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Option pricing in a conditional Bilateral Gamma model

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Abstract

We propose a conditional Bilateral Gamma model, in which the shape parameters of the Bilateral Gamma distribution have a Garch-like dynamics. After risk neutralization by means of a Bilateral Esscher Transform, the model admits a recursive procedure for the computation of the characteristic function of the underlying at maturity, à la Heston and Nandi (2000). We compare the calibration performance on SPX options with the models of Heston and Nandi (2000), Christoffersen, Heston and Jacobs (2006) and with a Dynamic Variance Gamma model introduced in Mercuri and Bellini (2011), obtaining promising results.

Keywords: *Bilateral Gamma, Garch, Bilateral Esscher Transform, semi-analytical pricing, SPX options.*

1 Introduction

Garch models, introduced in the seminal papers of Engle (1982) and Bollerslev (1986), are still very popular in the financial econometric literature since they capture the essential features of stock price dynamics in a very parsimonious way. Indeed, even the simplest Garch (1,1) model is able to capture the volatility clustering and the presence of paretian tails that is often observed in financial time series. It is then very natural to use Garch models also for option pricing; however, generally speaking, the literature on option pricing in Garch models is still quite limited, for example in comparison with the literature on continuous-time stochastic volatility models. From a theoretical point of view, a problem is that discrete-time models with continuous innovations are always incomplete, so it is necessary to specify the transformation between the actual probability P and the risk neutral probability Q; until now the literature mainly concentrated on the use of the Conditional Esscher transform, introduced by Gerber and

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Shiu (1994) and Buhlmann et al. (1996). Recently, several others change of measure have been considered; see for example Goovaerts and Laeven (2008) and Monfort and Pegoraro (2012) and the references therein.
Another problem is that Garch option pricing models do not produce closed formulas for option prices; for example in the seminal papers of Duan (1995) and Siu et al. (2004) option prices were computed by means of Monte Carlo simulations.

A major breakthrough occurred with the work of Heston and Nandi (2000), in which the authors showed a recursive procedure for the computation of the characteristic function of the price of the underlying at maturity, thus introducing a semianalytical procedure for option pricing based on the inversion of a Fourier transform, as in Carr and Madan (1999).
Since then, several papers appeared that considered option pricing in Garch models with different specifications, different shapes of the innovations, different choices of the martingale measure, different estimation approaches (historical, pure calibration, mixed). Siu et al. (2004) were the first to consider Gamma innovations, although their model did not allow semianalytical valuation. Christoffersen et al. (2006) considered Inverse Gaussian innovations; Mercuri (2007) considered a model with Gamma innovations that allowed the recursive computation of the characteristic function; later Mercuri (2008) considered the family of Tempered Stable innovations, further generalized in Menn and Rachev (2009) and Kim et al. (2010).

Gamma innovations are very tractable, since their characteristic function is very simple; however, their extreme asymmetry makes the model work not so well in calibration (see Mercuri, 2007 and Bellini and Mercuri, 2007); a similar problem is shared by the Inverse Gaussian innovations of Christoffersen et al. (2006). Recently Kuchler and Tappe (2008 a,b) suggested the use of Bilateral Gamma distributions, that are defined simply as the difference between two independent Gamma distributions. In this way, many of the analytical properties of the Gamma distribution are preserved, while it is possible to model also close-to symmetric datasets. Moreover, the Bilateral Gamma distribution admits as a special case the Variance Gamma distribution, introduced in Finance by Madan and Seneta (1990).

In this work we propose a dynamic, discrete time conditional Bilateral Gamma model, that tries to capture the volatility dynamics in a Garch-like fashion, while retaining the possibility of a simple semianalytical computation of option prices. The idea is to model separately the evolution of the shape parameters of the two Gamma distributions in the definition of the Bilateral Gamma.
Risk neutralization is achieved by means of a static Bilateral Esscher Transform, that modifies only the scale parameter of the Gamma distributions.
The model is compared with the Heston and Nandi model, with the CHJ model, and with a DVG model developed in Mercuri and Bellini (2011).

The paper is structured as follows: in Section 2 we recall the basic properties of the Bilateral Gamma distribution, in Section 3 we define our model, in Section 4 we briefly recall the competing models and in Section 5 we run the calibration comparison.

2 Bilateral Gamma distributions

The Bilateral Gamma distribution is defined as the difference of two independent Gamma distributions:

$$X \sim BG(a,b,c,d) \text{ if } X = Y - Z, \text{ with } Y \sim \Gamma\left(a,b\right) \text{ and } Z \sim \Gamma\left(c,d\right), \quad (1)$$

with $a, b, c, d > 0$. Bilateral Gamma distributions (shortly BG) have essentially the same analytical tractability of the Gamma distribution, while being much more flexible for modelling near-to-symmetric datasets. The m.g.f. of the BG distribution is given by

$$M_X(t) = \left(\frac{1}{1-bt}\right)^a \left(\frac{1}{1+dt}\right)^c, \ t \in \left(-\frac{1}{d}, \frac{1}{b}\right) \quad (2)$$

and its first moments are the following:

$$E[X] = ab - cd, \, \sigma^2[X] = ab^2 + cd^2 \quad (3)$$
$$\gamma(X) = \frac{2\left(ab^3 - cd^3\right)}{\sqrt{\left(ab+cd\right)^3}}, \, \kappa(X) = 3 + \frac{6\left(ab^4 + cd^4\right)}{\left(ab+cd\right)^2}$$

where $\gamma(X)$ and $\kappa(X)$ are the usual coefficients of asymmetry and kurtosis. The properties of BG distributions have been studied in detail in Kuchler and Tappe (2008b) and their use in financial modelling has been avocated in Kuchler and Tappe (2008a). Option pricing with BG distributions in continuous time models has been considered in Kuchler and Tappe (2009). The following properties of BG distributions are immediate consequences of the definition (see Kuchler and Tappe (2008b)):

Proposition 1 *Elementary properties of the Bilateral Gamma distribution:*
a) If $X \sim BG(a,b,c,d)$, then $kX \sim BG(a,kb,c,kd)$;
b) if $X_1 \sim BG(a_1,b,c_1,d)$ and $X_2 \sim BG(a_2,b,c_2,d)$, with X_1 and X_2 independent, then

$$X_1 + X_2 \sim BG(a_1+a_2, b, c_1+c_2, d);$$

c) the BG distribution is infinitely divisible;
d) the Lévy measure in the Lévy-Kintchine formula for the characteristic function of X is given by

$$F(dx) = \left(\frac{a}{x}e^{-\frac{x}{b}}1_{(0,+\infty)}(x) + \frac{c}{x}e^{-\frac{x}{d}}1_{(0,+\infty)}(x)\right)dx;$$

e) BG distributions are absolutely continuous and their density $f(x)$ satisfies the following symmetry relationship:

$$f(-x,a,b,c,d) = f(x,c,d,a,b), \ x \neq 0.$$

Moreover, Kuchler and Tappe provided an explicit expression for the density $f(x)$ of BG variables in terms of special functions and proved the following more difficult results:

Proposition 2 *Advanced properties of the Bilateral Gamma distribution:*
a) The BG distribution is strictly unimodal;
b) if $N < a + c \leq N + 1$*, then* $f \in C^N(R\setminus\{0\})$ *and* $f \in C^{N-1}(R)\setminus C^N(R)$*;*
c) $\lim_{x\to+\infty} \frac{\log(f(x))}{x} = -\frac{1}{b}$ *and* $\lim_{x\to-\infty} \frac{\log(f(x))}{x} = -\frac{1}{d}$*.*

The shapes of the BG densities are reported in Fig.1, respectively for $a+c = 0.9, 1.8$ and 2.5; from the item b) of the preceding Proposition it is evident how this affect smoothness in $x = 0$.
An important special case of the BG distribution that arises when $a = c$ is the Variance Gamma distribution, introduced in financial applications by Madan and Seneta (1990).

Insert Fig.1 about here

3 A Dynamic Bilateral Gamma model

In this Section we propose a discrete-time, conditional Bilateral Gamma model (DBG for short), in which the parameters of the conditional BG distribution evolve in a recursive, Garch-like fashion. We consider the following specification:

$$\begin{cases} X_n = Y_n - Z_n \\ Y_n|F_{n-1} \sim \Gamma(a_n, b) \\ Z_n|F_{n-1} \sim \Gamma(c_n, d) \\ a_n = \alpha_0 + \alpha_1 Y_{n-1} + \beta_1 a_{n-1} \\ c_n = \alpha_0 + \alpha_1 Z_{n-1} + \beta_1 c_{n-1} \end{cases} \tag{4}$$

where α_0, α_1, $\beta_1 > 0$, to ensure the positivity of the conditional shape parameters a_n and c_n. The logreturns X_n have a conditional BG distribution, whose scale parameters b and d are fixed and whose shape parameters a_n and c_n evolve according to the last two equations in (4). In order to introduce explicitly the risk premium per unit variance, we define

$$\lambda = b - b^2\theta,\ \nu = d^2\theta + d$$

and rewrite the first line of model (4) as follows:

$$X_n = Y_n - Z_n + r - \lambda a_n + \nu c_n.$$

In this way

$$\begin{aligned} E[X_n|F_{n-1}] &= r + (b^2\theta - b)a_n + (d^2\theta + d)c_n + a_n b - c_n d = \\ &= r + \theta(b^2 a_n + d^2 c_n) = r + \theta Var[X_n|F_{n-1}] \end{aligned}$$

so that the parameter θ plays the role of the risk premium per unit variance. The model depends on the six parameters $(\theta, b, d, \alpha_0, \alpha_1, \beta_1)$, disregarding the transient initial values of a_0, c_0, Y_0, Z_0. A risk neutral version is the following:

$$\begin{cases} \widetilde{X}_n = \widetilde{Y}_n - \widetilde{Z}_n + r - \widetilde{\lambda} a_n + \widetilde{\nu} c_n \\ \widetilde{Y}_n | F_{n-1} \sim \Gamma\left(a_n, \widetilde{b}\right) \\ \widetilde{Z}_n | F_{n-1} \sim \Gamma\left(c_n, \widetilde{d}\right) \\ a_n = \alpha_0 + \alpha_1 \widetilde{Y}_{n-1} + \beta_1 a_{n-1} \\ c_n = \alpha_0 + \alpha_1 \widetilde{Z}_{n-1} + \beta_1 c_{n-1} \end{cases} \tag{5}$$

where $\widetilde{b} = 1 - e^{-\widetilde{\lambda}}$ and $\widetilde{d} = e^{\widetilde{\nu}} - 1$. The change of measure from (4) to (5) modifies only the scale parameters b and d of the BG distribution, leaving the dynamics of the conditional shape parameters a_n and c_n unaffected; it can be seen as a product of two independent static Esscher transforms on Y_n and Z_n. The main advantage of models (4) and (5) is that it is possible to compute recursively the characteristic function of the underlying at maturity, as in the case of the Heston and Nandi (2000) model. Indeed, if

$$S_N = S_0 \exp(\sum_{n=1}^{N} X_n)$$

then

$$E[e^{it \log S_N} | F_n] = E[S_N^{it} | F_n] = S_n^{it} \exp\left(A_n\left(it\right) + B_n\left(it\right) a_{n+1} + C_n\left(it\right) c_{n+1}\right)$$

where the coefficients A_n, B_n and C_n can be computed recursively as follows:

$$\begin{cases} A_n\left(t\right) = rt + A_{n+1}\left(t\right) + \alpha_0 B_{n+1}\left(t\right) + \alpha_0 C_{n+1}\left(t\right) \\ B_n\left(t\right) = -\tilde{\lambda} t + \beta_1 B_{n+1}\left(t\right) - \ln\left(1 - \tilde{b} t - \alpha_1 \tilde{b} B_{n+1}\left(t\right)\right) \\ C_n\left(t\right) = \tilde{\nu} t + \beta_1 C_{n+1}\left(t\right) - \ln\left(1 + \tilde{d} t - \alpha_1 \tilde{d} C_{n+1}\left(t\right)\right) \end{cases} \tag{6}$$

(see the Appendix for the proof).

4 A brief recall of the competing models

In this section we briefly recall the competing models. The Heston and Nandi (2000) model (HN) is well-known; its specification under the P measure is the following:

$$\begin{cases} X_n = r + \lambda h_n + \sqrt{h_n} \varepsilon_n \\ h_n = \alpha_0 + \alpha_1 (\varepsilon_{n-1} - \gamma \sqrt{h_{n-1}})^2 + \beta_1 h_{n-1} \end{cases}$$

where ε_n are i.i.d. standard normal. The parameter λ represents the risk premium per unit variance and h_n is the conditional variance. The model depends on the 5 parameters $(\lambda, \gamma, \alpha_0, \alpha_1, \beta_1)$. Its main advantage is that it is possible to compute recursively the characteristic function of the price process; the main drawback is that the innovations are assumed to be conditionally normal, that does not match very well with the typical findings of conditional asymmetry and fat-tailedness. The risk neutralization is achieved by means of an Esscher transform, that results the following:

$$\begin{cases} \widetilde{X}_n = r - \frac{h_n}{2} + \sqrt{h_n}\varepsilon_n \\ h_n = \alpha_0 + \alpha_1(\varepsilon_{n-1} - \widetilde{\gamma}\sqrt{h_{n-1}})^2 + \beta_1 h_{n-1} \end{cases} \tag{7}$$

with

$$\widetilde{\gamma} = \gamma + \lambda + \frac{1}{2} \tag{8}$$

The Christoffersen, Heston and Jacobs (2006) model (CHJ) is an attempt to introduce conditional asymmetry by means of Inverse Gaussian (IG) innovations. The specification is the following:

$$\begin{cases} X_n = \eta Y_n + r + \lambda h_n \\ Y_n | F_{n-1} \sim IG(\delta_n) \text{ with } \delta_n = \frac{h_n}{\eta^2} \\ h_n = \alpha_0 + \beta_1 h_{n-1} + \alpha_1 Y_{n-1} + \gamma \frac{h_{n-1}^2}{Y_{n-1}} \end{cases} \tag{9}$$

We recall that the Inverse Gaussian distribution $IG(\delta)$ has support on the positive semiaxis with the following density:

$$f(x;\delta) = \frac{\delta}{\sqrt{2\pi x^3}} \exp(-\frac{1}{2}(\sqrt{x} - \frac{\delta}{\sqrt{x}})^2), \text{ with } \delta > 0,\ x > 0. \tag{10}$$

The CHJ model under the Q measure has a similar dynamics with different parameters $\widetilde{\eta}$, $\widetilde{\lambda}$, $\widetilde{\alpha}_0$, $\widetilde{\beta}_1$, $\widetilde{\alpha}_1$, $\widetilde{\gamma}$:

$$\begin{cases} X_n = \widetilde{\eta} Y_n + r + \widetilde{\lambda} h_n \\ Y_n | F_{n-1} \sim IG(\widetilde{\delta}_n) \text{ with } \widetilde{\delta}_n = \frac{\widetilde{h}_n}{\widetilde{\eta}^2} \\ \widetilde{h}_n = \widetilde{\alpha}_0 + \widetilde{\beta}_1 \widetilde{h}_{n-1} + \widetilde{\alpha}_1 Y_{n-1} + \widetilde{\gamma} \frac{h_{n-1}^2}{Y_{n-1}} \end{cases} \tag{11}$$

moreover, $\widetilde{\lambda}$ can be expressed as a function of the other 5 parameters.

The Dynamic Variance Gamma (DVG) model has been suggested in Mercuri and Bellini (2011). We recall that a Variance Gamma distribution is defined as a mixture of normals, with gamma-distributed variances. Our idea was to consider a dynamic, Garch-like equation for the gamma mixing density, in order to capture some degree of persistence of high volatility periods, exactly in the same way as in the usual Garch(1,1) models. The model is the following:

$$\begin{cases} X_n = r + \lambda h_n + \sigma\sqrt{V_n}\varepsilon_n \\ V_n | F_{n-1} \sim \Gamma(h_n, 1) \\ \varepsilon_n \sim N(0,1), \text{ i.i.d.} \\ h_n = \alpha_0 + \alpha_1 V_{n-1} + \beta_1 h_{n-1} \end{cases} \tag{12}$$

and the risk neutral version is given by

$$\begin{cases} X_n = r - \frac{\sigma^2}{2} V_n + \sigma\sqrt{V_n}\varepsilon_n \\ V_n | F_{n-1} \sim \Gamma(h_n, 1) \\ \varepsilon_n \sim N(0,1), \text{ i.i.d.} \\ h_n = \alpha_0 + \alpha_1 V_{n-1} + \beta_1 h_{n-1} \end{cases} \tag{13}$$

and it is identified by the four parameters $(\sigma, \alpha_0, \alpha_1, \beta_1)$.

5 Calibrations and comparisons

We compare the calibration performances of the DBG, CHJ, HN and DVG models. The dataset is composed by closing prices of European plain vanilla call and put options on SPX index, collected each Wednesday from 01/01/2009 to 31/12/2009. The first Wednesday of each month represents in-sample dates; the remaining Wednesdays are the out-of sample dates. The considered options have a time to maturity that ranges from 10 to 100 days and a moneyness that ranges between 90% and 110%. As it is customary in the literature on calibration on daily closing data, a preliminary verification of the Merton's no arbitrage bounds and convexity in the strike have been carried out, leading to the elimination of approximately 40% of the quotations originally present in the dataset, leaving 758 options. The riskfree rate r is taken from the Bloomberg LIBOR curve C079. The behaviour of the underlying SPX index in the relevant period and that of its implied volatility, measured by the VIX index, can be seen in Fig.2.

Insert Fig.2 about here

We calibrate the four considered models on the first Wednesday of each month by minimizing the in sample "dollar" root mean squared error (RMSE), defined as follows:

$$RMSE_{in} = \left(\frac{1}{N} \sum_{i=1}^{N} (C_i^{th} - C_i^{obs})^2 \right)^{\frac{1}{2}}$$

Insert Tables 1-4 about here

Tables 1-4 report the estimated parameters respectively for the DBG, CHJ, HN and DVG models, along with the total RMSE for each Wednesday of the in sample period. On the average, the DBG model is the best performer, as it is evident from Table 5. The total in sample RMSE of the CHJ, HN and DVG models are respectively 16%, 23,5% and 26,5% higher than the total in sample RMSE of the DBG model. Moreover, the DBG seems to overperform the other models in a quite systematic way; indeed from Table 5 it is possible to see that it is the best model in 8 of the 12 considered Wednesdays.

Insert Table 5 about here

We recall that the DBG and CHJ models depend on five parameters, while the HN and DVG only on four, so it is not surprising that the second best model on average is the conditional Inverse Gaussian CHJ model. This may be probably explained by the extreme asymmetry of the CHJ model, that fits less accurately the shape of the conditional risk-neutral distribution.

We then move to the comparison of out-of sample prices; that is, for each month, we compute option prices for the next Wednesdays using the parameters calibrated as before on the closing prices of the first Wednesday. In Table 6 we report the out of sample RMSE for the four models, and again we see that on the average the DBG model performs better, although in this case the error reduction with respect to the other model are lower, equal respectively to 15.2% for HN, 6.5% for CHJ and 7.5% for DVG. The DBG model is now the best only in half of the considered months. The out of sample RMSE is displayed in Fig.3-6.

Insert Table 6 about here

Insert Figures 3-6 about here

Since the CHJ model is the second best, we decided to conduct a deeper comparison between the DBG and the CHJ models.
In Table 7, we report the average out-of-sample RMSE for varying moneyness and time to maturity. We notice that the performance of the two models are similar when considering options with short time to maturity or moneyness higher than 1.05. However, the DBG model performs much better when considering options with moneyness between 0.95 and 1.05 (error reduction of 18.19%) and when considering time to maturity ranging from 31 to 60 days (error reduction of 13.21%). A graphical illustration is reported in Fig. 7. Again, the most likely explanation is the strong asymmetry of the Inverse Gaussian distribution.

Insert Table 7 about here

Insert Figure 7 about here

6 Appendix

As in Heston and Nandi (2000), we write the conditional characteristic function of the log price in the following way:

$$E[e^{it\log S_N}|F_{n+1}] = E[S_N^{it}|F_{n+1}] = S_{n+1}^{it}\exp\left(A_{n+1}(it) + B_{n+1}(it)\,a_{n+2} + C_{n+1}(it)\,c_{n+2}\right)$$

Using the iteration law of conditional expectations we can compute:

$$\begin{aligned} E[e^{it\log S_N}|F_n] &= E[E[e^{it\log S_N}|F_{n+1}]|F_n] \\ &= E[S_n^{it}\exp\left(it\tilde{X}_{n+1} + A_{n+1}(it) + B_{n+1}(it)\,a_{n+2} + C_{n+1}(it)\,c_{n+2}\right)|F_n] \end{aligned}$$

by replacing the dynamics of $\tilde{X}_{n+1}$, a_{n+2} and c_{n+2} we obtain:

$$\begin{aligned} E[e^{it\log S_N}|F_n] &= S_n^{it}\exp\left[itr + A_{n+1}(it) + \alpha_0 B_{n+1}(it) + \alpha_0 C_{n+1}(it)\right] \\ &* \exp\left[\left(B_{n+1}(it)\,\beta_1 - it\widetilde{\lambda}\right)a_{n+1} + \left(it\widetilde{\nu} + C_{n+1}(it)\,\beta_1\right)c_{n+1}\right] \\ &* E[\exp\left(\left(it + B_{n+1}(it)\,\alpha_1\right)\widetilde{Y}_{n+1} - \left(it - C_{n+1}(it)\,\alpha_1\right)\widetilde{Z}_{n+1}\right)|F_n] \end{aligned}$$

and introducing the characteristic function of the Bilateral Gamma distribution we get:

$$\begin{aligned} E[e^{it\log S_N}|F_n] &= S_n^{it}\exp\left[itr + A_{n+1}(it) + \alpha_0 B_{n+1}(it) + \alpha_0 C_{n+1}(it)\right] \\ &\quad * \exp\left[\left(-it\widetilde{\lambda} + B_{n+1}(it)\beta_1 - \ln\left(1 - it - B_{n+1}(it)\alpha_1\right)\right)a_{n+1}\right] \\ &\quad * \exp\left[\left(it\widetilde{\nu} + C_{n+1}(it)\beta_1 - \ln\left(1 + it - C_{n+1}(it)\alpha_1\right)\right)c_{n+1}\right] \end{aligned}$$

and by comparison we obtain the recursive system for the coefficients $A_n(t)$, $B_n(t)$ and $C_n(t)$.

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Date	$\widetilde{\lambda}$	$\widetilde{\nu}$	α_0	α_1	β_1	$RMSE_{in}$
07/01/2009	0.139	0.313	0.005	0.041	0.215	2.295
04/02/2009	0.117	0.146	0.010	0.010	0.210	1.676
04/03/2009	0.143	0.231	0.005	0.022	0.222	1.249
01/04/2009	0.148	0.237	0.004	0.030	0.205	1.692
06/05/2009	0.236	0.119	0.004	0.031	0.189	4.146
03/06/2009	0.096	0.173	0.005	0.017	0.181	3.613
01/07/2009	0.076	0.145	0.005	0.010	0.160	2.167
05/08/2009	0.043	0.052	0.023	$1.07*10^{-6}$	0.191	1.804
02/09/2009	0.098	0.148	0.006	0.013	0.184	2.247
07/10/2009	0.073	0.112	0.005	0.008	0.142	2.570
04/11/2009	0.071	0.141	0.005	0.011	0.185	2.935
02/12/2009	0.049	0.050	0.017	$1.62*10^{-6}$	0.120	3.352

Table 1: Estimated parameters for DBG model in sample period

Date	$\widetilde{\eta}$	$\widetilde{\alpha}_0$	$\widetilde{\alpha}_1$	$\widetilde{\beta}_1$	$\widetilde{\gamma}$	$RMSE_{in}$
07/01/2009	0.010	$7.32*10^{-6}$	$8.30*10^{-5}$	0.097	999.467	2.769
04/02/2009	0.012	$5.98*10^{-5}$	$8.33*10^{-5}$	0.099	999.844	0.914
04/03/2009	0.011	$2.36*10^{-5}$	$8.08*10^{-5}$	0.098	999.638	1.575
01/04/2009	0.010	$1.86*10^{-5}$	$6.87*10^{-5}$	0.096	999.373	2.602
06/05/2009	0.019	$6.85*10^{-9}$	$1.22*10^{-8}$	0.130	1007.968	5.711
03/06/2009	0.013	$4.79*10^{-5}$	$5.49*10^{-5}$	0.097	999.679	5.126
01/07/2009	0.019	$6.85*10^{-9}$	$1.22*10^{-8}$	0.130	1007.968	5.711
05/08/2009	0.012	$8.50*10^{-5}$	$1.37*10^{-10}$	0.086	999.531	1.500
02/09/2009	0.011	$1.88*10^{-5}$	$8.21*10^{-5}$	0.095	999.233	1.974
07/10/2009	0.008	$1.51*10^{-5}$	$4.47*10^{-5}$	0.034	991.117	2.917
04/11/2009	0.011	$1.03*10^{-5}$	$7.82*10^{-5}$	0.089	998.468	3.343
02/12/2009	0.016	$7.70*10^{-8}$	$2.26*10^{-12}$	0.132	1008.593	3.351

Table 2: Estimated parameters for CHJ model in sample period

Date	$\widetilde{\gamma}$	α_0	α_1	β_1	$RMSE_{in}$
07/01/2009	33.988	$4.00*10^{-14}$	$3.09*10^{-4}$	$4.80*10^{-4}$	5.609
04/02/2009	33.914	$6.93*10^{-5}$	$2.20*10^{-4}$	0.002	1.257
04/03/2009	33.927	$1.27*10^{-11}$	$2.48*10^{-4}$	0.002	2.140
01/04/2009	33.945	$2.34*10^{-14}$	$1.89*10^{-4}$	0.001	3.509
06/05/2009	33.857	$2.75*10^{-4}$	$1.80*10^{-9}$	0.001	5.509
03/06/2009	34.742	$2.34*10^{-14}$	$1.36*10^{-4}$	0.002	5.243
01/07/2009	33.915	$2.34*10^{-14}$	$1.15*10^{-4}$	0.002	3.205
05/08/2009	33.988	$1.50*10^{-5}$	$1.01*10^{-4}$	0.002	1.518
02/09/2009	33.915	$4.58*10^{-5}$	$1.68*10^{-4}$	0.002	2.820
07/10/2009	33.914	$1.63*10^{-9}$	$8.15*10^{-5}$	0.001	3.212
04/11/2009	33.913	$1.39*10^{-10}$	$1.04*10^{-4}$	0.002	3.843
02/12/2009	33.900	$2.34*10^{-5}$	$6.31*10^{-5}$	0.002	3.318

Table 3: Estimated parameters for HN model in sample period

Date	$\widetilde{\sigma}$	α_0	α_1	β_1	$RMSE_{in}$
07/01/2009	0.247	0.008	0.049	0.111	6.713
04/02/2009	0.116	0.018	0.107	0.322	1.734
04/03/2009	0.195	0.006	0.150	0.322	2.447
01/04/2009	0.171	0.004	0.347	0.304	3.467
06/05/2009	0.008	1.088	0.458	0.274	5.394
03/06/2009	0.014	$1.46*10^{-4}$	0.831	0.169	3.929
01/07/2009	0.010	0.016	0.742	0.252	2.744
05/08/2009	0.021	0.169	0.219	0.201	1.660
02/09/2009	0.143	0.009	$5.51*10^{-5}$	0.214	2.966
07/10/2009	0.138	0.005	0.002	0.162	2.799
04/11/2009	0.010	0.010	0.841	0.142	3.289
02/12/2009	0.005	0.141	0.822	0.150	3.221

Table 4: Estimated parameters for DVG model in sample period

Date	DBG	CHJ	HN	DVG
07/01/2009	2.295	2.769	5.609	6.713
04/02/2009	1.676	0.914	1.257	1.734
04/03/2009	1.249	1.575	2.140	2.447
01/04/2009	1.692	2.602	3.509	3.467
06/05/2009	4.146	5.711	5.509	5.394
03/06/2009	3.613	5.126	5.243	3.929
01/07/2009	2.167	2.492	3.205	2.744
05/08/2009	1.804	1.500	1.518	1.660
02/09/2009	2.247	1.974	2.820	2.966
07/10/2009	2.570	2.917	3.212	2.799
04/11/2009	2.935	3.343	3.843	3.289
02/12/2009	3.352	3.351	3.318	3.221
Tot. $RMSE_{in}$	2.690	3.197	3.654	3.522

Table 5: Comparison of the in sample RMSE in the different models

Period	From	To	DBG	CHJ	HN	DVG
I	14/01/2009	28/01/2009	9.739	9.794	11.971	11.414
II	11/02/2009	25/02/2009	5.269	5.012	5.345	5.561
III	11/03/2009	25/03/2009	6.202	6.965	6.934	6.190
IV	08/04/2009	29/04/2009	4.559	3.992	4.483	4.261
V	13/05/2009	27/05/2009	7.918	8.284	8.310	7.992
VI	10/06/2009	24/06/2009	4.066	4.783	5.117	6.726
VII	08/07/2009	29/07/2009	3.220	3.026	3.509	3.870
VIII	12/08/2009	26/08/2009	4.192	3.504	3.513	3.832
IX	09/09/2009	30/09/2009	5.884	7.182	7.913	6.992
X	14/10/2009	28/10/2009	3.009	3.859	4.654	4.802
XI	11/11/2009	25/11/2009	2.680	3.098	3.382	4.368
XII	09/12/2009	23/12/2009	2.597	2.293	2.711	3.059
Tot. $RMSE$			4.917	5.257	5.803	5.313

Table 6: Comparison of the out of sample RMSE in the different models

	DBG			CHJ		
Moneyness	[0.9, 0.95)	[0.95, 1.05]	(1.05, 1.1]	[0.9, 0.95)	[0.95, 1.05]	(1.05, 1.1]
$RMSE$	5.57	3.67	5.74	5.75	4.49	5.73
Time To Mat.	[10, 30]	[31, 60]	[61, 100]	[10, 30]	[31, 60]	[61, 100]
$RMSE$	3.65	3.68	6.18	3.45	4.25	6.58

Table 7: Comparison of the out of sample RMSE of the DBG and CHJ models for different moneyness and maturities

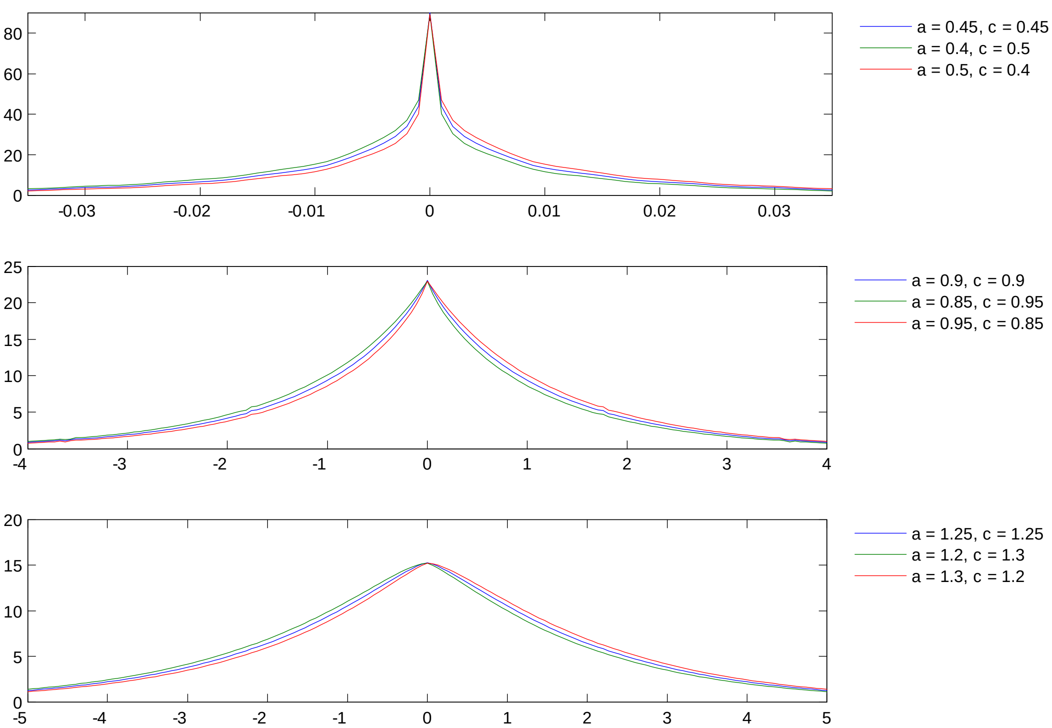

Figure 1: Bilateral Gamma densities for $b = d = 0.025$, varying a and c

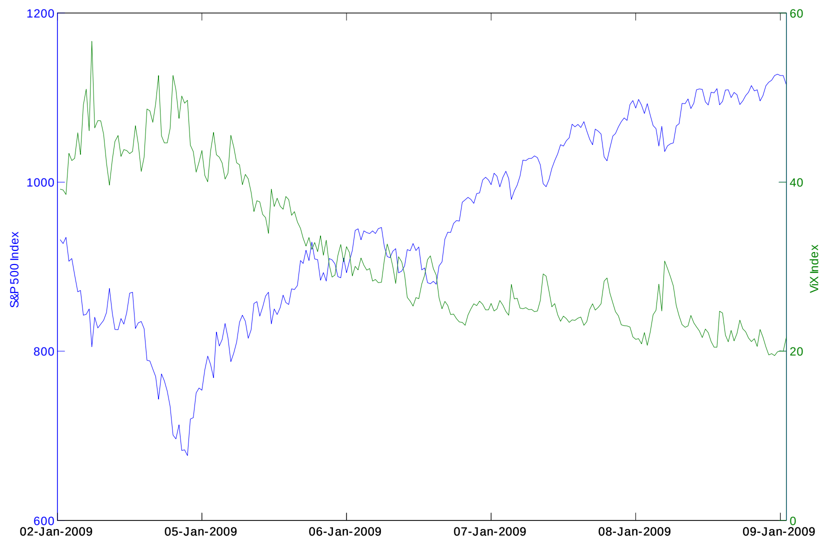

Figure 2: SPX Index and Vix Index

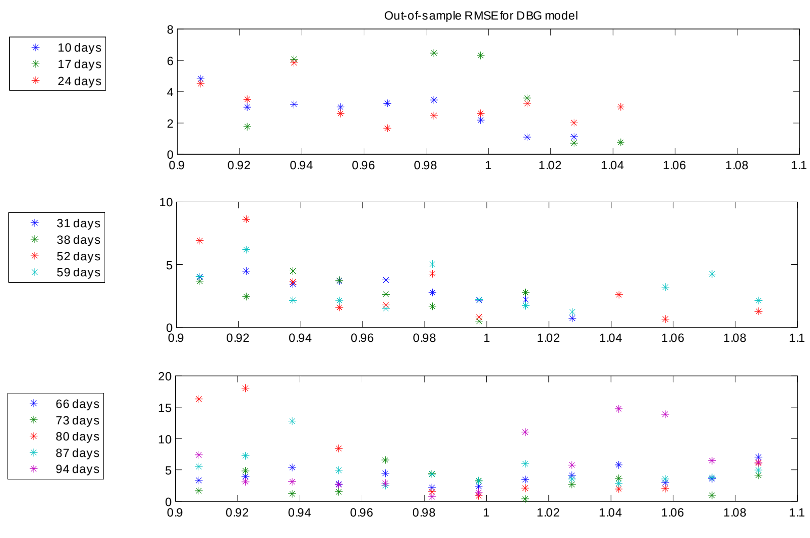

Figure 3: Out of sample $RMSE$ of the DBG model for different maturities and moneyness

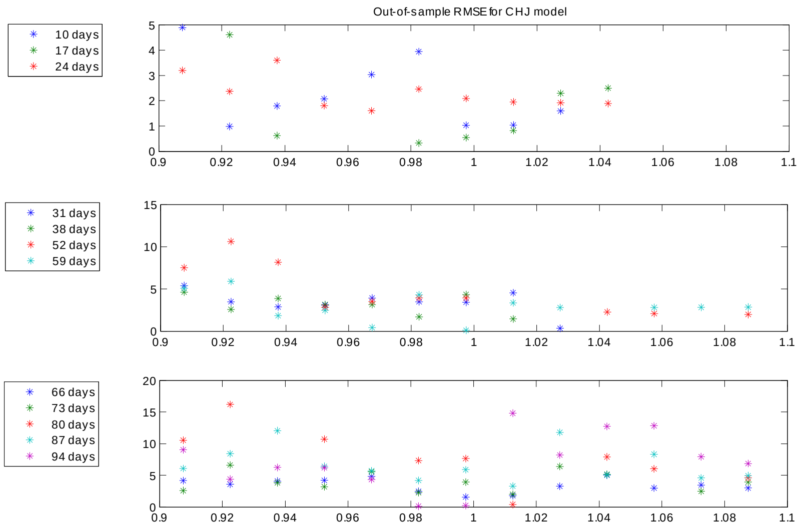

Figure 4: Out of sample $RMSE$ of the CHJ model for different maturities and moneyness

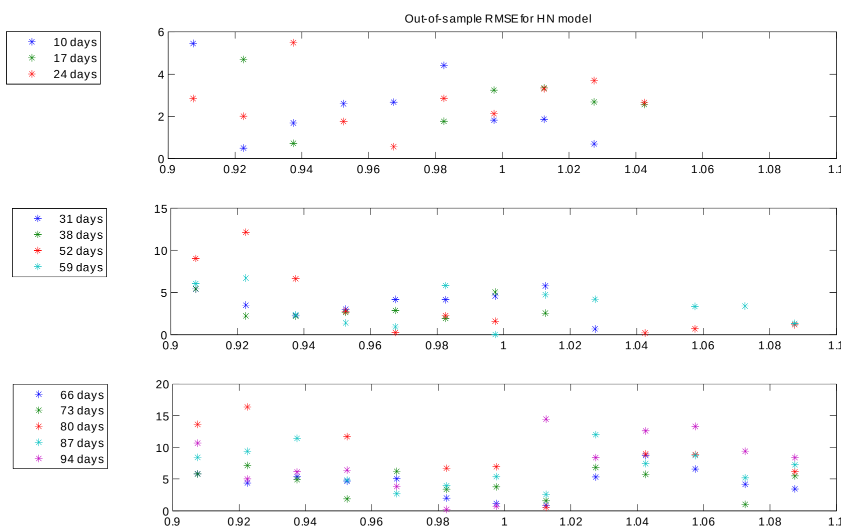

Figure 5: Out of sample $RMSE$ of the HN model for different maturities and moneyness

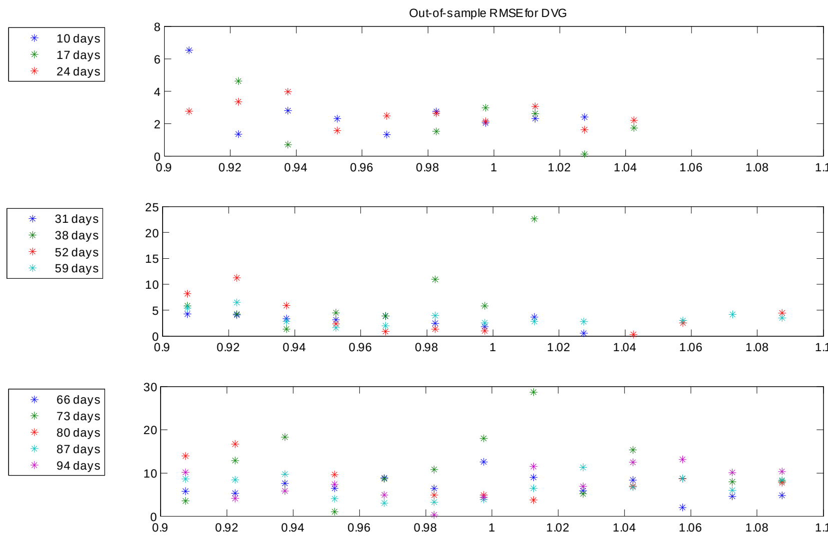

Figure 6: Out of sample $RMSE$ of the DVG model for different maturities and moneyness

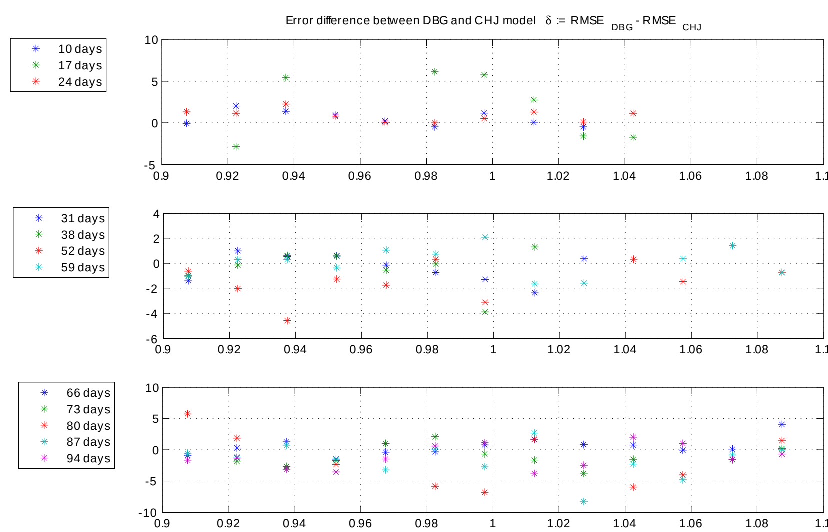

Figure 7: Difference of out of sample $RMSE$ between DBG and CHJ models.