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Progresses on some classical problems of the Calculus of Variations

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Chapter 1

Introduction

Outline of the thesis

In this chapter we introduce most of the problems that we will treat in detail later, we recall a (far from being complete) list of known results, and we state our main results. In the following chapters we will present the full proofs of the results.

For the most part, the results presented in this thesis are available in [5], [6], [7], and [8].

1.1 On the Euler-Lagrange equation

The basic problem (P) of the Calculus of Variations is the minimization of an integral functional, the so called *action*

$$I(u) = \int_\Omega L(x, u(x), \nabla u(x)) \, \mathrm{d}x,$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ lies in a suitable space X of trajectories and satisfies some appropriate boundary conditions. The integrand function $L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is called the *Lagrangian* function.

One of the first natural question which are related to our problem (P) is the existence of a minimizer in the space X. As one can guess, in this full generality, the answer is negative.

However, once the existence is proven, one can wonder whether this minimizer satisfies some auxiliary properties, e.g. necessary conditions such as the validity of the Euler-Lagrange equation along the minimizer. Moreover, it is quite natural to ask if there is any further regularity property, such as higher integrability and higher differentiability.

One of the first answers to the existence problem was given in the second decade of the 20th century: in the one-dimensional case ($n = 1$), L. Tonelli could prove that if one considers X as the space of absolutely continuous function $\mathrm{AC}[a, b]$, then a minimizer $\overline{u}$ does exist, provided L satisfies some kind of lower semicontinuity and it is convex and superlinear with respect to the third variable, i.e.,

$$L(x, u, \xi) \geq \varphi(|\xi|) \quad \text{for some } \varphi \text{ such that} \quad \lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty.$$

The proof of this (classical) result is now known as the *Direct Method* of the Calculus of Variations.

Assuming the existence of a minimizer $\overline{u}$, one can look for necessary conditions satisfied by $\overline{u}$. Following an idea which is quite standard in Analysis, one can explore a neighbourhood of the minimizer, that is, one can *do variations* and consider $\overline{u} + \varepsilon\eta$. In here, ε is a small positive parameter and η is a smooth function which vanishes at the boundary of Ω, so that $\overline{u} + \varepsilon\eta$ is a competitor for the minimization problem, since it belongs to X and $\overline{u} + \varepsilon\eta = \overline{u}$ on the boundary of Ω.

Being $\overline{u}$ a minimizer, $I(\overline{u} + \varepsilon\eta) - I(\overline{u}) \geq 0$ holds for any ε and any admissible variation η, and so

$$\int_\Omega \frac{L(x, \overline{u}(x) + \varepsilon\eta(x), \nabla\overline{u}(x) + \varepsilon\nabla\eta(x)) - L(x, \overline{u}(x), \nabla\overline{u}(x))}{\varepsilon} \, \mathrm{d}x \geq 0.$$

The integrand converges pointwise to

$$\langle \nabla_\xi L(x, \overline{u}(x), \nabla\overline{u}(x)), \nabla\eta(x) \rangle + L_u(x, \overline{u}(x), \nabla\overline{u}(x))\eta(x),$$

then, if one can pass to the limit $\varepsilon \to 0$ under the integral sign,

$$\int_\Omega [\langle \nabla_\xi L(x, \overline{u}(x), \nabla\overline{u}(x)), \nabla\eta(x) \rangle + L_u(x, \overline{u}(x), \nabla\overline{u}(x))\eta(x)] \, \mathrm{d}x \geq 0.$$

Replacing now η with $-\eta$, one finally has the so called *Euler-Lagrange equation*, (EL):

$$\int_\Omega [\langle \nabla_\xi L(x, \overline{u}(x), \nabla\overline{u}(x)), \nabla\eta(x) \rangle + L_u(x, \overline{u}(x), \nabla\overline{u}(x))\eta(x)] \, \mathrm{d}x = 0. \tag{1.1}$$

Notice that if one considers div as the weak divergence, (EL) can be written in a shorter form as

$$\mathrm{div}_x \nabla_\xi L(\cdot, \overline{u}, \nabla\overline{u}) = L_u(\cdot, \overline{u}, \nabla\overline{u}).$$

Anyway, as a first step one has to prove that equation (1.1) makes sense, or, in other words, that $\nabla_\xi L(\cdot, \overline{u}, \nabla\overline{u})$ and $L_u(\cdot, \overline{u}, \nabla\overline{u})$ are (at least locally) integrable.

In the following we will mainly focus on the term $\nabla_\xi L$ since we will assume $|L_u(x, u, \xi)| \leq KL(x, u, \xi)$ for a.e. x in Ω and for every (u, ξ) in $\mathbb{R} \times \mathbb{R}^n$.

1.1.1 The state of the art

It is well known that the Euler-Lagrange equation plays an important role in the minimization problems of the Calculus of Variations, in particular in the regularity theory for minimizers. In spite of this, we are still far from having a general theorem on its validity.

So far, without any further assumption on the Lagrangian L, the validity of the Euler-Lagrange equation has to be considered a conjecture.

Indeed, the result cannot be true in its full generality and further conditions on L are needed. In [2], J. M. Ball and V. J. Mizel present a Lagrangian whose derivative with respect to the "gradient variable" is not integrable and, therefore, (EL) cannot hold along the minimizer.

Notice that Ball and Mizel built their famous counterexample in the one-dimensional context, although the validity of necessary conditions of very general nature has been proved in this case.

As for the scalar case, the validity of the Euler-Lagrange equation has been proved only for particular cases: in the following, we are going to present some of these results.

As one can guess, in the multidimensional case $u : \mathbb{R}^n \to \mathbb{R}^m$, $m > 1$, (see [22] and the references cited there) things get even worse...

The conditions on the Lagrangian L which are usually imposed to prove the validity of the Euler-Lagrange equation can be of growth type: for instance, consider a Lagrangian $L(x, u, \xi) = f(\xi) + g(x, u)$, where x lies in a domain $\Omega \subset \mathbb{R}^n$, u is real and f grows at most polynomially, i.e.,

$$|f(\xi)| \leq M(1 + |\xi|^p), \text{ for some positive } p \text{ and } M. \tag{1.2}$$

In this case, classical results show that (EL) holds along the minimizer $\overline{u}$, provided $\overline{u}$ belongs to $W^{1,p}(\Omega)$. Indeed, from (1.2) and the convexity of f, it follows $|\nabla f(\xi)| \leq K(1 + |\xi|^{p-1})$, for some K. A slightly more accurate argument can be used to prove that $f(\xi)$ with exponential growth are also allowed.

However, superexponential growth is still a challenging problem, although some special results have been obtained: in [29], G. M. Lieberman considers the problem

$$\text{minimize} \int_\Omega \exp(|\nabla u(x)|^2)\ \mathrm{d}x$$

and proves that minimizers are classical solutions to the corresponding Euler-Lagrange equation.

Otherwise, one can try to get some regularity properties for the solution $\overline{u}$: for instance, if $\overline{u} \in W^{1,\infty}_{loc}(\Omega)$, then it is easy to prove the validity of the Euler-Lagrange equation. P. Marcellini, in [31], gains local Lipschitz continuity for solutions to variational problems requiring some growth assumptions on the Lagrangian L. A. Cellina, [13], instead, proves global Lipschitz continuity through conditions on the set Ω and on the boundary datum u_0. More recently, weaker assumptions on Ω and on u_0 were introduced by P. Bousquet and F. H. Clarke, [9], to get local Lipschitz continuity.

Another remarkable result is the recent one by M. Degiovanni and M. Marzocchi: in [23], they consider the functional

$$\int_\Omega L(\nabla u(x))\ \mathrm{d}x \ + \ \varphi(u),$$

where $u \in u_0 + W_0^{1,p}(\Omega)$, $1 \le p < \infty$ and $\varphi \in W^{-1,p'}(\Omega)$, and prove that any minimizer $\overline{u}$ satisfies the associated Euler-Lagrange equation

$$\int_\Omega \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta(x) \rangle\ \mathrm{d}x \ = \ -\varphi(\eta), \quad \forall \eta \in \mathcal{C}_0^\infty(\Omega).$$

Since there is no upper growth condition (main hypotheses on L are just convexity and regularity), at present this is the border of knowledge about the validity of the Euler-Lagrange equation in the regular case.

1.1.2 Without growth conditions

The lack of growth assumptions may lead one to say that Degiovanni and Marzocchi opened a new path in order to prove the validity of the Euler-Lagrange equation.

The main result of Chapter 2 is inspired by their work. In fact, we use some results presented in [23] as a main tool applied to the functional

$$I(u) = \int_\Omega [L(\nabla u(x)) + g(x, u(x))] \, \mathrm{d}x,$$

where L is a convex function and g is a Carathéodory map (see Appendix A.1) such that $u \mapsto g(x, u)$ is concave for almost every x in Ω and satisfies some growth assumptions. We prove that for any minimizer $\overline{u}$ of I, there exists a selection $\sigma(x)$ from the subdifferential $\partial g(x, \overline{u}(x))$ such that, for every $\eta \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta(x) \rangle \, \mathrm{d}x = -\int_\Omega \sigma(x) \eta(x) \, \mathrm{d}x.$$

Our result generalizes the case considered by Degiovanni and Marzocchi: in fact, the functional $\varphi \in W^{-1,p'}(\Omega)$ which appears in [23] is substituted by a more generic $u \mapsto \int g(x, u(x)) \, \mathrm{d}x$.

Functionals of this type, with the same concavity assumption, were considered by A. Cellina and G. Colombo, [15], but their purpose was to prove existence of solutions and the domain of integration was one-dimensional.

1.1.3 On the semi-classical Euler-Lagrange equation

Consider the problem

$$\text{minimize} \int_\Omega [f(|\nabla u(x)|) + g(x, u(x))] \, \mathrm{d}x \quad \text{on } u_0 + W_0^{1,1}(\Omega),$$

where f is a convex function defined on $\mathbb{R}^+$ and g is a Carathéodory function, differentiable with respect to u, and whose derivative g_u is also a Carathéodory function. The main point here is that we ask f to be convex but we do not require any differentiability properties on f.

As a consequence, one cannot speak about the "differential of f" and one only has the notion of *subdifferential* of f, which is a generalization of the gradient in the convex context, see e.g. [37] and Appendix A.3.1.

Definition 1.1. A vector x^* is said to be a *subgradient* of a convex function f at a point x if

$$f(z) \geq f(x) + \langle x^*, z - x \rangle, \ \forall z.$$

The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted by $\partial f(x)$.

Obviously, one cannot write the Euler-Lagrange equation in its classical form (1.1) and may therefore wonder which is the right statement. A suggestion comes from the *Pontryagin Maximum Principle* for optimal control problems, [34].

So as to show what we mean, consider a one-dimensional domain $[a, b]$ and express our variational problem in terms of an optimal control problem

$$\text{minimize} \int_a^b f(t, x(t), u(t))\mathrm{d}t \quad \text{and} \quad x(a) = x_0, \ x(b) = x_1, \tag{1.3}$$

where the state x and the control u are linked by the differential condition $x'(t) = u(t)$ and the set of the admissible controls U is the effective domain of $f(t, u, \cdot)$.

The *Pontryagin Maximum Principle* states that if $(\overline{x}, \overline{u})$ is a solution to (1.3), then there exist a non negative p_0 and a map p in $W^{1,1}([a, b])$ such that $(p_0, p) \neq (0, 0)$ and almost everywhere in $[a, b]$:

i) $\dfrac{d}{dt}p(t) = p_0\dfrac{\partial f}{\partial x}(t, \overline{x}(t), \overline{u}(t))$;

ii) $p(t)\overline{u}(t) - p_0 f(t, \overline{x}(t), \overline{u}(t)) = \max\limits_{u\in U} \big\{p(t)u - p_0 f(t, \overline{x}(t), u)\big\}$.

In the non-trivial case $p_0 \neq 0$, when we consider the problem of minimizing functionals of the form

$$\int_\Omega [f(\nabla u(x)) + g(x, u(x))] \ \mathrm{d}x,$$

where f is a convex function defined on $\mathbb{R}^n$, one can conjecture that the suitable form of the Euler-Lagrange equations satisfied by a solution $\overline{u}$ should be

$$\text{there exists } p \in (\mathrm{L}^1(\Omega))^n \text{ such that } \operatorname{div} p(\cdot) = g_u(\cdot, \overline{u}(\cdot)),$$

in the sense of distributions and, for a.e. x and every ξ in $\mathbb{R}^n$, we have

$$\langle p, \nabla\overline{u}(x)\rangle - [f(\nabla\overline{u}(x)) + g(x, \overline{u}(x))] \geq \langle p, \xi\rangle - [f(\xi) + g(x, \overline{u}(x))].$$

Equivalently, the condition can be expressed as

$$\exists\ p \in (\mathrm{L}^1(\Omega))^n, \text{ a selection from } \partial f(\nabla\overline{u}), \text{ such that } \operatorname{div} p(\cdot) = g_u(\cdot, \overline{u}(\cdot)).$$

In this form, this condition is the equivalent of the Pontryagin Maximum Principle and we call it the *Euler-Lagrange equation in semi-classical form.*

Some results about this topic are available in [10], [11], and [14].

1.1.4 Without differentiability assumptions

Let $\overline{u}$ be a locally bounded solution to the problem of minimizing

$$\int_\Omega L(x, u(x), \nabla u(x))\ \mathrm{d}x$$

on $u_0 + W_0^{1,1}(\Omega)$, where $L(x, u, \xi)$ is a Carathéodory function, differentiable with respect to u, and whose derivative L_u is also a Carathéodory function, and the map $\xi \mapsto L(x, u, \xi)$ is convex and defined on $\mathbb{R}^n$. We do not assume further regularity on L, with the exception of standard growth estimates.

On the same wave-length of Section 1.1.3, it has been conjectured that the suitable form of the Euler-Lagrange equations satisfied by $\overline{u}$ should be

$$\exists\, p \in (\mathrm{L}^1(\Omega))^n, \text{ a selection from } \partial_\xi L(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot)), \text{ such that}$$
$$\operatorname{div} p(\cdot) = L_u(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot))$$

in the sense of distributions. This fact has been proved in a few special cases: in [20] for maps of the form $L(u, \xi)$, jointly convex in (u, ξ), and, more recently, in [16] for maps $L(x, u, \xi) = f(|\xi|) + g(x, u)$, depending on ξ through its norm.

The proof introduced in [16] is elementary and it is based on the Riesz representation Theorem and on the Hahn-Banach Theorem. The result which is presented in Chapter 3 is a sequel to [16] and one can see that a modification of the same elementary proof allows us to prove the conjecture in its full generality. The proofs we present are self-contained.

Notice that this is not the most general problem about the validity of necessary conditions for minimization problems with f convex: our f is defined on $\mathbb{R}$ and in our result are not included problems with restrictions on ∇u, e.g. $|\nabla u| \leq 1$, that would require extended valued convex functions. This is a very active and difficult area of research and only a few results are available: [3], [4], and [19].

1.1.5 Beyond exponential growth

In this work we consider a higher integrability property of a solution $\overline{u}$ to the problem of minimizing

$$\int_\Omega L(x, u(x), \nabla u(x))\mathrm{dx}.$$

More precisely, our aim is to establish the local integrability of the map

$$|\nabla_\xi L\left(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot)\right)||\nabla\overline{u}(\cdot)|. \tag{1.4}$$

In fact, for Lagrangians $L(x,u,\cdot)$ growing faster than exponential, the integrability of $L\left(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot)\right)$ does not, in general, imply the integrability of $|\nabla_\xi L\left(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot)\right)|$ (see an example in [17]).

This inconvenience does not occur as soon as we are able to prove some additional regularity properties of the solution $\overline{u}$. Consider a problem of the type

$$\text{minimize} \int_\Omega L\big(|\nabla u(x)|\big)\ \mathrm{d}x :$$

in [38], the author, using a barrier and under smoothness conditions on the boundary and on the second derivative of L, proves that the gradient of the solution is bounded. In [31] and in [32], under general growth conditions, the authors, taking advantage of the regularity properties of solutions to elliptic equations, prove that the gradient of the minimizer is locally bounded. Clearly, a proof of regularity ($\nabla\overline{u}$ in L^∞) of the solution is also a proof of the higher integrability of the solution. In this sense, for the case $L(\xi) = \exp|\xi|^2$, special cases of higher integrability have been obtained by Lieberman, [29], and by H. Naito, [33]; Lieberman, in the same paper, considers also a more general Lagrangian, but assuming, among other regularity conditions, that the Euler-Lagrange equation admits a $\mathcal{C}^3$ solution.

On the one hand our result, that is, the local integrability of (1.4), is weaker than the local boundedness of $\nabla\overline{u}$, but on the other hand it holds for a larger class of functionals. Indeed, we do not assume the existence of a second derivative of $L(x,u,\cdot)$, nor we assume its strict convexity. Moreover, we also allow a dependence on x and on u.

However, the integrability of (1.4) is needed both to establish the validity of the Euler-Lagrange equation for the solution to this problem, i.e., in order to prove that the equation

$$\int_\Omega [\langle\nabla_\xi L(x,\overline{u}(x),\nabla\overline{u}(x)),\nabla\eta(x)\rangle + L_u(x,\overline{u}(x),\nabla\overline{u}(x))\eta(x)]\mathrm{dx} = 0$$

holds for every admissible variation η, and to prove additional regularity properties (higher differentiability) of the solution, as in [18].

In [17], the authors obtain a higher integrability result considering a Lagrangian of the kind $L = e^{f(|\nabla u|)} + g(x,u)$, where f and g are regular functions satisfying some growth assumptions and f is convex. In Chapter 4 we follow their steps and we are aimed by a twofold purpose: first we wish to present a more general result, suited for being used in the investigation of further regularity properties of the solution; second, we wish to use the higher integrability property to establish the validity of the Euler-Lagrange equation for a class of Lagrangians growing faster than exponential.

It is well known, in fact, (see e.g. [20]) that, so far, the validity of the Euler-Lagrange equation has been established for Lagrangians growing at most exponentially; some exceptions to this statement exist, beginning with the already mentioned Lieberman [29]; more recently, [23] and [8]. However, the few results proved so far for integrands having growth faster than exponential hold only for Lagrangians of a very special form.

The proof of the higher integrability result, that will be presented in Chapter 4, is independent on the validity of the Euler-Lagrange equation; this fact prompted us to try to use the higher integrability property to extend the validity of the Euler-Lagrange equation beyond exponential growth. A result along these lines is presented in the second part of Chapter 4: in it, we allow the growth of L with respect to ξ to be approximately up to $|\xi|^{|\xi|} \equiv \exp(|\xi| \log |\xi|)$.

1.2 On the non-occurrence of the Lavrentiev phenomenon

In Section 1.1 we recalled that L. Tonelli could prove the existence of a minimizer for a one-dimensional minimization problem over the space $\mathrm{AC}(a, b)$. It is natural to ask what would happen if one considers a different set X of trajectories: one cannot expect existence results in every space, but, at the same time, one would expect the infimum to be the same over different spaces. Indeed, in 1927, a remarkable paper by M. A. Lavrentiev, [28], presented an example of a variational functional over the interval (a, b), with boundary conditions $u(a) = \alpha$, $u(b) = \beta$, whose infimum over the set of absolutely continuous functions was strictly lower than the infimum of the same functional over the set of Lipschitzean functions satisfying the same boundary conditions. Since then, this phenomenon is called the *Lavrentiev phenomenon.* Some years later, in 1934, B. Manià, [30], published an example of a simpler functional exhibiting the Lavrentiev phenomenon.

Notice that the occurrence of this phenomenon prevents the possibility of computing the minimum, and the minimizer, by standard finite-element schemes: indeed, these methods base on Lipschitz continuous approximations.

In the autonomous case, which is the one we will treat, several authors presented sufficient conditions to prevent the occurrence of this phenomenon, through growth assumptions, as in [21] for $n = 1$, or through some regularity conditions on the Lagrangian: in 1993, G. Alberti and F. Serra Cassano, [1], did show that the phenomenon does not occur for autonomous integrands over a one-dimensional integration set. As for $n \geq 1$, G. Buttazzo and M. Belloni in [12, Remark 3.4] use an approximation technique in order to prove that the Lavrentiev phenomenon can never occur in the autonomous case: unfortunately such a method does not seem to preserve the boundary value. Finally, connections between the regularity of a solution and the non-occurrence of the Lavrentiev phenomenon have been pointed out e.g. in [25].

In [24], Lemma 2.1, L. Esposito, F. Leonetti, and G. Mingione prove that the phenomenon does not occur for functionals of the form

$$\int_\Omega f(\nabla v(x))\, \mathrm{d}x,$$

provided that Ω is the unit ball, f is a convex $C^2(\mathbb{R}^N)$ function and the growth of f is of the $(p-q)$ type, i.e., $m|z|^p \leq f(z) \leq L(1+|z|)^q$, with $2 \leq q < p < 2+q$; in addition, some further growth conditions on the first and second derivatives of f are assumed.

In Chapter 5, we show that the Lavrentiev phenomenon does not occur for functionals of the form

$$\int_\Omega L(|\nabla u(x)|)\, \mathrm{d}x,$$

where L is an arbitrary convex function, provided that both $\partial\Omega$ and u^0 are of class C^2. Our Theorem 5.1 contains neither regularity nor growth assumptions on the Lagrangian L, besides its being convex.

Notice that, when the integration set Ω is a subset of $\mathbb{R}^N$, the boundary condition is described by the inclusion $u - u^0 \in W_0^{1,1}(\Omega)$ and, in order for the problem of the occurrence of the Lavrentiev phenomenon to make sense, u^0 is a Lipschitzean function on $\overline{\Omega}$.

Finally, in Chapter 6 we present a modification of Manià's functional on $\Omega \subset \mathbb{R}^2$ with a linear boundary function u^0, exhibiting the Lavrentiev phenomenon.

Part I

On the Euler-Lagrange equation

Chapter 2

Without growth assumptions

This chapter is based on a joint work with M. Mazzola: [8], *On the validity of the Euler-Lagrange equation in a nonlinear case*, Nonlinear Analysis: Theory, Methods & Applications **73** (2010), no. 1, pp. 266-269.

2.1 Assumptions and main result

Let us consider the problem (P) of minimizing the integral functional

$$I(u) = \int_\Omega [L(\nabla u(x)) + g(x, u(x))] \, \mathrm{d}x \text{ on the set } u_0 + W_0^{1,p}(\Omega),$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $L : \mathbb{R}^N \to \mathbb{R}$ is a convex map and $1 \leq p < \infty$. We also suppose that the boundary datum u_0 lies in $W^{1,p}(\Omega) \cap \mathrm{L}^\infty_{loc}(\Omega)$ and satisfies $I(u_0) < +\infty$.

We observe that assuming u_0 to be locally bounded is non restrictive if the Lavrentiev phenomenon does not occur, as already remarked in [23].

In the following, we denote by p' the dual conjugate exponent of p, i.e., $1/p + 1/p' = 1$, and by p^* the Sobolev conjugate exponent of p, that is $p^* = \frac{Np}{N-p}$ if $p < N$.

As already remarked, for the properties of convex and concave functions we refer to [37]: we only recall here that, given a *concave* function $f : \mathbb{R}^N \to \mathbb{R}$, its subdifferential at the point $\xi \in \mathbb{R}^N$ is the set

$$\partial f(\xi) = \{s \in \mathbb{R}^N : f(x) \leq f(\xi) + \langle s, x - \xi \rangle \, , \; \forall x \in \mathbb{R}^N\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^N$.

In order to prove our result we will need the following growth assumption on the map g:

Assumption 2.1. *Let* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory map such that* $u \mapsto g(x,u)$ *is* concave *for almost every* $x \in \Omega$. *Moreover:*

- *if* $p < N$, *there exist* $\alpha_1 \in \mathrm{L}^{(p^*)'}(\Omega)$ *and* $\beta_1 \in \mathrm{L}^\infty(\Omega)$ *such that*

$$|g(x,u)| \leq \alpha_1(x) + \beta_1(x)|u|^{p^*}, \quad a.e.\ x \in \Omega,\ \forall u \in \mathbb{R};$$

- *if* $p = N$, *there exist* $r > 1$, $\alpha_2 \in \mathrm{L}^{r'}(\Omega)$ *and* $\beta_2 \in \mathrm{L}^\infty(\Omega)$ *such that*

$$|g(x,u)| \leq \alpha_2(x) + \beta_2(x)|u|^{r}, \quad a.e.\ x \in \Omega,\ \forall u \in \mathbb{R};$$

- *if* $p > N$, *there exist* $\alpha_3 \in \mathrm{L}^1(\Omega)$ *and* $\beta_3 : \mathbb{R} \to \mathbb{R}$ *non-decreasing such that*

$$|g(x,u)| \leq \alpha_3(x) + \beta_3(u), \quad a.e.\ x \in \Omega,\ \forall u \in \mathbb{R}.$$

We can now state our main result

Theorem 2.2. *If* L *is convex of class* $C^1(\mathbb{R}^N)$ *and* g *satisfies Assumption 2.1, then the Euler-Lagrange equation associated to problem (P) holds, that is, if* $\overline{u}$ *is a minimizer for* I, *then there exists a selection* $\sigma(\cdot)$ *from the set valued map* $x \mapsto \partial g(x, \overline{u}(x))$ *such that*

$$\int_\Omega \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta(x) \rangle \, \mathrm{d}x = -\int_\Omega \sigma(x)\eta(x)\, \mathrm{d}x \quad \forall \eta \in C_c^\infty(\Omega).$$

Remark: We observe that Assumption 2.1 is sufficient in order to prove our main result, but it does not guarantee the existence of a minimum for problem (P) in the case $p > N$.

2.2 Proof of the Theorem

For the sake of brevity, we will treat at the same time the cases $p < N$ and $p = N$, by assuming g to satisfy the following general condition:

$$|g(x,u)| \leq \alpha(x) + \beta(x)|u|^r, \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R}, \tag{2.1}$$

where $r = p^*$ if $p < N$, some $r \in (1, \infty)$ if $p = N$ and $\alpha \in \mathrm{L}^{r'}(\Omega)$, $\beta \in \mathrm{L}^\infty(\Omega)$.

Proof of Theorem 2.2. Given a set X, we denote

$$\|X\| = \max\{|x| : x \in X\}.$$

We prove the theorem in several steps.

a) When $p \leq N$, we show that if g satisfies (2.1), then there exist a constant $C > 0$ and a function $\beta' \in \mathrm{L}^\infty(\Omega)$ such that

$$\|\partial g(x,u)\| \leq 2\alpha(x) + \beta'(x)|u|^{r-1} + C, \text{ a.e. } x \in \Omega, \forall u \in \mathbb{R}.$$

Fix $(x,u) \in \Omega \times \mathbb{R}$ and let $y \in \partial g(x,u)$. From the concavity of the map $u \mapsto g(x,u)$, we have

$$g(x, u+h) \leq g(x,u) + yh, \ \forall h \in \mathbb{R}.$$

Define the constant

$$h_0 = -\frac{y}{|y|}(1+|u|^r)^{1/r}.$$

By the choice of h_0, we have

$$\begin{aligned} |yh_0| &\leq |g(x,u+h_0)| + |g(x,u)| \\ &\leq 2\alpha(x) + \beta(x)\left[|u+h_0|^r + |u|^r\right] \\ &\leq 2\alpha(x) + \beta'(x)\left[|h_0|^r + |u|^r\right] \\ &= 2\alpha(x) + \beta'(x)\left[1 + 2|u|^r\right]. \end{aligned}$$

Finally, using the inequality

$$(1+|\xi|^s)^{1-\frac{1}{s}} \leq 1 + |\xi|^{s-1}$$

and up to renaming β', we obtain

$$\begin{aligned} |y| &\leq \frac{2\alpha(x)}{(1+|u|^r)^{1/r}} + \beta'(x)\,(1+|u|^r)^{1-\frac{1}{r}} \\ &\leq 2\alpha(x) + \beta'(x)|u|^{r-1} + C. \end{aligned}$$

b) Assume that g satisfies Assumption 2.1. We claim that for any $u \in \mathrm{L}^r(\Omega)$ there exists a measurable selection $\sigma(\cdot)$ from the set valued map $x \mapsto \partial g(x, u(x))$ such that σ belongs to $\mathrm{L}^{r'}(\Omega)$ if $1 < r < \infty$ (that is case $p \leq N$) and to $\mathrm{L}^1(\Omega)$ if $r = \infty$ (case $p > N$).

Recalling that g is a Carathéodory map, we have that for any $u : \Omega \to \mathbb{R}$ measurable, the set valued map $x \mapsto \partial g(x, u(x))$ is also measurable (see Corollary 4.6 of [36]).

So, the Kuratowski and Ryll-Nardzewski theorem, [27], yields the existence of a measurable selection $\sigma(x)$ from the map $\partial g(x,u(x))$.

As for the integrability, by step a), we obtain that in the case $p \leq N$

$$|\sigma(x)| \leq \|\partial g(x,u(x))\| \leq 2\alpha(x) + \beta'(x)|u(x)|^{r-1} + C, \text{ a.e. } x \in \Omega.$$

Then, Assumption 2.1 let us conclude that σ belongs to $\mathrm{L}^{r'}(\Omega)$. When $p > N$, we use the boundedness of u in order to prove that σ is in $\mathrm{L}^1(\Omega)$.

c) Let $\overline{u}$ be a solution of problem (P) and let $\sigma(\cdot) \in \partial g(\cdot,\overline{u}(\cdot))$ be the selection given by step b). For $u \in u_0 + W_0^{1,p}(\Omega)$, define the functional

$$J\left(u\right) = \int_\Omega \left[L\left(\nabla u\left(x\right)\right) + \sigma\left(x\right)\cdot\left(u\left(x\right) - \overline{u}\left(x\right)\right) + g\left(x,\overline{u}\left(x\right)\right)\right] \mathrm{d}x.$$

We claim that $\overline{u}$ is a minimizer for J, too.

Indeed, let $v \in u_0 + W_0^{1,p}\left(\Omega\right)$. From the concavity of $u \mapsto g\left(x,u\right)$, we have

$$g\left(x,v\left(x\right)\right) \leq g\left(x,\overline{u}\left(x\right)\right) + \sigma\left(x\right)\cdot\big(v\left(x\right) - \overline{u}\left(x\right)\big), \quad \text{for a.e. } x \in \Omega;$$

then

$$J\left(\overline{u}\right) = I\left(\overline{u}\right) \leq I\left(v\right) \leq J\left(v\right).$$

d) Consider the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ such that

$$\Phi : \varphi \mapsto \int_\Omega \sigma\left(x\right)\varphi\left(x\right)\ \mathrm{d}x.$$

Since, by step b), σ belongs to $\mathrm{L}^{(p^*)'}(\Omega)$ if $p < N$, to $\mathrm{L}^{r'}(\Omega)$ if $p = N$ and to $\mathrm{L}^1(\Omega)$ if $p > N$, Sobolev embedding theorem let us conclude that Φ is continuous in all three cases.

Finally, Theorem 1.1 in [23] yields that the minimizer $\overline{u}$ of J satisfies the Euler-Lagrange equation, that is

$$\int_\Omega \langle \nabla L(\nabla\overline{u}(x)), \nabla\eta(x)\rangle\ \mathrm{d}x \ = \ -\Phi(\eta) \qquad \forall \eta \in \mathcal{C}_0^\infty(\Omega),$$

hence our claim. □

Chapter 3

Without differentiability assumptions

This chapter is based on a joint work with A. Cellina: [5], *The validity of the Euler-Lagrange equation*, Discrete and Continuous Dynamical Systems. Series A **28** (2010), no. 2, pp. 511-517.

3.1 Preliminaries and assumptions

Let u be a locally bounded solution to the problem of minimizing

$$\int_\Omega L(x, v(x), \nabla v(x))\ \mathrm{d}x \tag{3.1}$$

on $v_0 + W_0^{1,1}(\Omega)$, where $L(x, v, \xi)$ is a Carathéodory function, differentiable with respect to v, and whose derivative L_v is also a Carathéodory function. Assume also that the map $\xi \mapsto L(x, v, \xi)$ is convex and defined on $\mathbb{R}^n$.

We do not need further regularity on L, with the exception of standard growth estimates.

We consider $\mathbb{R}^N$ with the Euclidean norm $|\cdot|$ and we call $\mathbb{B}$ the unit ball. $\ell(A)$ is the N-dimensional Lebesgue measure of a set A.

Given a closed convex $K \subset \mathbb{R}^N$, by m_K we mean the unique point of K of minimal norm and by $\|K\|$ we mean $\sup\{|k| : k \in K\}$.

A set valued map K with values in the non-empty compact subsets of $\mathbb{R}^N$ is called *upper semicontinuous* at x_0 if for every ε there exists δ such that $|x - x_0| < \delta$ implies $K(x) \subset K(x_0) + \varepsilon\mathbb{B}$. In this chapter we shall also meet real valued upper and lower semicontinuous maps, with the usual definitions.

Given a function $L(x, v, \xi)$, convex with respect to ξ for each fixed (x, v), by $\partial_\xi L(x, v, \xi)$ we mean the *subdifferential* of L with respect to the variable ξ. Under the assumptions of the present work, $\partial_\xi L(x, v, \xi)$ is a non-empty compact convex subset of $\mathbb{R}^N$ and the map $\xi \mapsto \partial_\xi L(x, v, \xi)$ is (for fixed (x, v)) an upper semicontinuous set valued map. We shall assume further properties of this map in Assumption 3.1.

$I_A(\cdot)$ is the *indicator function*, in the sense of the Convex Analysis, of the set A and f^* is the *polar* or *Fenchel transform* of f, see Appendix A.3.2 and [37].

Ω is a bounded open subset of $\mathbb{R}^N$. Given a solution u, the shorthand notation $D_L(x)$ means the set $\partial_\xi L(x, u(x), \nabla u(x))$.

Assumption 3.1. i) *$L(x, v, \xi)$ is a Carathéodory function, differentiable with respect to v, and whose derivative L_v is also a Carathéodory function, and, for every pair (x, v), the map $\xi \mapsto L(x, v, \xi)$ is convex and defined on $\mathbb{R}^N$.*

ii) *There exist a convex function f and constants H_1 and H_2 such that*

$$\|\partial f(\xi)\| \leq H_1 f(\xi) + H_2 \tag{3.2}$$

and, for every U, there exist functions α_U, β_U and γ_U in $\mathrm{L}^1(\Omega)$ and positive constants h_U^1, h_U^2 and h_U^3, such that $|v| \leq U$ implies

$$\alpha_U(x) + h_U^1 f(\xi) \leq L(x, v, \xi) \tag{3.3}$$

$$\partial_\xi L(x, v, \xi) \leq \beta_U(x) + h_U^2 \partial f(\xi) \tag{3.4}$$

$$|L_u(x, v, \xi)| \leq \gamma_U(x) + h_U^3 f(\xi). \tag{3.5}$$

iii) *For every $\delta > 0$, there exists $\Omega_\delta \subset \Omega$ with $\ell(\Omega \setminus \Omega_\delta) < \delta$, such that the restriction of $\partial_\xi L(x, v, \xi)$ to $\Omega_\delta \times \mathbb{R} \times \mathbb{R}^N$ is upper semicontinuous.*

Notice that Assumption 3.1 ii) limits the growth of L in the variable ξ to be exponential. This growth limitation still holds, so far, for the proofs of the validity of the Euler-Lagrange equation for variational problems of general form, independently on whether there are additional differentiability assumptions or not.

We recall that the only exceptions, as far as we know, to this statement are the recent paper [23], where no growth limitations are assumed but functionals have a special form; our generalization (Chapter 2); and the result we are going to present in Chapter 4.

3.2 Main result

It is our purpose to prove the following

Theorem 3.2. *Let L satisfy Assumption 3.1. Let u be a locally bounded solution to Problem (3.1). Then,*

$$\textit{there exists } p \in (\mathrm{L}^1(\Omega))^N, \ \textit{ a selection from } \partial_\xi L(\cdot, u(\cdot)\nabla u(\cdot)),$$

such that

$$\operatorname{div} p(\cdot) = L_u(\cdot, u(\cdot), \nabla u(\cdot))$$

in the sense of distributions.

We shall need the following variant of the Riesz Representation Theorem.

Lemma 3.3. *Let D be a map from Ω to the closed convex non-empty subsets of $R\mathbb{B}$, such that $v \in (\mathrm{L}^\infty(\Omega))^N$ implies that the map $x \mapsto m_{[D(x)-v(x)]}$ is measurable; let $T : (\mathrm{L}^1(\Omega))^N \to \mathbb{R}$ be a linear functional satisfying*

$$T(\xi) \leq \int_\Omega (I_{D(x)})^*(\xi(x)) \, \mathrm{d}x.$$

Then, there exists $\tilde{p} \in (\mathrm{L}^\infty(\Omega))^N$, $\tilde{p}(x)$ a.e. in $D(x)$, that represents T, i.e., such that

$$T(\xi) = \int_\Omega \langle \tilde{p}(x), \xi(x) \rangle \, \mathrm{d}x. \tag{3.6}$$

Proof. a) Since $|(I_{D(x)})^*(\xi(x))| \leq \|D(x)\| |\xi(x)|$, we have that T is a bounded linear functional on $(\mathrm{L}^1(\Omega))^N$. Writing ξ as $\xi_1(x)e_1 + \ldots + \xi_N(x)e_N$ and applying the standard Riesz representation Theorem, we infer the existence of a function $\tilde{p} \in (\mathrm{L}^\infty(\Omega))^N$ that satisfies (3.6). To show that $\tilde{p}(x)$ is in $D(x)$ almost everywhere, assume that there exists a set $E \subset \Omega$ of positive measure such that, on E, $\tilde{p}(x) \notin D(x)$, i.e., $0 \notin D(x) - \tilde{p}(x)$. Setting $D^* := D(x) - \tilde{p}(x)$, we can equivalently say that $|m_{D^*(x)}| > 0$ on E.

Let $z(x)$ be the projection of minimal distance of $\tilde{p}(x)$ on $D(x)$, so that, $z(x) - \tilde{p}(x) = m_{D(x)-\tilde{p}(x)}$ or, $z(x) - \tilde{p}(x) = m_{D^*(x)}$. From the characterization of the projection of minimal distance, we obtain

$$\langle \tilde{p}(x) - z(x), z(x) \rangle \geq \langle \tilde{p}(x) - z(x), k \rangle, \quad \forall k \in D(x).$$

that can be rewritten as

$$\langle -m_{D^*}(x), \tilde{p}(x)\rangle \geq |m_{D^*}(x)|^2 + \langle -m_{D^*}(x), k\rangle, \quad \forall k \in D(x).$$

Hence, we have that, on E,

$$\langle -m_{D^*}(x), \tilde{p}(x)\rangle > \sup\left\{\langle -m_{D^*}(x), k\rangle : k \in D(x)\right\} = (I_{D(x)})^*(-m_{D^*}(x)).$$

b) Setting $\tilde{\xi} := -m_{D^*}\chi_E$, we have that $\tilde{\xi} \in (\mathrm{L}^1(\Omega))^N$ and

$$T(\tilde{\xi}) = \int_\Omega \langle \tilde{p}, \tilde{\xi}\rangle = \int_E \langle \tilde{p}, -m_{D^*}\rangle > \int_\Omega (I_{D(x)})^*(\tilde{\xi}) \geq T(\tilde{\xi}),$$

a contradiction. □

We shall also need the following propositions:

Proposition 3.4. *Let $x \mapsto K(x)$ be an upper semicontinuous set valued map. Then,*

i) *the real valued map $x \mapsto |m_{K(x)}|$ is* lower semicontinuous *and the real valued map $x \mapsto \|K(x)\|$ is* upper semicontinuous*;*

ii) *the real valued map $(x, \xi) \mapsto (I_{K(x)})^*(\xi)$ is* continuous *in ξ for each fixed x and* upper semicontinuous *in x for each fixed ξ.*

Proposition 3.5. *Let $x \mapsto K(x)$ be an upper semicontinuos set valued map with values in the closed convex subsets of $\mathbb{R}^N$. Then, $|m_{K(\cdot)}|$ continuous at x_0 implies that $m_{K(\cdot)}$ is continuous at x_0.*

Proof. Fix x_0, a point of continuity of $|m_{K(\cdot)}|$, and consider two cases: i) $0 \notin K(x_0)$ and, ii), $0 \in K(x_0)$.

i) Fix $\varepsilon > 0$, with $\varepsilon < 2\sqrt{2}|m_{K(x_0)}|$. Let $\sigma > 0$ be such that

$$\left(|m_{K(x_0)}| - \sigma\right)^2 = |m_{K(x_0)}|^2 - \frac{\varepsilon^2}{8}$$

and let η be such that

$$\frac{1}{2}\left(|m_{K(x_0)}|^2 + (|m_{K(x_0)}| + \eta)^2\right) = |m_{K(x_0)}|^2 + \frac{\varepsilon^2}{8}.$$

Let δ be such that $|x - x_0| < \delta$ implies that both $K(x) \subset K(x_0) + \sigma\mathbb{B}$ and $\left||m_{K(x_0)}| - |m_{K(x)}|\right| < \eta$. As a consequence, from the convexity of $K(x_0) + \sigma\mathbb{B}$, we obtain that

$$\frac{m_{K(x_0)} + m_{K(x)}}{2} \in K(x_0) + \sigma\mathbb{B},$$

so that
$$\left|\frac{m_{K(x_0)}+m_{K(x)}}{2}\right| \geq |m_{K(x_0)}|-\sigma.$$
From the identity
$$\left|\frac{m_{K(x_0)}-m_{K(x)}}{2}\right|^2 = \frac{1}{2}\left(|m_{K(x_0)}|^2+|m_{K(x)}|^2\right) - \left|\frac{m_{K(x_0)}+m_{K(x)}}{2}\right|^2,$$
we obtain
$$\begin{aligned}\left|\frac{m_{K(x_0)}-m_{K(x)}}{2}\right|^2 &\leq \frac{1}{2}\left(|m_{K(x_0)}|^2+|m_{K(x_0)}+\eta|^2\right) - (|m_{K(x_0)}|-\sigma)^2\\ &= |m_{K(x_0)}|^2+\frac{\varepsilon^2}{8}-|m_{K(x_0)}|^2+\frac{\varepsilon^2}{8}=\frac{\varepsilon^2}{4}.\end{aligned}$$

ii) Fix $\varepsilon>0$; for $\sigma>0$ such that $|x-x_0|<\sigma$ implies $K(x)\subset K(x_0)+\varepsilon\mathbb{B}$, we have that $m_{K(x)}-0\in\varepsilon\mathbb{B}$. □

Lemma 3.6. i) $v\in(\mathrm{L}^\infty(\Omega))^N$ *implies that the map*
$$x\mapsto m_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]}\ \textit{is in}\ (\mathrm{L}^\infty(\Omega))^N$$
and, ii)*, for* $\xi\in(\mathrm{L}^1(A))^N$*, the map*
$$x\mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]})^*(\xi(x))\ \textit{is in}\ \mathrm{L}^1(\Omega).$$

Proof. i) Fix ε. Let Ω' be the subset of Ω provided by Assumption 3.1 iii), with $\delta=\frac{\varepsilon}{4}$.

Applying Luzin's Theorem (see Appendix A.2), there exists $E\subset\Omega'$ with $\ell(\Omega'\setminus E)\leq\frac{\varepsilon}{4}$, such that $u|_E$, $v|_E$ and $\nabla u|_E$ are continuous so that, on E, the set valued map D_L is upper semicontinuous and, by Proposition 3.4, the real valued map $\|D_L\|$ is upper semicontinuous.

Hence, there exists $E'\subset E$, with $\ell(\Omega'\setminus E')\leq\frac{2}{4}\varepsilon$, such that the restriction of $\|D_L\|$ to E' is continuous. Then, the set valued map $x\mapsto\frac{1}{\|D_L(x)\|}D_L(x)$ is upper semicontinuous on E'.

In fact, let $x_n\in E'$ such that $x_n\to x_*$ and $w_n\in\frac{1}{\|D_L(x_n)\|}D_L(x_n)$ with $w_n\to w_*$. Then, $\|D_L(x_n)\|w_n\to\|D_L(x_*)\|w_*$ that belongs to $D_L(x_*)$, i.e., $w_*\in\frac{1}{\|D_L(x_*)\|}D_L(x_*)$.

We have obtained that the restriction to E' of the map $\frac{1}{\|D_L\|}D_L$ has closed graph, and it follows that it is upper semicontinuous. Then, so is the the restriction to E' of the set valued map $\frac{1}{\|D_L\|}D_L-v$.

Applying Proposition 3.4 i), we infer that the restriction to E' of the map $x \mapsto |m_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]}|$ is lower semicontinuous, hence, for a suitable $E'' \subset E'$ with $\ell(\Omega' \setminus E'') \leq \frac{3}{4}\varepsilon$, its restriction to E'' is continuous. By Proposition 3.5, the restriction to E'' of $m_{[\frac{1}{\|D_L\|}D_L-v]}$ is continuous, and $\ell(\Omega \setminus E'') \leq \varepsilon$. Being ε arbitrary, $m_{\frac{1}{\|D_L\|}D_L-v}$ is measurable on Ω and belongs to $(\mathrm{L}^\infty(\Omega))^N$.

ii) Consider a simple function $\xi_s = \sum \alpha_i \chi_{A_i}$, with $\cup A_i = E'$; we have

$$(I_{[\frac{1}{\|D_L(x)\|}D_L-v(x)]})^*(\xi_s(x)) = \sum (I_{[\frac{1}{\|D_L(x)\|]}D_L-v(x)})^*(\alpha_i)\chi_{A_i}(x):$$

by Proposition 3.4 ii), it is upper semicontinuous in x on each A_i, hence measurable on E'.

Let (ξ_ν) be a sequence of simple functions, converging to $\xi|_{E'}$. Fix $\tilde{x}$: again by Proposition 3.4,

$$(I_{[\frac{1}{\|D_L(\tilde{x})\|}D_L(\tilde{x})-v(\tilde{x})]})^*(\xi_\nu(\tilde{x})) \text{ converges to } (I_{[\frac{1}{\|D_L(\tilde{x})\|}D_L(\tilde{x})-v(\tilde{x})]})^*(\xi(\tilde{x})).$$

Moreover, each of the functions $x \mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L-v(x)]})^*(\xi_\nu(x))$ is measurable, and so is their pointwise limit $(I_{[\frac{1}{\|D_L(\cdot)\|}D_L(\cdot)-v(\cdot)]})^*(\xi(\cdot))$. Being ε arbitrary, we have that $x \mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]})^*(\xi(x))$ is measurable on Ω. Finally, $|(I_{[\frac{1}{\|D_L\|}D_L-v]})^*(\xi)| \leq |\xi|$. □

3.3 Proof of the Theorem

Proof of Theorem 3.2. a) Let u be a locally bounded solution to problem (3.1), let $\eta \in C_0^\infty(\Omega)$. Without loss of generality assume that $\sup|\eta| \leq 1$ and $\sup|\nabla\eta| \leq 1$.

Set $\omega = supp(\eta)$, let U^* such that $|u(x)| \leq U^*$ on ω, and set $U = U^* + 1$. From (3.2) we infer that, for $|z| \leq 1$, $f(\xi + z) \leq f(\xi)e^H$. Recalling the notation $D_L(x) = \partial_\xi L(x, u(x), \nabla u(x))$, we have that

$$\frac{1}{\varepsilon}\big[L(x, u(x) + \varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x)) - L(x, u(x), \nabla u(x))\big]$$
$$\to \big[\sup_{k \in D_L(x)} \langle k, \nabla\eta(x)\rangle\big] + L_u(x, u(x), \nabla u(x))\eta(x)$$

pointwise w.r.t. x.

Moreover,

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\big[L(x,u(x)+\varepsilon\eta(x),\nabla u(x)+\varepsilon\nabla\eta(x))-L(x,u(x),\nabla u(x))\big]\right| \\
&\qquad = \left|\frac{1}{\varepsilon}\big[L(x,u(x)+\varepsilon\eta(x),\nabla u(x)+\varepsilon\nabla\eta(x))+\dots\right. \\
&\qquad\qquad\qquad \left.\dots - L(x,u(x),\nabla u(x)+\varepsilon\nabla\eta(x))\big]\right| \\
&\qquad\quad + \left|\frac{1}{\varepsilon}\big[L(x,u(x),\nabla u(x)+\varepsilon\nabla\eta(x))-L(x,u(x),\nabla u(x))\big]\right| \\
&\qquad \le \big|L_u(x,u(x)+\theta_1\varepsilon\eta(x),\nabla u(x)+\varepsilon\nabla\eta(x))\eta(x)\big| \\
&\qquad\quad + \big|\sup\{\langle k,\nabla\eta(x)\rangle : k\in\partial_\xi L(x,u(x),\nabla u(x)+\theta_2\varepsilon\nabla\eta(x))\}\big|.
\end{aligned}$$

From (3.5), we have

$$\begin{aligned}
|L_u(x,u(x)&+\theta_1\varepsilon\eta(x),\nabla u(x)+\varepsilon\nabla\eta(x))\eta(x)| \\
&\le \gamma_U(x)+h_U^3 f(\nabla u(x)+\varepsilon\nabla\eta(x)) \\
&\le \gamma_U(x)+h_U^3 f(\nabla u(x))e^H.
\end{aligned}\tag{3.7}$$

Assumption (3.3) implies that $f(\nabla u)$ is integrable, so that the right hand side of (3.7) is an integrable function, independent of ε.

We also have:

$$\begin{aligned}
\big|\sup\{\langle k,\nabla\eta(x)\rangle &: k\in\partial_\xi L(x,u(x),\nabla u(x)+\theta_2\varepsilon\nabla\eta(x))\}\big| \\
&\le |\nabla\eta(x)|\big[\beta_U(x)+h_U^2\big|\partial f(\nabla u(x)+\theta_2\varepsilon\nabla\eta(x))\big|\big] \\
&\le |\nabla\eta(x)|\big[\beta_U(x)+h_U^2 K f(\nabla u(x)+\theta_2\varepsilon\nabla\eta(x))\big] \\
&\le |\nabla\eta(x)|\big[\beta_U(x)+h_U^2 H f(\nabla u(x))e^H\big],
\end{aligned}$$

an integrable function, independent of ε.

Hence, by dominated convergence,

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\int_\Omega L(x,u(x)+\varepsilon\eta(x),\nabla u(x)+\varepsilon\nabla\eta(x))\,\mathrm{d}x-\int_\Omega L(x,u(x),\nabla u(x))\,\mathrm{d}x\right] \\
&\qquad \to \int_\Omega \sup_{k\in D_L(x)}\langle k,\nabla\eta(x)\rangle\,\mathrm{d}x+\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\,\mathrm{d}x \\
&\qquad = \int_\Omega (I_{D_L(x)})^*(\nabla\eta(x))\,\mathrm{d}x+\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, we obtain

$$0\le \int_\Omega (I_{D_L(x)})^*(\nabla\eta(x))\,\mathrm{d}x+\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\,\mathrm{d}x$$

or,

$$-\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\;\mathrm{d}x \le \int_\Omega (I_{D_L(x)})^*(\nabla\eta)\;\mathrm{d}x$$
$$= \int_\Omega \sup_{\{k\in D_L(x)\}} \langle k,\nabla\eta\rangle = \int_\Omega \sup_{\{k\in D_L(x)\}} \langle \frac{k}{\|D_L(x)\|}, \|D_L(x)\|\nabla\eta\rangle. \tag{3.8}$$

b) From (3.4), (3.3) and (3.2), we have that

$$\begin{aligned}\|D_L(x)\| &\le \beta_U(x) + h_U^2 Hf(\nabla u(x))\\ &\le \beta_U(x) + h_U^2 H\frac{1}{h_U^1}(L(x,u,\nabla u(x)) - \alpha_U(x)),\end{aligned}$$

so that $\|D_L\| \in \mathrm{L}^1(\Omega)$; for every $\eta\in C_0^\infty(\Omega)$ we have that $\|D_L\|\nabla\eta$ is in $(\mathrm{L}^1(\Omega))^N$. Consider $\mathbb{L}$, the linear subspace of $(\mathrm{L}^1(\Omega))^N$ defined as

$$\mathbb{L} = \left\{\xi \in (\mathrm{L}^1(\Omega))^N : \exists\eta\in C_0^\infty(\Omega) : \xi = \|D_L(x)\|\nabla\eta\right\}$$

and, on $\mathbb{L}$, the linear functional

$$T(\xi) = -\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\;\mathrm{d}x.$$

We notice that T is well defined: assume that there exist η^1 and η^2 in $C_0^\infty(\Omega)$ such that $\xi = \|D_L\|\nabla\eta^1 = \|D_L\|\nabla\eta^2$. Then, from (3.8), we have

$$|-\int_\Omega L_u(x,u(x),\nabla u(x))\eta^1(x)\;\mathrm{d}x + \int_\Omega L_u(x,u(x),\nabla u(x))\eta^2(x)\;\mathrm{d}x| = 0,$$

so that T is well defined.

The map

$$\begin{aligned}\varrho(\xi) := &\int_\Omega \sup_{\{k\in D_L(x)\}} \langle\frac{k}{\|D_L(x)\|}, \|D_L(x)\|\nabla\eta\rangle\;\mathrm{d}x\\ = &\int_\Omega \sup_{\{h\in \frac{1}{\|D_L(x)\|}D_L(x)\}} \langle h, \|D_L(x)\|\nabla\eta\rangle\;\mathrm{d}x\\ = &\int_\Omega \big(I_{\frac{1}{\|D_L(x)\|}D_L(x)}\big)^*\big(\|D_L(x)\|\nabla\eta(x)\big)\;\mathrm{d}x\end{aligned}$$

appearing at the right hand side of (3.8) is defined on $\mathbb{L}$ as a convex, positively homogeneous map. It can be extended, preserving these properties, to $(\mathrm{L}^1(\Omega))^N$, since $\big(I_{\frac{1}{\|D_L(x)\|}D_L(x)}\big)^*(\xi(x)) \le |\xi(x)|$.

Hence, by the Hahn-Banach Theorem, the linear map T can be extended from $\mathbb{L}$ to the whole of $(\mathrm{L}^1(\Omega))^N$, still satisfying $|T(\xi)| \leq \varrho(\xi)$.

c) By Lemma 3.6, we can apply Lemma 3.3 to the map $D = \frac{1}{\|D_L\|} D_L$. Hence, we infer the existence of a $\tilde{p} \in (\mathrm{L}^\infty(\Omega))^N$, with $\tilde{p}(x) \in \frac{1}{\|D_L(x)\|} D_L(x)$ almost everywhere on Ω, i.e., $\tilde{p}(x) = \frac{1}{\|D_L(x)\|} p(x)$ with $p(x) \in D_L(x)$, representing the extension of T to $(\mathrm{L}^1(\Omega))^N$, in particular, representing T on $\mathbb{L}$.

Hence, for every $\eta \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} -\int_\Omega L_u(x, u(x), \nabla u(x))\eta(x) \, \mathrm{d}x &= \int_\Omega \langle \tilde{p}(x), \|D_L(x)\| \nabla \eta(x) \rangle \, \mathrm{d}x \\ &= \int_\Omega \langle p(x), \nabla \eta(x) \rangle \, \mathrm{d}x \end{aligned}$$

In other words, for every $\eta \in C_0^\infty(\Omega)$,

$$\int_\Omega \langle p(x), \nabla \eta(x) \rangle \, \mathrm{d}x + \int_\Omega L_u(x, u(x), \nabla u(x))\eta(x) \, \mathrm{d}x = 0.$$

The map $p(\cdot)$ is a selection from $\partial_\xi L(\cdot, u(\cdot), \nabla u(\cdot))$ defined on Ω, thus proving the Theorem. □

Part II

Higher Integrability

Chapter 4

Beyond exponential growth

This chapter is based on a joint work with A. Cellina and M. Mazzola: [7], *The higher integrability and the validity of the Euler-Lagrange equation for solutions to variational problems*, preprint (submitted 2011).

4.1 Assumptions and higher integrability results

Some results in this chapter will depend on the properties of the *polar* or *Legendre-Fenchel transform* L^* of a convex function L; for its definition, we refer to Appendix A.3.2 and [37].

We shall consider Lagrangians L satisfying the following convexity and regularity assumptions.

Assumption 4.1. *$L(x,u,\xi)$ is non negative and positive whenever $\xi \neq 0$, and the map $t \mapsto L(x,u,t\xi)$ is non-decreasing for $t \geq 0$. In addition, for every (x,u), the restriction to the set $|\xi| \geq 1$ of the mapping $\xi \mapsto L(x,u,\xi)$ is the restriction to the same set of a convex function. Moreover, $L(x,u,\xi)$ is $C^1(u \times \xi)$, for each fixed x, and measurable in x for each fixed $(u \times \xi)$ and it is such that, for every $\omega \subset\subset \Omega$ and U, there exist constants $M = M(\omega,U)$, $K = K(\omega,U)$ and, for every R, a function $\alpha_{\omega,U,R}$ in $\mathrm{L}^1(\omega)$ such that for almost every $x \in \omega$, for every $|u| \leq U$, we have*

- i) *for every $\xi \in \mathbb{R}^n$, $\left|\frac{\partial L(x,u,\xi)}{\partial u}\right| \leq KL(x,u,\xi)$;*
- ii) $\sup\{|\nabla_\xi L(x,u,\xi)| : |u| \leq U; |\xi| \leq R\} \leq \alpha_{\omega,U,R}(x)$*;*
- iii) $\langle \nabla_\xi L(x,u,\xi), \xi \rangle \geq M |\nabla_\xi L(x,u,\xi)|\, |\xi|$.

The higher integrability results will depend on the following Condition. In it, and for the remainder of the chapter, for an open $O \subset\subset \Omega$ and $\delta > 0$, we set $O_\delta = O + B(0, \delta)$. Explicit classes of Lagrangians satisfying Condition 4.2 will be provided by Theorem 4.5.

Condition 4.2. *For every open $O \subset\subset \Omega$, $\delta^0 > 0$ and U there exist: a constant $\delta \leq \delta^0$ such that $\overline{O}_\delta$ is in Ω; a Lipschitz continuous function $\eta \in C_c(O_\delta)$, with $\eta(x) \geq 0$ and $\eta(x) = 1$ on O, and constants $\tilde{K} = \tilde{K}(U, O_\delta) \geq 0$ and $\tilde{R} = \tilde{R}(U, O_\delta)$, such that: for every ξ with $|\xi| \geq \tilde{R}$, for every u with $|u| \leq U$, for almost every $x \in O_\delta$, for every $\varepsilon > 0$ sufficiently small, we have*

$$\log L(x, u - \varepsilon\eta u, \xi(1 - \varepsilon\eta) - \varepsilon u \nabla\eta) - \log L(x, u, \xi) \leq \varepsilon\tilde{K}. \tag{4.1}$$

Next Theorem infers the higher integrability result from the validity of Condition 4.2.

Theorem 4.3. *Let L satisfy Assumption 4.1 and Condition 4.2. Let $\overline{u}$ be a locally bounded solution to the problem of minimizing*

$$\int_\Omega L(x, u(x), \nabla u(x))\mathrm{d}x$$

on $u^0 + W_0^{1,1}(\Omega)$. Then,

$$|\nabla_\xi L(\cdot, \overline{u}(\cdot), \nabla\overline{u}(\cdot))||\nabla\overline{u}(\cdot)| \in \mathrm{L}^1_{loc}(\Omega).$$

Proof of Theorem 4.3. a) Fix $O \subset\subset \Omega$. It is enough to prove the existence of H_1 such that

$$\int_O \langle \nabla_\xi L(x, \tilde{u}(x), \nabla\tilde{u}(x)), \nabla\tilde{u}(x)\rangle \, \mathrm{d}x \leq H_1.$$

In fact, if this is true, taking O to be ω in Assumption 4.1, point iii) proves the claim.

Hence, let $O_{\delta^0} \subset\subset \Omega$, let U be a bound for $|\overline{u}|$ on O_{δ^0}. Let δ, η and the constants $\tilde{R}$ and $\tilde{K}$ be provided by Condition 4.2 (we assume $\tilde{R} \geq 1$).

Since $\overline{u}$ is a solution, for the variation $-\varepsilon\eta\overline{u}$, with $\varepsilon > 0$, we obtain

$$0 \leq \frac{1}{\varepsilon}\int_{O_\delta} \left[L(x, \overline{u} - \varepsilon\eta\overline{u}, \nabla\tilde{u}(1 - \varepsilon\eta) - \varepsilon\overline{u}\nabla\eta) - L(x, \tilde{u}, \nabla\tilde{u})\right] \mathrm{d}x. \tag{4.2}$$

We have

$$\begin{aligned} &L(x, \overline{u} - \varepsilon\eta\overline{u}, \nabla\tilde{u}(1 - \varepsilon\eta) - \varepsilon\overline{u}\nabla\eta) - L(x, \tilde{u}, \nabla\tilde{u}) = \\ &\qquad = \varepsilon\int_0^1 \left[\frac{\partial L}{\partial u}(-\eta\overline{u}) + \langle\nabla_\xi L, -\eta\nabla\tilde{u} - \overline{u}\nabla\eta\rangle\right] \mathrm{d}s, \end{aligned}$$

where $\frac{\partial L}{\partial u}$ and $\nabla_\xi L$ are computed at $(x, \overline{u} - s\varepsilon\eta\overline{u}, \nabla\tilde{u}(1 - s\varepsilon\eta) - s\varepsilon\overline{u}\nabla\eta)$; hence, as $\varepsilon \to 0$, by the continuity of the partial derivatives of L,

$$\begin{aligned} \frac{L\left(x, \overline{u} - \varepsilon\eta\overline{u}, \nabla\tilde{u}(1 - \varepsilon\eta) - \varepsilon\overline{u}\nabla\eta\right) - L(x, u, \nabla\tilde{u})}{\varepsilon} &\to \\ \to \frac{\partial L}{\partial u}(-\eta\overline{u}) + \langle \nabla_\xi L, -\eta\nabla\tilde{u} - \overline{u}\nabla\eta \rangle\,, \end{aligned} \tag{4.3}$$

pointwise in x, and with the r.h.s. computed at $(x, \overline{u}(x), \nabla\tilde{u}(x))$. Set ${O_\delta}^- = \{x \in O_\delta : |\nabla\tilde{u}(x)| < \tilde{R}\}$ and ${O_\delta}^+ = \{x \in O_\delta : |\nabla\tilde{u}(x)| \geq \tilde{R}\}$: on ${O_\delta}^-$, the left hand side of (4.3) is uniformly bounded, so that, for every ε and for some $\tilde{M}$, we have

$$\left| \frac{1}{\varepsilon} \int_{{O_\delta}^-} [L\left(x, \overline{u} - \varepsilon\eta\overline{u}, \nabla\tilde{u}(1 - \varepsilon\eta) - \varepsilon\overline{u}\nabla\eta\right) - L(x, \tilde{u}(x), \nabla\tilde{u}(x))]\mathrm{d}x \right| \leq \tilde{M}.$$

b) On ${O_\delta}^+$, consider the constant $\tilde{K}$: setting

$$\tilde{\ell}_\varepsilon(x) = \log L(x, \overline{u}(x) - \varepsilon\eta(x)\overline{u}(x), \nabla\tilde{u}(x)(1 - \varepsilon\eta(x)) - \varepsilon\overline{u}(x)\nabla\eta(x)),$$

from (4.2) we have

$$\begin{aligned} -\tilde{M} &\leq \int_{{O_\delta}^+} \left(\frac{e^{\tilde{\ell}_\varepsilon} - e^{\log L(x, \overline{u}(x), \nabla\tilde{u}(x))}}{\varepsilon} \right) \mathrm{d}x \\ &= \int_{{O_\delta}^+} \left(\frac{e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K} + \varepsilon\tilde{K}} - e^{\log L(x, \overline{u}(x), \nabla\tilde{u}(x))}}{\varepsilon} \right) \mathrm{d}x \\ &= \int_{{O_\delta}^+} e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \left[\frac{e^{\varepsilon\tilde{K}} - 1 + 1 - e^{\log L(x, \overline{u}(x), \nabla\tilde{u}(x)) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K}}}{\varepsilon} \right] \mathrm{d}x, \end{aligned}$$

i.e.,

$$\begin{aligned} \tilde{M} + \int_{{O_\delta}^+} e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \left[\frac{e^{\varepsilon\tilde{K}} - 1}{\varepsilon} \right] \mathrm{d}x & \\ \geq \int_{O_\delta^+} e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \left[\frac{e^{\log L(x, \overline{u}(x), \nabla\tilde{u}(x)) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K}} - 1}{\varepsilon} \right] \mathrm{d}x & \end{aligned} \tag{4.4}$$

Since, on ${O_\delta}^+$, $\tilde{\ell}_\varepsilon(x) - \varepsilon\tilde{K} \leq \log L(x, \overline{u}(x), \nabla\tilde{u}(x))$ and also $\frac{e^{\varepsilon\tilde{K}} - 1}{\varepsilon} \leq \tilde{K}e^{\tilde{K}}$, the left hand side of (4.4) is bounded by

$$\tilde{M} + \tilde{K}e^{\tilde{K}} \int_\Omega L(x, \overline{u}(x), \nabla\tilde{u}(x))\mathrm{d}x = H,$$

independent of ε.

c) Consider the right hand side. For fixed x we have

$$\log L\big(x, \tilde{u}(x), \nabla\tilde{u}(x)\big) - \tilde{\ell}_\varepsilon(x) \\ = -\varepsilon\left[\frac{1}{L}\frac{\partial L}{\partial u}(-\eta\overline{u}) + \frac{1}{L}\left\langle \nabla_\xi L, -\eta\nabla\tilde{u} - \overline{u}\nabla\eta\right\rangle\right] + o(\varepsilon)$$

so that, as $\varepsilon \to 0$, pointwise w.r.t. x,

$$\frac{e^{\log L(x,\tilde{u},\nabla\tilde{u}) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K}} - 1}{\varepsilon} \to \tilde{K} + \frac{1}{L}\frac{\partial L}{\partial u}\eta\overline{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\overline{u} + \overline{u}\nabla\eta\right\rangle. \quad (4.5)$$

In addition, by (4.1), $\log L\big(x, \tilde{u}(x), \nabla\tilde{u}(x)\big) - \tilde{\ell}_\varepsilon(x) + \varepsilon\tilde{K} \geq 0$, so that the left hand side of (4.5) is non negative and so is its limit,

$$\tilde{K} + \frac{1}{L}\frac{\partial L}{\partial u}\eta\overline{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\overline{u} + \overline{u}\nabla\eta\right\rangle \geq 0.$$

Finally, pointwise, $e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \to e^{\log L(x,\tilde{u},\nabla\tilde{u})}$. Hence, applying Fatou's lemma, we obtain

$$\int_{O_\delta^+} L(x,\tilde{u},\nabla\tilde{u})\left[\tilde{K} + \frac{1}{L}\frac{\partial L}{\partial u}\eta\overline{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \overline{u}\nabla\eta\right\rangle\right] \leq H,$$

i.e.,

$$\int_{O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\overline{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \overline{u}\nabla\eta\right\rangle\right] \leq H.$$

Since the integrand above is non-negative, we have obtained, in particular, that

$$\int_{O\cap O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\overline{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \overline{u}\nabla\eta\right\rangle\right] \leq H.$$

On O we have that $\eta \equiv 1$, $\tilde{u}$ is bounded and that, by i) of Assumption 4.1, there exists K such that

$$\left|\frac{\partial L(x,\overline{u}(x),\nabla\overline{u}(x))}{\partial u}\right| \leq KL(x,\overline{u}(x),\nabla\overline{u}(x));$$

hence there exists H^+ such that

$$\int_{O\cap O_\delta^+}\left\langle \nabla_\xi L\left(x,\overline{u}(x),\nabla\overline{u}(x)\right), \nabla\tilde{u}(x)\right\rangle \mathrm{d}x \leq H^+.$$

Consider $O\cap O_\delta^-$: by Assumption 4.1, ii), we have that

$$\left|\left\langle \nabla_\xi L\left(x,\overline{u}(x),\nabla\overline{u}(x)\right), \nabla\tilde{u}(x)\right\rangle\right| \leq \tilde{R}\cdot\alpha_{\omega,U,\tilde{R}}(x) \text{ on } O\cap O_\delta^-.$$

Hence we have obtained that the integral

$$\int_O \langle \nabla_\xi L\left(x, \overline{u}(x), \nabla \overline{u}(x)\right), \nabla \tilde{u}(x) \rangle \, \mathrm{d}x$$

is bounded, thus proving the theorem. □

It is easy to show that the Lagrangians of exponential growth satisfy Condition 4.2. However, the following result shows that this condition is satisfied by a substantially larger class of functions. We shell need the following

Definition 4.4. A convex C^1 function Λ is called a *comparison function* for L if for every U there exist constants K_0, K_1 and K_2 such that for almost every $x \in \omega$, $|u| \leq U$ and $|\xi| \geq 1$ imply

1) $\Lambda(|\xi|) \leq K_0 L(x, u, \xi)$;

2) $K_1 \Lambda'(|\xi|) \leq |\nabla_\xi L(x, u, \xi)| \leq K_2 \Lambda'(|\xi|)$.

We shall also refer to the following

Exponential growth condition. *For every open $O \subset\subset \Omega$ and U there exists a constant c such that, for almost every $x \in O$, $|u| \leq U$ and $|\xi| \geq 1$ imply $|\nabla_\xi L(x, u, \xi)| \leq cL(x, u, \xi)$.*

Theorem 4.5. *Let L satisfy Assumption 4.1. Assume that, either*

i) *L satisfies the Exponential growth condition, or*

ii) *for $|\xi| \geq 1$, the map $\log(L(x, u, \cdot))$ is convex; there exists a comparison function Λ such that $\mathbb{L}(\cdot) = \log(\Lambda(\cdot))$ is convex and such that $\mathbb{L}^*$ is defined on $\mathbb{R}$. Assume that*

$$\int^\infty \frac{1}{z\partial(\mathbb{L}^*)(z)} \mathrm{d}z < \infty. \tag{4.6}$$

Then, Condition 4.2 is satisfied.

Remark 4.6. For every sufficiently large z, $0 \notin \partial(\mathbb{L}^*)(z)$; if this was not the case, in fact, the map $z \mapsto \mathbb{L}^*(z)$ would be constant and $\mathbb{L}$, hence Λ, would be defined on a single point.

Remark 4.7. The map $t \mapsto \exp(|t|^p)$ for $p > 1$ satisfies condition ii) but not condition i); the map $t \mapsto \exp(\exp(t))$ satisfies neither condition i) nor condition ii).

In the proof of Theorem 4.5, we shall need the following preliminary result (Lemma 1 in [17]).

Lemma 4.8. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be upper semicontinuous, strictly increasing and such that $G(0) = \{0\}$. Assume that, for a selection g from G,*

$$\int^{\infty} g(1/s)\mathrm{d}s < \infty.$$

Then, the implicit Cauchy problem

$$\begin{cases} x(t) \in G(x'(t)) \\ x(0) = 0, \end{cases}$$

admits a solution $\tilde{x}$, positive on some interval $(0, \tau)$.

Proof of Theorem 4.5. Fix O, δ^0 and U. Let $0 < \delta \leq \delta^0$ be such that $\overline{O}_\delta$ is in Ω. For a fixed variation η, we shall use the following notation

$$\ell_\varepsilon(x, u, \xi) = \log L\left(x, u - \varepsilon\eta u, \xi(1 - \varepsilon\eta) - \varepsilon u \nabla\eta\right).$$

a) In case i), choose any Lipschitz continuous function $\eta \in C_c(O_\delta)$, with $\eta(x) \geq 0$ and $\eta(x) = 1$ on O, and set $\mu = \sup|\nabla\eta|$. Choose $\tilde{R} = \max\{4, 2U\mu\}$, so that, for $\varepsilon \leq \frac{1}{2}$, $|\xi| \geq \tilde{R}$ implies, for $s \in [0, 1]$, $|\xi(1 - \varepsilon\eta) - s\varepsilon u\nabla\eta| \geq 1$.

Fix u such that $|u| \leq U$ and notice that, for $s \in [0, 1]$, we have $|u - s\varepsilon\eta u| \leq U$. It holds

$$\begin{aligned} \ell_\varepsilon - \log L(x, u, \xi) \ &= \int_0^1 \frac{\partial \log L(x, u - s\varepsilon\eta u, \xi(1 - \varepsilon\eta) - \varepsilon u\nabla\eta)}{\partial u}(-\varepsilon\eta u)\mathrm{d}s \\ &+ \int_0^1 \langle \nabla_\xi \log L(x, u, \xi(1 - \varepsilon\eta) - s\varepsilon u\nabla\eta), -\varepsilon u\nabla\eta\rangle \,\mathrm{d}s \\ &+ \log L(x, u, \xi(1 - \varepsilon\eta)) - \log L(x, u, \xi). \end{aligned}$$

By Assumption 4.1, $t \mapsto L(x, u, t\xi)$ is non-decreasing with respect to t on $\{t \geq 0\}$, hence the third term at the right hand side is non positive. Moreover, $|\frac{\partial \log L}{\partial u}| = \frac{1}{L}|\frac{\partial L}{\partial u}| \leq K$ so that the first term is bounded by ε times a constant. By the Exponential growth assumption, $|\nabla_\xi \log L| = \frac{|\nabla_\xi L|}{L} \leq c$, hence, the same is true for the second term.

b) Consider case ii). From

$$\begin{aligned} &\ell_\varepsilon(x, u, \xi) - \log L(x, u, \xi) \\ &\qquad = \varepsilon(-\eta u) \int_0^1 \frac{1}{L}\frac{\partial L}{\partial u}\mathrm{d}s + \varepsilon \int_0^1 \left\langle \frac{\nabla_\xi L}{L}, -\eta\xi - u\nabla\eta \right\rangle \mathrm{d}s, \end{aligned}$$

where the first integrand is evaluated at $(x, u - s\varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u\nabla\eta)$ and the second at $(x, u, \xi(1-s\varepsilon\eta) - s\varepsilon u\nabla\eta)$, we obtain

$$\ell_\varepsilon(x,u,\xi) - \log L(x,u,\xi) \le \varepsilon\left[KU + \left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta \right\rangle\right], \quad (4.7)$$

where $\xi_\varepsilon = (1 - s_\varepsilon\varepsilon\eta)\,\xi - s_\varepsilon\varepsilon u\nabla\eta$, for some $0 \le s_\varepsilon \le 1$.

For $z \neq 0$, set

$$G(1/z) = \frac{1}{z}\frac{7U}{M\partial(\mathbb{L}^*)(z)}.$$

From the assumption of convexity, $\partial(\mathbb{L}^*)$ is non-increasing as a function of $\frac{1}{z}$, while $\frac{7U}{M\partial(\mathbb{L}^*)(z)}$ is non-decreasing as a function of $\frac{1}{z}$, so that G satisfies the assumptions of Lemma 1.

Consider $\tilde{x}$, the solution to $\tilde{x} \in G(\tilde{x}')$, provided by Lemma 1, defined and positive on $(0,\tau]$. Possibly decreasing τ, we can assume, without loss of generality, that

$$x'(t) \le 1, \text{ for all } t \in (0,\tau]. \quad (4.8)$$

Notice that, from the inclusion $x(t) \in G(x'(t))$, we infer that $x' > 0$ on $(0,\tau]$, hence that x is strictly increasing, so that x' is strictly increasing as well. Set $\delta_\tau = \min\{\tau, \delta\}$ and define η as follows let $d(x)$ be the distance from a point $x \in O_{\delta_\tau}$ to ∂O_{δ_τ} and set

$$\eta(x) = \inf\left\{\frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}(d(x)), 1\right\}$$

so that, in particular, $\eta = 1$ on O. Almost everywhere, d is differentiable with $|\nabla d| = 1$ and, at a point of differentiability, we have

$$\nabla\eta(x) = \begin{cases} 0 & \text{if } d(x) > \delta_\tau \\ \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'(d(x))\nabla d(x) & \text{if } d(x) < \delta_\tau. \end{cases}$$

Hence, almost everywhere, we have that $|\nabla\eta| \le \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'(\delta_\tau)$ and that, either $\nabla\eta = 0$, or

$$\begin{aligned}\eta(x) &= \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}\big(d(x)\big) = \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'\big(d(x)\big)\frac{7U}{M\partial(\mathbb{L}^*)\big(\frac{1}{\tilde{x}'(d(x))}\big)} \\ &= h\big(\tilde{x}(\delta_\tau)|\nabla\eta(x)|\big)|\nabla\eta(x)|, \end{aligned} \quad (4.9)$$

where we have set

$$h(z) = \frac{7U}{M\partial(\mathbb{L}^*)(\frac{1}{z})}, \quad (4.10)$$

an increasing function.

Consider the term $\varepsilon\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right),-\eta\xi-u\nabla\eta\right\rangle$ in (4.7) and set $\overline{\xi}=(1-\varepsilon\eta)\,\xi-\varepsilon u\nabla\eta$.

Set $\mu_1=\sup|\nabla\eta|$ and $\tilde{R}=2+U\mu_1$, so that, for $\varepsilon\leq\frac{1}{2}$, $|\xi|\geq\tilde{R}$ implies that both $|\xi_\varepsilon|\geq 1$ and $|\overline{\xi}|\geq 1$. For those x such that

$$\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right),-\eta\xi-u\nabla\eta\right\rangle\leq 0, \tag{4.11}$$

any $\tilde{K}\geq KU$ will do to prove the result. Moreover, since the mapping $\xi\mapsto L(x,u,\xi)$ is convex and attains its minimum in 0, for $0\leq\varepsilon\leq 1$ we have $\frac{d}{ds}L(x,u,\xi(1-s\varepsilon\eta))\leq 0$, i.e., $\langle\nabla_\xi L(x,u,\xi(1-s\varepsilon\eta)),-\eta\xi\rangle\leq 0$, so that

$$\left\langle\frac{\nabla_\xi L}{L}(x,u,\xi(1-s\varepsilon\eta)),-\eta\xi\right\rangle\leq 0;$$

from this, we infer that, when $\nabla\eta(x)=0$, (4.11) holds.

Hence, we are left to consider those x such that, at once,

$$\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right),-\eta\xi-u\nabla\eta\right\rangle>0\quad\text{and}\quad\eta(x)=|\nabla\eta(x)|h\big(\tilde{x}(\delta_\tau)|\nabla\eta(x)|\big).$$

Given any $v,w\in\mathbb{R}^n$, from the assumption of convexity of $\log L\left(x,u,\cdot\right)$, we obtain that its gradient is monotonic, i.e., that

$$(s_1-s_2)\left\langle\frac{\nabla_\xi L}{L}\left(x,u,v+s_1w\right)-\frac{\nabla_\xi L}{L}\left(x,u,v+s_2w\right),w\right\rangle\geq 0,$$

i.e., that the mapping $s\mapsto\left\langle\frac{\nabla_\xi L}{L}\left(x,u,v+sw\right),w\right\rangle$ is non decreasing. Hence, from the inequality

$$\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right),-\eta\xi-u\nabla\eta\right\rangle>0,$$

we obtain $\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),-\eta\xi-u\nabla\eta\right\rangle>0$, where $\overline{\xi}=(1-\varepsilon\eta)\,\xi-\varepsilon u\nabla\eta$.

We infer that

$$\begin{aligned}\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\xi\right\rangle &< \left\langle\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),-u\frac{\nabla\eta}{\eta}\right\rangle\\ &\leq \left|\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right)\right|\cdot U\frac{|\nabla\eta|}{\eta}.\end{aligned} \tag{4.12}$$

Recalling Assumption 4.1, iii), we have

$$\begin{aligned}\left\langle\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\xi\right\rangle &= \left\langle\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\overline{\xi}+\varepsilon\big(\eta\xi+u\nabla\eta\big)\right\rangle\\ &\geq \left|\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right)\right|\big[M|\overline{\xi}|-\varepsilon\eta|\xi|-\varepsilon U|\nabla\eta|\big].\end{aligned} \tag{4.13}$$

From inequalities (4.12) and (4.13), we infer

$$\begin{aligned} U\frac{|\nabla\eta|}{\eta} &> M|\bar{\xi}| - \varepsilon\eta|\xi| - \varepsilon U|\nabla\eta| \\ &\geq M[(1-\varepsilon\eta)\,|\xi| - \varepsilon U|\nabla\eta|] - \varepsilon\eta|\xi| - \varepsilon U|\nabla\eta|, \end{aligned}$$

i.e.,

$$U\frac{|\nabla\eta|}{\eta} + \varepsilon U[M+1]|\nabla\eta| > |\xi|[M(1-\varepsilon\eta) - \varepsilon\eta].$$

We are free to assume $M < 1$; taking $\varepsilon < \frac{M}{4}$, we finally have

$$\begin{aligned} 3U\frac{|\nabla\eta|}{\eta} &> U\frac{|\nabla\eta|}{\eta} + \varepsilon|\nabla\eta|U[M+1] \\ &> |\xi|[M(1-\varepsilon\eta) - \varepsilon\eta] \; > \; \frac{1}{2}M|\xi| \end{aligned} \tag{4.14}$$

and, recalling (4.9), we obtain

$$|\xi| < \frac{6U}{Mh(\tilde{x}(\delta_\tau)|\nabla\eta(x)|)}. \tag{4.15}$$

From Definition 4.4 we have

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta \right\rangle \leq \big(\eta|\xi| + U|\nabla\eta|\big)K_0K_2\mathrm{L}'\left(|\xi_\varepsilon|\right); \tag{4.16}$$

noticing that

$$|\xi_\varepsilon| \leq \frac{6U}{Mh\big(\tilde{x}(\delta_\tau)|\nabla\eta|\big)} + \varepsilon|\nabla\eta|U \tag{4.17}$$

and that $\mathbb{L}'$ is non-decreasing, from (4.7), (4.9), (4.14), (4.16) and (4.17), we obtain

$$\begin{aligned} &\ell_\varepsilon - \log L(x,u,\xi) \\ &\quad \leq \varepsilon KU + \varepsilon\left(\eta|\xi| + U|\nabla\eta|\right)K_0K_2\mathrm{L}'\left(|\xi_\varepsilon|\right) \\ &\quad \leq \varepsilon KU + \varepsilon|\nabla\eta|K_0K_2\;\mathrm{L}'\left(\frac{6U}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)} + \varepsilon|\nabla\eta|U\right)\left(\frac{6U}{M} + U\right). \end{aligned}$$

By (4.8), we have that $\tilde{x}(\delta_\tau)|\nabla\eta| \leq 1$; there exists σ such that, for $t < \sigma$, we have

$$h(1) \leq \frac{1}{MUt}$$

so that for $|\nabla\eta| < \sigma$, $h(\tilde{x}(\delta_\tau)|\nabla\eta|) \leq h(1) \leq \frac{1}{MU|\nabla\eta|}$. Then,

$$\frac{6}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)} + \varepsilon|\nabla\eta|U \leq \frac{7}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)}.$$

Hence, for those x such that $|\nabla\eta(x)| < \sigma$, recalling (4.10), we obtain

$$\begin{aligned}
\mathrm{L}'\left(\frac{6U}{M\cdot h\big(\tilde{x}(\delta_\tau)|\nabla\eta|\big)}+\varepsilon|\nabla\eta|U\right)|\nabla\eta| &\leq \mathrm{L}'\left(\frac{7U}{M\cdot h\big(\tilde{x}(\delta_\tau)|\nabla\eta|\big)}\right)|\nabla\eta| \\
&= \mathrm{L}'\left(\partial(\mathbb{L}^*)\left(\frac{1}{\tilde{x}(\delta_\tau)|\nabla\eta|}\right)\right)|\nabla\eta| \\
&= \frac{1}{\tilde{x}(\delta_\tau)},
\end{aligned}$$

a constant independent on ε, thus proving the result in this case.

It is left to consider those x such that $|\nabla\eta(x)| \geq \sigma$: in this case, from (4.15), we have $|\xi| \leq \frac{6U}{M\cdot h(\tilde{x}(\delta_\tau)\sigma)}$ and, from (4.7), the result follows from the boundedness of $|\nabla\eta|$. □

4.2 The validity of the Euler-Lagrange equation

The higher integrability property for a minimizer $\overline{u}$ is independent on the validity of the Euler-Lagrange equation. In the next theorem we wish to use this result in order to establish the validity of the Euler-Lagrange equation for a class of problems including Lagrangians having growth faster than exponential.

Theorem 4.9. *Let $L(x,u,\xi)$ satisfy Assumption 4.1 and assume that there exist a comparison function Λ and a constant $c>0$ such that, for $t\geq 1$, either*

i) $\frac{d}{dt}\mathbb{L}(t) \leq c$, *or*

ii) $\frac{d}{dt}\mathbb{L}(t) \leq c\,(1+\log t)$, $\mathbb{L}(\cdot)$ *is convex and* $\mathrm{Dom}(\mathbb{L}^*)$ *is open,*

where $\mathbb{L} = \log\Lambda$. Then, a locally bounded solution $\overline{u}$ to the problem of minimizing

$$\int_\Omega L\left(x,u\left(x\right),\nabla u\left(x\right)\right)\mathrm{d}x \quad \textit{for} \quad u\in u_0+W_0^{1,1}\left(\Omega\right)$$

satisfies the Euler-Lagrange equation.

Lagrangians of exponential growth satisfy i); the map $\Lambda(t)=t^t$ is not of exponential growth but satisfies ii): in this case, $\mathrm{Dom}(\mathbb{L}^*)=\mathbb{R}$.

In order to prove Theorem 4.9, we shall need the following Lemma.

Lemma 4.10. *Let* $L : \mathbb{R} \to \mathbb{R}$ *be convex and* C^1 *and such that* $\mathrm{Dom}(\partial L^*)$ *is open; let* δ^* *be any selection from* ∂L^**. Then, there exists a sequence of convex* C^2 *functions* L_m *such that*

i) $\mathrm{Dom}(L'_m) \supset [-m+1, m-1]$*;* $\forall x \in \mathrm{Dom}(L'_m)$*, we have* $|L'_m(x) - L'(x)| < \frac{1}{m}$*;*

ii) $L^*_m \in C^1(\mathrm{Dom}(\partial L^*))$*; for every* $[a,b] \subset \mathrm{Dom}(\partial L^*)$ *there exists a subsequence* $m(j)$ *such that* $(L^*_{m(j)})' \to \delta^*$ *pointwise a.e. on* $[a,b]$*.*

Proof. Ad i). By assumption, L' is a single-valued, continuous and non-decreasing function; hence, its inverse, ∂L^*, is strictly increasing, possibly multi-valued, defined on the image of L'. The selection δ^* (discontinuous at most on a set of measure zero) is strictly increasing and bounded on sets compactly contained in its domain.

Consider the interval $[-n, n]$, so that $[L'(-n), L'(n)]$ is a compact subset of the open set $\mathrm{Dom}(\partial L^*)$. Then, there exists a subsequence $n(m)$ such that both $L'(-n(m+1)) < L'(-n(m))$ and $L'(n(m+1)) > L'(n(m))$. Moreover, there exists $N(n(m))$, with $N(n(m)) \geq n(m)$ and $\frac{1}{N(n(m))} \leq \frac{1}{4}\min\{L'(-n(m)) - L'(-n(m+1)), L'(n(m+1)) - L'(n(m))\}$, such that

$$[L'(-n(m)) - \frac{1}{N(n(m))}, L'(n(m)) + \frac{1}{N(n(m))}] \subset \mathrm{Dom}(\partial L^*),$$

so that the map

$$(L^*_m)' = \rho_{N(n(m))} * \delta^*,$$

where $\rho_{N(n(m))}$ is a standard mollifier having support in $[-\frac{1}{N(n(m))}, \frac{1}{N(n(m))}]$, is well defined on $[L'(-n(m)), L'(n(m))]$ as a strictly increasing function. Its image is the interval $I(n(m)) = [(L^*_m)'(L'(-n(m)), (L^*_m)'(L'(n(m)))]$.

We claim that $I(n(m+1)) \supset [-n(m), n(m)] \supset [-m, m]$.

In fact, consider $\overline{p} = \frac{1}{2}(L'(-n(m+1)) + L'(-n(m)))$: for every p such that $|p - \overline{p}| \leq \frac{1}{N(n(m))}$, we have $\delta^*(p) < \delta^*(L'(-n(m))) = -n(m)$, so that $\rho_{N(n(m+1))} * \delta^*(\overline{p}) < -n(m)$. Analogously for $\frac{1}{2}(L'(n(m+1)) + L'(n(m)))$.

The map $(L^*_m)'$ is a C^1 and strictly increasing, hence invertible, function: on the interval $I(n(m))$ we set $L'_m = ((L^*_m)')^{-1}$. Fix arbitrarily m and $\overline{x} \in I(n(m))$. Set $\overline{y}_m = L'_m(\overline{x})$, so that

$$\overline{x} = \rho_{N(n(m))} * \delta^*(\overline{y}_m) = \int_{[\overline{y}_m - \frac{1}{N(n(m))}, \overline{y}_m + \frac{1}{N(n(m))}]} \rho_{N(n(m))}(\overline{y}_m - y)\delta^*(y)\, \mathrm{d}y.$$

We notice that $\overline{x} \in \mathrm{co}\{\delta^*(y) : |y - \overline{y}_m| \leq \frac{1}{N(n(m))}\}$: in fact, otherwise, there exists α such that, for every $y \in \{|y - \overline{y}_m| \leq \frac{1}{N(n(m))}\}$, we have $\alpha\overline{x} > \alpha\delta^*(y)$;

then,

$$\alpha\overline{x} = \int \alpha\overline{x} \cdot \rho_{N(n(m))}(\overline{y}_m - y) > \alpha \int \delta^*(y) \cdot \rho_{N(n(m))}(\overline{y}_m - y) = \alpha\overline{x}.$$

Hence, there are y_1 and y_2 such that $|y_i - \overline{y}_m| \leq \frac{1}{N(n(m))}$ and $\delta^*(y_1) \leq \overline{x} \leq \delta^*(y_2)$. By the monotonicity of L', the last inequality can be written as $y_1 \leq L'(\overline{x}) \leq y_2$, so that

$$\overline{y}_m - \frac{1}{N(n(m))} \leq y_1 \leq L'(\overline{x}) \leq y_2 \leq \overline{y}_m + \frac{1}{N(n(m))}.$$

We have obtained

$$|L'(\overline{x}) - L'_m(\overline{x})| \leq \frac{1}{n(m)} \leq \frac{1}{m}.$$

Ad ii). Fix arbitrarily $[a, b]$: we have that $(L^*_m)' \to \delta^*$ in $L^1([a, b])$. Hence, there exists a sequence $(L^*_{m(j)})'$ converging to δ^* pointwise a.e. on $[a, b]$.

□

The condition that $\mathrm{Dom}(\partial L^*)$ be open is not satisfied by a map like $L(t) = |t|$; however it is satisfied by the minimal area functional $L(t) = \sqrt{1 + |t|^2}$ and, *a fortiori*, by any L of superlinear growth.

Proof of Theorem 4.9. Fix $h_0 > 0$, set $\Lambda^* = \sup\{\Lambda'(s) : 0 \leq s \leq 1 + h_0\}$. We claim that, both in case i) and in case ii), there exists K such that $0 < h < h_0$ implies

$$\Lambda'(t+h) \leq K\left[1 + \Lambda(t) + t\Lambda'(t)\right]. \tag{4.18}$$

In fact, in case i), we have $\Lambda'(t+h) \leq \Lambda^*$ for $t \leq 1$, while, for $t > 1$, $\Lambda(t+h) \leq c\Lambda(t)e^{ch}$ and we infer that (4.18) holds. In case ii), again $\Lambda'(t+h) \leq \Lambda^*$ for $t \leq 1$, while, for $t > 1$, we have

$$\begin{aligned}
\Lambda'(t+h) &= \mathrm{L}'(t+h)\frac{\Lambda(t+h)}{\Lambda(t)}\Lambda(t) \\
&= \mathrm{L}'(t+h) \cdot \exp\left(\int_t^{t+h} \mathrm{L}'(s)\,\mathrm{d}s\right) \cdot \Lambda(t) \\
&\leq c\left[1 + \log(t+h_0)\right] \cdot \exp\left[c\int_t^{t+h_0} (1 + \log s)\,\mathrm{d}s\right] \cdot \Lambda(t) \\
&\leq c\left(1 + \log t + \frac{h_0}{t}\right) \cdot (t+h_0)^{ch_0} \cdot e^{[c\ t(\log(t+h_0) - \log t)]} \cdot \Lambda(t) \\
&\leq c_1(1 + \log t) \cdot t^{ch_0}e^{ch_0}\Lambda(t) \leq c_2 \cdot t\Lambda(t).
\end{aligned}$$

By assumption, $\log(\Lambda)$ is convex, so that there exists c_3 such that $\frac{\Lambda'(t)}{\Lambda(t)} \geq c_3$. Hence, $\Lambda'(t+h) \leq c_2 c_3 \cdot t\Lambda'(t)$, and (4.18) is established.

Next, we claim that, setting $t = |\nabla\overline{u}(x)|$ in the right hand side of (4.18), we obtain a function integrable on compact subsets of Ω. By i) of the comparison assumption, we have that $\Lambda(|\nabla\overline{u}|) \in \mathrm{L}^1_{loc}(\Omega)$. By ii) of the comparison assumption, to show that $|\nabla\overline{u}|\Lambda'(|\nabla\overline{u}|) \in \mathrm{L}^1_{loc}(\Omega)$, is enough to show that Theorem 4.3 holds, i.e., that the assumptions of Theorem 4.5 are satisfied. The assumptions are obviously satisfied in case i), so we consider case ii). We have to prove that

$$\int^{\infty} \frac{1}{z\partial(\mathbb{L}^*)(z)} \mathrm{d}z < \infty.$$

Since $\mathrm{Dom}(\partial\mathbb{L}^*)$ is open, Lemma 4.10 can be applied to $\mathbb{L}$. Consider the sequence $(\mathbb{L}_m)$; for any α, β $(\alpha \geq 1)$, by the change of variables $z = \mathbb{L}'_m(t)$, we have

$$\begin{aligned} \int_{\mathbb{L}'_m(\alpha)}^{\mathbb{L}'_m(\beta)} \frac{dz}{z(\mathbb{L}^*_m)'(z)} \mathrm{d}z &= \int_{\alpha}^{\beta} \frac{\mathrm{L}''_m(t)}{t \cdot \mathrm{L}'_m(t)} \mathrm{d}t \\ &= \left.\frac{\log \mathrm{L}'_m(t)}{t}\right|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \frac{\log \mathrm{L}'_m(t)}{t^2} \mathrm{d}t. \end{aligned} \tag{4.19}$$

By assumption ii),

$$\left.\frac{\log \mathbb{L}'(t)}{t}\right|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \frac{\log \mathbb{L}'(t)}{t^2} \mathrm{d}t \leq \left.\frac{\log(c(1+\log t))}{t}\right|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \frac{\log(c(1+\log t))}{t^2}\, dt$$

and there exists H such that, for every $\alpha \geq 1$, for every β, the right hand side is bounded by H. Whenever $m - 1 \geq \beta$, $\mathrm{Dom}(\mathbb{L}'_m) \supset [\alpha, \beta]$ and $|\mathbb{L}' - \mathbb{L}'_m| \leq \frac{1}{m}$, so that the right hand side of (4.19) is bounded by $H + 2$ (independent of α, β and m). Consider the subsequence $m(j)$ provided by ii) of Lemma 4.10. Fix any $a, b \in \mathbb{R}$; let α, β such that, for j sufficiently large, $[a, b] \subset [\mathbb{L}'_{m(j)}(\alpha), \mathbb{L}'_{m(j)}(\beta)]$. By ii) of Lemma 4.10 and by Fatou's Lemma,

$$\int_a^b \frac{1}{z\partial(\mathbb{L}^*)(z)} \mathrm{d}z = \int_a^b \frac{1}{z\delta^*(z)} \mathrm{d}z \leq \liminf \int_a^b \frac{dz}{z(\mathbb{L}^*_{m(j)})'(z)} \mathrm{d}z \leq H + 2,$$

so that (4.6) is satisfied and the integrability claim holds.

To establish the validity of the Euler-Lagrange equation, fix $\eta \in C^1_c(\Omega)$ (we assume $\sup \eta \leq 1$) and set $\tilde{h} = \sup |\nabla\eta|$ and $S = \mathrm{spt}(\eta)$. Since $\overline{u}$ is a solution, we have

$$\int_S \frac{L(x, \overline{u}(x) + \varepsilon\eta(x), \nabla\overline{u}(x) + \varepsilon\nabla\eta(x)) - L(x, \overline{u}(x), \nabla\overline{u}(x))}{\varepsilon} \mathrm{d}x \geq 0;$$

the integrand converges pointwise to

$$\langle \nabla_\xi L\left(x, \overline{u}, \nabla \overline{u}\right), \nabla \eta\rangle + \frac{\partial L}{\partial u}\left(x, \overline{u}, \nabla \overline{u}\right) \cdot \eta$$

and we wish to dominate the integrand by an integrable function.

We have

$$\begin{aligned}
&\left|\frac{L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \varepsilon \nabla \eta\right) - L\left(x, \overline{u}, \nabla \overline{u}\right)}{\varepsilon}\right| \\
&\qquad \leq \left|\frac{L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u}\right) - L\left(x, \overline{u}, \nabla \overline{u}\right)}{\varepsilon}\right| \\
&\qquad\qquad + \left|\frac{L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \varepsilon \nabla \eta\right) - L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u}\right)}{\varepsilon}\right| \\
&\qquad \leq \left|\frac{\partial L}{\partial u}\left(x, \overline{u} + \overline{s}\varepsilon\eta, \nabla \overline{u}\right) \cdot \eta\right| \\
&\qquad\qquad + \left|\left\langle \nabla_\xi L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \overline{t}\varepsilon \nabla \eta\right), \nabla \eta\right\rangle\right|.
\end{aligned}$$

By Assumption 4.1, i), the first term is bounded by $K\, L\left(x, \overline{u}, \nabla \overline{u}\right) \cdot e^K \cdot |\eta|$, an integrable function. Set $E = \{x : |\nabla \overline{u}(x)| \geq 1 + \tilde{h}\}$ and write the second term as

$$\begin{aligned}
&\left|\left\langle \nabla_\xi L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \overline{t}\varepsilon \nabla \eta\right), \nabla \eta\right\rangle\right| \chi_{S\setminus E} \\
&\qquad + \left|\left\langle \nabla_\xi L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \overline{t}\varepsilon \nabla \eta\right), \nabla \eta\right\rangle\right| \chi_E.
\end{aligned}$$

Set $U = \sup\{|\overline{u}(x)|\} + 1$. On $S \setminus E$, $|\nabla \overline{u} + \varepsilon \nabla \eta| \leq 1 + 2\tilde{h} = R$, hence by Assumption 4.1, ii), the first term is bounded by an integrable function.

On E, we have $|\nabla \overline{u} + \varepsilon \nabla \eta| \geq 1$, hence, by ii) of the comparison assumption and (4.18), whenever $\varepsilon \tilde{h} < h_0$,

$$\begin{aligned}
&\left|\left\langle \nabla_\xi L\left(x, \overline{u} + \varepsilon\eta, \nabla \overline{u} + \overline{t}\varepsilon \nabla \eta\right), \nabla \eta\right\rangle\right| \\
&\qquad \leq K_2 \Lambda'\left(|\nabla \overline{u}(x) + \overline{t}\varepsilon \nabla \eta|\right) \\
&\qquad \leq K_2 K\left[1 + \Lambda\left(|\nabla \overline{u}(x)|\right) + |\nabla \overline{u}(x)| \Lambda'\left(|\nabla \overline{u}(x)|\right)\right].
\end{aligned}$$

The last term is integrable, by our previous claim, and is independent of ε, so that we can pass to the limit under the integral sign. Finally, considering also $-\eta$, we obtain that

$$\int_\Omega [\langle \nabla_\xi L(x, \overline{u}(x), \nabla \overline{u}(x)), \nabla \eta(x)\rangle + L_u(x, \overline{u}(x), \nabla \overline{u}(x))\eta(x)] \mathrm{dx} = 0$$

for every admissible variation η. □

Part III

On the Lavrentiev phenomenon

Chapter 5

Non-occurrence of the Lavrentiev phenomenon

This chapter is based on a joint work with A. Cellina: [6], *On the non-occurrence of the Lavrentiev phenomenon*, preprint (submitted 2011).

5.1 Assumptions and main result

The purpose of the present chapter is to prove the following result, an approximation result that, in particular, guarantees the non-occurrence of the Lavrentiev phenomenon.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with $\partial\Omega \in C^2$; let $u^0 \in C^2(\overline{\Omega})$; let $L : [0,\infty) \to [0,\infty)$ be convex and such that $L(0) = 0$. Let $u \in u^0 + W^{1,1}(\Omega)$ be bounded on Ω and such that*

$$\int_\Omega L(|\nabla u(x)|)\ \mathrm{d}x < \infty.$$

Then, given $\varepsilon > 0$, there exists $u_\varepsilon \in u^0 + W^{1,1}(\Omega)$, with u_ε Lipschitzean on $\overline{\Omega}$, such that

$$\int_\Omega L(|\nabla u_\varepsilon(x)|)\ \mathrm{d}x \le \int_\Omega L(|\nabla u(x)|)\ \mathrm{d}x + \varepsilon.$$

Notice that neither regularity nor growth conditions are assumed on the Lagrangian L, besides its being convex. When u is a solution, the boundedness of u follows from the boundedness of u^0.

We shall use the following notation: $B(x, \delta)$ is the open ball centered at x of radius δ; the Lebesgue measure of a subset A of $\mathbb{R}^N$ is $|A|$; ω_N is the measure of the unit ball; the complement of Ω is $C\Omega$; $d(x) = \mathrm{dist}(x, C\Omega)$, a Lipschitzean function of Lipschitz constant 1; diam is the diameter of Ω; $\Omega_\delta = \{x \in \Omega : d(x) \leq \delta\}$; d_H is the Hausdorff distance; the normal to $\partial\Omega$ at the point y, pointing towards the interior of Ω, is $\nu(y)$; $T(y)$ is the tangent plane to $\partial\Omega$ at y and $T^1(y) = \{\tau \in T(y) : |\tau| = 1\}$. A vector $x \in \mathbb{R}^N$ will be often written as $(\hat{x}, x_N)$. The Hessian matrix of a function ϕ is H_ϕ. For the coarea Theorem and the notion of Jacobian of a map $g : \mathbb{R}^N \to \mathbb{R}^n$ we refer to [26].

With the above notations, we summarize the assumptions of Theorem 5.1 assuming that *there exists* $K > 1$ *such that:* $|\nabla u^0| \leq K$*;* $|H_{u^0}| \leq K$*; the map* $y \mapsto \nu(y)$ *is Lipschitzean of constant* K*. Moreover,* $d_H(T(y^1), T(y^2)) \leq K|y^2 - y^1|$*. In addition, there exists* $M \geq 1$ *such that for* $x \in \Omega$*,* $|u(x)| \leq M$*,* $|u^0(x)| \leq M$*.*

5.2 Preliminary results

In what follows, a constant h will be chosen; apart from further conditions, we shall always assume that $h > 3K$.

Definition 5.2. For $x \in \Omega$, set

$$w_+^h(x) = \min\{u^0(z) + h|z - x| : z \in \partial\Omega\}, \tag{5.1}$$

$$w_-^h(x) = \max\{u^0(z) - h|z - x| : z \in \partial\Omega\},$$

and

$$M^h(x) = \begin{cases} w_+^h(x) & \text{when } u(x) > w_+^h(x), \\ u(x) & \text{when } w_-^h(x) \leq u(x) \leq w_+^h(x), \\ w_-^h(x) & \text{when } u(x) < w_-^h(x). \end{cases}$$

Lemma 5.3. *Let* Ω *and* u^0 *be as in Theorem 5.1. Let* $y = y(x)$ *be a point where* $w_+^h(x) = u^0(y(x)) + h|y(x) - x|$*. Then*

i) $|y - x| \leq \frac{h+K}{h-K} d(x) \leq 2d(x)$ *and* $|w_+^h(x) - u^0(y(x))| \leq [K + h]d(x)$*, and*

ii) *(uniqueness) there exist* h^* *and* d^* *such that* $h \geq h^*$ *and* $d(x) \leq d^*$ *imply that* $y = y(x)$ *is uniquely defined and we have*

$$|y - x| = \frac{w_+^h(x) - u^0(y)}{h}.$$

The same inequalities hold for w^h_-*, provided that in* ii) *we read* $|y-x| = \frac{u^0(y)-w^h_-(x)}{h}$.

Proof. We shall prove the inequalities for w^h_+.

Ad i). Let $y^* \in \partial\Omega$ be such that $|y^*-x| = d(x)$. From the definition of w^h_+ we have that $u^0(y^*)+hd(x) \geq u^0(y)+h|y-x|$, hence $h|y-x| \leq hd(x)+|u^0(y^*)-u^0(y)| \leq hd(x)+K|y^*-y| \leq hd(x)+K[|y-x|+d(x)]$, so that

$$|y-x| \leq \left(\frac{h+K}{h-K}\right) d(x);$$

again from $u^0(y^*)+hd(x) \geq w^h_+(x)$ we infer

$$\begin{aligned} |w^h_+(x)-u^0(x)| &\leq |u^0(y^*)-u^0(x)|+hd(x) \\ &\leq K|y^*-x|+hd(x) = [K+h]d(x), \end{aligned}$$

thus proving i).

Ad ii). Whenever the minimum is attained at a point y, since y is a constrained minimum point, we must have

$$\nabla u^0(y) + h\frac{y-x}{|y-x|} = \nabla\left(u^0(y)+h|y-x|\right) = \lambda\nu(y),$$

so that, for any τ in $T(y)$,

$$\langle \nabla u^0(y), \tau\rangle = -h\langle \frac{y-x}{|y-x|}, \tau\rangle. \tag{5.2}$$

Assume that y^1 and y^2 are points where the minimum is attained; set $r = |x-y^2|-|x-y^1|$, so that $|r| \leq |y^2-y^1|$.

For any $\tau^i \in T(y^i)$, from (5.2) we infer

$$0 = \langle x-y^2+|x-y^2|\frac{\nabla u^0(y^2)}{h}, \tau^2\rangle = \langle x-y^1+|x-y^1|\frac{\nabla u^0(y^1)}{h}, \tau^1\rangle$$

so that

$$\begin{aligned} &\langle x-y^2+|x-y^2|\frac{\nabla u^0(y^2)}{h}, \tau^1\rangle - \langle x-y^1+|x-y^1|\frac{\nabla u^0(y^1)}{h}, \tau^1\rangle \qquad (5.3) \\ &\qquad = \langle x-y^2+|x-y^2|\frac{\nabla u^0(y^2)}{h}, \tau^1-\tau^2\rangle. \end{aligned}$$

There exists η^* such that: for any y^2 with $|y^2-y^1| \leq \eta^*$ there is $\tau \in T(y^1)$ (with τ depending on y^2) such that

$$\langle \frac{y^1-y^2}{|y^1-y^2|}, \tau\rangle \geq \frac{1}{2}.$$

We have

$$\langle x-y^2+|x-y^2|\frac{\nabla u^0(y^2)}{h},\tau\rangle-\langle x-y^1+|x-y^1|\frac{\nabla u^0(y^1)}{h},\tau\rangle$$

$$=\langle y^1-y^2+r\frac{\nabla u^0(y^2)}{h},\tau\rangle-|x-y^1|\langle\frac{\nabla u^0(y^1)-\nabla u^0(y^2))}{h},\tau\rangle$$

$$=|y^1-y^2|\langle\frac{y^1-y^2}{|y^1-y^2|},\tau\rangle+r\langle\frac{\nabla u^0(y^2)}{h},\tau\rangle-|x-y^1|\langle\frac{\nabla u^0(y^1)-\nabla u^0(y^2))}{h},\tau\rangle.$$

Set $d^1=\min\{\frac{\eta^*}{4},1\}$, so that $d(x)\le d^1$ implies $|y^1-y^2|\le\eta^*$ and, from equation (5.3), we obtain, for any $\tau^1\in T(y^1)$,

$$\langle x-y^2+|x-y^2|\frac{\nabla u^0(y^2)}{h},\tau^1-\tau^2\rangle\ge\frac{1}{2}|y^1-y^2|-3|y^2-y^1|\frac{K}{h}.$$

Consider the left hand side for $\tau^1=\tau$; choose $\tau^2\in T^1(y^2)$ so that $|\tau^2-\tau|\le K|y^1-y^2|$; we obtain

$$2d(x)\left(1+\frac{K}{h}\right)K|y^1-y^2|\ge\frac{1}{2}|y^1-y^2|-3|y^2-y^1|\frac{K}{h};$$

choosing $h=12K$ and $d^*=\min\{d^1,\frac{1}{20K}\}$, the previous inequality implies $|y^2-y^1|=0$.

It is easy to check that ∇w^h_+ is constant of norm h along the line segment joining y to x and is directed in the direction from y to x; hence we have the identity

$$|y-x|=\frac{w^h_+(x)-u^0(y)}{h}. \tag{5.4}$$

□

Lemma 5.4. *Let $v\in W^{1,1}(\Omega)$ be such that $|v(x)|\le M$ a.e. on Ω and, on $\Omega\setminus\Omega_\delta$, define the function*

$$\tilde v(x)=\frac{1}{|B(x,\delta)|}\int_{B(x,\delta)}v(z)\ dz.$$

Then: i) *$\tilde v$ is Lipschitzean of constant $NM\frac{1}{\delta}$ and,* ii) *$\tilde v$ is a.e. differentiable and, at a point x of differentiability, we have*

$$\nabla\tilde v(x)=\frac{1}{\omega_N\delta^N}\int_{B(0,\delta)}\nabla v(x-z)\ dz.$$

Proof. Ad i).

$$\begin{aligned} |\tilde{v}(x^2)-\tilde{v}(x^1)| &= \left| \frac{1}{|B(x^2,\delta)|}\int_{B(x^2,\delta)} v(z)\ \mathrm{d}z - \frac{1}{|B(x^1,\delta)|}\int_{B(x^1,\delta)} v(z)\ \mathrm{d}z \right| \\ &\le \frac{1}{\omega_N \delta^N} M |B(x^1,\delta)\triangle B(x^2,\delta)| \end{aligned}$$

and

$$|B(x^1,\delta)\triangle B(x^2,\delta)| \le 2\omega_N\delta^N \le \omega_N \delta^{N-1}|x^1-x^2|$$

when $|x^1-x^2|\ge 2\delta$, while, when $|x^1-x^2|<2\delta$,

$$|B(x^1,\delta)\triangle B(x^2,\delta)| \le \omega_N[(\delta+|x^1-x^2|)^N-\delta^N] \le N\omega_N\delta^{N-1}|x^1-x^2|$$

so that, in either case,

$$|\tilde{v}(x^2)-\tilde{v}(x^1)| \le NM\frac{1}{\delta}|x^1-x^2|.$$

Ad ii). From i) we have that there exists $\Omega_\delta^* \subset \Omega_\delta$ of full measure, such that $\tilde{v}$ is differentiable on Ω_δ^*. Hence, for $x\in\Omega_\delta^*$, there exists a vector $\nabla\tilde{v}(x)$ and a function $\varepsilon(h)$, $\varepsilon(h)\to 0$ as $h\to 0$, such that, for every h sufficiently small, we have $\tilde{v}(x+h)-\tilde{v}(x) = \langle\nabla\tilde{v}(x),h\rangle + |h|\varepsilon(h)$. Consider one coordinate direction e_i. On almost every line parallel to e_i, the map $t\mapsto v(x+te_i)$ is absolutely continuous; there exists Ω_δ^i of full measure such that $x\in\Omega_\delta^i$ and t small imply

$$\begin{aligned} \tilde{v}(x+te_i)-\tilde{v}(x) &= \frac{1}{\omega_N\delta^N}\int_{B(0,\delta)} v(x-z+te_i)-v(x-z)\ \mathrm{d}z \\ &= \frac{1}{\omega_N\delta^N}\int_{B(0,\delta)} [\int_0^1 \langle\nabla u(x-z+ste_i),te_i\rangle\ \mathrm{d}s]\ \mathrm{d}z \\ &= \frac{1}{\omega_N\delta^N}[\int_{B(0,\delta)} \langle\nabla v(x-z),te_i\rangle\ \mathrm{d}z \\ &\quad + \int_{B(0,\delta)} \mathrm{d}z \int_0^1 \langle\nabla u(x-z+ste_i)-\nabla v(x-z),te_i\rangle \mathrm{d}s] \\ &= \langle\frac{1}{\omega_N\delta^N}\int_{B(0,\delta)} \nabla v(x-z)\ \mathrm{d}z, te_i\rangle + r_i(t) \end{aligned}$$

and

$$\begin{aligned} r_i(t) = \frac{1}{\omega_N\delta^N}\int_0^1 [&\int_{[B(0,\delta)-ste_i]\setminus B(0,\delta)} \langle\nabla v(x-z),te_i\rangle\ \mathrm{d}z \\ &- \int_{B(0,\delta)\setminus[B(0,\delta)-ste_i]} \langle\nabla v(x-z),te_i\rangle\ \mathrm{d}z]\mathrm{d}s \end{aligned}$$

so that $\frac{r_i(t)}{|t|} \to 0$. Hence, for $x \in \Omega^*_\delta \cap [\cap_i \Omega^i_\delta]$, we have

$$\nabla \tilde{v}(x) = \frac{1}{\omega_N \delta^N} \int_{B(0,\delta)} \nabla v(x-z) \, \mathrm{d}z.$$

□

Lemma 5.5. *Assume that either i) g is measurable and such that $|g(x)| \leq Dd(x)$ or, ii), that g is Lipschitzean with Lipschitz constant D. Then, there exists D^* such that the function*

$$\tilde{g}(x) = \frac{1}{|B(x,d(x))|} \int_{B(x,d(x))} g(z) \, dz$$

is Lipschitzean of constant D^*.

Proof. Fix x^1 and x^2, let $d(x^2) \geq d(x^1)$, let y^1 and y^2 in $\partial\Omega$ be the nearest points to x^1 and x^2. From $|x^2 - y^2| \leq |x^2 - y^1| \leq |x^2 - x^1| + |x^1 - y^1|$, we obtain

$$|x^2 - x^1| \geq d(x^2) - d(x^1). \tag{5.5}$$

On the segment $[y^2, x^2]$, let x^{2*} be such that $d(x^{2*}) = |y^2 - x^{2*}| = d(x^1)$. We have

$$\begin{aligned} |x^1 - x^{2*}| &\leq |x^1 - x^2| + |x^2 - x^{2*}| \\ &= |x^1 - x^2| + (d(x^2) - d(x^1)) \\ &\leq 2|x^1 - x^2|. \end{aligned} \tag{5.6}$$

Ad i). We have

$$\begin{aligned} |\tilde{g}(x^2) - \tilde{g}(x^1)| &= \left| \frac{1}{|B(x^2,d(x^2))|} \int_{B(x^2,d(x^2))} g(z) \, \mathrm{d}z \right. \\ &\qquad \left. - \frac{1}{|B(x^1,d(x^1))|} \int_{B(x^1,d(x^1))} g(z) \, \mathrm{d}z \right| \\ &\leq \frac{1}{|B(x^2,d(x^2))|} \left| \int_{B(x^2,d(x^2))} g(z) \, \mathrm{d}z \right. \\ &\qquad \left. - \int_{B(x^1,d(x^1))} g(z) \, \mathrm{d}z \right| \\ &\quad + \int_{B(x^1,d(x^1))} |g(z)| \, \mathrm{d}z \left| \frac{1}{|B(x^2,d(x^2))|} \right. \\ &\qquad \left. - \frac{1}{|B(x^1,d(x^1))|} \right| \\ &= \alpha + \beta. \end{aligned}$$

Consider α.

$$\begin{aligned}\alpha &\leq \frac{1}{|B(x^2,d(x^2))|}\left\{|\int_{B(x^2,d(x^2))} g(z)\ \mathrm{d}z - \int_{B(x^{2*},d(x^{2*}))} g(z)\ \mathrm{d}z|\right.\\ &\quad \left. +|\int_{B(x^{2*},d(x^{2*}))} g(z)\ \mathrm{d}z - \int_{B(x^1,d(x^1))} g(z)\ \mathrm{d}z|\right\}\\ &= \frac{1}{|B(x^2,d(x^2))|}\{\alpha_1+\alpha_2\}.\end{aligned}$$

Since $B(x^{2*},d(x^{2*}))\subset B(x^2,d(x^2))$, we have

$$\begin{aligned}\alpha_1 &= |\int_{B(x^2,d(x^2))\setminus B(x^{2*},d(x^{2*}))} g(z)\ \mathrm{d}z|\\ &\leq \omega_N[(d(x^2))^N-(d(x^{2*}))^N]\cdot 2Dd(x^2)\\ &\leq 2D\omega_N P_N(d(x^2))^N[d(x^2)-d(x^{2*})]\\ &= 2D\omega_N P_N(d(x^2))^N[(x^2)-d(x^1)].\end{aligned}$$

Also,

$$\begin{aligned}\alpha_2 &\leq \int_{B(x^{2*},d(x^1))\triangle B(x^1,d(x^1))} |g(z)|\ \mathrm{d}z\\ &\leq 2Dd(x^1)|B(x^{2*},d(x^1))\triangle B(x^1,d(x^1))|,\end{aligned}$$

and we have: when $2d(x^1)\leq |x^1-x^{2*}|$, it follows

$$\begin{aligned}|B(x^{2*},d(x^1))\triangle B(x^1,d(x^1))| &= 2\omega_N(d(x^1))^N\\ &\leq \omega_N(d(x^1))^{N-1}|x^1-x^{2*}|;\end{aligned}$$

when $2d(x^1)>|x^1-x^{2*}|$,

$$\begin{aligned}|B(x^{2*},d(x^1))\triangle B(x^1,d(x^1))| &\leq \omega_N[(d(x^1)+|x^1-x^{2*}|)^N-d(x^1))^N]\\ &\leq \omega_N|x^1-x^{2*}|P_N(d(x^1)+|x^1-x^{2*}|)^{N-1}\\ &\leq \omega_N|x^1-x^{2*}|P_N(3d(x^1))^{N-1}.\end{aligned}$$

In either case,

$$|B(x^{2*},d(x^1))\triangle B(x^1,d(x^1))|\leq \omega_N|x^1-x^{2*}|3^{N-1}P_N(d(x^1))^{N-1}.$$

Hence, $\alpha_2\leq 2\cdot 3^{N-1}D\omega_N P_N(d(x^1))^N|x^1-x^{2*}|$, so that

$$\alpha\leq 2DP_N[(d(x^2)-d(x^1))+3^{N-1}|x^1-x^{2*}|].$$

From (5.5) and (5.6) we obtain

$$\alpha \leq 2DP_N(1 + 2 \cdot 3^{N-1})|x^1 - x^2|.$$

Consider β: we have

$$\int_{B(x^1,d(x^1))} |g(z)| \, \mathrm{d}z \leq \omega_N (d(x^1))^N \cdot 2Dd(x^1)$$

and

$$\begin{aligned}
|\frac{1}{|B(x^2,d(x^2))|} - \frac{1}{|B(x^1,d(x^1))|}| &= \frac{|B(x^2,d(x^2))| - |B(x^1,d(x^1))|}{|B(x^1,d(x^1))||B(x^2,d(x^2))|} \\
&= \frac{1}{\omega_N} \frac{(d(x^2))^N - (d(x^1))^N}{(d(x^2))^N (d(x^1))^N} \\
&\leq \frac{P_N}{\omega_N} \frac{(d(x^2) - d(x^1))}{d(x^2)(d(x^1))^N}
\end{aligned}$$

so that

$$\beta \leq 2DP_N(d(x^2) - d(x^1)) \leq 2DP_N|x^2 - x^1|.$$

We have obtained

$$|\tilde{g}(x^2) - \tilde{g}(x^1)| \leq 2DP_N(2 + 2 \cdot 3^N)|x^2 - x^1|.$$

Ad ii).

$$\begin{aligned}
|\tilde{g}(x^2) - \tilde{g}(x^1)| &= \Bigg| g(x^2) - g(x^1) \\
&\quad + \frac{1}{|B(x^2,d(x^2))|} \int_{B(x^2,d(x^2))} (g(z) - g(x^2)) \, \mathrm{d}z \\
&\quad - \frac{1}{|B(x^1,d(x^1))|} \int_{B(x^1,d(x^1))} (g(z) - g(x^1)) \, \mathrm{d}z \Bigg|.
\end{aligned}$$

a) When $|x^2 - x^1| \geq d(x^2) + d(x^1)$

$$\begin{aligned}
&\Bigg| \frac{1}{|B(x^2,d(x^2))|} \int_{B(x^2,d(x^2))} (g(z) - g(x^2)) \, \mathrm{d}z \\
&\qquad - \frac{1}{|B(x^1,d(x^1))|} \int_{B(x^1,d(x^1))} (g(z) - g(x^1)) \, \mathrm{d}z \Bigg| \\
&\leq Dd(x^2) + Dd(x^1) \leq D|x^2 - x^1|.
\end{aligned}$$

b) Let $|x^2 - x^1| \leq d(x^2) + d(x^1)$. We have

$$\begin{aligned}
|\tilde{g}(x^2) - \tilde{g}(x^1)| &\leq |g(x^2) - g(x^1)| \\
&\quad +|\frac{1}{\omega_N (d(x^2))^N} \int_{B(x^2,d(x^2))\setminus B(x^1,d(x^1))} (g(z) - g(x^2))\ \mathrm{d}z| \\
&\quad +|\frac{1}{\omega_N (d(x^1))^N} \int_{B(x^1,d(x^1))\setminus B(x^2,d(x^2))} (g(z) - g(x^1))\ \mathrm{d}z| \\
&\quad +|\int_{B(x^1,d(x^1))\cap B(x^2,d(x^2))} [\frac{1}{\omega_N (d(x^2))^N} (g(z) - g(x^2)) \\
&\qquad -\frac{1}{\omega_N (d(x^1))^N} (g(z) - g(x^1))]\ \mathrm{d}z| \\
&= |g(x^2) - g(x^1)| + \alpha + \beta + \gamma.
\end{aligned}$$

We have

$$|\alpha| \leq Dd(x^2)\frac{1}{(d(x^2))^N}[(d(x^2) + |x^2 - x^1|)^N - (d(x^1))^N];$$

since $d(x^2) - d(x^1) \leq |x^2 - x^1| \leq 2d(x^2)$, we obtain

$$|\alpha| \leq \frac{D}{(d(x^2))^{N-1}} 2|x^2 - x^1| P_N (3d(x^2))^{N-1} = 2P_N 3^{N-1} D|x^2 - x^1|.$$

Consider β; we have

$$|\beta| \leq \frac{D}{\omega_N (d(x^1))^{N-1}} |B(x^1, d(x^1)) \setminus B(x^2, d(x^2))|.$$

Since $B(x^1, d(x^2) - |x^2 - x^1|) \subset B(x^2, d(x^2))$, we infer

$$B(x^1, d(x^1)) \setminus B(x^2, d(x^2)) \subset B(x^1, d(x^1)) \setminus B(x^1, d(x^2) - |x^2 - x^1|),$$

hence

$$\begin{aligned}
|B(x^1, d(x^1)) \setminus B(x^2, d(x^2))| &\leq \omega_N[((d(x^1))^N - (d(x^2) - |x^2 - x^1|)^N] \\
&\leq \omega_N(d(x^1) - d(x^2) \\
&\qquad +|x^2 - x^1|) P_N (2d(x^1))^{N-1} \\
&\leq \omega_N |x^2 - x^1| P_N (2d(x^1))^{N-1}
\end{aligned}$$

and

$$|\beta| \leq DP_N 2^{N-1} |x^2 - x^1|.$$

Consider γ; write the absolute value of the integrand as

$$\begin{aligned}
&|\frac{1}{\omega_N(d(x^2))^N}(g(z)-g(x^2)) - \frac{1}{\omega_N(d(x^2))^N}(g(z)-g(x^1)) \\
&\qquad + \frac{1}{\omega_N(d(x^2))^N}(g(z)-g(x^1)) - \frac{1}{\omega_N(d(x^1))^N}(g(z)-g(x^1))| \\
&\quad = |\frac{1}{\omega_N(d(x^2))^N}(g(x^1)-g(x^2)) \\
&\qquad + (g(z)-g(x^1))(\frac{1}{\omega_N(d(x^2))^N} - \frac{1}{\omega_N(d(x^1))^N})|.
\end{aligned}$$

Since $|B(x^1, d(x^1)) \cap B(x^2, d(x^2))| \leq \omega_N(d(x^1))^N$, we obtain

$$\begin{aligned}
\gamma &\leq \left[\frac{D}{\omega_N(d(x^2))^N}|x^2-x^1| + Dd(x^1)\left(\frac{(d(x^2))^N-(d(x^1))^N}{\omega_N(d(x^1))^N(d(x^2))^N}\right)\right]\omega_N(d(x^1))^N \\
&\leq D|x^2-x^1| + D\frac{d(x^1)}{d(x^2)}P_N(d(x^2)-d(x^1)) \\
&\leq D(1+P_N)|x^2-x^1|.
\end{aligned}$$

$\square$

5.3 Differentiability results

Let $P \in \partial\Omega$; we choose as coordinate system (depending on P) the one that has the origin in P and the x_N axis in the direction of the normal to the inside of Ω, so that, for $i < N$, the x_i axis is on the tangent plane to P. On this system, $\partial\Omega$ is described locally by $x_N = \phi(\hat{x})$, with ϕ a smooth function such that $\phi(\hat{0}) = \nabla\phi(\hat{0}) = 0$; given $\Phi \leq 1$, we shall call $B_\Phi(P)$ the maximal open ball centered at $\hat{0}$ in $\mathbb{R}^{N-1}$ such that, for $\hat{x} \in B_\Phi$, we have $|\nabla\phi(\hat{x})| < \Phi$.

Set

$$\overline{\nu} = \begin{pmatrix} -\phi_{x_1} \\ \vdots \\ -\phi_{x_{N-1}} \\ 1 \end{pmatrix};\ \overline{\tau}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \phi_{x_1} \end{pmatrix};\ \ldots;\ \overline{\tau}_{N-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \phi_{x_{N-1}} \end{pmatrix}$$

and

$$\nu = \frac{\overline{\nu}}{|\overline{\nu}|};\ \tau_i = \frac{\overline{\tau}_i}{|\overline{\tau}_i|}.$$

Given a point $x \in \Omega$, as before we denote by $y(x)$ the point in $\partial\Omega$ where the minimum in (5.1) is attained. We shall consider the map $x \mapsto \hat{y}(x)$; $J(\hat{y})$ is the Jacobian of this map.

Lemma 5.6 (**Differentiability Lemma**). *For every η there exist $\tilde{h}$ and $\tilde{\Phi}$ such that $h \geq \tilde{h}$ and $\Phi \leq \tilde{\Phi}$ imply that the map $x \mapsto \hat{y}$ is well defined and differentiable on $\Omega_{\frac{3M}{h}}$, and we have*

$$\frac{1-\eta}{1+\eta} \leq J(\hat{y}) \leq \frac{\sqrt{1+\eta^2}}{(1-\eta)}.$$

Being the case $N = 2$ substantially simpler than the general case, we present it separately. In the proof of this lemma we shall consider partial derivatives evaluated at different points; it will be convenient to set f'_j to denote the partial derivative of the (scalar-valued) function f with respect to its j-th variable.

Proof. The case $N = 2$. a) We first claim that the map ∇w_+^h is a known function when computed at a generic point $(y_1, \phi(y_1)) \in \partial\Omega$. In fact, from $u^0(y_1, \phi(y_1)) \equiv w_+^h(y_1, \phi(y_1))$ we obtain

$$\frac{d}{dy_1} u^0(y_1, \phi(y_1)) = \langle \nabla u^0, \overline{\tau} \rangle = \frac{d}{dy_1} w_+^h(y_1, \phi(y_1)) = \langle \nabla w_+^h, \overline{\tau} \rangle$$

so that

$$\langle \nabla w_+^h(y_1, \phi(y_1)), \tau \rangle \equiv \langle \nabla u^0(y_1, \phi(y_1)), \tau \rangle;$$

since the norm of ∇w_+^h is h, we also have

$$\langle \nabla w_+^h(y_1, \phi(y_1)), \nu \rangle = \sqrt{h^2 - \langle \nabla u^0(y_1, \phi(y_1)), \tau \rangle^2}.$$

Let e_i be the coordinate directions; writing

$$e_1 = \langle \tau, e_1 \rangle \tau + \langle \nu, e_1 \rangle \nu; \qquad e_2 = \langle \tau, e_2 \rangle \tau + \langle \nu, e_2 \rangle \nu$$

we obtain the Cartesian coordinates of ∇w_+^h, i.e.,

$$\begin{pmatrix} (w_+^h)'_1 \\ (w_+^h)'_2 \end{pmatrix} = \begin{pmatrix} \langle \nabla w_+^h, e_1 \rangle \\ \langle \nabla w_+^h, e_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle \nabla w_+^h, \tau \rangle \langle \tau, e_1 \rangle + \langle \nabla w_+^h, \nu \rangle \langle \nu, e_1 \rangle \\ \langle \nabla w_+^h, \tau \rangle \langle \tau, e_2 \rangle + \langle \nabla w_+^h, \nu \rangle \langle \nu, e_2 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \nabla u^0, \tau \rangle \langle \tau, e_1 \rangle + \sqrt{h^2 - \langle \nabla u^0, \tau \rangle^2} \langle \nu, e_1 \rangle \\ \langle \nabla u^0, \tau \rangle \langle \tau, e_2 \rangle + \sqrt{h^2 - \langle \nabla u^0, \tau \rangle^2} \langle \nu, e_2 \rangle \end{pmatrix}.$$

In particular,

$$(w_+^h)'_1(y_1, \phi(y_1)) \equiv [\langle \nabla u^0, \tau \rangle \langle \tau, e_1 \rangle + \sqrt{h^2 - \langle \nabla u^0, \tau \rangle^2} \langle \nu, e_1 \rangle]|_{(y_1, \phi(y_1))}. \tag{5.7}$$

b) Consider h^* and d^* defined in Lemma 5.3. We can assume that $h^* \geq \frac{3M}{d^*}$. For every $h \geq h^*$ and $d(x) \leq d^*$, the map (depending on h) $x \mapsto y(x) = (y_1, \phi(y_1))$ is well defined. We claim that y_1 is a differentiable function of x.

Recalling that ∇w_+^h is constant along the line segment joining (x_1, x_2) and $(y_1, \phi(y_1))$, we obtain the identity

$$\nabla w_+^h(x_1, x_2) = \begin{pmatrix} \langle \nabla u^0, \tau\rangle\langle \tau, e_1\rangle + \sqrt{h^2 - \langle \nabla u^0, \tau\rangle^2}\langle \nu, e_1\rangle \\ \langle \nabla u^0, \tau\rangle\langle \tau, e_2\rangle + \sqrt{h^2 - \langle \nabla u^0, \tau\rangle^2}\langle \nu, e_2\rangle \end{pmatrix}, \tag{5.8}$$

where the right hand side is computed at the point $(y_1(x), \phi(y_1(x)))$.

The points x and y are related by the identity $x = y + |x-y|\frac{x-y}{|x-y|}$, i.e., from (5.4), by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \phi(y_1(x)) \end{pmatrix} + \frac{(w_+^h(x) - u^0(y_1(x), \phi(y_1)))}{h}\frac{\nabla w_+^h(y_1(x), \phi(y_1))}{h}.$$

In particular,

$$x_1 \equiv y_1 + \frac{1}{h^2}(w_+^h(x_1, x_2) - u^0(y_1, \phi(y_1))(w_+^h)_1'(y_1, \phi(y_1));$$

differentiating with respect to x_1 this identity, we have

$$1 \equiv (y_1)_{x_1} + \frac{1}{h^2}\Big\{[(w_+^h)_{x_1} - \big(\langle \nabla u^0, \overline{\tau}\rangle \cdot (y_1)_{x_1}\big)](w_+^h)_1'(y_1, \phi(y_1)) \\ + (w_+^h - u^0(y_1, \phi(y_1)))\langle \nabla((w_+^h)_1'), \overline{\tau}\rangle \cdot (y_1)_{x_1}\Big\}$$

and

$$0 \equiv (y_1)_{x_2} + \frac{1}{h^2}\Big\{[(w_+^h)_{x_2} - \big(\langle \nabla u^0, \overline{\tau}\rangle \cdot (y_1)_{x_2}\big)](w_+^h)_{x_1}(y_1, \phi(y_1)) \\ + (w_+^h - u^0(y_1, \phi(y_1)))\langle \nabla((w_+^h)_1'), \overline{\tau}\rangle \cdot (y_1)_{x_2}\Big\}.$$

From (5.8), we have $(w_+^h)_i'(y_1, \phi(y_1)) = (w_+^h)_i'(x_1, x_2)$ and we obtain

$$(y_1)_{x_1} = \frac{1 - \frac{1}{h^2}((w_+^h)_1')^2}{1 - \frac{1}{h^2}[\langle \nabla u^0, \overline{\tau}\rangle(w_+^h)_1' - (w_+^h - u^0(y_1, \phi(y_1)))\langle \nabla((w_+^h)_1'), \overline{\tau}\rangle]}$$

and

$$(y_1)_{x_2} = \frac{-\frac{1}{h^2}(w_+^h)_1'(w_+^h)_2'}{1 - \frac{1}{h^2}[\langle \nabla u^0, \overline{\tau}\rangle(w_+^h)_1' - (w_+^h - u^0(y_1, \phi(y_1)))\langle \nabla((w_+^h)_1'), \overline{\tau}\rangle]}.$$

c) We wish to estimate the Jacobian of the map $x \mapsto y_1(x)$. Differentiating (5.7),

$$\begin{aligned}\frac{d}{dy_1}(w^h_+)'_1(y_1,\phi(y_1)) &= \langle\nabla((w^h_+)'_1),\overline{\tau}\rangle\\ &= (\langle\nabla u^0,\tau\rangle)_{y_1}\langle\tau,e_1\rangle + \langle\nabla u^0,\tau\rangle(\langle\tau,e_1\rangle)_{y_1}\\ &\quad -\frac{\langle\nabla u^0,\tau\rangle(\langle\nabla u^0,\tau\rangle)_{y_1}}{\sqrt{h^2-\langle\nabla u^0,\tau\rangle^2}}\langle\nu,e_1\rangle\\ &\quad -\sqrt{h^2-\langle\nabla u^0,\tau\rangle^2}(\langle\nu,e_1\rangle)_{y_1}\\ &= A+B+C+D;\end{aligned}$$

also

$$(\frac{d}{dy_1}\langle\nabla u^0,\tau\rangle)\frac{1}{\sqrt{1+(\phi')^2}} = \tau^T H_{u^0}\tau + \frac{\phi''}{(1+(\phi')^2)^{\frac{3}{2}}}\langle\nabla u^0,\nu\rangle$$

and

$$(\langle\tau,e_1\rangle)_{y_1} = -\frac{\phi'\phi''}{(1+(\phi')^2)^{\frac{3}{2}}};\quad (\langle\nu,e_1\rangle)_{y_1} = -\frac{\phi''}{(1+(\phi')^2)^{\frac{3}{2}}}.$$

We have $|(w^h_+)'_1|\le h$ and $|\nabla u^0|\le K$; $|\langle\nabla u^0,\overline{\tau}\rangle|\le K\sqrt{1+(\phi'(y_1))^2}$. Recalling that $\Phi<1$ and $h>3K$,

$$|A|\le 2K+K^2;\quad |B|\le K^2;\quad |C|\le\frac{2K+K^2}{h}K\le K^2;\quad |D|\le hK,$$

so that

$$\left|\frac{d}{dy_1}(w^h_+)'_1(y_1,\phi(y_1))\right|\le K_1+Kh.$$

Recalling i) of Lemma 5.3, on the set $\Omega_{\frac{3M}{h}}$ we have $w^h_+(x)-u^0(y(x)) = h|x-y(x)|\le h\cdot 2\frac{3M}{h}=6M$, so that

$$\left|(w^h_+(x)-u^0(y_1,\phi(y_1)))\frac{1}{h^2}\left(\frac{d}{dy_1}(w^h_+)'_1\right)\right|\le 6M\frac{1}{h^2}(K_1+Kh);$$

in addition,

$$\left|\langle\nabla u^0,\overline{\tau}\rangle\frac{1}{h^2}(w^h_+)'_1\right|\le 2K\frac{1}{h^2}h;$$

we have obtained that the denominator satisfies

$$\begin{aligned}1+\frac{2}{h^2}&\ [3M(K_1+2Kh)]\\ &\ge\ 1-\frac{1}{h^2}[\langle\nabla u^0,\overline{\tau}\rangle(w^h_+)'_1-(w^h_+-u^0(y_1,\phi(y_1)))\langle\nabla((w^h_+)'_1),\overline{\tau}\rangle]\\ &\ge\ 1-\frac{2}{h^2}[3M(K_1+2Kh)].\end{aligned}$$

In addition, from (5.8), we have $\frac{1}{h}|w^h_{x_1}| \le \frac{K}{h}+\Phi$ and $|\frac{1}{h^2}w^h_{x_1}w^h_{x_2}| \le \frac{K}{h}+\Phi$ so that we can make either term arbitrarily small by choosing $\frac{1}{h}$ and Φ small.

d) Fix η. Fix $\tilde{h}$ so large and $\tilde{\Phi}$ so small that $h \ge \tilde{h}$ and $\Phi \le \tilde{\Phi}$ imply:

$$1-\eta \le 1-\frac{1}{h^2}[\langle \nabla u^0, \overline{\tau}\rangle (w^h_+)'_1 - (w^h_+ - u^0(y_1,\phi(y_1)))\langle \nabla((w^h_+)'_1), \overline{\tau}\rangle] \le 1+\eta;$$

$$\frac{1}{h^2}((w^h_+)'_1)^2 \le \eta \quad\text{and}\quad \left|\frac{1}{h^2}(w^h_+)'_1(w^h_+)'_2\right| \le \eta.$$

We obtain, for every $x \in \Omega_{\frac{3M}{h}}$,

$$\frac{1-\eta}{1+\eta} \le (y_1)_{x_1} \le \frac{1}{1-\eta}; \quad 0 \le |(y_1)_{x_2}| \le \eta$$

and

$$\frac{1-\eta}{1+\eta} \le J((y_1)(x)) = \sqrt{(y_1)^2_{x_1}+(y_1)^2_{x_2}} \le \frac{\sqrt{1+\eta^2}}{(1-\eta)}. \tag{5.9}$$

□

Proof. The general case. a) Consider a generic point $(\hat{y},\phi(\hat{y})) \in \partial\Omega$, so that $\tau_i = \tau_i(\hat{y})$ and $\nu = \nu(\hat{y})$: we claim that the map ∇w^h_+ is known when computed at $(\hat{y},\phi(\hat{y}))$. In fact, from $u^0(\hat{y},\phi(\hat{y})) \equiv w^h_+(\hat{y},\phi(\hat{y}))$, we obtain

$$\frac{d}{dy_i}u^0(\hat{y},\phi(\hat{y})) = \langle \nabla u^0, \overline{\tau}_i\rangle = \frac{d}{dy_i}w^h_+(\hat{y},\phi(\hat{y})) = \langle \nabla w^h_+, \overline{\tau}_i\rangle,$$

so that

$$\langle \nabla w^h_+(\hat{y},\phi(\hat{y})), \tau_i\rangle = \langle \nabla u^0(\hat{y},\phi(\hat{y})), \tau_i\rangle. \tag{5.10}$$

For a vector v in $\mathbb{R}^3$, let $P(v)$ be the projection of v on the tangent plane; write $v = \langle v,\nu\rangle\nu + \sum a_i\tau_i$, so that $\sum a_i\tau_i = P(v)$; we obtain, for the coefficients a_i, the system

$$\langle v, \tau_j\rangle = \sum_i a_i\langle \tau_i, \tau_j\rangle. \tag{5.11}$$

In particular, for the vector ∇w^h_+, we obtain

$$\nabla w^h_+ = \langle \nabla w^h_+, \nu\rangle\nu + \sum_{i=1}^{N-1} a_i\tau_i,$$

and so, from (5.10), (5.11) becomes

$$\langle \nabla u^0, \tau_j\rangle = \sum a_i\langle \tau_i, \tau_j\rangle. \tag{5.12}$$

The coefficient matrix $T = (\langle \tau_i, \tau_j \rangle)$ of system (5.12) converges to $(\delta_{i,j})$ as $\Phi \to 0$; hence, for every Φ small, system (5.12) is solvable.

We also have

$$h^2 = |\nabla w_+^h|^2 = \langle \nabla w_+^h, \nu \rangle^2 + (P(\nabla w_+^h))^2,$$

and we obtain

$$\langle \nabla w_+^h, \nu \rangle = \sqrt{h^2 - \left(\sum_{i=1}^{N-1} a_i^2 + \sum_{i \neq l} a_i a_l \langle \tau_i, \tau_l \rangle \right)}. \tag{5.13}$$

b) Equations (5.10) and (5.13) provide $\langle \nabla w_+^h, \tau_i \rangle$ and $\langle \nabla w_+^h, \nu \rangle$; in order to obtain the Cartesian coordinates of ∇w_+^h, write, for $j = 1, \ldots, N$,

$$e_j = \langle e_j, \nu \rangle \nu + \sum_{i=1}^{N-1} b_i^j \tau_i. \tag{5.14}$$

We have

$$(w_+^h)'_j = \langle \nabla w_+^h, e_j \rangle = \langle e_j, \nu \rangle \langle \nabla w_+^h, \nu \rangle + \sum_{i=1}^{N-1} b_i^j \langle \nabla w_+^h, \tau_i \rangle,$$

hence

$$\langle \nabla w_+^h, e_j \rangle (\hat{y}, \phi(\hat{y})) \equiv \Bigg[\sum_{i=1}^{N-1} b_i^j \langle \nabla u^0, \tau_i \rangle \tag{5.15}$$

$$+ \langle e_j, \nu \rangle \sqrt{h^2 - \left(\sum_{i=1}^{N-1} a_i^2 + \sum_{i \neq l} a_i a_l \langle \tau_i, \tau_l \rangle \right)} \Bigg] \Bigg|_{(\hat{y}, \phi(\hat{y}))}.$$

c) We have the identity

$$\begin{pmatrix} \hat{x} \\ x_N \end{pmatrix} = \begin{pmatrix} \hat{y}(x) \\ \phi(\hat{y}(x)) \end{pmatrix} + \frac{(w_+^h(x) - u^0(\hat{y}(x), \phi(\hat{y}(x))))}{h} \frac{\nabla w_+^h(\hat{y}(x), \phi(\hat{y}(x))}{h}.$$

Differentiate with respect to x_j the first $N-1$ lines and recall that $(w_+^h)'_j(x) = (w_+^h)'_j(\hat{y}, \phi(\hat{y}))$, to have

$$\delta_{i,j} = y^i_{x_j} + \frac{1}{h^2} \Bigg[\left((w_+^h)_{x_j} - \sum_l \langle \nabla u^0, \overline{\tau}_l \rangle (y^l)_{x_j} \right) (w_+^h)_{x_i}$$

$$+ (w_+^h - u^0) \sum_l \langle \nabla ((w_+^h)'_i), \overline{\tau}_l \rangle y^l_{x_j} \Bigg],$$

where $\langle \nabla((w_+^h)_i'), \overline{\tau}_l \rangle$, u^0 and $\langle \nabla u^0, \overline{\tau}_l \rangle$ are computed at the point $(\hat{y}, \phi(\hat{y}))$. Hence, for $i = 1, \ldots, N-1$ and $j = 1, \ldots, N$,

$$\begin{aligned} \delta_{i,j} - \frac{1}{h^2}(w_+^h)_i'(w_+^h)_j' &= y^i_{x_j} + \frac{1}{h^2}\Big\{ \sum_l [w_+^h \langle \nabla((w_+^h)_i'), \overline{\tau}_l \rangle. \\ &\quad - (w_+^h)_{x_i} \langle \nabla u^0, \overline{\tau}_l \rangle - u^0 \langle \nabla((w_+^h)_i'), \overline{\tau}_l \rangle] y^l_{x_j} \Big\}. \end{aligned} \tag{5.16}$$

System (5.16) has the form

$$\begin{pmatrix} 1+\sigma_{1,1} & \sigma_{1,2} & \ldots & \sigma_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N-1,1} & 1+\sigma_{N-1,2} & \ldots & \sigma_{N-1,N} \end{pmatrix} \tag{5.17}$$

$$= \begin{pmatrix} (1+\eta_{1,1}) & \ldots & \eta_{1,N-1} \\ \vdots & \ddots & \vdots \\ \eta_{N-1,1} & \ldots & (1+\eta_{N-1,N-1}) \end{pmatrix} \begin{pmatrix} y^1_{x_1} & y^1_{x_2} & \ldots & y^1_{x_N} \\ \vdots & \vdots & \ddots & \vdots \\ y^{N-1}_{x_1} & y^{N-1}_{x_2} & \ldots & y^{N-1}_{x_N} \end{pmatrix},$$

with

$$\eta_{i,l} = \frac{1}{h^2}[w_+^h \langle \nabla((w_+^h)_i'), \overline{\tau}_l \rangle - (w_+^h)_{x_i} \langle \nabla u^0, \overline{\tau}_l \rangle - u^0 \langle \nabla((w_+^h)_i'), \overline{\tau}_l \rangle].$$

We claim that system (5.17) is solvable in the unknowns $y^i_{x_j}$; for this it is enough to show that the $\eta_{i,l}$ can be made arbitrarily small.

d) The expression for $\eta_{i,l}$ contains second derivatives of the function w_+^h, computed at $(\hat{y}(x), \phi(\hat{y}(x)))$, that can be obtained differentiating (5.15); in turn, this requires the existence of the derivatives of a_i and of the b^i_j. We have the derivatives of a_i by differentiating the identity, obtained from (5.12),

$$\langle \nabla u^0(\hat{y}, \phi(\hat{y})), \overline{\tau}_j(\hat{y}) \rangle \equiv \sum a_i(\hat{y}) \langle \tau_i(\hat{y}), \overline{\tau}_j(\hat{y}) \rangle;$$

we have

$$\frac{\partial}{\partial y_l} \langle \nabla u^0, \overline{\tau}_j \rangle = (\overline{\tau}_j)^T H_{u^0} \overline{\tau}_l + u^0_{y_N} \phi_{y_j y_l} = \sum [(a_i)_{y_l} \langle \tau_i, \overline{\tau}_j \rangle + a_i \frac{\partial}{\partial y_l} \langle \tau_i, \overline{\tau}_j \rangle],$$

i.e.,

$$(\overline{\tau}_j)^T H_{u^0} \overline{\tau}_l + u^0_{y_N} \phi_{y_j y_l} - \sum_i a_i \frac{\partial}{\partial y_l} \langle \tau_i, \overline{\tau}_j \rangle = \sum_i (a_i)_{y_l} \langle \tau_i, \overline{\tau}_j \rangle. \tag{5.18}$$

Again, for all Φ sufficiently small, system (5.18) is solvable and $(a_i)_{y_j}$ exist.

Consider (5.14) and take scalar products with ∇u^0; since the left hand side is differentiable, so is the right hand side and we obtain

$$(\langle e_j, \nabla u^0\rangle)_{x_l} - [\langle \nu, e_j\rangle\langle \nu, \nabla u^0\rangle]_{x_l} = \sum_{r=1}^{N-1}(b_r^j\langle \tau_r, \nabla u^0\rangle)_{x_l}. \tag{5.19}$$

Finally, consider (5.15); since we have shown that the right hand side is differentiable, so is the left hand side and we obtain

$$\begin{aligned}\frac{\partial}{\partial y_l}\langle \nabla w_+^h, e_j\rangle(\hat{y}, \phi(\hat{y})) = (\frac{\partial}{\partial y_l}\langle e_j, \nu\rangle)&\sqrt{h^2 - \left(\sum_i a_i^2 + \sum_{i\neq j} a_i a_j\langle \tau_i, \tau_j\rangle\right)}\\ &+\langle e_j, \nu\rangle\frac{\partial}{\partial y_l}\sqrt{h^2 - \left(\sum_i a_i^2 + \sum_{i\neq j} a_i a_j\langle \tau_i, \tau_j\rangle\right)}\\ &+\sum_{i=1}^{N-1}(b_i^j\langle \nabla u^0, \tau_i\rangle)_{y_l}.\end{aligned} \tag{5.20}$$

e) Consider the following estimates as $\Phi \to 0$. We have that, as $\Phi \to 0$, for $j = 1, \ldots, N-1$, $\tau_j \to e_j$, while $\nu \to e_N$; from (5.12) we obtain

$$a_j \to \langle \nabla u^0(\hat{0}, 0), e_j\rangle = u^0_{y_j}(\hat{0}, 0),$$

so that

$$\langle \nabla w_+^h, \nu\rangle \longrightarrow \sqrt{h^2 - \sum_i(\langle \nabla u^0, e_i\rangle)^2}$$

and

$$\sqrt{h^2 - (\sum_i a_i^2 + \sum_{i\neq j} a_i a_j\langle \tau_i, \tau_j\rangle)} \longrightarrow \sqrt{h^2 - \sum_{i=1}^{N-1}(\langle \nabla u^0, e_i\rangle)^2}.$$

We also have

$$(\langle e_j, \nu\rangle) \to \begin{cases} 0 & \text{when } j \neq N \\ 1 & \text{when } j = N\end{cases}$$

and, from (5.14), we obtain $b_j^i \to \delta_{ij}$. Moreover,

$$(\langle e_j, \nu\rangle)_{y_l} \to \begin{cases} -\phi_{y_j y_l} & \text{when } j \neq N \\ 0 & \text{when } j = N,\end{cases}$$

$$\langle \nu, \nabla u^0\rangle \to u^0_{y_N}, \quad \text{and} \quad (\langle \nu, \nabla u^0\rangle)_{y_l} \to -\sum_{i=1}^{N-1}\phi_{y_i y_l}u^0_{y_i} + u^0_{y_N y_l}.$$

From (5.15) we infer

$$(w^h_+)_{x_j} = \langle \nabla w^h_+, e_j\rangle \to \begin{cases} \langle \nabla u^0, e_j\rangle & \text{for } j \neq N \\ \sqrt{h^2 - \sum_i \langle \nabla u^0, e_i\rangle^2} & \text{for } j = N. \end{cases} \tag{5.21}$$

From $\frac{\partial}{\partial y_l}\frac{\phi_{y_i}}{\sqrt{1+\phi^2_{y_i}}} \to \phi_{y_i y_l}$ we infer that $\frac{\partial}{\partial y_l}\langle \tau_i, \tau_j\rangle \to 0$; hence, solving system (5.18), we obtain

$$(a_j)_{y_l} \to (H_{u^0})_{j,l} + u^0_{y_N}\phi_{y_j y_l}$$

that implies that there exists H_1 such that, for all sufficiently small Φ and all h, $|(a_j)_{y_l}| \leq H_1$. Hence, there exists H_2 such that

$$\frac{\partial}{\partial y_l}\sqrt{h^2 - \left(\sum_i a_i^2 + \sum_{i\neq j} a_i a_j \langle \tau_i, \tau_j\rangle\right)}$$

$$= \frac{\sum_i 2a_i(a_i)_{y_l} + \sum_{i\neq j}[(a_i a_j)_{y_l}\langle \tau_i, \tau_j\rangle + a_i a_j(\langle \tau_i, \tau_j\rangle)_{y_l}]}{2\sqrt{h^2 - (\sum_i a_i^2 + \sum_{i\neq j} a_i a_j\langle \tau_i, \tau_j\rangle)}} \leq H_2.$$

From (5.19) we obtain

$$\sum_{r=1}^{N-1}(b^j_r\langle \tau_r, \nabla u^0\rangle)_{y_l} \to \begin{cases} u^0_{y_j y_l} + \phi_{y_j y_l} u^0_{y_N} & j \neq N \\ \sum_i \phi_{y_i y_l} u^0_{y_i} & j = N \end{cases}$$

that yields the existence of H_3 such that, for all Φ sufficiently small,

$$|\sum_{r=1}^{N-1}(b^j_r\langle \tau_r, \nabla u^0\rangle)_{y_l}| \leq H_3.$$

Then, from (5.20),

$$\left|\frac{\partial}{\partial y_l}\langle \nabla w^h_+, e_j\rangle(\hat{y}, \phi(\hat{y}))\right| \leq 2Kh + H_2 + H_3.$$

Since $|(w^h_+)_{x_j}| \leq h$, on the set $\Omega_{\frac{3M}{h}}$ we obtain

$$|\eta_{i,l}| \leq \frac{1}{h^2}[6M(2Kh + H_2 + H_3) + 2Kh].$$

f) Consider system (5.16) and notice that $i < N$: from (5.21) we obtain that each $\sigma_{i,j}$ can be made arbitrarily small by choosing $\frac{1}{h}$ and Φ small. From (5.16) we obtain that, as both Φ and $\frac{1}{h} \to 0$, $y^i_{x_j} \to \delta_{ij}$, with $i = 1, \ldots, N-1$ and $j = 1, \ldots, N$. The determinant of the minor of the matrix $(y^i_{x_j})$ obtained by suppressing the last column, $(y^i_{x_N})$, converges to 1, while the determinants of all the other square matrices, that must contain the last column, tend to 0. Hence, by the formula for the Jacobian ([26], p. 89), given η, we can find $\tilde{h} \geq h^*$ and $\tilde{\Phi}$ such that $h \geq \tilde{h}$ and $\Phi \leq \tilde{\Phi}$ imply that, for $x \in \Omega_{\frac{3M}{h}}$,

$$\frac{1-\eta}{1+\eta} \leq J(\hat{y}(x)) \leq \frac{\sqrt{1+\eta^2}}{(1-\eta)}.$$

□

5.4 Proof of Theorem 5.1

The Proof of Theorem 5.1 is partially based on the following fact: the problem of minimizing

$$\int_a^b L(|u'(t)|)\ \mathrm{d}t$$

on the set of $u : [a, b] \to \mathbb{R}^N$ absolutely continuous and satisfying $u(a) = \alpha$; $u(b) = \beta$, where L is a convex function defined on $\mathbb{R}$, admits the solution

$$\tilde{u}(t) = \alpha + \frac{\beta - \alpha}{b - a}(t - a).$$

We shall need the following Definition. In it, and for the remainder of this section, for $\xi \in B_\Phi(P)$, we set $y_\xi = \begin{pmatrix} \xi \\ \phi(\xi) \end{pmatrix}$.

Definition 5.7. For given h, Φ, δ, and for $P \in \partial\Omega$, set,

$$V^+_{h,\Phi,\delta}(P) = \Bigg\{ x \in \Omega : x = y_\xi + \ell \frac{\nabla w^h_+(y_\xi)}{h}; \\ \xi \in B_\Phi(P); \ell \in (0, \ell^*); d(y_\xi + \ell^* \frac{\nabla w^h_+(y_\xi)}{h}) = \delta \Bigg\}.$$

For a measurable subset Z of the ball $B_\Phi(P)$, set V^+_Z to be the subset of $V^+_{h,\Phi,\delta}(P)$ such that $\xi \in Z$.

Set

$$V^-_{h,\Phi,\delta}(P) = \Bigg\{ x \in \Omega : x = y_\xi - \ell \frac{\nabla w^h_-(y_\xi)}{h}; \\ \xi \in B_\Phi(P); \ell \in (0, \ell^*); d(y_\xi - \ell^* \frac{\nabla w^h_-(y_\xi)}{h}) = \delta \Bigg\}.$$

For a measurable subset Z of the ball $B_\Phi(P)$, set V^-_Z to be the subset of $V^-_{h,\Phi,\delta}(P)$ such that $\xi \in Z$.

Proof of Theorem 5.1. Fix ε. Set $\varepsilon^1 = \frac{\varepsilon}{4\int_\Omega L(|\nabla u(x)|)\,\mathrm{d}x}$ and let η $(0<\eta<1)$ be such that

$$\frac{(1+\eta)\sqrt{1+\eta^2}}{(1-\eta)^2} = (1+\varepsilon^1);$$

consider $\tilde h$, and $\tilde\Phi$ supplied by the Differentiability Lemma for this η; set $\tilde\delta = \frac{3M}{\tilde h}$; recall the function $M^{\tilde h}$ in Definition 5.2.

a) Set $\Omega^+ = \{x : u(x) > w_+^{\tilde h}(x)\}$, $\Omega^- = \{x : u(x) < w_-^{\tilde h}(x)\}$ and $\Omega^0 = \{x : w_-^{\tilde h}(x) \le u(x) \le w_+^{\tilde h}(x)\}$. Notice that $d(x)\ge\tilde\delta$ implies that $w_+^{\tilde h}(x) = u^0(y(x)) + w_+^{\tilde h}(x) - u^0(y(x)) \ge -M + \tilde h|y(x)-x| \ge 2M > M \ge u(x)$, so that $\Omega^+ \subset \Omega_{\tilde\delta}$. In the same way one obtains also $\Omega^- \subset \Omega_{\tilde\delta}$. Hence, the estimates on the Jacobian of the map $x\mapsto \hat y$, provided by the Differentiability Lemma, hold on Ω^+ and on Ω^-.

We have, almost everywhere in Ω,

$$|\nabla M^{\tilde h}| = \begin{cases} \tilde h & \text{for } x\in\Omega^-\cup\Omega^+ \\ |\nabla u| & \text{for } x\in\Omega^0, \end{cases}$$

so that

$$\int_\Omega L(|\nabla M^{\tilde h}(x)|)\,\mathrm{d}x = \int_{\Omega^-} L(\tilde h)\,\mathrm{d}x + \int_{\Omega^+} L(\tilde h)\,\mathrm{d}x + \int_{\Omega^0} L(|\nabla u|)\,\mathrm{d}x$$

b) We wish to show that

$$\int_\Omega L(|\nabla M^{\tilde h}(x)|)\,\mathrm{d}x \le \int_\Omega L(|\nabla u(x)|)\,\mathrm{d}x + \frac{\varepsilon}{2}; \tag{5.22}$$

it is enough to show that

$$\int_{\Omega^+} L(|\tilde h|)\,\mathrm{d}x = \int_{\Omega^+} L(|\nabla M^{\tilde h}(x)|)\,\mathrm{d}x \le \int_{\Omega^+} L(|\nabla u(x)|)\,\mathrm{d}x + \frac{\varepsilon}{4} \tag{5.23}$$

and

$$\int_{\Omega^-} L(|\tilde h|)\,\mathrm{d}x = \int_{\Omega^-} L(|\nabla M^{\tilde h}(x)|)\,\mathrm{d}x \le \int_{\Omega^-} L(|\nabla u(x)|)\,\mathrm{d}x + \frac{\varepsilon}{4}. \tag{5.24}$$

c) We hall prove (5.23), being (5.24) proved in the same way. Consider $\Delta = \{x\in\Omega : d(x) = \frac{\tilde\delta}{2}\}$: Δ is a compact subset of Ω. By ii) of Lemma 5.3, the collection of open sets, defined in Definition 5.7, $\{V^+_{\tilde h,\tilde\Phi,\tilde\delta}(P) : P\in\partial\Omega\}$ is a covering of Δ. Let $\{V^+_{\tilde h,\tilde\Phi,\tilde\delta}(P_j) : 1\le j\le J\}$ be a finite subcover. We are going to define measurable subsets Z_j of $B_{\tilde\Phi}(P_j)$: set $Z = Z_1 = B_{\tilde\Phi}(P_1)$; consider P_2 and set

$$Z_2 = \{\xi\in B_{\tilde\Phi}(P_2) : (y_\xi + \frac{\tilde\delta}{2}\frac{\nabla w_+^{\tilde h}(y_\xi)}{\tilde h}) \cap V^+_{Z_1} = \emptyset\}.$$

Having defined Z_j up to $\tilde{j}$, set

$$Z_{\tilde{j}+1} = \{\xi \in B_{\tilde{\Phi}}(P_{\tilde{j}+1}) : (y_\xi + \frac{\tilde{\delta}}{2}\frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}}) \cap V_{Z_j}^+ = \emptyset \text{ for } 1 \le j \le \tilde{j}\}.$$

Hence, every point in Δ belongs to one and only one $V_{Z_j}^+$ and, by the uniqueness in Lemma 5.3, so is for $\Omega_{\tilde{\delta}}$.

d) We claim that for every j,

$$\int_{\Omega^+ \cap V_{Z_j}^+} L(|\nabla M^{\tilde{h}}(x)|)\ \mathrm{d}x \le (1+\varepsilon) \int_{\Omega^+ \cap V_{Z_j}^+} L(|\nabla u(x)|)\ \mathrm{d}x.$$

Apply the Coarea Theorem, [26], to the set $\Omega^+ \cap V_{Z_j}^+$ and to the function $\hat{y}(x)$ to obtain

$$\int_{\Omega^+ \cap V_{Z_j}^+} L(|\nabla u(x)|)\ \mathrm{d}x = \int_{Z_j^+} [\int_{\{\hat{y}(x)=\xi\} \cap (\Omega^+ \cap V_{Z_j}^+)} \frac{L(|\nabla u(x)|)}{J(\hat{y}(x))}\ \mathrm{d}H^1]\ \mathrm{d}\xi; \tag{5.25}$$

consider the line segment

$$L_\xi = \{y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}} : \ell \in (0, \ell^*);\ d(y_\xi + \ell^* \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}}) = \tilde{\delta}\} :$$

we have that $\{\hat{y}(x) = \xi\} \cap (\Omega^+ \cap V_{Z_j}) = L_\xi \cap \Omega^+$. For almost every $\xi \in Z_j$ the maps

$$\tilde{u}_\xi(\ell) = u(y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}}),$$
$$\tilde{w}_+^{\tilde{h}}(\ell) = w_+^{\tilde{h}}(y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}})$$

are absolutely continuous, so that the set $S_\xi = \{\ell : \tilde{u}_\xi(\ell) > \tilde{w}_+^{\tilde{h}}(\ell)\}$ is a (possibly empty) open set. Then, there are at most countably many open intervals (a_j, b_j) such that $S_\xi = \cup(a_j, b_j)$ and $\tilde{u}_\xi(a_j) - \tilde{w}_+^{\tilde{h}}(a_j) = \tilde{u}_\xi(b_j) - \tilde{w}_+^{\tilde{h}}(b_j) = 0$ while, for $\ell \in (a_j, b_j)$, $\tilde{u}_\xi(\ell) > \tilde{w}_+^{\tilde{h}}(\ell)$. Fix one such (a_j, b_j). The problem of minimizing

$$\int_{a_j}^{b_j} L(|v'(\ell)|)\ \mathrm{d}\ell; \quad v(a_j) = \tilde{u}_\xi(a_j);\ v(b_j) = \tilde{u}_\xi(b_j)$$

admits the solution $\tilde{w}_+^{\tilde{h}}$, so that, in particular,

$$\begin{aligned} \int_{a_j}^{b_j} L(\tilde{h})\ \mathrm{d}\ell &\le \int_{a_j}^{b_j} L(|\tilde{u}'_\xi(\ell)|)\ \mathrm{d}\ell \\ &= \int_{a_j}^{b_j} L(|\langle \nabla u(y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}}), \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}} \rangle|)\ \mathrm{d}\ell. \end{aligned}$$

Recall that $|\frac{\nabla w^{\tilde{h}}(y_\xi)}{\tilde{h}}| = 1$; since L is non-decreasing, we obtain that

$$L(|\langle \nabla u(y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}}), \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}} \rangle|) \leq L(|\nabla u(y_\xi + \ell \frac{\nabla w_+^{\tilde{h}}(y_\xi)}{\tilde{h}})|),$$

hence that

$$\int_{a_j}^{b_j} L(\tilde{h})\ \mathrm{d}\ell \leq \int_{a_j}^{b_j} L(|\nabla u(y_\xi + \ell \frac{\nabla w^{\tilde{h}}(y_\xi)}{\tilde{h}}|)\ \mathrm{d}\ell \tag{5.26}$$

Since the restriction to $L_\xi \cap \Omega^+$ of the gradient of $M^{\tilde{h}}$ is $\nabla w^{\tilde{h}}(y_\xi)$ when ℓ belongs to the intervals (a_j, b_j), inequality (5.26) implies

$$\int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla M^{\tilde{h}}|)\ \mathrm{d}H^1 \leq \int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla u|)\ \mathrm{d}H^1. \tag{5.27}$$

By (5.25), (5.9) and (5.27),

$$\begin{aligned}
\int_{V_{Z_j}\cap\Omega^+} L(|\nabla u(x)|)\ \mathrm{d}x &= \int_{Z_j} [\int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} \frac{L(|\nabla u(x)|)}{J(\hat{y}(x))}\ \mathrm{d}H^1]\ \mathrm{d}\xi \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_{Z_j} [\int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla u(x)|)\ \mathrm{d}H^1]\ \mathrm{d}\xi \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_{Z_j} [\int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla M^{\tilde{h}}(x)|)\ \mathrm{d}H^1]\ \mathrm{d}\xi \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_{Z_j} [\int_{\{\hat{y}(x)=\xi\}\cap(V_{Z_j}\cap\Omega^+)} \frac{1-\eta}{1+\eta} \frac{L(|\nabla M^{\tilde{h}}(x)|)}{J(\hat{y}(x))}\ \mathrm{d}H^1]\ \mathrm{d}\xi \\
&= \frac{(1-\eta)^2}{(1+\eta)\sqrt{1+\eta^2}} \int_{V_{Z_j}\cap\Omega^+} L(|\nabla M^{\tilde{h}}(x)|)\ \mathrm{d}x.
\end{aligned}$$

We have obtained

$$\int_{V_{Z_j}\cap\Omega^+} L(|\nabla M^{\tilde{h}}(x)|)\ \mathrm{d}x \leq (1+\varepsilon^1) \int_{V_{Z_j}\cap\Omega^+} L(|\nabla u(x)|)\ \mathrm{d}x.$$

Summing over j, we have

$$\begin{aligned}
\int_{\Omega^+} L(|\nabla M^{\tilde{h}}(x)|)\ \mathrm{d}x &\leq (1+\varepsilon^1) \int_{\Omega^+} L(|\nabla u(x)|)\ \mathrm{d}x \\
&\leq \int_{\Omega^+} L(|\nabla u(x)|)\ \mathrm{d}x + \frac{\varepsilon}{2},
\end{aligned}$$

thus (5.22) is proved.

e) Write

$$M^{\tilde{h}}(x) = u^0(x) + [M^{\tilde{h}}(x) - M^{\tilde{h}}(y(x)) - u^0(x) + u^0(y(x)];$$

we have $|(M^{\tilde{h}}(x) - M^{\tilde{h}}(y(x)) - (u^0(x) - u^0(y(x)))| \leq (\tilde{h} + K)|y(x) - x| \leq 2(\tilde{h} + K)d(x)$, by i) of Lemma 5.3. Hence, $M^{\tilde{h}}$ is the sum of a Lipschitzean function and of a function g such that $|g(x)| \leq Dd(x)$.

Apply Lemma 5.5 to infer the existence of D^* such that

$$\tilde{M}^{\tilde{h}}(x) = \frac{1}{|B(x, d(x))|} \int_{B(x,d(x))} M^{\tilde{h}}(z) \; \mathrm{d}z$$

is Lipschitzean of constant D^*. Consider $L(D^*)$: there exists $\delta^* \leq \tilde{\delta}$ such that

$$\int_{\Omega_{\delta^*}} L(D^*) \; \mathrm{d}x < \frac{\varepsilon}{2}.$$

f) Having fixed δ^*, define the continuous function

$$u_\varepsilon(x) = \begin{cases} \frac{1}{|B(x,d(x))|} \int_{B(x,d(x))} M^{\tilde{h}}(z) \; \mathrm{d}z, & \text{when } d(x) \leq \delta^*, \\ \frac{1}{|B(x,\delta^*)|} \int_{B(x,\delta^*)} M^{\tilde{h}}(z) \; \mathrm{d}z, & \text{when } d(x) > \delta^*. \end{cases}$$

From e) and Lemma 5.4, we have that u_ε is Lipschitzean and, moreover, that $u_\varepsilon|_{\partial\Omega} = u^0|_{\partial\Omega}$. We claim that

$$\int_\Omega L(|\nabla u_\varepsilon(x)|) \; \mathrm{d}x \leq \int_\Omega L(|\nabla M^{\tilde{h}}|) \; \mathrm{d}x + \frac{\varepsilon}{2}.$$

Write $\Omega = \Omega_{\delta^*} \cup [\Omega \setminus \Omega_{\delta^*}]$. Consider the restriction of u_ε to $\Omega \setminus \Omega_{\delta^*}$. By ii) of Lemma 5.4 (applied to $\delta = \delta^*$), we have that, for a.e. $x \in \Omega \setminus \Omega_{\delta^*}$,

$$\nabla u_\varepsilon(x) = \frac{1}{\omega_N (\delta^*)^N} \int_{B(0,\delta^*)} \nabla M^{\tilde{h}}(x - z) \; \mathrm{d}z,$$

so that, by the convexity of $L(| \cdot |)$,

$$L(|\nabla u_\varepsilon(x)|) \leq \frac{1}{\omega_N (\delta^*)^N} \int_{B(0,\delta^*)} L(|\nabla M^{\tilde{h}}(x - z)|) \; \mathrm{d}z$$

and

$$\begin{aligned} \int_{\Omega \setminus \Omega_{\delta^*}} L(|\nabla u_\varepsilon(x)|) \; \mathrm{d}x &\leq \frac{1}{\omega_N (\delta^*)^N} \int_{B(0,\delta^*)} \mathrm{d}z \int_{\Omega \setminus \Omega_{\delta^*}} L(|\nabla M^{\tilde{h}}(x - z)|) \; \mathrm{d}x \\ &\leq \int_\Omega L(|\nabla M^{\tilde{h}}(x)|) \; \mathrm{d}x. \end{aligned}$$

By Lemma 5.5, u_ε is Lipschitzean of constant D^*. From our choice of δ^*, we have

$$\int_{\Omega_{\delta^*}} L(|\nabla u_\varepsilon(x)|)\ \mathrm{d}x \leq \int_{\Omega_{\delta^*}} L(D^*)\ \mathrm{d}x \leq \frac{\varepsilon}{2},$$

and so

$$\begin{aligned}\int_{\Omega} L(|\nabla u_\varepsilon(x)|)\ \mathrm{d}x &= \int_{\Omega_{\delta^*}} L(|\nabla u_\varepsilon(x)|)\ \mathrm{d}x + \int_{\Omega\setminus\Omega_{\delta^*}} L(|\nabla u_\varepsilon(x)|)\ \mathrm{d}x \\ &\leq \int_{\Omega} L(|\nabla M^{\tilde{h}}|)\ \mathrm{d}x + \frac{\varepsilon}{2},\end{aligned}$$

thus, by (5.22), proving the Theorem. □

Chapter 6

A two-dimensional Manià-type example

Set Ω be the square $Q = [-1/2, 1/2] \times [0, 1] \subset \mathbb{R}^2$ and let $u^0(x, y) = y$ be the boundary data. We wish to show the occurrence of the Lavrentiev phenomenon, i.e., that

$$\inf_{v \in \mathcal{W}_1} \int_Q f(x, y, v, \nabla v)\, \mathrm{d}x\mathrm{d}y < \inf_{v \in \mathcal{W}_\infty} \int_Q f(x, y, v, \nabla v)\, \mathrm{d}x\mathrm{d}y, \tag{6.1}$$

where

$$f(x, y, u, \nabla u) := \left\{ \left[(1 - 2|x|)\sqrt[3]{y} + 2|x|y\right]^3 - u^3(x, y) \right\}^2 \left\{ \frac{\partial u}{\partial y}(x, y) \right\}^6,$$

and $\mathcal{W}_p = \{u \in W^{1,p}(Q) : u|_{\partial Q} = u^0\}$, for $p \in [1, \infty]$. As one can easily see, the functional is non negative, the minimum over $\mathcal{W}_1$ is zero and it is attained at $u(x, y) = (1 - 2|x|)\sqrt[3]{y} + 2|x|y$.

In order to prove (6.1), we adapt the original proof by B. Manià, [30], to the two-dimensional case.

Let u be in $\mathcal{W}_\infty$. By regularity, for any fixed $x > 0$, one can choose $\alpha = \alpha(x)$ and $\beta = \beta(x)$ such that $\alpha(x) < \beta(x)$ and

$$\begin{aligned} u(x, \alpha(x)) &= \frac{1}{4}\left[(1 - 2x)\sqrt[3]{\alpha(x)} + 2x\alpha(x)\right]; \\ u(x, \beta(x)) &= \frac{1}{2}\left[(1 - 2x)\sqrt[3]{\beta(x)} + 2x\beta(x)\right]. \end{aligned}$$

Moreover, if one considers $x \in [1/8, 1/4]$, then

$$\begin{aligned} u(x,\beta(x)) - u(x,\alpha(x)) &= \frac{1}{2}[(1-2x)\sqrt[3]{\beta} + 2x\beta] - \frac{1}{4}[(1-2x)\sqrt[3]{\alpha} + 2x\alpha] \\ &\geq \frac{1}{4}\sqrt[3]{\beta(x)} + \frac{1}{8}\beta(x) - \frac{3}{16}\sqrt[3]{\alpha(x)} - \frac{1}{8}\alpha(x) \\ &\geq \frac{1}{16}\sqrt[3]{\beta(x)}. \end{aligned}$$

Using Jensen's inequality and the fact that $\beta(\cdot) < 1$,

$$\begin{aligned} &\int_{1/8}^{1/4} \mathrm{d}x \int_{\alpha(x)}^{\beta(x)} \left\{[(1-2x)\sqrt[3]{y} + 2xy]^3 - u^3(x,y)\right\}^2 \left\{\frac{\partial u}{\partial y}(x,y)\right\}^6 \mathrm{d}y \\ &\quad\geq \int_{1/8}^{1/4} \mathrm{d}x \int_{\alpha(x)}^{\beta(x)} \Big\{[(1-2x)\sqrt[3]{y} + 2xy]^3 \\ &\qquad\qquad - \frac{1}{8}[(1-2x)\sqrt[3]{y} + 2xy]^3\Big\}^2 \left\{\frac{\partial u}{\partial y}\right\}^6 \mathrm{d}y \\ &\quad\geq \frac{7^2}{8^2}\int_{1/8}^{1/4} \mathrm{d}x \int_{\alpha(x)}^{\beta(x)} y^2 \left\{\frac{\partial u}{\partial y}(x,y)\right\}^6 \mathrm{d}y \\ &\quad= \frac{7^2 3^5}{8^2 5^5}\int_{1/8}^{1/4} \mathrm{d}x \int_{\alpha^{3/5}(x)}^{\beta^{3/5}(x)} \left\{\frac{\partial u}{\partial y}(x,y(\xi))\right\}^6 \mathrm{d}\xi \\ &\quad= \frac{7^2 3^5}{8^2 5^5}\int_{1/8}^{1/4} \frac{\beta^{3/5}(x) - \alpha^{3/5}(x)}{\beta^{3/5}(x) - \alpha^{3/5}(x)} \int_{\alpha^{3/5}(x)}^{\beta^{3/5}(x)} \left\{\frac{\partial u}{\partial y}(x,y(\xi))\right\}^6 \mathrm{d}\xi \mathrm{d}x \\ &\quad\geq \frac{7^2 3^5}{8^2 5^5}\int_{1/8}^{1/4} [\beta^{3/5}(x) - \alpha^{3/5}(x)] \cdot \\ &\qquad\qquad \cdot\left(\frac{1}{\beta^{3/5}(x) - \alpha^{3/5}(x)}\int_{\alpha^{3/5}(x)}^{\beta^{3/5}(x)} \left\{\frac{\partial u}{\partial y}\right\} \mathrm{d}\xi\right)^6 \mathrm{d}x \\ &\quad\geq \frac{7^2 3^5}{8^2 5^5}\int_{1/8}^{1/4} \frac{1}{[\beta^{3/5}(x) - \alpha^{3/5}(x)]^5}[u(x,\beta(x)) - u(x,\alpha(x))]^6 \,\mathrm{d}x \\ &\quad\geq \frac{7^2 3^5}{8^2 5^5}\int_{1/8}^{1/4} \frac{1}{\beta^3(x)}[u(x,\beta(x)) - u(x,\alpha(x))]^6 \,\mathrm{d}x \\ &\quad\geq \frac{7^2 3^5}{8^2 5^5}\frac{1}{16}\int_{1/8}^{1/4} \frac{\sqrt[3]{\beta(x)}}{\beta^3(x)} \,\mathrm{d}x \\ &\quad\geq \frac{7^2 3^5}{8^2 5^5 2^4}\frac{1}{8}. \end{aligned}$$

Part IV

Appendix

Appendix A

A.1 Carathéodory functions

We recall Definition 3.5 and Remark 3.6 in [22].

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$. Then f is said to be a *Carathéodory function* if

- $x \mapsto f(x, \xi)$ is measurable for every fixed $\xi \in \mathbb{R}^N$;
- $\xi \mapsto f(x, \xi)$ is continuous for almost every fixed $x \in \Omega$.

In most of the uses of the above notion, we apply it to functions $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$, $f = f(x, u, \xi)$. When we speak of Carathéodory functions in this context, we consider the variable ξ as playing the role of (u, ξ) and $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^M$.

A.2 Luzin's Theorem

Theorem (Luzin). *Let $f : X \to \mathbb{R}$ be a measurable function defined on a Lebesgue measurable set $X \subset \mathbb{R}^n$, for which the Lebesgue measure $\ell(X)$ is finite. Then for each $\varepsilon > 0$ there exists a* compact *subset $K \subset X$ such that $\ell(X \setminus K) < \varepsilon$ and such that $f|_K$, the restriction of f to K, is* continuous *on K.*

For a proof, see for example [35] .

A.3 Convex functions - Some basic properties

What follows is arranged from [37]: Chapters 4, 12, and 23.

Let f be a function whose values are real or $\pm\infty$ and whose domain is a subset S of $\mathbb{R}^n$. The set

$$\operatorname{epi} f = \{(x,\mu) : x \in S, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

is called the *epigraph* of f and is denoted by $\operatorname{epi} f$. We define f to be a *convex* function on S if $\operatorname{epi} f$ is convex as a subset of $\mathbb{R}^{n+1}$. A *concave* function on S is a function whose negative is convex. An *affine* function on S is a function which is finite, convex and concave.

The *effective domain* of a convex function f on S, which we denote by $\operatorname{Dom} f$, is the projection on $\mathbb{R}^n$ of the epigraph of f:

$$\operatorname{Dom} f = \{x : \exists\, \mu \text{ s.t. } (x,\mu) \in \operatorname{epi} f\} = \{x : f(x) < +\infty\}\,.$$

A convex function f is said to be *proper* if its epigraph is non-empty and contains no vertical lines, i.e., if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x.

The following theorem states an important characterization of convex functions.

Theorem A.1. *Let f be a function from C to $(-\infty,+\infty]$, where C is a convex set. Then f is convex on C if and only if*

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y), \qquad \forall x,y \in C,\ \ \forall \lambda \in [0,1].$$

A.3.1 Subdifferential of convex functions

A vector x^* is said to be a *subgradient* of a convex function f at a point x if

$$f(z) \geq f(x) + \langle x^*, z-x\rangle, \quad \forall z.$$

This condition, which we refer to as the *subgradient inequality*, has a simple geometric meaning when f is finite at x: it says that the graph of the affine function $h(z) = f(x) + \langle x^*, z-x\rangle$ is a non-vertical supporting hyperplane to the convex set $\operatorname{epi} f$ at the point $(x, f(z))$.

The set of all subgradients of f at x is called the *subdifferential of f at x* and is denoted by $\partial f(x)$. The set valued mapping $\partial f : x \mapsto \partial f(x)$ is called the *subdifferential* of f. Obviously $\partial f(x)$ is a closed convex set and, in general, it may be empty, or it may consist of just one vector. If $\partial f(x)$ is not empty, f is said to be *subdifferentiable* at x.

For example, the Euclidean norm $f(x) = |x|$ is subdifferentiable at every $x \in \mathbb{R}^n$, although it is differentiable only at every $x \neq 0$. The set $\partial f(0)$ consists of all the vectors x^* such that $|z| \geq \langle x^*, z\rangle$ for every z. In other words, it is the Euclidean unit ball.

The following theorem shows how gradients and subgradients are related.

Theorem A.2. *Let f be a convex function, and let x be a point where f is finite. If f is differentiable at x, then $\nabla f(x)$ is the unique subgradient of f at x and, in particular,*

$$f(z) \geq f(x) + \langle \nabla f(x), z - x\rangle, \quad \forall z.$$

Vice versa, if $\partial f(x)$ consists of just one vector, then f is differentiable at x and $\partial f(x) = \{\nabla f(x)\}$.

A.3.2 Polars of convex functions

Given a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, we define

$$f^*(x^*) := \sup_x \{\langle x, x^*\rangle - f(x)\} = -\inf_x \{f(x) - \langle x, x^*\rangle\}.$$

This f^* is called the *conjugate* or the *polar* of f. It is actually the pointwise supremum of the affine functions $g(x^*) = \langle x, x^*\rangle - \mu$ such that (x, μ) belongs to the set epi f. Hence, f^* is another convex function, in fact a closed convex function. Since f is the pointwise supremum of the affine functions $h(x) = \langle x, x^*\rangle - \mu^*$ such that (x^*, μ^*) belongs to epi f^*, we have

$$f(x) = \sup_{x^*} \{\langle x, x^*\rangle - f^*(x^*)\} = -\inf_{x^*} \{f^*(x^*) - \langle x, x^*\rangle\}$$

But this says that the polar f^{**} of f^* is f.

Theorem A.3. *Let f be a convex function. The polar function f^* is then a closed convex function, proper if and only if f is proper. Moreover,* $(\mathrm{cl}\, f)^* = f^*$ *and* $f^{**} = \mathrm{cl}\, f$.

Taking polars clearly reverses functionals functional inequalities: $f_1 \leq f_2$ implies $f_1^* \leq f_2^*$.

The theory of conjugacy can be regarded as the theory of the "best" inequalities of the type

$$\langle x, y\rangle \leq f(x) + g(y), \ \forall x, \ \forall y,$$

where f and g are functions from $\mathbb{R}^n$ to $(-\infty, +\infty]$. Let W denote the set of all function pairs (f, g) for which this inequality is valid. The "best" pairs (f, g) in W are those for which the inequality cannot be tightened, i.e., those such that, if $(f', g') \in W$, $f' \leq f$ and $g' \leq g$, then $f' = f$ and $g' = g$. Clearly, one has $(f, g) \in W$ if and only if

$$g(y) \geq \sup_x \{\langle x, y\rangle - f(x)\} = f^*(y),\ \forall y,$$

or, equivalently

$$f(x) \geq \sup_y \{\langle x, y\rangle - g(y)\} = g^*(x),\ \forall x.$$

Therefore, the "best" pairs in W are precisely those such that $g = f^*$ and $f = g^*$. The "best" inequalities thus correspond to the pairs of mutually conjugate closed proper convex functions.

It is useful to remember, in particular, that the inequality

$$\langle x, x^*\rangle \leq f(x) + f^*(x^*),\ \forall x,\ \forall x^* \textit{ (Fenchel's inequality)},$$

holds for any proper convex function f and its conjugate f^*. The pairs (x, x^*) for which Fenchel's inequality is satisfied as an equation form the graph of the subdifferential ∂f:

Theorem A.4. *For any proper convex function f and any vector x, the following four conditions on a vector x^* are equivalent to each other:*

- $x^* \in \partial f(x)$;
- $\langle z, x^*\rangle - f(z)$ *achieves its supremum in z at $z = x$;*
- $f(x) + f^*(x^*) \leq \langle x, x^*\rangle$;
- $f(x) + f^*(x^*) = \langle x, x^*\rangle$.

If $(\mathrm{cl}\, f)(x) = f(x)$, three more conditions can be added to this list

- $x \in \partial f^*(x^*)$;
- $\langle x, z^*\rangle - f^*(z^*)$ *achieves its supremum in z^* at $z^* = x^*$;*
- $x^* \in \partial(\mathrm{cl}\, f)(x)$.

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