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Bochner–Riesz means of eigenfunction expansions and local Hardy spaces on manifolds with bounded geometry

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Contents

Introduction

This thesis deals with two quite diverse problems: the first is the pointwise convergence of Bochner-Riesz means of functions with some smoothness and the estimate of the dimension of the divergence set, the second focuses on the development of new function spaces of Hardy type, which are tailored to produce endpoint estimates for interesting operators, mainly related to the Laplace–Beltrami operator on noncompact Riemannian manifolds.

The Bochner-Riesz means are discussed in Chapters 1, 2 and 3, and the Hardy spaces occupy Chapters 4 and 5. Here we briefly describe the main results we have obtained, and illustrate the relationships with related results in the literature.

Bochner–Riesz means

The Fourier transform of an integrable function is not necessarily integrable, hence the inversion formula for this transform may involve integrals which are not absolutely convergent. In order to get around this difficulty, suitable summability methods are introduced. In particular, the Bochner-Riesz means of order β of functions in $\mathbb{R}^d$ are defined by the Fourier integrals

$$S_R^\beta f(x) = \int_{\{|\xi|<R\}} \left(1 - \left|\frac{\xi}{R}\right|^2\right)^\beta \widehat{f}(\xi) \exp(2\pi i \xi x) d\xi.$$

When $\beta = 0$ one obtains the spherical partial sums, which are a natural analogue of the partial sums of one-dimensional Fourier series.

A classical result in this field is the following:

Bochner-Riesz means with index $\beta > (d-1)\,|1/p - 1/2|$ *of functions in* $L^p\left(\mathbb{R}^d\right)$ *converge in norm and almost everywhere.*

The cases $p = 1, 2, +\infty$ are due to Bochner and the general case $1 \le p \le +\infty$ is due to Stein. See [SW, VII.5]. See also [Ca, CRV, Chr2, Le] for better results when $p > 2$, and [T] for the case $p < 2$. Indeed, the results for the norm convergence are different from the ones for pointwise almost everywhere convergence. Anyhow, in

dimension greater than one the precise range of indexes for which convergence holds is still unknown.
Since the harmonic analysis of radial functions reduces to a weighted one-dimensional analysis, the problem of convergence of Fourier expansions becomes simpler and more precise results are known:

Bochner-Riesz means with index $\beta > \max\{d\,|1/p - 1/2| - 1/2, 0\}$ *of radial functions in* $L^p(\mathbb{R}^d)$ *converge in norm and almost everywhere.*

See [P, K, MP] for extensions to symmetric spaces. For endpoint results in Lorentz spaces see also [CP, CCTV, CTV, RS]. Other classical results on convergence of Fourier series of functions with some smoothness are due to Beurling, and Salem and Zygmund:

If $\sum n^\gamma \left(|a(n)|^2 + |b(n)|^2\right) < +\infty$*, then the set of points of divergence of the series* $\sum a(n)\cos(nx) + b(n)\sin(x)$ *has outer* $(1-\gamma)$ *capacity zero if* $0 < \gamma < 1$*, or it has outer logarithmic capacity zero if* $\gamma = 1$*.*

See [Zy, XIII.11]. These results have been extended to multi-dimensional Bochner-Riesz means:

Bochner-Riesz means with index $\beta > \max\{(d-1)/2 - \gamma, 0\}$ *of functions with* $\gamma > 0$ *integrable derivatives in* $L^1(\mathbb{R}^d)$ *converge pointwise, with possible exception of sets of points with Hausdorff dimension at most* $d - \gamma$*. Similarly, Bochner-Riesz means with index* $\beta \geq 0$ *of functions with* γ *square integrable derivatives in* $L^2(\mathbb{R}^d)$ *converge pointwise, with possible exception of sets of points with Hausdorff dimension at most* $d - 2\gamma$*.*

See [Col1]. See also [CS2], and [BBCR] for related results on the convergence to the initial data of solutions to dispersive equations. In the first chapter of our dissertation we consider an analogue of the above results for functions with derivatives in $L^p(\mathbb{R}^d)$ with $1 < p < +\infty$. In particular, we prove the following:

Let $\beta > 0$ *and* $0 < \gamma < d$ *and denote by* $G^\gamma(x)$ *the Bessel kernel defined by the Fourier multiplier* $\widehat{G^\gamma}(\xi) = (1 + |\xi|^2)^{-\gamma/2}$*. If* $f(x) = G^\gamma * F(x)$ *with* $F \in L^p(\mathbb{R}^d)$*,* $1 < p < +\infty$*, then the Bochner-Riesz means* $S_R^\beta f(x)$ *converge pointwise to* $f(x)$ *as* $R \to +\infty$ *with a possible exception of a set of points* x *with Hausdorff dimension at most* $d - \gamma p$ *if one of the following conditions holds:*

(i) $\beta > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$,

or

(ii) $\beta > d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *and* $\beta + \gamma \geq \frac{d-1}{2}$.

These results are not exhaustive and the ranges of indexes are not optimal. Perhaps this reflects the fact that also the problem of convergence of Bochner-Riesz means in L^p classes is still open. However, at least for radial functions, we can prove some definitive results. In the second chapter we first prove an equiconvergence result between Bochner-Riesz means of trigonometric and Bessel expansions. In particular, this implies equiconvergence between Bochner-Riesz means of radial functions in one and several dimensions. When applied to functions in Sobolev classes, the equiconvergence result gives the following:

Assume $\beta \geq 0$, $\gamma \geq 0$, $1 \leq p \leq +\infty$, *and* $2d/\left(d+1+2\beta+2\gamma\right) < p < 2d/\left(d-1-2\beta\right)$. *Then the Bochner-Riesz means with index* β *of radial functions with* γ *derivatives in* $L^p\left(\mathbb{R}^d\right)$ *converge pointwise, with a possible exception of a set of points* Ω *with the following properties:*

(1) if $\gamma p \leq 1$, *then the Hausdorff dimension of* Ω *is at most* $d-\gamma p$*;*

(2) if $1 < \gamma p \leq d$, *then* Ω *either is empty or it reduces to the origin;*

(3) if $\gamma p > d$, *then* Ω *is empty.*

The above ranges of indexes are the best possible. Divergence of Bochner-Riesz means of radial functions occurs in spheres $\{|x| = r\}$, and sets of spheres of dimension $d-\gamma p$ in $\mathbb{R}^d$ correspond to sets of radii of dimension $1-\gamma p$ in $\mathbb{R}_+$. Observe that the above results are natural, since functions with γ derivatives in $L^p\left(\mathbb{R}^d\right)$ can be infinite on sets with dimension $d-\gamma p$, but not on larger sets. Also observe the asymmetry between $p < 2$ and $p > 2$. When $p < 2$, if the smoothness index γ increases, then the critical index $\beta = d\left(1/p-1/2\right)-1/2-\gamma$ for summability decreases, but when $p > 2$ the critical index $\beta = d\left(1/2-1/p\right)-1/2$ for summability is independent of the smoothness γ.

Indeed, we prove more precise results for Lorentz spaces and Fourier-Bessel expansions.

In the third chapter we prove an equiconvergence result between Bochner-Riesz means of Bessel expansions and expansions in eigenfunctions of Sturm-Liouville operators of the form

$$-A(x)^{-1}\frac{d}{dx}\Big(A(x)\frac{d}{dx}\Big).$$

In particular, when $A(x) = x^{d-1}$, this operator is the radial part of the Laplace operator in $\mathbb{R}^d$, and when $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, with suitable α and β, this operator is the radial component of the Laplace-Beltrami operator on noncompact rank one symmetric spaces. Hence, this gives equiconvergence between Bochner-Riesz means of radial functions in Euclidean and non-Euclidean spaces. Finally we apply this equiconvergence result to the study of the convergence of eigenfunction expansions of functions in Sobolev classes.

Hardy type spaces

The Hilbert transform on the real line is the prototype of a class of operators known as *singular integral operators*. These operators arise in quite diverse settings, ranging from classical Fourier Analysis to fluid dynamics. Their study was initiated around 1920, when Kolmogorov [Ko] and M. Riesz [R] proved their seminal results concerning the Hilbert transform, and it is still the source of many interesting problems. The Hilbert transform is bounded on $L^p(\mathbb{R})$ for all $p \in (1, \infty)$, but it is unbounded on $L^1(\mathbb{R})$ and on $L^\infty(\mathbb{R})$. In the aforementioned paper, Kolmogorov proved that the Hilbert transform is of weak type 1, a result which, together with the Marcinkiewicz interpolation theorem and a duality argument, allows us to recover the $L^p(\mathbb{R})$ boundedness of the Hilbert transform for all $p \in (1, \infty)$. A modern approach to the boundedness of the Hilbert transform relies on an estimate for $p = 1$ which involves the Hardy space $H^1(\mathbb{R})$. This approach also produces an endpoint estimate for $p = \infty$, i.e., the $L^\infty(\mathbb{R})$-$BMO(\mathbb{R})$ estimate.

The purpose of this thesis is to develop a theory of appropriate spaces of Hardy type, and of their duals, on quite general settings and obtain endpoint estimates for interesting singular integral operators on Riemannian manifolds.

Singular integral operators have been considered in a variety of contexts, including Lie groups, symmetric spaces and Riemannian manifolds [S, CD, ACDH, Ru, AMR], and, more generally, on measured metric spaces (M, d, ν), where (M, d) is a metric space and ν is a Borel measure on M satisfying some mild conditions. If ν is *doubling*, i.e., there exists a constant D such that

$$\nu(2B) \leq D\,\nu(B) \qquad \text{for every metric ball } B, \tag{0.0.1}$$

the theory of singular integral operators parallels the classical theory in $\mathbb{R}^n$ (see the excellent exposition in [Chr1] along with the classical paper [CW] and the book [St2]). However, if (0.0.1) fails, the classical approach is no longer applicable and new ideas are required.

Motivated by the classical Painlevé problem [Chr1], a line of research was initiated, aiming at the development of a theory of singular integral operators in the Euclidean space $\mathbb{R}^n$, endowed with a *locally nondoubling* measure ν of *polynomial growth*. This theory includes the spectacular contributions [NTV, V, To] (see also the references therein) and it is still in evolution.

In this thesis we mainly focus on the case where ν is locally doubling but globally nondoubling, i.e., for every positive number R there exists a constant D_R such that

$$\nu(2B) \leq D_R\,\nu(B) \qquad \text{for every } B \text{ of radius at most } R, \tag{0.0.2}$$

and $\sup_R D_R = +\infty$. The driving example we have in mind is the hyperbolic plane. If the Riemannian metric is suitably normalised, then the volume of hyperbolic balls of radius r is approximately r^2 when r is small but it is e^r when r is large, so that the hyperbolic measure is locally doubling but globally nondoubling.

The study of singular integral operators in this thesis is motivated by the problem of analysing operators naturally associated to the Laplace–Beltrami operator on noncompact Riemannian manifolds. Let M be a connected noncompact complete Riemannian manifold with Riemannian measure μ, (positive) Laplace–Beltrami operator $\mathcal{L}$ and gradient ∇. The closure of $\mathcal{L}$, initially defined on $C_c^\infty(M)$, is a self adjoint operator, which we still denote by $\mathcal{L}$. In the classical case where $M = \mathbb{R}^n$ the Riesz transform $\nabla(-\Delta)^{-1/2}$ and the purely imaginary powers $(-\Delta)^{iu}$, $u \in \mathbb{R}$, of the Laplacian Δ are prototypes of singular integral operators on $\mathbb{R}^n$. They are known to be bounded on $L^p(\mathbb{R}^n)$ for all p in $(1,\infty)$, and unbounded on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$ [St2]. L. Hörmander [Ho], and C. Fefferman and E.M. Stein [FeS]) proved that singular integral operators satisfying the so-called Hörmander integral condition are of weak type $(1,1)$ and bounded from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. These results apply, in particular, to $\nabla(-\Delta)^{-1/2}$ and $(-\Delta)^{iu}$. Recall that functions of the Laplacian may, at least formally, be reconstructed from $(-\Delta)^{iu}$ via a subordination formula involving the Mellin transform [St1, Co]. This is one of the reasons which make $(-\Delta)^{iu}$ an important example of singular integral operator.

In [S] R.S. Strichartz proposed to investigate the boundedness of the (translated) Riesz transform on Riemannian manifolds, and several multiplier results of Hörmander type for Laplacians on Lie groups [CGHM, HeS, Va] and symmetric spaces [CS, A1, A2, I1, I2, I3, MV] were proved. It is natural to investigate the boundedness of the operators $\nabla\mathcal{L}^{-1/2}$ (and its modified version $\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$ for $a > 0$) and $\mathcal{L}^{iu}$, and of more general functions of $\mathcal{L}$ that will be described below. It turns out that, at least locally, the Schwartz kernels of these operators behave much as the kernels of their Euclidean analogues. Thus, these operators deserve to be termed *singular integral operators on manifolds.*

The multiplier result for generators of semigroups proved in [St1, Co] applies to $\mathcal{L}^{iu}$ and gives the $L^p(M)$ boundedness of these operators for p in $(1,\infty)$. The $L^p(M)$

boundedness of $\nabla\mathcal{L}^{-1/2}$ for $p \neq 2$, and without additional assumptions on M, seems to be a challenging problem, and it is the object of a very active line of research (see, for instance, [CD, ACDH] and the references therein).

As far as endpoint estimates for $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$ are concerned, interesting results for $p = 1$ have been obtained in the case where μ is doubling and M satisfies some extra assumptions, such as appropriate on-diagonal estimate for the heat kernel [CD], or scaled Poincaré inequality [Ru, MRu, AMR]. It is noteworthy that if M is a Riemannian manifold of exponential volume growth, then weak type $(1,1)$ estimates for $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$ are known only when M is a Riemannian symmetric space of the noncompact type [A1, A2, I2, I3, MV]. Weak type $(1,1)$ estimates for a reasonably wide class of Riemannian manifolds with exponential volume growth seems to be out of reach. Thus, an intense research activity has recently been devoted to find analogues of the Hardy space $H^1(\mathbb{R}^n)$ in various settings, including Riemannian manifolds.

In this thesis we introduce and study certain spaces of Hardy type on Riemannian manifolds M with Ricci curvature bounded from below and positive injectivity radius, and we obtain endpoint estimates for Riesz transforms and classes of spectral multipliers of $\mathcal{L}$. Here are the details.

An improvement of a result of M. Taylor

In Chapter 4 we define a *local Hardy space* $\mathfrak{h}^1(M)$ of Goldberg type (see [G] for the original definition in the Euclidean case) on a measured metric space M enjoying the local doubling and the approximate midpoint properties and satisfying the uniform ball size condition (see Section 4.1 for the definitions). The space $\mathfrak{h}^1(M)$ is an atomic space, and atoms are either *standard atoms*, i.e., L^2 functions supported in balls of radius ≤ 1 (say) and satisfying a size condition and a cancellation conditions (the same as in the classical case), or global atoms, i.e., functions defined on balls of radius exactly equal to 1, that satisfy the same size condition as standard atoms, but do not have cancellation properties (see Definition 4.2.1 for details). We prove that the topological dual of $\mathfrak{h}^1(M)$ may be identified with a local space $\mathfrak{bmo}(M)$ of functions of bounded mean oscillation in an appropriate sense (see Sections 4.4 and 4.5), and that if $p \in (1,2)$, then $L^p(M)$ is a complex interpolation space between $\mathfrak{h}^1(M)$ and $L^2(M)$ (see Section 4.7).

This theory applies in particular to Riemannian manifolds M with Ricci curvature bounded from below and positive injectivity radius. Note that such manifolds may very well have exponential volume growth, so that they may not be homogeneous spaces in the sense of Coifman–Weiss.

Under much more restrictive assumptions on the geometry of M, a local Hardy type space has recently been introduced by M. Taylor [T3]. We show that, under these assumptions, Taylor's space agrees with the space $\mathfrak{h}^1(M)$ defined above. Taylor's result that $L^p(M)$, $1 < p < 2$, is an interpolation space between $\mathfrak{h}^1(M)$ and $L^2(M)$ is therefore a special case of Theorem 4.7.1 below. It is worth noticing that our methods are rather different from Taylor's, and some new ideas are required to deal with metric measured spaces possessing the aforementioned properties. Indeed, Taylor's assumptions imply a uniform local control on all derivatives of the Riemann tensor and are strong enough to transfer "any local issue" to the tangent space and to transform it in a purely local Euclidean issue. This is no longer possible under our more general assumptions. Taylor proved also $\mathfrak{h}^1(M)$-$\mathfrak{h}^1(M)$ estimates for certain spectral multipliers of $\mathcal{L}$ satisfying an appropriate Hörmander type condition at infinity. By slightly modifying arguments of Mauceri, Meda and Vallarino, in Section 4.9 we prove a similar result when M is a Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius. In the same section we show also that translated Riesz transforms $\nabla(a\mathcal{I} + \mathcal{L})^{-1/2}$, $a > 0$, are bounded from $\mathfrak{h}^1(M)$ to $L^1(M)$ as long as a is large enough, thereby complementing comparatively recent results of Russ [Ru].

New Hardy type spaces and SIO at infinity

Suppose that M is a Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. A challenging problem is to find a Banach space X, contained in $L^1(M)$, such that

(a) the (nontranslated!) Riesz transform $\nabla\mathcal{L}^{-1/2}$ and spectral multiplier operators satisfying conditions of Mihlin–Hörmander type, both at infinity and locally, are bounded from X to $L^1(M)$;

(b) $L^p(M)$, $1 < p < 2$, is an interpolation space between X and $L^2(M)$.

Note that the main difference between the kernels of $\nabla\mathcal{L}^{-1/2}$ and of $\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$, $a > 0$, is that the latter is, for a large enough, integrable at infinity, i.e., there exists a constant C such that

$$\int_{B(y,1)^c} |k_{\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}}(p,y)| \, \mathrm{d}\mu(p) \leq C \qquad \forall y \in M,$$

whereas a similar estimate for the kernel of $\nabla\mathcal{L}^{-1/2}$ fails (for instance in the case where M is the hyperbolic plane). Roughly speaking, the difference between $\nabla(a\mathcal{I}+$

$\mathcal{L})^{-1/2}$ and $\nabla\mathcal{L}^{-1/2}$ is somewhat similar to the difference between the Hilbert transform and the truncated Hilbert transform, whose convolution kernel agrees with that of the Hilbert transform near the origin and vanishes outside a compact interval of $\mathbb{R}$.

It is not hard to show that in the case where M is the hyperbolic plane, then $\nabla\mathcal{L}^{-1/2}$ is unbounded from $\mathfrak{h}^1(M)$ to $L^1(M)$, so that $\mathfrak{h}^1(M)$ cannot play the role of the space X above.

The material contained in Chapter 5 is the result of our attempts to find a good candidate for the space X. We must say that we have developed a good theory for manifolds satisfying the additional assumption that M has spectral gap, i.e., the bottom b of the spectrum of $\mathcal{L}$ is strictly positive. However, this theory parallels the one recently developed by Mauceri, Meda and Vallarino under the same assumptions, and therefore, though new, it is not really innovative with respect to the existing results in the literature.

However, the spaces we introduce have a potentially wider range of applications that those of Mauceri, Meda, and Vallarino. Indeed, the latter are based on the local Hardy spaces $H^1(M)$ of Carbonaro, Mauceri, and Meda that enjoy interesting interpolation properties only in the case where M possesses the isoperimetric property IP (equivalently, under the assumption of bounded geometry, if $b > 0$), whereas the former, especially their atomic definition (see Definition 5.4.4), make sense without assuming the IP. Unfortunately, so far we have not been able to develop all the implications of our theory, and we shall leave further investigations to the near future.

We now briefly describe the results contained in Chapter 5. We assume that M is a Riemannian manifold with Ricci curvature bounded from below, positive injectivity radius and spectral gap, i.e., the bottom b of the spectrum of $\mathcal{L}$ is strictly positive. It may be worth observing that under these assumptions the Riemannian measure is nondoubling and that the volume of geodesic balls in M grows exponentially with the radius. Recall also that for a Riemannian manifold satisfying the above assumptions there are positive constants α, β and C such that

$$\mu\big(B(p,r)\big) \leq C\, r^{\alpha}\, \mathrm{e}^{2\beta\, r} \qquad \forall r \in [1,\infty) \quad \forall p \in M, \tag{0.0.3}$$

where $\mu\big(B(p,r)\big)$ denotes the Riemannian volume of the geodesic ball with centre p and radius r.

Notable examples of such manifolds are Cartan–Hadamard manifolds with Ricci curvature bounded from above and below by two negative constants, in particular, nonamenable connected unimodular Lie groups equipped with a left invariant Riemannian distance, symmetric spaces of the noncompact type with the Killing metric and Damek–Ricci spaces.

Mauceri, Meda, and Vallarino [MMV2, MMV3] introduced a sequence of new spaces

$$X^1(M), X^2(M), X^3(M), \ldots$$

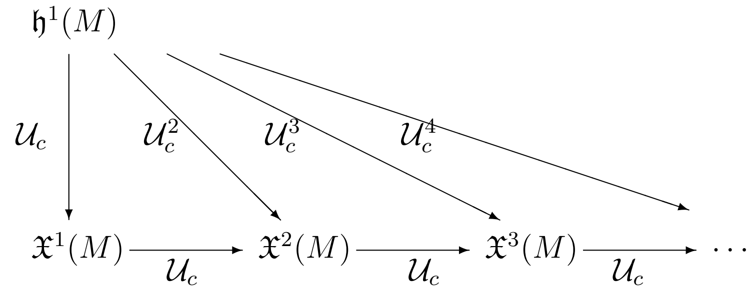
each of which is isomorphic as a Banach space to the local Hardy space $H^1(M)$, defined in [CMM1], and possesses the properties (a) and (b) above. Here we show that we may define a sequence of spaces of Hardy type

$$\mathfrak{X}^1(M), \mathfrak{X}^2(M), \mathfrak{X}^3(M), \ldots$$

each of which is isomorphic to $\mathfrak{h}^1(M)$ as a Banach space, which may play the role of the space X above. Since the theory of $\mathfrak{X}^k(M)$ is parallel to that of the spaces $X^k(M)$, we omit the lengthy details thereof and just state the main results. The space $\mathfrak{X}^k(M)$ is defined as follows. Denote by $\mathcal{U}_c$ the operator $\mathcal{L}\,(c\mathcal{I} + \mathcal{L})^{-1}$, where c is a positive constant, large enough. It is straightforward to check that $\mathcal{U}_c$ is a bounded injective operator on $L^1(M) + L^2(M)$. Denote by $\mathfrak{X}^k(M)$ the range of the restriction of $\mathcal{U}_c^k$ to $\mathfrak{h}^1(M)$, endowed with the norm

$$\|f\|_{\mathfrak{X}^k} = \|\mathcal{U}_c^{-k} f\|_{\mathfrak{h}^1}.$$

By definition, each arrow of the following commutative diagram is an isometric isomorphism of Banach spaces.



By a result of R. Brooks, $b \leq \beta^2$. We shall prove (see Theorem 5.3.2) that some *strongly singular* spectral multipliers (see Definition 5.3.1) are bounded from $\mathfrak{h}^1(M)$ to $L^1(M)$ when $b < \beta^2$, and from $\mathfrak{X}^k(M)$ to $L^1(M)$, for k large enough, when $b = \beta^2$. Moreover, we shall prove that if $b = \beta^2$, then the Riesz transform $\nabla\mathcal{L}^{-1/2}$ is bounded from $\mathfrak{X}^k(M)$ to $L^1(M)$, for k large enough.

We also show that $\mathfrak{X}^k(M)$ has an atomic decomposition in terms of special atoms, which generalise the special atoms introduced in [MMV3]. This atomic space, which agrees with the previously defined $\mathfrak{X}^k(M)$ when $b > 0$, makes sense even without the assumption that M has spectral gap. Then, an important issue in this case is whether $L^p(M)$, $1 < p < 2$, is an interpolation space between $\mathfrak{X}^k(M)$ and $L^2(M)$. We know that this is not the case when $M = \mathbb{R}$. It would be interesting to find manifolds M without spectral gap for which the aforementioned interpolation property holds.

Part I

Bochner-Riesz means

Chapter 1

Pointwise convergence of Bochner-Riesz means

In this chapter we define and study the Bochner-Riesz means of functions in Sobolev spaces. In particular, we determine sufficient conditions for the pointwise convergence and we estimate the Hausdorff dimension of the divergence set.

First we introduce the main object of our analysis and we recall the basic definitions we shall need.
Given a function $f \in L^1(\mathbb{R}^d)$, its *Fourier transform* is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \xi \cdot x}\, dx$$

and the *Fourier inversion formula* is given by

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi.$$

The last integral may not be absolutely convergent, since in general $\widehat{f}$ is not integrable. Therefore, in order to recover a function f from its Fourier transform, it is necessary to introduce some summability methods. The simplest way is to consider the *spherical partial sums* S_R defined by

$$S_R * f(x) = \int_{|\xi| \leq R} \widehat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi,$$

or equivalently

$$(S_R * f)\widehat{\ }(\xi) = \chi_{\{|\xi| \leq R\}}\, \widehat{f}(\xi).$$

A classical problem in Fourier analysis consists in determining if

$$\lim_{R \to +\infty} S_R * f = f$$

both in norm and pointwise, everywhere or almost everywhere.

The problem of convergence in L^p-norm in dimension $d = 1$ was solved by Riesz: he proved that $\lim_{R\to+\infty} \|S_R * f - f\|_p = 0$ for all $f \in L^p(\mathbb{R})$ if and only if $1 < p < +\infty$. In higher dimensions, Fefferman showed that convergence in L^p-norm holds if and only if $p = 2$.

As for the problem of almost everywhere convergence, the Carleson-Hunt theorem states that if $f \in L^p(\mathbb{R})$, $1 < p < +\infty$, then the spherical sums of f converge almost everywhere. The problem remains open for $d \geq 2$. For the related problem of localization see [B], [Col2], [CS2]. This results in higher dimensions suggest to consider summability methods whose multipliers are more regular than the characteristic function of a ball, such as the Bochner-Riesz means.

The *Bochner-Riesz kernel* $S_R^\beta(x)$ of order β, with $\beta > 0$, is defined in terms of its Fourier transform by

$$\widehat{S_R^\beta}(\xi) = \Big(1 - \Big|\frac{\xi}{R}\Big|^2\Big)_+^\beta.$$

This kernel can be written explicitly in terms of Bessel functions:

$$S_R^\beta(x) = \pi^{-\beta}\,\Gamma(\beta+1) R^{d/2-\beta} |x|^{-\beta-d/2} J_{\beta+d/2}(2\pi R|x|).$$

By the asymptotic formula of Bessel functions this kernel has a decay

$$|S_R^\beta(x)| \leq c\, R^d (1 + R|x|)^{-\beta-(d+1)/2}.$$

The *Bochner-Riesz means* of order β of functions on $\mathbb{R}^d$ are defined by

$$S_R^\beta * f(x) = \int_{\mathbb{R}^d} \Big(1 - \Big|\frac{\xi}{R}\Big|^2\Big)_+^\beta \widehat{f}(\xi)\, e^{2\pi i\xi\cdot x}\, d\xi.$$

Observe that if $\beta = 0$ we obtain the spherical sums.

The *Bessel kernel* $G^\gamma(x)$, with $\gamma > 0$, is defined as that function whose Fourier transform is

$$\widehat{G^\gamma}(\xi) = (1 + |\xi|^2)^{-\gamma/2}.$$

It is known that G^γ is a positive and integrable function. Indeed, it is asymptotic to $c\,|x|^{\gamma-d}$ when $x \to 0$ and it has an exponential decay at infinity. See [St3] and Lemma 1.0.2 below.

Define also the *Riesz kernel* $I^\gamma(x)$, with $0 < \gamma < d$, as

$$I^\gamma(x) = |x|^{\gamma-d}.$$

If $\gamma > 0$ and $p > 1$, the *Bessel capacity* of a set $E \subset \mathbb{R}^d$ is defined by

$$B_{\gamma,p}(E) = \inf\{\|f\|_p^p \;:\; G^\gamma * f(x) \geq 1 \text{ on } E\}.$$

The *Riesz capacity* $R_{\gamma,p}$ is defined in a similar way, by replacing G^γ with I^γ. It follows directly from the definitions that $R_{\gamma,p}(E) \le C\, B_{\gamma,p}(E)$. Actually, it is also true that the Bessel and Riesz capacities have the same null sets (see [Z, p. 67]).
It can be proved that when the $d-\gamma p$ Hausdorff measure of E is finite, then $B_{\gamma,p}(E) = 0$. Conversely, if $B_{\gamma,p}(E) = 0$, then for every $\epsilon > 0$ the $d - \gamma p + \epsilon$ Hausdorff measure of E is 0. (See [Z, Th 2.6.16]).

The problem of almost everywhere convergence of Bochner-Riesz means has been widely studied. The following results are well known, see [SW]. If $\mathrm{Re}(\beta) > (d-1)/2$, the critical index, then the Bochner-Riesz maximal operator $S^\beta_* f(x) = \sup_{R>0} |S^\beta_R * f|(x)$ is of weak type $(1,1)$ since it is pointwise dominated by the Hardy-Littlewood maximal operator. If $p = 2$ and $\mathrm{Re}(\beta) > 0$, then $S^\beta_* f$ is bounded on $L^2(\mathbb{R}^d)$. Therefore, by complex interpolation, the Bochner-Riesz maximal operator is bounded on $L^p(\mathbb{R}^d)$, with $1 \le p \le +\infty$ and $\beta > (d-1)|1/p - 1/2|$. From these results for the maximal operator the almost everywhere convergence follows: if f is in $L^p(\mathbb{R}^d)$, with $1 \le p \le +\infty$ and $\beta > (d-1)|1/p - 1/2|$, then $\lim_{R\to+\infty} S^\beta_R * f(x) = f(x)$ a.e..
This result is not optimal. Indeed, Carbery in [Ca] established the pointwise convergence of the Bochner-Riesz means when $2 \le p < 2d/(d-1-2\,\beta)$ and $d = 2$. The same result for $d \ge 3$ was obtained by Christ in [Chr2], under the extra assumption that $\beta \ge (d-1)/2(d+1)$. See also [Le]. Finally, Carbery, Rubio de Francia, and Vega in [CRV] removed the restriction on β, showing that $S^\beta_* f$ is bounded on the weighted space $L^2(|x|^{-\lambda})$ if $d(1-2/p) \le \lambda < 1 + 2\beta \le d$ and observing that $L^p \subset L^2 + L^2(|x|^{-\lambda})$, and it has been shown by Rubio de Francia that the Bochner-Riesz means of index β are not defined in $L^p(\mathbb{R}^d)$ when $p \ge 2d/(d-1-2\,\beta)$. Therefore the problem of almost everywhere convergence of Bochner-Riesz means when $p \ge 2$ is essentially solved. On the other hand, as far as we know, sharp results when $p < 2$ are not known. Anyhow, see [T].
Here we consider the pointwise convergence for more regular functions, in particular functions in Sobolev classes. As shown by Colzani in [Col1], the Bochner-Riesz means of functions with γ integrable derivatives with $\beta + \gamma > (d-1)/2$, may diverge only in a set of points of Hausdorff dimension at most $d - \gamma$. Here we generalise this result to the case of functions with γ derivatives in $L^p(\mathbb{R}^d)$, $1 < p < \infty$. In particular, we obtain conditions on β, γ, p and d that ensure the pointwise convergence up to a set with Hausdorff dimension at most $d - \gamma\, p$. Since the functions we are considering may be infinite precisely on sets of dimension $d - \gamma\, p$, the estimate for the dimension of the divergence set is best possible. We prove this in two different cases: when

$\beta > (d-1)|1/p - 1/2|$ or when $\beta + \gamma \geq (d-1)/2$ and $\beta > d(1/2 - 1/p) - 1/2$.

More precisely, we obtain the following result.

Theorem 1.0.1. *Let $1 < p < +\infty$, $0 < \gamma < d$ and $\beta > 0$, and assume that one of the following properties holds:*

(i) $\beta > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$

or

(ii) $\beta > d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *and* $\beta + \gamma \geq \frac{d-1}{2}$.

Then for every function $F \in L^p(\mathbb{R}^d)$ there exists a function $H \in L^p(\mathbb{R}^d)$ with $\|H\|_p \leq C\,\|F\|_p$ and such that

$$S_*^\beta(G^\gamma * F)(x) = \sup_{R>0} |S_R^\beta * G^\gamma * F|(x) \leq I^\gamma * H(x).$$

We split the proof into a series of lemmas. The first lemma describes the behaviour of G^γ near the origin and at infinity.

Lemma 1.0.2. *If $0 < \gamma < d$, then*

$$G^\gamma(x) \leq \begin{cases} C\,|x|^{\gamma-d} & \text{if} \quad |x| \to 0 \\ e^{-c|x|} & \text{if} \quad |x| \to +\infty. \end{cases}$$

Proof. There is an explicit formula of this kernel in terms of the Bessel functions of the third order. Anyhow, there is a more elementary approach. By the identity for the Gamma function,

$$(1 + |\xi|^2)^{-\gamma/2} = \Gamma(\gamma/2)^{-1} \int_0^\infty t^{\gamma/2-1}\, e^{-t(1+|\xi|^2)}\, dt,$$

it follows that the Bessel kernel is a superposition of heat kernels,

$$\Gamma(\gamma/2)^{-1} \int_0^{+\infty} \{(4\pi t)^{-d/2} e^{-|x|^2/4t}\} e^{-t} t^{\gamma/2-1} dt.$$

The lemma easily follows by estimating the size of the above integral. For more details see for example [St3]. □

Now we prove that the convolutions in the statement of the theorem are well defined. We shall denote by $L^{p,\infty}(\mathbb{R}^d)$ the Weak-L^p spaces. As it is well known, $L^p(\mathbb{R}^d) \subset L^{p,\infty}(\mathbb{R}^d)$.

Lemma 1.0.3. *Let* $1 < p < +\infty$, $0 < \gamma < d$, $\beta > \max\{0, d(1/2 - 1/p) - 1/2\}$. *Then for every* $F \in L^p(\mathbb{R}^d)$ *the convolution* $S_R^\beta * G^\gamma * F$ *is well defined and it is commutative and associative:*

$$S_R^\beta * (G^\gamma * F) = (S_R^\beta * G^\gamma) * F = G^\gamma * (S_R^\beta * F).$$

Proof. This follows by applying twice Young's inequality for convolutions in Weak-L^p spaces: *given* $1 < p, q < \infty$ *with* $1/p + 1/q > 1$, *if* $f \in L^1(\mathbb{R}^d)$, $g \in L^{p,\infty}(\mathbb{R}^d)$ *and* $h \in L^{q,\infty}(\mathbb{R}^d)$, *then the convolution* $f * g * h$ *is well defined and it is commutative and associative.*
The Bochner-Riesz kernel $S_R^\beta(x)$ has decay $c\, R^d(1 + R|x|)^{-\beta-(d+1)/2}$, hence $S_R^\beta(x)$ is in $L^{q,\infty}(\mathbb{R}^d)$, with $1/q = (2\beta + d + 1)/2d$. Moreover, by the previous lemma, $G^\gamma(x)$ is an integrable function. Hence, in order to prove that $S_R^\beta * G^\gamma * F$ is well defined it suffices to require that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{2\beta + d + 1}{2d} > 1,$$

that is to say

$$\beta > d\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2}.$$

□

To prove the theorem we shall make use of the following lemmas.

Lemma 1.0.4. *Let* $1 < p < +\infty$ *and* $\beta > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. *Then the maximal operator* $S_*^\beta f(x) = \sup_{R>0} |S_R^\beta * f|(x)$ *is bounded on* $L^p(\mathbb{R}^d)$.

Proof. This is a classical result of Stein, see [SW, Thm 5.1]. □

Lemma 1.0.5. *If* $\beta + \gamma \geq (d-1)/2$ *with* $0 < \gamma < d$, *then*

$$\sup_{R>0} |S_R^\beta * G^\gamma|(x) \leq C|x|^{\gamma - d}.$$

Proof. By definition

$$S_R^\beta * G^\gamma(x) = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-\gamma/2} \Big(1 - |R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i \xi \cdot x} d\xi.$$

Introduce a partition of unity $\{\phi, \psi\}$ on $\mathbb{R}_+$ with ϕ and ψ smooth and nonnegative, $\phi(\rho) + \psi(\rho) = 1$ and

$$\phi(\rho) = 1 \text{ if } 0 \leq \rho \leq 1/3 \text{ and } \phi(\rho) = 0 \text{ if } \rho > 2/3,$$

$$\psi(\rho) = 1 \text{ if } 2/3 \leq \rho \leq 1 \text{ and } \psi(\rho) = 0 \text{ if } \rho < 1/3.$$

Then

$$S_R^\beta * G^\gamma(x) = \int_{\mathbb{R}^d} \phi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi$$
$$+ \int_{\mathbb{R}^d} \psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi.$$

To estimate the first integral set

$$\widehat{K}(\xi) = \phi(|\xi|)(1-|\xi|^2)_+^\beta.$$

Then, if $K_R(x) = R^dK(Rx)$, we can rewrite the first integral as

$$\int_{\mathbb{R}^d} \phi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi = K_R * G^\gamma(x).$$

Let us now recall the definition of the *Hardy-Littlewood maximal function* $\mathcal{M}f$: if f is a locally integrable function, then its Hardy-Littlewood maximal function is

$$\mathcal{M}f(x) = \sup_{r>0}\frac{1}{|B_r|}\int_{B_r} |f(x-y)|\,dy,$$

where B_r denotes the ball of radius r centred at the origin.
Since the kernel $K(x)$ is bounded and rapidly decreasing at infinity, one has

$$\sup_{R>0}|K_R * G^\gamma(x)| \le C\mathcal{M}G^\gamma(x).$$

Since $G^\gamma(x) \le C|x|^{\gamma-d}$ and since the Hardy-Littlewood maximal function of a radial homogeneous function is radial homogeneous, it also follows that

$$\mathcal{M}G^\gamma(x) \le C\mathcal{M}|x|^{\gamma-d} = C|x|^{\gamma-d}.$$

Now we have to estimate the second integral. A crude estimate gives

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi\Big|\\
&\le \int_{\mathbb{R}^d} \psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta d\xi\\
&= C\int_0^R \psi(R^{-1}\rho)(1+\rho^2)^{-\gamma/2}\Big(1-(R^{-1}\rho)^2\Big)^\beta \rho^{d-1}d\rho\\
&= CR^{d-\gamma}\int_0^1 \psi(\rho)(R^{-2}+\rho^2)^{-\gamma/2}(1-\rho^2)^\beta\rho^{d-1}d\rho\\
&\le CR^{d-\gamma}\\
&= C(R|x|)^{d-\gamma}|x|^{\gamma-d}.
\end{aligned}$$

If $R|x| \le 3$, the desired estimate follows.

To estimate the integral when $R|x| > 3$, we introduce another smooth cut-off function $0 \le \chi(\rho) \le 1$ such that $\chi(\rho) = 1$ if $0 \le \rho \le 1 - 2/R|x|$ and $\chi(\rho) = 0$ if $\rho \ge 1 - 1/R|x|$. Moreover we require that for all $j = 0, 1, 2, \ldots,$

$$\Big|\frac{d^j}{d\rho^j}\chi(\rho)\Big| \le c(j)(R|x|)^j.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^{\beta} e^{2\pi i\xi\cdot x}d\xi \\
&= \int_{\mathbb{R}^d}(1-\chi(R^{-1}|\xi|))\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^{\beta} e^{2\pi i\xi\cdot x}d\xi \\
&+ \int_{\mathbb{R}^d}\chi(R^{-1}|\xi|)\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^{\beta} e^{2\pi i\xi\cdot x}d\xi.
\end{aligned}$$

In polar coordinates the first integral becomes

$$\begin{aligned}
&\int_{\mathbb{R}^d}(1-\chi(R^{-1}|\xi|))\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^{\beta} e^{2\pi i\xi\cdot x}d\xi \\
&= \int_0^R \rho^{d-1}(1-\chi(R^{-1}\rho))\psi(R^{-1}\rho)(1+\rho^2)^{-\gamma/2}\Big(1-(R^{-1}\rho)^2\Big)^{\beta}\int_{|\theta|=1} e^{2\pi i\rho x\cdot\theta}d\theta d\rho \\
&= R^{d-\gamma}\int_0^1 \rho^{d-1}(1-\chi(\rho))\psi(\rho)(R^{-2}+\rho^2)^{-\gamma/2}(1+\rho)^{\beta}(1-\rho)^{\beta}\int_{|\theta|=1} e^{2\pi iR\rho x\cdot\theta}d\theta d\rho.
\end{aligned}$$

It is well known that

$$\Big|\int_{|\theta|=1} e^{2\pi ix\cdot\theta}d\theta\Big| \le C|x|^{-(d-1)/2}.$$

This is a standard estimate for oscillatory integrals with non-degenerate critical points. Anyhow, this estimate immediately follows from the decay of Bessel functions and the explicit formula

$$\int_{|\theta|=1} e^{2\pi ix\cdot\theta}d\theta = 2\pi|x|^{(2-d)/2}J_{(d-2)/2}(2\pi|x|).$$

See, for example, [St2, p.347].

Therefore, if $\beta+\gamma \ge (d-1)/2$ and $R|x| > 3$, we get

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d}(1-\chi(R^{-1}|\xi|))\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^{\beta} e^{2\pi i\xi\cdot x}d\xi\Big| \\
&\le C\ R^{(d+1)/2-\gamma}|x|^{-(d-1)/2}\int_0^1(1-\chi(\rho))(1-\rho)^{\beta}d\rho \\
&\le C\ R^{(d+1)/2-\gamma}|x|^{-(d-1)/2}(R|x|)^{-\beta-1} \\
&= C\ (R|x|)^{(d-1)/2-\beta-\gamma}|x|^{\gamma-d} \\
&\le C\ |x|^{\gamma-d}.
\end{aligned}$$

The last inequality holds provided that $\beta + \gamma \geq (d-1)/2$.
To estimate the second integral, we introduce the Laplacian $\Delta_\xi = -\sum_{j=1}^d \partial^2/\partial\xi_j^2$. Since the Laplacian is self-adjoint and $\Delta_\xi^k\{e^{2\pi i\xi\cdot x}\} = |2\pi x|^{2k}e^{2\pi i\xi\cdot x}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi(R^{-1}|\xi|)\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi \\
&= \int_{\mathbb{R}^d} \chi(R^{-1}|\xi|)\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta \Delta_\xi^k\Big[\frac{e^{2\pi i\xi\cdot x}}{|2\pi x|^{2k}}\Big]d\xi \\
&= |2\pi x|^{-2k}R^{-\gamma}\int_{\mathbb{R}^d} \Delta_\xi^k[g(R^{-1}\xi)]e^{2\pi i\xi\cdot x}d\xi,
\end{aligned}$$

where we have set

$$g(z) = \chi(|z|)\psi(|z|)(R^{-2}+|z|^2)^{-\gamma/2}(1-|z|^2)_+^\beta.$$

Now, if we denote by Δ_ρ^k the radial part of the Laplacian and we set $g_0(|z|) = g(z)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi(R^{-1}|\xi|)\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi \\
&= |2\pi x|^{-2k}R^{-\gamma-2k}\int_{\mathbb{R}^d} (\Delta_\xi^k g)(R^{-1}\xi)e^{2\pi i\xi\cdot x}d\xi \\
&= |2\pi x|^{-2k}R^{-\gamma-2k}\int_0^{+\infty} (\Delta_\rho^k g_0)(R^{-1}\rho)\rho^{d-1}\int_{|\theta|=1} e^{2\pi i\rho x\cdot\theta}d\theta\ d\rho \\
&= |2\pi x|^{-2k}R^{d-\gamma-2k}\int_0^{+\infty} (\Delta_\rho^k g_0)(\rho)\rho^{d-1}\int_{|\theta|=1} e^{2\pi iR\rho x\cdot\theta}d\theta\ d\rho.
\end{aligned}$$

Then, recalling the properties of the cut-off functions $\chi(\rho)$ and $\psi(\rho)$, if $k > (\beta-1)/2$ we finally get

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \chi(R^{-1}|\xi|)\psi(R^{-1}|\xi|)(1+|\xi|^2)^{-\gamma/2}\Big(1-|R^{-1}\xi|^2\Big)_+^\beta e^{2\pi i\xi\cdot x}d\xi\Big| \\
&\leq C|x|^{-2k-(d-1)/2}R^{(d+1)/2-\gamma-2k}\int_0^{+\infty} \big|\Delta_\rho^k g_0(\rho)\big|\,\rho^{(d-1)/2}\ d\rho \\
&\leq C|x|^{-2k-(d-1)/2}R^{(d+1)/2-\gamma-2k}\int_{1/3}^{1-1/R|x|} (1-\rho)^{\beta-2k}d\rho \\
&\leq C|x|^{-2k-(d-1)/2}R^{(d+1)/2-\gamma-2k}(R|x|)^{-\beta+2k-1} \\
&= C|x|^{\gamma-d}(R|x|)^{(d-1)/2-\beta-\gamma} \\
&\leq C|x|^{\gamma-d}.
\end{aligned}$$

As before, the last inequality follows from the fact that $\beta + \gamma \geq (d-1)/2$.

□

Proof of Theorem 1.0.1. (*i*) First assume that $\beta > (d-1)\Big|\frac{1}{p}-\frac{1}{2}\Big|$. Observe that this hypothesis is stronger than the assumption on β in Lemma 1.0.3, therefore we can define $S_*^\beta(G^\gamma * F)$. Since the Bessel kernel G^γ is positive and it is dominated by $|x|^{\gamma-d}$, we can estimate the maximal function as

$$\sup_{R>0} |S_R^\beta * G^\gamma * F(x)| \leq C\, I^\gamma * \sup_{R>0} |S_R^\beta * F|(x).$$

As stated in Lemma 1.0.4, if $\beta > (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$ and $F \in L^p(\mathbb{R}^d)$, then also $\sup_{R>0} |S_R^\beta * F| \in L^p(\mathbb{R}^d)$ and $\|\sup_{R>0} |S_R^\beta * F|\|_p \leq C\,\|F\|_p$.

(*ii*) Now assume that $\beta > d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}$ and $\beta+\gamma \geq \frac{d-1}{2}$. Then, by Lemma 1.0.5,

$$\begin{aligned}\sup_{R>0} |S_R^\beta * G^\gamma * F(x)| &\leq |F| * \sup_{R>0} |S_R^\beta * G^\gamma|(x)\\ &\leq C\, I^\gamma * |F|(x).\end{aligned}$$

□

Corollary 1.0.6. *Let $\beta > 0$ and $0 < \gamma < d$. If $f(x) = G^\gamma * F(x)$ with $F \in L^p(\mathbb{R}^d)$, $1 < p < +\infty$, then $S_R^\beta * f(x)$ converges pointwise to $f(x)$ as R tends to infinity with a possible exception of a set of points x with Hausdorff dimension at most $d - \gamma p$ if one of the following conditions holds:*

(i) $\beta > (d-1)\left|\dfrac{1}{p} - \dfrac{1}{2}\right|$

or

(ii) $\beta > d\left(\dfrac{1}{2} - \dfrac{1}{p}\right) - \dfrac{1}{2}$ *and* $\beta + \gamma \geq \dfrac{d-1}{2}$.

Proof. Let $\{F_n\}$ be a sequence of functions in the Schwartz class which converges to F in the metric of $L^p(\mathbb{R}^d)$ and let $f_n(x) = G^\gamma * F_n(x)$.
Since $f_n(x)$ is in the Schwartz class, $\lim_{R\to+\infty} S_R^\beta * f_n(x) = f_n(x)$ and for every $t > 0$,

$$\begin{aligned}&\{x : \limsup_{R\to+\infty} |S_R^\beta * f(x) - f(x)| > t\}\\ &\subseteq \{x : \sup_{R>0} |S_R^\beta * f(x) - S_R^\beta * f_n(x)| > t/2\} \cup \{x : |f_n(x) - f(x)| > t/2\}.\end{aligned}$$

The Bessel capacity of the second term can be estimated by

$$\begin{aligned}&B_{\gamma,p}(\{x : |f_n(x) - f(x)| > t/2\})\\ &\leq B_{\gamma,p}(\{x : G^\gamma * |F_n - F|(x) > t/2\})\\ &\leq \left(\frac{t}{2}\right)^{-p} \|F_n - F\|_p^p.\end{aligned}$$

Now we estimate the Bessel capacity of the first term in the case (i): when $\beta > (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$

$$\begin{aligned}
&B_{\gamma,p}(\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * f(x) - S_R^\beta * f_n(x)| > t/2\})\\
&\le B_{\gamma,p}(\{x\,:\,G^\gamma * \sup_{R>0}|S_R^\beta * (F-F_n)|(x) > t/2\})\\
&\le \left(\frac{t}{2}\right)^{-p}\left\|\sup_{R>0}|S_R^\beta * (F-F_n)|\right\|_p^p\\
&\le C\left(\frac{t}{2}\right)^{-p}\|F_n - F\|_p^p.
\end{aligned}$$

Since $\|F_n - F\|_p \to 0$ as $n\to+\infty$, we obtain

$$B_{\gamma,p}(\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * f(x) - f(x)| > t\}) = 0. \tag{1.0.1}$$

In the case (ii) when $\beta \ge d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}$ and $\beta+\gamma\ge\frac{d-1}{2}$ we estimate the Riesz capacity:

$$\begin{aligned}
&R_{\gamma,p}(\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * (f-f_n)|(x) > t/2\})\\
&\le R_{\gamma,p}(\{x\,:\,I^\gamma * |(F-F_n)|(x) > (Ct)/2\})\\
&\le \left(\frac{Ct}{2}\right)^{-p}\|F_n - F\|_p^p.
\end{aligned}$$

and the last term tends to zero as $n\to+\infty$.

Since the Riesz and Bessel capacities have the same null sets (see [Z, p. 67]), in both cases we get

$$\begin{aligned}
&B_{\gamma,p}(\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * f(x) - f(x)| > 0\})\\
&\le \sum_{k=1}^{\infty} B_{\gamma,p}(\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * f(x) - f(x)| > 1/k\})\\
&= 0.
\end{aligned}$$

Therefore, applying [Z, Th 2.6.16]), we obtain that the Hausdorff dimension of the set $\{x\,:\,\limsup_{R\to+\infty}|S_R^\beta * f(x) - f(x)| > 0\}$ is at most $d-\gamma p$. □

Observe that at least for the above range of indexes the corollary is sharp since the function $f(x) = G^\gamma * F(x)$ with $F\in L^p(\mathbb{R}^d)$ may be infinite on sets with Hausdorff dimension $d-\gamma p$.

We want to remark again that, as we said in the introduction, our analysis is not exhaustive and some of the results stated are not best possible. In particular, using the estimates of the maximal Bochner-Riesz operator in [Ca], [Chr2], [Le] and [T], one can easily improve part (i) of Theorem 1.0.1 and Corollary 1.0.6. Anyhow, even these improvements are partial and not definitive.

Chapter 2

Bochner-Riesz means of radial functions

In this chapter we establish an equiconvergence result between Bochner-Riesz means in one and several dimensions and we determine the Hausdorff dimension of the divergence set of Bochner-Riesz means of radial functions in Sobolev classes. Actually we shall prove more precise results for Lorentz spaces and Fourier-Bessel expansions.

2.1 An equiconvergence result

Before stating and proving the main result of this section, we recall some facts about Lorentz spaces.

The non-increasing rearrangement of a function $f(x)$ on a measure space $(\mathbb{X}, d\mu(x))$ is a decreasing function $f^*(t)$ on $0 < t < +\infty$ with the same distribution function of $|f(x)|$,

$$f^*(t) = \inf\left\{s > 0,\ \mu\left\{|f(x)| > s\right\} \leq t\right\}.$$

The Lorentz spaces $L^{p,q}(\mathbb{X}, d\mu(x))$, $0 < p < +\infty$ and $0 < q \leq +\infty$, are defined by the quasi-norm

$$\|f\|_{p,q} = \left\{\frac{q}{p}\int_0^{+\infty}\left(t^{1/p}f^*(t)\right)^q\frac{dt}{t}\right\}^{1/q}, \qquad 0 < q < +\infty$$

$$\|f\|_{p,\infty} = \sup_{t>0}\left\{t^{1/p}f^*(t)\right\}.$$

Lorentz spaces with equal indexes $p = q$ are the classical Lebesgue spaces $L^p(\mathbb{X}, d\mu(x))$, while the spaces $L^{p,\infty}(\mathbb{X}, d\mu(x))$ coincide with the Weak-$L^p(\mathbb{X}, d\mu(x))$ spaces. Moreover, $L^{p,q}(\mathbb{X}, d\mu(x))$ is imbedded into $L^{p,r}(\mathbb{X}, d\mu(x))$ if $q < r$. With the normalizing factor q/p the norm of a characteristic function $\chi_\Omega(x)$ is independent of q, $\|\chi_\Omega\|_{p,q} = \mu(\Omega)^{1/p}$, and $L^{p,1}(\mathbb{X}, d\mu(x))$ and $L^{p,\infty}(\mathbb{X}, d\mu(x)) = \text{Weak-}L^p(\mathbb{X}, d\mu(x))$ are the smallest and largest rearrangement invariant Banach function space where characteristic functions of measurable sets Ω have norms $\mu(\Omega)^{1/p}$. Finally, there is a duality between the spaces $L^{p,q}(\mathbb{X}, d\mu(x))$ and $L^{r,s}(\mathbb{X}, d\mu(x))$, with $1/p + 1/r = 1$ and $1/q + 1/s = 1$:

$$\left| \int_{\mathbb{X}} f(x)g(x)d\mu(x)dx \right| \leq \int_0^{+\infty} f^*(t)g^*(t)dt$$
$$\leq \left(\frac{p}{q}\right)^{1/q} \left(\frac{r}{s}\right)^{1/s} \left\{\frac{q}{p}\int_0^{+\infty} \left(t^{1/p}f^*(t)\right)^q \frac{dt}{t}\right\}^{1/q} \left\{\frac{s}{r}\int_0^{+\infty} \left(t^{1/r}g^*(t)\right)^s \frac{dt}{t}\right\}^{1/s}.$$

Trigonometric expansions of radial functions in $\mathbb{R}^d$ are particular cases of Fourier-Bessel expansions, and in the sequel we shall deal with this slightly more general context.

The Bessel functions $(tx)^{-\alpha} J_\alpha(tx)$ are the analytic eigenfunctions of the radial component of the Laplace operator,

$$\left(-\frac{d^2}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx}\right) \frac{J_\alpha(tx)}{(tx)^\alpha} = t^2 \frac{J_\alpha(tx)}{(tx)^\alpha}.$$

For $\alpha \geq -1/2$ the Fourier-Bessel transform and its inversion formula are

$$\mathcal{F}_\alpha f(t) = \int_0^{+\infty} f(y)\frac{J_\alpha(ty)}{(ty)^\alpha} y^{2\alpha+1} dy,$$
$$f(x) = \int_0^{+\infty} \mathcal{F}_\alpha f(t) \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt.$$

The parameter $2\alpha+2$ plays the role of space dimension. When $\alpha = -1/2$, then $J_{-1/2}(z) = \sqrt{2/(\pi z)}\cos(z)$ and the Fourier-Bessel transform reduces to the cosine transform,

$$\mathcal{F}_{-1/2} f(t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y)\cos(ty)\, dy,$$
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \mathcal{F}_{-1/2} f(t) \cos(tx)\, dt.$$

The Bochner-Riesz means of Fourier-Bessel expansions are

$$S_R^\beta f(x) = \int_0^R \left(1-(t/R)^2\right)^\beta \mathcal{F}_\alpha f(t) \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt$$
$$= \int_0^{+\infty} \left(\int_0^R \left(1-(t/R)^2\right)^\beta \frac{J_\alpha(tx)}{(tx)^\alpha}\frac{J_\alpha(ty)}{(ty)^\alpha}(ty)^{2\alpha+1}\, dt\right) f(y)\, dy.$$

Bessel functions have simple asymptotic expansions in terms of trigonometric functions. Using these asymptotic expansions, we shall prove that the means $S_R^\beta f(x)$

are equiconvergent with the Bochner-Riesz means of the cosine expansion of a suitable truncated of the function $f(x)$. Fix an interval $0 < \varepsilon < \eta < +\infty$ and let $\chi(x)$ be a smooth cut-off with $\chi(x) = 1$ if $\varepsilon/2 < x < 2\eta$ and $\chi(x) = 0$ if $x < \varepsilon/3$ or $x > 3\eta$. Define

$$\begin{aligned} T_R^\beta f(x) &= \sqrt{\frac{2}{\pi}} \int_0^R \left(1 - (t/R)^2\right)^\beta \mathcal{F}_{-1/2}(\chi f)(t) \cos(tx)\, dt \\ &= \int_0^{+\infty} \left(\frac{2}{\pi} \chi(y) \int_0^R \left(1 - (t/R)^2\right)^\beta \cos(tx)\cos(ty)\, dt \right) f(y) dy. \end{aligned}$$

The following is an equiconvergence result between $S_R^\beta f(x)$ and $T_R^\beta f(x)$.

Theorem 2.1.1. *Let $\alpha \geq -1/2$, $\beta \geq 0$, $\lambda = \min\{\alpha + 1/2, \beta\}$, and assume that*

$$\int_0^{+\infty} |f(x)| \frac{x^{\alpha+\lambda+1/2}}{(1+x)^{\beta+\lambda+1}} dx < +\infty.$$

Then the means $S_R^\beta f(x)$ and $T_R^\beta f(x)$ are equiconvergent in $\varepsilon < x < \eta$:

$$\lim_{R \to +\infty} \left\{ \sup_{0<\varepsilon<x<\eta<+\infty} \left| S_R^\beta f(x) - T_R^\beta f(x) \right| \right\} = 0.$$

Proof. The case $\beta = 0$ is already in [CCTV, Thm 2.3] and the case $\beta > 0$ is implicitly but essentially contained in [CTV]. The idea is that the main term in the asymptotic expansion of the kernel associated to the operator S_R^β is independent of α and it coincides with the kernel of the operator T_R^β and, under appropriate assumptions, the contribution of the remainder in the asymptotic expansion of the kernel is negligible. Write

$$\begin{aligned} S_R^\beta f(x) &= \int_0^{+\infty} S_R^\beta(x, y) f(y)\, dy, \\ T_R^\beta f(x) &= \int_0^{+\infty} T_R^\beta(x, y) f(y)\, dy. \end{aligned}$$

Then the theorem follows from the following claims.

(1) If $g(x)$ is a smooth function with compact support and if $\varepsilon < x < \eta$, then

$$\lim_{R \to +\infty} \left\{ S_R^\beta g(x) \right\} = \lim_{R \to +\infty} \left\{ T_R^\beta g(x) \right\} = g(x).$$

(2) If $\varepsilon < x < \eta$, then

$$\left| S_R^\beta(x, y) - T_R^\beta(x, y) \right| \leq c \frac{y^{\alpha+\lambda+1/2}}{(1+y)^{\beta+\lambda+1}}.$$

The first claim is the classical Fourier inversion formula for smooth functions. For a short and elementary proof see [CCTV]. The proof of the second claim is contained in the following lemmas. □

Lemma 2.1.2. *The kernel of the operator* T_R^β *is*

$$\begin{aligned} T_R^\beta(x,y) &= \frac{2}{\pi}\chi(y)\int_0^R \left(1-(t/R)^2\right)^\beta \cos(tx)\cos(ty)\,dt \\ &= \chi(y)\frac{2^\beta\Gamma(\beta+1)}{\sqrt{2\pi}}R\left(\frac{J_{\beta+1/2}(R|x-y|)}{(R|x-y|)^{\beta+1/2}}+\frac{J_{\beta+1/2}(R|x+y|)}{(R|x+y|)^{\beta+1/2}}\right). \end{aligned}$$

Proof. This follows from the integral representation of Bessel functions:

$$J_{\beta+1/2}(z) = 2^{1/2-\beta}\pi^{-1/2}\Gamma(\beta+1)^{-1}z^{-\beta-1/2}\int_0^z \left(z^2-t^2\right)^\beta \cos(t)\,dt.$$

□

Lemma 2.1.3. *The kernel of the operator* S_R^β *is*

$$S_R^\beta(x,y) = \int_0^R \left(1-(t/R)^2\right)^\beta \frac{J_\alpha(tx)}{(tx)^\alpha}\frac{J_\alpha(ty)}{(ty)^\alpha}(ty)^{2\alpha+1}\,dt.$$

This kernel satisfies the estimates:

(1) $\left|S_R^\beta(x,y)\right| \le cR^{2\alpha+2}y^{2\alpha+1}(1+R|x-y|)^{-\alpha-\beta-3/2}$,

(2) $\left|S_R^\beta(x,y)\right| \le cR^{-\beta}x^{-\alpha-\beta-3/2}y^{\alpha+1/2}$ *if* $2y<x$,

(3) $\left|S_R^\beta(x,y)\right| \le cR^{-\beta}x^{-\alpha-1/2}y^{\alpha-\beta-1/2}$ *if* $2x<y$,

(4) $\left|S_R^\beta(x,y)-\frac{2^\beta\Gamma(\beta+1)}{\sqrt{2\pi}}R\frac{J_{\beta+1/2}(R|x-y|)}{(R|x-y|)^{\beta+1/2}}\right| \le c$ *if* $0<\varepsilon<x,y<\eta<+\infty$.

Proof. This has been proved in [CTV]. Here we just hint at the proof of (4), which is the main ingredient in what follows. The asymptotic expansion of Bessel functions is

$$J_\alpha(z) = \sqrt{2/\pi z}\cos(z-\alpha\pi/2-\pi/4)+\ldots.$$

Hence the asymptotic expansion of the kernel $S_R^\beta(x,y)$ is

$$\begin{aligned} &\int_0^R \left(1-(t/R)^2\right)^\beta \frac{J_\alpha(tx)}{(tx)^\alpha}\frac{J_\alpha(ty)}{(ty)^\alpha}(ty)^{2\alpha+1}\,dt \\ &= x^{-\alpha}y^{\alpha+1}R^2\int_0^1 s\left(1-s^2\right)^\beta J_\alpha(Rxs)J_\alpha(Rys)\,ds \\ &= \pi^{-1}(y/x)^{\alpha+1/2}R\int_0^1 \left(1-s^2\right)^\beta\cos(R(x-y)s)\,ds \\ &\qquad + \pi^{-1}(y/x)^{\alpha+1/2}R\int_0^1\left(1-s^2\right)^\beta \sin(R(x+y)s-\alpha\pi)\,ds+\ldots \\ &= \frac{2^\beta\Gamma(\beta+1)}{\sqrt{2\pi}}\left(\frac{y}{x}\right)^{\alpha+1/2}R\frac{J_{\beta+1/2}(R|x-y|)}{(R|x-y|)^{\beta+1/2}}+\ldots. \end{aligned}$$

The terms with $x+y$ are more oscillating than the ones with $x-y$, hence they are less singular. By the way, when α is a half integer, the asymptotic expansions of $J_\alpha(z)$ and $S_R^\beta(x,y)$ are finite equalities, and the desired estimates are easily verified. Finally, when $\varepsilon < x, y < \eta$ then one can easily get rid of the factor $(y/x)^{\alpha+1/2}$:

$$\begin{aligned}
&\left|\left(\frac{y}{x}\right)^{\alpha+1/2} R\frac{J_{\beta+1/2}(R|x-y|)}{(R|x-y|)^{\beta+1/2}} - R\frac{J_{\beta+1/2}(R|x-y|)}{(R|x-y|)^{\beta+1/2}}\right| \\
&\quad = R\left|(y/x)^{\alpha+1/2}-1\right|\left|(R|x-y|)^{-\beta-1/2} J_{\beta+1/2}(R|x-y|)\right| \\
&\quad \le c\,(R|x-y|)^{-\beta+1/2}\left|J_{\beta+1/2}(R|x-y|)\right| \\
&\quad \le \begin{cases} c\,(R|x-y|) & \text{if } R|x-y| \le 1, \\ c\,(R|x-y|)^{-\beta} & \text{if } R|x-y| \ge 1. \end{cases}
\end{aligned}$$

□

Since the interest is in the limit $R \to +\infty$, in the sequel R will be assumed large.

Lemma 2.1.4. *There exists a constant c such that for every $\varepsilon < x < \eta$,*

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| \le c\frac{y^{\alpha+\lambda+1/2}}{(1+y)^{\beta+\lambda+1}}.$$

Proof. If $0 < y < 1/R$ then, by Lemma 2.1.3 (1),

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| = \left|S_R^\beta(x,y)\right| \le c\,R^{\alpha-\beta+1/2}y^{2\alpha+1} \le c\,y^{\alpha+\lambda+1/2}.$$

If $1/R < y < \varepsilon/3$ then, by Lemma 2.1.3 (2),

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| = \left|S_R^\beta(x,y)\right| \le c\,R^{-\beta}y^{\alpha+1/2} \le c\,y^{\alpha+\lambda+1/2}.$$

If $\varepsilon/3 < y < \varepsilon/2$ then, by Lemma 2.1.2 and Lemma 2.1.3 (2), and the estimate $\left|J_{\beta+1/2}(z)\right| \le cz^{-1/2}$,

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| \le \left|S_R^\beta(x,y)\right| + \left|T_R^\beta(x,y)\right| \le c\,R^{-\beta}. \tag{2.1.1}$$

If $\varepsilon/2 < y < 2\eta$ then, by Lemma 2.1.2 and Lemma 2.1.3 (4),

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| \le c. \tag{2.1.2}$$

If $2\eta < y < 3\eta$ then, by Lemma 2.1.2 and Lemma 2.1.3 (3),

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| \le \left|S_R^\beta(x,y)\right| + \left|T_R^\beta(x,y)\right| \le c\,R^{-\beta}. \tag{2.1.3}$$

Finally, if $y > 3\eta$ then, by Lemma 2.1.3 (3)

$$\left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| = \left|S_R^\beta(x,y)\right| \le c\,R^{-\beta}y^{\alpha-\beta-1/2}.$$

□

This completes the proof of the equiconvergence theorem. Indeed, the factor $R^{-\beta}$ in the proof of Lemma 2.1.4 suggests the possibility of an improvement. In order to obtain this improvement we shall give more precise estimates in terms of Lorentz norms.

Lemma 2.1.5. *If $\beta > 0$ and $p = (4\alpha + 4) / (2\alpha + 2\beta + 3)$, then*

$$\sup_{\varepsilon<x<\eta} \left\{ \int_0^{\varepsilon/3} \left| S_R^\beta(x,y) - T_R^\beta(x,y) \right| |f(y)| \, dy \right\} \le c \, \|f\|_{p,\infty} ,$$
$$\sup_{\varepsilon<x<\eta} \left\{ \int_{3\eta}^{+\infty} \left| S_R^\beta(x,y) - T_R^\beta(x,y) \right| |f(y)| \, dy \right\} \le c \, R^{-\beta} \, \|f\|_{p,\infty} .$$

Proof. If $\varepsilon < x < \eta$ and $0 < y \le R^{-1} \le \varepsilon/3$, by Lemma 2.1.3 (1) and the duality between $L^{p,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ and $L^{r,1}(\mathbb{R}_+, x^{2\alpha+1}dx)$ with $1/p + 1/r = 1$,

$$\begin{aligned}
\int_0^{1/R} \left| S_R^\beta(x,y) - T_R^\beta(x,y) \right| |f(y)| \, dy &= \int_0^{1/R} \left| S_R^\beta(x,y) \right| |f(y)| \, dy \\
&\le c \, R^{\alpha-\beta+1/2} \int_0^{1/R} |f(y)| \, y^{2\alpha+1} dy \\
&\le c \, R^{\alpha-\beta+1/2} \left\| \chi_{(0,1/R)} \right\|_{r,1} \|f\|_{p,\infty} \\
&\le c \, R^{\alpha-\beta+1/2-(2\alpha+2)/r} \, \|f\|_{p,\infty} \\
&\le c \, \|f\|_{p,\infty} .
\end{aligned}$$

Similarly, if $\varepsilon < x < \eta$, $R^{-1} \le y \le \varepsilon/3$, $1/p + 1/r = 1$, by Lemma 2.1.3 (2),

$$\begin{aligned}
\int_{1/R}^{\varepsilon/3} \left| S_R^\beta(x,y) - T_R^\beta(x,y) \right| |f(y)| \, dy &= \int_{1/R}^{\varepsilon/3} \left| S_R^\beta(x,y) \right| |f(y)| \, dy \\
&\le c \, R^{-\beta} \int_{1/R}^{+\infty} y^{-\alpha-1/2} \, |f(y)| \, y^{2\alpha+1} dy \\
&\le c \, R^{-\beta} \left\| y^{-\alpha-1/2} \chi_{(1/R,+\infty)} \right\|_{r,1} \|f\|_{p,\infty} .
\end{aligned}$$

Moreover,

$$\begin{aligned}
R^{-\beta} \left\| y^{-\alpha-1/2} \chi_{(1/R,+\infty)} \right\|_{r,1} &\le R^{-\beta} \sum_{k=0}^{+\infty} \left(2^k/R\right)^{-\alpha-1/2} \left\| \chi_{\left(2^k/R, 2^{k+1}/R\right)} \right\|_{r,1} \\
&\le c R^{\alpha-\beta+1/2-(2\alpha+2)/r} \sum_{k=0}^{+\infty} 2^{-k(\alpha+1/2-(2\alpha+2)/r)} \\
&\le c \sum_{k=0}^{+\infty} 2^{-\beta k}.
\end{aligned}$$

If $\beta > 0$, then the above series converges.
Finally, if $\varepsilon < x < \eta$, $\eta \le y < +\infty$, $1/p + 1/r = 1$, by Lemma 2.1.3 (3),

$$\begin{aligned}\int_{3\eta}^{+\infty} \left|S_R^\beta(x,y) - T_R^\beta(x,y)\right| |f(y)|\, dy &= \int_{3\eta}^{+\infty} \left|S_R^\beta(x,y)\right| |f(y)|\, dy \\ &\le cR^{-\beta} \int_{\eta}^{+\infty} y^{-\alpha-\beta-3/2} |f(y)|\, y^{2\alpha+1} dy \\ &= cR^{-\beta} \left\|y^{-\alpha-\beta-3/2} \chi_{(\eta,+\infty)}\right\|_{r,1} \|f\|_{p,\infty}.\end{aligned}$$

Moreover, as before,

$$\begin{aligned}\left\|y^{-\alpha-\beta-3/2}\chi_{(\eta,+\infty)}\right\|_{r,1} &\le \sum_{k=0}^{+\infty} \left(2^k\eta\right)^{-\alpha-\beta-3/2} \left\|\chi_{\left(2^k\eta, 2^{k+1}\eta\right)}\right\|_{r,1} \\ &\le c \sum_{k=0}^{+\infty} 2^{-k(\alpha+\beta+3/2-(2\alpha+2)/r)} \\ &= c \sum_{k=0}^{+\infty} 2^{-k(2\beta+1)}.\end{aligned}$$

and the above series converges. □

Theorem 2.1.6. *Assume that $\alpha \ge -1/2$, $\beta \ge 0$ and that one of the following holds:*

(1) $p = q = 1$ *and* $\beta \ge \alpha + 1/2$;

(2) $1 < p \le +\infty$, $\dfrac{4\alpha+4}{2\alpha+2\beta+3} < p < \dfrac{4\alpha+4}{2\alpha-2\beta+1}$, $q \le +\infty$;

(3) $\beta > 0$, $p = \dfrac{4\alpha+4}{2\alpha+2\beta+3}$, $q \le +\infty$;

(4) $p = \dfrac{4\alpha+4}{2\alpha-2\beta+1}$, $q = 1$;

(5) $p = \dfrac{4\alpha+4}{2\alpha+2\beta+3}$, $q = 1$.

If $f(x)$ is in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$, and in the case (3) if $f(x)$ is in the closure of test functions in $L^{p,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$, then the means $S_R^\beta f(x)$ and $T_R^\beta f(x)$ are equiconvergent in $\varepsilon < x < \eta$.

Proof. The cases (1), (2), (4) and (5) follow from Theorem 2.1.1, while the endpoint result (3) follows from Lemma 2.1.5.
If $p = q = 1$ and $\beta \ge \alpha + 1/2$, then

$$\int_0^{+\infty} |f(x)| \frac{x^{2\alpha+1}}{(1+x)^{\beta+\alpha+3/2}}\, dx \le \int_0^{+\infty} |f(x)|\, x^{2\alpha+1} dx = \|f\|_1.$$

Now assume that $(4\alpha+4)/(2\alpha+2\beta+3) < p < (4\alpha+4)/(2\alpha-2\beta+1)$. By the duality between $L^{p,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ and $L^{r,1}(\mathbb{R}_+, x^{2\alpha+1}dx)$ with $1/p+1/r=1$,

$$\int_0^{+\infty} |f(x)| \frac{x^{\alpha+\lambda+1/2}}{(1+x)^{\beta+\lambda+1}}\, dx \le \|f\|_{p,\infty} \left\| \frac{x^{\lambda-\alpha-1/2}}{(1+x)^{\beta+\lambda+1}} \right\|_{r,1}.$$

Write

$$\frac{x^{\lambda-\alpha-1/2}}{(1+x)^{\beta+\lambda+1}} = \chi_{(0,1]} \frac{x^{\lambda-\alpha-1/2}}{(1+x)^{\beta+\lambda+1}} + \chi_{(1,+\infty)} \frac{x^{\lambda-\alpha-1/2}}{(1+x)^{\beta+\lambda+1}}.$$

Then

$$\begin{aligned}\left\|x^{\lambda-\alpha-1/2}\chi_{(0,1)}\right\|_{r,1} &\le \sum_{k=0}^{+\infty} 2^{-k(\lambda-\alpha-1/2)} \left\|\chi_{(2^{-(k+1)},2^{-k})}\right\|_{r,1} \\ &\le c \sum_{k=0}^{+\infty} 2^{-k(\lambda-\alpha-1/2+(2\alpha+2)/r)}.\end{aligned}$$

If $p > (4\alpha+4)/(2\alpha+2\beta+3)$, then the above series converges.
Moreover,

$$\begin{aligned}\left\|x^{-\alpha-\beta-3/2}\chi_{(1,+\infty)}\right\|_{r,1} &\le \sum_{k=0}^{+\infty} 2^{-k(\alpha+\beta+3/2)} \left\|\chi_{(2^k,2^{k+1})}\right\|_{r,1} \\ &\le c \sum_{k=0}^{+\infty} 2^{-k(\alpha+\beta+3/2-(2\alpha+2)/r)}.\end{aligned}$$

If $p < (4\alpha+4)/(2\alpha-2\beta+1)$, then the above series converges.
Similarly, if $p = (4\alpha+4)/(2\alpha-2\beta+1)$ and $q=1$, then if $1/p+1/r=1$

$$\int_0^{+\infty} |f(x)| \frac{x^{\alpha+\lambda+1/2}}{(1+x)^{\beta+\lambda+1}} dx \le \|f\|_{p,1} \left\| \frac{x^{\lambda-\alpha-1/2}}{(1+x)^{\beta+\lambda+1}} \right\|_{r,\infty}$$

and $x^{\lambda-\alpha-1/2}/(1+x)^{\beta+\lambda+1}$ is in $L^{r,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ with $r=(4\alpha+4)/(2\alpha+2\beta+3)$.
In the same way we obtain (5).
Finally, (3) is a consequence of Lemma 2.1.5: indeed, by applying Lemma 2.1.5 and the estimates (2.1.1), (2.1.2), (2.1.3) in Theorem 2.1.1 we get

$$\sup_{\varepsilon<x<\eta} \sup_{R>1} \left\{ \left| S_R^\beta f(x) - T_R^\beta f(x) \right| \right\} \le c \|f\|_{p,\infty}.$$

Therefore, if $f(x)$ is in the closure of test functions in $L^{p,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$, the equiconvergence follows.

□

It can be proved that when $p < (4\alpha+4)/(2\alpha+2\beta+3)$ there exist functions in $L^p(\mathbb{R}_+, x^{2\alpha+1}dx)$ with Bochner-Riesz means of order β diverging everywhere, while when $p \ge (4\alpha+4)/(2\alpha-2\beta+1)$ these means are not even defined as tempered distributions. See [CCTV] and [CTV], and the remarks in the following section.

2.2 Bochner-Riesz means of radial functions in Sobolev spaces

The differential operator $\Delta_\alpha = -\frac{d^2}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx}$ is the radial component of the Laplace operator in dimension $2\alpha+2$. This operator has Fourier-Bessel transform t^2, that is $\mathcal{F}_\alpha \Delta_\alpha g(t) = t^2 \mathcal{F}_\alpha g(t)$. This suggests to define the fractional integral operators $(I+\Delta_\alpha)^{-\gamma/2}$ by

$$(I+\Delta_\alpha)^{-\gamma/2} g(x) = \int_0^{+\infty} \left(1+t^2\right)^{-\gamma/2} \mathcal{F}_\alpha g(t) \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt.$$

If $\mathbb{X}$ is a Banach function space on $0 < x < +\infty$, the Sobolev space $W^\gamma(\mathbb{X})$ is the space of all distributions $f(x) = (I+\Delta_\alpha)^{-\gamma/2} g(x)$, with $g(x)$ in $\mathbb{X}$ and with norm $\|f\|_{W^\gamma(\mathbb{X})} = \|g\|_{\mathbb{X}}$. In particular, if $\Delta_\alpha^j f(x)$ is in $\mathbb{X}$ for all $j = 0, 1, ..., n$, then $f(x)$ is in $W^{2n}(\mathbb{X})$. In what follows, $\mathbb{X}$ will be the Lorentz space $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$. The set of divergence $D(\beta, f)$ of Bochner-Riesz means is defined by

$$D(\beta,f) = \left\{0 \le x < +\infty,\ \lim_{R\to+\infty} \left\{S_R^\beta f(x)\right\} \textit{ does not exists}\right\}.$$

Theorem 2.2.1. *Assume that $\alpha \ge -1/2$, $\beta \ge 0$ and that one of the following holds:*

(1) $p = q = 1$ and $\beta + \gamma \ge \alpha + 1/2 > 0$, or $\beta + \gamma > 0$ if $\alpha = -1/2$;

(2) $1 < p \le +\infty$, $\frac{4\alpha+4}{2\alpha+2\beta+2\gamma+3} < p < \frac{4\alpha+4}{2\alpha-2\beta+1}$, $q \le +\infty$;

(3) $\beta + \gamma > 0$, $p = \frac{4\alpha+4}{2\alpha+2\beta+2\gamma+3}$, $q \le +\infty$;

(4) $p = (4\alpha+4)/(2\alpha+3)$, $q = 1$ and $\beta+\gamma = 0$, or $q \le +\infty$ and $\beta+\gamma > 0$,

(5) $p = \frac{4\alpha+4}{2\alpha-2\beta+1}$, $q = 1$.

Let $f(x) = (I+\Delta_\alpha)^{-\gamma/2} g(x)$ with $g(x)$ in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$, and in the case (3) *with $g(x)$ in the closure of test functions in $L^{p,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$.*

(A) If $0 \le \gamma \le 1/p$, then the divergence set of $S_R^\beta f(x)$ has Hausdorff dimension at most $1-\gamma p$.

(B) If $1/p < \gamma \le (2\alpha+2)/p$, then the divergence set either is empty or reduces to the origin.

(C) If $\gamma > (2\alpha + 2)/p$ or $\gamma = (2\alpha + 2)/p$ and $q = 1$, then convergence holds everywhere.

Observe that when $p < 2$ the critical index for summability β improves when the index of smoothness γ increases, but when $p > 2$ then smoothness does not lower the critical index. The case $p = 1$ and $\gamma = 0$ in (1) is the classical result of Bochner and it follows from Theorem 2.1.1. The case $p = 1$ and $\gamma > 0$ follows from the case $1 < p < +\infty$. Indeed, a function with γ derivatives in $L^{1,1}(\mathbb{R}_+, x^{2\alpha+1}dx)$ has $0 \leq \delta < \gamma$ derivatives in $L^{r,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ with $r = (2\alpha + 2)/(2\alpha + 2 + \delta - \gamma)$. See Lemma 2.2.2 below. It then suffices to consider $p > 1$. Finally, it suffices to prove that $D(\beta, f) \cap (\varepsilon, \eta)$ has Hausdorff dimension at most $1 - \gamma p$ for every $0 < \varepsilon < \eta < +\infty$. In order to prove the theorem, first we shall prove that, under the above assumptions, the $2\alpha + 2$ dimensional means $S_R^\beta f(x)$ are equiconvergent in $\varepsilon < x < \eta$ with the 1 dimensional means $T_R^\beta f(x)$, then we shall prove that the divergence set of $T_R^\beta f(x)$ has dimension at most $1 - \gamma p$. As before, in order to simplify the exposition we split the proof into a series of lemmas.

Lemma 2.2.2. *The following properties hold:*

(1) If $1 < p < r < +\infty$, $1 \leq q \leq +\infty$, and if $\gamma = (2\alpha + 2)(1/p - 1/r)$, then the operator $(I + \Delta_\alpha)^{-\gamma/2}$ is bounded from the Lorentz space $\mathbb{L}^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$ into $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx) \cap L^{r,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$.

*(2) If $1 < p < r < +\infty$, $1 \leq q \leq +\infty$, and if $\gamma - 1 = (2\alpha + 2)(1/p - 1/r)$, then the operator $(d/dx)(I + \Delta_\alpha)^{-\gamma/2}$ is bounded from $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$ into $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$
$\cap L^{r,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$.*

Proof. The properties of fractional integrals for Euclidean Lebesgue spaces are well known and the corresponding properties for Lorentz spaces follow by interpolation. See [St3] and [SW]. Anyhow, let us sketch these proofs. By the addition formula for Bessel functions,

$$\frac{J_\alpha(tx)}{(tx)^\alpha}\frac{J_\alpha(ty)}{(ty)^\alpha} = c(\alpha)\int_0^\pi \frac{J_\alpha\left(t\sqrt{x^2+y^2-2xy\cos(\vartheta)}\right)}{\left(t\sqrt{x^2+y^2-2xy\cos(\vartheta)}\right)^\alpha}\sin^{2\alpha}(\vartheta)\,d\vartheta.$$

This allows to define the convolution

$$f * g(x) = \int_0^{+\infty}\left(c(\alpha)\int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos(\vartheta)}\right)\sin^{2\alpha}(\vartheta)\,d\vartheta\right)g(y)y^{2\alpha+1}dy.$$

With this definition, $\mathcal{F}_\alpha (f * g)(t) = \mathcal{F}_\alpha f(t) \cdot \mathcal{F}_\alpha g(t)$. In particular, the fractional integral operator $(I + \Delta_\alpha)^{-\gamma/2}$ is a convolution operator with kernel

$$G^\gamma(x) = \int_0^{+\infty} \left(1+t^2\right)^{-\gamma/2} \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt,$$

that is,

$$(I + \Delta_\alpha)^{-\gamma/2} g(x) = G^\gamma * g(x)$$

$$= \int_0^{+\infty} \left(c(\alpha) \int_0^\pi G^\gamma \left(\sqrt{x^2+y^2-2xy\cos(\vartheta)} \right) \sin^{2\alpha}(\vartheta)\, d\vartheta \right) g(y) y^{2\alpha+1} dy.$$

The Fourier-Bessel transform of the Gaussian $t \to \exp(-st^2)$ is the Gaussian $x \to (2s)^{-\alpha-1} \exp(-x^2/4s)$, and the kernel $G^\gamma(x)$ is superposition of Gaussians,

$$\begin{aligned} G^\gamma(x) &= \int_0^{+\infty} \left(1+t^2\right)^{-\gamma/2} \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt \\ &= \int_0^{+\infty} \left(\Gamma(\gamma/2)^{-1} \int_0^{+\infty} s^{\gamma/2-1} \exp\left(-s\left(1+t^2\right)\right) ds \right) \frac{J_\alpha(tx)}{(tx)^\alpha} t^{2\alpha+1} dt \\ &= 2^{-\alpha-1} \Gamma(\gamma/2)^{-1} \int_0^{+\infty} s^{\gamma/2-\alpha-2} \exp(-s) \exp\left(-x^2/4s\right) ds. \end{aligned}$$

Then this kernel is positive and smooth in $0 < x < +\infty$, singular at the origin, $G^\gamma(x) \approx cx^{\gamma-2\alpha-2}$, with an exponential decay at infinity. In particular, this kernel is in $L^1(\mathbb{R}_+, x^{2\alpha+1}dx) \cap L^{r,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ with $\gamma = (2\alpha+2)(1-1/r)$. Then the convolution with $G^\gamma(x)$ maps $L^1(\mathbb{R}_+, x^{2\alpha+1}dx)$ into $L^{r,\infty}(\mathbb{R}_+, x^{2\alpha+1}dx)$ and, if $1/r + 1/s = 1$, then it maps $L^{s,1}(\mathbb{R}_+, x^{2\alpha+1}dx)$ into $L^\infty(\mathbb{R}_+, x^{2\alpha+1}dx)$. Finally, (1) follows by interpolation between $(1, r)$ and (s, ∞). The proof of (2) is similar. The operator $(d/dx)(I + \Delta_\alpha)^{-\gamma/2}$ is not a convolution,

$$\begin{aligned} \frac{d}{dx}(I+\Delta_\alpha)^{-\gamma/2} g(x) = \int_0^{+\infty} \Bigg(c(\alpha) \int_0^\pi \frac{dG^\gamma}{dx} \left(\sqrt{x^2+y^2-2xy\cos(\vartheta)} \right) \\ \times \frac{x - y\cos(\vartheta)}{\sqrt{x^2+y^2-2xy\cos(\vartheta)}} \sin^{2\alpha}(\vartheta)\, d\vartheta \Bigg) g(y) y^{2\alpha+1} dy. \end{aligned}$$

However this operator is dominated by the convolution

$$\left| \frac{d}{dx}(I+\Delta_\alpha)^{-\gamma/2} g(x) \right|$$

$$\leq \int_0^{+\infty} \left(c(\alpha) \int_0^\pi \left| \frac{dG^\gamma}{dx} \left(\sqrt{x^2+y^2-2xy\cos(\vartheta)} \right) \right| \sin^{2\alpha}(\vartheta)\, d\vartheta \right) |g(y)|\, y^{2\alpha+1} dy.$$

It follows from the representation of $G^\gamma(x)$ as superposition of Gaussians that $d\,G^\gamma(x)/dx = c\,x\,G^{\gamma-2}(x)$. In particular, $|d\,G^\gamma(x)/dx|$ has the singularity $c\,x^{\gamma-2\alpha-3}$ at the origin and it has an exponential decay at infinity. Hence $(d/dx)(I+\Delta_\alpha)^{-\gamma/2}$ has the same mapping properties of $(I+\Delta_\alpha)^{-(\gamma-1)/2}$. □

Lemma 2.2.3. *If α, β, γ, p, q, and $f(x)$ satisfy the assumptions in Theorem 2.2.1, then for every $\varepsilon < x < \eta$,*

$$\lim_{R\to+\infty}\left\{S_R^\beta f(x) - T_R^\beta f(x)\right\} = 0.$$

Proof. The case $\gamma = 0$ is Theorem 2.1.6. The case $\gamma > 0$ follows from the case $\gamma = 0$ and the imbedding properties of fractional integral operators. Indeed, by Lemma 2.2.2, if $1 < p < r < +\infty$ and $\gamma = (2\alpha+2)(1/p - 1/r)$, and if $g(x)$ is in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$, then $(I+\Delta_\alpha)^{-\gamma/2} g(x)$ is in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx) \cap L^{r,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$. If $p \geq (4\alpha+4)/(2\alpha+2\beta+3)$ then Theorem 2.1.6 applies. If $p < (4\alpha+4)/(2\alpha+2\beta+3)$, define $p < r < (4\alpha+4)/(2\alpha+3)$ so that $\beta = (2\alpha+2)(1/r - 1/2) - 1/2$. Then again Theorem 2.1.6 applies. □

In order to prove (A), it then suffices to show that if $g(x)$ is in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$, then $\lim_{R\to+\infty}\left\{T_R^\beta\left(\chi(I+\Delta_\alpha)^{-\gamma/2} g\right)(x)\right\}$ exists up to a set with Hausdorff dimension at most $1 - \gamma p$.

Lemma 2.2.4. *Let $\left(I+\Delta_{-1/2}\right)^{-\gamma/2}$ and $(I+\Delta_\alpha)^{-\gamma/2}$ be the fractional integral operators associated to the Laplacian in dimension 1 and $2\alpha+2$. Also let $\chi(x)$ be a smooth function with support in $0 < \varepsilon/3 \leq x \leq 3\eta < +\infty$. Finally, let $1 < p < +\infty$, $1 \leq q \leq +\infty$, and $\gamma \geq 0$. Then for every function $g(x)$ in $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$ there exists a function $h(x)$ in $L^{p,q}(\mathbb{R}_+, dx)$ such that*

$$\chi(x)(I+\Delta_\alpha)^{-\gamma/2} g(x) = \left(I+\Delta_{-1/2}\right)^{-\gamma/2} h(x).$$

Proof. The meaning of the lemma is quite simple. On the support of the cut-off $\chi(x)$ the measures dx and $x^{2\alpha+1}dx$ are comparable, hence on this support the associated Sobolev classes coincide. The details of the proof are more complicated. It suffices to prove the boundedness from $L^{p,q}(\mathbb{R}_+, x^{2\alpha+1}dx)$ into $L^{p,q}(\mathbb{R}_+, dx)$ of the operator

$$P^\gamma g(x) = \left(I+\Delta_{-1/2}\right)^{\gamma/2}\chi(x)(I+\Delta_\alpha)^{-\gamma/2} g(x).$$

First assume that $\gamma = 2n$ is an even integer. For $k = 1, 2, 3, \ldots$ and for some rational functions $a_j(x)$ and $b_j(x)$,

$$\frac{d^{2k}}{dx^{2k}} = \left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx} - 1\right)^k + \sum_{j=0}^{2k-1} a_j(x)\frac{d^j}{dx^j}$$

$$\frac{d^{2k-1}}{dx^{2k-1}} = \frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx} - 1\right)^{k-1} + \sum_{j=0}^{2k-2} b_j(x)\frac{d^j}{dx^j}.$$

Hence, expanding $\left(I+\Delta_{-1/2}\right)^n$ and introducing some commutators, one can write

$$P^{2n}g(x)=\sum_{j=0}^{n}A_j(x)\left(I+\Delta_\alpha\right)^{-j}g(x)+\sum_{j=1}^{n}B_j(x)\left(d/dx\right)\left(I+\Delta_\alpha\right)^{-j}g(x).$$

By Lemma 2.2.2, the operators $\left(I+\Delta_\alpha\right)^{-j}$ and $\left(d/dx\right)\left(I+\Delta_\alpha\right)^{-j}$ are bounded on $L^{p,q}\left(\mathbb{R}_+,x^{2\alpha+1}dx\right)$. Since the functions $A_j(x)$ and $B_j(x)$ are smooth with support contained in the support of $\chi(x)$, the associated operators also map $L^{p,q}\left(\mathbb{R}_+,x^{2\alpha+1}dx\right)$ into $L^{p,q}\left(\mathbb{R}_+,dx\right)$. Next assume $\gamma=2n+i\tau$ with n integer and τ real. By Hörmander multiplier theorem, $\left(I+\Delta_{-1/2}\right)^{i\tau}$ is bounded on $L^{p,q}\left(\mathbb{R}_+,dx\right)$ and $\left(I+\Delta_\alpha\right)^{-i\tau}$ is bounded on $L^{p,q}\left(\mathbb{R}_+,x^{2\alpha+1}dx\right)$, hence $P^{2n+i\tau}=\left(I+\Delta_{-1/2}\right)^{i\tau}P^{2n}\left(I+\Delta_\alpha\right)^{-i\tau}$ is bounded from $L^{p,q}\left(\mathbb{R}_+,x^{2\alpha+1}dx\right)$ into $L^{p,q}\left(\mathbb{R}_+,dx\right)$, with a norm of polynomial growth in τ. Finally, the case $0<\gamma<2n$ follows by complex interpolation between $0+i\tau$ and $2n+i\tau$. □

Lemma 2.2.5. *Let $0\le\gamma\le 1/p$, $1<p<+\infty$ and $1\le q\le+\infty$. The (γ,p,q) capacity of a set $X\subseteq\mathbb{R}_+$ is defined by*

$$C\left(\gamma,p,q,X\right)=\inf\left\{\|h\|^p_{L^{p,q}(\mathbb{R}_+,dx)},\ \left(I+\Delta_{-1/2}\right)^{-\gamma/2}h(x)\ge 1\ for\ every\ x\ in\ X\right\}.$$

If $h(x)$ is in $L^{p,q}\left(\mathbb{R}_+,dx\right)$, $1<p<+\infty$, $1\le q<+\infty$, and in the case $q=+\infty$ if $h(x)$ is in the closure of test functions in this space, then the limit $\lim_{R\to+\infty}\left\{T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h\left(x\right)\right\}$ exists finite in a set whose complement has (γ,p,q) capacity 0 if $0\le\gamma p\le 1$.

Proof. Observe that when $\gamma=0$, then the $(0,p,q)$ capacity coincide with Lebesgue measure. Also observe that for every $h(x)$ in $L^{p,q}\left(\mathbb{R}_+,dx\right)$, then

$$C\left(\gamma,p,q,\left\{\left|\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h(x)\right|\ge t\right\}\right)\le t^{-p}\|h\|^p_{L^{p,q}(\mathbb{R}_+,dx)}.$$

The existence of $\lim_{R\to+\infty}\left\{T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h\left(x\right)\right\}$ follows from the boundedness of the maximal operator $\sup_{R>0}\left\{\left|T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h\left(x\right)\right|\right\}$. Since the operators T_R^β and $\left(I+\Delta_{-1/2}\right)^{-\gamma/2}$ commute and since the fractional integral $\left(I+\Delta_{-1/2}\right)^{-\gamma/2}$ is a positive operator, one has

$$\sup_{R>0}\left\{\left|T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h\left(x\right)\right|\right\}\le\left(I+\Delta_{-1/2}\right)^{-\gamma/2}\left(\sup_{R>0}\left\{\left|T_R^\beta h\right|\right\}\right)(x).$$

Moreover,

$$\left\|\sup_{R>0}\left\{\left|T_R^\beta h\right|\right\}\right\|_{L^{p,q}(\mathbb{R}_+,dx)}\le c\|h\|_{L^{p,q}(\mathbb{R}_+,dx)}.$$

The boundedness of the maximal partial sum operator $\sup_{R>0}\left\{\left|T_R^\beta h(x)\right|\right\}$ when $\beta=0$ and $1<p<+\infty$ is the Carleson-Hunt theorem. The case $\beta>0$ is simpler. By Lemma 2.1.2 and the estimate $\left|J_{\beta+1/2}(z)\right|\leq c\,|z|^{-1/2}$,

$$\left|T_R^\beta(x,y)\right|\leq cR\,(1+R\,|x-y|)^{-\beta-1}.$$

This estimate for the kernel of the operator T_R^β imply that when $\beta>0$ the maximal operator $\sup_{R>0}\left\{\left|T_R^\beta h(x)\right|\right\}$ is dominated by the Hardy-Littlewood maximal operator and this last operator is bounded on $L^{p,q}(\mathbb{R}_+,dx)$. For every function $w(x)$ smooth with compact support in $\mathbb{R}_+$ and every x,

$$\lim_{R\to+\infty}\left\{T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}w(x)\right\}=\left(I+\Delta_{-1/2}\right)^{-\gamma/2}w(x).$$

Hence, for every $\varepsilon>0$,

$$\begin{aligned}X(\varepsilon)&=\left\{x:\ \limsup_{R\to+\infty}\left\{\left|T_R^\beta\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h(x)-\left(I+\Delta_{-1/2}\right)^{-\gamma/2}h(x)\right|\right\}>\varepsilon\right\}\\&\subseteq\left\{x:\ \left(I+\Delta_{-1/2}\right)^{-\gamma/2}\sup_{R>0}\left\{\left|T_R^\beta(h-w)(x)\right|\right\}>\varepsilon/2\right\}\\&\quad\cup\left\{x:\ \left(I+\Delta_{-1/2}\right)^{-\gamma/2}|(h-w)(x)|>\varepsilon/2\right\}.\end{aligned}$$

Hence,

$$C(\gamma,p,q,X(\varepsilon))\leq c\,\varepsilon^{-p}\,\|h-w\|_{L^{p,q}(\mathbb{R}_+,dx)}.$$

Since $\|h-w\|_{L^{p,q}(\mathbb{R}_+,dx)}$ can be chosen arbitrarily small, $C(\gamma,p,q,X(\varepsilon))=0$. Finally, since capacity is subadditive, also $C(\gamma,p,q,\cup_{\varepsilon>0}X(\varepsilon))=0$. □

In order to conclude the proof of (A), it suffices to recall that sets with (γ,p,q) capacity 0 have Hausdorff dimension at most $1-\gamma p$. The case $q=p$ of Lebesgue spaces is well known. See e.g. [Z]. The proof for Lorentz spaces is the same.

(B) is a consequence of Theorem 2.1.1 and the following lemma.

Lemma 2.2.6. *(1) If $g(x)$ is in $L^{p,q}(\mathbb{R}_+,x^{2\alpha+1}dx)$ and if $\gamma>1/p$, then the function $(I+\Delta_\alpha)^{-\gamma/2}g(x)$ is bounded in $x>\varepsilon>0$ and Hölder continuous of order $\gamma-1/p$.*

(2) If $f(x)$ is Hölder continuous of positive order, then

$$\lim_{R\to+\infty}\left\{T_R^\beta f(x)\right\}=f(x).$$

Proof. By Lemma 2.2.4, $\chi(x)\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g\left(x\right)=\left(I+\Delta_{-1/2}\right)^{-\gamma/2} h\left(x\right)$ with $h(x)$ in $L^{p,q}\left(\mathbb{R}_{+},dx\right)$, and, by the Sobolev imbedding theorem, functions with γ derivatives in $L^{p,q}\left(\mathbb{R}_{+},dx\right)$ are Hölder continuous of order $\gamma-1/p$. Finally, the pointwise convergence of $T_{R}^{\beta}\left(I+\Delta_{-1/2}\right)^{-\gamma/2} h\left(x\right)$ follows by the Dini criterion for convergence of Fourier series and integrals. □

Finally, in order to prove (C) it suffices to estimate the norm of the linear functional which associates to a function $g(x)$ in $L^{p,q}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$ the number $S_{R}^{\beta}\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g(0)$.

Lemma 2.2.7. *Assume that $\gamma=\left(2\alpha+2\right)/p$ and $q=1$ or $\gamma>\left(2\alpha+2\right)/p$ and $q\leq+\infty$ and that the assumptions of Theorem 2.2.1 holds.*
If $g(x)$ is in $L^{p,q}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$, then

$$\lim_{R\rightarrow+\infty} S_{R}^{\beta}\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g\left(0\right)=\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g(0).$$

Proof. Let

$$S_{R}^{\beta}G^{\gamma}\left(x\right)=\int_{0}^{R}\left(1-\left(t/R\right)^{2}\right)^{\beta}\left(1+t^{2}\right)^{-\gamma/2}\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt.$$

By the convolution structure of Fourier-Bessel expansions,

$$S_{R}^{\beta}\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g\left(0\right)=\int_{0}^{+\infty} S_{R}^{\beta}G^{\gamma}\left(x\right) g(x)\, x^{2\alpha+1}dx.$$

Then, by duality between Lorentz spaces and the Banach-Steinhaus theorem, $\lim_{R\rightarrow+\infty} S_{R}^{\beta}\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g\left(0\right)=\left(I+\Delta_{\alpha}\right)^{-\gamma/2} g(0)$ for every $g(x)$ in the closure of test functions in $L^{p,q}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$ if and only if $\sup_{R>0}\left\|S_{R}^{\beta}G^{\gamma}\right\|_{r,s}<+\infty$, with $1/p+1/r=1$ and $1/q+1/s=1$. Under the stated assumptions on α, β, γ, p, q, the function $G^{\gamma}(x)$ is in $L^{1}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)\cap L^{k,\infty}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$ with $k=(2\alpha+2)/(2\alpha+2-\gamma)$, hence it is also in $L^{r,s}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$. Moreover, the operators S_{R}^{β} are uniformly bounded on $L^{r,s}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$. See [RS] for the case $\beta=0$, and [CTV] for the case $\beta>0$. □

The following remarks show that the ranges of indexes α, β, p, q, in the above theorem are best possible.

Remark 2.2.8. Functions with γ derivatives in $L^{p,\infty}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$ may be infinite on sets with Hausdorff dimension $1-\gamma p$. Moreover, if $\gamma<\left(2\alpha+2\right)/p$ these functions may be unbounded at the origin. Hence the dimension of divergence sets cannot be decreased.

Remark 2.2.9. When $p<2$ the function $f(x)$ which has Fourier-Bessel transform $\left(1+t^{2}\right)^{-\gamma/2}t^{(2\alpha+2)(1/p-1)}$ shows that the indexes for Bochner-Riesz summability in Theorem 2.2.1 are best possible. Indeed, the homogeneous function $t^{(2\alpha+2)(1/p-1)}$ has an homogeneous Fourier-Bessel transform $cx^{-(2\alpha+2)/p}$, so that $\left(I+\Delta_{\alpha}\right)^{\gamma/2}f\left(x\right)=cx^{-(2\alpha+2)/p}$. Hence this function $f(x)$ is in $W^{\gamma}\left(L^{p,\infty}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)\right)$. Let $\varphi(t)$ be a smooth function with $\varphi(t)=1$ if $t\leq 1/3$ and $\varphi(t)=0$ if $t\geq 2/3$. Then

$$\begin{aligned}S_{R}^{\beta}f\left(x\right)=&\int_{0}^{2R/3}\varphi\left(t/R\right)\left(1-\left(t/R\right)^{2}\right)^{\beta}\mathcal{F}_{\alpha}f\left(t\right)\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt\\&+\int_{R/3}^{R}\left(1-\varphi\left(t/R\right)\right)\left(1-\left(t/R\right)^{2}\right)^{\beta}\mathcal{F}_{\alpha}f\left(t\right)\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt.\end{aligned}$$

The multiplier $\varphi\left(t\right)\left(1-t^{2}\right)^{\beta}$ is smooth and for almost every x,

$$\lim_{R\to+\infty}\left\{\int_{0}^{2R/3}\varphi\left(t/R\right)\left(1-\left(t/R\right)^{2}\right)^{\beta}\mathcal{F}_{\alpha}f\left(t\right)\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt\right\}=f(x).$$

Moreover, by the asymptotic expansion of $\mathcal{F}_{\alpha}f\left(t\right)$ and of Bessel functions, for an appropriate function $\Phi\left(t\right)$, phases ζ and ϑ, and constant c, one has

$$\begin{aligned}&\int_{R/3}^{R}\left(1-\varphi\left(t/R\right)\right)\left(1-\left(t/R\right)^{2}\right)^{\beta}\mathcal{F}_{\alpha}f\left(t\right)\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt\\&\qquad\approx R^{(2\alpha+2)(1/p-1/2)-\gamma+1/2}x^{-\alpha-1/2}\int_{1/3}^{1}\Phi\left(t\right)\left(1-t\right)^{\beta}\cos\left(Rxt-\zeta\right)dt\\&\qquad\approx cR^{(2\alpha+2)(1/p-1/2)-\beta-\gamma-1/2}x^{-\alpha-\beta-3/2}\cos\left(Rx-\vartheta\right).\end{aligned}$$

In particular, if $\beta+\gamma=\left(2\alpha+2\right)\left(1/p-1/2\right)-1/2$ this term is bounded but does not converge when $R\to\infty$.

Remark 2.2.10. When $p>2$ an argument of Rubio de Francia shows that the Bochner-Riesz means of index $\beta\leq\left(2\alpha+2\right)\left(1/2-1/p\right)-1/2$ of functions in $W^{\gamma}\left(L^{p}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)\right)$ are not tempered distributions. Suppose the contrary. Then by duality the operator S_{R}^{β} is also bounded from the space of test functions into $W^{-\gamma}\left(L^{r}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)\right)$ with $1/p+1/r=1$, and $\left(I+\Delta_{\alpha}\right)^{-\gamma/2}S_{R}^{\beta}$ is bounded from the space of test functions into $\mathbb{L}^{r}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$. On the other hand, if $f(x)$ is a test function with $\mathcal{F}_{\alpha}f\left(t\right)=\left(1+t^{2}\right)^{\gamma/2}$ for all $t\leq R$, then

$$\left(I+\Delta_{\alpha}\right)^{-\gamma/2}S_{R}^{\beta}f\left(x\right)=\int_{0}^{R}\left(1-\left(t/R\right)^{2}\right)^{\beta}\frac{J_{\alpha}\left(tx\right)}{\left(tx\right)^{\alpha}}t^{2\alpha+1}dt=c\,S_{R}^{\beta}\left(x,0\right).$$

By Lemma 2.1.3 (1), as $|x|\to+\infty$,

$$\left|S_{R}^{\beta}\left(x,0\right)\right|\approx c\,R^{\alpha-\beta+1/2}\left|x\right|^{-\alpha-\beta-3/2}.$$

This function is in $L^{r,\infty}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$, but not in $L^{r}\left(\mathbb{R}_{+},x^{2\alpha+1}dx\right)$.

Remark 2.2.11. Bochner-Riesz means with negative index in Sobolev spaces have been considered in [BC].

Chapter 3

Sturm-Liouville expansions

The results obtained in the previous chapter can be generalised. In particular, in this chapter we state some equiconvergence results between Bochner-Riesz means of expansions in eigenfunctions of suitable Sturm-Liouville operators and we determine the Hausdorff dimension of the divergence set of Bochner-Riesz means of radial functions in Sobolev classes on Euclidean and non-Euclidean spaces.

Hereafter we consider Bessel expansions and expansions in eigenfunctions of Sturm-Liouville operators on $0 < x < +\infty$ of the form

$$\mathcal{L} = -A^{-1}(x)\frac{d}{dx}\left(A(x)\frac{d}{dx}\right).$$

This operator is formally self adjoint with respect to the measure $A(x)dx$. For example, when $A(x) = x^{d-1}$ then $\mathcal{L}$ is the radial component of the Laplacian in $\mathbb{R}^d$, when $A(x) = x^{2\alpha+1}$ then $\mathcal{L}$ is the Bessel operator of the previous section, and when $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, with suitable α and β, then $\mathcal{L}$ is the radial component of the Laplace Beltrami operator on non-compact rank one symmetric spaces. In what follows we assume that the operator $\mathcal{L}$ is a perturbation of the Bessel operator. More precisely, we assume the following:

(1) $A(x)$ is continuous in $0 \le x < +\infty$, positive, non-decreasing and smooth in $0 < x < +\infty$, and $\lim_{x\to+\infty} A(x) = +\infty$.

(2) $A'(x)/A(x)$ is decreasing in $0 < x < +\infty$ and there exists $\alpha > -1/2$ and a smooth odd function $B(x)$ such that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x).$$

In particular, $A(x) \approx c\,x^{2\alpha+1}$ as $x \to 0^+$, and with a change of variable one can assume that $c = 1$. Set $\lim_{x\to+\infty} A'(x)/A(x) = 2\rho$. Observe that when $A(x) = x^{2\alpha+1}$,

that is the case of Fourier-Bessel expansions, then $\rho = 0$.
For every $t \in \mathbb{C}$ the Cauchy problem

$$\begin{cases} -A^{-1}(x)\dfrac{d}{dx}\left(A(x)\dfrac{d}{dx}u(x)\right) = (t^2+\rho^2)\,u(x) \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$$

has a unique solution $\varphi_t(x)$ defined in $0 \le x < +\infty$, and if t is real then $|\varphi_t(x)| \le 1$. Moreover, there exists a function $c(t)$ such that for every test function one can define the Fourier transform and an inversion formula:

$$\mathcal{F}f\,(t) = \int_0^{+\infty} f(y)\,\varphi_t(y)\,A(y)\,dy, \quad f\,(x) = \int_0^{+\infty} \mathcal{F}f\,(t)\,\varphi_t(x)\frac{dt}{2\pi\,|c(t)|^2}.$$

By means of the Liouville transformation $\sqrt{A(x)}u(x) = v(x)$, the equation $\mathcal{L}u(x) = (t^2+\rho^2)\,u(x)$ becomes

$$\left(\frac{d^2}{dx^2} + t^2\right)v(x) = q(x)v(x),$$

with

$$q(x) = \frac{1}{2}\frac{d}{dx}\left(\frac{A'(x)}{A(x)}\right) + \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 - \rho^2.$$

We also assume that

(3) There exists $a \ge 0$ such that

$$q(x) = \frac{a^2 - 1/4}{x^2} + \zeta(x),$$

with $\int_1^{+\infty} |\zeta(x)|\,x\log(x)dx < +\infty$ if $a = 0$, and $\int_1^{+\infty} |\zeta(x)|\,xdx < +\infty$ if $a > 0$. Under these assumptions, when $x \to +\infty$ then $\sqrt{A(x)}\varphi_0(x) \approx cx^{1/2}\log(x)$, or $\sqrt{A(x)}\varphi_0(x) \approx cx^{1/2+b}$ with $|b| = a$ and $b > -1/2$.

(4) If $-1/2 < b < 0$ and $\sqrt{A(x)}\varphi_0(x) \approx cx^{1/2+b}$ as $x \to +\infty$, we assume that $\int_1^{+\infty} |\zeta(x)|\,x^{2|b|+1}dx < +\infty$, while if $b = 0$ and $\sqrt{A(x)}\varphi_0(x) \approx cx^{1/2}$, we assume that $\int_1^{+\infty} |\zeta(x)|\,x\log^2(x)dx < +\infty$.

This last assumption occurs only if $\rho = 0$. It turns out that, as in the case of Bessel expansions, the constant $2\alpha + 2$ plays the role of the dimensions of the space at 0, while when $\rho = 0$, then $2b + 2$ plays the role of the dimensions of the space at $+\infty$.

As we said, an explicit example relevant to the harmonic analysis on hyperbolic spaces is $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$. In this case $\varphi_t(x)$ is a Jacobi function,

$$\varphi_t(x) = F\left(\frac{\alpha+\beta+1-it}{2}; \frac{\alpha+\beta+1+it}{2}; \alpha+1; -\sinh^2(t)\right),$$

and $c(t)$ is the Harish-Chandra function

$$c(t) = \frac{\Gamma(\alpha+1)\,\Gamma(it/2)\,\Gamma((1+it)/2)}{2\sqrt{\pi}\,\Gamma((\alpha-\beta+1+it)/2)\,\Gamma((\alpha+\beta+1+it)/2)}.$$

One can easily check that $A'(x)/A(x)$ is decreasing if and only if $\alpha \geq \beta$, and

$$2\rho = \lim_{x\to+\infty} \frac{A'(x)}{A(x)} = \lim_{x\to+\infty} \frac{(2\alpha+2\beta+2)\cosh^2(x) - 2\beta - 1}{\sinh(x)\cosh(x)} = 2\alpha+2\beta+2.$$

The Bochner-Riesz means with index β of Sturm-Liouville expansions are

$$\begin{aligned} W_R^\beta f(x) &= \int_0^R \left(1-(t/R)^2\right)^\beta \mathcal{F}f(t)\,\varphi_t(x) \frac{dt}{2\pi\,|c(t)|^2} \\ &= \int_0^{+\infty} \left(A(y) \int_0^R \left(1-(t/R)^2\right)^\beta \varphi_t(x)\varphi_t(y) \frac{dt}{2\pi\,|c(t)|^2} \right) f(y)\,dy. \end{aligned}$$

As we said, $\mathcal{L}$ is a perturbation of the Bessel operator, and the eigenfunctions $\varphi_t(x)$ have asymptotic expansions in terms of Bessel functions. This suggests the possibility of equiconvergence between Sturm-Liouville and cosine expansions. The following is a generalization of Theorem 2.1.1.

Theorem 3.0.12. *Let $\lambda = \min\{\alpha+1/2, \beta\}$ and assume that*

$$\int_0^{+\infty} |f(x)| \frac{\sqrt{A(x)}\,x^\lambda}{(1+x)^{\lambda+\beta+1}} dx < +\infty.$$

Then the means $W_R^\beta f(x)$ and $T_R^\beta f(x)$ are equiconvergent in $\varepsilon < x < \eta$ as $R \to +\infty$,

$$\lim_{R\to+\infty} \left\{ \left| W_R^\beta f(x) - T_R^\beta f(x) \right| \right\} = 0.$$

Proof. This theorem contains Theorem 2.1.1. However, the proof of this theorem is based on Theorem 2.1.1.
The case $\beta = 0$ is already in [BG]. The proof of the case $\beta > 0$ is similar. With the notation of the previous chapter, define

$$\begin{aligned} V_R^\beta f(x) &= \sqrt{x^{2\alpha+1}/A(x)}\, S_R^\beta \left(\sqrt{A(y)/y^{2\alpha+1}} f(y) \right)(x) \\ &= \int_0^{+\infty} \left(\sqrt{\frac{A(y)}{A(x)}} \int_0^R \left(1-(t/R)^2\right)^\beta \sqrt{tx} J_\alpha(tx) \sqrt{ty} J_\alpha(ty)\,dt \right) f(y)\,dy. \end{aligned}$$

It suffices to show that the Sturm-Liouville expansions $W_R^\beta f(x)$ are equiconvergent with the Bessel expansions $V_R^\beta f(x)$, and that these Bessel expansions $V_R^\beta f(x)$ are equiconvergent with the trigonometric expansions $T_R^\beta f(x)$. We do this in the next lemmas. □

Lemma 3.0.13. *Let $V_R^\beta(x,y)$ and $W_R^\beta(x,y)$ be the kernels of the operators V_R^β and W_R^β,*

$$V_R^\beta(x,y) = \sqrt{A(y)/A(x)} \int_0^R \left(1-(t/R)^2\right)^\beta \sqrt{tx} J_\alpha(tx) \sqrt{ty} J_\alpha(ty)\,dt,$$

$$W_R^\beta(x,y) = A(y) \int_0^R \left(1-(t/R)^2\right)^\beta \varphi_t(x)\varphi_t(y) \frac{dt}{2\pi\,|c(t)|^2}.$$

Then there exists a constant c such that for every $\varepsilon < x < \eta$,

$$\left| V_R^\beta(x,y) - W_R^\beta(x,y) \right| \le c\sqrt{A(y)}\, y^\lambda\, (1+y)^{-\lambda-\beta-1}.$$

Proof. The following proof relies heavily on [BG]. For every $t \in \mathbb{C} - \{0\}$ the differential equation

$$\mathcal{L}u(x) = \left(t^2+\rho^2\right) u(x)$$

has a unique solution $\Phi_t(x)$ over $(0,+\infty)$ twice continuously differentiable and satisfying the condition at infinity

$$\sqrt{A(x)}\Phi_t(x) = e^{itx}\left(1+\mathcal{R}(t,x)\right),$$

with $\mathcal{R}(t,x) \to 0$ and $\partial\mathcal{R}(t,x)/\partial x \to 0$ as $t \to +\infty$. For any t in $\mathbb{C} - \{0\}$, $\Phi_t(x)$ and $\Phi_{-t}(x)$ are two independent solutions of $\mathcal{L}u = (t^2+\rho^2)\,u$, and the Harish-Chandra function is defined precisely as the coefficient $c(t)$ that realizes the identity

$$\varphi_t(x) = c(t)\,\Phi_t(x) + c(-t)\,\Phi_{-t}(x).$$

Assume $0 < y < x$. For t real, $c(-t) = \overline{c(t)}$. Hence,

$$\begin{aligned} W_R^\beta(x,y) &= A(y) \int_0^R \left(1-(t/R)^2\right)^\beta \varphi_t(x)\,\varphi_t(y)\,\frac{dt}{2\pi\,|c(t)|^2} \\ &= A(y) \int_0^R \left(1-(t/R)^2\right)^\beta \left(c(t)\,\Phi_t(x) + c(-t)\,\Phi_{-t}(x)\right)\varphi_t(y)\,\frac{dt}{2\pi\,|c(t)|^2} \\ &= A(y) \int_{-R}^R \left(1-(t/R)^2\right)^\beta \Phi_{-t}(x)\,\varphi_t(y)\,\frac{dt}{2\pi c(t)}. \end{aligned}$$

The above formula holds also in the particular case $A(x) = x^{2\alpha+1}$, and in this case the functions $\Phi_{-t}(x)$, $\varphi_t(y)$, and $c(t)$ are replaced respectively by

$$\begin{aligned} \boldsymbol{\Phi}_{-t}(x) &= \sqrt{\frac{\pi t}{2}} e^{i\frac{\pi}{4}(2\alpha+1)} x^{-\alpha} H_\alpha^{(2)}(tx), \\ \boldsymbol{\varphi}_t(y) &= 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(ty)}{(ty)^\alpha}, \\ \mathbf{c}(t) &= \frac{2^\alpha \Gamma(\alpha+1)\, e^{-i\frac{\pi}{4}(2\alpha+1)}}{\sqrt{2\pi}\, t^{\alpha+\frac{1}{2}}}. \end{aligned}$$

Here $H_{\alpha}^{(2)}(z)$ is the Bessel function of third kind, or Hankel function of order α. Thus

$$S_R^{\beta}(x,y) = y^{2\alpha+1}\int_{-R}^{R}\left(1-(t/R)^2\right)^{\beta}\mathbf{\Phi}_{-t}(x)\,\boldsymbol{\varphi}_t(y)\,\frac{dt}{2\pi\mathbf{c}(t)}.$$

Therefore

$$V_R^{\beta}(x,y) = \sqrt{x^{2\alpha+1}y^{2\alpha+1}A(y)/A(x)}\int_{-R}^{R}\left(1-(t/R)^2\right)^{\beta}\mathbf{\Phi}_{-t}(x)\,\boldsymbol{\varphi}_t(y)\,\frac{dt}{2\pi\mathbf{c}(t)}.$$

It then follows that

$$W_R^{\beta}(x,y) - V_R^{\beta}(x,y) = \sqrt{A(y)/A(x)}\int_{-R}^{R}\left(1-(t/R)^2\right)^{\beta}$$
$$\times\left(\sqrt{A(y)}\varphi_t(y)\sqrt{A(x)}\frac{\Phi_{-t}(x)}{2\pi c(t)} - \sqrt{y^{2\alpha+1}}\boldsymbol{\varphi}_t(y)\sqrt{x^{2\alpha+1}}\frac{\mathbf{\Phi}_{-t}(x)}{2\pi\mathbf{c}(t)}\right)dt.$$

In our hypotheses, the eigenfunctions $\varphi_t(y)$ and $\boldsymbol{\varphi}_t(y)$ are entire in t, while the functions $\Phi_{-t}(x)/c(t)$ and $\mathbf{\Phi}_{-t}(x)/\mathbf{c}(t)$ are continuous in $\{\operatorname{Im} t \le 0\}$ and analytic in $\{\operatorname{Im} t < 0\}$. See Theorems 1.19 and 2.4 in [BG]. We can therefore estimate the above integral after a modification of the path of integration. If $\omega = \left\{Re^{i\theta} : -\pi \le \theta \le 0\right\}$, then

$$W_R^{\beta}(x,y) - V_R^{\beta}(x,y) = \sqrt{A(y)/A(x)}\int_{\omega}\left(1-(t/R)^2\right)^{\beta}$$
$$\times\left(\sqrt{A(y)}\varphi_t(y)\sqrt{A(x)}\frac{\Phi_{-t}(x)}{2\pi c(t)} - \sqrt{y^{2\alpha+1}}\boldsymbol{\varphi}_t(y)\sqrt{x^{2\alpha+1}}\frac{\mathbf{\Phi}_{-t}(x)}{2\pi\mathbf{c}(t)}\right)dt.$$

Now observe that, under the hypotheses (1), (2), (3) and (4) above, by Theorems 1.2, 1.17 and 2.1 in [BG], the following estimates hold uniformly in $|t| \ge 1$, $x > \varepsilon$, $y > 0$,

$$\sqrt{A(y)}\varphi_t(y) = \sqrt{y^{2\alpha+1}}\boldsymbol{\varphi}_t(y) + R_0(t,y), \quad |R_0(t,y)| \le c\,|t|^{-\alpha-3/2}e^{|\operatorname{Im}(ty)|},$$

$$\sqrt{A(x)}\Phi_{-t}(x) = e^{-itx}\left(1+\mathcal{R}_1(t,x)\right), \quad |\mathcal{R}_1(t,x)| \le c\,|t|^{-1},$$

$$\sqrt{x^{2\alpha+1}}\mathbf{\Phi}_{-t}(x) = e^{-itx}\left(1+\mathcal{R}_2(t,x)\right), \quad |\mathcal{R}_2(t,x)| \le c\,|t|^{-1},$$

$$c(t)^{-1} = \mathbf{c}(t)^{-1}\left(1+E(t)\right), \quad |E(t)| \le c\,|t|^{-1}.$$

Therefore

$$W_R^{\beta}(x,y) - V_R^{\beta}(x,y) = \frac{1}{2\pi}\sqrt{A(y)/A(x)}\left(I_1+I_2+I_3\right),$$

where

$$I_1 = \int_{\omega}\left(1-(t/R)^2\right)^{\beta}\sqrt{y^{2\alpha+1}}\boldsymbol{\varphi}_t(y)\,e^{-itx}\left(\mathcal{R}_1(t,x)-\mathcal{R}_2(t,x)\right)\mathbf{c}(t)^{-1}\,dt,$$
$$I_2 = \int_{\omega}\left(1-(t/R)^2\right)^{\beta}\sqrt{y^{2\alpha+1}}\boldsymbol{\varphi}_t(y)\,e^{-itx}\left(1+\mathcal{R}_1(t,x)\right)\mathbf{c}(t)^{-1}E(t)\,dt,$$
$$I_3 = \int_{\omega}\left(1-(t/R)^2\right)^{\beta}R_0(t,y)\,e^{-itx}\left(1+\mathcal{R}_1(t,x)\right)\mathbf{c}(t)^{-1}\left(1+E(t)\right)dt.$$

The desired estimate now follows by taking absolute values inside the integral sign, along with well known estimates for Bessel functions,

$$\left|\varphi_t(y)\right| \le c\,(1+|ty|)^{-\alpha-1/2}\, e^{|\mathrm{Im}t|y}, \quad y>0,\ t\in\mathbb{C}-\{0\}.$$

Let us show the case of I_1, the other two cases being similar:

$$\begin{aligned}|I_1| &\le c\left(\frac{Ry}{1+Ry}\right)^{\alpha+1/2}\int_0^\pi \left|1-e^{-2i\theta}\right|^\beta e^{-R(x-y)\sin\theta}d\theta \\ &\le c\left(\frac{Ry}{1+Ry}\right)^{\alpha+1/2}(1+R(x-y))^{-\beta-1}.\end{aligned}$$

If, on the other hand, $\varepsilon<x<y$, then switching variables,

$$\begin{aligned}\left|W_R^\beta(x,y)-V_R^\beta(x,y)\right| &= \frac{A(y)}{A(x)}\left|W_R^\beta(y,x)-V_R^\beta(y,x)\right| \\ &\le c\left(\frac{Ry}{1+Ry}\right)^{\alpha+1/2}\sqrt{A(y)/A(x)}\,(1+R(y-x))^{-\beta-1}.\end{aligned}$$

In particular, for every $\varepsilon<x<+\infty$ and $0<y<+\infty$,

$$\left|V_R^\beta(x,y)-W_R^\beta(x,y)\right| \le c\left(\frac{Ry}{1+Ry}\right)^{\alpha+1/2}\frac{\sqrt{A(y)/A(x)}}{(1+R|x-y|)^{\beta+1}}.$$

It remains to show that if $\varepsilon<x<\eta$ and $0<y<+\infty$ and if R is large, then

$$\left(\frac{Ry}{1+Ry}\right)^{\alpha+1/2}\frac{\sqrt{A(y)/A(x)}}{(1+R|x-y|)^{\beta+1}} \le c\,\frac{\sqrt{A(y)}\,y^\lambda}{(1+y)^{\lambda+\beta+1}}.$$

This inequality is elementary. It suffices to consider separately the cases $0<y\le 1/R$, $1/R\le y\le x/2$, $x/2\le y\le 2x$, and $y\ge 2x$. □

Lemma 3.0.14. *For every $\varepsilon<x<\eta$ and every function $f(x)$ satisfying the assumptions of Theorem 3.0.12,*

$$\lim_{R\to+\infty}\left\{\left|V_R^\beta f(x)-T_R^\beta f(x)\right|\right\}=0.$$

Proof. Define

$$\begin{aligned}U_R^\beta f(x) &= \sqrt{x^{2\alpha+1}/A(x)}\,T_R^\beta\left(\sqrt{A(y)/y^{2\alpha+1}}f(y)\right)(x) \\ &= \int_0^{+\infty}\sqrt{\frac{x^{2\alpha+1}A(y)}{y^{2\alpha+1}A(x)}}\left(\frac{2}{\pi}\chi(y)\int_0^R\left(1-(t/R)^2\right)^\beta\cos(tx)\cos(ty)\,dt\right)f(y)dy\end{aligned}$$

and the associated kernel

$$U_R^\beta(x,y)=\sqrt{\frac{x^{2\alpha+1}A(y)}{y^{2\alpha+1}A(x)}}\left(\frac{2}{\pi}\chi(y)\int_0^R\left(1-(t/R)^2\right)^\beta\cos(tx)\cos(ty)\,dt\right).$$

Therefore, under the assumptions of Theorem 3.0.12, the means $V_R^\beta f(x)$ and $U_R^\beta f(x)$ are equiconvergent, since Theorem 2.1.1 applies to the function $\sqrt{A(x)/x^{2\alpha+1}}f(x)$. To conclude the proof, observe that

$$U_R^\beta(x,y) - T_R^\beta(x,y) = \left(\sqrt{\frac{x^{2\alpha+1}A(y)}{y^{2\alpha+1}A(x)}} - 1\right) T_R^\beta(x,y).$$

By the mean value theorem, for $\varepsilon/3 < y < 3\eta$

$$\left|\sqrt{\frac{x^{2\alpha+1}A(y)}{y^{2\alpha+1}A(x)}} - 1\right| \le C\,|x-y|,$$

and since

$$\left|T_R^\beta(x,y)\right| \le cR\left(1+R\,|x-y|\right)^{-\beta-1},$$

we can conclude that

$$\left|U_R^\beta(x,y) - T_R^\beta(x,y)\right| \le C.$$

If g is a smooth function with compact support and $\varepsilon < x < \eta$, then

$$\lim_{R\to+\infty} U_R^\beta g(x) = \lim_{R\to+\infty} T_R^\beta g(x) = g(x),$$

thus for every function f satisfying the assumptions of Theorem 3.0.12 the equiconvergence between $U_R^\beta f(x)$ and $T_R^\beta f(x)$ follows. □

Corollary 3.0.15. *For every $f(x)$ in $L^p(\mathbb{R}_+, A(x)dx)$, $1 \le p \le +\infty$, and every $\varepsilon < x < \eta$ the means $W_R^\beta f(x)$ and $T_R^\beta f(x)$ are equiconvergent under the following assumptions:*

(1) If $\rho = 0$ and $\dfrac{4\alpha+4}{2\alpha+2\beta+3} < p < \dfrac{4b+4}{2b-2\beta+1}$,

(2) If $\rho > 0$ and $\dfrac{4\alpha+4}{2\alpha+2\beta+3} < p \le 2$.

Proof. This follows from the previous theorem by applying Hölder's inequality with $1/p + 1/q = 1$:

$$\int_0^{+\infty} |f(x)| \frac{\sqrt{A(x)}x^\lambda}{(1+x)^{\lambda+\beta+1}}$$
$$\le \left(\int_0^{+\infty} |f(x)|^p A(x)dx\right)^{1/p} \left(\int_0^{+\infty} \left|\frac{x^\lambda}{\sqrt{A(x)}(1+x)^{\lambda+\beta+1}}\right|^q A(x)dx\right)^{1/q}.$$

□

The fractional powers of the differential operator $\mathcal{L} = -A^{-1}(x)\frac{d}{dx}\left(A(x)\frac{d}{dx}\right)$ are defined spectrally by

$$(I+\mathcal{L})^{-\gamma/2} g(x) = \int_0^{+\infty} \left(1+\rho^2+t^2\right)^{-\gamma/2} \mathcal{F}f(t)\,\varphi_t(x)\frac{dt}{2\pi\,|c(t)|^2}.$$

The Sobolev spaces $\mathbb{W}^{\gamma,p}(\mathbb{R}_+, A(x)dx)$, $\gamma \geq 0$ and $1 \leq p \leq +\infty$, are the spaces of all distributions $f(x) = (I+\mathcal{L})^{-\gamma/2} g(x)$, with $g(x)$ in $L^p(\mathbb{R}_+, A(x)dx)$ and with norm $\|f\|_{\mathbb{W}^{\gamma,p}} = \|g\|_{L^p}$. In particular, $f(x)$ is in $\mathbb{W}^{2n,p}(\mathbb{R}_+, A(x)dx)$ if and only if $\mathcal{L}^j f(x)$ is in $L^p(\mathbb{R}_+, A(x)dx)$ for all $j = 0, 1, ..., n$.

The set of divergence of Bochner-Riesz means is defined by

$$D(\beta, f) = \left\{0 \leq x < +\infty,\ \lim_{R\to+\infty}\left\{W_R^\beta f(x)\right\}\ \textit{does not exists}\right\}.$$

In what follows we shall assume $\rho > 0$, although we suspect that the next result still holds when $\rho = 0$: indeed, this is the case when $A(x) = x^{2\alpha+1}$, which corresponds to the Fourier-Bessel expansions, as shown in Theorem 2.2.1.

Theorem 3.0.16. *Let $f(x) = (I+\mathcal{L})^{-\gamma/2} g(x)$ with $g(x)$ in $L^p(\mathbb{R}_+, A(x)dx)$, $1 \leq p \leq +\infty$, $\beta \geq 0$, and assume that $\rho > 0$ and*

$$\frac{4\alpha+4}{2\alpha+2\beta+2\gamma+3} < p \leq 2.$$

(A) If $0 \leq \gamma \leq 1/p$, then the divergence set of $W_R^\beta f(x)$ has Hausdorff dimension at most $1-\gamma p$.

(B) If $1/p < \gamma \leq (2\alpha+2)/p$, then the divergence set either is empty or reduces to the origin.

(C) If $\gamma > (2\alpha+2)/p$ and $1 \leq p \leq 2$, then convergence holds everywhere.

Proof. Observe that, as in Theorem 2.2.1 with $p \leq 2$, if the smoothness index γ increases, then the critical index for summability β decreases. In order to prove the theorem, it suffices to show that, under the above assumptions, the means $W_R^\beta f(x)$ are equiconvergent in $\varepsilon < x < \eta$ with the one-dimensional means $T_R^\beta f(x)$, that functions with γ derivatives in $L^p(\mathbb{R}_+, A(x)dx)$ can be defined up to sets with Hausdorff dimension $1-\gamma p$, and that the divergence set of $T_R^\beta f(x)$ has dimension at most $1-\gamma p$. □

Lemma 3.0.17. *Assume that $\rho > 0$ and $(4\alpha+4)/(2\alpha+2\beta+2\gamma+3) < p \leq 2$. If $f(x) = (I+\mathcal{L})^{-\gamma/2} g(x)$ with $g(x)$ in $L^p(\mathbb{R}_+, A(x)dx)$, then the means $W_R^\beta f(x)$ and $T_R^\beta f(x)$ are equiconvergent in $0 < \varepsilon < x < \eta < +\infty$.*

Proof. The case $\gamma = 0$ is Corollary 3.0.15. The case $\gamma > 0$ follows from the case $\gamma = 0$ and the imbedding properties of the fractional integral operators. Indeed, if $1 = p < q < +\infty$ and if $\gamma > (2\alpha + 2)(1/p - 1/q)$, or if $1 < p < q < +\infty$ and if $\gamma \geq (2\alpha + 2)(1/p - 1/q)$, then the operator $(I + \mathcal{L}_\alpha)^{-\gamma/2}$ is bounded from $L^p(\mathbb{R}_+, A(x)dx)$ into $L^p(\mathbb{R}_+, A(x)dx) \cap L^q(\mathbb{R}_+, A(x)dx)$. This is well-known in the Euclidean case (see [St3]); for a proof of this property for Sturm-Liouville expansions see [BX]. $\square$

Now the proof of Theorem 3.0.16 is analogous to that of Theorem 2.2.1 with the operator $\mathcal{L}_\alpha$ in place of Δ_α. We omit the details.
As we said, we suspect that the conclusions of Theorem 3.0.16 still hold when $\rho = 0$ and $\dfrac{4\alpha + 4}{2\alpha + 2\beta + 2\gamma + 3} < p < \dfrac{4b + 4}{2b - 2\beta + 1}$.

Part II

Hardy spaces

Chapter 4

Local Hardy type spaces

In this chapter we define and develop the basic theory of a *local* Hardy space $\mathfrak{h}^1(M)$ in the setting of a measured metric space (M, d, μ) possessing the local doubling property, the approximate midpoint property and satisfying the uniform ball size condition.

This chapter is rather long. It may be helpful to briefly describe its content here. The basic geometric assumptions on the measured metric space M are given in Section 4.1. The local Hardy space $\mathfrak{h}^1(M)$ is defined in Section 4.2. In fact, a two-parameter family of spaces $\mathfrak{h}_b^{1,p}(M)$ is introduced: here b is a positive scale parameter, and p is an index in $(1, \infty]$. Under a natural assumption on the scale parameter b, depending on geometric constants arising in the definition of the approximate midpoint property, the spaces $\mathfrak{h}_b^{1,p}(M)$ are independent of b and p, and will therefore be denoted simply by $\mathfrak{h}^1(M)$. To prove that $\mathfrak{h}^1(M)$ generalises the space introduced by Taylor in [T3], in Section 4.3 we prove that $\mathfrak{h}^1(M)$ admits a "ionic" decomposition. In fact, our version of the ionic decomposition is more general than Taylor's for two reasons: it works in a setting much larger than Taylor's, and it involves more general ions. In Section 4.4 we define the local $\mathfrak{bmo}(M)$ space, and in Section 4.5 we show that the dual of $\mathfrak{h}^1(M)$ is isomorphic to $\mathfrak{bmo}(M)$. The proof of the duality is rather standard. One of the main result of this chapter is that $L^p(M)$, $1 < p < 2$, is the complex interpolation space between $\mathfrak{h}^1(M)$ and $L^2(M)$. This is proved in Section 4.7 by adapting the original argument of Fefferman and Stein [FeS], and uses a preliminary relative distributional inequality, proved in Section 4.6. This inequality appears to be rather nontrivial in our setting. In Section 4.7 we also show that the interpolation result is false if we take the local Hardy space $H_1^1(M)$

introduced by Carbonaro, Mauceri, and Meda in [CMM1].

Applications to the study of the translated Riesz transform and of spectral multipliers of the Laplace–Beltrami operator on manifolds with Ricci curvature bounded from below and positive injectivity radius will be given in Section 4.9.

4.1 Notation, terminology and geometric assumptions

Suppose that (M, d, μ) is a measured metric space, and denote by $\mathcal{B}$ the family of all balls on M. We assume that $\mu(M) > 0$ and that every ball has finite measure. For each B in $\mathcal{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, we denote by $k\,B$ the ball with centre c_B and radius $k\,r_B$. For each s in $\mathbb{R}^+$, we denote by $\mathcal{B}_s$ the family of all balls B in $\mathcal{B}$ such that $r_B \leq s$.

We now introduce some properties which (M, d, μ) may or may not have.

We say that M possesses the *local doubling property* (LDP) if for every s in $\mathbb{R}^+$ there exists a constant D_s such that

$$\mu\big(2B\big) \leq D_s\, \mu\big(B\big) \qquad \forall B \in \mathcal{B}_s.$$

Remark 4.1.1. The LDP implies that for each $\tau \geq 1$ and for each s in $\mathbb{R}^+$ there exists a constant C such that

$$\mu\big(B'\big) \leq C\, \mu(B) \tag{4.1.1}$$

for each pair of balls B and B', with $B \subset B'$, B in $\mathcal{B}_s$, and $r_{B'} \leq \tau\, r_B$. We shall denote by $D_{\tau,s}$ the smallest constant for which (4.1.1) holds. In particular, if (4.1.1) holds (with the same constant) for all balls B in $\mathcal{B}$, then μ is doubling and we shall denote by $D_{\tau,\infty}$ the smallest constant for which (4.1.1) holds.

We say that M possesses the *approximate midpoint property* (AMP) if there exist R_0 in $[0, \infty)$ and β in $[1/2, 1)$ such that for every pair of points x and y in M with $d(x, y) > R_0$ there exists a point z in M such that $d(x, z) < \beta\, d(x, y)$ and $d(y, z) < \beta\, d(x, y)$. This is clearly equivalent to the requirement that there exists a ball B containing x and y such that $r_B < \beta\, d(x, y)$.

If M is a measured metric space for which $\beta = 1/2$ and $R_0 = 0$, then we say that M possesses the midpoint property (MP). Typically graphs enjoy the AMP, but rarely a segment in a graph has a midpoint. On the other hand, every connected Riemannian manifold possesses the MP.

We say that M satisfies the *uniform ball size condition* (UBSC) if

$$\inf\big\{\mu\big(B(p,r)\big) : p \in M\big\} > 0 \qquad \text{and} \qquad \sup\big\{\mu\big(B(p,r)\big) : p \in M\big\} < \infty.$$

We say that M possesses the *isoperimetric property* (IP) if there exist κ_0 and C in $\mathbb{R}^+$ such that for every bounded open set A

$$\mu\Big(\big\{x \in A : d(x, A^c) \leq \kappa\big\}\Big) \geq C\,\kappa\,\mu(A) \qquad \forall \kappa \in (0, \kappa_0]. \tag{4.1.2}$$

Hereafter in this chapter we assume that M possesses the local doubling property (LDP), the approximate midpoint property (AMP) and satisfies the uniform ball size condition (UBSC).

Given a positive number η, a set of points $\mathfrak{M}$ in M is a η-discretisation of M if it is maximal with respect to the following property:

$$\min\{d(z,w) : z, w \in \mathfrak{M}, z \neq w\} > \eta \qquad \text{and} \qquad d(\mathfrak{M}, x) \leq \eta \qquad \forall x \in M.$$

It is straightforward to show that η-discretisations exist for every η. Given a ball B in M, denote by $\mathfrak{M}_B$ the set of all points z in $\mathfrak{M}$ such that $B(z, 2\eta) \cap B \neq \emptyset$, and by $\sharp\mathfrak{M}_B$ its cardinality.

Lemma 4.1.2. *Let (M, d, μ) be a measured metric space with LDP, AMP and UBSC. Suppose that c is a positive number and let $\mathfrak{M}$ be a $c/2$-discretisation of M. The following hold:*

(i) the family $\{B(z,c) : z \in \mathfrak{M}\}$ is a locally uniformly finite covering of M;

(ii) for every $b > c$ there exists a constant C, depending only on c and b, such that for every ball B of radius b

$$\sharp\mathfrak{M}_B \leq C\,\mu(B).$$

Proof. First we prove (i). For each $x \in M$ set

$$\mathfrak{M}_x := \{z \in \mathfrak{M} : x \in B(z,c)\}.$$

We need to show that $\sharp\mathfrak{M}_x$ is uniformly bounded with respect to x. For z in $\mathfrak{M}_x$ we have

$$B(z,c) \subset B(x,2c) \subset B(z,3c).$$

By the LDP (4.1.1)

$$\mu(B(z,3c)) \leq D_{12,c/4}\,\mu(B(z,c/4)),$$

whence

$$\mu(B(z,c/4)) \geq \frac{1}{D_{12,c/4}}\,\mu(B(x,2c)).$$

Since $B(z,c/4) \subset B(x,2c)$ and the balls $B(z,c/4)$ are pairwise disjoint,

$$\begin{aligned}\mu(B(x,2c)) &\geq \mu\Big(\bigcup_{z\in\mathfrak{M}_x} B(z,c/4)\Big) = \sum_{z\in\mathfrak{M}_x} \mu(B(z,c/4)) \\ &\geq \frac{\sharp\mathfrak{M}_x}{D_{12,c/4}}\,\mu(B(x,2c)),\end{aligned}$$

whence $\sharp\mathfrak{M}_x \leq D_{12,c/4}$, as required.

Now we prove (ii). First we show that $\sharp\mathfrak{M}_B \leq C\,\mu(B')$, where B' denotes the ball with centre c_B and radius $b+2c$.

Indeed, by (i) there exists a positive integer K such that

$$\sum_{z\in\mathfrak{M}_B} \mathbf{1}_{B(z,c)} \leq K\,\mathbf{1}_{B'}.$$

Integrating both sides we see that

$$\sum_{z\in\mathfrak{M}_B} \mu(B(z,c)) \leq K\,\mu(B').$$

By the UBSC there exists a positive constant δ, depending on c, such that $\mu(B(z,c)) \geq \delta$ for every z in M. Therefore

$$\delta\,\sharp\mathfrak{M}_B \leq \sum_{z\in\mathfrak{M}_B} \mu(B(z,c)) \leq K\,\mu(B').$$

By the LDP (4.1.1)

$$\mu(B') \leq D_{3,b}\;\mu(B),$$

whence $\sharp\mathfrak{M}_B \leq C\,\mu(B)$, with C depending only on b and c. □

4.2 The local Hardy space $\mathfrak{h}^1(M)$

Definition 4.2.1. Suppose that p is in $(1,\infty]$ and let p' be the index conjugate to p. Suppose that b is a positive number. A *standard p-atom at scale b* is a function a in $L^1(M)$ supported in a ball B in $\mathcal{B}_b$ satisfying the following conditions:

(i) *size condition*:
$\|a\|_\infty \leq \mu(B)^{-1}$ if $p=\infty$ and $\|a\|_p \leq \mu(B)^{-1/p'}$ if $p\in(1,\infty)$;

(ii) *cancellation condition*:
$\int_B a\,\mathrm{d}\mu = 0.$

A *global p-atom at scale b* is a function a in $L^1(M)$ supported in a ball B of radius *exactly equal to b* satisfying the size condition above (but possibly not the cancellation condition!). Standard and global p-atoms will be referred to simply as *p-atoms*.

Definition 4.2.2. Let b be a positive number. The *local atomic Hardy space* $\mathfrak{h}_b^{1,p}(M)$ is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} \lambda_j \, a_j, \tag{4.2.1}$$

where the a_j's are p-atoms at scale b and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. The norm $\|f\|_{\mathfrak{h}_b^{1,p}}$ of f is the infimum of $\sum_{j=1}^{\infty} |\lambda_j|$ over all decompositions (4.2.1) of f.

We shall prove that $\mathfrak{h}_b^{1,p}(M)$ is independent of p and b, and the space $\mathfrak{h}_1^{1,p}(M)$ will therefore be denoted simply by $\mathfrak{h}^1(M)$.

Note that if M is the Euclidean space, then $\mathfrak{h}^1(M)$ **is strictly contained** in the local Hardy space of Goldberg [G]. Indeed, global atoms in the Goldberg space may have support contained in balls of any radius, whereas in our case their support is contained in balls of radius exactly equal to one.

We shall prove that in the case where M is a Riemannian manifold with *strongly bounded geometry* the space $\mathfrak{h}^1(M)$ agrees with the local Hardy space recently introduced by M. Taylor [T3]. All Riemannian manifolds possess the AMP. The subclass of those which possess the LDP and satisfy the UBSC is by far larger than the class of manifolds considered by Taylor. Thus, our theory extends that of Taylor significantly.

The definition of the space $\mathfrak{h}^1(M)$ is similar to that of the atomic Hardy space $H^1(M)$, introduced by A. Carbonaro, G. Mauceri, and S. Meda [CMM1, CMM2], the only difference being that atoms are only standard atoms. As a consequence, functions in $H^1(M)$ have vanishing integral, a property not enjoyed by functions in $\mathfrak{h}^1(M)$. Thus, trivially, $H^1(M)$ is properly contained in $\mathfrak{h}^1(M)$. Note, however, that some key properties of $H^1(M)$, such as the fact that $L^p(M)$, $1 < p < 2$, is an interpolation space between $H^1(M)$ and $L^2(M)$, have been proved only under the assumption that (M, d, μ) possesses the isoperimetric property (4.1.2). One of the advantages of considering $\mathfrak{h}^1(M)$ is that we shall be able to prove a similar interpolation property without assuming the isoperimetric property. However, notice that we assume the UBSC, which is not needed for $H^1(M)$.

The following lemma produces an economical decomposition of atoms supported in "big" balls as finite linear combinations of atoms supported in smaller balls. This result extends to global atoms the economical decomposition for standard atoms proved in [MMV3, Lemma 6.1]; see also [CMM1, Prop 4.3 (i)] for a "less economical" decomposition. We observe that the method of [CMM1, Prop 4.3 (i)] gives a worse control of the norm of atoms at big scales in terms of Hardy spaces at smaller scales than that adopted in [MMV3, Lemma 6.1]. However, it does not require the UBSC, and uses only the LDP and the AMP. It is worth investigating to what extent we can develop our theory without assuming the UBSC.

Lemma 4.2.3. *Suppose that p is in $(1,\infty]$, b and c are numbers such that $R_0/(1-\beta) < c < b$ (R_0 and β are as in the definition of the AMP). Then there exist a constant C and a nonnegative integer N, depending only on M, b and c, such that for each ball B of radius b and each p-atom a at scale b supported in B there exist at most N p-atoms at scale c, $a_1,\ldots,a_N$, and N constants $\lambda_1,\ldots,\lambda_N$ such that $|\lambda_j| \leq C$,*

$$a = \sum_{j=1}^{N} \lambda_j a_j \qquad \text{and} \qquad \|a\|_{\mathfrak{h}_c^{1,p}} \leq C.$$

Proof. The result is known for standard p-atoms (see [CMM1, Prop 4.3 (i)]). Thus, we need to prove the lemma only for global p-atoms.

Suppose that a is a global p-atom. Let $\mathfrak{M}$ be a $c/2$-discretisation of M. By Lemma 4.1.2(i) the family $\{B(z,c) : z \in \mathfrak{M}\}$ is a covering of M that is locally uniformly finite. Denote by $z_1,\ldots,z_N$ the set of all points z in $\mathfrak{M}$ such that $B(z,c) \cap B \neq \emptyset$. Note that $N \leq C\,\mu(B)$ by Lemma 4.1.2(ii). Denote by B_j the ball with centre z_j and radius c, and by $\{\psi_j : j = 1,\ldots,N\}$ the partition of unity on B subordinated to the covering $\{B_j : j = 1,\ldots,N\}$, defined by

$$\psi_j = \frac{\mathbf{1}_{B_j}}{\sum_{k=1}^{N} \mathbf{1}_{B_k}}.$$

Denote by a_j the function $a\,\psi_j$. Thus,

$$a = \sum_{j=1}^{N} \psi_j\, a = \sum_{j=1}^{N} a_j.$$

Next, for every j in $\{1,\ldots,N\}$ define

$$b_j = \frac{a_j}{\|a_j\|_p\, \mu(B_j)^{1/p'}},$$

where p' is the index conjugate to p. Clearly, b_j is a global p-atom at scale c. Therefore $\|b_j\|_{\mathfrak{h}_c^{1,p}} \leq 1$, whence

$$\|a_j\|_{\mathfrak{h}_c^{1,p}} \leq \|a_j\|_p\, \mu(B_j)^{1/p'}. \tag{4.2.2}$$

We have the decomposition

$$a = \sum_{j=1}^{N} \lambda_j \, b_j,$$

with $\lambda_j = \|a_j\|_p \, \mu(B_j)^{1/p'}$. By the LDP there exists a constant C, depending only on M, b and c, such that $\mu(B_j) \leq C \, \mu(B)$, so that

$$|\lambda_j| \leq \|a\|_p \, \mu(B_j)^{1/p'} \leq \big(\mu(B_j)/\mu(B)\big)^{1/p'} \leq C^{1/p'}.$$

Now we use (4.2.2) and the fact that $\mu(B_j) \leq C$ for $j = 1, \ldots, N$ by the UBSC, and obtain that

$$\begin{aligned} \|a\|_{\mathfrak{h}_c^{1,p}} &\leq \sum_{j=1}^{N} \|a_j\|_{\mathfrak{h}_c^{1,p}} \\ &\leq \sum_{j=1}^{N} \|a_j\|_p \, \mu(B_j)^{1/p'} \\ &\leq C \sum_{j=1}^{N} \|a_j\|_p. \end{aligned}$$

Then we use Hölder's inequality, the fact that $N \leq C \, \mu(B)$ and the bounded overlap property of the family $\{B_j : j = 1, \ldots, N\}$ to conclude that

$$\begin{aligned} \|a\|_{\mathfrak{h}_c^{1,p}} &\leq C \, N^{1/p'} \Big(\sum_{j=1}^{N} \|a_j\|_p^p \Big)^{1/p} \\ &\leq C \, \mu(B)^{1/p'} \, \|a\|_p \\ &\leq C. \end{aligned} \tag{4.2.3}$$

The last inequality holds because a is a global p-atom supported in B. □

Proposition 4.2.4. *Suppose that p is in $(1, \infty]$ and b and c are in $\mathbb{R}^+$ with $R_0/(1-\beta) < c < b$ (R_0 and β are as in the definition of the AMP). A function f is in $\mathfrak{h}_c^{1,p}(M)$ if and only if f is in $\mathfrak{h}_b^{1,p}(M)$. Furthermore, there exist constants C_1 and C_2 such that*

$$C_1 \, \|f\|_{\mathfrak{h}_b^{1,p}} \leq \|f\|_{\mathfrak{h}_c^{1,p}} \leq C_2 \, \|f\|_{\mathfrak{h}_b^{1,p}} \qquad \forall f \in \mathfrak{h}_c^{1,p}(M).$$

Proof. We begin by showing that $\mathfrak{h}_c^{1,p}(M) \subset \mathfrak{h}_b^{1,p}(M)$. If a is a p-atom at scale c with support contained in B, then

$$a \left(\frac{\mu(B)}{\mu((b/c)B)} \right)^{1/p'}$$

is a p-atom at scale b, and

$$\begin{aligned}\|a\|_{\mathfrak{h}_b^{1,p}} &\leq \Big(\frac{\mu((b/c)B)}{\mu(B)}\Big)^{1/p'} \\ &\leq D_{b/c,c}^{1/p'}.\end{aligned}$$

This implies that if f belongs to $\mathfrak{h}_c^{1,p}(M)$, then f is in $\mathfrak{h}_b^{1,p}(M)$ and $\|f\|_{\mathfrak{h}_b^{1,p}} \leq D_{b/c,c}^{1/p'} \|f\|_{\mathfrak{h}_c^{1,p}}$.

The reverse inclusion follows directly from Lemma 4.2.3. □

Remark 4.2.5. Suppose that p is in $(1,\infty]$. Then for every b and c such that $R_0/(1-\beta) < c < b$ the spaces $\mathfrak{h}_b^{1,p}(M)$ and $\mathfrak{h}_c^{1,p}(M)$ are isomorphic (in fact, they contain the same functions) by Proposition 4.2.4. Hereafter we shall denote the space $\mathfrak{h}_1^{1,p}(M)$, endowed with any of the equivalent norms defined above, simply by $\mathfrak{h}^{1,p}(M)$.

In Section 4.5 we shall prove that $\mathfrak{h}^{1,p}(M)$ does not depend on the parameter p in $(1,\infty)$, and we shall denote all the spaces $\mathfrak{h}^{1,p}(M)$ simply by $\mathfrak{h}^1(M)$.

4.3 The local ionic space $\mathfrak{h}_I^1(M)$

In this section we show that $\mathfrak{h}^1(M)$ admits a "ionic decomposition". Specifically, we shall define a "ionic" Hardy space $\mathfrak{h}_I^1(M)$. The space $\mathfrak{h}_I^1(M)$ is defined much as $\mathfrak{h}^1(M)$, but with ions in place of atoms. It will be clear from the definition that every atom is an ion, but not conversely. In fact, we shall consider a one-parameter family of different types of ions. When this parameter is equal to one, and M is a Riemannian manifold with strongly bounded geometry (in a sense explained later), then $\mathfrak{h}_I^1(M)$ is the local Hardy space introduced by Taylor in [T3].

Definition 4.3.1. Suppose that p is in $(1,\infty]$ and let p' be the index conjugate to p. Suppose that α is in $\mathbb{R}^+$. A (p,α)-*ion* is a function g in $L^1(M)$ supported in a ball B of radius r with the following properties:

(i) $\|g\|_\infty \leq \mu(B)^{-1}$ if $p=\infty$ and $\|g\|_p \leq \mu(B)^{-1/p'}$ if $p \in (1,\infty)$;

(ii) $\Big|\int_B g \,\mathrm{d}\mu\Big| \leq r^\alpha$.

A $(p,1)$-ion will be simply called a p-ion.

Note that Taylor considered ∞-ions only.

Definition 4.3.2. Suppose that b and α are in $\mathbb{R}^+$. The *local ionic Hardy space* $\mathfrak{h}^{1,p,\alpha}_{I,b}(M)$ is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} \mu_j \, g_j, \tag{4.3.1}$$

where the g_j's are (p,α)-ions *supported in balls of radius at most* b and $\sum_{j=1}^{\infty} |\mu_j| < \infty$. The norm $\|f\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}}$ of f is the infimum of $\sum_{j=1}^{\infty} |\mu_j|$ over all decompositions (4.3.1) of f.

If $\alpha = 1$, then we denote $\mathfrak{h}^{1,p,\alpha}_{I,b}(M)$ simply by $\mathfrak{h}^{1,p}_{I,b}(M)$.

We shall prove that the spaces $\mathfrak{h}^{1,p,\alpha}_{I,b}(M)$ are, in fact, independent of α. Indeed, we shall show that all these spaces coincide with the atomic spaces $\mathfrak{h}^{1,p}_{b}(M)$ and the corresponding norms are equivalent. We shall make use of the following remark.

Remark 4.3.3. If $\alpha \geq 1$, then it is easy to show that $\mathfrak{h}^{1,p,\alpha}_{I,b}(M) \subset \mathfrak{h}^{1,p}_{I,b}(M)$ and $\|f\|_{\mathfrak{h}^{1,p}_{I,b}} \leq \|f\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}}$ for every f in $\mathfrak{h}^{1,p,\alpha}_{I,b}(M)$.

Indeed, consider a (p,α)-ion g supported in a ball of radius r. If $r \geq 1$, then the size condition implies that

$$\Big|\int_B g \,\mathrm{d}\mu\Big| \leq \|g\|_p \, \mu(B)^{1/p'} \leq 1 \leq r.$$

If $r < 1$, then

$$\Big|\int_B g \,\mathrm{d}\mu\Big| \leq r^\alpha \leq r.$$

Hence g is a p-ion. The inclusion $\mathfrak{h}^{1,p,\alpha}_{I,b}(M) \subset \mathfrak{h}^{1,p}_{I,b}(M)$ and the desired norm inequality follow.

Theorem 4.3.4. *Suppose that* $p \in (1,\infty]$, $\alpha \in \mathbb{R}^+$ *and* $b > R_0/(1-\beta)$. *The spaces* $\mathfrak{h}^{1,p,\alpha}_{I,b}(M)$ *and* $\mathfrak{h}^{1,p}_{b}(M)$ *coincide as vector spaces. Furthermore, there exist constants* C_1 *and* C_2 *such that*

$$C_1 \, \|f\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}} \leq \|f\|_{\mathfrak{h}^{1,p}_{b}} \leq C_2 \, \|f\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}} \qquad \forall f \in \mathfrak{h}^{1,p}_{b}(M).$$

Proof. Fix $\alpha > 0$. First we prove that $\mathfrak{h}^{1,p}_{b}(M) \subset \mathfrak{h}^{1,p,\alpha}_{I,b}(M)$, by showing that each p-atom at scale b is a multiple of a (p,α)-ion supported in the same ball.

Indeed, clearly each standard p-atom is a (p,α)-ion. Now, suppose that a is a global p-atom supported in a ball B of radius b. Then the size condition implies that

$$\Big|\int_B a \,\mathrm{d}\mu\Big| \leq \|a\|_p \, \mu(B)^{1/p'} \leq 1. \tag{4.3.2}$$

If $b \geq 1$, then $\left|\int_B a \, \mathrm{d}\mu\right| \leq b^\alpha$ and a is a (p,α)-ion at scale b. If $b < 1$, then it is clear that $b^\alpha a$ is a (p,α)-ion at scale b. Therefore $\|a\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}} \leq 1/b^\alpha$. Thus, $\mathfrak{h}^{1,p}_b(M) \subset \mathfrak{h}^{1,p,\alpha}_{I,b}(M)$ and

$$\|f\|_{\mathfrak{h}^{1,p,\alpha}_{I,b}} \leq \max(1, b^{-\alpha}) \, \|f\|_{\mathfrak{h}^{1,p}_b} \qquad \forall f \in \mathfrak{h}^{1,p}_b(M).$$

To prove the reverse inclusion, we show that there exists a constant C such that each (p,α)-ion g supported in a ball $B \in \mathcal{B}_b$ is in $\mathfrak{h}^{1,p}_b(M)$ and $\|g\|_{\mathfrak{h}^{1,p}_b} \leq C$.

By Remark 4.3.3 if $\alpha \geq 1$, then $\mathfrak{h}^{1,p,\alpha}_{I,b}(M) \subset \mathfrak{h}^{1,p}_{I,b}(M)$. Therefore, it suffices to prove this containment when $\alpha \leq 1$. We decompose g as

$$g = a + h,$$

where

$$a = g - \frac{\chi_B}{\mu(B)} \int_B g \, \mathrm{d}\mu \qquad \text{and} \qquad h = \frac{\chi_B}{\mu(B)} \int_B g \, \mathrm{d}\mu.$$

Then a is a multiple of a standard p-atom at scale b. Indeed, clearly $\int_B a \, \mathrm{d}\mu = 0$ and

$$\begin{aligned} \|a\|_p &\leq \|g\|_p + \left|\int_B g \, \mathrm{d}\mu\right| \frac{\|\chi_B\|_p}{\mu(B)} \\ &\leq \mu(B)^{-1/p'} + r^\alpha \mu(B)^{-1/p'} \\ &\leq (1 + b^\alpha) \, \mu(B)^{-1/p'}, \end{aligned}$$

so that $\|a\|_{\mathfrak{h}^{1,p}_b} \leq (1+b^\alpha)$. Now we decompose h as a finite combination of $\mathfrak{h}^{1,p}_b$-atoms. Set $N := [\log_2(b/r)]$ and write

$$h = \sum_{i=1}^{N+2} h_i,$$

where

$$h_i = \left[\frac{\chi_{2^{i-1}B}}{\mu(2^{i-1}B)} - \frac{\chi_{2^i B}}{\mu(2^i B)}\right] \int_B g \, \mathrm{d}\mu \quad i = 1, ..., N+1$$

and

$$h_{N+2} = \frac{\chi_{2^{N+1}B}}{\mu(2^{N+1}B)} \int_B g \, \mathrm{d}\mu.$$

A straightforward computation shows that for all $i = 1, ..., N+1$,

$$\begin{aligned} \int_M \left|\frac{\chi_{2^{i-1}B}}{\mu(2^{i-1}B)} - \frac{\chi_{2^i B}}{\mu(2^{i-1}B)}\right|^p \mathrm{d}\mu &= \frac{\mu(2^i B \setminus 2^{i-1}B)}{\mu(2^i B)^p} + \frac{\mu(2^{i-1}B)}{\mu(2^{i-1}B)^p} \\ &\leq \mu(2^i B)^{1-p} + \mu(2^{i-1}B)^{1-p} \\ &\leq 2 \, \mu(2^{i-1}B)^{1-p}. \end{aligned}$$

Therefore

$$\begin{aligned}\|h_i\|_p &\leq r^\alpha\, 2^{1/p}\, \mu(2^{i-1}B)^{-1/p'}\\ &\leq r^\alpha\, 2^{1/p}\, D_{2b,2}^{1/p'}\, \mu(2^iB)^{-1/p'}.\end{aligned}$$

Since $2^iB \in \mathcal{B}_b$ for all $i = 1, ..., N$, $h_i/[(2D_{2b,2})^{1/p'}\, r^\alpha]$ is a standard p-atom, so that $\|h_i\|_{\mathfrak{h}_b^{1,p}} \leq C\, r^\alpha$.

The functions h_{N+1} and h_{N+2} are supported in the ball $2^{N+1}B$, which has radius greater than b, and

$$\|h_{N+1}\|_p \leq C\, b^\alpha\, \mu(2^{N+1}B)^{-1/p'},$$

$$\|h_{N+2}\|_p \leq b^\alpha\, \mu(2^{N+1}B)^{-1/p'}.$$

Observe that the radius of $2^{N+1}B$ is $< 2b$. Then, by Lemma 4.2.3, there exists a constant D, depending only on M and b, such that $\|h_i\|_{\mathfrak{h}^{1,p}} \leq D$ for $i = N+1, N+2$. By combining these estimates we get

$$\begin{aligned}\|h\|_{\mathfrak{h}_b^{1,p}} &\leq \sum_{i=1}^{N+2} \|h_i\|_{\mathfrak{h}_b^{1,p}}\\ &\leq C\, N\, r^\alpha + 2D\\ &\leq C\, r^\alpha\, \log_2 \frac{b}{r} + 2D\\ &\leq \tilde{C} + 2D,\end{aligned}$$

Therefore, each (α, p)-ion g is in $\mathfrak{h}_b^{1,p}(M)$ and $\|g\|_{\mathfrak{h}^{1,p}} \leq C$, where the constant C depends only on M, b and α, as required. $\square$

We have already mentioned that the spaces $\mathfrak{h}_b^{1,p}(M)$ will be proved to be independent of the parameters p and b. Then, by Theorem 4.3.4, for p in $(1, \infty]$, $b > R_0/(1-\beta)$ and α in $\mathbb{R}^+$, the spaces $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ coincide with equivalence of the norms.

Remark 4.3.5. We shall denote by $\mathfrak{h}_I^1(M)$ all the spaces $\mathfrak{h}_{I,1}^{1,p,\alpha}(M)$, endowed with any of the equivalent norms defined above.

4.4 The space $\mathfrak{bmo}(M)$

Suppose that q is in $[1, \infty)$ and b is in $\mathbb{R}^+$. For each locally integrable function f define the *local sharp maximal function* $f_b^{\sharp,q}$ by

$$f_b^{\sharp,q}(x) = \sup_{B \in \mathcal{B}_b(x)} \Big(\frac{1}{\mu(B)} \int_B |f - f_B|^q \, \mathrm{d}\mu \Big)^{1/q} \qquad \forall x \in M,$$

where f_B denotes the average of f over B and $\mathcal{B}_b(x)$ denotes the family of all balls in $\mathcal{B}_b$ centred at the point x. Define also the *modified local sharp maximal function* $N_b^q(f)$ by

$$N_b^q(f)(x) := f_b^{\sharp,q}(x) + \Big[\frac{1}{\mu(B_b(x))}\int_{B_b(x)} |f|^q \,\mathrm{d}\mu\Big]^{1/q} \qquad \forall x \in M,$$

where $B_b(x)$ denotes the ball with centre x and radius exactly b. Denote by $\mathfrak{bmo}_b^q(M)$ the space of all locally integrable functions f such that $N_b^q(f)$ is in $L^\infty(M)$, endowed with the norm

$$\|f\|_{\mathfrak{bmo}_b^q} = \|N_b^q(f)\|_\infty.$$

The space $\mathfrak{bmo}_b^q(M)$ is related to the space $BMO_b^q(M)$, introduced in [CMM1]. The latter is the Banach space of all locally integrable functions f (modulo constants) such that

$$\|f\|_{BMO_b^q} = \|f_b^{\sharp,q}\|_\infty < \infty.$$

As shown in [CMM1], the spaces $BMO_b^q(M)$ do not depend on the parameters q and b and we denote them all by $BMO(M)$.

Remark 4.4.1. Given f in $\mathfrak{bmo}_b^q(M)$, we have

$$\|f_b^{\sharp,q}\|_\infty \leq \|N_b^q(f)\|_\infty = \|f\|_{\mathfrak{bmo}_b^q}.$$

Denote by $[f]$ the equivalence class in $BMO_b^q(M)$ which contains f. By the estimate above, the linear map $\iota : \mathfrak{bmo}_b^q(M) \to BMO_b^q(M)$, defined by $\iota(f) = [f]$, is continuous, i.e.,

$$\|\iota(f)\|_{BMO_b^q} \leq \|f\|_{\mathfrak{bmo}_b^q} \qquad \forall f \in \mathfrak{bmo}_b^q(M). \tag{4.4.1}$$

In the following proposition we show that the space $\mathfrak{bmo}_b^q(M)$ does not depend on the parameter b.

Proposition 4.4.2. *Suppose that q is in $[1,\infty)$ and $R_0/(1-\beta) < c < b$. The following hold:*

(i) $\mathfrak{bmo}_b^q(M)$ *and* $\mathfrak{bmo}_c^q(M)$ *coincide and their norms are equivalent;*

(ii) $\mathfrak{bmo}_1^q(M)$ *and* $\mathfrak{bmo}_1^1(M)$ *coincide and their norms are equivalent (here we assume implicitly that $R_0/(1-\beta) < 1$).*

Proof. First we prove (i). Suppose that f is in $\mathfrak{bmo}_b^q(M)$. Since $c < b$, $f_c^{\sharp,q}(x) \leq f_b^{\sharp,q}(x)$. Moreover, for each $x \in M$

$$\begin{aligned}\frac{1}{\mu(B_c(x))}\int_{B_c(x)} |f|^q \,\mathrm{d}\mu &\leq \frac{1}{\mu(B_c(x))}\int_{B_b(x)} |f|^q \,\mathrm{d}\mu \\ &= \frac{\mu(B_b(x))}{\mu(B_c(x))}\,\frac{1}{\mu(B_b(x))}\int_{B_b(x)} |f|^q \,\mathrm{d}\mu \\ &\leq D_{b/c,c}\,\frac{1}{\mu(B_b(x))}\int_{B_b(x)} |f|^q \,\mathrm{d}\mu,\end{aligned}$$

(see (4.1.1)). Therefore $N_c^q(f)(x) \leq D_{b/c,c}\, N_b^q(f)(x)$. Thus f is in $\mathfrak{bmo}_c^q(M)$ and $\|f\|_{\mathfrak{bmo}_c^q} \leq D_{b/c,c}\,\|f\|_{\mathfrak{bmo}_b^q}$.

To prove the reverse inequality, observe that, by [CMM1, Prop 5.1], there exists a constant C_1 depending only on b, c and M, such that

$$\|f_b^{\sharp,q}\|_\infty \leq C_1\,\|f_c^{\sharp,q}\|_\infty \qquad \forall f \in \mathfrak{bmo}_c^q(M).$$

Now suppose that B_b is a ball of radius b. Then

$$\Big(\frac{1}{\mu(B_b)}\int_{B_b} |f|^q \,\mathrm{d}\mu\Big)^{1/q} = \frac{1}{\mu(B_b)^{1/q}} \sup_{\|\phi\|_{L^{q'}(B_b)}\leq 1}\Big|\int_{B_b} f\,\phi \,\mathrm{d}\mu\Big|,$$

where q' is the exponent conjugate to q. If ϕ is a function in $L^{q'}(B_b)$ with $\|\phi\|_{L^{q'}(B_b)} \leq 1$, then $\phi/\mu(B_b)^{1/q}$ is a q'-global atom at scale b. Therefore, by Lemma 4.2.3 there exist N q'-global atoms $a_1, \ldots, a_N$ at scale c supported in balls B_j such that $\phi/\mu(B_b)^{1/q} = \sum_{j=1}^N \lambda_j a_j$, with $|\lambda_j| \leq C$ and $\|a_j\|_{q'} = \mu(B_j)^{-1/q}$, where C and N are constants which depend only on b, c and M. Thus, by Hölder's inequality,

$$\begin{aligned}\frac{1}{\mu(B_b)^{1/q}}\Big|\int_{B_b} f\,\phi\,\mathrm{d}\mu\Big| &= \Big|\sum_{j=1}^N \lambda_j \int_{B_j} f\; a_j \,\mathrm{d}\mu\Big| \\ &\leq C\,\sum_{j=1}^N \Big(\int_{B_j} |f|^q \,\mathrm{d}\mu\Big)^{1/q}\|a_j\|_{q'} \\ &\leq C\,\sum_{j=1}^N \Big(\frac{1}{\mu(B_j)}\int_{B_j} |f|^q \,\mathrm{d}\mu\Big)^{1/q} \\ &\leq C\;N\,\|f\|_{\mathfrak{bmo}_c^q}.\end{aligned}$$

The above estimates imply that $\|f\|_{\mathfrak{bmo}_b^q} \leq (C_1 + CN)\,\|f\|_{\mathfrak{bmo}_c^q}$, as required to conclude the proof of (i).

Next we prove (ii). Recall that ι is a contractive map between $\mathfrak{bmo}^1(M)$ and $BMO^1(M)$ and that the spaces $BMO^1(M)$ and $BMO^q(M)$ agree (with equivalence of norms) for all q in $(1,\infty)$ (see [CMM1, Corollary 5.5]). Therefore there exists a constant C such that

$$\|\iota(f)\|_{BMO^q} \leq C\,\|\iota(f)\|_{BMO^1} \leq C\,\|f\|_{\mathfrak{bmo}^1} \qquad \forall f \in \mathfrak{bmo}^1(M).$$

Thus, for every B in $\mathcal{B}_1$,

$$\Big(\frac{1}{\mu(B)}\int_B |f-f_B|^q \,\mathrm{d}\mu\Big)^{1/q} \leq C\,\|f\|_{\mathfrak{bmo}^1} \qquad \forall f \in \mathfrak{bmo}^1(M).$$

Now suppose that B_1 is a ball of radius 1. By the triangle inequality

$$\begin{aligned}\Big(\frac{1}{\mu(B_1)}\int_{B_1} |f|^q \,\mathrm{d}\mu\Big)^{1/q} &\leq \Big(\frac{1}{\mu(B_1)}\int_{B_1} |f-f_{B_1}|^q \,\mathrm{d}\mu\Big)^{1/q} + |f_{B_1}| \\ &\leq \|\iota(f)\|_{BMO^q} + |f|_{B_1} \\ &\leq C\,\|\iota(f)\|_{BMO^1} + \frac{1}{\mu(B_1)}\int_{B_1}|f|\,\mathrm{d}\mu \\ &\leq (C+1)\,\|f\|_{\mathfrak{bmo}^1}.\end{aligned}$$

These estimates imply that

$$\|f\|_{\mathfrak{bmo}^q} \leq (2\,C+1)\|f\|_{\mathfrak{bmo}^1} \qquad \forall f \in \mathfrak{bmo}^1(M).$$

Furthermore, for q in $(1,\infty)$ and for every locally integrable f

$$N^1(f)(x) \leq N^q(f)(x) \qquad \forall x \in M,$$

so that $\|f\|_{\mathfrak{bmo}^1} \leq \|f\|_{\mathfrak{bmo}^q}$. Thus, $f \in \mathfrak{bmo}^1(M)$ implies $f \in \mathfrak{bmo}^q(M)$ for all q in $(1,\infty)$.

The proof of (ii) is complete. $\square$

Remark 4.4.3. In view of the observation above, all the spaces $\mathfrak{bmo}^q_b(M)$, $b > R_0/(1-\beta)$, q in $[1,\infty)$, coincide. We shall denote them simply by $\mathfrak{bmo}(M)$, endowed with any of the equivalent norms $\|\cdot\|_{\mathfrak{bmo}^q_b}$. This remark will be important in the proof of the duality between $\mathfrak{h}^1(M)$ and $\mathfrak{bmo}(M)$.

4.5 Duality

In this section we shall prove that the topological dual of $\mathfrak{h}^{1,p}(M)$ is isomorphic to $\mathfrak{bmo}^{p'}(M)$, where p' denotes the index conjugate to p.

We need more notations and preliminary observations. Suppose that p is in $[1,\infty)$. For each closed ball B in M, we denote by $L^p(B)$ the space of all functions in $L^p(M)$ which are supported in B. The union of all spaces $L^p(B)$ as B varies over all balls coincides with the space $L^p_c(M)$ of all functions in $L^p(M)$ with compact support. Fix a reference point o in M and for each positive integer k denote by B_k the ball centred at o with radius k. A convenient way of topologising $L^p_c(M)$ is to interpret $L^p_c(M)$ as the strict inductive limit of the spaces $L^p_c(B_k)$ (see [Bou, II, p. 33] for the definition of the strict inductive limit topology). We denote by X^p the space $L^p_c(M)$ with this topology, and write X^p_k for $L^p_c(B_k)$. The next lemma is well known, and it is included here for the sake of completeness.

Lemma 4.5.1. *Suppose that p is in $[1,\infty)$. The topological dual of X^p is $L^{p'}_{loc}(M)$, where p' denotes the index conjugate to p.*

Proof. Suppose that $g \in L^{p'}_{\mathrm{loc}}(M)$ and define the linear functional Λ_g on X^p by

$$\Lambda_g(f) = \int_M f\, g \,\mathrm{d}\mu.$$

Since X^p is the strict inductive limit of X^p_k, to show that Λ_g is continuous on X^p it suffices to prove that the restriction of Λ_g to X^p_k is a continuous functional on X^p_k for each k. Given $f \in X^p_k$, by Hölder's inequality we have that

$$|\Lambda_g(f)| = \Big|\int_{B_k} f\, g \,\mathrm{d}\mu\Big| \leq \|f\|_{L^p(B_k)} \|g\|_{L^{p'}(B_k)},$$

whence Λ_g is continuous on X^p_k.

To show the reverse inclusion, suppose that Λ is a continuous linear functional on X^p. Then for each k the restriction of Λ to X^p_k, which we denote by Λ_k, is a continuous linear functional. Since the dual of X^p_k is $L^{p'}(B_k)$, for each k there exists a unique function $g_k \in L^{p'}(B_k)$ such that

$$\Lambda_k(f) = \int_{B_k} f\, g_k \,\mathrm{d}\mu \qquad \forall f \in X^p_k.$$

Since $X^p_k \subset X^p_{k+1}$,

$$\int_{B_k} f\, g_k \,\mathrm{d}\mu = \int_{B_k} f\, g_{k+1} \,\mathrm{d}\mu \qquad \forall f \in X^p_k.$$

This implies that g_k coincides with g_{k+1} on B_k for each k. Therefore we can define on M a function g by requiring that g coincides with g_k on B_k for each k. It follows that g is in $L^{p'}_{\mathrm{loc}}(M)$; moreover, since for each $f \in X^p$ there exists $\hat{k}$ such that $f \in X^p_{\hat{k}}$ we get

$$\Lambda(f) = \Lambda_{\hat{k}}(f) = \int_{B_{\hat{k}}} f\, g_{\hat{k}} \,\mathrm{d}\mu = \int_M f\, g \,\mathrm{d}\mu.$$

$\square$

We denote by $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ the subspace of $\mathfrak{h}^{1,p}(M)$ consisting of all finite linear combinations of p-atoms. Clearly, $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ is dense in $\mathfrak{h}^{1,p}(M)$ with respect to the norm of $\mathfrak{h}^{1,p}(M)$. A natural norm on $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ is defined as follows:

$$\|f\|_{\mathfrak{h}^{1,p}_{\text{fin}}} = \inf\Big\{\sum_{j=1}^{N} |c_j| : f = \sum_{j=1}^{N} c_j\, a_j,\ a_j \text{ is a } p\text{-atom, } N \in \mathbb{N}^+\Big\}. \tag{4.5.1}$$

Note that the infimum is taken over *finite* linear combinations of atoms. Obviously,

$$\|f\|_{\mathfrak{h}^{1,p}} \leq \|f\|_{\mathfrak{h}^{1,p}_{\text{fin}}} \qquad \forall f \in \mathfrak{h}^{1,p}_{\text{fin}}(M). \tag{4.5.2}$$

Remark 4.5.2. Observe also that $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ and $L^p_c(M)$ agree as vector spaces. Indeed, on the one hand each function in $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ has finite L^p-norm and is compactly supported, hence it belongs to $L^p_c(M)$; on the other hand, let g be in $L^p_c(M)$; then, there exists a ball B of radius $r_B \geq 1$ such that g is supported in B. Set

$$a = \frac{g}{\|g\|_p \mu(B)^{1/p'}},$$

where p' is the exponent conjugate to p. Then a is a global p-atom at scale r_B and, by Lemma 4.2.3, a can be written as a finite linear combination of p-atoms at scale 1. This means that a is in $\mathfrak{h}^{1,p}_{\text{fin}}(M)$ and so is g.

We remark that

$$\|f^{s,q}\|_\infty + \sup\Big[\frac{1}{\mu(B_1(x))}\int_{B_1(x)} |f|^q \,\mathrm{d}\mu\Big]^{1/q},$$

where

$$f^{s,q}(x) = \sup_{B\in\mathcal{B}_1(x)} \inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(B)}\int_B |f-c|^q \,\mathrm{d}\mu\Big)^{1/q} \qquad \forall x \in M,$$

is an equivalent norm on $\mathfrak{bmo}^q(M)$. The proof is straightforward and it is omitted. We shall write f^s, instead of $f^{s,1}$.

Lemma 4.5.3. *If $f \in \mathfrak{bmo}^q(M)$, then $|f| \in \mathfrak{bmo}^q(M)$ and $\||f|\|_{\mathfrak{bmo}^q} \leq 2\|f\|_{\mathfrak{bmo}^q}$.*

Proof. It is straightforward to check that $f^{s,q}(x) \leq f^{\sharp,q}(x) \leq 2\, f^{s,q}(x)$ for all x in M. Therefore

$$\begin{aligned} |f|^{\sharp,q}(x) &\leq 2\ |f|^{s,q}(x) \\ &\leq 2 \sup_{B\in\mathcal{B}_1(x)} \Big(\frac{1}{\mu(B)}\int_B \Big||f| - |f_B|\Big|^q \,\mathrm{d}\mu\Big)^{1/q} \\ &\leq 2 \sup_{B\in\mathcal{B}_1(x)} \Big(\frac{1}{\mu(B)}\int_B |f - f_B|^q \,\mathrm{d}\mu\Big)^{1/q} = 2\ f^{\sharp,q}(x), \end{aligned}$$

whence

$$
\begin{aligned}
N^q(|f|)(x) &= |f|^{\sharp,q}(x) + \Big(\frac{1}{\mu(B_1(x))}\int_{B_1(x)} |f|^q \,\mathrm{d}\mu\Big)^{1/q} \\
&\leq 2\, f^{\sharp,q}(x) + \Big(\frac{1}{\mu(B_1(x))}\int_{B_1(x)} |f|^q \,\mathrm{d}\mu\Big)^{1/q} \\
&\leq 2\, N^q(f)(x).
\end{aligned}
$$

□

Next we identify the dual of $\mathfrak{h}^1(M)$ with $\mathfrak{bmo}(M)$. The proof follows the lines of the classical result of Coifman and Weiss [CW] in the case of spaces of homogeneous type, and of [CMM1]. We give all the details for the sake of completeness.

Theorem 4.5.4. *Suppose that p is in $(1,\infty)$ and let p' be the index conjugate to p. The following hold:*

(i) for every g in $\mathfrak{bmo}^{p'}(M)$ the functional F, initially defined on $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$ by the rule

$$F(f) = \int_M f\, g \,\mathrm{d}\mu,$$

has a unique bounded extension to $\mathfrak{h}^{1,p}(M)$. Furthermore

$$|||F||| \leq 4\, \|g\|_{\mathfrak{bmo}^{p'}},$$

where $|||F|||$ denotes the norm of F as a continuous linear functional on $\mathfrak{h}^{1,p}(M)$.

(ii) for every continuous linear functional F on $\mathfrak{h}^{1,p}(M)$ there exists a function g_F in $\mathfrak{bmo}^{p'}(M)$ such that $\|g_F\|_{\mathfrak{bmo}^{p'}} \leq 3\, |||F|||$ and

$$F(f) = \int_M f\, g_F \,\mathrm{d}\mu \qquad \forall f \in \mathfrak{h}^{1,p}_{\mathrm{fin}}(M).$$

Proof. Observe that for every g in $\mathfrak{bmo}^{p'}(M)$ and every finite linear combination f of p-atoms the integral $\int_M f\, g \,\mathrm{d}\mu$ is convergent. Therefore the functional F is well defined on $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$.

To prove (i) it suffices to show that for every g in $\mathfrak{bmo}^{p'}(M)$ the inequality

$$\Big|\int_M f\, g \,\mathrm{d}\mu\Big| \leq 4\, \|f\|_{\mathfrak{h}^{1,p}} \|g\|_{\mathfrak{bmo}^{p'}} \tag{4.5.3}$$

holds for each f in the dense subspace $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M) \subset \mathfrak{h}^{1,p}(M)$. First of all we prove (4.5.3) for f in $\mathfrak{h}^{1,p}(M)$ under the extra assumption that g is bounded. Given f

in $\mathfrak{h}^{1,p}(M)$ we can write $f = \sum_j \lambda_j a_j$, where the a_j's are p-atoms. Since the sum $\sum_j \lambda_j a_j$ converges in the $L^1(M)$-norm, we get

$$\int_M f\, g \,\mathrm{d}\mu = \sum_j \lambda_j \int_M g\, a_j \,\mathrm{d}\mu.$$

If a_j is a standard p-atom supported in a ball B_j, using the cancellation property we can write

$$\int_M g\, a_j \,\mathrm{d}\mu = \int_M (g - g_{B_j})\, a_j \,\mathrm{d}\mu.$$

Therefore, by the size condition,

$$\begin{aligned}
\Big|\int_M g\, a_j \,\mathrm{d}\mu\Big| &\le \Big[\frac{1}{\mu(B_j)}\int_{B_j} |g - g_{B_j}|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \mu(B_j)^{1/p'} \|a_j\|_p \\
&\le \Big[\frac{1}{\mu(B_j)}\int_{B_j} |g - g_{B_j}|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \\
&\le \|g\|_{BMO^{p'}} \\
&\le \|g\|_{\mathfrak{bmo}^{p'}}.
\end{aligned}$$

If a_j is a global p-atom supported in the ball B_j of radius 1, we get

$$\begin{aligned}
\Big|\int_M g\, a_j \,\mathrm{d}\mu\Big| &= \Big|\int_M (g - g_{B_j})\, a_j \,\mathrm{d}\mu + \int_M g_{B_j}\, a_j \,\mathrm{d}\mu\Big| \\
&\le \Big[\frac{1}{\mu(B_j)}\int_{B_j} |g - g_{B_j}|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} + \frac{1}{\mu(B_j)}\int_{B_j} |g| \,\mathrm{d}\mu \; \|a_j\|_1 \\
&\le \Big[\frac{1}{\mu(B_j)}\int_{B_j} |g - g_{B_j}|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} + \Big[\frac{1}{\mu(B_j)}\int_{B_j} |g|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \\
&\le \|g\|_{\mathfrak{bmo}^{p'}}.
\end{aligned}$$

Therefore

$$\Big|\int_M f\, g \,\mathrm{d}\mu\Big| \le \sum_j |\lambda_j|\, \|g\|_{\mathfrak{bmo}^{p'}} \le \|f\|_{\mathfrak{h}^{1,p}} \|g\|_{\mathfrak{bmo}^{p'}},$$

so that the inequality (4.5.3) holds for all bounded functions g and for all f in $\mathfrak{h}^{1,p}(M)$.

Now we assume that $g \in \mathfrak{bmo}^{p'}(M)$ is real-valued and $f \in \mathfrak{h}^{1,p}_{\text{fin}}(M)$. We consider the truncated functions g_k, defined by

$$g_k(x) = \begin{cases} k & \text{if } g(x) \ge k \\ g(x) & \text{if } |g(x)| \le k \\ -k & \text{if } g(x) \le -k. \end{cases}$$

Observe that each g_k can be written as $g_k = \psi_k \circ g$, where ψ_k are functions of a real variable given by

$$\psi_k(t) = \begin{cases} k & \text{if } t \geq k \\ t & \text{if } |t| \leq k \\ -k & \text{if } t \leq -k. \end{cases}$$

Since each ψ_k is a Lipschitz function with Lipschitz constant 1, for each ball B in $\mathcal{B}_1$

$$\begin{aligned} \Big(\frac{1}{\mu(B)}\int_B |g_k - (g_k)_B|^{p'}\,\mathrm{d}\mu\Big)^{1/p'} &= \Big(\frac{1}{\mu(B)}\int_B |\psi_k \circ g - (\psi_k \circ g)_B|^{p'}\,\mathrm{d}\mu\Big)^{1/p'} \\ &\leq 2\,\Big(\frac{1}{\mu(B)}\int_B |\psi_k \circ g - \psi_k(g_B)|^{p'}\,\mathrm{d}\mu\Big)^{1/p'} \\ &\leq 2\,\Big(\frac{1}{\mu(B)}\int_B |g - g_B|^{p'}\,\mathrm{d}\mu\Big)^{1/p'}. \end{aligned}$$

Moreover, noting that $|g_k| \leq |g|$, for each ball B_1 of radius 1 we get

$$\Big(\frac{1}{\mu(B_1)}\int_{B_1} |g_k|^{p'}\,\mathrm{d}\mu\Big)^{1/p'} \leq \Big(\frac{1}{\mu(B_1)}\int_{B_1} |g|^{p'}\,\mathrm{d}\mu\Big)^{1/p'}.$$

This implies that $N^{p'}(g_k) \leq 2\; N^{p'}(g)$, hence

$$\|g_k\|_{\mathfrak{bmo}^{p'}} \leq 2\;\|g\|_{\mathfrak{bmo}^{p'}}. \tag{4.5.4}$$

Since each g_k is bounded, the case just proved and (4.5.4) imply that

$$\Big|\int_M f\,g_k\,\mathrm{d}\mu\Big| \leq 2\,\|f\|_{\mathfrak{h}^{1,p}}\|g\|_{\mathfrak{bmo}^{p'}}.$$

Since f is a finite linear combination of p-atoms, we have $|fg_k| \leq |fg| \in L^1(M)$ and $fg_k \to fg$ almost everywhere as $k \to +\infty$. Thus the dominated convergence theorem gives

$$\Big|\int_M f\,g\,\mathrm{d}\mu\Big| = \lim_{k\to+\infty}\Big|\int_M f\,g_k\,\mathrm{d}\mu\Big| \leq 2\,\|f\|_{\mathfrak{h}^{1,p}}\|g\|_{\mathfrak{bmo}^{p'}}.$$

If g is complex-valued, we apply this estimate to the real and imaginary part and, since $\|\mathrm{Re}(g)\|_{\mathfrak{bmo}^{p'}} \leq \|g\|_{\mathfrak{bmo}^{p'}}$ and $\|\mathrm{Im}(g)\|_{\mathfrak{bmo}^{p'}} \leq \|g\|_{\mathfrak{bmo}^{p'}}$, we get

$$\Big|\int_M f\,g\,\mathrm{d}\mu\Big| \leq 4\,\|f\|_{\mathfrak{h}^{1,p}}\|g\|_{\mathfrak{bmo}^{p'}}$$

for all $g \in \mathfrak{bmo}^{p'}(M)$ and $f \in \mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$. Since $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$ is a dense subspace of $\mathfrak{h}^{1,p}(M)$ with respect to the $h^{1,p}(M)$-norm, the desired estimate follows.

Now we prove (ii). First we prove that if F is a continuous linear functional on $\mathfrak{h}^{1,p}(M)$, then F may be represented by a function in $\mathfrak{bmo}^{p'}(M)$.
Since F is a continuous linear functional on $\mathfrak{h}^{1,p}(M)$, for every p-atom a

$$|Fa| \leq |||F||| \, \|a\|_{\mathfrak{h}^{1,p}} \leq |||F|||,$$

because each p-atom has $\mathfrak{h}^{1,p}(M)$-norm at most 1. Thus

$$\sup\{|Fa| : a \text{ is a } \mathfrak{h}^{1,p}\text{-atom}\} \leq |||F|||.$$

For each ball B of radius $r_B \geq 1$, and each f in $L^p(B)$ such that $\|f\|_p = 1$, the function $f/\mu(B)^{1/p'}$ is a global p-atom at scale r_B. Thus, by Lemma 4.2.3 we obtain that $\|f/\mu(B)^{1/p'}\|_{\mathfrak{h}^{1,p}} \leq C$, where C is as in the statement of Lemma 4.2.3. This implies the estimate

$$|Ff| \leq C \, |||F||| \, \mu(B)^{1/p'}$$

for every function $f \in L^p(B)$ such that $\|f\|_p = 1$. Hence the restriction of F to X^p_k is a bounded linear functional on X^p_k for each k. Therefore F is a continuous linear functional on X^p. Since the dual of X^p is the space $L^{p'}_{\mathrm{loc}}(M)$ by Lemma 4.5.1, there exists a function g_F in $L^{p'}_{\mathrm{loc}}(M)$ such that

$$Ff = \int_M f \, g_F \, \mathrm{d}\mu \qquad \forall f \in X^p. \tag{4.5.5}$$

In particular, this holds whenever f is a p-atom.

To conclude the proof it suffices to prove that g_F belongs to $\mathfrak{bmo}^{p'}(M)$ and that

$$\|g_F\|_{\mathfrak{bmo}^{p'}} \leq 3 \, |||F|||. \tag{4.5.6}$$

Suppose that B is a ball of radius at most 1, and observe that

$$\Big[\int_B |g_F - (g_F)_B|^{p'} \, \mathrm{d}\mu\Big]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \Big|\int_B \varphi \, \big(g_F - (g_F)_B\big) \, \mathrm{d}\mu\Big|.$$

But

$$\begin{aligned}\int_B \varphi \, \big(g_F - (g_F)_B\big) \, \mathrm{d}\mu &= \int_B \big(\varphi - \varphi_B\big) \, \big(g_F - (g_F)_B\big) \, \mathrm{d}\mu \\ &= \int_B \big(\varphi - \varphi_B\big) \, g_F \, \mathrm{d}\mu,\end{aligned}$$

and since $\|\varphi\|_{L^p(B)} = 1$

$$|\varphi_B| \leq \Big[\frac{1}{\mu(B)} \int_B |\varphi|^p \, \mathrm{d}\mu\Big]^{1/p} \leq \mu(B)^{-1/p}.$$

Moreover,

$$\begin{aligned}\|\varphi - \varphi_B\|_{L^p(B)} &\leq \|\varphi\|_{L^p(B)} + |\varphi_B|\, \mu(B)^{1/p} \\ &\leq 2,\end{aligned}$$

so that the function $(\varphi - \varphi_B)/(2\,\mu(B)^{1/p'})$ is a standard p-atom. Therefore

$$\Big|\int_B (\varphi - \varphi_B)\; g_F \,\mathrm{d}\mu\Big| \leq 2\, \|\!|F|\!\|\, \mu(B)^{1/p'}.$$

Combining the above, we conclude that for every ball B of radius at most 1

$$\Big[\frac{1}{\mu(B)} \int_B |g_F - (g_F)_B|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \leq 2\, \|\!|F|\!\|,$$

Now take a ball B of radius exactly equal to 1. We have

$$\Big[\int_B |g_F|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \Big|\int_B \varphi\, g_F \,\mathrm{d}\mu\Big|.$$

The function $\varphi/\mu(B)^{1/p'}$ is a global p-atom at scale 1, thus

$$\Big|\int_B \varphi\; g_F \,\mathrm{d}\mu\Big| \leq \|\!|F|\!\|\, \mu(B)^{1/p'}.$$

Therefore, for every ball B of radius 1

$$\Big[\frac{1}{\mu(B)} \int_B |g_F|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \leq \|\!|F|\!\|.$$

Combining these estimates, (4.5.6) follows. This concludes the proof of (ii) and of the theorem. □

In view of the last result, we are now able to prove that all the spaces $\mathfrak{h}^{1,p}(M)$, with p in $(1,\infty)$, coincide. Indeed, suppose that $1 < r < p < \infty$. Then $(\mathfrak{h}^{1,r}(M))^* = (\mathfrak{h}^{1,p}(M))^*$, since $\mathfrak{bmo}^{r'}(M) = \mathfrak{bmo}^{p'}(M)$. Moreover, the identity is a continuous injection of $\mathfrak{h}^{1,p}(M)$ into $\mathfrak{h}^{1,r}(M)$ and $\mathfrak{h}^{1,p}(M)$ is a dense subspace of $\mathfrak{h}^{1,r}(M)$, therefore the Hahn-Banach theorem implies that $\mathfrak{h}^{1,r}(M) = \mathfrak{h}^{1,p}(M)$.

4.6 Estimates for the operator N

The purpose of this section is to establish a basic $L^p(M)$ estimate for the operator N, which acts on a locally integrable function f by

$$N(f)(x) = f^\sharp(x) + N_0 f(x) \qquad \forall x \in M,$$

where $f^\sharp$ is the *local centred sharp maximal function* given by the formula

$$f^\sharp(x) = \sup_{B\in\mathcal{B}_1(x)} \frac{1}{\mu(B)} \int_B |f - f_B| \,\mathrm{d}\mu$$

and

$$N_0 f(x) = \frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f| \, \mathrm{d}\mu.$$

The main result of this section, Theorem 4.6.1 below, will be the key to prove a basic interpolation results for $\mathfrak{h}^1(M)$ in the next section.

For each locally integrable function f, define the *local centred Hardy-Littlewood maximal function* $\mathcal{M}f$ as

$$\mathcal{M}f(x) = \sup_{0<r\leq 1} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

The operator $\mathcal{M}$ is bounded on $L^p(M)$ for every $p \in (1, \infty]$ and of weak type 1. The proof is analogous to that in the Euclidean case. Clearly $N(f)(x) \leq 3\,\mathcal{M}f(x)$, so that the L^p-boundedness of $\mathcal{M}$ implies that for $1 < p < \infty$

$$\|f\|_{L^p(M)} \geq C \, \|Nf\|_{L^p(M)} \qquad \forall f \in L^p(M).$$

In the next theorem we prove a reverse inequality.

Theorem 4.6.1. *Suppose that $p \in (1, \infty)$. Then there exists a constant C such that*

$$\|f\|_{L^p(M)} \leq C \, \|Nf\|_{L^p(M)}$$

for every $f \in L^1_{loc}(M)$ such that $Nf \in L^p(M)$.

We recall ([CMM1, Thm. 7.3]) that if M possesses the isoperimetric property IP (see Section 4.1), then for each p in $(1, \infty)$ there exists a constant C such that

$$\|f\|_{L^p(M)} \leq C \, \|f^\sharp\|_{L^p(M)} \qquad \forall f \in L^p(M). \tag{4.6.1}$$

Observe that the isoperimetric property is necessary for this estimate to hold. For instance, (4.6.1) is false for $M = \mathbb{R}^n$, as shown in [I1]. The inequality in Theorem 4.6.1 is weaker than (4.6.1), but it does not require any extra geometric assumption.

The proof of Theorem 4.6.1 will make use of the so-called dyadic cubes introduced by G. David and M. Christ [Chr, Da] on spaces of homogeneous type. Since Christ's construction requires only the local doubling property, we can adapt this theory to our setting, as shown in [CMM1].

Theorem 4.6.2. ([CMM1, Thm. 3.2]) *There exist a collection of open subsets $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ and constants δ in $(0,1)$, a_0, C_1 in $\mathbb{R}^+$ such that*

(i) $\bigcup_\alpha Q^k_\alpha$ is a set of full measure in M for each k in $\mathbb{Z}$;

(ii) *if* $\ell \geq k$, *then either* $Q^\ell_\beta \subset Q^k_\alpha$ *or* $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$;

(iii) *for each* (k, α) *and each* $\ell < k$ *there is a unique* β *such that* $Q^k_\alpha \subset Q^\ell_\beta$;

(iv) $\mathrm{diam}(Q^k_\alpha) \leq C_1\, \delta^k$;

(v) *each* Q^k_α *contains some ball* $B(z^k_\alpha, a_0\, \delta^k)$.

Note that (iv) and (v) imply that for every integer k and each α in I_k

$$B(z^k_\alpha, a_0\, \delta^k) \subset Q^k_\alpha \subset B(z^k_\alpha, C_1\, \delta^k).$$

We shall denote by $\mathcal{Q}^k$ the class of all dyadic cubes of "resolution" k, i.e., the family of cubes $\{Q^k_\alpha : \alpha \in I_k\}$, and by $\mathcal{Q}$ the set of all dyadic cubes.

We shall need the following additional properties of dyadic cubes.

Proposition 4.6.3. ([CMM1, Prop. 3.4]) *Suppose that* b *is in* $\mathbb{R}^+$ *and that* ν *is in* $\mathbb{Z}$, *and let* C_1 *and* δ *be as in Theorem 4.6.2. The following hold:*

(i) *suppose that* Q *is in* $\mathcal{Q}^k$ *for some* $k \geq \nu$, *and that* B *is a ball such that* $c_B \in Q$. *If* $r_B \geq C_1\, \delta^k$, *then*

$$\mu(B \cap Q) = \mu(Q); \tag{4.6.2}$$

if $r_B < C_1\, \delta^k$, *then*

$$\mu(B \cap Q) \geq D^{-1}_{C_1/(a_0\delta), \delta^\nu}\, \mu(B); \tag{4.6.3}$$

(ii) *suppose that* τ *is in* $[2, \infty)$. *For each* Q *in* $\mathcal{Q}$ *the space* $\big(Q, d_{|Q}, \mu_{|Q}\big)$ *is of homogeneous type. Denote by* $D^Q_{\tau,\infty}$ *its doubling constant (see Remark 4.1.1 for the definition). Then*

$$\sup\Big\{D^Q_{\tau,\infty} : Q \in \bigcup_{k=\nu}^{\infty} \mathcal{Q}^k\Big\} \leq D_{\tau, C_1\delta^\nu}\, D_{C_1/(a_0\delta), \delta^\nu};$$

(iii) *for each ball* B *in* $\mathcal{B}_b$, *let* k *be the integer such that* $\delta^k \leq r_B < \delta^{k-1}$, *and and let* $\widetilde{B}$ *denote the ball with centre* c_B *and radius* $\big(1 + C_1\big)\, r_B$. *Then* $\widetilde{B}$ *contains all dyadic cubes in* $\mathcal{Q}^k$ *that intersect* B *and*

$$\mu(\widetilde{B}) \leq D_{1+C_1, b}\, \mu(B);$$

(iv) *suppose that* B *is in* $\mathcal{B}_b$, *and that* k *is an integer such that* $\delta^k \leq r_B < \delta^{k-1}$. *Then there are at most* $D_{(1+C_1)/(a_0\delta), b}$ *dyadic cubes in* $\mathcal{Q}^k$ *that intersect* B.

In particular, property (ii) states that, when the resolution k is fixed, all the cubes in $\mathcal{Q}^k$ are spaces of homogeneous type with doubling constants uniformly bounded from above. More precisely, if we set $C_{\tau,k} := D_{\tau,C_1\delta^k}\, D_{C_1/(a_0\delta),\delta^k}$, we get

$$D^Q_{\tau,\infty} \leq C_{\tau,k} \tag{4.6.4}$$

for each cube Q in $\mathcal{Q}^k$.

For each locally integrable function f and each dyadic cube Q the *noncentred Hardy–Littlewood maximal function* $\mathcal{M}^Q f$ is defined by

$$\mathcal{M}^Q f(x) = \sup_{B: B\cap Q \ni x} \frac{1}{\mu(B\cap Q)} \int_{B\cap Q} |f| \,\mathrm{d}\mu \qquad \forall x \in Q,$$

where each B is a ball in $\mathcal{B}$ whose centre belongs to Q.
The operator $\mathcal{M}^Q$ is bounded on $L^p(Q)$ for every p in $(1,\infty]$ and of weak type 1 uniformly in the resolution of Q, i.e., there exists a constant C_0, such that

$$\mu\big(\{x \in Q \,:\, \mathcal{M}^Q f(x) > \lambda\}\big) \leq \frac{C_0}{\lambda}\, \|f\|_{L^1(Q)}; \tag{4.6.5}$$

note that C_0 depends only on the doubling constant of $(Q, d_{|Q}, \mu_{|Q})$, which is uniformly bounded when Q varies in the set of all dyadic cubes of the same resolution, by (ii) above.

For each locally integrable function f and each dyadic cube Q we define the *noncentred sharp maximal* function $f^{\sharp,Q}(x)$ on Q as

$$f^{\sharp,Q}(x) = \sup_{B: B\cap Q \ni x} \frac{1}{\mu(B\cap Q)} \int_{B\cap Q} |f - f_{B\cap Q}| \,\mathrm{d}\mu \qquad \forall x \in Q,$$

where B is a ball in $\mathcal{B}$ whose centre belongs to Q and

$$f_{B\cap Q} = \frac{1}{\mu(B\cap Q)} \int_{B\cap Q} f \,\mathrm{d}\mu.$$

We split the proof of Theorem 4.6.1 into a series of lemmas.

Lemma 4.6.4. *Suppose that k is in $\mathbb{Z}$. Then there exist constants A and $B > 1$ such that for every $\beta > B$, $\gamma > 0$, f in $L^1_{loc}(M)$, and Q in $\mathcal{Q}^k$*

$$\mu\big(\{x \in Q \,:\, \mathcal{M}^Q f(x) > \beta\lambda,\ f^{\sharp,Q} \leq \gamma\lambda\}\big) \leq A\,\frac{\gamma}{\beta}\,\mu\big(\{x \in Q \,:\, \mathcal{M}^Q f(x) > \lambda\}\big)$$

for every $\lambda > \frac{C_0}{\mu(Q)}\|f\|_{L^1(Q)}$, where C_0 is as in (4.6.5).

Proof. Set

$$\lambda_0 = \frac{C_0}{\mu(Q)}\|f\|_{L^1(Q)},$$

$E_\lambda = \{x \in Q : \mathcal{M}^Q f(x) > \lambda\}$, $F_\lambda = \{x \in Q : f^{\sharp,Q}(x) \leq \lambda\}$ and $G_\lambda^{\beta,\gamma} = E_{\beta\lambda} \cap F_{\gamma\lambda}$. Since $\lambda > \lambda_0$, $\mu(E_\lambda) < \mu(Q)$, so that E_λ is a proper subset in Q. Since E_λ is open and Q is a space of homogeneous type, we can apply a Whitney type covering lemma [CW, Thm 3.2] (with 1 in place of C and K therein), and obtain a sequence $\{B_i \cap Q\}$ of balls in Q, where $B_i \in \mathcal{B}$, such that:

(i) $E_\lambda = \bigcup_i (B_i \cap Q)$;

(ii) there exists a constant $K_0 = K_0(k)$ such that no point of E_λ belongs to more than K_0 balls $B_i \cap Q$;

(iii) $(3B_i \cap Q) \cap ((E_\lambda)^c \cap Q) \neq \emptyset$.

Note that K_0 does not depend on the particular cube Q in $\mathcal{Q}^k$ because K_0 depends only on the doubling constant of the space of homogeneous type and for cubes of the same resolution the doubling constants are uniformly bounded from above.

Suppose that $\beta \geq 1$. Then $G_\lambda^{\beta,\gamma} \subset E_{\beta\lambda} \subset E_\lambda$, so that

$$\mu(G_\lambda^{\beta,\gamma}) = \mu\Big(G_\lambda^{\beta,\gamma} \cap \Big(\bigcup_i (B_i \cap Q)\Big)\Big) = \mu\Big(\bigcup_i (G_\lambda^{\beta,\gamma} \cap B_i)\Big) \leq \sum_i \mu(G_\lambda^{\beta,\gamma} \cap B_i).$$

If $G_\lambda^{\beta,\gamma} \cap B_i = \emptyset$ for some index i, we simply ignore the ball B_i; otherwise, there exists at least a point $y_i \in G_\lambda^{\beta,\gamma} \cap B_i$, whence $f^{\sharp,Q}(y_i) \leq \gamma\lambda$.

We claim that there exists $B_0 > 1$ such that $E_{\beta\lambda} \cap B_i$ is contained in $\{x \in Q : \mathcal{M}^Q(f\chi_{5B_i})(x) > (\beta/B_0)\lambda\}$ for all $\beta \geq B_0$.

The claim will imply that

$$\mu(G_\lambda^{\beta,\gamma} \cap B_i) \leq \mu(E_{\beta\lambda} \cap B_i) \leq \mu(\{\mathcal{M}^Q(f\chi_{5B_i}) > (\beta/B_0)\lambda\}).$$

To prove the claim, we consider the centred Hardy–Littlewood maximal function on the cube Q defined by

$$\widetilde{\mathcal{M}}^Q f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x)\cap Q} |f| \, \mathrm{d}\mu \qquad \forall x \in Q.$$

Since the restriction of μ to each cube Q is a doubling measure,

$$\mathcal{M}^Q f(x) \leq C_{2,k}\, \widetilde{\mathcal{M}}^Q f(x) \qquad \forall x \in Q,$$

where $C_{2,k}$ is the constant that appears in (4.6.4) (it depends only on the resolution of Q).

Suppose that $x \in E_{\beta\lambda} \cap B_i$ and $\beta \geq C_{2,k}$. We need to prove that

$$\mathcal{M}^Q(f\chi_{5B_i}) > \frac{\beta}{C_{2,k}}\lambda.$$

Clearly, $\widetilde{\mathcal{M}}^Q f(x) > \frac{\beta}{C_{2,k}}\lambda$, so that there exists a ball $B_r(x)$ such that

$$\frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x)\cap Q} |f| \, \mathrm{d}\mu > \frac{\beta}{C_{2,k}}\lambda.$$

Condition (iii) above implies that there exists a point x_i in $3B_i \cap Q$ such that

$$\mathcal{M}^Q f(x_i) \leq \lambda. \tag{4.6.6}$$

Since we have assumed that $\beta \geq C_{2,k}$, $x_i \notin B_r(x)$, for otherwise

$$\mathcal{M}^Q f(x_i) \geq \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x)\cap Q} |f| \, \mathrm{d}\mu > \frac{\beta}{C_{2,k}}\lambda \geq \lambda.$$

Since x_i is in $3B_i \setminus B_r(x)$, $r < 4\, r_{B_i}$. Hence $B_r(x) \subset 5B_i$ and

$$\frac{\beta}{C_{2,k}}\lambda < \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x)\cap Q} |f|\chi_{5B_i} \, \mathrm{d}\mu \leq \mathcal{M}^Q(f\chi_{5B_i})(x).$$

This concludes the proof of the claim (with $B_0 = C_{2,k}$).
Now we observe that

$$\mathcal{M}^Q(f\chi_{5B_i})(x) \leq \mathcal{M}^Q((f - f_{5B_i\cap Q})\chi_{5B_i})(x) + |f_{5B_i\cap Q}|.$$

Since x_i is in $3B_i \cap Q$ and $\mathcal{M}^Q f(x_i) \leq \lambda$ by (4.6.6),

$$|f_{5B_i\cap Q}| \leq \frac{1}{\mu(5B_i \cap Q)} \int_{5B_i\cap Q} |f| \, \mathrm{d}\mu \leq \mathcal{M}^Q f(x_i) \leq \lambda.$$

Therefore, if $\beta > 2\, C_{2,k}$, then $|f_{5B_i\cap Q}| < \frac{\beta}{2\, C_{2,k}}\lambda$. This estimate, together with the weak type 1 inequality for $\mathcal{M}^Q$ and the assumption that $f^{\sharp,Q}(y_i) \leq \gamma\lambda$, implies that, if $\beta > 2\, C_{2,k}$,

$$\begin{aligned}
\mu(\{\mathcal{M}^Q(f\chi_{5B_i}) > (\beta/C_{2,k})\lambda\}) &\leq \mu(\{\mathcal{M}^Q((f - f_{5B_i\cap Q})\chi_{5B_i}) > (\beta/2\, C_{2,k})\lambda\}) \\
&\leq \frac{2\, C_{2,k}\, C_0}{\beta\, \lambda} \int_Q |f - f_{5B_i\cap Q}|\chi_{5B_i} \, \mathrm{d}\mu \\
&\leq \frac{2\, C_{2,k}\, C_0}{\beta\, \lambda}\, \mu(5B_i \cap Q)\, f^{\sharp,Q}(y_i) \\
&\leq 2\, C_{2,k}\, C_0 \frac{\gamma}{\beta}\, \mu(5B_i \cap Q).
\end{aligned}$$

Thus, we have proved that

$$\mu(G_\lambda^{\beta,\gamma} \cap B_i) \leq 2\, C_{2,k}\, C_0 \frac{\gamma}{\beta}\, \mu(5B_i \cap Q),$$

so that, using the doubling property on Q and condition (ii) , we get

$$\begin{aligned}
\mu(G_\lambda^{\beta,\gamma}) &\leq 2\, C_{2,k}\, C_0 \frac{\gamma}{\beta} \sum_i \mu(5B_i \cap Q) \\
&\leq 2\, C_{2,k}\, C_0\, C_{5,k} \frac{\gamma}{\beta} \sum_i \mu(B_i \cap Q) \\
&\leq 2\, C_{2,k}\, C_0\, C_{5,k}\, K_0 \frac{\gamma}{\beta}\, \mu(E_\lambda),
\end{aligned}$$

as required (with $B = 2\, C_{2,k}$ and $A = 2\, C_{2,k}\, C_0\, C_{5,k}\, K_0$). □

Lemma 4.6.5. *For each k in $\mathbb{Z}$ there exists a constant $C = C(k)$ such that for each cube Q in $\mathcal{Q}^k$*

$$\|f\|^p_{L^p(Q)} \leq C \left(\|f^{\sharp,Q}\|^p_{L^p(Q)} + \|f\|^p_{L^1(Q)} \right).$$

Proof. Since $\mathcal{M}^Q f \geq |f|$ almost everywhere, it suffices to show that

$$\|\mathcal{M}^Q f\|^p_{L^p(Q)} \leq C \left(\|f^{\sharp,Q}\|^p_{L^p(Q)} + \|f\|^p_{L^1(Q)} \right),$$

We set $E_\lambda = \{x \in Q : \mathcal{M}^Q f(x) > \lambda\}$ and $\lambda_0 = \frac{C_0}{\mu(Q)} \|f\|_{L^1(Q)}$, as in Lemma 4.6.4. Note that,

$$\begin{aligned}
\|\mathcal{M}^Q f\|^p_{L^p(Q)} &= p \int_0^\infty \lambda^{p-1}\, \mu\big(E_\lambda\big)\, \mathrm{d}\lambda \\
&= p\, \beta^p \int_0^\infty \lambda^{p-1}\, \mu\big(E_{\beta\lambda}\big)\, \mathrm{d}\lambda \\
&= p\, \beta^p \int_0^{\lambda_0} \lambda^{p-1}\, \mu\big(E_{\beta\lambda}\big)\, \mathrm{d}\lambda + p\, \beta^p \int_{\lambda_0}^{+\infty} \lambda^{p-1}\, \mu\big(E_{\beta\lambda}\big)\, \mathrm{d}\lambda \\
&=: I_1 + I_2.
\end{aligned}$$

We choose $\beta > 2C_{2,k}$. Since the maximal operator $\mathcal{M}^Q$ is of weak type 1,

$$\begin{aligned}
I_1 &\leq C_0\, p\, \beta^{p-1} \|f\|_{L^1(Q)} \int_0^{\lambda_0} \lambda^{p-2}\, \mathrm{d}\lambda \\
&= C_0\, \frac{p}{p-1}\, \beta^{p-1}\, \|f\|_{L^1(Q)}\, \lambda_0^{p-1} \\
&= C_0^p\, \frac{p}{p-1}\, \beta^{p-1} \mu(Q)^{1-p} \|f\|^p_{L^1(Q)}.
\end{aligned}$$

Given $\gamma > 0$ we write I_2 as the sum of

$$p\, \beta^p \int_{\lambda_0}^{+\infty} \lambda^{p-1}\, \mu\big(E_{\beta\lambda} \cap \{f^{\sharp,Q} \leq \gamma\lambda\}\big)\, \mathrm{d}\lambda$$

and

$$p\,\beta^p \int_{\lambda_0}^{+\infty} \lambda^{p-1}\,\mu\big(E_{\beta\lambda} \cap \{f^{\sharp,Q} > \gamma\lambda\}\big)\,\mathrm{d}\lambda.$$

Then, by Lemma 4.6.4,

$$\begin{aligned}
I_2 &\leq A\,p\,\beta^{p-1}\gamma \int_{\lambda_0}^{+\infty} \lambda^{p-1}\,\mu\big(E_\lambda\big)\,\mathrm{d}\lambda + p\,\beta^p \int_{\lambda_0}^{+\infty} \lambda^{p-1}\,\mu\big(\{f^{\sharp,Q} > \gamma\lambda\}\big)\,\mathrm{d}\lambda \\
&= A\,p\,\beta^{p-1}\gamma \int_{\lambda_0}^{+\infty} \lambda^{p-1}\,\mu\big(E_\lambda\big)\,\mathrm{d}\lambda + p\,\frac{\beta^p}{\gamma^p} \int_{\gamma\lambda_0}^{+\infty} \lambda^{p-1}\,\mu\big(\{f^{\sharp,Q} > \lambda\}\big)\,\mathrm{d}\lambda \\
&\leq A\,\beta^{p-1}\gamma\,\|\mathcal{M}^Q f\|^p_{L^p(Q)} + \frac{\beta^p}{\gamma^p}\,\|f^{\sharp,Q}\|^p_{L^p(Q)}.
\end{aligned}$$

Therefore

$$(1 - A\,\beta^{p-1}\gamma)\|\mathcal{M}^Q f\|^p_{L^p(Q)} \leq \frac{C_0^p\,p}{p-1}\,\beta^{p-1}\mu(Q)^{1-p}\|f\|^p_{L^1(Q)} + \frac{\beta^p}{\gamma^p}\|f^{\sharp,Q}\|^p_{L^p(Q)}.$$

Now, we choose γ small enough so that $1 - A\,\beta^{p-1}\gamma > 0$; for instance, we set $\gamma = 1/(2A\,\beta^{p-1})$ and we get

$$\|\mathcal{M}^Q f\|^p_{L^p(Q)} \leq \frac{2\,C_0^p\,p}{p-1}\,\beta^{p-1}\mu(Q)^{1-p}\|f\|^p_{L^1(Q)} + 2^{p+1}\,A^p\,\beta^{p^2}\,\|f^{\sharp,Q}\|^p_{L^p(Q)}.$$

Since each cube Q of resolution k contains a ball of radius $a_0\delta^k$, the UBSC implies that there exists a constant D, depending only on k, such that $\mu(Q) \geq D$ for each cube Q in $\mathcal{Q}^k$. Therefore

$$\|\mathcal{M}^Q\|^p_{L^p(Q)} \leq C\,\Big(\|f^{\sharp,Q}\|^p_{L^p(Q)} + \|f\|^p_{L^1(Q)}\Big),$$

where $C = \max\Big(2\,C_0^p\,\frac{p}{p-1}\,\beta^{p-1}D^{1-p}, 2^{p+1}\,A^p\,\beta^{p^2}\Big)$, as required. □

Lemma 4.6.6. *For each k in $\mathbb{Z}$ there exists a constant $C = C(k)$ such that for each cube Q in $\mathcal{Q}^k$*

$$\|f\|_{L^1(Q)} \leq C\,\|N_0 f\|_{L^1(Q)}.$$

Proof. For the sake of definiteness, suppose that Q is the dyadic cube Q^k_α. Then Q is contained in the ball $B(z^k_\alpha, C_1\,\delta^k)$. Denote by $\tilde{B}$ the ball centred at z^k_α and of radius $1 + C_1\,\delta^k$. Then, $B_1(x) \subset \tilde{B}$ for each x in Q, so that, using Tonelli's theorem and the UBSC, we get

$$\begin{aligned}
\|N_0 f\|_{L^1(Q)} &= \int_Q \frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f(y)| \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(x) \\
&= \int_Q \frac{1}{\mu(B_1(x))} \int_{\tilde{B}} \chi_{B_1(x)}(y) |f(y)| \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(x) \\
&= \int_{\tilde{B}} |f(y)| \int_Q \frac{\chi_{B_1(x)}(y)}{\mu(B_1(x))} \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\
&\geq C \int_{\tilde{B}} |f(y)| \int_Q \chi_{B_1(x)}(y) \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\
&= C \int_{\tilde{B}} |f(y)| \int_Q \chi_{B_1(y)}(x) \,\mathrm{d}\mu(x) \,\mathrm{d}\mu(y) \\
&= C \int_{\tilde{B}} |f(y)| \mu(Q \cap B_1(y)) \,\mathrm{d}\mu(y) \\
&\geq C \int_Q |f(y)| \mu(Q \cap B_1(y)) \,\mathrm{d}\mu(y),
\end{aligned}$$

where the last inequality follows from the fact that $Q \subset \tilde{B}$. To estimate $\mu(Q \cap B_1(y))$ from below we consider the cases where $C_1 \delta^k \leq 1$ and $C_1 \delta^k > 1$ separately.
Suppose that $C_1 \delta^k \leq 1$. Since the diameter of Q is at most $C_1 \, \delta^k$ by Theorem 4.6.2 (iv), $Q \cap B_1(y) = Q$. Moreover, Q contains the ball $B(z_\alpha^k, a_0 \, \delta^k)$, so that, by the UBSC, $\mu(Q \cap B_1(y)) = \mu(Q) > \mu(B(z_\alpha^k, a_0 \, \delta^k)) \geq C$.

Now assume that $C_1 \delta^k > 1$. By Proposition 4.6.3 (i) and the UBSC, we obtain that $\mu(B_1(y) \cap Q) \geq D^{-1}_{C_1/(a_0 \delta), \delta^k} \, \mu(B_1(y)) \geq C$. In both cases, the constant C depends only on the resolution k. Therefore

$$\|N_0 f\|_{L^1(Q)} \geq C \int_Q |f(y)| \,\mathrm{d}\mu(y) = C \, \|f\|_{L^1(Q)},$$

as required. □

Lemma 4.6.7. *Suppose that k is an integer $> [\log_\delta(1/(2C_1))]$, where δ and C_1 are as in Theorem 4.6.2. Then there exists a constant $C = C(k)$ such that for each cube Q in $\mathcal{Q}^k$*

$$f^{\sharp,Q}(x) \leq C \, f^\sharp(x) \qquad \forall x \in Q.$$

Proof. For each b in $\mathbb{R}^+$ we define the noncentred sharp function $\widetilde{f}_b^\sharp$ of a locally integrable function f as

$$\widetilde{f}_b^\sharp(x) = \sup_B \frac{1}{\mu(B)} \int_B |f - f_B| \,\mathrm{d}\mu \qquad \forall x \in M,$$

where the supremum is taken over all balls in $\mathcal{B}_b$ that contain x.
We first show that for each integer k there exists a constant $C = C(k)$ such that,

for each cube Q in $\mathcal{Q}^k$, $f^{\sharp,Q}(x) \leq C\, \widetilde{f}^{\sharp}_{C_1\delta^k}(x)$ for any x in Q.

Fix k in $\mathbb{Z}$ and choose Q in $\mathcal{Q}^k$. Take x in Q and suppose that B is a ball whose centre belongs to Q and such that $x \in B \cap Q$. We consider the cases where $C_1\delta^k \leq r_B$ and $C_1\delta^k > r_B$ separately.

If $r_B < C_1\delta^k$, the triangle inequality gives

$$\begin{aligned} \frac{1}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_{B\cap Q}|\,\mathrm{d}\mu &\leq \frac{1}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_B|\,\mathrm{d}\mu + |f_B - f_{B\cap Q}| \\ &\leq \frac{2}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_B|\,\mathrm{d}\mu \\ &\leq \frac{2\,D_{C_1/(a_0\delta),\delta^k}}{\mu(B)}\int_B |f - f_B|\,\mathrm{d}\mu, \end{aligned}$$

where the last inequality follows from the fact that

$$\mu(B\cap Q) \geq D^{-1}_{C_1/(a_0\delta),\delta^k}\,\mu(B)$$

by Proposition 4.6.3 (i). Since the ball B belongs to $\mathcal{B}_{C_1\delta^k}$, the right hand side of the formula above is majorised by $2\,D_{C_1/(a_0\delta),\delta^k}\,\widetilde{f}^{\sharp}_{C_1\delta^k}(x)$.

Now assume that $r_B \geq C_1\delta^k$. Without loss of generality, we may suppose that Q is the dyadic cube Q^k_α. Since $\mathrm{diam}(Q) \leq C_1\,\delta^k$ by Theorem 4.6.2 (iv) , $Q \cap B = Q$; moreover, Theorem 4.6.2 (v) implies that

$$B(z^k_\alpha, a_0\,\delta^k) \subset Q \subset B(z^k_\alpha, C_1\,\delta^k).$$

Denote by $\bar{B}$ the ball $B(z^k_\alpha, C_1\,\delta^k)$. Then, by the triangle inequality,

$$\begin{aligned} \frac{1}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_{B\cap Q}|\,\mathrm{d}\mu &\leq \frac{1}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_{\bar{B}}|\,\mathrm{d}\mu + |f_{\bar{B}} - f_{B\cap Q}| \\ &\leq \frac{2}{\mu(B\cap Q)}\int_{B\cap Q} |f - f_{\bar{B}}|\,\mathrm{d}\mu \\ &\leq \frac{2}{\mu(B(z^k_\alpha, a_0\,\delta^k))}\int_{\bar{B}} |f - f_{\bar{B}}|\,\mathrm{d}\mu. \end{aligned}$$

Now, the local doubling property implies that

$$\mu(\bar{B}) \leq D_{C_1/a_0, a_0\delta^k}\,\mu(B(z^k_\alpha, a_0\,\delta^k));$$

hence, the right hand side can be estimated from above by

$$\frac{2\,D_{C_1/a_0,a_0\delta^k}}{\mu(\bar{B})}\int_{\bar{B}} |f - f_{\bar{B}}|\,\mathrm{d}\mu,$$

which may be majorised by $2\,D_{C_1/a_0,a_0\delta^k}\,\widetilde{f}^{\sharp}_{C_1\delta^k}(x)$, since the ball $\bar{B}$ has radius $C_1\,\delta^k$. By taking the supremum over all balls B containing x and whose centre belongs to Q, we get

$$f^{\sharp,Q}(x) \leq 2\,D_{C_1/(a_0\delta),\max(1,a_0)\delta^k}\,\widetilde{f}^{\sharp}_{C_1\delta^k}(x) \qquad \forall x \in Q.$$

Since the local doubling property ensures that for each b in $\mathbb{R}^+$ there exists a constant $C = C(b)$ such that $\widetilde{f_b^\sharp} \leq C\, f_{2b}^\sharp$, we can deduce that

$$f^{\sharp,Q}(x) \leq C\, f^\sharp_{2C_1\delta^k}(x) \qquad \forall x \in Q.$$

Now, if we choose the integer k large enough so that $2C_1\delta^k \leq 1$, i.e., $k > [\log_\delta(1/2C_1)]$, we get $f^\sharp_{2C_1\delta^k} \leq f^\sharp$, which gives the desired conclusion. □

Now we are ready to prove the main result of this section.

Proof of Theorem 4.6.1. Consider the collection of dyadic cubes $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$, as in Theorem 4.6.2. Fix a resolution $k > [\log_\delta(1/2C_1)]$. Since the dyadic cubes of a fixed resolution are pairwise disjoint and their union is a set of full measure in M,

$$\begin{aligned}
\|f\|^p_{L^p(M)} &= \sum_\alpha \|f\|^p_{L^p(Q^k_\alpha)} \\
&\leq C \Big\{ \sum_\alpha \|f^{\sharp,Q^k_\alpha}\|^p_{L^p(Q^k_\alpha)} + \sum_\alpha \|f\|^p_{L^1(Q^k_\alpha)} \Big\} \\
&\leq C \Big\{ \sum_\alpha \|f^\sharp\|^p_{L^p(Q^k_\alpha)} + \sum_\alpha \|N_0 f\|^p_{L^1(Q^k_\alpha)} \Big\},
\end{aligned}$$

where the first inequality follows from Lemma 4.6.5 and the second is a consequence of Lemma 4.6.6 and Lemma 4.6.7. Since $Q^k_\alpha \subset B(z^k_\alpha, C_1\,\delta^k)$ for all α in I_k, each Q^k_α has finite measure depending only on k, by the UBSC. Therefore there exists a constant C such that $\|N_0 f\|^p_{L^1(Q^k_\alpha)} \leq C\, \|N_0 f\|^p_{L^p(Q^k_\alpha)}$. Thus,

$$\begin{aligned}
\|f\|^p_{L^p(M)} &\leq C\, \{\|f^\sharp\|^p_{L^p(M)} + \|N_0 f\|^p_{L^p(M)}\} \\
&\leq C\, \|N f\|^p_{L^p(M)}\,,
\end{aligned}$$

as required. □

4.7 Interpolation

Suppose that X and Y are Banach spaces, and that θ is in $(0,1)$. We denote by S the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\}$, and by $\overline{S}$ its closure. We consider the class $\mathcal{F}(X,Y)$ of all functions $F : \overline{S} \to X + Y$ with the following properties:

1. F is continuous and bounded in $\overline{S}$ and analytic in S;

2. the functions $t \mapsto F(it)$ and $t \mapsto F(1+it)$ are continuous from $\mathbb{R}$ into X and Y respectively;

3. $\lim_{|t|\to+\infty} \|F(it)\|_X = 0$ and $\lim_{|t|\to+\infty} \|F(1+it)\|_Y = 0$.

We endow $\mathcal{F}(X,Y)$ with the norm

$$\|F\|_{\mathcal{F}(X,Y)} = \sup\big\{\max(\|F(it)\|_X, \|F(1+it)\|_Y), t \in \mathbb{R}\big\}.$$

We define the complex interpolation space $(X,Y)_{[\theta]}$ between X and Y with parameter θ by

$$(X,Y)_{[\theta]} = \{F(\theta) \,:\, F \in \mathcal{F}(X,Y)\},$$

endowed with the norm

$$\|f\|_{(X,Y)_{[\theta]}} = \inf\big\{\|F\|_{\mathcal{F}(X,Y)} \,:\, F \in \mathcal{F}(X,Y) \,\text{and}\, F(\theta) = f\big\}.$$

For more on the complex interpolation method see, for instance, [BL].

Theorem 4.7.1. *Suppose that θ is in $(0,1)$. The following hold:*

(i) if p_θ is $2/(1-\theta)$, then $\big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]} = L^{p_\theta}(M)$;

(ii) if p_θ is $2/(2-\theta)$, then $\big(\mathfrak{h}^1(M), L^2(M)\big)_{[\theta]} = L^{p_\theta}(M)$.

Proof. First we prove (i). Since $L^\infty(M)$ is included in $\mathfrak{bmo}(M)$, it follows that

$$\big(L^2(M), L^\infty(M)\big)_{[\theta]} \subset \big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]};$$

since $\big(L^2(M), L^\infty(M)\big)_{[\theta]} = L^{p_\theta}(M)$, the inclusion $L^{p_\theta}(M) \subset \big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]}$ follows.

Now we prove the reverse inclusion $\big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]} \subset L^{p_\theta}(M)$. Suppose that f is in the interpolation space $\big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]}$. Then, given $\epsilon > 0$ there exists a function F in $\mathcal{F}(L^2(M), \mathfrak{bmo}(M))$ such that $F(\theta) = f$ and

$$\|F\|_{\mathcal{F}(L^2,\mathfrak{bmo})} \leq \|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}} + \epsilon.$$

Let ϕ be any measurable function which associates to any point x in M a ball $\phi(x)$ in $\mathcal{B}_1(x)$. Furthermore, let $\eta : M \times M \to \mathbb{C}$ be any measurable function such that $|\eta| = 1$. We consider the linear operators $S^{\phi,\eta}$ and T^η which act on a function f in $L^2(M)$ as follows:

$$S^{\phi,\eta} f(x) = \frac{1}{\mu\big(\phi(x)\big)} \int_{\phi(x)} \big[f - f_{\phi(x)}\big]\, \eta(x,\cdot)\,\mathrm{d}\mu \qquad \forall x \in M$$

and

$$T^\eta f(x) = \frac{1}{\mu\big(B_1(x)\big)} \int_{B_1(x)} f\, \eta(x,\cdot)\,\mathrm{d}\mu \qquad \forall x \in M.$$

Then

$$\sup_{\phi,\eta} |S^{\phi,\eta} f| = f^\sharp \qquad \text{and} \qquad \sup_{\eta} |T^\eta f| = N_0 f. \tag{4.7.1}$$

For each ϕ and η as before, consider the functions $S^{\phi,\eta}F$ and $T^\eta F$, where F is in the space $\mathcal{F}(L^2(M), \mathfrak{bmo}(M))$.
We claim that $S^{\phi,\eta}F$ and $T^\eta F$ belong to the class $\mathcal{F}(L^2(M), L^\infty(M))$ and that

$$\|S^{\phi,\eta}F\|_{\mathcal{F}(L^2,L^\infty)} \leq C\,\|F\|_{\mathcal{F}(L^2,\mathfrak{bmo})}$$

and

$$\|T^\eta F\|_{\mathcal{F}(L^2,L^\infty)} \leq C\,\|F\|_{\mathcal{F}(L^2,\mathfrak{bmo})}.$$

Indeed, recall that $g^\sharp \leq 2\,\mathcal{M}g$ and that $\mathcal{M}$ is bounded on $L^2(M)$. Thus,

$$\|S^{\phi,\eta}F(it)\|_2 \leq \|F(it)^\sharp\|_2 \leq 2\,\|\mathcal{M}F(it)\|_2 \leq C\,\|F(it)\|_2.$$

Note that the constant C in the above inequality does not depend on ϕ and η. Moreover,

$$\|S^{\phi,\eta}F(1+it)\|_\infty \leq \|F(1+it)^\sharp\|_\infty \leq \|F(1+it)\|_{\mathfrak{bmo}}.$$

Similarly,

$$\|T^\eta F(it)\|_2 \leq \|\mathcal{M}F(it)\|_2 \leq C\,\|F(it)\|_2;$$

and

$$\|T^\eta F(1+it)\|_\infty \leq \|N_0 F(1+it)\|_\infty \leq \|F(1+it)\|_{\mathfrak{bmo}},$$

where C is independent of η.
Hence

$$\begin{aligned}\|S^{\phi,\eta}f\|_{p_\theta} &= \|S^{\phi,\eta}F(\theta)\|_{(L^2,L^\infty)_{[\theta]}}\\ &\leq \|S^{\phi,\eta}F\|_{\mathcal{F}(L^2,L^\infty)}\\ &\leq C\,\|F\|_{\mathcal{F}(L^2,\mathfrak{bmo})}\\ &\leq C\big(\|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}} + \epsilon\big).\end{aligned}$$

By taking the infimum over all $\epsilon > 0$ we get

$$\|S^{\phi,\eta}f\|_{p_\theta} \leq C\,\|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}}.$$

Now, by taking the supremum over all ϕ and η we obtain the estimate

$$\|f^\sharp\|_{p_\theta} \leq C\,\|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}}. \tag{4.7.2}$$

Similarly, we get

$$\|T^\eta f\|_{p_\theta} \leq C\|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}}$$

and taking the supremum over all functions η we have

$$\|N_0 f\|_{p_\theta} \leq C \, \|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}}. \tag{4.7.3}$$

Now, applying Theorem 4.6.1 and combining (4.7.2) and (4.7.3) we may conclude that

$$\begin{aligned} \|f\|_{p_\theta} &\leq C \, \|Nf\|_{p_\theta} \\ &\leq C \left(\|f^\sharp\|_{p_\theta} + \|N_0 f\|_{p_\theta} \right) \\ &\leq C \, \|f\|_{(L^2,\mathfrak{bmo})_{[\theta]}} \qquad \forall f \in \big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]} \end{aligned}$$

and the required inclusion $\big(L^2(M), \mathfrak{bmo}(M)\big)_{[\theta]} \subset L^{p_\theta}(M)$ follows.

Now (ii) follows from (i) and the duality theorem [BL, Corollary 4.5.2]. □

We end this section by proving that if in Theorem 4.7.1 (ii) we replace $\mathfrak{h}^1(\mathbb{R})$ with the local Hardy space $H^1_1(\mathbb{R})$ of Carbonaro, Mauceri, and Meda, then the conclusion is false. Recall, once more, that here $H^1_1(\mathbb{R})$ does not denote the classical Hardy space on $\mathbb{R}$, but the smaller atomic space obtained by requiring that the supports of atoms are contained in intervals of length at most (say) 2.

Theorem 4.7.2. *If $p_\theta = 2/(2-\theta)$, then $\big(H^1_1(\mathbb{R}), L^2(\mathbb{R})\big)_{[\theta]} \subsetneqq L^{p_\theta}(\mathbb{R})$.*

Proof. Clearly $\big(H^1_1(\mathbb{R}), L^2(\mathbb{R})\big)_{[\theta]} \subset L^{p_\theta}(\mathbb{R})$, for $H^1_1(\mathbb{R})$ is included in $L^1(\mathbb{R})$ and

$$\big(L^1(\mathbb{R}), L^2(\mathbb{R})\big)_{[\theta]} = L^{p_\theta}(\mathbb{R}).$$

For each complex number z, denote by k_z the function

$$k_z(x) = (1+x^2)^{-z/2} \qquad \forall x \in \mathbb{R},$$

and by T_z the associated convolution operator.

If $\operatorname{Re} z > 1$, then k_z is integrable, so that T_z is bounded on $L^p(\mathbb{R})$ for each p in $[1,\infty]$; in particular, T_z is bounded on $L^2(\mathbb{R})$.

We claim that T_z is bounded from $H^1_1(\mathbb{R})$ to $L^1(\mathbb{R})$ for all z such that $\operatorname{Re} z > 0$.

Deferring momentarily the proof of the claim, we show that the required conclusion follows from it. We argue by contradiction. Suppose that $\big(H^1_1(\mathbb{R}), L^2(\mathbb{R})\big)_{[\theta]} = L^{p_\theta}(\mathbb{R})$. Then, the interpolation theorem for analytic families of operators of Cwikel and Janson (see [CJ]) implies that for each $z \in (\theta,\infty)$ the operator T_z is bounded on $L^{p_\theta}(\mathbb{R})$. Since θ is in $(0,1)$, it is always possible to choose z in $(0,1)$. We show that for such z the operator T_z is unbounded on $L^p(\mathbb{R})$ for each p in $[1,\infty)$. Indeed, since the kernel k_z is positive, if the operator T_z were bounded on $L^p(\mathbb{R})$, then k_z should be integrable (see [Ho]), but this is clearly false.

Therefore, to conclude the proof of the theorem it suffices to prove the claim. In view of [MSV, Thm 4.1], it is enough to show that

$$\sup\{\|T_z a\|_1 : a \text{ is an } H^1_1(\mathbb{R})\text{-atom}\} < \infty.$$

Let a be an $H^1(\mathbb{R})$-atom supported in an interval I of length ≤ 2. Since T_z is translation invariant, we may assume that I is centred at 0. By the cancellation property of a

$$\begin{aligned} k_z * a(x) &= \int_I k_z(x-y)\, a(y)\, \mathrm{d}y \\ &= \int_I [k_z(x-y) - k_z(x)]\, a(y)\, \mathrm{d}y, \end{aligned}$$

so that

$$\|T_z a\|_1 = \int_{\mathbb{R}} |k_z * a(x)|\, \mathrm{d}x \leq \int_I |a(y)| \int_{\mathbb{R}} |k_z(x-y) - k_z(x)|\, \mathrm{d}x\, \mathrm{d}y.$$

Now, by Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}} |k_z(x-y) - k_z(x)|\, \mathrm{d}x &= \int_{\mathbb{R}} \left| \int_0^1 \frac{d}{dt} k_z(x-ty)\, \mathrm{d}t \right| \mathrm{d}x \\ &\leq \int_0^1 \int_{\mathbb{R}} |(k_z)'(x-ty)|\, |y|\, \mathrm{d}x\, \mathrm{d}t \\ &\leq \int_{\mathbb{R}} \sup_{t\in(0,1)} |(k_z)'(x-ty)|\, \mathrm{d}x, \end{aligned}$$

where the last inequality follows from the fact that $|y| \leq 1$ since $I \subseteq [-1,1]$. Observe that

$$(k_z)'(x) = -\frac{zx}{(1+x^2)^{z/2+1}}.$$

It is straightforward to show that there exists a constant $C = C(z)$ such that

$$\sup_{t\in(0,1)} |(k_z)'(x-ty)| \leq \frac{C}{(1+|x|)^{1+\operatorname{Re} z}} \qquad \forall x \in \mathbb{R} \quad \forall y \in [-1,1].$$

Combining the above estimates, we get

$$\begin{aligned} \|T_z a\|_1 &\leq C\, \|a\|_1 \int_{\mathbb{R}} \frac{\mathrm{d}x}{(1+|x|)^{1+\operatorname{Re} z}} \\ &\leq C, \end{aligned}$$

where the constant C depends only on z. This concludes the proof of the claim, and of the theorem. □

4.8 On the $\mathfrak{h}^1 - L^1$ boundedness of operators

One of the reasons which make $\mathfrak{h}^1(M)$ useful is that to prove that a linear operator T maps $\mathfrak{h}^1(M)$ to a Banach space X it suffices to prove that T is uniformly bounded on atoms. This extends to the space $\mathfrak{h}^1(M)$ the analogous result for $H^1(M)$ (see [MSV]).

Theorem 4.8.1. *Suppose that p is in $(1,\infty)$ and that T is a $L^1(M)$-valued linear operator defined on $\mathfrak{h}^{1,p}_{fin}(M)$ with the property that*

$$A := \sup\{\|Ta\|_1 : a \text{ is a } p\text{-atom}\} < \infty.$$

Then there exists a unique bounded operator $\widetilde{T}$ from $\mathfrak{h}^1(M)$ to $L^1(M)$ which extends T.

Proof. Suppose that B is a ball of radius $r_B \geq 1$. For each $f \in L^p(B)$ such that $\|f\|_p = 1$ set $a = \mu(B)^{-1/p'} f$, where p' denotes the index conjugate to p. Then a is a p-atom at scale r_B and by Lemma 4.2.3 there exist global p-atoms at scale 1, $a_1, \ldots, a_N$ such that $a = \sum_{j=1}^N c_j a_j$, with $|c_j| \leq C$, where C and N are constants, which depend only on r_B and M. Thus we get

$$\begin{aligned}\|Tf\|_1 &= \|T(\mu(B)^{1/p'} a)\|_1 \\ &\leq \mu(B)^{1/p'} \sum_{j=1}^N |c_j| \, \|Ta_j\|_1 \\ &\leq C\, N\, A\, \mu(B)^{1/p'}\end{aligned}$$

for every $f \in L^p(B)$ such that $\|f\|_p = 1$. In particular, the restriction of T to X^p_k is bounded from X^p_k to $L^1(M)$ for each k. Therefore, T is bounded from X^p to $L^1(M)$. It follows that the transpose operator T^* is bounded from $L^\infty(M)$ to the dual of X^p, which can be identified with the space $L^{p'}_{\text{loc}}(M)$ by Lemma 4.5.1. Therefore, for every f in $L^\infty(M)$ and every p-atom a we have

$$\langle Ta, f\rangle = \langle a, T^* f\rangle = \int_M a\, T^* f \, \mathrm{d}\mu,$$

so that

$$\Big|\int_M a\, T^* f \, \mathrm{d}\mu\Big| = |\, \langle Ta, f\rangle \,| \leq \|Ta\|_1 \|f\|_\infty \leq A\, \|f\|_\infty.$$

Now we show that $T^* f$ belongs to $\mathfrak{bmo}(M)$ and that

$$\|T^* f\|_{\mathfrak{bmo}} \leq 3\, A\, \|f\|_\infty \qquad \forall f \in L^\infty(M).$$

Suppose that B is a ball of radius at most 1; we have

$$\Big[\int_B |T^*f - (T^*f)_B|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \Big|\int_B \varphi\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu\Big|.$$

But

$$\begin{aligned}\int_B \varphi\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu &= \int_B \big(\varphi - \varphi_B\big)\,\big(T^*f - (T^*f)_B\big)\,\mathrm{d}\mu \\ &= \int_B \big(\varphi - \varphi_B\big)\,T^*f\,\mathrm{d}\mu,\end{aligned}$$

and since $\|\varphi\|_{L^p(B)} = 1$

$$\big|\varphi_B\big| \leq \Big[\frac{1}{\mu(B)}\int_B |\varphi|^p \,\mathrm{d}\mu\Big]^{1/p} \leq \mu(B)^{-1/p}.$$

Then

$$\begin{aligned}\|\varphi - \varphi_B\|_{L^p(B)} &\leq \|\varphi\|_{L^p(B)} + |\varphi_B|\,\mu(B)^{1/p} \\ &\leq 2,\end{aligned}$$

so that $(\varphi - \varphi_B)/(2\,\mu(B)^{1/p})$ is a standard p-atom. Therefore

$$\Big|\int_B (\varphi - \varphi_B)\; T^*f\,\mathrm{d}\mu\Big| \leq 2\,A\,\|f\|_\infty\,\mu(B)^{1/p},$$

and hence we conclude that for every ball B of radius at most 1

$$\Big[\frac{1}{\mu(B)}\int_B |T^*f - (T^*f)_B|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \leq 2\,A\,\|f\|_\infty.$$

Now take a ball B of radius exactly equal to 1. We have

$$\Big[\int_B |T^*f|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \Big|\int_B \varphi\,T^*f\,\mathrm{d}\mu\Big|.$$

The function $\varphi/\mu(B)^{1/p}$ is a global p-atom, thus

$$\Big|\int_B \varphi\;T^*f\,\mathrm{d}\mu\Big| \leq A\,\|f\|_\infty\,\mu(B)^{1/p}.$$

Therefore, for every ball B of radius 1

$$\Big[\frac{1}{\mu(B)}\int_B |T^*f|^{p'} \,\mathrm{d}\mu\Big]^{1/p'} \leq A\,\|f\|_\infty.$$

Combining the above estimates, we get

$$\|T^*f\|_{\mathfrak{bmo}} \leq \|T^*f\|_{\mathfrak{bmo}^{p'}} \leq 3\,A\,\|f\|_\infty \qquad \forall f \in L^\infty(M),$$

as required.

Now we prove that T extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. We recall that X^p and $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$ coincide as vector spaces. For every g in $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$ and every f in $L^\infty(M)$

$$\begin{aligned} |\langle Tg, f\rangle| &= |\langle g, T^* f\rangle| \\ &\leq C\, \|g\|_{\mathfrak{h}^1} \|T^* f\|_{\mathfrak{bmo}} \\ &\leq 3\, C\, A\, \|g\|_{\mathfrak{h}^1} \|f\|_\infty. \end{aligned}$$

By taking the supremum of both sides over all functions f in $L^\infty(M)$ with $\|f\|_\infty = 1$, we obtain that

$$\|Tg\|_1 \leq 3\, C\, A\, \|g\|_{\mathfrak{h}^1} \qquad \forall g \in \mathfrak{h}^{1,p}_{\mathrm{fin}}(M).$$

Since $\mathfrak{h}^{1,p}_{\mathrm{fin}}(M)$ is dense in $\mathfrak{h}^1(M)$ with respect to the norm of $\mathfrak{h}^1(M)$, the required conclusion follows by a density argument. □

Suppose that T is a bounded linear operator on $L^2(M)$. Then T is automatically defined on $\mathfrak{h}^{1,2}_{\mathrm{fin}}(M)$. If we assume that

$$A := \sup\{\|Ta\|_1 : a \textit{ is a } 2\textit{-atom}\} < \infty,$$

then, by the previous theorem, the restriction of T to $\mathfrak{h}^{1,2}_{\mathrm{fin}}(M)$ has a unique bounded extension to an operator $\widetilde{T}$ from $\mathfrak{h}^1(M)$ to $L^1(M)$. We wonder if the operators T and $\widetilde{T}$ are consistent, i.e., if they agree on the intersection $\mathfrak{h}^1(M) \cap L^2(M)$ of their domains. As in the case of the same problem on the space $H^1(M)$ (see [MSV, Prop 4.2]), the answer is in the affirmative, as shown in the next proposition.

Proposition 4.8.2. *Suppose that T is a bounded linear operator on $L^2(M)$ and that*

$$A := \sup\{\|Ta\|_1 : a \text{ is a 2-atom}\} < \infty.$$

Denote by $\widetilde{T}$ the unique bounded extension of the restriction of T to $\mathfrak{h}^{1,2}_{\mathit{fin}}(M)$ to an operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then the operators T and $\widetilde{T}$ coincide on $\mathfrak{h}^1(M) \cap L^2(M)$.

Proof. Assume that f is in $L^2(M) \cap L^\infty(M)$ and that g is in $L^2_c(M)$. Denote by T^* the transpose operator of T (as an operator on $L^2(M)$). Then

$$\int_M g\, T^* f \,\mathrm{d}\mu = \int_M Tg\, f \,\mathrm{d}\mu.$$

Since g is in $\mathfrak{h}^{1,2}_{\mathrm{fin}}(M)$ and the operators T and $\widetilde{T}$ agree on $\mathfrak{h}^{1,2}_{\mathrm{fin}}(M)$, we get

$$\int_M Tg\, f \,\mathrm{d}\mu = \int_M \widetilde{T}g\, f \,\mathrm{d}\mu.$$

Denote by $(\widetilde{T})^*$ the transpose of $\widetilde{T}$ as an operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then

$$\int_M \widetilde{T}g\, f \,\mathrm{d}\mu = \left\langle g, (\widetilde{T})^* f \right\rangle.$$

Since $(\widetilde{T})^* f$ is in $\mathfrak{bmo}(M)$ and g is in $\mathfrak{h}^{1,2}_{\mathrm{fin}}(M)$, we can write the last scalar product $\left\langle g, (\widetilde{T})^* f \right\rangle$ (with respect to the duality between $\mathfrak{h}^1(M)$ and $\mathfrak{bmo}(M)$) as

$$\left\langle g, (\widetilde{T})^* f \right\rangle = \int_M g\, (\widetilde{T})^* f \,\mathrm{d}\mu.$$

Thus, combining the above equalities, we obtain that

$$\int_M g\, [T^* f - (\widetilde{T})^* f] \,\mathrm{d}\mu = 0 \qquad \forall g \in L^2_c(M),$$

i.e., for all g in X^2. This implies that $T^* f - (\widetilde{T})^* f = 0$ is in the dual space of X^2, i.e., in $L^2_{\mathrm{loc}}(M)$. Thus $T^* f = (\widetilde{T})^* f$ almost everywhere.
Now, suppose that f is in $L^2(M) \cap L^\infty(M)$ and that g is in $\mathfrak{h}^1(M) \cap L^2(M)$. Then

$$\begin{aligned} \int_M Tg\, f \,\mathrm{d}\mu &= \int_M g\, T^* f \,\mathrm{d}\mu \\ &= \int_M g\, (\widetilde{T})^* f \,\mathrm{d}\mu \\ &= \int_M \widetilde{T}g\, f \,\mathrm{d}\mu. \end{aligned}$$

Thus we have obtained that

$$\int_M [Tg - \widetilde{T}g]\, f \,\mathrm{d}\mu = 0$$

for an arbitrary f in $L^2(M) \cap L^\infty(M)$. This implies that $Tg = \widetilde{T}g$ for all g in $\mathfrak{h}^1(M) \cap L^2(M)$. □

4.9 Applications to SIO

The purpose of this section is to show that the Hardy space $\mathfrak{h}^1(M)$ may be used to obtain endpoint estimates for interesting singular integral operators on Riemannian manifolds.

Hereafter in this chapter, we assume that M is a complete connected noncompact n-dimensional Riemannian manifold with *bounded geometry*, that is with Ricci curvature bounded from below and positive injectivity radius. We view M as a measured metric space with respect to the Riemannian distance and measure. It is well known that manifolds with bounded geometry satisfy the UBSC (see, for instance,

[CMP], where complete references are given), and, as a consequence of the Bishop-Gromov comparison theorem (see, for instance, [Gr1], [Ch, Thm III.4.5], [BC]), they possess the LDP. Furthermore, the AMP property clearly holds. Thus, the theory of local Hardy spaces $\mathfrak{h}^1(M)$ developed in the previous chapters applies to this setting. Denote by $-\mathcal{L}$ the Laplace–Beltrami operator on M: $\mathcal{L}$ is a symmetric operator on $C_c^\infty(M)$ and its closure is a self adjoint operator on $L^2(M)$ which we still denote by $\mathcal{L}$.

For the sake of definiteness here we shall focus on two examples of SIO, which have attracted the attention of quite a large number of researchers, and which are commonly indicated as paradigmatic: the (translated) Riesz transforms and spectral multipliers of $\mathcal{L}$ satisfying a Mihlin type condition at infinity.

The latter operators are treated in [T3] and in [MMV2]. A comparison between the results obtained therein and our result is in order. We extend the result in [T3] by relaxing significantly the assumptions on the geometry of M. Indeed, in [T3] a uniform control of all the covariant derivatives of the Riemann tensor is assumed. In [MMV2] the Riemannian manifold M is assumed to have bounded geometry in the same sense as here, but an additional hypothesis is made, i.e., that the bottom b of the L^2 spectrum of M is strictly positive. This assumption rules out, for instance, all Riemannian manifolds of polynomial volume growth [Br]. The reason for this additional assumption is that the local Hardy space $H_1^1(M)$ used in [MMV2] is known to interpolate with $L^2(M)$ to give $L^p(M)$, $1 < p < 2$, only when $b > 0$. This is a real limitation, for we have proved in Theorem 4.7.2 that the interpolation space between $L^2(\mathbb{R})$ and the local Hardy space $H_1^1(\mathbb{R})$ is *strictly contained* in $L^p(\mathbb{R})$, $1 < p < 2$. The interpolation result proved in Section 4.7 makes all these problems disappear if we use $\mathfrak{h}^1(\mathbb{R})$ instead of $H_1^1(\mathbb{R})$.

Recall that the (translated) Riesz transforms are defined by $R_a := \nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$, where ∇ denotes the Riemannian gradient and a is a positive number. The problem of establishing endpoint estimates for R_a when $p = 1$ in the setting of noncompact Riemannian manifolds has been widely studied. In particular, T. Coulhon and X.T. Doung [CD] proved that if M is locally doubling, of exponential growth, and supports an L^2-scaled Poincaré inequality, then R_a is of weak type 1. Russ [Ru] complemented this result by showing that, for a large enough, R_a is bounded from the atomic Hardy space $H_1^1(M)$ to $L^1(M)$. Note, however, that Russ' result is known to interpolate with $L^2(M)$ to give $L^p(M)$ estimates only when M has bounded geometry and spectral gap (see [CMM1] and the remarks above).

Here we prove, under the assumption that M has bounded geometry, that if a is suitably large, then R_a is bounded from $\mathfrak{h}^1(M)$ to $L^1(M)$. This result complements the analogous result in [CMM1].

4.9.1 Spectral multipliers

First we define the class of symbols which will be needed in the statement of Theorem 4.9.2.

Definition 4.9.1. Suppose that J is a positive integer and that W is in $\mathbb{R}^+$. Denote by $\mathbf{S}_W$ the strip $\{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) \in (-W, W)\}$ and by $H^\infty(\mathbf{S}_W; J)$ the vector space of all bounded *even* holomorphic functions f in $\mathbf{S}_W$ for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C\,(1 + |\zeta|)^{-j} \qquad \forall \zeta \in \mathbf{S}_W \quad \forall j \in \{0, 1, \ldots, J\}. \tag{4.9.1}$$

We denote by $\|f\|_{\mathbf{S}_W;J}$ the infimum of all constants C for which (4.9.1) holds.
If $\|f\|_{\mathbf{S}_W;J} < \infty$ we say that f satisfies a *Mihlin condition of order J at infinity on* $\mathbf{S}_W$.

Denote by ω an even function in $C_c^\infty(\mathbb{R})$ which is supported in $[-3/4, 3/4]$, is equal to 1 in $[-1/4, 1/4]$, and satisfies

$$\sum_{j \in \mathbb{Z}} \omega(t - j) = 1 \qquad \forall t \in \mathbb{R}.$$

Denote by $\mathcal{D}$ the operator $\sqrt{\mathcal{L} - b}$, where b denotes the bottom of the L^2 spectrum of $\mathcal{L}$. Clearly spectral multipliers of $\mathcal{L}$ may equivalently be expressed as spectral multipliers of $\mathcal{D}$ (with a different multiplier). Recall that the heat semigroup is the one-parameter family $\{\mathcal{H}_t\}_{t \geq 0}$ defined, at least on $L^2(M)$, by

$$\mathcal{H}_t f := \mathrm{e}^{-t\mathcal{L}} f \qquad \forall f \in L^2(M).$$

It is well known that $\mathcal{H}_t$ extends to a contraction semigroup on $L^p(M)$ for all $p \in [1, \infty]$. Furthermore, since M has Ricci curvature bounded from below, the heat semigroup $\{\mathcal{H}^t\}$ satisfies the following ultracontractivity estimate [Gr1, Section 7.5]

$$\left|\!\left|\!\left| \mathcal{H}^t \right|\!\right|\!\right|_{1;2} \leq C\,\mathrm{e}^{-bt}\, t^{-n/4}\,(1 + t)^{n/4 - \delta/2} \qquad \forall t \in \mathbb{R}^+ \tag{4.9.2}$$

for some δ in $[0, \infty)$.

The following result should be compared with [T3, Proposition B.5].

Theorem 4.9.2. *Assume that* α *and* β *are as in (0.0.3), and* δ *as in (4.9.2). Denote by* N *the integer* $[\![n/2+1]\!]+1$. *Suppose that* J *is an integer* $> \max\big(N+2+\alpha/2-\delta, N+1/2\big)$. *Then there exists a constant* C *such that*

$$|\!|\!| m(\mathcal{D}) |\!|\!|_{\mathfrak{h}^1} \leq C \, \|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty\big(\mathbf{S}_\beta; J\big).$$

Proof. We claim that it suffices to prove that for each 2-atom a at scale 1, the function $m(\mathcal{D})\, a$ may be written as the sum of 2-atoms supported in balls of $\mathcal{B}_1$, with ℓ^1 norm of the coefficients controlled by $C\,\|m\|_{\mathbf{S}_\beta;J}$.

Indeed, suppose that f is a function in $\mathfrak{h}^1(M)$ and that $f = \sum_j \lambda_j\, a_j$ is an atomic decomposition of f with $\|f\|_{\mathfrak{h}^1} \geq \sum_j |\lambda_j| - \varepsilon$. Since for each 2-atom a we have $\|m(\mathcal{D})a\|_1 \leq C\,\|m\|_{\mathbf{S}_\beta;J}$, by Theorem 4.8.1 $m(\mathcal{D})$ extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then $m(\mathcal{D})f = \sum_j \lambda_j\, m(\mathcal{D})a_j$, where the series is convergent in $L^1(M)$. But the partial sums of the series $\sum_j \lambda_j\, m(\mathcal{D})a_j$ is a Cauchy sequence in $\mathfrak{h}^1(M)$, hence the series is convergent in $\mathfrak{h}^1(M)$, and the sum must be the function $m(\mathcal{D})f$. Therefore

$$\begin{aligned}
\|m(\mathcal{D})f\|_{\mathfrak{h}^1} &\leq \sum_j |\lambda_j|\, \|m(\mathcal{D})a_j\|_{\mathfrak{h}^1} \\
&\leq C\, \|m\|_{\mathbf{S}_\beta;J} \sum_j |\lambda_j| \\
&\leq C\, \|m\|_{\mathbf{S}_\beta;J}\, (\|f\|_{\mathfrak{h}^1} + \varepsilon),
\end{aligned}$$

and the required conclusion follows by taking the infimum of both sides with respect to all admissible decompositions of f.

It has already been shown in the proof of [MMV2, Thm 3.4] that the claim holds for standard atoms. Therefore it suffices to prove it for global atoms.

As in the proof of [MMV2, Thm 3.4], we split the operator $m(\mathcal{D})$ into the sum of two operators and analyse them separately. The functions $\widehat{\omega} * m$ and $m - \widehat{\omega} * m$ (ω is the cut-off function defined above) are bounded. Define the operators $\mathcal{S}$ and $\mathcal{T}$ spectrally by

$$\mathcal{S} = (\widehat{\omega} * m)(\mathcal{D}) \qquad \text{and} \qquad \mathcal{T} = (m - \widehat{\omega} * m)(\mathcal{D}).$$

Thus $m(\mathcal{D}) = \mathcal{S} + \mathcal{T}$.

Suppose that a is a global 2-atom supported in $B(p,1)$ for some p in M. Observe that the function $\widehat{\omega} * m$ is bounded and

$$\|\widehat{\omega} * m\|_\infty \leq C\,\|m\|_\infty \leq C\,\|m\|_{\mathbf{S}_\beta;J}. \tag{4.9.3}$$

Therefore, $(\widehat{\omega} * m)(\mathcal{D})$ is bounded on $L^2(M)$ by the spectral theorem, and

$$|||(\widehat{\omega} * m)(\mathcal{D})|||_2 \leq \|\widehat{\omega} * m\|_\infty \leq C \|m\|_{\mathbf{S}_\beta;J}.$$

We have used (4.9.3) in the second inequality above. Observe that the support of the kernel of the operator $(\widehat{\omega} * m)(\mathcal{D})$ is contained in $\{(x,y) : d(x,y) \leq 1\}$, for $\mathcal{L}$ possesses the finite propagation speed property, hence the function $(\widehat{\omega} * m)(\mathcal{D})a$ is supported in the ball with centre p and radius 2. Moreover,

$$\begin{aligned}\|(\widehat{\omega} * m)(\mathcal{D})a\|_2 &\leq C\, |||(\widehat{\omega} * m)(\mathcal{D})|||_2 \, \|a\|_2 \\ &\leq C \|m\|_{\mathbf{S}_\beta;J}\, \mu(B(p,1))^{-1/2} \\ &\leq C \|m\|_{\mathbf{S}_\beta;J}\, \mu(B(p,2))^{-1/2}.\end{aligned}$$

We have used the LDP in the last inequality. Thus, $(\widehat{\omega} * m)(\mathcal{D})a$ is a constant multiple of a global atom at scale 2 and, by Lemma 4.2.3,

$$\|(\widehat{\omega} * m)(\mathcal{D})a\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathbf{S}_\beta;J}.$$

Now we analyse $\mathcal{T}a$. In the proof of [MMV2, Thm 3.4] it is shown that if b is a standard 2-atom, then $\mathcal{T}b$ may be decomposed as

$$\mathcal{T}b = \sum_{j=1}^{\infty} \lambda_j \, b_j,$$

where b_j is an atom at scale $j+2$, and

$$|\lambda_j| \leq C \|m\|_{\mathbf{S}_\beta;J}\, j^{N+\alpha/2-J-\delta} \qquad \forall j = 1,2,3,\ldots \tag{4.9.4}$$

A close examination of the proof reveals that the cancellation property of b is used to show that the atoms b_j also have this property, but it is not required in the proof of (4.9.4). Thus, by arguing as in Step IV in the proof of [MMV2, Thm 3.4], we may conclude that $\mathcal{T}a$ may be written as

$$\mathcal{T}a = \sum_{j=1}^{\infty} \lambda_j \, a_j,$$

where a_j is a global 2-atom at scale $j+2$ and λ_j satisfies estimate (4.9.4). Now Lemma 4.2.3 (and its version for Riemannian manifolds [MMV2, Lemma 5.7]) imply that there exists a constant C such that

$$\|a_j\|_{\mathfrak{h}^1} \leq C\, j \qquad j = 1,2,3,\ldots$$

Therefore

$$\|\mathcal{T}a\|_{\mathfrak{h}^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \sum_{j=1}^{\infty} j^{1+N+\alpha/2-J-\delta} \leq C\,\|m\|_{\mathbf{S}_\beta;J},$$

where C is independent of a. Hence $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(M)$ to $\mathfrak{h}^1(M)$.

So far, we have proved that there exists a constant C such that for every global atom a

$$\|\mathcal{S}a\|_{\mathfrak{h}^1} + \|\mathcal{T}a\|_{\mathfrak{h}^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J}.$$

Hence

$$\|m(\mathcal{D})a\|_{\mathfrak{h}^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J}.$$

The required conclusion follows from the claim at the beginning of the proof. □

4.9.2 The translated Riesz transform

We shall need the following local estimate for the space derivative of the heat kernel.

Lemma 4.9.3. *There exists $\eta > 0$ such that for all $y \in M$, $t > 0$*

$$\int_{d(x,y)\geq\sqrt{t}} |\nabla_x h_s(x,y)|\,\mathrm{d}\mu(x) \leq \begin{cases} C e^{-\eta t/s} s^{-1/2} & \forall s \in (0,1] \\ \\ C e^{-\eta t/s} e^{cs} s^{-1/2} & \forall s \in (1,\infty) \end{cases}.$$

This result is stated in [CD], though its proof is given in full detail only in the case where M is globally doubling. However, it is not hard to modify the argument to produce a proof of Lemma 4.9.3. The proof hinges on upper estimates for the heat kernel and its time derivatives (see [Gr1, Gr2, D]) and on weighted estimates for the space derivative of the heat kernel ([CD]).

Theorem 4.9.4. *There exists $a > 0$ such that the translated Riesz transform $\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$ is bounded from $\mathfrak{h}^1(M)$ to $L^1(M)$.*

Proof. We know that if a is large enough, then $\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$ is bounded from $H^1(M)$ to $L^1(M)$ by [Ru]. Therefore it suffices to show that the kernel k of $\nabla(a\mathcal{I}+\mathcal{L})^{-1/2}$ satisfies the condition

$$\sup_{y\in M} \int_{B(y,2)^c} |k(x,y)|\ \mathrm{d}\mu(x) < \infty \tag{4.9.5}$$

and then apply [CMM3, Prop 4.5]. The kernel is given, off the diagonal, by

$$k(x,y) = \int_0^{+\infty} \frac{\mathrm{e}^{-as}}{\sqrt{s}} \nabla_x h_s(x,y)\ \mathrm{d}s.$$

By Fubini's theorem, we obtain

$$\begin{aligned}\int_{B(y,2)^c} |k(x,y)| \ \mathrm{d}\mu(x) &= \int_{B(y,2)^c} \left| \int_0^{+\infty} \frac{\mathrm{e}^{-as}}{\sqrt{s}} \nabla_x h_s(x,y) \ \mathrm{d}s \right| \mathrm{d}\mu(x) \\ &\leq \int_0^{+\infty} \frac{\mathrm{e}^{-as}}{\sqrt{s}} \int_{d(x,y)\geq 2} |\nabla_x h_s(x,y)| \,\mathrm{d}\mu(x) \,\mathrm{d}s \\ &:= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_0^1 \frac{\mathrm{e}^{-as}}{\sqrt{s}} \int_{d(x,y)\geq 2} |\nabla_x h_s(x,y)| \,\mathrm{d}\mu(x) \,\mathrm{d}s$$

and

$$I_2 = \int_1^{+\infty} \frac{\mathrm{e}^{-as}}{\sqrt{s}} \int_{d(x,y)\geq 2} |\nabla_x h_s(x,y)| \,\mathrm{d}\mu(x) \,\mathrm{d}s.$$

Now we apply Lemma 4.9.3 to estimate the inner integrals. We get

$$I_1 \leq C \int_0^1 \frac{\mathrm{e}^{-as-4\eta/s}}{s} \,\mathrm{d}s \leq C \int_0^1 \frac{\mathrm{e}^{-4\eta/s}}{s^2} \,\mathrm{d}s = C\, \frac{\mathrm{e}^{-4\eta}}{4\eta},$$

and

$$I_2 \leq C \int_1^{+\infty} \frac{\mathrm{e}^{-(a-c)s-4\eta/s}}{s} \,\mathrm{d}s \leq C \int_1^{+\infty} \mathrm{e}^{-(a-c)s} \,\mathrm{d}s.$$

Note that the last integral converges only when $a > c$. Therefore (4.9.5) holds if $a > c$ and for such a the operator $\nabla(a\mathcal{I} + \mathcal{L})^{-1/2}$ extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$, as required. □

Chapter 5

Hardy type spaces for SIO at infinity

5.1 Notation

Basic assumptions 5.1.1. Hereafter, we assume that M is an *unbounded* connected noncompact complete Riemannian manifold that possesses the following properties:

(i) the injectivity radius $\mathrm{Inj}(M)$ is positive;

(ii) there exists a positive number κ such that the Ricci curvature satisfies the bound
$$\mathrm{Ric}(M) \geq -\kappa^2;$$

(iii) the bottom b of the $L^2(M)$ spectrum of $\mathcal{L}$ is strictly positive.

It is known that for manifolds with Ricci curvature bounded from below the assumption $b > 0$ is equivalent to the isoperimetric property (4.1.2), and that this property implies that M has exponential volume growth, therefore μ is nondoubling. For the proof, see [CMM1].
Set $\beta = \limsup_{r\to\infty} \big[\log \mu\big(B(o,r)\big)\big]/(2r)$, where o is any reference point in M. By a result of R. Brooks $b \leq \beta^2$ [Br].

Denote by $\{\mathcal{H}^t\}$ the *heat semigroup* acting on $L^1(M) + L^2(M)$, which possesses the following properties:

(a) the restriction of $\{\mathcal{H}^t\}$ to $L^1(M)$ is a strongly continuous semigroup of contractions;

(b) the restriction of $\{\mathcal{H}^t\}$ to $L^2(M)$ is strongly continuous, and has *spectral gap* $b > 0$, i.e.,

$$\|\mathcal{H}^t f\|_2 \leq \mathrm{e}^{-bt}\,\|f\|_2 \qquad \forall f \in L^2(M) \quad \forall t \in \mathbb{R}^+;$$

(c) $\{\mathcal{H}^t\}$ is *ultracontractive*, i.e., for every t in $\mathbb{R}^+$ the operator $\mathcal{H}^t$ maps $L^1(M)$ into $L^\infty(M)$.

Since M has Ricci curvature bounded from below, the heat semigroup $\{\mathcal{H}^t\}$ satisfies the following ultracontractivity estimate [Gr1, Section 7.5]

$$\left\vert\!\left\vert\!\left\vert \mathcal{H}^t \right\vert\!\right\vert\!\right\vert_{1;2} \leq C\,\mathrm{e}^{-bt}\,t^{-n/4}\,(1+t)^{n/4-\delta/2} \qquad \forall t \in \mathbb{R}^+ \tag{5.1.1}$$

for some δ in $[0,\infty)$.

We recall some further properties of the heat semigroup $\{\mathcal{H}^t\}$:

(i) $\{\mathcal{H}^t\}$ is a strongly continuous semigroup of contractions on $L^1(M) + L^2(M)$;

(ii) since for each p in $[1,2]$ the space $L^p(M)$ is continuously embedded in $L^1(M) + L^2(M)$, we may consider the restriction $\mathcal{H}^t_p$ of the operator $\mathcal{H}^t$ to $L^p(M)$. Then $\{\mathcal{H}^t_p\}$ is strongly continuous on $L^p(M)$, and satisfies the estimate

$$\|\mathcal{H}^t_p f\|_p \leq \mathrm{e}^{-2b\,(1-1/p)\,t}\,\|f\|_p \qquad \forall f \in L^p(M) \quad \forall t \in \mathbb{R}^+; \tag{5.1.2}$$

(iii) for each t in $\mathbb{R}^+$ the operator $\mathcal{H}^t$ maps $L^1(M)$ into $L^1(M) \cap L^2(M)$. By interpolation $\mathcal{H}^t$ maps $L^1(M)$ into $L^p(M)$ for each p in $[1,2]$.

5.2 New local Hardy spaces

Denote by $-\mathcal{G}$ the infinitesimal generator of $\{\mathcal{H}^t\}$ on $L^1(M) + L^2(M)$. Since $\{\mathcal{H}^t\}$ is contractive on $L^1(M) + L^2(M)$, the spectrum of $\mathcal{G}$ is contained in the right half plane. Then, for every σ in $\mathbb{R}^+$ we may consider the resolvent operator $(\sigma\mathcal{I} + \mathcal{G})^{-1}$ of $\{\mathcal{H}^t\}$, that we denote by $\mathcal{R}_\sigma$. We denote by $\mathcal{R}_{\sigma,p}$ the restriction of $\mathcal{R}_\sigma$ to $L^p(M)$, and by $-\mathcal{G}_p$ the generator of $\{\mathcal{H}^t_p\}$. Obviously $\mathcal{R}_{\sigma,p}$ is the resolvent of $\{\mathcal{H}^t_p\}$ and $-\mathcal{G}_p$ is the restriction of $-\mathcal{G}$ to $\mathrm{Dom}(\mathcal{G}_p)$, which coincides with $\mathcal{R}_\sigma\big(L^p(M)\big)$.

For every σ in $\mathbb{R}^+$ denote by $\mathcal{U}_\sigma$ the operator $\mathcal{G}\mathcal{R}_\sigma$. Observe that

$$\mathcal{U}_\sigma = \mathcal{I} - \sigma\,\mathcal{R}_\sigma,$$

so that $\mathcal{U}_\sigma$ is bounded on $L^1(M) + L^2(M)$, and its restriction $\mathcal{U}_{\sigma,p}$ to $L^p(M)$ is bounded on $L^p(M)$ for every $p \in [1,2]$. Moreover $\mathcal{U}_\sigma$ and $\mathcal{H}^t$ commute for every t in $\mathbb{R}^+$.

Proposition 5.2.1. *For each positive integer k the following hold:*

(i) if p is in $(1, 2]$, then the operator $\mathcal{U}^k_{\sigma,p}$ is an isomorphism of $L^p(M)$;

(ii) the operator $\mathcal{U}^k_\sigma$ is injective on $L^1(M) + L^2(M)$;

(iii) if σ_1 and σ_2 are in $(\beta^2 - b, \infty)$, then the operator $\mathcal{U}_{\sigma_1}^{-1}\mathcal{U}_{\sigma_2}$, initially defined on $L^2(M)$, extends to an isomorphism of $\mathfrak{h}^1(M)$.

Proof. Statements (i) and (ii) have been proved in [MMV2, Proposition 2.4]. The proof of (iii) follows the same line as the proof of [MMV2, Theorem 3.5], but uses Theorem 4.9.2 in place of Theorem 3.4 therein. □

Definition 5.2.2. For each positive integer k and for each σ in $\mathbb{R}^+$ we denote by $\mathfrak{X}^k_\sigma(M)$ the Banach space of all $L^1(M)$ functions f such that $\mathcal{U}_\sigma^{-k} f$ is in $\mathfrak{h}^1(M)$, endowed with the norm

$$\|f\|_{\mathfrak{X}^k_\sigma(M)} = \|\mathcal{U}_\sigma^{-k} f\|_{\mathfrak{h}^1(M)}.$$

By definition, $\mathcal{U}_\sigma^{-k}$ is an isometric isomorphism between $\mathfrak{X}^k_\sigma(M)$ and $\mathfrak{h}^1(M)$. Note that the definition of $\mathfrak{X}^k_\sigma(M)$ is similar to that of the space $X^k_\sigma(M)$ in [MMV2]. The space $\mathfrak{X}^k_\sigma(M)$ is isomorphic to $\mathfrak{h}^1(M)$ via $\mathcal{U}_\sigma^{-k}$, whereas $X^k_\sigma(M)$ is isomorphic to the smaller space $H^1(M)$ (this is the local Hardy space defined in [CMM1]) via the same operator $\mathcal{U}_\sigma^{-k}$.

Remark 5.2.3. It follows directly from the definition that the space $\mathfrak{X}^k_\sigma(M)$ is continuously included in $X^k_\sigma(M)$. Moreover $\mathfrak{X}^k_\sigma(M)$ is continuously included in $L^1(M)$.

Indeed, suppose that f is in $\mathfrak{X}^k_\sigma(M)$. Then

$$\begin{aligned}
\|f\|_1 &= \big\|\mathcal{U}^k_\sigma \mathcal{U}_\sigma^{-k} f\big\|_1 \\
&\leq \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|\mathcal{U}_\sigma^{-k} f\big\|_1 \\
&\leq \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|\mathcal{U}_\sigma^{-k} f\big\|_{\mathfrak{h}^1} \\
&= \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|f\big\|_{\mathfrak{X}^k_\sigma},
\end{aligned}$$

as required. Note that the last inequality is a consequence of the fact that $\mathfrak{h}^1(M)$ is continuously included in $L^1(M)$.

Definition 5.2.4. For each positive integer k, and for each σ in $\mathbb{R}^+$ we denote by $\mathfrak{Y}^k_\sigma(M)$ the Banach dual of $\mathfrak{X}^k_\sigma(M)$.

Remark 5.2.5. Since $\mathcal{U}_\sigma^{-k}$ is an isometric isomorphism between $\mathfrak{X}^k_\sigma(M)$ and $\mathfrak{h}^1(M)$, its adjoint map $\big(\mathcal{U}_\sigma^{-k}\big)^*$ is an isometric isomorphism between the dual of $\mathfrak{h}^1(M)$, i.e., $\mathfrak{bmo}(M)$, and $\mathfrak{Y}^k_\sigma(M)$. Hence

$$\big\|\big(\mathcal{U}_\sigma^{-k}\big)^* f\big\|_{\mathfrak{Y}^k_\sigma} = \|f\|_{\mathfrak{bmo}}.$$

We shall prove an interpolation result for the spaces $\mathfrak{X}^k_\sigma(M)$ and their duals. We recall the following result.

Proposition 5.2.6 ([MMV2, Prop. 2.13])**.** *Suppose that (X^0, X^1) and (Y^0, Y^1) are interpolation pairs of Banach spaces and that $\mathcal{T}$ is a bounded linear map from $X^0 + X^1$ to $Y^0 + Y^1$, such that the restrictions $\mathcal{T} : X^0 \to Y^0$ and $\mathcal{T} : X^1 \to Y^1$ are isomorphisms. Then for every θ in $(0,1)$ the restriction $\mathcal{T} : X_\theta \to Y_\theta$ is an isomorphism.*

Proposition 5.2.7. *Suppose that σ is in $\mathbb{R}^+$, k is a positive integer, and θ is in $(0,1)$. The following hold:*

(i) if $1/p = 1 - \theta/2$, then $\big(\mathfrak{X}^k_\sigma(M), L^2(M)\big)_{[\theta]} = L^p(M)$ with equivalent norms;

(ii) if $1/q = (1-\theta)/2$, then $\big(L^2(M), \mathfrak{Y}^k_\sigma(M)\big)_{[\theta]} = L^q(M)$ with equivalent norms.

Proof. First we prove (i). We recall that $\mathcal{U}^k_\sigma$ is an isomorphism of $L^2(M)$ and between $\mathfrak{h}^1(M)$ and $\mathfrak{X}^k_\sigma(M)$ and that $\mathcal{U}^k_\sigma$ is injective on $L^1(M) + L^2(M)$. Therefore $\mathcal{U}^k_\sigma$ is an isomorphism of $\mathfrak{h}^1(M) + L^2(M)$ onto $\mathfrak{X}^k_\sigma(M) + L^2(M)$. Then we may apply Proposition 5.2.6 with $\mathcal{U}^k_\sigma$ in place of $\mathcal{T}$, $X^0 = \mathfrak{h}^1(M)$, $Y^0 = \mathfrak{X}^k_\sigma(M)$, $X^1 = L^2(M) = Y^1$ and we obtain that $\mathcal{U}^k_\sigma$ is an isomorphism between $\big(\mathfrak{h}^1(M), L^2(M)\big)_{[\theta]}$ and $\big(\mathfrak{X}^k_\sigma(M), L^2(M)\big)_{[\theta]}$. By Theorem 4.7.1

$$\big(\mathfrak{h}^1(M), L^2(M)\big)_{[\theta]} = L^p(M).$$

Therefore, $\mathcal{U}^k_\sigma$ restricted to $L^p(M)$ is an isomorphism between tha spaces $L^p(M)$ and $\big(\mathfrak{X}^k_\sigma(M), L^2(M)\big)_{[\theta]}$. But the restriction of $\mathcal{U}^k_\sigma$ to $L^p(M)$ is just $\mathcal{U}^k_{\sigma,p}$, which is an isomorphism of $L^p(M)$ by Proposition 5.2.1. Hence $\big(\mathfrak{X}^k_\sigma(M), L^2(M)\big)_{[\theta]} = L^p(M)$ and their norms are equivalent.

Now (ii) follows from (i) by the duality theorem. □

An important consequence of Theorem 4.9.2 is that, for fixed k, the spaces $\mathfrak{X}^k_\sigma(M)$ do not depend on the parameter σ, as σ varies in the range $(\beta^2 - b, \infty)$.

Theorem 5.2.8. *The following hold:*

(i) if σ_1 and σ_2 are in $(\beta^2 - b, \infty)$, then $\mathfrak{X}^k_{\sigma_1}(M)$ and $\mathfrak{X}^k_{\sigma_2}(M)$ agree as vector spaces, and their norms are equivalent;

(ii) if σ is in $(\beta^2 - b, \infty)$, then $\mathfrak{h}^1(M) \supset \mathfrak{X}^1_\sigma(M) \supset \mathfrak{X}^2_\sigma(M) \supset \cdots$ with continuous inclusions.

Proof. First we prove (i). Consider the operator $\mathcal{T}_{\sigma_1,\sigma_2}$, defined on $L^2(M)$ by

$$\mathcal{T}_{\sigma_1,\sigma_2} = \mathcal{U}_{\sigma_1}^{-1}\mathcal{U}_{\sigma_2}.$$

Since $\mathcal{T}_{\sigma_1,\sigma_2}$ extends to an isomorphism of $\mathfrak{h}^1(M)$ by Proposition 5.2.1 and clearly $\mathcal{U}_{\sigma_1}\mathcal{T}_{\sigma_1,\sigma_2}\mathcal{U}_{\sigma_2}^{-1} = \mathcal{I}$, the identity is an isomorphism between $\mathfrak{X}^1_{\sigma_1}(M)$ and $\mathfrak{X}^1_{\sigma_2}(M)$, as required to conclude the proof of (i) in the case where $k = 1$. The proof in the case where $k \geq 2$ is similar, and is omitted.

Note that (ii) is equivalent to the boundedness of $\mathcal{U}_\sigma$ on $\mathfrak{h}^1(M)$. Since $\mathcal{U}_\sigma = \mathcal{I} - \sigma\,(\sigma\mathcal{I} + \mathcal{L})^{-1}$, it suffices to prove that the resolvent operator $(\sigma\mathcal{I} + \mathcal{L})^{-1}$ is bounded on $\mathfrak{h}^1(M)$. This follows from the fact that the associated spectral multiplier $\xi \mapsto (\sigma + b + \xi^2)^{-1}$ satisfies the hypotheses of Theorem 4.9.2, and (ii) follows. □

If σ_1 and σ_2 are in $(\beta^2 - b, \infty)$, then the spaces $\mathfrak{Y}^k_{\sigma_1}(M)$ and $\mathfrak{Y}^k_{\sigma_2}(M)$ agree, with equivalent norms.

Definition 5.2.9. Suppose that k is a positive integer. The spaces $\mathfrak{X}^k_{\beta^2}(M)$ and $\mathfrak{Y}^k_{\beta^2}(M)$ will be denoted simply by $\mathfrak{X}^k(M)$ and $\mathfrak{Y}^k(M)$ respectively.

5.3 Spectral multipliers and Riesz transforms on manifolds

In this section we apply the results of Section 4.9.1 to spectral multipliers and to the first order Riesz transform associated to the Laplace–Beltrami operator. In order to deal with a wider class of operators, we also define a larger space of functions that may be singular also at the points $\pm iW$.

Definition 5.3.1. Suppose that J is a positive integer, that τ is in $[0, \infty)$, and that W is in $\mathbb{R}^+$. The space $H(\mathbf{S}_W; J, \tau)$ is the vector space of all holomorphic *even* functions f in the strip $\mathbf{S}_W$ for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C \max\bigl(|\zeta^2 + W^2|^{-\tau-j}, |\zeta|^{-j}\bigr) \qquad \forall \zeta \in \mathbf{S}_W \quad \forall j \in \{0, 1, \ldots, J\}. \quad (5.3.1)$$

We denote by $\|f\|_{\mathbf{S}_W;J,\tau}$ the infimum of all constants C for which (5.3.1) holds.

For notational convenience, denote by $\mathcal{D}$ the operator $\sqrt{\mathcal{L} - b}$, defined via the spectral theorem.

Theorem 5.3.2. *Assume that α and β are as in (0.0.3), and δ as in (5.1.1). Suppose that τ is in $[0,\infty)$, that J and k are integers, with $k > \tau + J$ and $J > \max\big(N+2+\alpha/2-\delta, N+1/2\big)$, where N denotes the integer $[\![n/2+1]\!]+1$. The following hold:*

(i) if $b < \beta^2$, then there exists a constant C such that

$$|\!|\!|m(\mathcal{D})|\!|\!|_{\mathfrak{h}^1;L^1} \leq C\,\|m\|_{\mathbf{S}_{\beta;J,\tau}} \qquad \forall m \in H(\mathbf{S}_\beta;J,\tau)$$

and

$$|\!|\!|m(\mathcal{D})^t|\!|\!|_{L^\infty;\mathfrak{bmo}} \leq C\,\|m\|_{\mathbf{S}_{\beta;J,\tau}} \qquad \forall m \in H(\mathbf{S}_\beta;J,\tau),$$

where $m(\mathcal{D})^t$ denotes the transpose operator of $m(\mathcal{D})$;

(ii) if $b = \beta^2$, then there exists a constant C such that

$$|\!|\!|m(\mathcal{D})|\!|\!|_{\mathfrak{X}^k;\mathfrak{h}^1} \leq C\,\|m\|_{\mathbf{S}_{\beta;J,\tau}} \qquad \forall m \in H(\mathbf{S}_\beta;J,\tau)$$

and

$$|\!|\!|m(\mathcal{D})^t|\!|\!|_{\mathfrak{bmo};\mathfrak{Y}^k} \leq C\,\|m\|_{\mathbf{S}_{\beta;J,\tau}} \qquad \forall m \in H(\mathbf{S}_\beta;J,\tau),$$

where $m(\mathcal{D})^t$ denotes the transpose operator of $m(\mathcal{D})$.

Proof. The proof is almost *verbatim* the same as the proof of [MMV2, Theorem 4.3], and it is omitted. □

We conclude this section with an endpoint result for the first order Riesz transform.

Theorem 5.3.3. *Assume that α and β are as in (0.0.3), and δ as in (5.1.1). Suppose that $b = \beta^2$ and that k is an integer $> \max\big(N+2+\alpha/2-\delta, N+1/2\big)$, where N denotes the integer $[\![n/2+1]\!]+1$. Then the first order Riesz transform $\nabla\mathcal{L}^{-1/2}$ is bounded from $\mathfrak{X}^k(M)$ to $L^1(M)$.*

Proof. The proof is, *mutatis mutandis*, similar to that of [MMV2, Theorem 4.6], and it is omitted. □

5.4 Atomic decomposition of $\mathfrak{X}^k(M)$

The purpose of this section is to illustrate an atomic decomposition of $\mathfrak{X}^k(M)$. An atom A in $\mathfrak{X}^k(M)$ will be a standard atom in $\mathfrak{h}^1(M)$ satisfying an additional infinite dimensional cancellation condition, expressed as orthogonality of A to the space of k-harmonic functions in a neighbourhood of the support of A.

Definition 5.4.1. Suppose that k is a positive integer and that B is a ball in M. We say that a function V in $L^2(M)$ is *k-harmonic* on $\overline{B}$ if $\mathcal{L}^k V$ is zero (in the sense of distributions) in a neighbourhood of $\overline{B}$. We shall denote by P_B^k the space of k-harmonic functions on $\overline{B}$. Furthermore, let Q_B^k denote the space of *k-quasi-harmonic* functions on $\bar{B}$, i.e., the subspace of $L^2(M)$ consisting of all the functions V such that $\mathcal{L}^k V$ is constant (in the sense of distributions) in a neighbourhood of $\bar{B}$.

Remark 5.4.2. Observe that P_B^k coincides with the space of $L^2(M)$ functions V that are smooth in a neighbourhood of $\overline{B}$ and such that $\mathcal{L}^k V$ is zero therein. Indeed, each V in P_B^k is in $L^2(M)$ and $\mathcal{L}^k V$ is zero in the sense of distributions in a neighbourhood of $\overline{B}$. It follows from elliptic regularity that V is smooth on this neighbourhood. A similar remark applies to Q_B^k.

As a direct consequence of the definition of P_B^k we have the following inclusions:

$$P_B^1 \subset P_B^2 \subset \cdots ; \qquad (P_B^1)^\perp \supset (P_B^2)^\perp \supset \cdots$$

Recall that for each ball B in M we denote by $L^2(B)$ the space of all $L^2(M)$ functions supported in the ball $\overline{B}$.

The following result is the analogue for the space P_B^k of [MMV3, Prop. 3.3].

Proposition 5.4.3. *Suppose that k is a positive integer, and that B is a ball in M. The following hold:*

(i) $(P_B^k)^\perp = \big\{F \in L^2(M) : \mathcal{L}^{-k} F \in L^2(B)\big\}$;

(ii) $\mathcal{L}^{-k}\big((P_B^k)^\perp\big)$ *is contained in* $L^2(B) \cap \mathrm{Dom}(\mathcal{L}^k)$. *Furthermore, functions in* $(P_B^k)^\perp$ *have support contained in* $\overline{B}$*;*

(iii) $\mathcal{U}_{\beta^2}^{-k}\big((P_B^k)^\perp\big)$ *is contained in* $L^2(B)$.

Proof. We begin by proving (i). First we show that $(P_B^k)^\perp$ is contained in $\big\{F \in L^2(M) : \mathcal{L}^{-k} F \in L^2(B)\big\}$.

Suppose that F is in $(P_B^k)^\perp$. Therefore, F is in $L^2(M)$ and, since $\mathcal{L}^{-k}$ is bounded, also $\mathcal{L}^{-k}F$ is $L^2(M)$. To show that the support of $\mathcal{L}^{-k}F$ is contained in $\overline{B}$ it suffices to prove that $(\mathcal{L}^{-k}F, \mathbf{1}_{B'}) = 0$ for every ball B' contained in $(\overline{B})^c$. Since $\mathcal{L}$ is self adjoint,

$$(\mathcal{L}^{-k}F, \mathbf{1}_{B'}) = (F, \mathcal{L}^{-k}\mathbf{1}_{B'}).$$

The function $\mathcal{L}^{-k}\mathbf{1}_{B'}$ is in P_B^k, hence the last inner product vanishes, as required.

Next we prove that $\big\{F \in L^2(M) : \mathcal{L}^{-k}F \in L^2(B)\big\}$ is contained in P_B^k. Suppose that $\mathcal{L}^{-k}F$ is in $L^2(B)$. Observe that $\mathcal{L}^{-k}F$ is in $\mathrm{Dom}(\mathcal{L}^k)$ and that $F = \mathcal{L}^k\mathcal{L}^{-k}F$.

Suppose now that V is in P_B^k. Then V is smooth in a neighbourhood of $\overline{B}$ by Remark 5.4.2, and, since $\mathcal{L}$ is self adjoint,

$$(F, V) = (\mathcal{L}^k \mathcal{L}^{-k} F, V) = (\mathcal{L}^{-k} F, \mathcal{L}^k V) = 0.$$

The last equality follows from the fact that $\mathcal{L}^k V = 0$ in a neighbourhood of $\overline{B}$.

Next we prove (ii). Clearly if F is in $L^2(M)$, then $\mathcal{L}^{-k} F$ is in $\mathrm{Dom}(\mathcal{L}^k)$ by abstract set theory. Moreover $\mathcal{L}^{-k} F$ is in $L^2(B)$ by (i), and the first statement of (ii) follows.

To prove the second statement of (ii), observe that the support of $\mathcal{L}^{-k} F$ is contained in $\overline{B}$, hence so is the support of $\mathcal{L}^k \mathcal{L}^{-k} F$, i.e., of F.

Finally, we prove (iii). Observe that $\mathcal{U}_{\beta^2}^{-k} = (\beta^2 \mathcal{I} + \mathcal{L})^k \, \mathcal{L}^{-k}$. Since $\mathcal{L}^{-k}\big((P_B^k)^\perp\big)$ is contained in $L^2(B) \cap \mathrm{Dom}(\mathcal{L}^k)$ by (ii), the statement follows from the fact that $\mathcal{L}^j\big(L^2(B) \cap \mathrm{Dom}(\mathcal{L}^k)\big)$ is contained in $L^2(B)$ for all j in $\{0, 1, \ldots, k\}$. $\square$

Definition 5.4.4. Suppose that k is a positive integer. A *standard $\mathfrak{X}^k$-atom* (at scale 1) associated to the ball B of radius ≤ 1 is a function A in $L^2(M)$, supported in B, such that

(i) A is in $(Q_B^k)^\perp$;

(ii) $\|A\|_2 \leq \mu(B)^{-1/2}$.

A *global $\mathfrak{X}^k$-atom* (at scale 1) associated to the ball B of radius 1 is a function A in $L^2(M)$, supported in B, such that

(i) A is in $(P_B^k)^\perp$;

(ii) $\|A\|_2 \leq \mu(B)^{-1/2}$.

An *$\mathfrak{X}^k$-atom* is either a standard $\mathfrak{X}^k$-atom or a global $\mathfrak{X}^k$-atom.

Clearly, a global $\mathfrak{X}^k$-atom is also a standard global atom (and it is also a standard atom: indeed, condition (i) implies that the integral of A vanishes, since χ_{2B} is in P_B^k).

Remark 5.4.5. Note that if A is a global $\mathfrak{X}^k$-atom supported in a ball B of radius 1, then $\mathcal{L}^{-k} A / |||\mathcal{L}^{-k}|||_2$ is a global atom with support contained in $\overline{B}$.

Indeed A is in $(P_B^k)^\perp$, so that $\mathcal{L}^{-k} A$ is in $L^2(B)$ by Proposition 5.4.3 (i). Moreover

$$\begin{aligned} \|\mathcal{L}^{-k} A\|_2 &\leq |||\mathcal{L}^{-k}|||_2 \, \|A\|_2 \\ &\leq |||\mathcal{L}^{-k}|||_2 \, \mu(B)^{-1/2}. \end{aligned}$$

Note also that a global $\mathfrak{X}^k$-atom A is in $\mathfrak{X}^k(M)$ and

$$\|A\|_{\mathfrak{X}^k} \leq \|\mathcal{U}_{\beta^2}^{-k}\|_2. \tag{5.4.1}$$

Indeed, since A is in $(P_B^k)^\perp$, the function $\mathcal{U}_{\beta^2}^{-k}A$ is in $L^2(B)$ by Proposition 5.4.3 (iii) and

$$\begin{aligned}\|\mathcal{U}_{\beta^2}^{-k}A\|_2 &\leq \|\mathcal{U}_{\beta^2}^{-k}\|_2 \, \|A\|_2 \\ &\leq \|\mathcal{U}_{\beta^2}^{-k}\|_2 \, \mu(B)^{-1/2}.\end{aligned}$$

Therefore $\mathcal{U}_{\beta^2}^{-k}A/\|\mathcal{U}_{\beta^2}^{-k}\|_2$ is a global atom, and the required estimate follows from the definition of $\mathfrak{X}^k$.

Definition 5.4.6. Suppose that k is a positive integer. The space $\mathfrak{X}^k_{\mathrm{at}}(M)$ is the space of all functions F in $\mathfrak{h}^1(M)$ that admit a decomposition of the form $F = \sum_j \lambda_j A_j$, where $\{\lambda_j\}$ is a sequence in ℓ^1 and $\{A_j\}$ is a sequence of $\mathfrak{X}^k$-atoms supported in balls B_j in $\mathcal{B}_1$. Atoms supported in balls in $\mathcal{B}_1$ will be called *admissible*. We endow $\mathfrak{X}^k_{\mathrm{at}}(M)$ with the norm

$$\|F\|_{\mathfrak{X}^k_{\mathrm{at}}} = \inf\Big\{\sum_j |\lambda_j| : \ F = \sum_j \lambda_j A_j, \quad A_j \text{ admissible } \mathfrak{X}^k\text{-atoms}\Big\}.$$

We shall prove that, under an additional hypothesis on M, $\mathfrak{X}^k(M) = \mathfrak{X}^k_{\mathrm{at}}(M)$ with equivalent norms.

Definition 5.4.7. We say that M has C^l *bounded geometry* if the injectivity radius is positive and the following hold:

- if $l = 0$, then the Ricci tensor is bounded from below;
- if l is positive, then the covariant derivatives $\nabla^j \mathrm{Ric}$ of the Ricci tensor are uniformly bounded on M for all $j \in \{0, \cdots, l\}$.

The main result of this section is the following.

Theorem 5.4.8. *Suppose that k is a positive integer and that M has C^{2k-2} bounded geometry. Then $\mathfrak{X}^k(M)$ and $\mathfrak{X}^k_{\mathrm{at}}(M)$ agree as vector spaces and there exists a constant C such that*

$$C\,\|F\|_{\mathfrak{X}^k_{\mathrm{at}}} \leq \|F\|_{\mathfrak{X}^k} \leq \big\|\mathcal{U}_{\beta^2}^{-k}\big\|_2 \, \|F\|_{\mathfrak{X}^k_{\mathrm{at}}} \qquad \forall F \in \mathfrak{X}^k(M). \tag{5.4.2}$$

Proof. The proof follows the same lines as that of [MMV3, Theorem 4.3], and it is omitted. □

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