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Higher-gradient Theories for Fluids and Concentrated Effects

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Alle mie famiglie passate, presenti e future,
per il loro sostegno incondizionato.

Contents

Introduction

To build up models for natural phenomena is an attitude of the human being, needed in order both to develop an understanding of what is going on around us and to design strategies to master it. Mathematics has gained a prominent position amongst the tools which can be used to perform such a modeling. Indeed, the evolution of technology has always been demanding in its request for precise previsions, in any field of application. The great, and sometimes surprising, achievements obtained by the interaction of physics, natural sciences and engineering with mathematics, resulted in a continuous growth of each of the disciplines involved in this process.

Nowadays, any kind of application is requiring new mathematical models, also because computers essentially read mathematics. The interaction with scientific calculus and numerical simulations also emphasizes an important fact: mathematical models cannot perfectly reproduce reality, but this is not a major drawback, provided that the achieved approximation be good enough.

Continuum Mechanics is an important field of applied mathematics, based on approximating the essentially granular matter with a continuum, which has provided a number of models, successfully exploited in physics and engineering. The main mathematical ingredient in Continuum Mechanics are differential equations, since their solutions represent the expected configuration or evolution of the physical system under consideration. Hence, we understand why three fundamental steps within continuum mechanical modeling are: to write the right equations, to analyze the properties of their solutions, and to solve them, mostly by means of numerical simulations. The theory of differential equations has produced a variety of approaches to them; those evolving perspectives have always improved the possibility of setting up continuum mechanical models, capturing new applications and phenomena. Moreover, they helped in constructing more flexible frameworks for Continuum Mechanics.

Within the present thesis, I want to consider the Virtual Power framework for Continuum Mechanics, which has gained considerable attention after a seminal paper by Germain [13], mainly in connection with its applicability to non-classical models for materials. That framework is intimately related to the weak (or variational) form of partial differential equations, whose introduction provided great enhancements in their mathematical theory, and is also the basis for the modern numerical simulations. I introduce, in Chapter 1, a geometrical approach to possibly infinite dimensional dynamical systems, based on the theory of Banach manifolds, which has not yet been fully exploited in Continuum Mechanics, though it has been used in some particular cases, as explained in [1]. This theory generalizes the Virtual Power framework, being even more flexible and allowing for the construction of continuum mechanical models on non-Euclidean domains.

Higher-gradient continuum mechanical theories has been considered by Toupin [32] in the context of elasticity, and by Germain [13] within the Virtual Power framework. More recently,

Gurtin [18] applied them to single-crystal viscoplasticity, and Fried and Gurtin [12] proposed a second-gradient model for viscous fluids. The importance of [12] relies also in having set these models into a more general and logically well-established framework, as they were, until then, merely *ad hoc* extensions of the differential problems related to classical models. The key feature of second-gradient theories is the presence of multiple length scales and the possibility of encompass non-standard interactions. In two papers with Marzocchi and Musesti ([16],[17]), using the Virtual Power framework, I studied the mathematical properties of a general linear isotropic incompressible second-gradient fluid. Constitutive prescriptions for these fluids are presented in Chapter 3, together with the constraints imposed by thermodynamical considerations on some new material parameters.

The key features of the analyzed model are the possibility of describing the adherence interaction of a three-dimensional fluid with one-dimensional structures immersed in it, and also of including concentrated interactions, thanks to the non-classical structure of the power expended by both the internal and the external stresses which act on the fluid. A presentation of higher-gradient theories is contained in Chapter 2. They are introduced, using the general framework proposed in Chapter 1, as a particular class of continuum mechanical models, arising from precise assumptions on the kinematics of the descriptors of the system. That assumptions are encoded in the fact that descriptors belong to a suitable function space, which constitutes a Banach manifold, and higher-order powers are defined as integral representations of elements of the cotangent bundle on that Banach manifold. Exploiting equivalent integral representations for powers of arbitrary order, the appearance of boundary interactions with a non-standard structure is described.

In Chapter 4, the differential problems associated with the pressure-driven flow of a second-order linear liquid, which adheres to a one-dimensional structure, is considered. Existence and uniqueness of solution are established, also for the situation in which the one-dimensional structure drags the three-dimensional fluid, producing the motion. Finally, some examples are provided in Chapter 5, in order to give explicit solutions, to show how the concentrated stresses, if present, can be computed, and to suggest possible interpretations for the physical meaning of the higher-order material parameters.

Chapter 1

Continuum Mechanics and the Principle of Virtual Powers

Continuum Mechanics is intended to give a way to describe the possible configurations, kinematics and dynamics of objects geometrically described by "infinitely many" points. Once settled configurations and kinematics, the Principle of Virtual Powers is a tool (amongst many available ones) to mathematically synthesize the dynamics of the object. Though the history of the usage of that principle has not been analyzed in detail, the contemporary attention to it has certainly originated by [13] and also by [3], and in [11] and [19] it is systematically employed to set up continuum mechanical theories.

As it will become clear later, the Virtual Powers approach to Continuum Mechanics corresponds to a nice geometrical approach to dynamical systems, which will be briefly described in the following section. The striking advantages of such a perspective consists both in a reconciliation under the same mathematical structure of Continuum Mechanics, Rational Mechanics, Lagrangian Field Theories and other branches of Mathematical Physics, and in a formulation of the related Differential Equations which corresponds to that required by the modern tools used in Functional and Numerical Analysis.

1.1 Geometrical approach to dynamical systems

A *dynamical system* is a triple $(\Omega, \mathscr{D}, \mathcal{S})$ where Ω is a set named *underlying space*, the *phase space* $\mathscr{D}$ is a product of sets of functions, whose domain is Ω, endowed with the structure of differentiable Banach manifold,[(1)] and $\mathcal{S}$ is a section of the cotangent bundle on $\mathscr{D}$. The section $\mathcal{S}$ evaluated at $u \in \mathscr{D}$ will be denoted $\mathcal{S}_u$. Elements of $\mathscr{D}$ are called *configurations*, their components are the *descriptors* of the system, and elements of the tangent space at any $u \in \mathscr{D}$ are called *virtual velocities*. We can characterize *equilibrium configurations* for the dynamical system $(\Omega, \mathscr{D}, \mathcal{S})$ as points u of $\mathscr{D}$ at which $\mathcal{S}$ is *vanishing* (or *vertical*), that is $\langle \mathcal{S}_u, v \rangle = 0$ for any virtual velocity $v \in T_u\mathscr{D}$.

Notice that I use the adjective 'virtual' to indicate that virtual velocities are not only far from being the actual velocity field of the system, but often far from being velocities at all, since their physical units depend on the units of the descriptors. Moreover, the term 'velocity'

[(1)] For basic facts about differentiable Banach manifolds see Appendix A.

is used to convey the idea of changing, and it is borrowed from the particular, though very important, problem of mechanical equilibrium for a system of point masses.

Dynamics concerns the determination of paths in the phase space, which are parametrized by time, describing feasible evolutions of the system. Equilibrium configurations correspond to constant paths, while feasible evolutions $u(t)$ are characterized by the differential equation:

$$\langle \dot{u}, v \rangle(t) = \langle \mathcal{S}_u, v \rangle(t) \tag{1.1}$$

for any virtual velocity $v \in T_{u(t)}\mathscr{D}$ at any instant t. A major issue is to clarify the meaning of both the time derivative (denoted by a superimposed dot) and of the phrase 'at any instant'.

Indeed, equation (1.1) suggests an interpretation of $\dot{u}$ as an element of the cotangent space $T_u^*\mathscr{D}$, whereas the definition of the time derivative as the derivative along a path, traced on $\mathscr{D}$ and parametrized by t, places $\dot{u}$ in the tangent space $T_u\mathscr{D}$. Hence, in order to give an unambiguous meaning to (1.1), we need a bijection between the tangent space and its topological dual, the cotangent space. If the tangent space is a Hilbert space, Riesz' isometry provides such a correspondence, but also when the tangent space is a reflexive Banach space, there is a canonical bijection into its dual space (see [29, Chap. 6]). By means of such *duality application*, we can identify a unique element in $T_u^*\mathscr{D}$ representing the tangent vector $\dot{u}$, and make use of it in equation (1.1).

Nevertheless, it is often useful to restate evolutionary problems, whose solutions are non-constant paths on $\mathscr{D}$, as static problems, whose solutions are constant paths, in a larger space. This can be done including the time variable in the underlying space, which becomes of the form $\hat{\Omega} \times [t_I, t_F]$, with $[t_I, t_F] \subseteq \mathbb{R}$. The phase space $\mathscr{D}$ shall be accordingly updated, and the section $\mathcal{S}$ will contain *inertial terms*, which encode the need for evolution. In this way, the differential problem associated with the dynamical system becomes:

$$\langle \mathcal{S}_u, v \rangle = 0 \tag{1.2}$$

for any virtual velocity $v \in T_u\mathscr{D}$. Since now the underlying space includes the temporal domain, also evolutionary solutions are recovered.

Within the subsequent section I will settle Continuum Mechanics in the described framework, but now I want to show how it encompasses Rational Mechanics and Lagrangian Field Theories. As to the former, let E be a discrete set with N elements; the set of functions, which associates to each point its position in $\mathbb{R}^3$ and its linear momentum, is isomorphic to $(\mathbb{R}^3)^{2N}$. Let $\mathscr{D}$ be the set of differentiable functions $(\mathsf{q}, \mathsf{p}) : [t_I, t_F] \to (\mathbb{R}^3)^{2N}$, then, solutions to the Newton's equations, for the system of point masses $(m_i)_{i=1}^N$, are equilibrium configurations for the dynamical system $(E \times [t_I, t_F], \mathscr{D}, \mathcal{S})$, where the first component of $\mathcal{S}$ is p, and the second one is the sum of internal and external forces and reactions, to which is added also the inertial term, which is proportional to $-(\dot{\mathsf{q}}, \dot{\mathsf{p}})$.

As to Lagrangian Field Theories, Ω is often the Minkowski space-time (a Lorentzian manifold), $\mathscr{D}$ is the functional space to which the field belongs, and $\mathcal{S}$ is the first variation of the Lagrangian action which defines the theory. The condition (1.2) corresponds to the Euler–Lagrange equations associated with the action.

1.1.1 Observer's changes and covariance

The *covariance of the physical laws* is a principle, requiring the physics of a system to be invariant with respect to changes in the frame of reference used by an observer to describe it. It is an essential requirement which must be satisfied when constructing any physical theory.

Given a dynamical system $(\Omega, \mathscr{D}, \mathcal{S})$, assume that we can describe Ω by means of a (maybe local) coordinate system and let A denote the expression of any mathematical object of the theory in that coordinate system. Assume also that there is a group $\mathcal{G}$ of transformations which associate to the original coordinate system a new one and, given $\phi \in \mathcal{G}$, denote by A^ϕ the expression of A in this new coordinate system. The dynamical system $(\Omega, \mathscr{D}, \mathcal{S})$ is said to be $\mathcal{G}$-*covariant* if, for any $u \in \mathscr{D}$ and any $v \in T_u\mathscr{D}$,

$$\langle \mathcal{S}^\phi_{u^\phi}, v^\phi \rangle^\phi = \langle \mathcal{S}_u, v \rangle ,$$

for any ϕ belonging to the group $\mathcal{G}$ of frame transformations.

If there exists a coordinate system on Ω, we can always consider the trivial group of transformations $\{\mathrm{Id}\}$, containing only the identity map, but $\{\mathrm{Id}\}$-covariance wouldn't require anything. As an opposite situation, we could consider a group of transformations which spans the whole set of possible coordinate system on Ω. This for instance the choice in General Relativity, but it is certainly not the most popular choice in Continuum Mechanics. Indeed, one can argue that there is a proper subset of the set of frames of reference, the set of so-called *inertial frames*, amongst which observers should choose their frame. Hence, most of the times, the group $\mathcal{G}$ spans only a set of "nicer" coordinate systems.

1.2 Continuum Mechanics

While speaking about Continuum Mechanics, two different aspects come into play: the mechanics *of* a continuum and the mechanics *on* a continuum. If the *on* side is to be emphasized, the natural choice is that of an Eulerian perspective, where the descriptors of the state of the system are fields living on a fixed continuous space-time. While that underlying space is the common domain of all the fields, they can take values in quite different sets, describing different objects. If the *of* side is stressed, a Lagrangian perspective is taken, the underlying space describes the continuum and its topological properties, while the *placement* of the continuum into the real space is a descriptor with a prominent kinematical role.

In both cases, the phase space $\mathscr{D}$ is the product of sets of functions whose components are descriptors of the continuum. Since a continuity assumption affects also the motion, the kinematics is essentially encoded in the tangent space of the phase space. Given the aims of Continuum Mechanics, it is very natural to assume $\mathscr{D}$ to be a Banach space, and hence $T_u\mathscr{D}$ turns out to be isomorphic to $\mathscr{D}$ for any u. If, moreover, $\mathscr{D}$ is a Hilbert space the isomorphism is a canonical one. Notice that the choice of descriptors implies a choice of the virtual fields: kinematics is encoded in the descriptors.

The dynamics of the system which is to be modeled is encoded in the definition of $\mathcal{S}$. A key concept, used while modeling it, is that of *power expenditure*: any action which is likely to produce a change in the actual configuration expends a nonzero power on the virtual velocity which represents the "direction" of such a change. An equilibrium configuration is such that the powers expended on any virtual velocity, by anything which can expend power, sum up to zero. From a mathematical point of view, linear forms on the space of virtual velocities are the objects which expend power, and the amount of the expended power is the value of the linear form calculated on a virtual velocity. Such linear forms represent interactions.

The Principle of Virtual Powers is now the name which is given to (1.2) when $\langle \mathcal{S}_u, v \rangle$ represents the total power expended by the system in the configuration u on the virtual velocity v. Usually, the total power is the difference between the power of the *internal*

interactions and the power of the *external* interactions, where the distinction comes from viewing the system as opposed to the environment. The internal power contains mutual interactions between parts of the system, while the external power represents the actions which can be performed on the system by the environment. Notice that (1.2) is a balance of powers only when we are looking for the mechanical equilibrium of a system, while, when we include the temporal domain in Ω, it becomes a balance of works (usually referred to as the Principle of Virtual Works), and it would be a balance of something else if the application required so.

The modeling of internal and external interactions will be treated in subsequent chapters, while now I want to emphasize the extended notion of kinematics which naturally arises from the present approach. In fact, while the original notion of kinematics is related to the possible evolutions of the position of point masses or of the placement of a continuous body in the physical space, here any descriptor carries its own kinematics, since it has its own possible evolutions, corresponding to paths traced on $\mathscr{D}$. Moreover, the evolution can be with respect to a parameter which is not necessarily the time. Finally, recall that, given a configuration $u \in \mathscr{D}$, the possible directions of the evolution are the elements of the tangent space $T_u\mathscr{D}$, which describes the kinematical constraints.

1.2.1 Inertial terms

In order to encompass evolutionary problems, we have to include in the balance (1.2) the power expended by the linear form associated with the time derivative of the descriptor. When the time variable is included in Ω by the very structure of the problem under consideration (*e.g.* in Relativistic Field Theories), time-evolution terms arise in a natural way. On the other hand, if we want to make an evolution equation out of a static problem with associated dynamical system $(\hat{\Omega}, \hat{\mathscr{D}}, \hat{\mathcal{S}})$, we have to consider equation (1.1); then, in order to put it in the form (1.2), we have to set $\Omega := [t_I, t_F] \times \hat{\Omega}$ and $\mathscr{D} := \mathscr{B}([t_I, t_F]; \hat{\mathscr{D}})$, where $\mathscr{B}$ is a suitable Banach space, and we have to extend $\mathcal{S}_u := \hat{\mathcal{S}}_u - \dot{u}$, which naturally belongs to $T^*_u\hat{\mathscr{D}}$, to a continuous linear form on $T^*_u\mathscr{D}$. Such an extension is usually straightforward in concrete situations.

There are some particular cases in which such terms produce the so-called *inertial forces*. It essentially happens when we have a placement field and the linear momentum field as relevant descriptors: given the link between them, expressed by the evolution equation for the placement, the time-evolution term in the equation for the linear momentum becomes the Newtonian 'mass times acceleration', which is often described as 'minus the inertial forces'.

A particular class of interactions, typical of the Eulerian perspective, which has an inertial origin, is that of advective terms: when a field describes a property which is carried about by the moving matter, it is transported by the associated velocity field, and it evolves in time also in the absence of any other interaction, if its gradient field is not everywhere orthogonal to the velocity field.

1.2.2 Thermodynamics brought in

Thinking about physics, we are often led to separate mechanical facts from thermodynamical ones. But, if thermodynamics is thought of as a theory involving quantities and laws which describe the global behavior of a system of many many particles, somehow discarding the details of their mechanical behavior at a microscopic level, then I would say that Continuum Mechanics as a whole is a thermodynamical theory.

Indeed, the points (or infinitesimal elements) of a continuous body are usually described as small enough to assume that any relevant descriptor takes a unique value on each point, but big enough to contain a number of particles which allows for a statistical treatment of their physical properties. Hence, mechanical and thermodynamical descriptors of a continuum deserve an identical mathematical treatment, being simply different components of a $u \in \mathscr{D}$.

It is then clear that the Principle of Virtual Powers can produce thermodynamical balances, contributing necessary additional equations to many continuous models. This perspective is introduced *e.g.* in [30] and in [19], where it is also explained how thermodynamical imbalances are obtained, and how the latter can be used to select constitutive prescriptions by the Coleman-Noll procedure [6].

1.3 Examples and further possibilities

In this section I want to show that some well-established continuum mechanical theories do fit into the proposed framework. Then I will enlighten the flexibility of that approach in defining theories on non-Euclidean domains, which still represent a non-standard setting for Continuum Mechanics.

1.3.1 Hyperelasticity and compressible fluids

As a first example I will consider a hyperelastic material. The variational structure of the problem, considered *e.g.* in [4] or [7], allows for a straightforward reinterpretation of it in the dynamical system setting. The information about the internal stresses is encoded in the stored elastic energy density $\psi(\nabla \mathsf{u})$, where ψ is a real-valued function on $\mathrm{Mat}_3(\mathbb{R})$, and the vector field $\mathsf{u}(\mathsf{x})$ represent the displacement of the material point with respect to its reference configuration x. For simplicity, we can assume that the external stresses are described by a volumetric force density $\mathsf{f}(\mathsf{x})$.

The underlying space Ω is an open bounded subset of $\mathbb{R}^3$, whose elements are material points. Considering the static problem, it requires the minimization of the functional

$$\int_\Omega [\psi(\nabla \mathsf{u}) - \mathsf{f} \cdot \mathsf{u}] \tag{1.3}$$

on a suitable function space, which will be our phase space $\mathscr{D}$, to which u belongs. It is then straightforward to see that the Euler–Lagrange equations associated with (1.3) are

$$\int_\Omega \left.\frac{\partial \psi(\mathsf{A})}{\partial \mathsf{A}}\right|_{\nabla \mathsf{u}} \cdot \nabla \mathsf{v} = \int_\Omega \mathsf{f} \cdot \mathsf{v} \tag{1.4}$$

for any $\mathsf{v} \in T_{\mathsf{u}}\mathscr{D}$. They represent the balance of an internal power expenditure (the left-hand side) with the external one (the right-hand side).

It remains to identify the function space $\mathscr{D}$. It will be strictly related to the constitutive assumption on ψ, since, in order to prove the existence of a solution, the quantity

$$\int_\Omega \left.\frac{\partial \psi(\mathsf{A})}{\partial \mathsf{A}}\right|_{\nabla \mathsf{u}} \cdot \nabla \mathsf{u}$$

should represent, roughly speaking, the norm of $\mathsf{u} \in \mathscr{D}$. As an example, assume, for $p \geqslant 2$,

$$\psi(\nabla \mathsf{u}) = \frac{1}{p}|\nabla \mathsf{u}|^p \,,$$

and Dirichlet boundary conditions: then $\mathscr{D}$ should be the Sobolev space $W_0^{1,p}(\Omega;\mathbb{R}^3)$.[(2)]

As a second example I will consider the theory of compressible, viscous and heat conducting fluids, in the barotropic regime, as presented in [10]. Given a bounded domain $\Omega \subseteq \mathbb{R}^3$, the state of the fluid at any instant $t \in [0,T]$ is described by the density $\rho(t,\mathsf{x})$, the Eulerian velocity field $\mathsf{u}(t,\mathsf{x})$ and the absolute temperature $\vartheta(t,\mathsf{x})$. Barotropicity implies that the pressure p is a given function of ρ and ϑ, and the following constitutive relation is assumed:

$$p(\rho,\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho)\,. \tag{1.5}$$

Other relevant quantities are:

- the specific internal energy $e(\rho,\vartheta)$, which is the sum of an "elastic" part $P_e(\rho)$, related to intermolecular forces, and a "thermal" part $Q(\vartheta)$;
- the heat flux q, for which the Fourier's law $\mathsf{q} = -\kappa(\vartheta)\nabla\vartheta$, with the conductivity $\kappa > 0$, is assumed;
- the Newtonian viscosity tensor

  $$\mathsf{V} := \mu(\nabla\mathsf{u} + (\nabla\mathsf{u})^\dagger) + \lambda \operatorname{div}\mathsf{u}\mathsf{I}\,,$$

  where the viscosity coefficients μ and λ depend on the absolute temperature;
- the given external force density $\mathsf{f}(t,\mathsf{x})$ and heat source $g(t,\mathsf{x})$.

As to the boundary conditions, adherence to the fixed boundary and thermal insulation are assumed, so that $\mathsf{u} = 0$ and $\mathsf{q} = 0$ on $\partial\Omega$.

According to this setting, the equation of motion, which will be supplemented by suitable initial conditions, are

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathsf{u}) = 0\,, \tag{1.6}$$

$$\frac{\partial}{\partial t}(\rho\mathsf{u}) + \operatorname{div}(\rho\mathsf{u}\otimes\mathsf{u}) + \nabla p = \operatorname{div}\mathsf{V} + \rho\mathsf{f}\,, \tag{1.7}$$

$$\frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho e\mathsf{u}) + \operatorname{div}\mathsf{q} = \mathsf{V}\cdot\nabla\mathsf{u} - p\operatorname{div}\mathsf{u} + \rho g\,. \tag{1.8}$$

A direct consequence of the stated assumptions is that the third equation can be replaced by

$$\frac{\partial}{\partial t}(\rho Q(\vartheta)) + \operatorname{div}(\rho Q(\vartheta)\mathsf{u}) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = \mathsf{V}\cdot\nabla\mathsf{u} - \vartheta p_\vartheta \operatorname{div}\mathsf{u} + \rho g\,. \tag{1.9}$$

In [10], the existence of a global solution for (1.6)-(1.7)-(1.9), with the stated boundary conditions and suitable initial conditions, is proved. What I want to clarify now is how to interpret the problem as a dynamical system, and the equations as a balance of powers. The underlying space can be easily recognized in $[0,T]\times\Omega$, and an element $u \in \mathscr{D}$ is a triple $(\rho,\mathsf{u},\vartheta)$, but, in order to identify $\mathscr{D}$ as a Banach space, we have to consider the weak form of

[(2)] For the theory of Sobolev spaces see [2] or [33].

the equations of motion:

$$\int_0^T \int_\Omega \left(\rho \frac{\partial \phi}{\partial t} + \rho \mathsf{u} \cdot \nabla \phi \right) = 0 \,, \tag{1.10}$$

$$\int_0^T \int_\Omega \left(\frac{\partial}{\partial t}(\rho \mathsf{u}) \cdot \mathsf{v} + \operatorname{div}(\rho \mathsf{u} \otimes \mathsf{u}) \cdot \mathsf{v} + [\mathsf{V} - p\mathsf{I}] \cdot \nabla \mathsf{v} \right) = \int_0^T \int_\Omega \rho \mathsf{f} \cdot \mathsf{v} \,, \tag{1.11}$$

$$\int_0^T \int_\Omega \left((\rho Q(\vartheta)) \cdot \frac{\partial \sigma}{\partial t} + \rho Q(\vartheta) \mathsf{u} \cdot \nabla \sigma + \mathcal{K}(\vartheta) \Delta \sigma \right) = \int_0^T \int_\Omega (\vartheta p_\vartheta \operatorname{div} \mathsf{u} - \mathsf{V} \cdot \nabla \mathsf{u} - \rho g) \cdot \sigma \,, \tag{1.12}$$

where $\mathcal{K}(\vartheta) := \int_0^\vartheta \kappa(s)\, ds$, and the triple $(\phi, \mathsf{v}, \sigma)$ is the generalized virtual velocity, belonging to $T_u \mathscr{D}$.

It is easy, by the form of equation (1.11), to see that it represents the balance of power expenditures on the virtual field v belonging to $L^2([0,T]; H^1_0(\Omega; \mathbb{R}^3))$; it is also easy to recognize the internal power (defined by $\mathsf{V} - p\mathsf{I}$), the external one (defined by $\rho\mathsf{f}$) and the inertial contribution, containing the time derivative and the advective term. It is less easy to guess the proper functional space to which ρ and ϑ should belong; but, interpreting (1.6) and (1.9) at first in the sense of distributions and following the analysis in [10], it turns out that

$$\rho \in L^\infty([0,T]; L^\gamma(\Omega)) \cap C([0,T]; L^1(\Omega)) \,,$$

$$\vartheta \in L^\alpha([0,T]; L^\alpha(\Omega)) \,.$$

for suitable exponents α and γ, related to growth assumption on κ and p_ϑ respectively. Then we can recognize that equations (1.10) and (1.12) represent the balance of generalized power expenditures on the virtual fields ϕ and σ.

1.3.2 Continuum mechanics on manifolds and non-smooth spaces

Though it is a hard task to generalize the classical continuum mechanical models to situations where the body itself or the space in which the body moves are not Euclidean manifolds, recent applications in material science, biology, environmental physics and multiscale and/or multiphysics systems, are more and more involving such *non-smooth* spaces. The dynamical system approach to Continuum Mechanics has a great potential with respect to that issue.

In fact, the modeler's attention is switched from the topological properties of the underlying space to the topological properties of the phase space $\mathscr{D}$; this because phase spaces with a similar structure can be defined on underlying spaces with very different structures, and hence a continuum mechanical model based on the properties of $\mathscr{D}$ can be adapted to very different situations. Here I just want to mention a very important example of that operation.

Many continuum mechanical models, especially those with a variational structure, require $\mathscr{D}$ to be a Sobolev space. Now it is well-known that the original definition of such spaces, given on Euclidean domains, can be easily generalized to compact Riemannian manifolds. This gives an immediate generalization for many theories, but we can go much further. Indeed, in recent years, various possible definitions of Sobolev spaces on metric measure spaces have been proposed (the review articles [20] and [21] provide very clear introductions to this rapidly evolving field). A *metric measure space* is a topological space endowed with a distance function and a Borel measure which define the metric and the measurable structure respectively, and those structures can be interrelated in various ways. It turns out that the interrelation

between the Lebesgue measure and the Euclidean distance, on which differentiability notions and Sobolev spaces are built in classical settings, can be partly reproduced in much weaker situations, such as subriemannian manifolds, singular manifolds (Alexandrov spaces) and even on some fractal set. This fact is the basis for the generalization to non-smooth underlying spaces of continuum mechanical theories based on Sobolev spaces.

Chapter 2

Higher-order power expenditures and concentrated interactions

Within this chapter I will focus on what is known in the literature as *higher-order gradient power expenditures*. I will show how this concept is related to a kinematical prescription involving the phase space, and I will describe how it can allow for a treatment of concentrated interactions, together with non-standard boundary interactions.

2.1 Integral representation of the power

In order to characterize a continuous medium, the power of the internal interactions is to be modeled. In great generality, such a power is a section of the cotangent bundle on the phase space $\mathscr{D}$, but it is usually better to deal with an *integral representation* of it. Indeed, powers has often been defined through their integral representation, on the basis of dynamical, rather than kinematical, considerations. In spite of that tradition, I want to emphasize that kinematical prescriptions often suggest a preferred integral representation for the power expenditures. This also reveals a constitutive side of kinematics. The argument is the following: if we declare the functional space of descriptors, usually isomorphic to that of virtual velocities, we will most likely find a theorem in Functional Analysis providing us an integral representation for the action of the elements of its topological dual, which will be the powers.

I will now focus on an example which represents a very common situation in Continuum Mechanics: let Ω be an open subset of $\mathbb{R}^n$, and let $\mathscr{D}$ be the Sobolev space $H^k(\Omega)$, for some $k \in \mathbb{N}$ fixed. Then, since $\mathscr{D}$ is a Hilbert space, for any $u \in \mathscr{D}$, $T_u\mathscr{D} = \mathscr{D}$, and, for any $F \in T_u^*\mathscr{D} = (H^k(\Omega))'$ there is a unique $f \in H^k(\Omega)$ such that

$$\langle F, v\rangle = \sum_{i=0}^{k} \int_\Omega \nabla^i f \cdot \nabla^i v \,, \tag{2.1}$$

for any $v \in H^k(\Omega)$; here $\nabla^i v$ denotes the i-th distributional gradient of v. It is also clear that, given for $i = 0, \dots, k$ some function $A^{(i)} \in L^2(\Omega; \mathbb{R}^{n^i})$, there is a unique $F \in (H^k(\Omega))'$ such that

$$\langle F, v\rangle = \sum_{i=0}^{k} \int_\Omega A^{(i)} \cdot \nabla^i v \,, \tag{2.2}$$

for any $v \in H^k(\Omega)$. It is this last relation which motivates the definition of a *higher-order power*.

Definition 2.1. We say that a section $\mathcal{P}$ of the cotangent bundle $T^*\mathscr{D}$ is a *power of order* k, or k-*power*, if there are $\{A^{(i)}\}_{i=0}^k$ vector fields on $\mathscr{D}$, with values in $L^2(\Omega;\mathbb{R}^{n^i})$, such that

$$\langle \mathcal{P}_u, v\rangle = \sum_{i=0}^{k} \int_\Omega A_u^{(i)} \cdot \nabla^i v \,, \tag{2.3}$$

for any $v \in T_u\mathscr{D}$, where $A_u^{(i)}$ denotes the vector field evaluated at $u \in \mathscr{D}$.

If we now assume that $\mathscr{D} = T_u\mathscr{D} = H^{k_1}(\Omega) \times \ldots \times H^{k_D}(\Omega)$, $D \in \mathbb{N}$, the previous definition can be generalized.

Definition 2.2. We call a $(k_1, \ldots, k_D)$-*power* a section $\mathcal{P}$ of the cotangent bundle $T^*\mathscr{D}$ such that, for any $v = (v_s)_{s=1}^D \in T_u\mathscr{D}$,

$$\langle \mathcal{P}_u, v\rangle = \sum_{s=1}^{D} \sum_{i=0}^{k_s} \int_\Omega A_u^{(i,s)} \cdot \nabla^i v_s \,, \tag{2.4}$$

for some vector fields $A^{(i,s)}$ on $\mathscr{D}$, with values in $L^2(\Omega;\mathbb{R}^{n^i})$.

The last definition is relevant if one needs different Sobolev regularities for the different descriptors which enter $\mathscr{D}$; on the other hand, if $k_1 = \ldots = k_D = k$, I will write 'k-power' for '$(k, \ldots, k)$-power'.

Obviously, there are a lot of continuum mechanical models for which $\mathscr{D}$ is formed by more complicated Hilbert spaces, or even with Banach spaces. In such cases there can still be canonical integral representations for the power expenditures, but it is hard (and not very useful) to classify them introducing new notions of order of the power. There is also a little chance, if compared to the possibility given by Definition 2.2, to find a continuum mechanical theory completely described by a k-power. Nevertheless, this is the case for second-gradient materials, which have been widely studied in recent years, and one of which is the subject of the following chapters. Within such theories we have $n = D = 3$ (or equal to 2) and a $(2, 2, 2)$-power acting on a sole vectorial descriptor, which usually represents a displacement or a velocity field. I will present below some additional results about these particular cases.

2.1.1 Galilean covariance, frame indifference, and consequences

Let me now carry out the consequences of some invariance principles within a continuum mechanical theory defined by a k-power. As underlying space we take $[t_I, t_F] \times \hat{\Omega}$ with $\hat{\Omega} \subseteq \mathbb{R}^3$ open, and as descriptor we have a velocity vector field $\mathsf{u} \in L^2([t_I, t_F]; H^k(\hat{\Omega};\mathbb{R}^3))$. The k-power is therefore defined by

$$\langle \mathcal{P}_\mathsf{u}, \mathsf{v}\rangle = \sum_{i=0}^{k} \int_{t_I}^{t_F} \int_{\hat{\Omega}} \mathsf{A}_\mathsf{u}^{(i)} \cdot \nabla^i \mathsf{v} \,, \tag{2.5}$$

for any $\mathsf{v} \in L^2([t_I, t_F]; H^k(\hat{\Omega};\mathbb{R}^3))$, where $\{\mathsf{A}^{(i)}\}_{i=0}^k$ are vector fields on $L^2([t_I, t_F]; H^k(\hat{\Omega};\mathbb{R}^3))$, with values in $L^2([t_I, t_F]; L^2(\hat{\Omega}; (\mathbb{R}^3)^{i+1}))$. Notice that ∇ represents only the gradient with

respect to the spatial variables, and that (2.5) does not perfectly fit into Definition 2.2, but we can still speak about a k-power, thanks to the fact that the L^2 regularity in time keeps the form of the power very similar to that of a genuine k-power.

Given an orthonormal Euclidean basis on $\mathbb{R}^4$, we obtain a frame of reference considering times along the direction of the first base vector, and $\hat{\Omega}$ as a subset of the orthogonal complement. We stipulate that any "nice" frame of reference can be obtained by direct isometries of the three-dimensional subspace containing $\hat{\Omega}$ (that is translations and rotations, but not reflections with respect to any plane), by translations of the time-axis, and by a rectilinear motion of the new origin, with uniform velocity s. Those transformations form the Galilean group G of *inertial* frame transformations.

Take now $\phi \in G$, and let (t, x) be the vector of the components of a point $P \in \Omega$ with respect to the original basis fixed on $\mathbb{R}^4$. We have

$$(t, \mathsf{x})^\phi = (t + a, \mathsf{Q}^\dagger(\mathsf{x} - t\mathsf{s}) + \mathsf{a}) \,,$$

for some $a \in \mathbb{R}$, $\mathsf{a}, \mathsf{s} \in \mathbb{R}^3$ representing different kinds of translations, and some $\mathsf{Q} \in SO_3(\mathbb{R})$ representing the rotation of the basis (I denote by $\mathsf{Q}^\dagger$ the transpose of Q). If we consider only transformations of the previous kind, with $a = 0$ and $\mathsf{s} = \mathsf{0}$, we obtain the Euclidean subgroup $E \subseteq G$. Since the virtual velocity field at any instant t is a section of the tangent bundle $T\hat{\Omega}$, it is an Euclidean contravariant vector field; moreover, since we consider the Euclidean connection on $T\hat{\Omega}$, at any instant, the k-th gradient $\nabla^k \mathsf{v}$ is a tensor field on $\hat{\Omega}$ with k covariant components and one contravariant component, so that we get the following transformation laws:

$$\mathsf{v}^\phi(t^\phi, \mathsf{x}^\phi) = \mathsf{Q}^\dagger \mathsf{v}(t, \mathsf{x}) \,, \qquad [\nabla^k \mathsf{v}]^\phi(t^\phi, \mathsf{x}^\phi) = \mathsf{Q}^\dagger [\nabla^k \mathsf{v}(t, \mathsf{x})] \mathsf{Q}^k \,. \tag{2.6}$$

The requirement of Euclidean covariance for the power expenditure (2.5) becomes

$$\sum_{i=0}^{k} \int_{t_I}^{t_F} \int_{\hat{\Omega}^\phi} [\mathsf{A}_\mathsf{u}^{(i)} \cdot \nabla^i \mathsf{v}]^\phi \, d\mathcal{L}^3(\mathsf{x}^\phi) \, d\mathcal{L}^1(t) = \sum_{i=0}^{k} \int_{t_I}^{t_F} \int_{\hat{\Omega}} \mathsf{A}_\mathsf{u}^{(i)} \cdot \nabla^i \mathsf{v} \, d\mathcal{L}^3(\mathsf{x}) \, d\mathcal{L}^1(t)$$

for any $\phi \in E$, since it is a scalar field; hence, taking into account the transformation laws (2.6), we have

$$\sum_{i=0}^{k} \int_{t_I}^{t_F} \int_{\hat{\Omega}} [\mathsf{A}_\mathsf{u}^{(i)}]^\phi \cdot [\mathsf{Q}^\dagger (\nabla^k \mathsf{v}) \mathsf{Q}^k] \, d\mathcal{L}^3(\mathsf{x}) \, d\mathcal{L}^1(t) = \sum_{i=0}^{k} \int_{t_I}^{t_F} \int_{\hat{\Omega}} \mathsf{A}_\mathsf{u}^{(i)} \cdot \nabla^i \mathsf{v} \, d\mathcal{L}^3(\mathsf{x}) \, d\mathcal{L}^1(t)$$

which implies $[\mathsf{A}_\mathsf{u}^{(i)}]^\phi = \mathsf{Q}^\dagger (\mathsf{A}_\mathsf{u}^{(i)}) \mathsf{Q}^k$, that is, each $\mathsf{A}_\mathsf{u}^{(i)}$ must be, at any time, a tensor field on $\hat{\Omega}$ with k covariant components and one contravariant component. This is what Euclidean covariance implies, and it must be satisfied when proposing constitutive prescriptions for the power, by assigning the dependence of each $\mathsf{A}_\mathsf{u}^{(i)}$ on the base point $\mathsf{u} \in \mathscr{D}$. Thanks to the peculiar role of the time variable, Galilean covariance does not add relevant constraints to those implied by Euclidean covariance, hence I will not insist on them.

Within the context of Continuum Mechanics it is customary to require a much stronger invariance, called *frame indifference*. It is the invariance of the power expenditure (2.5) with respect to any frame transformation of the form

$$(t, \mathsf{x})^\phi = (t, \mathsf{Q}_t^\dagger \mathsf{x} + \mathsf{a}_t) \,. \tag{2.7}$$

Notice that a_t and Q_t depend on the time instant t, and hence the frame of reference produced by (2.7) is generically non-inertial, though many inertial frames can be described by a transformation of that kind. I will denote by R the group of transformations defined by (2.7), which are called *rigid motions*, since they correspond to possible evolutions of a rigid body.

Definition 2.3. Given a scalar field f, a vector field b and tensor fields B_i of order $i+1$, we say that they are *frame indifferent* (that is, respectively, R-invariant, R-contravariant, i times R-covariant and one time R-contravariant) if

$$\begin{aligned} f^\sigma(t,\mathsf{x}^\sigma) &= f(t,\mathsf{x})\,,\\ \mathsf{b}^\sigma(t,\mathsf{x}^\sigma) &= \mathsf{Q}_t^\dagger \mathsf{b}(t,\mathsf{x})\,,\\ \mathsf{B}_i^\sigma(t,\mathsf{x}^\sigma) &= \mathsf{Q}_t^\dagger \mathsf{B}_i(t,\mathsf{x})\mathsf{Q}_t^i\,, \end{aligned}$$

for any $\sigma \in R$.

Obviously, Euclidean covariant or contravariant tensors need not be covariant or contravariant with respect to the action of the group R of all rigid motions. Indeed, given $\sigma \in R$, we have the following transformation rules for a velocity field and its gradient:

$$\mathsf{v}^\sigma(t,\mathsf{x}^\sigma) = \mathsf{Q}_t^\dagger \mathsf{v}(t,\mathsf{x}) + \dot{\mathsf{a}}_t + \dot{\mathsf{Q}}_t^\dagger \mathsf{x}\,, \tag{2.8}$$

$$[\nabla \mathsf{v}]^\sigma(t,\mathsf{x}^\sigma) = \mathsf{Q}_t^\dagger [\nabla \mathsf{v}(t,\mathsf{x})]\mathsf{Q}_t + \dot{\mathsf{Q}}_t^\dagger \mathsf{Q}_t\,. \tag{2.9}$$

Notice that $\mathsf{w}_\sigma := \mathsf{Q}_t\dot{\mathsf{a}}_t + \mathsf{Q}_t\dot{\mathsf{Q}}_t^\dagger \mathsf{x}$ is a *rigid velocity field*, associated with the rigid motion σ; moreover, differentiating the relation $\mathsf{Q}_t^\dagger\mathsf{Q}_t = \mathsf{I}$, we get

$$\dot{\mathsf{Q}}_t^\dagger\mathsf{Q}_t + \mathsf{Q}_t^\dagger\dot{\mathsf{Q}}_t = 0\,,$$

so that $\Omega_t := \dot{\mathsf{Q}}_t^\dagger\mathsf{Q}_t$ is skewsymmetric, and, similarly, we get also $\mathsf{Q}_t\dot{\mathsf{Q}}_t^\dagger$ skewsymmetric. On the other hand, for $k \geqslant 2$, the k-th gradients of the velocity field are frame indifferent.

Based on the previous considerations, we obtain the following results.

Proposition 2.1. *Fix $u \in \mathscr{D}$; given the power expenditure (2.5), assume that the Euclidean tensor fields $\{\mathsf{A}_\mathsf{u}^{(i)}\}_{i=0}^k$ are frame indifferent. Then the power expenditure (2.5) is frame indifferent if and only if $\langle \mathcal{P}_\mathsf{u}, \mathsf{w}\rangle = 0$ for any rigid velocity field w.*

Proof. Let $\sigma \in R$ be a rigid motion and let w_σ be the associated rigid velocity field. By the frame indifference of each $\mathsf{A}_\mathsf{u}^{(i)}$, we have

$$\sum_{i=0}^k \int_{t_I}^{t_F}\int_{\hat\Omega} [\mathsf{A}_\mathsf{u}^{(i)}]^\sigma \cdot [\nabla^i \mathsf{v}]^\sigma = \sum_{i=0}^k \int_{t_I}^{t_F}\int_{\hat\Omega} \mathsf{A}_\mathsf{u}^{(i)} \cdot \nabla^i \mathsf{v} + \int_{t_I}^{t_F}\int_{\hat\Omega} \mathsf{A}_\mathsf{u}^{(0)} \cdot \mathsf{w}_\sigma + \int_{t_I}^{t_F}\int_{\hat\Omega} \mathsf{A}_\mathsf{u}^{(1)} \cdot \nabla \mathsf{w}_\sigma\,,$$

that is

$$\langle \mathcal{P}_\mathsf{u}^\sigma, \mathsf{v}^\sigma\rangle^\sigma = \langle \mathcal{P}_\mathsf{u}, \mathsf{v}\rangle + \langle \mathcal{P}_\mathsf{u}, \mathsf{w}_\sigma\rangle\,,$$

from which the assertion follows. □

Proposition 2.2. *Fixed $u \in \mathscr{D}$, if the power expenditure (2.5) and the Euclidean tensor fields $\{\mathsf{A}_\mathsf{u}^{(i)}\}_{i=0}^k$ are frame indifferent, then $\mathsf{A}_\mathsf{u}^{(0)} = 0$, and $(\mathsf{A}_\mathsf{u}^{(1)})^\dagger = \mathsf{A}_\mathsf{u}^{(1)}$ almost everywhere on Ω.*

Proof. By the previous proposition, we can choose any rigid velocity field of the form $\mathsf{w} = \bar{\mathsf{w}}$ constant, and we have

$$0 = \langle \mathcal{P}_{\mathsf{u}}, \bar{\mathsf{w}} \rangle = \int_{t_I}^{t_F} \int_{\hat{\Omega}} \mathsf{A}_{\mathsf{u}}^{(0)} \cdot \bar{\mathsf{w}} \, ;$$

by the arbitrariness of $\bar{\mathsf{w}}$, we get $\mathsf{A}_{\mathsf{u}}^{(0)} = 0$ almost everywhere on Ω.

Taking now any uniformly rotating rigid velocity $\mathsf{w} = \mathsf{W}\mathsf{x}$, with $\mathsf{W} \in \mathrm{Mat}_n(\mathbb{R})$ skewsymmetric, we get

$$0 = \langle \mathcal{P}_{\mathsf{u}}, \mathsf{W}\mathsf{x} \rangle = \int_{t_I}^{t_F} \int_{\hat{\Omega}} \mathsf{A}_{\mathsf{u}}^{(1)} \cdot \mathsf{W} \, ;$$

by the arbitrariness of W, we get $(\mathsf{A}_{\mathsf{u}}^{(1)})^\dagger = \mathsf{A}_{\mathsf{u}}^{(1)}$ almost everywhere on Ω. □

2.1.2 Interactions at the boundary

Now I will introduce an equivalent representation for the k-power expenditure

$$\sum_{i=0}^{k} \int_{\hat{\Omega}} \mathsf{A}_{\mathsf{u}}^{(i)} \cdot \nabla^i \mathsf{v} \tag{2.10}$$

in terms of boundary integrals. In order to obtain such a representation I need to make some regularity assumptions on $\partial\hat{\Omega}$, and to assume $\mathsf{A}_{\mathsf{u}}^{(i)} \in H^i(\hat{\Omega}; (\mathbb{R}^3)^{i+1})$.

Proposition 2.3. *Let $\hat{\Omega} \subseteq \mathbb{R}^3$ be an open bounded domain with smooth boundary and let n denote the unit outer normal to $\partial\hat{\Omega}$; then there exist Euclidean tensor fields $\hat{\mathsf{b}}_{\mathsf{u}}$, $\hat{\mathsf{t}}_{\mathsf{u}}$ and, if $k \geqslant 2$, $\{\hat{\mathsf{M}}_{\mathsf{u}}^{(s)}\}_{s=0}^{k-2}$, such that*

$$\sum_{i=0}^{k} \int_{\hat{\Omega}} \mathsf{A}_{\mathsf{u}}^{(i)} \cdot \nabla^i \mathsf{v} = \int_{\hat{\Omega}} \hat{\mathsf{b}}_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\hat{\Omega}} \hat{\mathsf{t}}_{\mathsf{u}} \cdot \mathsf{v} + \sum_{s=0}^{k-2} \int_{\partial\hat{\Omega}} \hat{\mathsf{M}}_{\mathsf{u}}^{(s)} \cdot \frac{\partial}{\partial n} (\nabla^s \mathsf{v}) \tag{2.11}$$

and such that

$$\hat{\mathsf{b}}_{\mathsf{u}} = \sum_{i=0}^{k} (-\operatorname{div})^i \mathsf{A}_{\mathsf{u}}^{(i)} \, , \tag{2.12}$$

$$\hat{\mathsf{t}}_{\mathsf{u}} = \sum_{i=1}^{k} \sum_{j=1}^{i} (-\operatorname{div}_{\mathcal{S}})^{(j-1)} \{ [(-\operatorname{div})^{(i-j)} \mathsf{A}_{\mathsf{u}}^{(i)}] \mathsf{n} \} \, , \tag{2.13}$$

$$\hat{\mathsf{M}}_{\mathsf{u}}^{(s)} = \sum_{i=s}^{k-2} \sum_{j=0}^{i-s} \{ (-\operatorname{div}_{\mathcal{S}})^{(j)} [[(-\operatorname{div})^{(i-j-s)} \mathsf{A}_{\mathsf{u}}^{(i+2)}] \mathsf{n}] \} \mathsf{n} \, , \tag{2.14}$$

where div *is the divergence operator, $\frac{\partial}{\partial n}$ is the normal derivative, and* $\operatorname{div}_{\mathcal{S}}$ *is the surface-divergence operator.*[(1)]

Proof. The proof repeatedly exploits integration by parts, divergence theorem, the decomposition on $\partial\hat{\Omega}$ of the gradient operator in surface and normal part, and the surface-divergence

[(1)] Definitions and basic properties are collected in Appendix B.

theorem B.1. By means of such manipulations, it is easy to verify (2.11) for $k = 0, 1$; moreover, I will show by induction on l that, given two tensor fields B and S (with the right tensorial order), for any $l \geqslant 2$, we have

$$\int_{\partial\hat{\Omega}} \mathsf{S} \cdot \nabla^l \mathsf{v} = \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^l \mathsf{S} \cdot \mathsf{v} + \sum_{s=0}^{l-1} \int_{\partial\hat{\Omega}} [(-\operatorname{div}_{\mathcal{S}})^{l-1-s} \mathsf{S}]\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \,, \tag{2.15}$$

and

$$\begin{aligned}\int_{\hat{\Omega}} \mathsf{B} \cdot \nabla^l \mathsf{v} = \int_{\hat{\Omega}} (-\operatorname{div})^l \mathsf{B} \cdot \mathsf{v} + \sum_{j=1}^{l} \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^{(j-1)} \{[(-\operatorname{div})^{(l-j)} \mathsf{B}]\mathsf{n}\} \cdot \mathsf{v} \\ + \sum_{s=0}^{l-2} \sum_{j=1}^{l-s-1} \int_{\partial\hat{\Omega}} \{(-\operatorname{div}_{\mathcal{S}})^{(j-1)} [[(-\operatorname{div})^{(l-j-1-s)} \mathsf{B}]\mathsf{n}]\}\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \,.\end{aligned} \tag{2.16}$$

For $l = 2$ it is immediate to verify (2.15) and (2.16). Assuming that they hold for some $l \geqslant 2$, we have

$$\begin{aligned}&\int_{\partial\hat{\Omega}} \hat{\mathsf{S}} \cdot \nabla^{l+1} \mathsf{v} = \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}} \hat{\mathsf{S}}) \cdot \nabla^l \mathsf{v} + \int_{\partial\hat{\Omega}} \hat{\mathsf{S}}\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^l \mathsf{v}) \\ &= \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^l (-\operatorname{div}_{\mathcal{S}} \hat{\mathsf{S}}) \cdot \mathsf{v} + \sum_{s=0}^{l-1} \int_{\partial\hat{\Omega}} [(-\operatorname{div}_{\mathcal{S}})^{l-1-s} (-\operatorname{div}_{\mathcal{S}} \hat{\mathsf{S}})]\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) + \int_{\partial\hat{\Omega}} \hat{\mathsf{S}}\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^l \mathsf{v}) \\ &= \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^{l+1} \hat{\mathsf{S}} \cdot \mathsf{v} + \sum_{s=0}^{l} \int_{\partial\hat{\Omega}} [(-\operatorname{div}_{\mathcal{S}})^{l-s} \hat{\mathsf{S}}]\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \,,\end{aligned}$$

which establishes (2.15); consequently, substituting $(\hat{\mathsf{B}}\mathsf{n})$ for S,

$$\begin{aligned}&\int_{\hat{\Omega}} \hat{\mathsf{B}} \cdot \nabla^{l+1} \mathsf{v} = \int_{\hat{\Omega}} (-\operatorname{div} \hat{\mathsf{B}}) \cdot \nabla^l \mathsf{v} + \int_{\partial\hat{\Omega}} \hat{\mathsf{B}}\mathsf{n} \cdot \nabla^l \mathsf{v} \\ &\quad = \int_{\hat{\Omega}} (-\operatorname{div})^l (-\operatorname{div} \hat{\mathsf{B}}) \cdot \mathsf{v} + \sum_{j=1}^{l} \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^{(j-1)} \{[(-\operatorname{div})^{(l-j)} (-\operatorname{div} \hat{\mathsf{B}})]\mathsf{n}\} \cdot \mathsf{v} \\ &\qquad + \sum_{s=0}^{l-2} \sum_{j=1}^{l-s-1} \int_{\partial\hat{\Omega}} \{(-\operatorname{div}_{\mathcal{S}})^{(j-1)} [[(-\operatorname{div})^{(l-j-1-s)} (-\operatorname{div} \hat{\mathsf{B}})]\mathsf{n}]\}\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \\ &\qquad\quad + \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^l (\hat{\mathsf{B}}\mathsf{n}) \cdot \mathsf{v} + \sum_{s=0}^{l-1} \int_{\partial\hat{\Omega}} [(-\operatorname{div}_{\mathcal{S}})^{l-1-s} (\hat{\mathsf{B}}\mathsf{n})]\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \\ &\qquad\quad = \int_{\hat{\Omega}} (-\operatorname{div})^{l+1} \hat{\mathsf{B}} \cdot \mathsf{v} + \sum_{j=1}^{l+1} \int_{\partial\hat{\Omega}} (-\operatorname{div}_{\mathcal{S}})^{(j-1)} \{[(-\operatorname{div})^{(l+1-j)} \hat{\mathsf{B}}]\mathsf{n}\} \cdot \mathsf{v} \\ &\qquad\qquad + \sum_{s=0}^{l-1} \sum_{j=1}^{l-s} \int_{\partial\hat{\Omega}} \{(-\operatorname{div}_{\mathcal{S}})^{(j-1)} [[(-\operatorname{div})^{(l-j-s)} \hat{\mathsf{B}}]\mathsf{n}]\}\mathsf{n} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) \,,\end{aligned}$$

and also (2.16) is established.

Now it only remains to apply (2.16), substituting i for l and $\mathsf{A}_{\mathsf{u}}^{(i)}$ for B, and then to sum up the identities for $i = 0, \dots, k$, to obtain the representation (2.11). □

I will now consider what happens if the $\partial\hat\Omega$ is only piecewise smooth.

Proposition 2.4. *Let $\hat\Omega \subseteq \mathbb{R}^3$ be an open bounded domain with piecewise smooth boundary, let n denote the unit outer normal to the regular part of $\partial\hat\Omega$, and let $\mathcal{E}$ denote the singular part of $\partial\hat\Omega$; then there exist Euclidean tensor fields $\hat{\mathsf{b}}_\mathsf{u}$, $\hat{\mathsf{t}}_\mathsf{u}$, and, if $k \geqslant 2$, $\{\hat{\mathsf{M}}^{(s)}_\mathsf{u}\}_{s=0}^{k-2}$ and $\{\hat{\mathsf{K}}^{(s)}_\mathsf{u}\}_{s=0}^{k-2}$, such that*

$$\sum_{i=0}^{k}\int_{\hat\Omega}\mathsf{A}^{(i)}_\mathsf{u}\cdot\nabla^i\mathsf{v} = \int_{\hat\Omega}\hat{\mathsf{b}}_\mathsf{u}\cdot\mathsf{v} + \int_{\partial\hat\Omega}\hat{\mathsf{t}}_\mathsf{u}\cdot\mathsf{v} + \sum_{s=0}^{k-2}\int_{\partial\hat\Omega}\hat{\mathsf{M}}^{(s)}_\mathsf{u}\cdot\frac{\partial}{\partial n}(\nabla^s\mathsf{v}) + \sum_{s=0}^{k-2}\int_{\mathcal{E}}\hat{\mathsf{K}}^{(s)}_\mathsf{u}\cdot\nabla^s\mathsf{v} \tag{2.17}$$

and such that $\hat{\mathsf{b}}_\mathsf{u}$, $\hat{\mathsf{t}}_\mathsf{u}$, $\{\hat{\mathsf{M}}^{(s)}_\mathsf{u}\}_{s=0}^{k-2}$ satisfy relations (2.12), (2.13), (2.14) respectively, and

$$\begin{aligned}\hat{\mathsf{K}}^{(s)}_\mathsf{u} = \sum_{i=s}^{k-2}\sum_{j=0}^{i-s}\{(-\operatorname{div}_\mathcal{S})^{(j)}[[(-\operatorname{div})^{(i-j-s)}\mathsf{A}^{(i+2)}_\mathsf{u}]\mathsf{n}_a]\}\mathsf{e}_a \\ + \sum_{i=s}^{k-2}\sum_{j=0}^{i-s}\{(-\operatorname{div}_\mathcal{S})^{(j)}[[(-\operatorname{div})^{(i-j-s)}\mathsf{A}^{(i+2)}_\mathsf{u}]\mathsf{n}_b]\}\mathsf{e}_b\,,\end{aligned} \tag{2.18}$$

where n_a and n_b are the limits of n coming from the two sides of an edge in $\mathcal{E}$, and e_a and e_b are unit vector fields orthogonal to $\mathcal{E}$ and to n_a and n_b respectively, and pointing outward the a- and b-face respectively.

Proof. Since $\partial\hat\Omega$ is piecewise smooth, we have $\partial\hat\Omega = \bigcup_k \mathcal{S}_k$ with $\mathcal{S}_k$ a smooth subsurface whose unit outer normal will be denoted by n_k, and $\mathcal{E} = \bigcup_k \partial\mathcal{S}_k$. Hence, for any $s \geqslant 1$ and any tensor field T, decomposing the gradient on the surfaces as $\nabla = \nabla_\mathcal{S} + (\mathsf{n}_k\otimes\mathsf{n}_k)\nabla$,

$$\int_{\partial\hat\Omega}\mathsf{T}\mathsf{n}\cdot\nabla^s\mathsf{v} = \sum_k\int_{\mathcal{S}_k}\mathsf{T}\mathsf{n}_k\cdot\nabla^s\mathsf{v} = \sum_k\int_{\mathcal{S}_k}\mathsf{T}\mathsf{n}_k\cdot\nabla_\mathcal{S}(\nabla^{s-1}\mathsf{v}) + \sum_k\int_{\mathcal{S}_k}\mathsf{T}\mathsf{n}_k\mathsf{n}_k\cdot\frac{\partial}{\partial n}(\nabla^{s-1}\mathsf{v})\,.$$

Denote now by e_k the outer unit normal to $\partial\mathcal{S}_k$ orthogonal to n_k; applying the surface-divergence theorem on each $\mathcal{S}_k$ it follows

$$\begin{aligned}\sum_k\int_{\mathcal{S}_k}\mathsf{T}\mathsf{n}_k\cdot\nabla_\mathcal{S}(\nabla^{s-1}\mathsf{v}) &= -\sum_k\int_{\mathcal{S}_k}\operatorname{div}_\mathcal{S}(\mathsf{T}\mathsf{n}_k)\cdot\nabla^{s-1}\mathsf{v} + \sum_k\int_{\partial\mathcal{S}_k}\mathsf{T}\mathsf{n}_k\mathsf{e}_k\cdot\nabla^{s-1}\mathsf{v} \\ &= -\int_{\partial\hat\Omega}\operatorname{div}_\mathcal{S}(\mathsf{T}\mathsf{n})\cdot\nabla^{s-1}\mathsf{v} + \int_\mathcal{E}(\mathsf{T}\mathsf{n}_a\mathsf{e}_a + \mathsf{T}\mathsf{n}_b\mathsf{e}_b)\cdot\nabla^{s-1}\mathsf{v}\,,\end{aligned}$$

where $\mathsf{n}_a, \mathsf{e}_a$ and $\mathsf{n}_b, \mathsf{e}_b$ are related to the two faces $\mathcal{S}_a$ and $\mathcal{S}_b$ which meet along any edge in $\mathcal{E}$.

Hence we have

$$\int_{\partial\hat\Omega}\mathsf{T}\mathsf{n}\cdot\nabla^s\mathsf{v} = -\int_{\partial\hat\Omega}\operatorname{div}_\mathcal{S}(\mathsf{T}\mathsf{n})\cdot\nabla^{s-1}\mathsf{v} + \int_{\partial\hat\Omega}\mathsf{T}\mathsf{n}\mathsf{n}\cdot\frac{\partial}{\partial n}(\nabla^{s-1}\mathsf{v}) + \int_\mathcal{E}(\mathsf{T}\mathsf{n}_a\mathsf{e}_a + \mathsf{T}\mathsf{n}_b\mathsf{e}_b)\cdot\nabla^{s-1}\mathsf{v}\,.$$

Once established the last relation, the proof is similar to that of Proposition 2.3. □

If we consider the case $k = 2$, we obtain the following representation for a second-order power:

$$\sum_{i=0}^{k}\int_{\hat\Omega}\mathsf{A}^{(i)}_\mathsf{u}\cdot\nabla^i\mathsf{v} = \int_{\hat\Omega}\hat b_\mathsf{u}\cdot\mathsf{v} + \int_{\partial\hat\Omega}\hat{\mathsf{t}}_\mathsf{u}\cdot\mathsf{v} + \int_{\partial\hat\Omega}\hat{\mathsf{m}}_\mathsf{u}\cdot\frac{\partial\mathsf{v}}{\partial n} + \int_\mathcal{E}\hat{\mathsf{k}}_\mathsf{u}\cdot\mathsf{v} \tag{2.19}$$

with

$$\hat{\mathsf{b}}_{\mathsf{u}} = \mathsf{A}_{\mathsf{u}}^{(0)} - \operatorname{div} \mathsf{A}_{\mathsf{u}}^{(1)} + \operatorname{div}\operatorname{div} \mathsf{A}_{\mathsf{u}}^{(2)}\,, \tag{2.20}$$

$$\hat{\mathsf{t}}_{\mathsf{u}} = [\mathsf{A}_{\mathsf{u}}^{(1)} - \operatorname{div} \mathsf{A}_{\mathsf{u}}^{(2)}]\mathsf{n} + \operatorname{div}_{\mathcal{S}}[\mathsf{A}_{\mathsf{u}}^{(2)}\mathsf{n}]\,, \tag{2.21}$$

$$\hat{\mathsf{m}}_{\mathsf{u}} = \mathsf{A}_{\mathsf{u}}^{(2)}\mathsf{n}\mathsf{n}\,, \tag{2.22}$$

$$\hat{\mathsf{k}}_{\mathsf{u}} = \mathsf{A}_{\mathsf{u}}^{(2)}\mathsf{n}_a\mathsf{e}_a + \mathsf{A}_{\mathsf{u}}^{(2)}\mathsf{n}_b\mathsf{e}_b\,. \tag{2.23}$$

This corresponds to the results in [12] and [31].

Thanks to Propositions 2.3–2.4, we see that, once an integral representation for internal interactions is given in terms of volume integrals, it is possible to split it in bulk and surface contributions. This splitting, applied to higher-order powers, produces a number of surface terms which, containing the derivatives of n, are related to the geometry of the boundary in a way which is forbidden in the classical theories, based on first-order powers. Moreover, those terms expend power at the boundary on the normal derivative of the virtual velocity and of its gradients.

Remark 2.1. Relations (2.11) and (2.17) are representations for a k-power, initially defined by tensor fields on $\hat{\Omega}$, through some *interaction fields* which are not defined on $\hat{\Omega}$, with the exception of $\hat{\mathsf{b}}_{\mathsf{u}}$. In particular, $\hat{\mathsf{t}}_{\mathsf{u}}$ and $\{\hat{\mathsf{M}}_{\mathsf{u}}^{(s)}\}_{s=0}^{k-2}$ are defined on $\partial\hat{\Omega}$, and $\{\hat{\mathsf{K}}_{\mathsf{u}}^{(s)}\}_{s=0}^{k-2}$ are defined only on $\mathcal{E}$. Notice that equations (2.11) and (2.17) have been proved only for tensors $\mathsf{A}_{\mathsf{u}}^{(i)}$ regular enough, but we cannot in general expect this regularity.

Nevertheless, the integral representation (2.10) is equivalent to an interaction-fields representation

$$\int_{\hat{\Omega}} \hat{\mathsf{b}}_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\hat{\Omega}} \hat{\mathsf{t}}_{\mathsf{u}} \cdot \mathsf{v} + \sum_{s=0}^{k-2} \int_{\partial\hat{\Omega}} \hat{\mathsf{M}}_{\mathsf{u}}^{(s)} \cdot \frac{\partial}{\partial n}(\nabla^s \mathsf{v}) + \sum_{s=0}^{k-2} \int_{\mathcal{E}} \hat{\mathsf{K}}_{\mathsf{u}}^{(s)} \cdot \nabla^s \mathsf{v}\,. \tag{2.24}$$

in more general situations. Indeed, if (2.24) defines a linear continuous form on the space of virtual velocities, Riesz's representation theorem implies the existence of tensor fields $\mathsf{A}_{\mathsf{u}}^{(i)} \in L^2(\hat{\Omega}; (\mathbb{R}^3)^{i+1})$ such that (2.17) holds. Now, since trace operators[(2)] are linear and continuous, we easily see that it is enough to require

$$\begin{aligned} &\hat{\mathsf{b}}_{\mathsf{u}} \in (H^k(\hat{\Omega}; \mathbb{R}^3))'\,, && \hat{\mathsf{M}}_{\mathsf{u}}^{(s)} \in (H^{k-s-\frac{3}{2}}(\partial\hat{\Omega}; (\mathbb{R}^3)^{s+1}))'\,, \\ &\hat{\mathsf{t}}_{\mathsf{u}} \in (H^{k-\frac{1}{2}}(\partial\hat{\Omega}; \mathbb{R}^3))'\,, && \hat{\mathsf{K}}_{\mathsf{u}}^{(s)} \in (H^{k-s-1}(\mathcal{E}; (\mathbb{R}^3)^{s+1}))'\,, \end{aligned} \tag{2.25}$$

in order to obtain the interaction-fields representation of a k-power expenditure. ♢

2.2 Representation of the internal contact interactions

In the previous section I gave a representation for a k-power, initially defined by tensor fields on $\hat{\Omega}$, through some interaction fields, which are not all defined on $\hat{\Omega}$. A natural question is whether it be possible to give a representation akin to (2.17) for any subregion contained in $\hat{\Omega}$, obtaining interaction fields defined on the whole closure of $\hat{\Omega}$.

Many authors contributed to the study of such representations, and the main ingredient is to require, on a suitable class of subbodies, the same kind of balances which are imposed on

[(2)] For the theory of traces of Sobolev functions and the related trace spaces see [2] or [33].

the whole body. The definition of the class of subbodies is still not universal, and a discussion on it would involve notions of Geometric Measure Theory which I do not want to introduce. I will just state, in the present context, a result [8, Theorem 7] which provides a representation for second-order power expenditures on any $M \subseteq \Omega$ *with curvature measure*[3]. The simplest example of such a set is a cube: the curvature of its surface is a measure, since it can be seen as a Dirac's delta function with support along the edges of the cube.

Definition 2.4. Given the integral representation (2.10) for a power expenditure, its restriction to $M \subseteq \hat{\Omega}$ is

$$\langle \mathcal{P}_{\mathsf{u}}|_M, \mathsf{v}\rangle = \sum_{i=0}^{k} \int_M \mathsf{A}_{\mathsf{u}}^{(i)} \cdot \nabla^i \mathsf{v}\,. \tag{2.26}$$

Moreover, the restriction of a power $\mathcal{P}_{\mathsf{u}}|_M$ is a *contact power* if $\langle \mathcal{P}_{\mathsf{u}}|_M, \mathsf{v}\rangle = 0$ for any virtual velocity v with compact support contained in M.

In what follows, the notation $\mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{B}]$ means, in components, $(\mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{B}])_i = \sum_{j,k}(\mathsf{A}_{\mathsf{u}}^{(2)})_{ijk}\mathsf{B}_{jk}$.

Theorem 2.5. *Let $\mathcal{P}_{\mathsf{u}}|_M$ be a contact power of order 2. Then, for any $M \subseteq \hat{\Omega}$ with curvature measure, we have*

$$\langle \mathcal{P}_{\mathsf{u}}|_M, \mathsf{v}\rangle = \int_{\partial_* M} \{[\mathsf{A}_{\mathsf{u}}^{(1)} - \operatorname{div} \mathsf{A}_{\mathsf{u}}^{(2)}]\mathsf{n} + \operatorname{div}_{\mathcal{S}}[\mathsf{A}_{\mathsf{u}}^{(2)}\mathsf{n}]\} \cdot \mathsf{v} + \int_{\partial_* M} \mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{n}\otimes\mathsf{n}] \cdot \frac{\partial \mathsf{v}}{\partial n} + \int_{\partial_* M} \mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{U}] \cdot \mathsf{v}\, d\lambda_M\,, \tag{2.27}$$

where $\partial_* M$ is the measure theoretic boundary of M, $\mathsf{U}(\mathsf{x})$ is a unimodular matrix for λ_M-almost every $\mathsf{x} \in \partial_* M$, and λ_M is a measure, possibly singular with respect to the surface measure.

It is clear that the possibility of developing interactions which are concentrated along edges is related to the nature of the last integral in (2.27). If M has a piecewise smooth boundary, λ_M turns out to be exactly the one-dimensional Hausdorff measure concentrated on $\mathcal{E}$, and, with the notations of Proposition 2.4, we have

$$\mathsf{U} = \mathsf{n}_a \otimes \mathsf{e}_a + \mathsf{n}_b \otimes \mathsf{e}_b\,. \tag{2.28}$$

The previous result has a nice corollary, pointed out also in [31], related to the existence of second-order materials which cannot develop interactions concentrated along edges of any subbody.

Corollary 2.6. *Let $\mathcal{P}_{\mathsf{u}}|_M$ be a contact power of order 2, and let I denote the identity matrix. If the tensor field $\mathsf{A}_{\mathsf{u}}^{(2)}$ is of the form $\beta(\mathsf{g}\otimes\mathsf{I})$ for some vector g and some constant β, then we have $\mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{U}] = 0$, and there is no concentrated interaction along the edges of any subbody M with curvature measure.*

Proof. Assume that the boundary of M is piecewise smooth, so that (2.28) holds; since, by construction, $\mathsf{n}_a \perp \mathsf{e}_a$ and $\mathsf{n}_b \perp \mathsf{e}_b$, it is clear that the trace of U is equal to zero. Hence

$$\mathsf{A}_{\mathsf{u}}^{(2)}[\mathsf{U}] = \beta(\mathsf{g}\otimes\mathsf{I})[\mathsf{U}] = \beta(\mathsf{I}\cdot\mathsf{U})\mathsf{g} = 0\,.$$

It is enough to note that, also in general cases, U is traceless, and the proof is complete. □

[3] For the precise definition see [8, Definition 14].

Chapter 3

Second-order isotropic linear liquids

In the sequel I will always consider incompressible fluids, often referred to as *liquids*. This means that the volume of any portion of the fluid is preserved during the evolution, and it is equivalent to require the Eulerian velocity field u to be solenoidal.

The usual theory of Newtonian liquids aims at modeling viscosity. Since during a rigid motion viscous and ideal fluids display the same behavior, the action of viscous interactions must be related to the shear between adjacent fluid layers, and opposes to such a shear. Hence we say that viscous interactions should be proportional to the symmetric part of the gradient of the velocity field u, and, moreover, they act on the velocity field in order to modify its gradient. This argument classically leads to assume an internal power expenditure of the form

$$\int_\Omega \mathsf{T}_\mathsf{u} \cdot \nabla \mathsf{v} = \int_\Omega \operatorname{div} \mathsf{T}_\mathsf{u} \cdot \mathsf{v} + \int_{\partial\Omega} \mathsf{T}_\mathsf{u} \mathsf{n} \cdot \mathsf{v}\,,$$

where $\Omega \subseteq \mathbb{R}^3$ is the space domain. It should be now clear that it corresponds to a natural representation of linear forms on the Sobolev space $H^1(\Omega;\mathbb{R}^3)$: indeed this is the correct functional setting in order to develop both the existence theory and the finite element approximation of the flow equations for Newtonian liquids.

What if we require the space of velocity field to be H^2 instead of H^1?
There are at least three reasons for such an assumption:

- the velocity field would be continuous, and hence the adherence to one-dimensional structures (or even to points) could be modeled;
- new kinds of boundary interactions could be considered;
- the well-posedness of the corresponding PDEs could be established.

I will then set up a theory for incompressible linear viscous fluids, requiring both the real velocity field u and the virtual velocities v to belong either to $H^2(\Omega;\mathbb{R}^3)$, if steady flows are considered, or to $L^2([0,T];H^2(\Omega;\mathbb{R}^3))$, when evolutionary problems are studied. Since clearly I need second-order power expenditures, I will name such fluids *second-order liquids*.

Now, physical situations are characterized by the balance of the expenditures of the internal and the external power, $\mathcal{P}^{\text{in}}$ and $\mathcal{P}^{\text{ex}}$, and then the following fundamental principle provides the equation to be solved in order to describe the dynamics of the system.

Principle of Virtual Powers. *The motion described by the velocity field* u *is* feasible *if, for every virtual velocity field* v*, we have*

$$\langle \mathcal{P}^{\text{in}}_{\mathsf{u}}, \mathsf{v} \rangle = \langle \mathcal{P}^{\text{ex}}_{\mathsf{u}}, \mathsf{v} \rangle \,, \tag{3.1}$$

for any instant in a time interval.

According to Proposition 2.4 and Remark 2.1, the expenditures of $\mathcal{P}^{\text{in}}$ and $\mathcal{P}^{\text{ex}}$ for second-order fluids can take the equivalent forms:

$$\langle \mathcal{P}^{\text{in}}_{\mathsf{u}}, \mathsf{v} \rangle = \sum_{i=0}^{2} \int_{\Omega} \mathsf{A}^{(i)}_{\mathsf{u}} \cdot \nabla^i \mathsf{v} = \int_{\Omega} \hat{b}_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\Omega} \hat{\mathsf{t}}_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\Omega} \hat{\mathsf{m}}_{\mathsf{u}} \cdot \frac{\partial \mathsf{v}}{\partial n} + \int_{\mathcal{E}} \hat{\mathsf{k}}_{\mathsf{u}} \cdot \mathsf{v} \,,$$

$$\langle \mathcal{P}^{\text{ex}}_{\mathsf{u}}, \mathsf{v} \rangle = \sum_{i=0}^{2} \int_{\Omega} \mathsf{E}^{(i)}_{\mathsf{u}} \cdot \nabla^i \mathsf{v} = \int_{\Omega} b_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\Omega} \mathsf{t}_{\mathsf{u}} \cdot \mathsf{v} + \int_{\partial\Omega} \mathsf{m}_{\mathsf{u}} \cdot \frac{\partial \mathsf{v}}{\partial n} + \int_{\mathcal{E}} \mathsf{k}_{\mathsf{u}} \cdot \mathsf{v} \,.$$

Clearly, the classical viscous interactions are included in that powers, but new interactions come into play. Some of them have been described as hyperviscous ones, but their physical nature is far from being understood.

Notice that I model as second-order powers both $\mathcal{P}^{\text{in}}$ and $\mathcal{P}^{\text{ex}}$, since, when we model the action of the environment on a system, it is important to make the external world act in a way that *can be experienced* by the system. As to the external power expenditure, using the interaction-fields representation, I introduce:

- an external volume interaction field b_{u};
- external traction and hypertraction fields t_{u} and m_{u} as surface interactions on appropriate smooth parts of $\partial\Omega$;
- the field k_{u} as concentrated interaction on non-smooth parts of the boundary, if any.

We can most of the times drop the dependence of the external volume interaction b_{u} on the base point u, and it is useful to single out its conservative contribution, which plays the role of a pressure term in an incompressible theory. Hence we have

$$\mathsf{b}_{\mathsf{u}} = \mathsf{d} + \nabla f \,,$$

where f is a scalar field and d includes all non-conservative contributions (*e.g.* Coriolis forces) to the volume interactions of the medium. Statements about the surface interactions, concerning boundary conditions, will appear in Section 3.3, while in what follows I will not employ concentrated interactions at the boundary of the whole body Ω. Applying D'Alembert's principle, I also include within the external power expenditure the inertial term

$$-\int_M \rho\, \dot{\mathsf{u}} \cdot \mathsf{v} := -\int_M \rho \left(\frac{\partial \mathsf{u}}{\partial t} + (\mathsf{u} \cdot \nabla)\mathsf{u} \right) \cdot \mathsf{v} \,,$$

which will make the differential problem a nonlinear one.

I will now turn to the task of specifying a model of homogeneous fluid by means of constitutive prescriptions on the internal power expenditure; however, I want to emphasize that also the prescription of the order of the power is a matter of choice, related to the interactions which one wants to model.

3.1 Mechanical constraints

The descriptor of the state at any instant t in the time interval $[0,T] \subset \mathbb{R}$ is the Eulerian velocity field $\mathsf{u}(t,x)$, and the *incompressibility condition*, together with the assumption of homogeneity, allows us to set the mass density $\rho = 1$ identically, giving the first constraint on the velocity:

$$\forall t \in [0,T] : \ \operatorname{div} \mathsf{u} = 0 \,. \tag{3.2}$$

Further prescriptions are related to internal and external powers.

Since viscous interactions should not act during a rigid motion, and since I want to model generalized viscous fluids, I require the second-order internal power expenditure to vanish on any rigid velocity field. Then, by Propositions 2.1–2.2, its general form becomes

$$\langle \mathcal{P}^{\mathrm{in}}_{\mathsf{u}}, \mathsf{v} \rangle = \int_\Omega \mathsf{T} \cdot \nabla \mathsf{v} + \int_M \mathbf{G} \cdot \nabla\nabla \mathsf{v} \,,$$

where T is a symmetric tensor field of order 2 and $\mathbf{G}$ a tensor field of order 3. Linearity and isotropy of the fluid are encoded in the dependence of the tensor fields T and $\mathbf{G}$ on u. It is well-known that, within incompressible theories,

$$\mathsf{T}_{ij} = \mu(\mathsf{u}_{i,j} + \mathsf{u}_{j,i}) - p\,\delta_{ij} \,;$$

besides, in [28, Theorem 1.1] it has been shown that

$$\begin{aligned} \mathbf{G}_{ijk} =& \eta_1 \mathsf{u}_{i,jk} + \eta_2(\mathsf{u}_{j,ki} + \mathsf{u}_{k,ij} - \mathsf{u}_{i,ss}\delta_{jk}) \\ &+ \eta_3(\mathsf{u}_{j,ss}\delta_{ki} + \mathsf{u}_{k,ss}\delta_{ij} - 4\mathsf{u}_{i,ss}\delta_{jk}) - \mathsf{p}_k \delta_{ij} \,, \end{aligned}$$

where $\mu, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$ and δ_{ij} is the usual Kronecker symbol. The fields p and p, respectively a scalar and a vector one, enter the definition of the *pressure*, whose role in incompressible theories reduces to that of a Lagrange multiplier of the constraint (3.2).

Defining the *symmetric part* of a tensor X of order m as

$$\operatorname{Sym} \mathsf{X} := \frac{1}{m!} \sum_\sigma \mathsf{X}_{\sigma(i_1 \dots i_m)} \,,$$

where σ runs over the group of permutations of m elements, and setting $\mathsf{I} = (\delta_{ij})$, the previous relations can be written in intrinsic notation as

$$\mathsf{T} = 2\mu \operatorname{Sym} \nabla \mathsf{u} - p\,\mathsf{I} \,,$$

$$\begin{aligned} \mathbf{G} =& (\eta_1 - \eta_2)\nabla\nabla \mathsf{u} + 3\eta_2 \operatorname{Sym} \nabla\nabla \mathsf{u} \\ &- (\eta_2 + 5\eta_3)\Delta \mathsf{u} \otimes \mathsf{I} + 3\eta_3 \operatorname{Sym}(\Delta \mathsf{u} \otimes \mathsf{I}) - \mathsf{I} \otimes \mathsf{p} \,. \end{aligned}$$

Following these definitions, since also the virtual velocities undergo the constraint (3.2), we can write the internal power expenditure for a linear isotropic incompressible fluid as

$$\begin{aligned} \langle \mathcal{P}^{\mathrm{in}}_{\mathsf{u}}, \mathsf{v} \rangle = 2\mu \int_\Omega \operatorname{Sym} \nabla \mathsf{u} \cdot \nabla \mathsf{v} + (\eta_1 - \eta_2) \int_\Omega \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} \\ + 3\eta_2 \int_\Omega \operatorname{Sym} \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} - (\eta_2 + 4\eta_3) \int_\Omega \Delta \mathsf{u} \cdot \Delta \mathsf{v} \,. \end{aligned}$$

3.2 Thermodynamical constraints

By thermodynamical considerations, we need the instantaneous dissipation to be non-negative for any flow; hence we require the *dissipation inequality*

$$\mathsf{T} \cdot \nabla \mathsf{v} + \mathbf{G} \cdot \nabla\nabla \mathsf{v} \geqslant 0 \tag{3.3}$$

to be satisfied for every velocity v. This inequality, specialized for our model, reads

$$2\mu |\operatorname{Sym} \nabla \mathsf{v}|^2 + (\eta_1 - \eta_2)|\nabla\nabla \mathsf{v}|^2 + 3\eta_2 |\operatorname{Sym} \nabla\nabla \mathsf{v}|^2 - (\eta_2 + 4\eta_3)|\Delta \mathsf{v}|^2 \geqslant 0$$

for every virtual velocity v.

Since the first- and second-order derivatives of v can be independently set equal to zero, the dissipation inequality will be satisfied if and only if $\mu \geqslant 0$ and

$$\Gamma := \nabla\nabla \mathsf{v} \cdot \mathbf{G}[\nabla\nabla \mathsf{v}] = \eta_1 \mathsf{v}_{i,jk} \mathsf{v}_{i,jk} + \eta_2 (2\mathsf{v}_{k,ij} \mathsf{v}_{i,jk} - \mathsf{v}_{i,rr} \mathsf{v}_{i,ss}) - 4\eta_3 \mathsf{v}_{i,rr} \mathsf{v}_{i,ss} \geqslant 0$$

for every virtual velocity v. This last requirement is equivalent to the following conditions on the coefficients η_1, η_2 and η_3.

Proposition 3.1. *We have $\Gamma \geqslant 0$ for every virtual velocity v if and only if*

$$\eta_1 + 2\eta_2 \geqslant 0\,, \quad \eta_1 - \eta_2 \geqslant 0\,, \quad \eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \geqslant 0\,. \tag{3.4}$$

Proof. Let us identify the 18 independent components of $\nabla\nabla \mathsf{v}$ with an element $\mathsf{x} \in \mathbb{R}^{18}$ according to the following table:

$$\begin{array}{lllll}
x_1 = \mathsf{v}_{1,11} & x_2 = \mathsf{v}_{1,22} & x_3 = \mathsf{v}_{1,33} & x_4 = \mathsf{v}_{2,12} & x_5 = \mathsf{v}_{3,13} \\
x_6 = \mathsf{v}_{2,22} & x_7 = \mathsf{v}_{2,33} & x_8 = \mathsf{v}_{2,11} & x_9 = \mathsf{v}_{3,23} & x_{10} = \mathsf{v}_{1,12} \\
x_{11} = \mathsf{v}_{3,33} & x_{12} = \mathsf{v}_{3,11} & x_{13} = \mathsf{v}_{3,22} & x_{14} = \mathsf{v}_{1,13} & x_{15} = \mathsf{v}_{2,23} \\
x_{16} = \mathsf{v}_{1,23} & x_{17} = \mathsf{v}_{2,13} & x_{18} = \mathsf{v}_{3,12}\,. &&
\end{array}$$

Then we can write

$$\Gamma = \mathsf{x} \cdot (\eta_1 \mathsf{A} + \mathsf{B})\mathsf{x}\,, \tag{3.5}$$

where $\mathsf{A} = \operatorname{diag}(A_5, A_5, A_5, A_3)$, $\mathsf{B} = \operatorname{diag}(B_5, B_5, B_5, B_3)$ and

$$A_5 = \operatorname{diag}(1,1,1,2,2), \quad A_3 = \operatorname{diag}(2,2,2),$$

$$B_5 = \begin{bmatrix}
\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 0 \\
-\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 2\eta_2 & 0 \\
-\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 2\eta_2 \\
0 & 2\eta_2 & 0 & 2\eta_2 & 0 \\
0 & 0 & 2\eta_2 & 0 & 2\eta_2
\end{bmatrix}, \quad B_3 = \eta_2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The quadratic form (3.5) is positive definite if and only if its eigenvalues are all positive. Since A is positive definite, this is tantamount to say that the eigenvalues of $\eta_1 \mathsf{I} + \mathsf{A}^{-1}\mathsf{B}$ are positive definite.

Since $\mathsf{A}^{-1}\mathsf{B} = \mathrm{diag}(A_5^{-1}B_5, A_5^{-1}B_5, A_5^{-1}B_5, A_3^{-1}B_3)$ and

$$A_5^{-1}B_5 = \begin{bmatrix} \eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 2\eta_2 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 2\eta_2 \\ 0 & \eta_2 & 0 & \eta_2 & 0 \\ 0 & 0 & \eta_2 & 0 & \eta_2 \end{bmatrix},$$

$$A_3^{-1}B_3 = \eta_2 \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix},$$

a straightforward calculation shows that the eigenvalues of $\mathsf{A}^{-1}\mathsf{B}$ are

$$\lambda_{1,2} = -\eta_2 - 6\eta_3 \pm 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2}\,,\ \lambda_{3,4} = \pm\eta_2\,,\ \lambda_5 = 2\eta_2\,,\ \lambda_6 = -\frac{\eta_2}{2}\,.$$

Hence $\mathit{\Gamma} \geqslant 0$ for every velocity field if and only if $\eta_1 + \lambda_{\min} \geqslant 0$, where $\lambda_{\min}$ is the minimal eigenvalue. Since

$$\begin{cases} \lambda_{\min} = -\eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} & \text{if } \eta_2 + 4\eta_3 \geqslant 0 \\ \lambda_{\min} = -\eta_2 & \text{if } \eta_2 + 4\eta_3 \leqslant 0 \text{ and } \eta_2 \geqslant 0 \\ \lambda_{\min} = 2\eta_2 & \text{if } \eta_2 + 4\eta_3 \leqslant 0 \text{ and } \eta_2 \leqslant 0\,, \end{cases}$$

one has the global conditions (3.4). □

3.3 Interactions at the boundary

In order to have a picture in mind, we can think of Ω as a pipe containing the fluid, with solid walls and some open ends. I want to impose *no-slip* conditions on a part of the boundary denoted by $\mathcal{S}_W$, representing the wall of the pipe, for which I require $\mathcal{H}^2(\mathcal{S}_W) > 0$ (where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure). Then, I partition the remaining part of the boundary into two regions, $\mathcal{S}_D$ and $\mathcal{S}_F$, on which different conditions will be imposed.

The prescriptions of the external interaction fields $\mathsf{t_u}$ and $\mathsf{m_u}$ are related to such boundary conditions:

- assuming vanishing hypertraction and adherence to *fixed boundaries*, we have $\mathsf{m_u} = 0$ and $\mathsf{u} = 0$ on $\mathcal{S}_W$;
- imposing on $\mathcal{S}_D$ only a constant and uniform normal pressure gradient which drives the fluid flow, means $\mathsf{m_u} = 0$ and $\mathsf{t_u} = q\mathsf{n}$, $q \in \mathbb{R}$, on $\mathcal{S}_D$;
- prescribing no external action on the remaining part of the boundary, gives $\mathsf{m_u} = 0$ and $\mathsf{t_u} = 0$ on $\mathcal{S}_F$.

In the following chapter, I consider also the adherence of the fluid to one-dimensional immersed structures, but it does not appear within the boundary conditions, since it will be encoded in the definition of an appropriate functional space.

Chapter 4

Analysis of the differential problems

I now want to investigate existence and uniqueness of solutions for both the stationary and the evolutionary motion of a second-order incompressible fluid in a bounded Lipschitz domain Ω, in which a one-dimensional structure Λ is immersed. The assumption of the balance principle (3.1) in integral form leads directly to an interpretation of the functions as defined up to negligible sets (with zero Lebesgue measure). Indeed, both the real velocity field u and the virtual velocity v will belong to Hilbert spaces, whose definition is given in what follows.

4.1 The functional setting

We can construct a Hilbert subspace X of the Sobolev space $H^2(\Omega;\mathbb{R}^3)$ in the following way: we set

$$V := \left\{ \mathsf{v}|_\Omega \,:\, \mathsf{v} \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)\,,\ \operatorname{div}\mathsf{v} = 0 \right\}$$

and denote with H and H^2_d the completions of V in $L^2(\Omega;\mathbb{R}^3)$ and $H^2(\Omega;\mathbb{R}^3)$ respectively. Since $H^2(\Omega;\mathbb{R}^3)$ is continuously embedded in $C^0(\overline{\Omega};\mathbb{R}^3)$, we can define in an obvious way the closed subspace

$$Y := \left\{ \mathsf{v} \in H^2(\Omega;\mathbb{R}^3) \,:\, \mathsf{v} = 0 \ \text{ on } \overline{\mathcal{S}_W} \cup \Lambda \right\} ;$$

finally we set $X := H^2_d \cap Y$ endowed with the $H^2(\Omega;\mathbb{R}^3)$ norm

$$\|\mathsf{v}\|^2_X := \int_\Omega |\mathsf{v}|^2 + \int_\Omega |\nabla\mathsf{v}|^2 + \int_\Omega |\nabla\nabla\mathsf{v}|^2,$$

that encodes the natural regularity requested by the problem.

It is apparent that velocity fields belonging to the space X vanish on $\overline{\mathcal{S}_W} \cup \Lambda$ (while their derivatives in general do not); hence we can model adherence to walls and to one-dimensional immersed structures in a way consistent with the boundary conditions of Section 3.3. It remains to specify the role of the time variable t, which clearly enters the problem *via* the time derivative of u. As a first step we could take $\mathsf{u} \in L^2([0,T];X)$, so that $\mathsf{u}(t,\cdot) \in X$ for almost every $t \in [0,T]$; but we will see that, if u is a solution of our problem, then

$$\mathsf{u} \in L^2([0,T];X) \cap C^0([0,T];H) \cap H^1([0,T];X') =: \mathfrak{X}\,.$$

Given the representations of internal and external powers introduced in the previous chapter, the Principle of Virtual Powers, which corresponds to the equation of motion, becomes the following statement.

Problem 1. *Find* $\mathsf{u} \in \mathfrak{X}$ *such that*

$$2\mu \int_0^T \int_\Omega \operatorname{Sym} \nabla \mathsf{u} \cdot \nabla \mathsf{v} + (\eta_1 - \eta_2) \int_0^T \int_\Omega \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} + 3\eta_2 \int_0^T \int_\Omega \operatorname{Sym} \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} - (\eta_2 + 4\eta_3) \int_0^T \int_\Omega \Delta \mathsf{u} \cdot \Delta \mathsf{v} \int_0^T \int_\Omega \left(\frac{\partial \mathsf{u}}{\partial t} + (\mathsf{u} \cdot \nabla)\mathsf{u} \right) \cdot \mathsf{v} = \int_0^T \int_{\mathcal{S}_D} q\mathsf{n} \cdot \mathsf{v} + \int_0^T \int_\Omega \mathsf{d} \cdot \mathsf{v}\,, \quad (4.1)$$

for every $\mathsf{v} \in \mathfrak{X}$.

When stating the *evolutionary problem*, I will add an initial condition, while, to obtain the *stationary problem*, it is enough to set the time derivative of u equal to zero, to erase the time integrals, and to change $\mathfrak{X}$ into X in the previous statement.

In equation (4.1) the only nonlinearity is the convective term $(\mathsf{u} \cdot \nabla)\mathsf{u}$; clearly, it is the source of the main difficulties in solving our problem. I will work it out *via* a topological method in which compactness is the key tool; therefore, let me introduce some considerations about the compactness properties of that term.

Consider the bilinear function

$$F : \begin{cases} H^2 \times H^2 & \to & L^2 \\ (\mathsf{u}, \mathsf{v}) & \mapsto & (\mathsf{u} \cdot \nabla)\mathsf{v} \end{cases};$$

by Hölder's inequality, since $H^2(\Omega; \mathbb{R}^3)$ is embedded in $L^\infty(\Omega; \mathbb{R}^3)$, we have[(1)]

$$\|F(\mathsf{u}, \mathsf{v})\|_{L^2} = \|(\mathsf{u} \cdot \nabla)\mathsf{v}\|_{L^2} \leqslant \|\mathsf{u}\|_{L^\infty} \|\nabla \mathsf{v}\|_{L^2} \leqslant c_0 \|\mathsf{u}\|_{H^2} \|\mathsf{v}\|_{H^2} \quad (4.2)$$

for any $\mathsf{u}, \mathsf{v} \in H^2(\Omega; \mathbb{R}^3)$; hence F is continuous.

Theorem 4.1. *The* Navier operator $K_0(\mathsf{u}) := F(\mathsf{u}, \mathsf{u})$ *is compact from* X *to* X'.

Proof. As already noticed, the bilinear function F is continuous and so is K_0, by composition with the function $\{\, \mathsf{u} \mapsto (\mathsf{u}, \mathsf{u}) \,\}$. Moreover, by virtue of (4.2), it is bounded on bounded subsets of X.

Since $X \subseteq H^2(\Omega; \mathbb{R}^3)$ is compactly embedded in $L^2(\Omega; \mathbb{R}^3)$, we can identify $L^2(\Omega; \mathbb{R}^3)$ with its dual space and apply Schauder's theorem [5, Theorem 6.4] to obtain $L^2(\Omega; \mathbb{R}^3)$ compactly embedded in X'. □

Remark 4.1. The operator K_0 is also compact from any $W \subseteq H^1(\Omega)$ to W'; in fact the immersion $H^1(\Omega) \to L^q(\Omega)$ is compact for $q \in [1, 6[$ and such it is the dual $L^{q'}(\Omega) \to (H^1(\Omega))'$. We can take $q = 4$, $q' = \frac{4}{3}$, make the extension

$$F : \begin{cases} H^1 \times H^1 & \to & L^{\frac{4}{3}} \\ (\mathsf{u}, \mathsf{v}) & \mapsto & (\mathsf{u} \cdot \nabla)\mathsf{v} \end{cases}$$

and accordingly define K_0 on W. We have

$$\|(\mathsf{u} \cdot \nabla)\mathsf{v}\|_{L^{\frac{4}{3}}} \leqslant \|\mathsf{u}\|_{L^4} \|\nabla \mathsf{v}\|_{L^2} \leqslant \tilde{c}_0 \|\nabla \mathsf{u}\|_{L^2} \|\nabla \mathsf{v}\|_{L^2} < +\infty$$

for any $\mathsf{u}, \mathsf{v} \in W$ and, following the arguments of the previous theorem, we obtain the compactness of K_0 from W to W'. ♢

(1) Throughout this chapter c_i, $i \in \mathbb{N}$, denotes a positive constant depending only upon the geometry of Ω.

When considering the evolutionary problem, we need K_0 to be a compact operator from $L^2([0,T];X)$ into $L^2([0,T];X')$. The following lemma, proved in [26, Chap. 1, Sec. 5.2], will be useful.

Lemma 4.2. *Given three Banach spaces $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}_1$ with $\mathcal{B}_0$ and $\mathcal{B}_1$ reflexive and $\mathcal{B}_0$ compactly embedded in $\mathcal{B}$, and for $T \in (0,+\infty)$ and $p_0, p_1 \in (1,+\infty)$ fixed, set*

$$W := \left\{ \mathsf{v} : \ \mathsf{v} \in L^{p_0}([0,T];\mathcal{B}_0)\,, \quad \frac{\partial \mathsf{v}}{\partial t} \in L^{p_1}([0,T];\mathcal{B}_1) \right\}.$$

Then W is a Banach space compactly embedded in $L^{p_0}([0,T];\mathcal{B})$.

Theorem 4.3. *The Navier operator K_0 is compact from the space*

$$\mathfrak{X} = L^2([0,T];X) \cap C^0([0,T];H) \cap H^1([0,T];X')$$

into $L^2([0,T];X')$.

Proof. Since X is compactly embedded in $L^3(\Omega;\mathbb{R}^3)$ and so is $L^{\frac{3}{2}}(\Omega;\mathbb{R}^3)$ into X', we can apply Lemma 4.2 with $X \subset L^3(\Omega;\mathbb{R}^3) \subset X'$ and $p_0 = p_1 = 2$. It remains to show that K_0, as an operator with range in $L^2([0,T];L^{\frac{3}{2}}(\Omega;\mathbb{R}^3))$, is bounded on bounded subsets of its domain. The estimate

$$\int_0^T \|F(\mathsf{u},\mathsf{v})\|^2_{L^{\frac{3}{2}}}\,dt \leqslant \int_0^T \|\nabla \mathsf{v}\|^2_{L^6}\,\|\mathsf{u}\|^2_{L^2} \leqslant c_1^2 \int_0^T \|\mathsf{v}\|^2_{H^2}\,\|\mathsf{u}\|^2_{L^2}$$

gives the needed property since u belongs to $L^\infty([0,T];H)$. □

4.2 The stationary problem

I will first solve the stationary version of Problem 1, in which time dependence is suppressed. Stationary solutions represent an important class in Fluid Mechanics for many applications; moreover, the treatment of the evolutionary problem follows a similar argument.

Let us define the bilinear form $a : X \times X \to \mathbb{R}$ as follows:

$$a(\mathsf{u},\mathsf{v}) := 2\mu \int_\Omega \operatorname{Sym}\nabla \mathsf{u} \cdot \nabla \mathsf{v} + (\eta_1 - \eta_2) \int_\Omega \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} \\ + 3\eta_2 \int_\Omega \operatorname{Sym}\nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} - (\eta_2 + 4\eta_3) \int_\Omega \Delta \mathsf{u} \cdot \Delta \mathsf{v}\,.$$

In view of the dissipation inequality (3.3) and Proposition 3.1, I will assume a slightly stronger hypothesis on the sign of the coefficients.

Proposition 4.4. *Provided that*

$$\mu > 0\,, \quad \eta_1 + 2\eta_2 > 0\,, \quad \eta_1 - \eta_2 > 0\,, \quad \eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} > 0\,,$$

the bilinear form $a(\mathsf{u},\mathsf{v})$ is continuous and coercive on X.

Proof. The continuity of a is apparent. With the notation of Proposition 3.1, we have:

$$a(\mathsf{u},\mathsf{u}) = 2\mu \int_\Omega |\operatorname{Sym}\nabla\mathsf{u}|^2 + \int_\Omega \Gamma \geqslant 2\mu \,\|\operatorname{Sym}\nabla\mathsf{u}\|_{L^2}^2 + (\eta_1 + \lambda_{\min})\,\|\nabla\nabla\mathsf{u}\|_{L^2}^2 \;;$$

by an application of Korn's inequality [22, Lemma 6.2], there exists $\kappa > 0$ such that

$$a(\mathsf{u},\mathsf{u}) \geqslant \kappa(\|\mathsf{u}\|_{L^2}^2 + \|\nabla\mathsf{u}\|_{L^2}^2) + (\eta_1 + \lambda_{\min})\,\|\nabla\nabla\mathsf{u}\|_{L^2}^2 \;.$$

Setting

$$\nu := \min\{\,\kappa\,,\ \eta_1 + \lambda_{\min}\,\} > 0\,,$$

we have $a(\mathsf{u},\mathsf{u}) \geqslant \nu\,\|\mathsf{u}\|_X^2$. □

Consider now the trilinear form $b : H^2 \times H^2 \times H^2 \to \mathbb{R}$ given by

$$b(\mathsf{u},\mathsf{v},\mathsf{w}) := \int_\Omega F(\mathsf{u},\mathsf{v})\cdot\mathsf{w} = \int_\Omega (\mathsf{u}\cdot\nabla)\mathsf{v}\cdot\mathsf{w}\,,$$

which is indeed continuous since

$$|b(\mathsf{u},\mathsf{v},\mathsf{w})| \leqslant \|F(\mathsf{u},\mathsf{v})\|_{L^2}\,\|\mathsf{w}\|_{L^2} \leqslant c_0\,\|\mathsf{u}\|_{H^2}\,\|\mathsf{v}\|_{H^2}\,\|\mathsf{w}\|_{H^2}$$

for every $\mathsf{u},\mathsf{v},\mathsf{w} \in H^2(\Omega;\mathbb{R}^3)$.

Lemma 4.5. *For every $\mathsf{u} \in X$ we have $b(\mathsf{u},\mathsf{u},\mathsf{u}) = 0$.*

Proof. By standard formulae in tensor calculus we get the assertion for $\mathsf{u} \in V$ and we can extend it by a density argument. □

In the course of the proof of the existence result for solutions of the stationary problem, also the following theorem [9, Corollary 8.1] will be applied.

Theorem (Fixed Point). *Let X be a Banach space and $\Phi : X \to X$ a compact operator. Then either $\Phi(\mathsf{u}) = \mathsf{u}$ has a solution, or the set*

$$S = \Big\{\, \mathsf{u} \in X :\ \Phi(\mathsf{u}) = \lambda\mathsf{u} \quad \textit{for some } \lambda > 1 \,\Big\}$$

is unbounded.

Theorem 4.6. *There exists $\mathsf{u} \in X$ such that, for every $\mathsf{v} \in X$,*

$$a(\mathsf{u},\mathsf{v}) + b(\mathsf{u},\mathsf{u},\mathsf{v}) = \langle\,\varphi,\mathsf{v}\,\rangle\,, \tag{4.3}$$

where $\varphi \in X'$ is the linear form defined by

$$\langle\,\varphi,\mathsf{v}\,\rangle = \int_{\mathcal{S}_D} q\mathsf{n}\cdot\mathsf{v} + \int_\Omega \mathsf{d}\cdot\mathsf{v}\,.$$

Proof. By the Lax-Milgram theorem, the function $L : X \to X'$ defined by

$$\forall \mathsf{v} \in X : \ \langle\, L(\mathsf{u}), \mathsf{v} \,\rangle = a(\mathsf{u}, \mathsf{v})$$

is a homeomorphism. We have

$$L(\mathsf{u}) + K_0(\mathsf{u}) = \varphi \quad \text{in } X' \tag{4.4}$$

and then

$$\mathsf{u} = L^{-1}\left(\varphi - K_0(\mathsf{u})\right) =: \Phi(\mathsf{u}).$$

Assume that $\mathsf{u} \in X$ is a solution of (4.4); it means that for every $\mathsf{v} \in X$

$$\langle\, L(\mathsf{u}), \mathsf{v} \,\rangle + \langle\, K_0(\mathsf{u}), \mathsf{v} \,\rangle = \langle\, \varphi, \mathsf{v} \,\rangle.$$

Taking $\mathsf{v} = \mathsf{u}$ and applying Lemma 4.5, we can write

$$\nu \left\|\mathsf{u}\right\|_X^2 \leqslant a(\mathsf{u}, \mathsf{u}) \leqslant |b(\mathsf{u}, \mathsf{u}, \mathsf{u})| + |\langle\, \varphi, \mathsf{u} \,\rangle| \leqslant \left\|\varphi\right\|_{X'} \left\|\mathsf{u}\right\|_X \ ;$$

from which we have the *a priori* estimate

$$\left\|\mathsf{u}\right\|_X \leqslant \frac{1}{\nu} \left\|\varphi\right\|_{X'} =: R < +\infty \,. \tag{4.5}$$

Take now $\lambda > 1$ and assume that $\Phi(\mathsf{u}) = \lambda \mathsf{u}$; it means that

$$\langle\, L(\lambda \mathsf{u}), \mathsf{u} \,\rangle + \langle\, K_0(\mathsf{u}), \mathsf{u} \,\rangle = \langle\, \varphi, \mathsf{u} \,\rangle$$

and, following the argument leading to (4.5), we obtain $\left\|\mathsf{u}\right\|_X \leqslant \lambda^{-1} R < R$.

Hence the set S introduced in the Fixed Point theorem is bounded and there exists a fixed point $\mathsf{u} \in X$ for Φ. We can then conclude that such u is a stationary solution for Problem 1. □

Remark 4.2. Considering the Sobolev constant c_0 introduced in (4.2), it is easy to prove that, if the condition $c_0 \left\|\varphi\right\|_{X'} < \nu^2$ is satisfied, then the solution of the stationary problem is unique. This condition resembles that of a low Reynolds number: indeed it entails also the exponential stability of the unique stationary solution of the evolutionary problem treated below. ♢

4.3 The evolutionary problem

I now come to analyze the evolutionary problem. Take $\mathsf{u} \in L^2([0,T]; X)$, whose norm is

$$\left\|\mathsf{u}\right\|_{L^2([0,T];X)}^2 := \int_0^T \left\|\mathsf{u}(s)\right\|_X^2 \, ds \,,$$

and take $\varphi \in L^2([0,T]; X')$ defined by

$$\langle\, \varphi, \mathsf{v} \,\rangle = \int_0^T \int_{\mathcal{S}_D} q\mathsf{n} \cdot \mathsf{v} + \int_0^T \int_\Omega \mathsf{d} \cdot \mathsf{v} \,.$$

Theorem 4.7. *For every* $\mathsf{u}_0 \in H$ *there exists* $\mathsf{u} \in L^2([0,T];X)$ *such that*

$$\mathsf{u}(0) = \mathsf{u}_0, \tag{4.6}$$

$$\int_0^T \left(\int_\Omega \frac{\partial \mathsf{u}}{\partial t} \cdot \mathsf{v} + a(\mathsf{u},\mathsf{v}) + b(\mathsf{u},\mathsf{u},\mathsf{v}) \right) = \langle\, \varphi, \mathsf{v} \,\rangle, \tag{4.7}$$

for every $\mathsf{v} \in L^2([0,T];X)$.

Remark 4.3. Notice that the time derivative of u has to be interpreted as a linear form in $L^2([0,T];X')$, whose representation enters equation (4.7), and thus we will take $\mathsf{u} \in H^1([0,T];X')$. A key role in the evolutionary problem is played by the initial datum u_0 which belongs to H. At first the initial condition (4.6) should be understood in X', but we will see that it actually holds in H, as u belongs to $C^0([0,T];H)$. After these considerations we can take $\mathsf{u} \in \mathfrak{X}$. ♢

Proof. In order to proceed we need some estimates; first we set $\mathsf{v} = \mathsf{u}$ in (4.7) and apply Lemma 4.5 obtaining that

$$\int_\Omega \frac{\partial \mathsf{u}}{\partial t} \cdot \mathsf{u} + a(\mathsf{u},\mathsf{u}) = \langle\, \varphi, \mathsf{u} \,\rangle$$

for almost every $t \in [0,T]$. Integrating in time we get:

$$\frac{1}{2} \int_0^t \frac{d}{ds} \|\mathsf{u}(s)\|_{L^2}^2 \, ds + \int_0^t a(\mathsf{u},\mathsf{u})\, ds = \int_0^t \langle\, \varphi, \mathsf{u} \,\rangle \, ds\,;$$

hence, applying also Young's inequality,

$$\frac{1}{2} \|\mathsf{u}(t)\|_{L^2}^2 + \nu \int_0^t \|\mathsf{u}\|_X^2 \, ds \leqslant \frac{1}{2} \|\mathsf{u}_0\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\mathsf{u}\|_X^2 \, ds + c_2 \int_0^t \|\varphi\|_{X'}^2 \, ds\,,$$

that gives the first estimate for a.e. $t \in [0,T]$:

$$\|\mathsf{u}(t)\|_{L^2}^2 + \nu \int_0^t \|\mathsf{u}\|_X^2 \, ds \leqslant \|\mathsf{u}_0\|_{L^2}^2 + 2c_2 \int_0^t \|\varphi\|_{X'}^2 \, ds =: M\,.$$

This *a priori* bound tells us that any solution u of our problem belongs to a bounded subset of $L^2([0,T];X) \cap L^\infty([0,T];H)$. We now need the following theorem whose proof is given in [27, Chap. 3, Sec. 4.4].

Theorem (J. L. Lions). *Let* X *and* H *be two Hilbert spaces, with* X *dense and continuously embedded in* H*; identify* H *with its dual in such a way that* $X \subset H \subset X'$ *and fix* $T > 0$. *Consider a bilinear form* $a_t(\mathsf{u},\mathsf{v}) : X \times X \to \mathbb{R}$ *such that:*

i) the function $t \mapsto a_t(\mathsf{u},\mathsf{v})$ *is measurable for every* $\mathsf{u},\mathsf{v} \in X$ *;*

ii) $|a_t(\mathsf{u},\mathsf{v})| \leqslant C_1 \|\mathsf{u}\|_X \|\mathsf{v}\|_X$ *for a.e.* $t \in [0,T]$, *for every* $\mathsf{u},\mathsf{v} \in X$*;*

iii) $a_t(\mathsf{v},\mathsf{v}) \geqslant \alpha \|\mathsf{v}\|_X^2 - C_2 \|\mathsf{v}\|_H^2$ *for a.e.* $t \in [0,T]$, *for every* $\mathsf{v} \in X$*;*

where $\alpha > 0$, C_1 *and* C_2 *are constants.*

Then, for every $\mathsf{f} \in L^2([0,T];X')$ *and for every* $\mathsf{u}_0 \in H$, *there exists only one* u *such that*

$$\mathsf{u} \in L^2([0,T];X) \cap C^0([0,T];H) \cap H^1([0,T];X')$$
$$\mathsf{u}(0) = \mathsf{u}_0$$
$$\langle\, \frac{\partial \mathsf{u}}{\partial t}(t), \mathsf{v} \,\rangle + a_t(\mathsf{u}(t),\mathsf{v}) = \langle\, \mathsf{f}(t), \mathsf{v} \,\rangle \quad \text{for a.e. } t \in [0,T], \text{ for every } \mathsf{v} \in X.$$

It is easy to see that the spaces X, H and the bilinear form $a_t(\mathsf{u},\mathsf{v}) := a(\mathsf{u}(t),\mathsf{v}(t))$ fulfill the hypotheses of the previous theorem; then the function

$$L : \begin{cases} \mathfrak{X} & \to & H \times L^2([0,T];X') \\ \mathsf{u} & \mapsto & \left(\mathsf{u}(0),\ \dfrac{\partial \mathsf{u}}{\partial t}(t) + a(\mathsf{u}(t),\cdot\,) \right) \end{cases}$$

is a homeomorphism.

We can now write equations (4.6)–(4.7) in $H \times L^2([0,T];X')$ as

$$L(\mathsf{u}) + (\,0\,,\, K_0(\mathsf{u})\,) = (\,\mathsf{u}_0\,,\,\varphi\,),$$

from which

$$\mathsf{u} = L^{-1}(\,\mathsf{u}_0\,,\,\varphi - K_0(\mathsf{u})\,) =: \Phi(\mathsf{u}).$$

By Theorem 4.3 and composition arguments, Φ turns out to be a compact operator, and we can then apply the Fixed Point theorem in the Banach space $\mathfrak{X}$. Arguing as in the previous section, we take $\lambda > 1$ and assume $\Phi(\mathsf{u}) = \lambda\mathsf{u}$; in particular $\mathsf{u}_0 = \lambda\mathsf{u}(0)$ and

$$\frac{\lambda}{2}\int_0^t \frac{d}{ds}\,\|\mathsf{u}(s)\|_{L^2}^2\,ds + \lambda\int_0^t a(\mathsf{u},\mathsf{u})\,ds = \int_0^t \langle\,\varphi,\mathsf{u}\,\rangle\,ds\,.$$

We then obtain

$$\frac{\lambda}{2}\,\|\mathsf{u}(t)\|_{L^2}^2 + \lambda\nu\int_0^t \|\mathsf{u}\|_X^2\,ds \leqslant \frac{\lambda}{2}\,\left\|\lambda^{-1}\mathsf{u}_0\right\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \|\mathsf{u}\|_X^2\,ds + c_2\int_0^t \|\varphi\|_{X'}^2\,ds\,,$$

that gives

$$\lambda\,\|\mathsf{u}(t)\|_{L^2}^2 + (2\lambda-1)\nu\int_0^t \|\mathsf{u}\|_X^2\,ds \leqslant \lambda^{-1}\,\|\mathsf{u}_0\|_{L^2}^2 + 2c_2\int_0^t \|\varphi\|_{X'}^2\,ds < M\,.$$

Since $2\lambda - 1 > \lambda$, we can write

$$\|\mathsf{u}(t)\|_{L^2}^2 + \nu\int_0^t \|\mathsf{u}\|_X^2\,ds < \lambda^{-1}M$$

and the set S in the Fixed Point theorem is bounded in $L^2([0,T];X)\cap L^\infty([0,T];H)$. In order to complete the proof, it remains to show that S is bounded also in $H^1([0,T];X')$.

If there exists $\lambda > 1$ such that $\Phi(\mathsf{u}) = \lambda\mathsf{u}$, then we have that

$$\frac{\partial \mathsf{u}}{\partial t} = -a(\mathsf{u},\cdot\,) - \frac{1}{\lambda}K_0(\mathsf{u}) + \frac{1}{\lambda}\varphi \quad \text{in } L^2([0,T];X')$$

and that

$$\left\|\frac{\partial \mathsf{u}}{\partial t}\right\|_{X'} \leqslant \|a(\mathsf{u},\cdot\,)\|_{X'} + \frac{1}{\lambda}\,\|K_0(\mathsf{u})\|_{X'} + \frac{1}{\lambda}\,\|\varphi\|_{X'}\,.$$

By the continuity of a and the embeddings mentioned in the proof of Theorem 4.3 we can write:

$$\left\|\frac{\partial \mathsf{u}}{\partial t}\right\|_{X'} \leqslant c_3\,\|\mathsf{u}\|_X + \frac{c_4}{\lambda}\,\|K_0(\mathsf{u})\|_{L^{\frac{3}{2}}} + \frac{1}{\lambda}\,\|\varphi\|_{X'}\,.$$

We know that u belongs to a bounded subset of $L^2([0,T];X)$ and $\varphi \in L^2([0,T];X')$; we deduce that

$$\int_0^T \left\| \frac{\partial \mathsf{u}}{\partial t} \right\|_{X'}^2 \leqslant c_5 + \frac{2c_4^2}{\lambda^2} \int_0^T \|K_0(\mathsf{u})\|_{L^{\frac{3}{2}}}^2 < N$$

for a fixed $N > 0$, since K_0 maps bounded subsets of $L^2([0,T];X) \cap L^\infty([0,T];H)$ to bounded subsets of $L^2([0,T];L^{\frac{3}{2}}(\Omega;\mathbb{R}^3))$.

The last bound shows that $\frac{\partial \mathsf{u}}{\partial t}$ belongs to a bounded subset of $L^2([0,T];X')$ and implies that S is bounded in $\mathfrak{X}$. Hence there exists a fixed point $\mathsf{u} \in \mathfrak{X}$ for Φ and this is a solution for the Cauchy problem (4.6)–(4.7). □

Thanks to the H^2-regularity of the solution we have found, which in particular guarantees the L^∞-regularity in three dimensions, an important uniqueness result can be proved.

Theorem 4.8. *There exists a unique solution $\mathsf{u} \in \mathfrak{X}$ of the Cauchy problem (4.6)–(4.7).*

Proof. Let u_1 and u_2 be solutions of equation (4.7) with the same initial datum and set $\mathsf{w} := \mathsf{u}_1 - \mathsf{u}_2$. We easily obtain, by equations (4.6)–(4.7) and Lemma 4.5,

$$\begin{aligned} \mathsf{w}(0) &= 0, \\ \int_\Omega \frac{\partial \mathsf{w}}{\partial t} \cdot \mathsf{w} + a(\mathsf{w},\mathsf{w}) + b(\mathsf{w},\mathsf{u}_2,\mathsf{w}) &= 0\,, \end{aligned}$$

from which, integrating in time and applying the coercivity of a, we can deduce that

$$\frac{1}{2}\|\mathsf{w}(t)\|_{L^2}^2 + \nu \int_0^t \|\mathsf{w}\|_X^2\, ds \leqslant \int_0^t |b(\mathsf{w},\mathsf{u}_2,\mathsf{w})|\, ds\,.$$

By Young's inequality,

$$\begin{aligned} \|\mathsf{w}(t)\|_{L^2}^2 + 2\nu \int_0^t \|\mathsf{w}\|_X^2\, ds &\leqslant 2\int_0^t \|\mathsf{w}\|_{L^\infty} \|\nabla \mathsf{u}_2\|_{L^2} \|\mathsf{w}\|_{L^2}\, ds \\ &\leqslant 2c_0 \int_0^t \|\mathsf{w}\|_X \|\mathsf{u}_2\|_X \|\mathsf{w}\|_{L^2}\, ds \leqslant 2\nu \int_0^t \|\mathsf{w}\|_X^2\, ds + \frac{c_0^2}{2\nu} \int_0^t \|\mathsf{u}_2\|_X^2 \|\mathsf{w}\|_{L^2}^2\, ds\,; \end{aligned}$$

hence we have

$$\|\mathsf{w}(t)\|_{L^2}^2 \leqslant \frac{c_0^2}{2\nu} \int_0^t \|\mathsf{u}_2(s)\|_X^2 \|\mathsf{w}(s)\|_{L^2}^2\, ds$$

and by Gronwall's lemma we conclude that $\|\mathsf{w}(t)\|_{L^2}^2 = 0$ for every $t \in [0,T]$, that is $\mathsf{u}_1 = \mathsf{u}_2$. □

Remark 4.4. Notice that, if $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$, the bilinear form a is no longer coercive on X; nevertheless, all the previous results are still valid, if we replace X with the space

$$\hat{X} := \{\, \mathsf{v} \in \{\text{completion of } V \text{ in } H^1(\Omega;\mathbb{R}^3)\} : \Delta \mathsf{v} \in L^2(\Omega;\mathbb{R}^3) \text{ and } \mathsf{v} = 0 \text{ on } \overline{\mathcal{S}_W} \cup \Lambda \,\}\,,$$

on which the bilinear form a remains coercive, a key feature for existence theorems.

Moreover, standard regularity theorems for elliptic second-order PDEs, guarantee that functions belonging to $\hat{X}$ are essentially bounded [14, Theorem 8.16] and Hölder continuous [14, Theorem 8.22]. Essential boundedness provides the needed estimates for uniqueness of the solution, while the continuity in space of the solutions allows to assign their value on a one-dimensional set contained in the interior of Ω. ♢

4.4 Immersed structures dragging the fluid

I now want to study the flow of a second-gradient linear liquid dragged by one-dimensional immersed structures in an arbitrary time interval $[0,T]$ when the three-dimensional region occupied by the fluid Ω is a bounded domain with Lipschitz boundary which is fixed in time. Since here the emphasis is on the drag, I impose the boundary conditions stated in Section 3.3, with $\mathcal{S}_W = \partial\Omega$. For the sake of simplicity I consider only one rigid body Λ_0 immersed in the fluid, where Λ_0 is a connected one-dimensional closed subset of Ω such that $\Lambda_0 \cap \partial\Omega = \varnothing$. Moreover, I denote by $\Lambda(t)$ the image at time t of Λ_0 under the rigid displacement φ, *i.e.* $\Lambda(t) = \varphi(\Lambda_0, t)$ and $\Lambda(0) = \Lambda_0$. I assume the existence of a family of C^2-diffeomorphisms $\psi_t : \Omega \to \Omega$ such that $\psi_t(\Lambda_0) = \Lambda(t)$ and any ψ_t reduces to the identity map in a neighborhood of $\partial\Omega$. This assumption will be fulfilled if the one-dimensional structure never reaches the boundary of the domain, namely $\Lambda(t) \cap \partial\Omega = \varnothing$ for every $t \in [0,T]$. Finally, to avoid technicalities, I assume that φ and ψ_t enjoy a C^1 dependence on time. The introduction of such diffeomorphisms will enable us to map the linearized parabolic problem with a moving domain into an equivalent problem, easier to work out, where $\Lambda(t) = \Lambda_0$ for all the time.

I construct, for every $t \in [0,T]$, a Hilbert subspace X_t of the Sobolev space $H^2(\Omega;\mathbb{R}^3)$ in the following way: recall that I set

$$V := \left\{ \mathsf{v}|_\Omega \,:\, \mathsf{v} \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)\,,\ \operatorname{div}\mathsf{v} = 0 \right\} \tag{4.8}$$

and denote with H and H^2_d the completions of V in $L^2(\Omega;\mathbb{R}^3)$ and $H^2(\Omega;\mathbb{R}^3)$ respectively. Since $H^2(\Omega;\mathbb{R}^3)$ is continuously embedded in $C^0(\overline{\Omega};\mathbb{R}^3)$, we can define the closed subspace

$$X_t := \left\{ \mathsf{v} \in H^2(\Omega;\mathbb{R}^3) \,:\, \mathsf{v} = 0 \ \text{ on } \partial\Omega \cup \Lambda(t) \right\} \cap H^2_d \tag{4.9}$$

endowed with the H^2-norm.

Now I can redefine the space

$$\mathfrak{X} := L^2([0,T];X_t) \cap C^0([0,T];H) \cap H^1([0,T];X_t') \tag{4.10}$$

of divergence-free virtual velocities on Ω which belong to X_t for almost every $t \in [0,T]$, and whose first derivatives with respect to t belong to the dual space X_t' of X_t for a.e. $t \in [0,T]$. Moreover, it is apparent that velocity fields belonging to the space $\mathfrak{X}$ vanish on $[0,T] \times \partial\Omega$ and on the surface $\{\,(t,\mathsf{x}) : t \in [0,T], \mathsf{x} \in \Lambda(t)\,\}$.

Consider again the continuous bilinear form on $H^2(\Omega;\mathbb{R}^3)$ defined by

$$\begin{aligned} a_t(\mathsf{u},\mathsf{v}) := 2\mu \int_\Omega \operatorname{Sym}\nabla\mathsf{u}\cdot\nabla\mathsf{v} \\ + (\eta_1-\eta_2)\int_\Omega \nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} + 3\eta_2\int_\Omega \operatorname{Sym}\nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} \\ - (\eta_2+4\eta_3)\int_\Omega \Delta\mathsf{u}\cdot\Delta\mathsf{v}\,. \end{aligned} \tag{4.11}$$

Since $a_t(\mathsf{u},\mathsf{u})$ represents the dissipation at time t of the flow u, the classical Korn's inequality together with the constraints (3.4) on the coefficients (taken with strict inequalities) guarantee the coercivity of this bilinear form on the space X_t.

In order to prove the existence of a solution for the dynamical problem, assume to have an *interpolator* $\hat{\mathsf{u}}$, namely a divergence-free velocity field which vanishes on $\partial\Omega$ and equals the

velocity of the rigid body on $\Lambda(t)$, and to look for a solution of the type $\mathsf{u}+\hat{\mathsf{u}}$ where $\mathsf{u}\in\mathfrak{X}$. Then, substituting into the balance (3.1) and using the properties of $\hat{\mathsf{u}}$, it is straightforward to see that u must satisfy the following problem.

Problem 2. *Given* $\mathsf{u}_0\in H$ *and* $\hat{\mathsf{u}}$ *as above, find* $\mathsf{u}\in\mathfrak{X}$ *such that* $\mathsf{u}|_{t=0}=\mathsf{u}_0$ *and*

$$\int_0^T\left(\int_\Omega\frac{\partial\mathsf{u}}{\partial t}\cdot\mathsf{v}+a_t(\mathsf{u},\mathsf{v})\right)+\int_0^T\left(b(\mathsf{u},\mathsf{u},\mathsf{v})+b(\hat{\mathsf{u}},\mathsf{u},\mathsf{v})+b(\mathsf{u},\hat{\mathsf{u}},\mathsf{v})\right)$$
$$+\int_0^T\left(\int_\Omega\frac{\partial\hat{\mathsf{u}}}{\partial t}\cdot\mathsf{v}+a_t(\hat{\mathsf{u}},\mathsf{v})+b(\hat{\mathsf{u}},\hat{\mathsf{u}},\mathsf{v})\right)=0 \tag{4.12}$$

for every $\mathsf{v}\in\mathfrak{X}$.

In order to show that Problem 2 has a solution $\mathsf{u}\in\mathfrak{X}$ (and that it is the sole one), I need a further "smallness" condition on the interpolator $\hat{\mathsf{u}}$, that is, for a suitable $\beta>0$,

$$|b(\mathsf{v},\hat{\mathsf{u}},\mathsf{v})|\leqslant\beta\,\|\mathsf{v}\|^2_{H^2}\,, \tag{4.13}$$

which we suppose, for the time being, to be fulfilled. The problem of finding an interpolator will be considered in a while.

Now I will solve Problem 2. The first step is to define the linearized parabolic operator

$$L:\begin{cases}\mathfrak{X} & \to & H\times L^2([0,T];X_t')\\ \mathsf{u} & \mapsto & \left(\mathsf{u}|_{t=0}\,,\ \dfrac{\partial\mathsf{u}}{\partial t}(t)+a_t(\mathsf{u}(t),\cdot\,)\right)\end{cases} \tag{4.14}$$

and to establish that it is a homeomorphism. This fact guarantees that, at least when the convective term can be neglected (*e.g.* in the case of low Reynolds numbers), the problem admits a unique solution. Such a result can be obtained by the same argument used in the previous section, once we have noticed that the time-varying spaces X_t turn all into X_0 when we apply ψ_t^{-1} to Ω, and the coercivity of the bilinear form a_t is transferred to a new bilinear form $\tilde{a}$, as explained in [25]. After this transformation, we obtain a parabolic problem on the domain $\{\,(t,\psi_t^{-1}(\mathsf{x})):t\in[0,T],\mathsf{x}\in\Omega\,\}$, which enjoys existence and uniqueness of solution in the space

$$L^2([0,T];X_0)\cap C^0([0,T];H)\cap H^1([0,T];X_0')\,; \tag{4.15}$$

hence L is a homeomorphism.

As a second step, one has to deal with the nonlinear term $K(\mathsf{u}):=(\mathsf{u}\cdot\nabla)\mathsf{u}$. Its compactness properties have been successfully exploited in the previous sections and, since the same arguments apply also here, we need no additional effort to establish the existence of a solution $\mathsf{u}\in\mathfrak{X}$ for Problem 2. Uniqueness follows again by the proof of Theorem 4.8.

The construction of a good interpolator $\hat{\mathsf{u}}$ is not trivial, because in order to prove the existence of solutions we need a family of interpolators of the velocity assigned on $\Lambda(t)$ and $\partial\Omega$ among which, for any $\beta>0$, a $\hat{\mathsf{u}}$ can be selected such that (4.13) holds for every $\mathsf{v}\in X_t$. In order to obtain a divergence-free interpolator $\mathsf{g}(t,\mathsf{x})$, we consider, at any fixed time t, a neighborhood $R_t\subset\Omega$ of $\Lambda(t)$ with ∂R_t smooth. On R_t the field g is taken equal to the velocity of the rigid motion φ; since this is a divergence-free velocity and since we want g to vanish on $\partial\Omega$, we get

$$\int_{\partial(\Omega\setminus R_t)}\mathsf{g}\cdot\mathsf{n}=\int_{\partial\Omega}\mathsf{g}\cdot\mathsf{n}-\int_{\partial R_t}\mathsf{g}\cdot\mathsf{n}=0\,, \tag{4.16}$$

where n is the outer unit normal to $\partial(\Omega \setminus R_t)$. The existence of such a g is granted by the following lemma, which can be easily proved, adapting the argument in [15, Lemma 2.2].

Lemma 4.9. *Let A be an open connected subset of $\mathbb{R}^3$. For each $\tilde{\mathsf{g}} \in H^{\frac{3}{2}}(\partial A; \mathbb{R}^3)$ satisfying*

$$\int_{\partial A} \tilde{\mathsf{g}} \cdot \mathsf{n} = 0\,,$$

there exists a function $\mathsf{g} \in H^2(A; \mathbb{R}^3)$ such that $\operatorname{div} \mathsf{g} = 0$ in A and $\mathsf{g} = \tilde{\mathsf{g}}$ on ∂A.

Carefully considering what happens on ∂R_t, it turns out that the function

$$\mathsf{h} := \begin{cases} \varphi & \text{on } R_t \\ \mathsf{g} & \text{on } \Omega \setminus R_t \end{cases}$$

belongs to $H^2(\Omega; \mathbb{R}^3)$. Once h is found, we can construct $\hat{\mathsf{u}}$ satisfying (4.13) via two steps: we take R_t small enough, and then we apply Lemma 7.1 of [26, p. 103] on $\Omega \setminus R_t$ thanks to the regularity of ∂R_t.

Chapter 5

The role of higher-order material parameters

The aim of this chapter is to analyze the effects of the higher-order material parameters η_1, η_2, η_3 in some examples, in order to gain some insight into their meaning both from a physical and a mathematical point of view. It will become clear that η_1 is strictly related to the presence of concentrated interactions, while η_3 is only responsible for higher-order dissipation phenomena. The role of η_2 is less clear.

5.1 Dragged flow in a cylinder

Consider a fluid placed in the cavity between two infinitely long coaxial cylinders of radii $R_1 < R_2$; the flow is driven imposing a motion with constant velocity U of the inner cylinder along the axial direction e_z. Looking for cylindrically symmetric stationary solutions $u(r)\mathsf{e}_z$, where r is the cylindrical radius, in the case of Newtonian liquids with viscosity μ and the usual adherence condition on the walls, one easily finds

$$u(r) = U \frac{\log R_2 - \log r}{\log R_2 - \log R_1} \,. \tag{5.1}$$

It is clear that this solution has no continuous extension to the case $R_1 = 0$.

However, with a second-order linear isotropic viscous liquid the analogous problem gives the family of solutions

$$u(r) = \alpha_1 + \alpha_2 I_0\left(\frac{r}{L}\right) + \alpha_3 \log\left(\frac{r}{L}\right) + \alpha_4 K_0\left(\frac{r}{L}\right) \,, \tag{5.2}$$

where $\alpha_i, i = 1 \dots 4$, are constants (depending on R_1, R_2) fixed by the boundary conditions, I_0, K_0 are Bessel functions of imaginary argument [24, Sec. 5.7], and the parameter

$$L := \sqrt{(\eta_1 - \eta_2 - 4\eta_3)/\mu} \tag{5.3}$$

arises from the higher-oreder terms. If we now set $R_1 = 0$ the solution remains bounded provided that $\alpha_3 = \alpha_4$, since

$$\lim_{r\to 0} \left(\log\left(\frac{r}{L}\right) + K_0\left(\frac{r}{L}\right)\right) < +\infty \,; \tag{5.4}$$

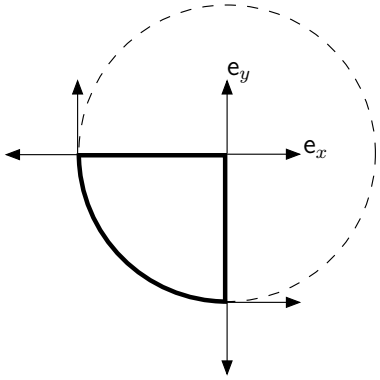

Figure 5.1: Section of the cylindrical sector.

besides, one can still meet the prescribed boundary conditions by a suitable choice of the constants.

Actually, true boundary conditions can now be imposed only at $r = R_2$, while, at $r = 0$, I will impose the value of the velocity field to be equal to U. This is an example of the constraint on the space of velocities which is considered in the previous chapter. Hence, being n the outer unit normal to the cylinder, the three conditions are:

$$u(0) = U\,, \tag{5.5}$$

$$u(R_2) = 0\,, \tag{5.6}$$

$$(\mathbf{G}\mathsf{n})\mathsf{n}|_{r=R_2} = 0\,. \tag{5.7}$$

While the first two conditions are clear and do not contain explicitly the higher-order parameters, condition (5.7) has to be specialized. On the surface of the cylinder we have

$$(\mathbf{G}\mathsf{n})\mathsf{n} = \eta_1 \frac{\partial^2 u}{\partial r^2} - (\eta_2 + 4\eta_3)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

which implies

$$\left[\eta_1 \frac{\partial^2 u}{\partial r^2} - (\eta_2 + 4\eta_3)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_{r=R_2} = 0\,. \tag{5.8}$$

Now, recall that the thermodynamical constraints (3.4) force η_2 to vanish if $\eta_1 = 0$, while η_3 can be any non-positive real number. On the other hand, in order to prove Proposition 4.4, η_1 cannot vanish, but if this is the case, with $\eta_3 < 0$, the fluid is still a non-simple one, thermodynamics is not violated, and we still have a unique solution according to Remark 4.4.

So far there is no clear distinction between the effects of the presence of the three parameters. Let us now calculate the concentrated force on the edges of a $\pi/2$-wide cylindrical sector, whose section is depicted in Fig. 5.1. Given the geometry, we have to evaluate the concentrated traction on the edges; with the notation of Proposition 2.4, we get

$$\hat{\mathsf{K}}_{\mathsf{u}} = \mathbf{G}(\mathsf{e}_x \otimes \mathsf{e}_y + \mathsf{e}_y \otimes \mathsf{e}_x) = 2\mathbf{G}(\mathsf{e}_x \otimes \mathsf{e}_y)\,;$$

indeed

$$\mathbf{G}_{112} = \mathbf{G}_{212} = 0\,, \quad \text{and} \quad \mathbf{G}_{312} = \eta_1 \frac{\partial^2 u}{\partial x \partial y}\,.$$

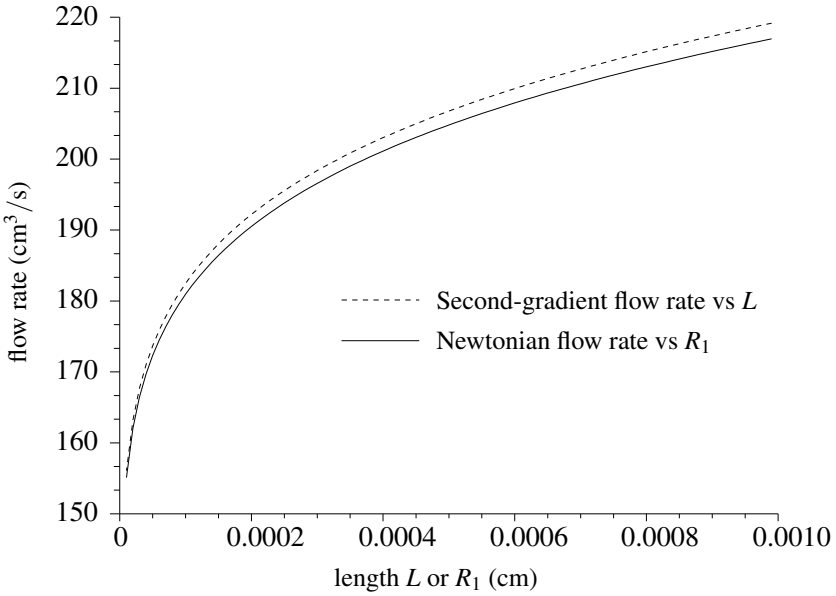

Figure 5.2: Comparison of the flow rates.

Hence it is clear that the parameter η_3 does not play any role in the determination of the concentrated force $\hat{\mathsf{K}}_{\mathsf{u}}$. Moreover, if $\eta_1 = 0$, no concentrated force can appear.

This result is in agreement with the general feature discussed in Section 2.2, and the results in [31, Sec. 4]. Indeed, the hyperstress takes the form

$$\mathbf{G} = -5\eta_3 \Delta \mathsf{u} \otimes \mathsf{I} + 3\eta_3 \operatorname{Sym}(\Delta \mathsf{u} \otimes \mathsf{I}) + \mathsf{I} \otimes \mathsf{p}\,,$$

whose active part, given the incompressibility constraint, is $\mathbf{G}_0 = -4\eta_3 \Delta \mathsf{u} \otimes \mathsf{I}$, which, by Corollary 2.6, cannot develop concentrated interactions.

Remark 5.1. The results of the present section show that the adherence to a one-dimensional object *should not* be viewed as a concentrated interaction; indeed, we can constitutively choose to have a fluid which can interact with one-dimensional objects without developing concentrated stresses. This is in fact the case when $\eta_1 = \eta_2 = 0$ and $\eta_3 < 0$, and that situation corresponds to the particular choice, introduced in Remark 4.4, of the functional space to which our velocity field belong.

Such a result is a bit surprising. But we have to properly understand the meaning of *concentrated interactions*; in fact, what we get in the case $\eta_1 = \eta_2 = 0$, $\eta_3 < 0$ is that the adherence to a one-dimensional structure can be represented, on any material surface containing it, by diffused surface interactions. On the other hand, when $\eta_1 > 0$, the representation of internal stresses can have a concentrated part on the singular edges of the surface of some subbody, even though there are no concentrated interactions on any part of the boundary or the interior of the whole body. ♢

As to the meaning of the length-scale L, an interesting feature appears. Computing the dependence upon R_1 of the flow rate through an annular section in the Newtonian case, and comparing it with the dependence upon L of the flow rate of our second-gradient fluid when the inner radius is equal to zero (Fig. 5.2), one can observe that the relative difference between

the two values computed at $R_1 = L$ is very small, indeed negligible when $R_1, L \ll 1$. This fact suggests that the parameter L represents a sort of *effective thickness* in the limiting case.

5.2 Pressure-driven flows

I will finally describe analytic expressions for the pressure-driven flow of a second-order liquid in a pipe with squared or circular section.

Consider a pipe with squared section in the (x, y)-plane, a velocity field $\mathsf{v} = w(x, y)\mathsf{e}_z$, and a constant and uniform pressure gradient $C\mathsf{e}_z$ whic drives the flow. The differential equation for the steady flow of a second-order liquid with viscosity μ, becomes

$$\mu\Delta w - \xi\Delta\Delta w = C\,, \tag{5.9}$$

with $w = 0$ and $\frac{\partial^2 w}{\partial n^2} = 0$ on the boundary of the pipe. Notice that Δ is the bidimensional Laplace operator in the (x, y)-plane. The previous equation can be written as

$$-\Delta(w - L^2\Delta w) = -\frac{C}{\mu}\,, \qquad L^2 := \frac{\xi}{\mu}\,,$$

and, since $w - L^2\Delta w = 0$ on the boundary, we can set $u = w - L^2\Delta w$, obtaining

$$\begin{cases} -\Delta u = -C/\mu =: \widetilde{C} \\ u = 0 \quad \text{on the boundary.} \end{cases} \tag{5.10}$$

Take now the section of the pipe to be $[0, \pi] \times [0, \pi]$, so that we can expand on the basis given by the eigenfunctions of the Laplace operator on that square:

$$X_{h,k} = \frac{4}{\pi^2} \sin hx \sin ky,$$

with eigenvalues $\lambda_{h,k} = h^2 + k^2$, with $h, k \in \mathbb{N}$. We have

$$\langle \widetilde{C}, X_{h,k} \rangle = \begin{cases} f_{h,k} := \frac{8\widetilde{C}}{\pi^2} hk & \text{for } h, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the solution of (5.10) is

$$\tilde{u} = \sum_{h,k \text{ odd}} \frac{f_{h,k}}{\lambda_{h,k}} X_{h,k}. \tag{5.11}$$

It remains to solve

$$-L^2\Delta w + w = \tilde{u}\,.$$

The operator $(-L^2\Delta + 1)$ has again $X_{h,k}$ as eigenfunctions, with eigenvalues $L^2\lambda_{h,k} + 1$; then we get

$$\begin{aligned} w(x, y) = \sum_{h,k \text{ odd}} \frac{f_{h,k}}{\lambda_{h,k}(L^2\lambda_{h,k} + 1)} X_{h,k} = \\ = -\frac{32C}{\mu\pi^4} \sum_{h,k \text{ odd}} \frac{hk}{(h^2 + k^2)(L^2h^2 + L^2k^2 + 1)} \sin hx \sin ky\,. \end{aligned} \tag{5.12}$$

Now we can evaluate the concentrated traction along the edges of the pipe. As in the previous section we have

$$\hat{\mathsf{K}}_{\mathsf{u}} = 2\mathbf{G}(\mathsf{e}_x \otimes \mathsf{e}_y)\,;$$

and

$$\mathbf{G}_{112} = \mathbf{G}_{212} = 0\,, \qquad \mathbf{G}_{312} = \eta_1 \frac{\partial^2 w}{\partial x \partial y}\,.$$

Once more, we see that a concentrated stress can appear only if $\eta_1 > 0$, while a second-order liquid with $\eta_1 = 0$ cannot develop such a concentration even if the boundary of the domain has a singular part.

Let us compute, with $\eta_1 > 0$, the concentrated stress density in the origin:

$$\hat{\mathsf{K}}_{\mathsf{u}}(0,0) = 2\eta_1 \frac{\partial^2 w}{\partial x \partial y}(0,0)\mathsf{e}_z = -2\eta_1 \frac{32C}{\mu\pi^4} \sum_{h,k \text{ odd}} \frac{h^2k^2}{(h^2+k^2)(L^2h^2+L^2k^2+1)}\mathsf{e}_z \neq \mathsf{0}\,.$$

Thanks to the symmetry of the problem, we get the same result on the remaining edges.

Take now an incompressible fluid which moves in a cylindrical pipe of radius R; the motion is again produced imposing a uniform pressure gradient $C\mathsf{e}_z$ ($C \in \mathbb{R}$) on the $z = 0$ circular section, being the axis of the cylinder along the z direction e_z. Looking for cylindrically symmetric stationary solutions $u(r)\mathsf{e}_z$, where $r^2 = x^2 + y^2$, in the case of Newtonian liquids with viscosity μ and the usual adherence condition, one finds

$$u(r) = \frac{C}{4\mu}(r^2 - R^2)$$

for the axial component of the velocity field.

With a second-order fluid the analogous problem gives the family of solutions

$$u(r) = \frac{C}{4\mu}r^2 + \alpha_1 + \alpha_2 I_0\left(\frac{r}{L}\right) + \alpha_3 \log r + \alpha_4 K_0\left(\frac{r}{L}\right),$$

where, as in the previous section, $\alpha_i, i = 1 \dots 4$, are constants fixed by the boundary conditions, L is the length-scale arising from the higher-order terms and I_0, K_0 are Bessel functions. Regularity assumptions for $r = 0$ imply $\alpha_3 = \alpha_4 = 0$, and the boundary conditions become

$$u(R) = 0\,, \tag{5.13}$$

$$(\mathbf{G}\mathsf{n})\mathsf{n}|_{r=R} = 0\,, \tag{5.14}$$

that is

$$\frac{C}{4\mu}R^2 + \alpha_1 + \alpha_2 I_0\left(\frac{R}{L}\right) = 0\,, \tag{5.15}$$

$$(\eta_1 - \eta_2 - 4\eta_3)\left(\frac{\alpha_2}{2L^2}\left[I_0\left(\frac{R}{L}\right) + I_2\left(\frac{R}{L}\right)\right]\right) - (\eta_2 + 4\eta_3)\left(\frac{C}{\mu} + \frac{\alpha_2}{LR}I_1\left(\frac{R}{L}\right)\right) = 0\,, \tag{5.16}$$

from which

$$\alpha_1 = -\frac{C}{4\mu}R^2 - \alpha_2 I_0\left(\frac{R}{L}\right), \tag{5.17}$$

$$\alpha_2 = \frac{C}{\mu}\frac{\eta_2 + 4\eta_3}{\frac{\eta_1-\eta_2-4\eta_3}{2L^2}\left[I_0\left(\frac{R}{L}\right) + I_2\left(\frac{R}{L}\right)\right] - \frac{\eta_2+4\eta_3}{LR}I_1\left(\frac{R}{L}\right)}\,. \tag{5.18}$$

Since the modified Bessel functions I_n grow exponentially fast, we see that, when L is small compared to the radius R of the cylinder, the second-order solution is very similar to the classical one.

Appendix A

Differentiable Banach manifolds

Within the present section I collected basic definitions and properties of Banach manifolds, which essentially are differentiable manifolds whose local charts are isomorphic to an open subset of a Banach space, not necessarily of $\mathbb{R}^n$, so that the more familiar theory of differentiable manifolds is recovered as a particular case. For a complete treatment of the subject the reader is referred to [23] and [1].

Definition A.1. Let S be a set. A *chart* on S is a bijection φ from a subset U of S to an open subset of a Banach space. We sometimes denote φ by (U, φ), to indicate the domain U of φ. A C^k *atlas* on S is a family of charts $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ such that:

(i) $S = \bigcup_{i\in I} U_i$;

(i) Any two charts in $\mathcal{A}$ are *compatible* in the sense that the overlap maps between members of $\mathcal{A}$ are C^k diffeomorphisms: for any two charts (U_i, φ_i) and (U_j, φ_j) with $U_i \cap U_j \neq \varnothing$, we define the *overlap map* on the intersection as $\varphi_{ji} := \varphi_i \circ \varphi_j^{-1}$, and we require $\varphi_i(U_i \cap U_j)$ to be open and φ_{ji} to be a C^k diffeomorphisms.

Definition A.2. Two C^k atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^k atlas. A C^k *differentiable structure* $\mathcal{D}$ on S is an equivalence class of atlases on S. The union of the atlases in $\mathcal{D}$,

$$\mathcal{A}_{\mathcal{D}} := \bigcup \{\mathcal{A} : \mathcal{A} \in \mathcal{D}\}$$

is the *maximal atlas* of $\mathcal{D}$, and a chart $(U, \varphi) \in \mathcal{A}_{\mathcal{D}}$ is an *admissible local chart*.

A *differentiable manifold* M is a pair $(S, \mathcal{D})$, where S is a set and $\mathcal{D}$ is a C^k differentiable structure on S. If a covering by charts takes values in a Banach space B, then B is called the *model space* and we say that M is a C^k Banach manifold modeled on B.

It is now clear that any Banach space B is a Banach manifold, whose differentiable structure is given by the atlas with the sole identity chart.

Definition A.3. Let $(S_1, \mathcal{D}_1)$ and $(S_2, \mathcal{D}_2)$ be two manifolds. The *product manifold*

$$(S_1 \times S_2, \mathcal{D}_1 \times \mathcal{D}_2)$$

consists of the set $S_1 \times S_2$ together with the differentiable structure $\mathcal{D}_1 \times \mathcal{D}_2$ generated by the atlas

$$\{\, (U_1 \times U_2, \varphi_1 \times \varphi_2) : (U_i, \varphi_i) \text{ is a chart of } (S_i, \mathcal{D}_i),\ i = 1, 2 \,\} \ .$$

Definition A.4. Let M be a manifold and $m \in M$. A *curve at* m is a C^1 map $c : I \to M$ from an interval $I \subseteq \mathbb{R}$ into M with $0 \in I$ and $c(0) = m$. Let c_1 and c_2 be two curves at m and (U, φ) an admissible chart with $m \in U$. Then we say c_1 and c_2 are *tangent at* m *with respect to* φ if and only if $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$.

Proposition A.1. *Let c_1 and c_2 be two curves at $m \in M$. Suppose (U_i, φ_i) are admissible charts with $m \in U_i$, $i = 1, 2$. Then c_1 and c_2 are tangent at m with respect to φ_1 if and only if they are tangent at m with respect to φ_2.*

By the previous proposition, tangency at $m \in M$ is an equivalence relation among curves at m, and an equivalence class with respect to tangency at m of such curves will be denoted by $[\, c\,]_m$, where c is a representative of the class.

Definition A.5. For a manifold M and $m \in M$ the *tangent space to* M *at* m is

$$T_m M := \{\, [\, c\,]_m : c \text{ is a curve at } m \,\};$$

given $A \subseteq M$ we set

$$TM|_A := \bigcup_{m \in A} T_m M\,,$$

and $TM := TM|_M$ is the *tangent bundle* on M. The mapping $\tau_M : TM \to M$, defined by $\tau_M([\, c\,]_m) = m$, is the *tangent bundle projection* on M.

Proposition A.2. *Let U be an open subset of the Banach space B, and let c be a curve at $u \in U$. Then there is a unique $b \in B$ such that the curve $c_{u,b}$ defined by $c_{u,b}(t) = u + tb$ (with t belonging to an interval I such that $c_{u,b}(I) \subset U$) is tangent to c at u.*

In particular TU can be identified with $U \times B$, and $T_u U$ is isomorphic to B for any $u \in U$.

Proof. By definition, the differential $Dc(0)$ is the unique linear map from $\mathbb{R}$ to B such that the curve $g : R \to B$, given by $g(t) = u + tDc(0)$ is tangent to c at $t = 0$. If $b = Dc(0)$, then $g = c_{u,b}$. Hence, the map $i : U \times B \to TU$ defined by $i(u, b) = [\, c_{u,b}\,]_u$, turns out to be a bijection; moreover, we can induce a manifold structure on TU by means of i, and this concludes the proof. □

I will now define cotangent spaces and bundles in the particular case which is of interest within the present work.

Definition A.6. Let U be an open subset of the Banach space B. Then, for any $u \in U$, the topological dual of $T_u U \cong B$ defines the *cotangent space* at u, denoted by $T_u^* U$, and

$$T^* U := \bigcup_{u \in U} T_u^* U$$

is the *cotangent bundle* on U.

Appendix B

Differential operators on a surface

Given a smooth surface $\mathcal{S}$ embedded in $\mathbb{R}^3$, we can define the projection operator P on $\mathcal{S}$, and its composition with the gradient and the divergence operator. Let n be the unit outer normal to $\mathcal{S}$; for any vector field a and any tensor field A, we have:

$$\begin{aligned}
\mathsf{P} &:= \mathsf{I} - \mathsf{n} \otimes \mathsf{n} \,, \\
\nabla_{\mathcal{S}} \mathsf{a} &:= (\nabla \mathsf{a})\mathsf{P} \,, \\
\operatorname{div}_{\mathcal{S}} \mathsf{a} := \operatorname{tr}(\nabla_{\mathcal{S}} \mathsf{a}) &= \mathsf{P} \cdot \nabla \mathsf{a} = \operatorname{div} \mathsf{a} - \mathsf{n} \cdot (\nabla \mathsf{a}) \mathsf{n} \,, \\
(\operatorname{div}_{\mathcal{S}} \mathsf{A})_i &:= A_{ij,k} P_{kj} \,,
\end{aligned}$$

where $\operatorname{tr}(\mathsf{A})$ denotes the trace of the matrix A. Moreover, the normal derivative is defined as

$$\frac{\partial \mathsf{a}}{\partial n} := (\nabla \mathsf{a}) \mathsf{n} \,;$$

the curvature tensor $\mathsf{C}(\mathsf{x})$ and the mean curvature $k(\mathsf{x})$ of $\mathcal{S}$ are defined by

$$\begin{aligned}
\mathsf{C}(\mathsf{x}) &:= -\nabla_{\mathcal{S}} \mathsf{n} = -(\nabla \mathsf{n}) \mathsf{P} \,, \\
k(\mathsf{x}) &:= \frac{1}{2} \operatorname{tr} \mathsf{C} = -\frac{1}{2} \operatorname{div}_{\mathcal{S}} \mathsf{n} \,.
\end{aligned}$$

for any $\mathsf{x} \in \mathcal{S}$.

With these definitions we have the following important result.

Surface-divergence Theorem B.1. *Let $\boldsymbol{\tau}$ be a vector field tangent to $\mathcal{S}$, and let $\mathcal{U} \subseteq \mathcal{S}$ be a subsurface with $\boldsymbol{\nu}$ unit outer normal to $\partial\mathcal{U}$; then*

$$\int_{\partial \mathcal{U}} \boldsymbol{\tau} \cdot \boldsymbol{\nu} = \int_{\mathcal{U}} \operatorname{div}_{\mathcal{S}} \boldsymbol{\tau} \,. \tag{B.1}$$

By direct application of the previous theorem to suitable tangential fields, we get the following corollary.

Corollary B.2. *For any tensor field A, any vector field a and any virtual velocity v, we have:*

$$\operatorname{div}_{\mathcal{S}}(\mathsf{AP}) = \operatorname{div}_{\mathcal{S}} \mathsf{A} + 2k\mathsf{An} \,, \tag{B.2}$$

$$\operatorname{div}_{\mathcal{S}}(\mathsf{A}^{\dagger}\mathsf{a}) = \mathsf{a} \cdot \operatorname{div}_{\mathcal{S}} \mathsf{A} + \mathsf{A} \cdot \nabla_{\mathcal{S}} \mathsf{a} \,, \tag{B.3}$$

$$\int_{\mathcal{S}} \mathsf{A} \cdot \nabla_{\mathcal{S}} \mathsf{v} = -\int_{\mathcal{S}} (\operatorname{div}_{\mathcal{S}} \mathsf{A} + 2k\mathsf{An}) \cdot \mathsf{v} \,. \tag{B.4}$$

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