lambda\,\theta)|^2)u(\lambda\,\theta) \\ &= \sum_{k=1}^{\infty} \zeta_k(\lambda)\psi_k(\theta), \end{aligned}$$

where $\lambda = |x| \in (0, R]$, $\theta = x/|x| \in \mathbb{S}^{N-1}$, and

$$\begin{aligned} \varphi_k(\lambda) &= \int_{\mathbb{S}^{N-1}} u(\lambda\,\theta)\overline{\psi_k(\theta)}\, dS(\theta), \\ \zeta_k(\lambda) &= \int_{\mathbb{S}^{N-1}} \big(h(\lambda\,\theta) + g(\lambda\,\theta, |u(\lambda\,\theta)|^2)\big)u(\lambda\,\theta)\overline{\psi_k(\theta)}\, dS(\theta). \end{aligned} \tag{2.23}$$

Equations (1.18) and (2.21) imply that, for every k,

$$-\varphi_k''(\lambda) - \frac{N-1}{\lambda}\varphi_k'(\lambda) + \frac{\mu_k(\mathbf{A}, a)}{\lambda^2}\varphi_k(\lambda) = \zeta_k(\lambda), \quad \text{in } (0, \tilde{r}).$$

A direct calculation shows that, for some $c_1^k(R), c_2^k(R) \in \mathbb{R}$,

$$\begin{aligned}\varphi_k(\lambda) = \lambda^{\sigma_k^+}\left(c_1^k(R) + \int_\lambda^R \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-}\zeta_k(s)\,ds\right)\\ + \lambda^{\sigma_k^-}\left(c_2^k(R) + \int_\lambda^R \frac{s^{-\sigma_k^- + 1}}{\sigma_k^- - \sigma_k^+}\zeta_k(s)\,ds\right),\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}\sigma_k^+ &= -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)},\\ \sigma_k^- &= -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}.\end{aligned} \tag{2.25}$$

In view of Lemma 2.8, there exist $j_0, m \in \mathbb{N}$, $j_0, m \geq 1$ such that m is the multiplicity of the eigenvalue $\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a)$ and

$$\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = \sigma_i^+, \quad i = j_0, \ldots, j_0 + m - 1. \tag{2.26}$$

The Parseval identity yields

$$H(\lambda) = \int_{\mathbb{S}^{N-1}} |u(\lambda\,\theta)|^2\,dS(\theta) = \sum_{k=1}^{\infty} |\varphi_k(\lambda)|^2, \quad \text{for all } 0 < \lambda \leq R. \tag{2.27}$$

Let us assume by contradiction that $\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0$. Then, (2.26) and (2.27) imply that

$$\lim_{\lambda\to 0^+} \lambda^{-\sigma_i^+}\varphi_i(\lambda) = 0 \qquad \text{for any } i \in \{j_0, \ldots, j_0 + m - 1\}\ . \tag{2.28}$$

We claim that the functions

$$s \mapsto \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s), \quad s \mapsto \frac{s^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s), \tag{2.29}$$

belong to $L^1(0, R)$ for any $i \in \{j_0, \ldots, j_0 + m - 1\}$. To this purpose, we define

$$Z_i(s) = \int_{B_s} |h(x) + g(x, |u(x)|^2)||u(x)||\psi_i(x/|x|)|\,dx$$

for any $s \in (0, \tilde{r})$ and for any $i \in \{j_0, \ldots, j_0 + m - 1\}$. We observe that Z_i is an absolutely continuous function whose derivative, defined for almost every $s \in (0, \tilde{r})$, is given by

$$Z_i'(s) = s^{N-1}\int_{\mathbb{S}^{N-1}} |h(s\theta) + g(s\,\theta, |u(s,\theta)|^2)||u(s\theta)||\psi_i(\theta)|\,dS(\theta)$$

for a.e. $s \in (0, \tilde r)$. Integrating by parts, we obtain

$$\begin{aligned}\int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}|\zeta_i(s)|\,ds &\le \int_\lambda^R \frac{s^{-\sigma_i^+ +2-N}}{\sigma_i^+ - \sigma_i^-} Z_i'(s)\,ds \\ &= \left[\frac{s^{-\sigma_i^+ +2-N}}{\sigma_i^+ - \sigma_i^-} Z_i(s)\right]_\lambda^R - \int_\lambda^R \frac{2-N-\sigma_i^+}{\sigma_i^+ - \sigma_i^-} s^{-\sigma_i^+ +1-N} Z_i(s)\,ds.\end{aligned} \tag{2.30}$$

From (2.4) and (1.20)

$$\begin{aligned}&|Z_i(s)| \\ &\le \left(\int_{B_s}|h+g(x,|u|^2)||u|^2dx\right)^{\frac12}\left(\int_{B_s}|h+g(x,|u|^2)|\big|\psi_i\big(\tfrac{x}{|x|}\big)\big|^2dx\right)^{\frac12} \\ &\le \left[\frac{\eta_0(s)+C_g\left(\frac{\omega_{N-1}}{N}\right)^{\frac2N}s^2+C_g\|u\|^{2^*-2}_{L^{2^*}(B_s)}}{\overline{C}}s^{N-2}\left(D(s)+\tfrac{N-2}{2}H(s)\right)\right]^{\frac12}\times \\ &\quad\times s^{\frac{N-2}{2}}\Big[\tfrac{\eta_0(s)}{N-2}\int_{\mathbb{S}^{N-1}}\left(|(\nabla_{\mathbb{S}^{N-1}}+i\mathbf{A})\psi_i(\theta)|^2-a(\theta)|\psi_i(\theta)|^2\right)dS(\theta) \\ &\qquad\qquad+\tfrac{N-2}{2}\eta_0(s)\int_{\mathbb{S}^{N-1}}|\psi_i(\theta)|^2dS(\theta) \\ &\qquad\qquad+\frac{C_g}{N^{2/2^*}}\|\psi_i\|^2_{L^{2^*}(\mathbb{S}^{N-1})}\left(\left(\tfrac{\omega_{N-1}}{N}\right)^{\frac2N}s^2+\|u\|^{2^*-2}_{L^{2^*}(B_s)}\right)\Big]^{\frac12} \\ &\le \widetilde C_1(i)\sqrt{\mathcal{N}(s)+\tfrac{N-2}{2}}\left(\eta_0(s)+s^2+s^{\frac{2(q-2^*)}{q}}\right)s^{N-2}\sqrt{H(s)} \\ &\le \widetilde C_1(i)\left(\sup_{(0,\tilde r/2)}\sqrt{\mathcal{N}+\tfrac{N-2}{2}}\right)s^{N-2}\widetilde\eta(s)\sqrt{H(s)}\end{aligned} \tag{2.31}$$

for all $s \in (0, \tilde r/2)$ for some constant $\widetilde C_1(i) > 0$ depending on $\overline{C}$, C_g, N, u, q, and ψ_i, where

$$\widetilde\eta(s) := \eta_0(s) + s^2 + s^{\frac{2(q-2^*)}{q}}.$$

We notice that, by assumption (1.15),

$$\frac{\widetilde\eta(s)}{s} \in L^1(0,\tilde r)$$

and, by Lemma 2.6,

$$\sup_{(0,\tilde r/2)}\sqrt{\mathcal{N}+\frac{N-2}{2}} < +\infty.$$

Inserting (2.31) into (2.30) we obtain

$$\begin{aligned}\int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}|\zeta_i(s)|\,ds &\le \widetilde C_2(i)\frac{\sqrt{H(R)}}{R^{\sigma_i^+}}\widetilde\eta(R) \\ &\quad+\widetilde C_2(i)\frac{\sqrt{H(\lambda)}}{\lambda^{\sigma_i^+}}\widetilde\eta(\lambda)+\widetilde C_3(i)\int_\lambda^R\frac{\sqrt{H(s)}}{s^{\sigma_i^+}}\frac{\widetilde\eta(s)}{s}\,ds\end{aligned} \tag{2.32}$$

and using (1.15), (2.19), the integrability of the first function in (2.29) follows. The integrability of the second function also follows since $\sigma_i^- < \sigma_i^+$. Hence

$$\lambda^{\sigma_i^+}\left(c_1^i(R) + \int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) = o(\lambda^{\sigma_i^-}) \quad \text{as } \lambda \to 0^+,$$

and then, since $\frac{u}{|x|} \in L^2(B_R, \mathbb{C})$ and $\frac{|x|^{\sigma_i^-}}{|x|} \notin L^2(B_R, \mathbb{C})$, we conclude that there must be

$$c_2^i(R) = -\int_0^R \frac{s^{-\sigma_i^- +1}}{\sigma_i^- - \sigma_i^+}\,\zeta_i(s)\,ds. \tag{2.33}$$

Using (2.31) and (2.19), we then deduce that

$$\begin{aligned}
&\left|\lambda^{\sigma_i^-}\left(c_2^i(R) + \int_\lambda^R \frac{s^{-\sigma_i^- +1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s)\,ds\right)\right| \\
&\quad = \left|\lambda^{\sigma_i^-}\left(\int_0^\lambda \frac{s^{-\sigma_i^- +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right)\right| \\
&\quad \leq \lambda^{\sigma_i^-}\int_0^\lambda \frac{s^{-\sigma_i^- +2-N}}{\sigma_i^+ - \sigma_i^-} Z_i'(s)\,ds \\
&\quad = \frac{\lambda^{2-N}}{\sigma_i^+ - \sigma_i^-} Z_i(\lambda) - \lambda^{\sigma_i^-}\int_0^\lambda \frac{2-N-\sigma_i^-}{\sigma_i^+ - \sigma_i^-} s^{-\sigma_i^- +1-N} Z_i(s)\,ds \\
&\quad = O\left(\lambda^{\sigma_i^+}\left[\tilde{\eta}(\lambda) + \int_0^\lambda \frac{\tilde{\eta}(s)}{s}\,ds\right]\right) = o(\lambda^{\sigma_i^+})
\end{aligned} \tag{2.34}$$

as $\lambda \to 0^+$. From (2.24), (2.28), and (2.34), we obtain that

$$c_1^i(R) + \int_0^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds = 0 \quad \text{for all } R \in (0, \tilde{r}/2). \tag{2.35}$$

Since $H \in C^1(0, \tilde{r})$ and since we are assuming by contradiction that

$$\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) = 0,$$

we may select a sequence $\{R_n\}_{n\in\mathbb{N}} \subset (0, \tilde{r}/2)$ decreasing to zero such that

$$\frac{\sqrt{H(R_n)}}{R_n^\gamma} = \max_{s\in[0,R_n]} \frac{\sqrt{H(s)}}{s^\gamma}.$$

Applying Lemma 2.10 with $\lambda_n = R_n$, we find $i_0 \in \{j_0, \ldots, j_0 + m - 1\}$ such that, up to a subsequence,

$$\lim_{n\to+\infty} \frac{\varphi_{i_0}(R_n)}{\sqrt{H(R_n)}} \neq 0. \tag{2.36}$$

We are now going to reach a contradiction with (2.35) by choosing $i = i_0$, $R = R_n$ and $n \in \mathbb{N}$ sufficiently large. By (2.35), (2.32), (2.36) and (2.19), we have

$$
\begin{aligned}
|c_1^{i_0}(R_n)| &= \left| \int_0^{R_n} \frac{s^{-\sigma_{i_0}^+ +1}}{\sigma_{i_0}^+ - \sigma_{i_0}^-} \zeta_{i_0}(s)\, ds \right| \\
&\le \widetilde{C}_2(i_0) \frac{\sqrt{H(R_n)}}{R_n^\gamma} \widetilde{\eta}(R_n) + \widetilde{C}_3(i_0) \int_0^{R_n} \frac{\sqrt{H(s)}}{s^\gamma} \frac{\widetilde{\eta}(s)}{s}\, ds \\
&\le \widetilde{C}_2(i_0) \left| \frac{\sqrt{H(R_n)}}{\varphi_{i_0}(R_n)} \right| \left| \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right| \widetilde{\eta}(R_n) \\
&+ \widetilde{C}_3(i_0) \left| \frac{\sqrt{H(R_n)}}{\varphi_{i_0}(R_n)} \right| \left| \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right| \int_0^{R_n} \frac{\widetilde{\eta}(s)}{s}\, ds \\
&= o\left(\frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right)
\end{aligned}
\tag{2.37}
$$

as $n \to +\infty$. By (2.24) with $k = i_0$, $R = R_n$ and $\lambda = R_n$, we obtain

$$
\frac{\varphi_{i_0}(R_n)}{R_n^{\sigma_{i_0}^+}} = c_1^{i_0}(R_n) + c_2^{i_0}(R_n) R_n^{\sigma_{i_0}^- - \sigma_{i_0}^+}. \tag{2.38}
$$

By (2.33), (2.31) and (2.36) we have that

$$
\begin{aligned}
|c_2^{i_0}(R_n) R_n^{\sigma_{i_0}^- - \sigma_{i_0}^+}| &= R_n^{\sigma_{i_0}^- - \sigma_{i_0}^+} \left| \int_0^{R_n} \frac{s^{-\sigma_{i_0}^- +1}}{\sigma_{i_0}^- - \sigma_{i_0}^+} \zeta_{i_0}(s)\, ds \right| \\
&\le \widetilde{C}_2(i_0) \frac{\sqrt{H(R_n)}}{R_n^\gamma} \widetilde{\eta}(R_n) + \widetilde{C}_4(i_0) R_n^{\sigma_{i_0}^- - \sigma_{i_0}^+} \int_0^{R_n} \frac{\sqrt{H(s)}}{s^{\sigma_{i_0}^-}} \frac{\widetilde{\eta}(s)}{s}\, ds \\
&= \widetilde{C}_2(i_0) \left| \frac{\sqrt{H(R_n)}}{\varphi_{i_0}(R_n)} \right| \left| \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right| \widetilde{\eta}(R_n) \\
&\quad + \widetilde{C}_4(i_0) R_n^{\sigma_{i_0}^- - \sigma_{i_0}^+} \int_0^{R_n} \frac{\sqrt{H(s)}}{s^{\sigma_{i_0}^+}} s^{\sigma_{i_0}^+ - \sigma_{i_0}^-} \frac{\widetilde{\eta}(s)}{s}\, ds \\
&\le \widetilde{C}_2(i_0) \left| \frac{\sqrt{H(R_n)}}{\varphi_{i_0}(R_n)} \right| \left| \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right| \widetilde{\eta}(R_n) \\
&\quad + \widetilde{C}_4(i_0) \left| \frac{\sqrt{H(R_n)}}{\varphi_{i_0}(R_n)} \right| \left| \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right| \int_0^{R_n} \frac{\widetilde{\eta}(s)}{s}\, ds \\
&= o\left(\frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right).
\end{aligned}
\tag{2.39}
$$

Inserting (2.39) into (2.38) we obtain

$$
c_1^{i_0}(R_n) = \frac{\varphi_{i_0}(R_n)}{R_n^\gamma} + o\left(\frac{\varphi_{i_0}(R_n)}{R_n^\gamma} \right)
$$

as $n \to +\infty$, thus contradicting (2.37). □

The proof of Theorem 1.1 can be now obtained by proceeding similarly to [5, Theorem 1.3] with small changes but for completeness we report it below.

Proof of Theorem 1.1. Identity (1.24) follows from part (i) of Lemma 2.8, thus there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that

$$\gamma := \lim_{r\to 0^+} \mathcal{N}_{u,h,f}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}.$$

Let m be the multiplicity of $\mu_{k_0}(\mathbf{A}, a)$, so that, for some $j_0 \in \mathbb{N}$ such that $j_0 \geq 1$, $j_0 \leq k_0 \leq j_0 + m - 1$, $\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a)$ and let $\{\psi_i : j_0 \leq i \leq j_0 + m - 1\}$ be an $L^2(\mathbb{S}^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A}, a)$. Let $\lambda_n > 0$, $n \in \mathbb{N}$ such that $\lim_{n\to+\infty} \lambda_n = 0$. Then, from part (ii) of Lemma 2.8 and Lemmas 2.9 and 2.11, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and m real numbers $\beta_{j_0}, \ldots, \beta_{j_0+m-1} \in \mathbb{R}$ such that $(\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_0+m-1}) \neq (0, 0, \ldots, 0)$ and

$$\lambda_{n_k}^{-\gamma} u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1}, \mathbb{C}) \quad \text{as } k \to +\infty \tag{2.40}$$

and

$$\lambda_{n_k}^{1-\gamma} \nabla u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i (\gamma\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)) \text{ in } C^{0,\tau}(\mathbb{S}^{N-1}, \mathbb{C}^N) \tag{2.41}$$

as $k \to +\infty$ for any $\tau \in (0, 1)$. We now show that the β_i's depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$.

Let $R > 0$ be such that $\overline{B}_R \subset \Omega$ and let φ_i and ζ_i as in (2.23). Then by (2.22) and (2.40) it follows that, for any $i = j_0, \ldots, j_0 + m - 1$,

$$\begin{aligned} \lambda_{n_k}^{-\gamma} \varphi_i(\lambda_{n_k}) &= \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_k}\theta)}{\lambda_{n_k}^{\gamma}} \overline{\psi_i(\theta)}\, dS(\theta) \\ &\to \sum_{j=j_0}^{j_0+m-1} \beta_j \int_{\mathbb{S}^{N-1}} \psi_j(\theta)\overline{\psi_i(\theta)}\, dS(\theta) = \beta_i \end{aligned} \tag{2.42}$$

as $k \to +\infty$. As showed in the proof of Lemma 2.11, for any $\lambda \in (0, R]$ and $i = j_0, \ldots, j_0 + m - 1$ we have

$$\begin{aligned} \varphi_i(\lambda) &= \lambda^{\sigma_i^+}\left(c_1^i(R) + \int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\, ds\right) \\ &\quad + \lambda^{\sigma_i^-}\left(\int_0^\lambda \frac{s^{-\sigma_i^- +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\, ds\right) \\ &= \lambda^{\sigma_i^+}\left(c_1^i(R) + \int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\, ds\right) + o(\lambda^{\sigma_i^+}) \quad \text{as } \lambda \to 0^+, \end{aligned} \tag{2.43}$$

for some $c_1^i(R) \in \mathbb{R}$, where $\sigma_i^{\pm}$ are as in (2.25) and $\sigma_i^+ = \gamma$. Choosing $\lambda = R$ in the first line of (2.43), we obtain

$$c_1^i(R) = R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{s^{-\sigma_i^-+1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds.$$

Using the last identity and letting $\lambda \to 0^+$ in (2.43) it follows that

$$\lambda^{-\gamma}\varphi_i(\lambda) \to R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{s^{-\sigma_i^-+1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds + \int_0^R \frac{s^{-\sigma_i^++1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds \quad \text{as } \lambda \to 0^+,$$

and hence by (2.42)

$$\begin{aligned}\beta_i &= R^{-\gamma}\int_{\mathbb{S}^{N-1}} u(R\theta)\overline{\psi_i(\theta)}\,dS(\theta)\\ &- R^{-2\gamma-N+2}\int_0^R \frac{s^{\gamma+N-1}}{2\gamma+N-2}\times\\ &\qquad\times\bigg(\int_{\mathbb{S}^{N-1}}\big(h(s\,\theta)+g(s\theta,|u(s\theta)|^2)\big)u(s\,\theta)\overline{\psi_i(\theta)}\,dS(\theta)\bigg)ds\\ &+\int_0^R \frac{s^{1-\gamma}}{2\gamma+N-2}\bigg(\int_{\mathbb{S}^{N-1}}\big(h(s\,\theta)+g(s\theta,|u(s\theta)|^2)\big)u(s\,\theta)\overline{\psi_i(\theta)}\,dS(\theta)\bigg)ds\ .\end{aligned}$$

We just proved that the β_i's depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. This proves that the convergences in (2.40) and (2.41) actually hold as $\lambda \to 0^+$ thus completing the proof of the theorem. □

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