TESTING FOR PREFERENCE CHANGE IN MARKETING OR OPINION RESEARCH

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SUMMARY

Let us suppose we have two important products or parties or opinions indicated respectively by A and B and a pool of small choices indicated by C and we are interested in understanding if the percentage of preferences for A and the percentage of preferences for B are unchanged, increased or decreased after some event (f.i. advertising). If the sample size is big, we propose to use a two stage hypotheses test proposed by Duncan [Miller, 1981] and improved by Pollastri [2008]. The test considered is based on the exact distribution of the absolute maximum [Zenga, 1979] and on the exact distribution of the absolute minimum [Pollastri-Tornaghi, 2004] of the components of a Bivariate Correlated Normal. Tables of the critical values are reported. The test proposed allows to accept one of the nine hypotheses about the invariance or increasing or decreasing of the percentage of A combined with the three movements of the percentage of B.

Keywords: *trinomial distribution, Bivariate Correlated Normal, two stage hypotheses test, absolute maximum and minimum.*

1. INTRODUCTION

One might be interested in comparing the parameters p_1, p_2, p_3 of a trinomial model in two different time periods or in two different situations. Also, it might be interesting to know how a proportion in the population has changed over time, that is, whether that proportion has increased, or whether it has decreased.

For instance, we need to check if an advertising campaign has changed the preferences for the two leading products. Another example may be constituted from the changes in preferences for the two main parties one year after the elections.

To this aim, several confidence regions concerning multinomial proportions have already been proposed by Quesenberry and Hurst [1964], Goodman [1965], Sison and Glaz [1995], Thompson [1995] and many other scholars. Generally this methods are based on confidence ellipsoid. There is a clear connection between such methods and the problem of testing hypothesis about the trinomial probabilities. However, by following one of the above-mentioned procedures, it is generally not possible to distinguish between the alternative hypotheses as we propose in the present paper.

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In this paper we address the problem of adapting the two-stage hypotheses test proposed by Duncan [Miller, 1981] and improved by Pollastri [2008] in order to test nine hypotheses describing all the feasible changes in the trinomial parameters. In Section 2, we provide a brief overview of the procedure proposed by Duncan. Section 3 summarizes the improvements recently proposed by Pollastri, while Section 4 deals with the application of the procedure to the trinomial case. A practical example of the applications of the test is provided in Section 5. Finally, we draw our conclusions in Section 6.

2. OVERVIEW OF DUNCAN'S PROCEDURE

As reported by Miller [1981], Duncan's procedure is a two stage procedure, based on Bonferroni's inequality, that can be used to compare the means of a variable in two different situations. The procedure tests the hypotheses that the means of a *Bivariate Correlated Normal* (B.C.N.) random variable are both equal to fixed values against all the possible alternatives.

Suppose a simple random sample of size n is drawn from a B.C.N. random variable $\mathbf{Y} = (Y_1, Y_2)$ such that

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} ; \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right] .$$

Further, suppose that the variances σ_1^2 and σ_2^2 are known or that the sample is large enough for their estimates to be used instead of the real values. If one assumes that the means of Y_1 and Y_2 are $\gamma_1, \gamma_2 \in R$, then the null hypotheses to be tested are given by

$$H_0 \colon (\mu_1 = \gamma_1) \cap (\mu_2 = \gamma_2)$$

and the alternative hypotheses are:

1. $(\mu_1 = \gamma_1) \cap (\mu_2 > \gamma_2)$;
2. $(\mu_1 = \gamma_1) \cap (\mu_2 < \gamma_2)$;
3. $(\mu_1 > \gamma_1) \cap (\mu_2 > \gamma_2)$;
4. $(\mu_1 > \gamma_1) \cap (\mu_2 < \gamma_2)$;
5. $(\mu_1 > \gamma_1) \cap (\mu_2 = \gamma_2)$;
6. $(\mu_1 < \gamma_1) \cap (\mu_2 = \gamma_2)$;
7. $(\mu_1 < \gamma_1) \cap (\mu_2 > \gamma_2)$;
8. $(\mu_1 < \gamma_1) \cap (\mu_2 < \gamma_2)$.

Let $\overline{y}_1$ and $\overline{y}_2$ denote the estimates of the means obtained from a simple random sample in which we observe (y_{1i}, y_{2i}), $i = 1, \ldots, n$. Then, under the null hypotheses, the statistics

$$X_1 = \frac{\overline{Y}_1 - \gamma_1}{\sigma_1/\sqrt{n}} \quad \text{and} \quad X_2 = \frac{\overline{Y}_2 - \gamma_2}{\sigma_2/\sqrt{n}} \,,$$

have the *Standardized Bivariate Correlated Normal* (S.B.C.N) distribution with correlation coefficient ρ, that is,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right] .$$

Duncan, in order to test the hypotheses $\mu_1 = \mu_2 = 0$ against the alternatives from A. to H. where $\gamma_1 = \gamma_2 = 0$, proposed the following two stage procedure.

Stage 1. The critical value c' is computed assuming that $|X_1|$ and $|X_2|$ are independent random variables. Hence, c' is such that

$$\begin{aligned} & P\left[\max\{\,|X_1|, |X_2|\,\} \le c'\right] \\ = & P\left[|X_1| \le c'\right] \cdot P\left[|X_2| \le c'\right] \\ = & 1 - \alpha \,. \end{aligned}$$

For a given level of significance α, one must compare the value of the statistic $\max\{\,|X_1|, |X_2|\,\}$ with the critical value c'.

If $\max\{\,|X_1|, |X_2|\,\} \le c'$, H_0 is accepted and the procedure stops.

If $\max\{\,|X_1|, |X_2|\,\} = |X_{(i)}| > c'$ ($i = 1$ or 2), one concludes that $\mu_{(i)} > 0$ or $\mu_{(i)} < 0$ respectively if $X_{(i)} > 0$ or $X_{(i)} < 0$ and proceeds to stage 2.

Stage 2. The statistic $\min\{\,|X_1|, |X_2|\,\} = |X_{(j)}|$ ($j = 1$ or 2, $j \ne i$) must now be compared with the $(1-\alpha/2) \times 100$-th percentile of the standard normal distribution, denoted by $z_{1-\alpha/2}$.

If $|X_{(j)}| < z_{1-\alpha/2}$, then one concludes that $\mu_{(j)} = 0$.

If $|X_{(j)}| > z_{1-\alpha/2}$, one concludes that $\mu_{(j)} > 0$ or $\mu_{(j)} < 0$ respectively if $X_{(j)} > 0$ or $X_{(j)} < 0$.

4. RECENT IMPROVEMENTS

Recently, Pollastri [2008] proposed an improvement of Duncan's procedure based on the exact distribution of the absolute maximum (studied by Zenga, 1979), and of the absolute minimum (studied by Pollastri and Tornaghi, 2004) of the components of the S.B.C.N. random variable. The new stepwise test allows to accept the null hypotheses with a fixed probability error while Duncan's procedure is conservative. Moreover, the procedure proposed by Pollastri is more powerful than Duncan's one.

In the present paper we adapt the above procedure to the observed frequencies of a trinomial distribution in order to test the null hypothesis of invariance of the parameters against the nine hypotheses describing the changes in the preferences.

The density function of $T = \max\{\,|X_1|, |X_2|\,\}$ is a mixture of two Arctangent density functions with parameters

$$a_1 = \sqrt{\frac{1+\rho}{1-\rho}} \quad \text{and} \quad a_2 = \sqrt{\frac{1-\rho}{1+\rho}}\,,$$

and with proportions

$$\pi_1 = \frac{2}{\pi}\arctan(a_1) \quad \text{and} \quad \pi_2 = \frac{2}{\pi}\arctan(a_2)\,.$$

Hence, the density function of the variable T is:

$$f_T(t) = g(t;a_1)\,\frac{\arctan(a_1)}{\pi/2} + g(t;a_2)\,\frac{\arctan(a_2)}{\pi/2}\,, \tag{1}$$

where $g(t;a_i)$ is the Arctangent density function with parameter a_i, that is:

$$g(x;a_i) = \begin{cases} \dfrac{e^{-\frac{1}{2}x^2}}{\arctan(a_i)} \displaystyle\int_0^{a_i x} e^{-\frac{1}{2}y^2}\,dy & \text{for } x \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

The density function of $V = \min\{\,|X_1|, |X_2|\,\}$ is given by

$$f_V(x) = 2\,(2\phi(x)) - f_T(x) \qquad \text{for } x \geq 0\,, \tag{3}$$

which is a linear combination of the density function of a *Folded Standard Normal* random variable and of the density function of the random variable T.

These results, as well as the whole procedure proposed by Pollastri [2008], are useful also when dealing with the multinomial distribution. Indeed, there is a parallel between the trinomial distribution and a S.B.C.N distribution.
Let (X_1, X_2, X_3) have the multinomial distribution with parameters n and (p_1, p_2, p_3). Each random variable X_i has the binomial distribution with parameters n and p_i. Therefore, $E[X_i] = np_i$ and $Var(X_i) = np_i(1-p_i)$. It follows that, by standardizing X_i, one obtains the random variable

$$Z_i = \frac{X_i - np_i}{\sqrt{np_i(1-p_i)}}\,. \tag{4}$$

Consider now the random variable $\mathbf{Z} = (Z_1, Z_2)$. It can be demonstrated that, for large values of n, $\mathbf{Z}$ has the Standardized Bivariate Correlated Normal distribution. In particular,

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]\,,$$

where

$$\rho = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}\,.$$

Thus, the d.f. of the random variables $T = \max\{\,|Z_1|, |Z_2|\,\}$ and $V = \min\{\,|Z_1|, |Z_2|\,\}$ are the ones already described in Equation (1) and in Equation (3) respectively. The two stage test procedure proposed by Pollastri can be applied to the trinomial case as described in the following section.

4. APPLICATION TO THE TRINOMIAL CASE

Let $h(\alpha, |\rho|)$ be the $(1-\alpha)\times 100$-th percentile of the random variable T, that is, $P\{T < h(\alpha, |\rho|)\} = 1-\alpha$. The values of $h(\alpha, |\rho|)$ for some levels of α and $|\rho|$ were determined by Pollastri [2008] and are given in the appendix (table 2).

Let $k(\alpha, |\rho|)$ be the $(1-\alpha)\times 100$-th percentile of the random variable V, that is, $P\{V < k(\alpha, |\rho|)\} = 1-\alpha$. The values of $k(\alpha, |\rho|)$ for some fixed α and $|\rho|$ were also determined by Pollastri and are reported in the appendix (table 3).

The improved Duncan's procedure proposed by Pollastri may be used to test the null hypotheses

$$H_0\colon (p_1 = p_1^*) \cap (p_2 = p_2^*)$$

against the alternatives:

1. $(p_1 = p_1^*) \cap (p_2 > p_2^*)$;
2. $(p_1 = p_1^*) \cap (p_2 < p_2^*)$;
3. $(p_1 > p_1^*) \cap (p_2 > p_2^*)$;
4. $(p_1 > p_1^*) \cap (p_2 < p_2^*)$;
5. $(p_1 > p_1^*) \cap (p_2 = p_2^*)$;
6. $(p_1 < p_1^*) \cap (p_2 = p_2^*)$;
7. $(p_1 < p_1^*) \cap (p_2 > p_2^*)$;
8. $(p_1 < p_1^*) \cap (p_2 < p_2^*)$.

Note that, as in the case of the S.C.B.N. random variable, the value of Z_1 and Z_2 must be computed under the null hypotheses, so that

$$Z_i = \frac{X_i - np_i^*}{\sqrt{np_i^*(1-p_i^*)}} \qquad \text{for } i = 1, 2\,.$$

Further, given that the real values of the multinomial parameters p_i are unknown, one needs to estimate the correlation coefficient between p_1 and p_2 using the point estimates $\hat{p}_i$, that is,

$$\hat{\rho} = -\sqrt{\frac{\hat{p}_1\hat{p}_2}{(1-\hat{p}_1)(1-\hat{p}_2)}}\,.$$

The two stage test is performed as follows.

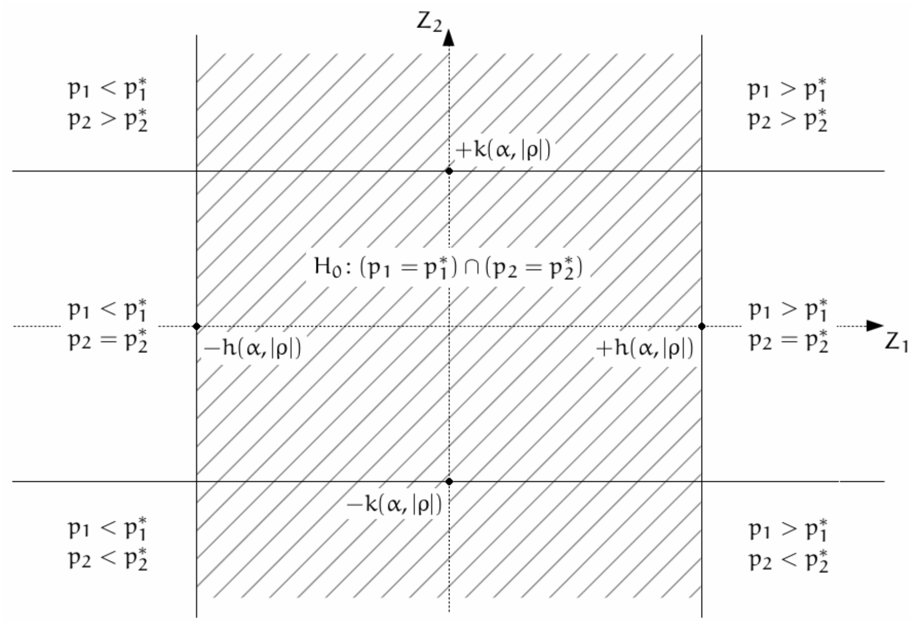

FIGURE 1: Acceptance regions for the modified Duncan's test.

Stage 1. For a given level of significance α, the value of the random variable $T = \max\{\,|Z_1|, |Z_2|\,\}$ must be compared with the critical value $h(\alpha, |\rho|)$.

If $\max\{\,|Z_1|, |Z_2|\,\} \leq h(\alpha, |\rho|)$, H_0 is accepted and the procedure stops.

If $\max\{\,|Z_1|, |Z_2|\,\} = |Z_{(i)}| > h(\alpha, |\rho|)$ ($i = 1$ or 2), one concludes that $p_{(i)} > p^*_{(i)}$ or $p_{(i)} < p^*_{(i)}$ respectively if $Z_{(i)} > 0$ or $Z_{(i)} < 0$ and proceeds to stage 2.

Stage 2. The statistic $\min\{\,|Z_1|, |Z_2|\,\} = |Z_{(j)}|$ ($j = 1$ or 2, $j \neq i$) must now be compared with the critical value $k(\alpha, |\rho|)$.

If $|Z_{(j)}| \leq k(\alpha, |\rho|)$, one concludes that $p_{(j)} = p^*_{(j)}$.

If $|Z_{(j)}| > k(\alpha, |\rho|)$, then one concludes that $p_{(j)} > p^*_{(j)}$ or $p_{(j)} < p^*_{(j)}$ respectively if $Z_{(j)} > 0$ or $Z_{(j)} < 0$.

Figure 1 shows the acceptance regions for the modified Duncan's test when it is assumed that $\max\{\,|Z_1|, |Z_2|\,\} = |Z_1|$. In particular, the region of acceptance for H_0 is highlighted, while there are six different rejection regions which correspond to the alternative hypotheses from C. to H. Indeed, the alternative hypotheses A. and B. are not considered if $|Z_1| > |Z_2|$. Conversely, if $|Z_2| > |Z_1|$ the hypotheses E. and F. are not taken into consideration.

Figure 2 provides a flowchart of the modified Duncan's procedure set out above.

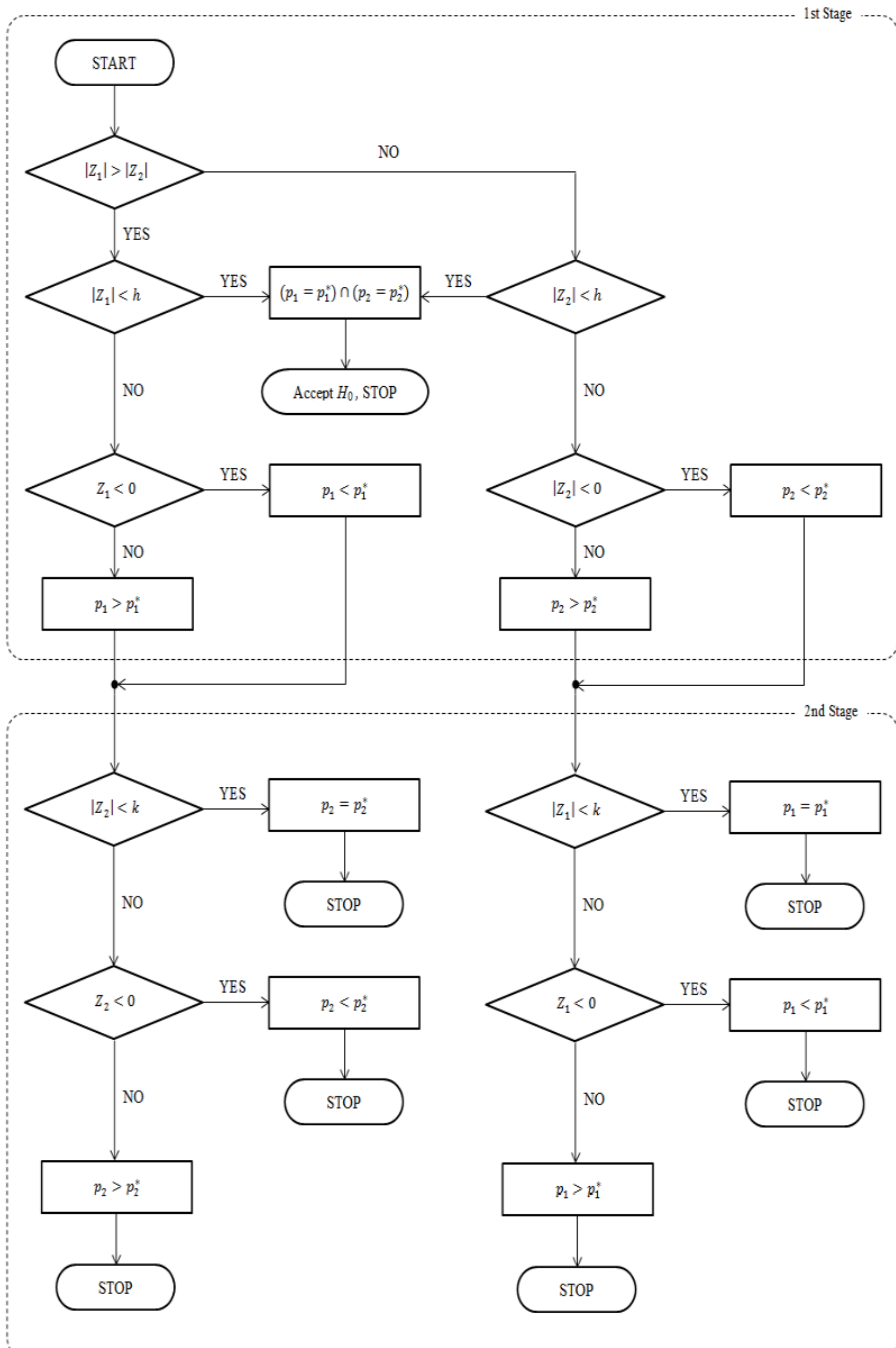

FIGURE 2: Flowchart of the modified Duncan's procedure.

Although the procedure takes into consideration only two parameters, namely p_1 and p_2, it is also possible to gather information about the remaining parameter, p_3. This is due to the fact that $\sum_{i=1}^{3} p_i = 1$.

Therefore, if the null hypotheses

$$H_0\colon (p_1 = p_1^*) \cap (p_2 = p_2^*)$$

are accepted, it is obviously possible to conclude that p_3 is equal to $(1 - p_1^* - p_2^*)$. Similarly, in some cases, information about the third parameter can be gathered even when the null hypotheses are rejected. For example, if the procedure leads to conclude that p_1 and p_2 have both increased over time, the only plausible explanation is that, at the same time, p_3 has decreased.

The procedure can be applied only if the model under investigation is trinomial. However, in some cases, a multinomial model with more than three parameters can be reduced to a trinomial distribution by collapsing the number of categories to three. For instance, as it will be shown in the example below, a multiple choice question with more than three possible answers can be studied through the trinomial distribution. This can be done by taking into consideration only the extreme ends of the scale and by collapsing the remaining choices in a single category, which might be labelled as the "indifferent" or "moderate" category.

4. EXAMPLE

Italy is the only G8 country without its own nuclear power plants, having closed its last reactors in 1990. In February 2008 *Eurobarometer* carried out a study with the objective of measuring the attitudes of European citizens towards nuclear energy. Results from the survey showed that Italians primarily had rather moderate opinions about nuclear energy. In particular, 17 percent of respondents were found to be totally opposed to energy production by nuclear power stations, while 12 percent of them stated to be totally in favour of it. The largest segment of the poll (71 percent) confirmed to have a moderate opinion by declaring to be "fairly opposed" or "fairly in favour" of nuclear energy or, in some cases, by not taking any position.

In March 2008, the Italian government policy towards nuclear underwent a change and a new nuclear build program was planned. It would therefore be interesting to assess whether the Italians changed their minds in the meantime, for at least two reasons. First, from that date the government has kicked off an awareness campaign to nuclear power which has become an ever-present topic in public debates.* Second, it is well known that citizens in countries that have an active nuclear energy program are considerably more likely to support nuclear energy than citizens in other countries.

Assume that a recent survey found the following results: 121 out of $1,036$ respondents claimed to be "totally in favour" of nuclear energy while 147 of them confirmed to be totally opposed to it. Again, the majority of respondents (768) showed to have a moderate opinion.

*See, as an example, the website http://www.forumnucleare.it/.

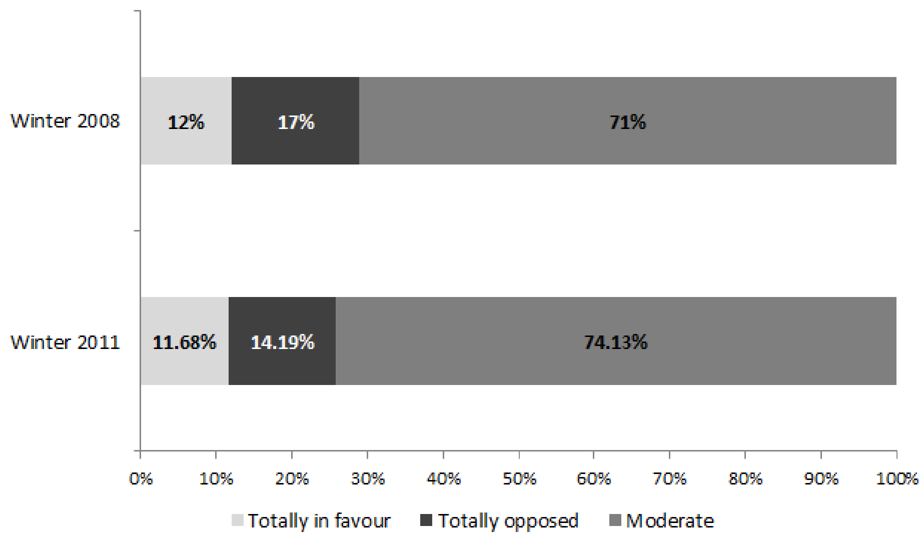

FIGURE 3: Proportions of people in Italy who are totally favourable, totally opposed or have a moderate opinion about nuclear power. A comparison between winter 2008 and winter 2011.

Figure 3 gives an idea of the change in the proportions comparing figures from *Eurobarometer*'s survey in winter 2008, below, with the point estimates of the parameters calculated on data from the present example (winter 2011), above.

In order to assess whether the shares of the population have actually changed over time and the direction of the change, it is possible to set up a modified Duncan's two stage test. This can be done by applying the procedure proposed by Pollastri [2008] to test multinomial parameters.

Let p_1 denote the proportion of Italians that are totally in favour of nuclear energy and let p_2 represent the ratio of individuals who claimed to be totally opposed to it. In such framework, of course, p_3 denotes the proportion of individuals who confirmed to have a moderate opinion. The focus is put on people who positioned themselves on the extreme ends of the scale, that is, the multinomial parameters of interest are p_1 and p_2.

Formally, assuming that the launch of a new nuclear energy program did not affect public opinion, one tests the null hypotheses

$$H_0\colon (p_1 = 0.12) \cap (p_2 = 0.17)$$

against the alternatives:

1. $(p_1 = 0.12) \cap (p_2 > 0.17)$;
2. $(p_1 = 0.12) \cap (p_2 < 0.17)$;
3. $(p_1 > 0.12) \cap (p_2 > 0.17)$;

TABLE 1: Computation of standardized frequencies Z_i for $i = 1, 2$.

Category	p_i	x_i	np_i	Z_i
Totally in favour	0.12	121	124.32	−0.3174
Totally opposed	0.17	147	176.12	−2.4085
Moderate opinion	0.71	768	735.56	−
Total	1	1036	1036	

4. $(p_1 > 0.12) \cap (p_2 < 0.17)$;
5. $(p_1 > 0.12) \cap (p_2 = 0.17)$;
6. $(p_1 < 0.12) \cap (p_2 = 0.17)$;
7. $(p_1 < 0.12) \cap (p_2 > 0.17)$;
8. $(p_1 < 0.12) \cap (p_2 < 0.17)$.

First, it is necessary to compute the standardized frequencies

$$Z_i = \frac{X_i - np_i^*}{\sqrt{np_i^*(1-p_i^*)}} \qquad \text{for } i = 1, 2\,,$$

where $p_1^* = 0.12$ and $p_2^* = 0.17$. Computations are summarized in Table 1. Also, one needs to estimate the correlation coefficient, ρ, between p_1 and p_2 using the point estimates

$$\hat{p}_1 = \frac{121}{1036} = 0.1168 \quad \text{and} \quad \hat{p}_2 = \frac{147}{1036} = 0.1419\,.$$

In this case,

$$\hat{\rho} = -\sqrt{\frac{\hat{p}_1\hat{p}_2}{(1-\hat{p}_1)(1-\hat{p}_2)}} = -0.148\,.$$

The two stage test procedure is carried out as follows.

Stage 1. The statistic to be used in this stage is $T = \max\{\,|Z_1|, |Z_2|\,\}$. From Table 1 it is possible to observe that, for this example, the value of T is given by $|Z_2| = 2.4085$.

The critical value approach is used to draw conclusions about the null hypotheses. With $\alpha = .05$ and $|\rho| \simeq 0.15$, the critical value $h(\alpha, |\rho|)$ for the test statistic is equal to 2.2335.

Since $\max\{\,|Z_1|, |Z_2|\,\} = |Z_2| > h(\alpha, |\rho|)$, H_0 is rejected and one concludes that $p_2 \neq 0.17$. In particular, since Z_2 is negative, one concludes that $p_2 < 0.17$. This means that the proportion of individuals who claimed to be totally opposed to nuclear energy has decreased over time. The launch of a new nuclear energy program in 2008 has generated consequences which affected public opinion by reducing the disagreement with this form of energy production.

Stage 2. The test statistic $\min\{\,|Z_1|,|Z_2|\,\} = |Z_1| = 0.3174$ must now be compared with the critical value $k(\alpha,|\rho|)$. At the usual probability level where $\alpha = .05$ and with $|\rho| \simeq 0.15$, the critical value for the present example is equal to 1.2314.

Since $|Z_1| < k(\alpha,|\rho|)$, one concludes that $p_1 = 0.12$. In other words, the change in the government policy towards nuclear did not affect the proportion of individuals who claimed to be totally in favour of nuclear energy.

The overall conclusion that can be drawn is that, since winter 2008, support for nuclear energy in Italian public opinion has not increased. People did not become more inclined to be totally favourable to energy production by nuclear power stations. However, disagreement has fallen sharply. The proportion of people totally opposed to nuclear energy is decreased from 17 percent in 2008 to about 14 percent in 2011.

Although the proportion of the moderates in the population was not taken into consideration by the procedure, it is possible to draw some conclusions about it thanks to the fact that $p_1 + p_2 + p_3 = 1$. In particular, given that p_1 remained constant while p_2 has decreased, it can be concluded that there has been an increase in p_3. This means that, since winter 2008, Italians became slightly more likely to have no opinion about nuclear energy. Since there has been a 3 percentage points decline in the proportion of people totally opposed to nuclear power, which left p_1 unchanged, there must have been an increase in p_3 of the same magnitude. Some people, who previously were extremely opposed to nuclear power, now have become more moderate.

This can be confirmed by performing the procedure again in order to test, for example, the null hypotheses

$$H_0\colon (p_1 = 0.12) \cap (p_3 = 0.71)\,.$$

One can easily find that, in this case, $\rho = -0.615$ and $Z_3 = 2.2211$, which also corresponds to the critical value of $T = \max\{\,|Z_1|,|Z_3|\,\}$. With $\alpha = .05$ and $|\rho| \simeq 0.6$, the critical value $h(\alpha,|\rho|)$ for such statistic is equal to 2.1977. Since the value of T is greater than the critical value, H_0 is rejected, and since $Z_3 > 0$, one concludes that $p_3 > 0.71$. Then, the critical value $k(0.05, 0.6) = 1.4647$ must be compared with the value of the statistic $V = \min\{\,|Z_1|,|Z_3|\,\} = |Z_1| = 0.3174$. Since the value of V is lower than the critical value, one accepts the hypothesis that $p_1 = 0.12$, thus reaching the same conclusion as before.

6. CONCLUSIONS

In this article we proposed to employ a two stage test of hypotheses, based on the proposal of Duncan and improved by Pollastri, in order to test hypotheses about the trinomial probabilities. The main advantage of such procedure is that it tests the null hypothesis against a series of mutually exclusive and exhaustive alternatives, thus allowing to assess not only whether a parameter has changed over time, but also the direction of the change. Therefore, the test is particularly suitable for those situations in which the researcher must decide if the preferences for different products or political parties remained unchanged or have increased or decreased after some events (e.g. advertising, election campaign, etc). Given two main products or parties, A and B, and a pool of small alternative, C, the test proposed allows to accept one out of nine hypotheses about the invariance or increasing or decreasing of the percentage of A combined with the three movements of the percentage of B. Moreover, compared to Duncan's test, the procedure here proposed is more powerful and it allows to accept the null hypotheses with a fixed probability error while the original Duncan's procedure is conservative.

APPENDIX

In this appendix we provide two tables with the critical values of $h(\alpha, |\rho|)$ and $k(\alpha, |\rho|)$ for some levels of α and $|\rho|$. Also, we report a table with the values of ρ for several different combinations of p_i and p_j.

TABLE 2: Values of the $(1-\alpha)\times 100$-th percentile $h(\alpha, |\rho|)$ of the distribution of $\max\{\,|Z_1|, |Z_2|\,\}$.

	α				
$\|\rho\|$	0.01	0.02	0.05	0.10	0.15
0.000	2.8018	2.5717	2.2354	1.9483	1.7618
0.025	2.8018	2.5717	2.2354	1.9482	1.7617
0.050	2.8017	2.5716	2.2352	1.9480	1.7614
0.075	2.8016	2.5714	2.2350	1.9476	1.7610
0.100	2.8015	2.5712	2.2346	1.9471	1.7603
0.125	2.8013	2.5709	2.2341	1.9464	1.7595
0.150	2.8011	2.5706	2.2335	1.9456	1.7585
0.175	2.8008	2.5702	2.2328	1.9446	1.7574
0.200	2.8005	2.5697	2.2320	1.9434	1.7560
0.225	2.8001	2.5691	2.2311	1.9421	1.7545
0.250	2.7996	2.5684	2.2300	1.9407	1.7527
0.275	2.7991	2.5677	2.2288	1.9390	1.7508
0.300	2.7985	2.5668	2.2275	1.9372	1.7487
0.325	2.7978	2.5659	2.2260	1.9352	1.7463
0.350	2.7970	2.5648	2.2244	1.9330	1.7438
0.375	2.7961	2.5636	2.2226	1.9307	1.7410
0.400	2.7952	2.5623	2.2207	1.9281	1.7381
0.425	2.7940	2.5608	2.2186	1.9254	1.7349

TABLE 2: continued on next page

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	α				
$\|\rho\|$	0.01	0.02	0.05	0.10	0.15
0.450	2.7928	2.5592	2.2163	1.9224	1.7314
0.475	2.7914	2.5574	2.2138	1.9192	1.7277
0.500	2.7898	2.5554	2.2111	1.9157	1.7237
0.525	2.7881	2.5532	2.2081	1.9120	1.7195
0.550	2.7861	2.5508	2.2049	1.9080	1.7149
0.575	2.7839	2.5482	2.2014	1.9037	1.7101
0.600	2.7815	2.5452	2.1977	1.8991	1.7049
0.625	2.7788	2.5420	2.1936	1.8941	1.6993
0.650	2.7758	2.5384	2.1891	1.8888	1.6934
0.675	2.7724	2.5345	2.1842	1.8830	1.6870
0.700	2.7686	2.5301	2.1788	1.8767	1.6801
0.725	2.7643	2.5252	2.1730	1.8699	1.6726
0.750	2.7595	2.5198	2.1665	1.8625	1.6645
0.775	2.7541	2.5137	2.1594	1.8544	1.6558
0.800	2.7479	2.5068	2.1514	1.8455	1.6461
0.825	2.7409	2.4990	2.1425	1.8355	1.6355
0.850	2.7327	2.4900	2.1324	1.8244	1.6237
0.875	2.7230	2.4796	2.1208	1.8118	1.6103
0.900	2.7115	2.4673	2.1072	1.7971	1.5949
0.925	2.6973	2.4521	2.0908	1.7796	1.5767
0.950	2.6787	2.4327	2.0700	1.7577	1.5540
0.975	2.6518	2.4048	2.0407	1.7272	1.5228
0.990	2.6253	2.3775	2.0126	1.6983	1.4934
1.000	2.5790	2.3306	1.9650	1.6502	1.4449

TABLE 2: concluded from previous page

TABLE 3: Values of the $(1-\alpha)\times 100$-th percentile $k(\alpha, |\rho|)$ of the distribution of $\min\{\,|Z_1|, |Z_2|\,\}$.

	α				
$\|\rho\|$	0.01	0.02	0.05	0.10	0.15
0.000	1.6449	1.4705	1.2170	1.0022	0.8645
0.025	1.6457	1.4712	1.2174	1.0025	0.8647
0.050	1.6483	1.4731	1.2186	1.0032	0.8652
0.075	1.6527	1.4763	1.2205	1.0044	0.8660
0.100	1.6588	1.4809	1.2233	1.0061	0.8672
0.125	1.6665	1.4867	1.2269	1.0083	0.8687
0.150	1.6760	1.4939	1.2314	1.0110	0.8706

TABLE 3: continued on next page

TABLE 3: continued from previous page

	α				
$\lvert\rho\rvert$	0.01	0.02	0.05	0.10	0.15
0.175	1.6871	1.5023	1.2366	1.0143	0.8729
0.200	1.6997	1.5120	1.2428	1.0181	0.8755
0.225	1.7139	1.5231	1.2498	1.0225	0.8786
0.250	1.7295	1.5353	1.2578	1.0274	0.8820
0.275	1.7464	1.5488	1.2666	1.0330	0.8860
0.300	1.7644	1.5635	1.2765	1.0393	0.8904
0.325	1.7836	1.5793	1.2873	1.0463	0.8953
0.350	1.8036	1.5961	1.2990	1.0540	0.9007
0.375	1.8244	1.6139	1.3117	1.0624	0.9068
0.400	1.8459	1.6325	1.3254	1.0717	0.9135
0.425	1.8679	1.6519	1.3400	1.0818	0.9208
0.450	1.8903	1.6719	1.3556	1.0928	0.9289
0.475	1.9131	1.6924	1.3720	1.1047	0.9377
0.500	1.9361	1.7135	1.3892	1.1175	0.9474
0.525	1.9593	1.7349	1.4071	1.1312	0.9579
0.550	1.9828	1.7567	1.4257	1.1458	0.9694
0.575	2.0065	1.7788	1.4449	1.1614	0.9818
0.600	2.0304	1.8012	1.4647	1.1779	0.9953
0.625	2.0545	1.8238	1.4850	1.1952	1.0098
0.650	2.0788	1.8468	1.5057	1.2133	1.0253
0.675	2.1035	1.8701	1.5269	1.2322	1.0419
0.700	2.1285	1.8937	1.5486	1.2518	1.0595
0.725	2.1539	1.9178	1.5707	1.2720	1.0780
0.750	2.1798	1.9424	1.5934	1.2930	1.0975
0.775	2.2062	1.9676	1.6167	1.3147	1.1179
0.800	2.2334	1.9934	1.6407	1.3372	1.1393
0.825	2.2613	2.0202	1.6657	1.3606	1.1617
0.850	2.2904	2.0480	1.6918	1.3852	1.1853
0.875	2.3208	2.0773	1.7193	1.4112	1.2104
0.900	2.3531	2.1084	1.7487	1.4392	1.2375
0.925	2.3880	2.1422	1.7809	1.4700	1.2674
0.950	2.4271	2.1803	1.8175	1.5052	1.3016
0.975	2.4746	2.2268	1.8624	1.5488	1.3444
0.990	2.5134	2.2652	1.9000	1.5856	1.3806
1.000	2.5670	2.3194	1.9537	1.6389	1.4337

TABLE 3: concluded from previous page

TABLE 4: Values of ρ for several different combinations of p_i and p_j.

	p_i																	
p_j	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9
0.05	-0.053	-0.076	-0.096	-0.115	-0.132	-0.15	-0.168	-0.187	-0.208	-0.229	-0.254	-0.281	-0.313	-0.35	-0.397	-0.459	-0.546	-0.688
0.1	-0.076	-0.111	-0.14	-0.167	-0.192	-0.218	-0.245	-0.272	-0.302	-0.333	-0.369	-0.408	-0.454	-0.509	-0.577	-0.667	-0.793	
0.15	-0.096	-0.14	-0.176	-0.21	-0.243	-0.275	-0.308	-0.343	-0.38	-0.42	-0.464	-0.514	-0.572	-0.642	-0.728	-0.84		
0.2	-0.115	-0.167	-0.21	-0.25	-0.289	-0.327	-0.367	-0.408	-0.452	-0.5	-0.553	-0.612	-0.681	-0.764	-0.866			
0.25	-0.132	-0.192	-0.243	-0.289	-0.333	-0.378	-0.424	-0.471	-0.522	-0.577	-0.638	-0.707	-0.787	-0.882				
0.3	-0.15	-0.218	-0.275	-0.327	-0.378	-0.429	-0.48	-0.535	-0.592	-0.655	-0.724	-0.802	-0.892					
0.35	-0.168	-0.245	-0.308	-0.367	-0.424	-0.48	-0.538	-0.599	-0.664	-0.734	-0.811	-0.899						
0.4	-0.187	-0.272	-0.343	-0.408	-0.471	-0.535	-0.599	-0.667	-0.739	-0.816	-0.903							
0.45	-0.208	-0.302	-0.38	-0.452	-0.522	-0.592	-0.664	-0.739	-0.818	-0.905								
0.5	-0.229	-0.333	-0.42	-0.5	-0.577	-0.655	-0.734	-0.816	-0.905									
0.55	-0.254	-0.369	-0.464	-0.553	-0.638	-0.724	-0.811	-0.903										
0.6	-0.281	-0.408	-0.514	-0.612	-0.707	-0.802	-0.899											
0.65	-0.313	-0.454	-0.572	-0.681	-0.787	-0.892												
0.7	-0.35	-0.509	-0.642	-0.764	-0.882													
0.75	-0.397	-0.577	-0.728	-0.866														
0.8	-0.459	-0.667	-0.84															
0.85	-0.546	-0.793																
0.9	-0.688																	

RIASSUNTO

Si supponga di essere interessati alle preferenze di una popolazione riguardo due prodotti (o opinioni o partiti), indicati rispettivamente con A e B, e un insieme di piccole scelte inglobate in un pool C. Si voglia verificare se, dopo un evento quale, ad esempio, una campagna promozionale, la proporzione di preferenze per A e per B sono aumentate, diminuite o rimaste inalterate. Al fine di verificare le ipotese di cui sopra, si estragga un campione con riposizione di ampiezza considerevole. Nel presente lavoro si propone di avvalersi di un test proposto inizialmente da Duncan e migliorato da Pollastri nel 2008. La procedura si basa sulla distribuzione del massimo assoluto (Zenga, 1979) e del minimo assoluto (Pollastri - Tornaghi, 2004) delle componenti di una variabile casuale Normale Bivariata a componenti correlate. Si riportano le tavole dei valori critici. Il test proposto permette di accettare una delle nove ipotesi circa linvarianza, lincremento o la diminuzione della proporzione di preferenze per A combinata con i possibili movimenti della percentuale di scelte per B.

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