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Nonatomic Games with Limited Anonymity*

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Abstract

After a brief survey of the literature about the existence of a Nash Equilibrium in the class of the nonatomic games, we prove the existence of an equilibrium in the class of the nonatomic games where the players' payoff depends over the average strategy of finitely many convex and disjoint subsets of players. Finally, several applications are shown, in the context of the economics of science and namely about the problem of the topic choice made by the set of the researchers, represented as a continuum.

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1 Introduction

Game theory studies the multipersonal decision problems, where a player's action has an effect on the other player's payoff and viceversa, and a player's action influences the other's decision. This framework seems intuitively well working in all the cases where the number of players is not too large, such as in the oligopoly, in the war games, in the bargaining games and so on. What happens with many players, that is in large games? For instance, as the number of firms in the Cournot game increases, the equilibrium tends, under very general conditions, to the competitive one (at the limit). What has to be remarked is that the typical outcome features of the Cournot game progressively disappear, that is the aggregate and individual productions are less and less restricted and the price converges to its competitive level: indeed, we obtain (at the limit) the competitive equilibrium. Thus, when the number of players increases (however, we have only countably many players), the strategic interaction effect decreases and, always at the limit, it becomes negligible. Nevertheless, it remains true that the aggregate behavior of a subset of players large enough (at the limit, an infinite subset) has an effect on the others' choice, at equilibrium. This is particularly evident, in a game with positive externalities (resp. with partial rivalry) like in Konishi, Le Breton and Weber (1997a, 1997b), i.e. a game where the payoff of player i depends positively (resp. negatively) on the number of players that plays in the same manner of i. A typical example of a game with partial rivalry is the problem of the city traffic: when one has to decide what road to intake with her car, she takes into account how much traffic is present over any alternative. Traffic is the distribution of all other drivers on the road network at the same time. Many other examples can be found in the context of the use of a congestioned good, such as internet, roads, electricity networks, when agents take into account congestion. Congestion, as traffic, can be measured by the agents' distribution on the networks at a given instant.

With countably many players, we can define a sequence of n-players games. At any point of the sequence, the strategic interaction effect exists and it is non negligeable. Only at the limit it disappear, as previously said. Even for a very large but finite n, any rational player is able to distinguish and to evaluate it, playing accordingly. Therefore, in a sense, at the limit (and only here), the game becomes "odd". Nevertheless, there are situations in the real world where the number of players is naturally very high and each player knows that she has no effect on the others. In this case it seems me not conceptually correct to model a n-player game and then computing the limit of the finite game outcome to approach the real world situation we are studying. What I wish to underline is that the game is naturally "odd". Examples can be the voting decision, the determination of an equilibrium price in a market, the topic choice for the whole set of young researchers in economics. In all these cases, the player's choice effect on the others is negligible if taken isolately.

When the situation we wish to model is such that the strategic interaction between individuals is extremely poor because of the great number of players, we can use the so called nonatomic games. The class of the nonatomic games allows to deal with problems where there is a continuum (i.e. uncountably many) of players. More precisely, a nonatomic game is a game where the set of players is endowed with a nonatomic measure. Indeed, the nonatomic games allows us to model several situations where individual weight in the "competition" is almost nil, but where the aggregate choice of a "large number", a mass, of players is relevant. In an election a single vote is normally not relevant to determine the winner, and so the voters' utility. On the contrary, the vote of the subset of young voters or that of the old voters may have a great impact on the voting result.

In the General Economic Equilibrium literature there are several papers that model the price formation and the trading in a large economy as a non atomic game. In particular, Dubey and Shapley (1994) give some interesting extensions in the non atomic framework of the finite market game proposed by Shapley and Shubik (1977), considering two different ways of payment (by paper money and by a valuable commodity). The fundamental feature of this approach is that players do not take the prices as given. They have an initial endowments of goods, they place it on the goods markets receiving the right to an amount of money when the prices will be fixed. Then they demand an amount of each commodity (this is the players' strategy), depending on their preferences, bidding an amount of money. Prices are formed in such a way that clear all the markets, i.e. the price of each good equals the ratio between the aggregate bid for that good and the aggregate endowment of it. Since the obtained bundle will depend on the formed prices, the players' utility, i.e. the players' payoff, will depend on the others' aggregate behavior. Under fairly weak conditions, Dubey and Shapley find that the Walrasian equilibria class and their strategic equilibria class are equivalent.

Also, Godognato and Ghosal (2001) extend in the non atomic game framework the Forges and Minelli (1997) paper "Self-Fulfilling Mechanisms and

Rational Expectation" in an exchange economy with private information.

The proof of the equilibrium existence for the nonatomic games (general and in pure strategies), came only in 1973 with the paper of Schmeidler. Thereafter, other authors gave proofs of the existence in pure strategies, following different approaches, such as Rath (1992) and Mas Colell (1984).

The existence in pure strategies depends on the property of anonymity. Roughly speaking anonymity means that every player (or agent) is symmetric to all the others. The single player's characteristics are not relevant in determining his impact. Therefore every player is interchangeable. Such a property has to be differently formalized in different contexts. Here are two examples. In the already mentioned finite games with positive externalities considered by Konishi, Le Breton and Weber (1997a), the strategic interaction is captured only by the fact that the player i's payoff is a (increasing) function of the number of players that play the alternative chosen by i; considering the number of players is a way to formalize the idea of anonymity: what matters is not who is the player that chooses a certain alternative, but only how many players choose it.

The second example comes from the social choice theory. We are interested in finding a good aggregator of preferences, a social welfare functional. Consider the majority voting rule: it is anonymous since if you permute in any possible way the preference profile, the result does not change because, again, what matters is the number of people that prefers a particular alternative and not the characteristics of such people.

In the nonatomic games the anonymity property (necessary to prove to existence in pure strategies) is formalized making the payoff functions dependent on the players' distribution over the alternatives. More precisely, the payoff functions depend on the Lebesgue integral of the strategy profile over the players' set. It is clear that in this way, we are interested only on the aggregate behavior i.e. on the measure of the players' set which chooses a particular alternative.

A further interest of the nonatomic games is that the existence of a Nash equilibrium, even in pure strategies, does not require strong hypothesis on the payoff functions. Only continuity with respect to the Lebesgue integral of the strategy profile is required.

In this paper I wish to briefly make a review or a little survey over the mentioned three different approaches. After, starting for a remark of Schmeidler (1973), following the Rath's approach, I extend the existence result (in pure strategies) to the class of games where payoffs depend on the average strategy of a finite number of convex and disjoint subsets of players, defined *a priori*. This generalization allows to greatly extend the domain of applicability of the nonatomic games to a quite large set of situations at little mathematical cost. This depends on the fact that it allows to reduce the anonymity in these games: belonging to a particular subset may be a feature that gives a different relative weight to different players. One may imagine several examples: for instance the "common" researcher's topic choice may depend on the distribution of the "stars" and on the distribution of the other "common" researchers; still, the driver's route choice may depend on the distribution of several groups that are moving on the same road network, such as car drivers and heavy truck drivers.

Since the partition of the players' set, unique for all the players, has to be defined *a priori*, we cannot deal, for instance, with games where the player's choice depends primarily on the choices of the players close to him[1], i.e., where the player t's choice depends on the actions of the players belonging to a neighborhood of t.

In the following, I will refer to the nonatomic games belonging to the generalized class defined above, as nonatomic games with "limited anonymity".

Finally, I will present three examples of nonatomic games with "limited anonymity" that may help to evaluate some features of the equilibria arising.

2 Three proofs of existence

In this section I present the three approaches to prove the existence of a Nash equilibrium in a normal form nonatomic game. The first will be the Schmeidler's proof, followed by that of Rath and that of Mas Colell. I will simply present quite informally the framework and the results, whereas I will present a formal proof in the next section to derive the extension described in the introduction.

The common framework and notation for the three proofs is the following. Specifications and particularities will be remarked for each one.

$T = [0, 1]$ is, without loss of generality, the set of players endowed with the Lebesgue measure λ.

There are n possible alternatives (actions) each of them represented by a unit vectors in $\Re^n$, that is, the vector e_i is the unit vector with 1 as i-th

[1] We think to a sort of influence with a limited extent of diffusion.

coordinate and zero otherwise and it is associated with the i-th alternative. Therefore the set of alternatives is $E = \{e_1, ..., e_n\}$.

The convex hull $\widehat{E} = conv(\{e_1, ..., e_n\})$ is the set of the all possible mixed strategies. Therefore the pure strategy e_i is a particular mixed strategy, as usual.

A mixed strategy profile is a measurable function $\widehat{f} : T \to \widehat{E}$ that associates to each player $t \in T$ an element of $\widehat{E}$ denoted as the n-vector $(\widehat{f}^1, ..., \widehat{f}^n)$ where $\widehat{f}^i$ is the real valued component from T to $[0, 1]$. In other words, we associate to each t the probability of playing each alternative. $\widehat{F}$ is the set of all possible strategy profiles.

A pure strategy profile is a measurable function $f : T \to E$ that associates to each $t \in T$ an element of E denoted as the n-vector $(f^1, ..., f^n)$ where f^i is the integer valued component from T to $\{0, 1\}$. F is the set of all possible strategy profiles.

We denote $S = \{\int_T f(t)d\lambda | f \in F\}$ the set of all Lebesgue integrals of the strategy profile defined as $(\int_T f^1(t)d\lambda, ..., \int_T f^n(t)d\lambda)$, or, in words, the Lebesgue integral of the vector f is the integral of all its coordinates. Finally, we denote with s an element of S. It is important to remark that S can be identified by the simplex in $\Re^n$.

2.1 Schmeidler (1973)

Schmeidler proves two results: the existence of a Nash equilibrium and the existence of a Nash equilibrium in pure strategies for the nonatomic games.

The set $\widehat{F}$, the set of all mixed strategy profiles, is endowed with the L_1 weak topology. This set is a compact, convex subset of a locally convex linear topological space.

Now, we define an auxiliary function $v(\cdot, \cdot) : T \times \widehat{F} \to \Re^n$. Its component $v^i(t, \widehat{f})$ describes the utility of player $t \in T$ playing e_i when almost every player chooses $\widehat{f}$, i.e. each player plays his mixed strategy, and player t chooses the pure strategy e_i. So the payoff of player t is defined as

$$u_t(\widehat{f}) = \widehat{f}(t) \cdot v(t, \widehat{f})$$

or the inner product in $\Re^n$. Thus, the payoff, playing the mixed strategy, is simply an expected payoff obtained using the probability distribution represented by the mixed strategy.

By now we have described the normal form of the game.

We need two assumptions.

1. For all $t \in T$, $v(t, \cdot)$ is continuous on $\widehat{F}$.

2. For all $\widehat{f}$ in $\widehat{F}$ and $i, j = 1, ..., n$ the set $\{t \in T | v^i(t, \widehat{f}) > v^j(t, \widehat{f})\}$ is measurable. In words, this is the set of all those players that prefer the pure strategy e_i to the pure strategy e_j, given $\widehat{f}$.

Definition 1 *A strategy profile $\widehat{f}$ is a Nash Equilibrium iff*
$\forall m \in \widehat{E} \qquad u_t(\widehat{f}) \geq m \cdot v(t, \widehat{f})$ *a.e.*

Definition 2 *A strategy profile $p \in \widehat{F}$ is called pure if and only if almost each player chooses a pure strategy.*

Now we can state the two results of Schmeidler.

Theorem 3 *A nonatomic game in normal form fulfilling conditions (1) and (2) has a Nash Equilibrium.*

Theorem 4 *If, in addition to the conditions of Theorem 3, a.e., $v(t, \widehat{f})$ depends only on $\int_T \widehat{f}$, then there is a Nash equilibrium in pure strategies.*

The proof of the Theorem 3 is, as usual, based on a fixed point argument.

Schmeidler first defines the best reply correspondence for the player t and given the strategy profile $\widehat{f}$ as

$$B(t, \widehat{f}) = \{m \in \widehat{E} | \forall m' \in \widehat{E} : m \cdot v(t, \widehat{f}) \geq m' \cdot v(t, \widehat{f})\}$$

The best reply correspondence is convex valued and, since $v(\cdot, \cdot)$ is continuous on $\widehat{F}$, it is non empty and has closed graph. The main interest of the proof lies in the following correspondence $\alpha : \widehat{F} \to \widehat{F}$ defined as:

$$\alpha(\widehat{f}) = \{\widehat{g} \in \widehat{F} | \text{ a.e. } \widehat{g}(t) \in B(t, \widehat{f})\}$$

Such a function associates to each mixed strategy profile the set of the mixed strategy profiles with the property that for almost every player, the mixed strategy played by player t belongs to the best reply correspondence of t, given the strategy profile $\widehat{f}$. That is $\alpha(\widehat{f})$ is the set of the best responses profile of all agents facing the profile $\widehat{f}$, or in other words, the set of the best deviation profiles given $\widehat{f}$. It is clear from now that if the best deviation profile to $\widehat{f}$ is $\widehat{f}$, then $\widehat{f}$ is a Nash equilibrium. Hence, we want the function $\alpha(\widehat{f})$ having a fixed point. Indeed Schmeidler shows that $\alpha(\widehat{f})$ is non empty, convex and it has closed graph. Therefore, by the Fan-Glicksberg fixed point theorem a fixed point exists and the proof is done.

Theorem 4 is a corollary of Theorem 3.

We have to show that there exists a pure strategy profile p that has the same Lebesgue integral of the mixed strategy profile of equilibrium $\widehat{f}^*$ and such a pure strategy profile belongs for almost all players to the respective best reply correspondence, given $\widehat{f}^*$. In other words, we need that the effect on the payoff of the two strategies is the same, since we assumed that the payoffs depend on $\int_T \widehat{f}$, and that the player is indifferent between playing the pure strategy or the mixed strategy of equilibrium, since both p and $\widehat{f}^*$ belong to $B(t, \widehat{f}^*)$. Roughly speaking, p is an "alternative" as good as $\widehat{f}^*$ for almost all players.

Since $B(t, \widehat{f}^*)$ is convex valued, if more than one pure strategy belongs to $B(\cdot, \cdot)$ therefore all the mixed strategies (=convex combinations) that assign positive probabilities only to these alternatives, belong to $B(\cdot, \cdot)$. More formally,

$$B(t, \widehat{f}^*) = conv(\{e_i | e_i \in B(t, \widehat{f}^*)\})$$

To show this, we use an intuitive argument. Suppose that $B(t, \widehat{f}^*) = \{e_1, e_2\}$. By the definition of integration of correspondence, we know that $\int_T \{e_1, e_2\} d\lambda = \{(a, 1-a, 0, ..., 0)$ for all $a \in [0,1]\}$. This is noting else that the convex hull of $\{e_1, e_2\}$. Now, if we integrate the set $\{(a, 1-a, 0, ..., 0)$ for all $a \in [0,1]\}$ we obtain the same set, because a linear combination of two elements of another linear combination belongs to this last linear combination.

Indeed we have:

$$\int_T B(t, \widehat{f}^*) = \int_T \{e_i | e_i \in B(t, \widehat{f}^*)\}$$

By definition, the Lebesgue integral of a correspondence is the set of Lebesgue integrals of all integrable selections belonging to the correspondence. Therefore we have that:

$$\int_T B(t,\widehat{f}^*) = \left\{ \int_T p \ \mid \ p \in \widehat{F} \text{ and a.e. } p(t) \in B(t,\widehat{f}^*) \right\}$$

$$\int_T \{e_i | e_i \in B(t,\widehat{f}^*)\} = \left\{ \int_T p \ \mid \ p \in \widehat{F} \text{ and a.e. } p(t) \in \{e_i | e_i \in B(t,\widehat{f}^*)\} \right\}$$

Clearly a selection of the set $\{e_i | e_i \in B(t,\widehat{f}^*)\}$ is a profile of pure strategy. Recall that $\int_T \widehat{f}^* \in \int_T B(t,\widehat{f}^*)$, since $\widehat{f}^*$ is a selection of $B(t,\widehat{f}^*)$. Combining the two equalities above, we have that $\int_T \widehat{f}^* \in \left\{ \int_T p \ \mid \ p \in \widehat{F} \text{ and a.e. } p(t) \in \{e_i | e_i \in B(t,\widehat{f}^*)\} \right\}$ and therefore there exists a pure strategy profile with the same integral of $\widehat{f}^*$ that also belongs to $B(t,\widehat{f}^*)$ for almost all t.

The measurability condition (2) ensures that everything is integrable.

The main result is surely the existence in pure strategies. Notice that it is ensured when the payoff function depends on the average strategy played by the other players, i.e., on their distribution over the alternatives. An interpretation of mixed strategies in the finite game is simply obtainable by this feature. Hence, we can imagine a mixed strategy of a finite game as the distribution of a continuum of players over the alternatives in a nonatomic game.

2.2 Rath (1992)

Rath proves directly the existence of a pure strategy Nash equilibrium, assuming that the payoff functions depend on the average response of the other players. This simply means, as in Schmeidler, that the payoff functions depend on the Lebesgue integral of the strategy profile. What differs from the Schmeidler's proof is the fact that here the existence in pure strategies is proved directly, i.e., without passing through the mixed strategies profile . Not only, the proof is much simpler, is based on the best reply correspondences and simply applies the Kakutani's fixed point theorem.

Since we consider only the pure strategy existence, we consider only the class of anonymous nonatomic games, i.e., as in Theorem 4, the class of nonatomic games whose payoff functions depend only on the Lebesgue integral of the (pure) strategy profile. Indeed, we consider the class of payoff function $U = \{u : T \times E \times S \to \Re\}$, real valued and continuous on $E \times S$. This restriction is the assumption 1 in the Schmeidler framework.

Notice that, as in Theorem 4, the Rath's payoff functions are defined on S, the set of the Lebesgue integrals of the strategy profiles[2]. Moreover, here we have to make apparent the dependance from the chosen alternative, while with the Schmeidler framework in mixed strategies this dependance was implicit on the definition of the payoff function as a inner product, or as an expected payoff function.

Definition 5 *A Nash equilibrium of a game is a pure strategy profile $f \in F$ such that for almost every t, $u(t, f(t), \int_T f) \geq u(t, e_i, \int_T f)$ $\forall e_i \in E$.*

We need the following assumption, completely parallel to the assumption 2.

For any $s \in S$ and $e_i, e_j \in E$, the set $\{t \in T | u(t, e_i, s) > u(t, e_j, s)\}$ is measurable.

Given this framework, the main result is:

Theorem 6 *Under the assumption above, every nonatomic game with payoff function belonging to the class U, has a Nash Equilibrium in pure strategies.*

As mentioned above, the proof is based on a fixed point argument and namely calls for the Kakutani's fixed point theorem. Since from the beginning only the anonymous nonatomic games are considered, the proof is greatly simplified. The identification of S with the simplex in $\Re^n$ allows us to consider only a finite dimension Euclidean space.

The first step is to define the best reply correspondence $B : T \times S \to E$ by

$$B(t, s) = \{a \in E | u(t, a, s) \geq u(t, e_i, s) \ \forall e_i \in E\}$$

Notice that s, representing the Lebesgue integral of a strategy profile, is the distribution of the players over the alternatives. Given such a distribution, $B(t, s)$ is the set of the best responses for t.

[2]Notice also that the set of Lebesgue integrals of mixed strategy profiles coincide with the set of Lebesgue integrals of pure strategy profiles.

Given the continuity of the utility function, this correspondence is non empty and has closed graph.

The second and interesting step of the proof is to define the correspondence $\Gamma : S \to S$ by

$$\Gamma(s) = \int_T B(t,s)d\lambda$$

If the correspondence Γ had a fixed point, then it would exist a pure strategy profile f^* such that $\int_T f^* d\lambda = s^* \in \int_T B(t,s^*)d\lambda$. This would mean that $f^*$is a selection of $B(t,s^*)$ or that for almost all t, the strategy $f^*(t)$ belongs to the best reply of t. Then the profile f^* would be a Nash Equilibrium. In fact, Rath shows that $\Gamma(s)$ is non empty and convex and has closed graph. Therefore, by the Kakutani's theorem, $\Gamma(s)$ has a fixed point.

We will use this kind of proof to show the extension mentioned in the introduction and discussed in the next section.Mas-Colell (1984).

2.3 Mas-Colell (1984)

The Mas-Colell's work is quite different from the previous two because he directly considers the (probability) distribution of the players rather than the Lebesgue integral of the strategy profile as an argument of the payoff functions. Therefore the kind of proof is quite different and it is not based on the properties of the Lebesgue integration.

As Rath, he shows the existence of an equilibrium in pure strategies.

Let us present the framework.

E is always the set of actions[3]. We consider the set of all possible probability distribution on E. Clearly, such a set is the simplex in $\Re^n$, i.e. the set S[4].

A player is completely characterized by a continuous utility function

$$u : E \times S \to \Re$$

[3] Actually, Mas-Colell is much more general, allowing for any non empty and compact metric space.

[4] Again, the author is much more general and considers the set of the Borel probability measures on the action space, endowed with the weak convergence topology. Here we make this simplification to unify our presentation of the three approaches to prove the Nash Equilibrium existence in the nonatomic games.

This is another way to allow each player have a possible different utility function. In the following we name a player by his utility function. In particular, given an action $e_i \in E$ and a distribution $s \in S$, $u(e_i, s)$ is the utility enjoyed by the player. Notice that using the concept of distribution, we consider anonymous nonatomic games, exactly as in Schmeidler (Theorem 4) and in Rath.

U_E is the space of all continuous utility function $u(\cdot, \cdot)$ endowed with the supremum norm. This represent also the space of players characteristics.

A game with a continuum of players is then characterized by a Borel measure μ on U_E. Notice that here the "number" of players does not matter. The probability distribution is normalized and works for all set of players. What matters are the characteristics of the players or their heterogeneity.

Definition 7 *Given a game μ, a Borel measure τ on $U_E \times E$ is a Nash Equilibrium distribution if, denoting τ_U, τ_E the marginals of τ on U_E and E respectively, we have*

(i) $\tau_U = \mu$

(ii) $\tau(\{(u, a) | u(a, \tau_E) \geq u(a', \tau_E) \ \forall e_i \in E\}) = 1$

Let us discuss on this definition.

The point (i) requires simply that all player characteristics be taken into account in the equilibrium distribution. The point (ii) requires that for almost each characteristic, there is a best reply alternative $a \in E$ given the marginal distribution of the players over the alternative set. We can also read such a condition as "the probability of the set of the pairs (u, a) such that $u(a, \tau_E) \geq u(e_i, \tau_E) \ \forall e_i \in E$ is one".

The main result is:

Theorem 8 *Given a game μ on U_E there exists a Nash Equilibrium distribution.*

The proof is an application of the Ky Fan fixed point theorem.

Let us denote by Ω the set of probability measures on $U_E \times E$ with the property that $\tau_U = \mu$, i.e., the set of all the distribution that verifies the condition (i).

Given $\tau \in \Omega$, $B_\tau = \{(u, a) | u(a, \tau_E) \geq u(e_i, \tau_E) \ \forall e_i \in E\}$ is the set of all the pairs (u, a) considered above. Now a correspondence $\Phi : \Omega \to \Omega$ is defined by

$$\Phi(\tau) = \{\tau' \in \Omega | \tau'(B_\tau) = 1\}$$

Such a correspondence draws all the joint distributions that verify the condition (i) and (ii) given the joint distribution τ. It is the equivalent in this context of the $\alpha(\cdot)$ function of Schmeidler: it associates to each distribution τ the (set of the) best deviation distribution from τ. Clearly, if a fixed point exists, the distribution τ is a Nash Equilibrium distribution. In fact, such a correspondence in shown convex valued, upper hemicontinuous and compact valued. Therefore there exists a fixed point by the Ky Fan theorem.

Notice how such framework is very general and relatively low demanding to show the existence of a Nash Equilibrium in pure strategies. What is determinant is the way to define the Nash Equilibrium distribution. Thereafter the procedure is quite usual.

3 An extension

It is apparent in what precedes that the existence of a pure strategy equilibrium is a consequence of the payoff function dependence over the average strategy of the players, i.e., over the distribution of the players on the alternatives. In other words, the equilibrium existence in pure strategies depends on the anonymity assumption.

Nevertheless, anonymity is not always a good requirement if we are interested in modelling settings where there are different groups of players that have different impacts on the payoff function and so on the choice of a given player. One may imagine several examples: for instance the researcher's topic choice may depend on the distribution of the stars and on the distribution of the other researchers (a situation examined in the next section); still, the driver's route choice may depend of the distribution of several groups that are moving at his same time, such as students, workers, employees etc.

Inside each group there is no reason to give up anonymity, but, between the groups, having complete anonymity impedes to model correctly the setting. A way to introduce a "limited anonymity" is to consider that the payoff functions depend over several average responses, one for each group. In this way is possible to consider the different weight that a particular group decision has on the choice of a given player.

It is worth to remark that the group definition has to be a fixed partition, i.e. it cannot be player-dependent. For instance, it is not possible, in the context of this extension, to consider situations where the player t's payoff depends mainly on the strategies adopted by the players close to him, i.e.

belonging to a symmetric neighborhood $N_t \subset T$ of t. In this case, we would have a binary partition $\{N_t, T \backslash N_t\}$, *for each* t.

In the following I will state the framework and prove the existence of a such equilibrium in pure strategies. The proof is an extension of the Rath's proof.

Consider a set $T = [0, 1]$ that represents the set of all players in the game. Such a set is endowed with the atomless Lebesgue measure λ. Consider k real numbers in T, denoted as $\tau_1 < ... < \tau_k$. Let τ_0 be 0 and τ_k be 1 (the boundaries of the T interval).

Denote $T_1 = [0, \tau_1]$ and T_h the subset $]\tau_{h-1}, \tau_h]$. By construction, we have that $\bigcup_{h=1}^{k} T_h = T$. Therefore, the T_h subsets represent the groups of players discussed above.

The set of alternatives is the set of the unit vectors in $\Re^n$ where the vector e_i has one at the i-th coordinate.

A pure strategy profile is a measurable function $f : T \to E$ which associates an alternative to each player.

Since, if a function is integrable on T, then it is integrable on any T_h, we denote with $S_h = \{\int_{T_h} f d\lambda | f \in F\}$ for $h = 1, ..., k$ the set of the Lebesgue integrals for any possible strategy profile f. We denote an element of S_h as s_h. Notice that $S_1 \times ... \times S_k$ is a compact and convex subset of $\Re^{k \times n}$ and that $S_h = \{(s_h^1, ..., s_h^n) \in \Re^+ | \sum_{i=1}^{n} s_h^i = \tau_h - \tau_{h-1}\}$ or the $(\tau_h - \tau_{h-1})-$simplex in $\Re^n$.

We define the payoff function as

$$u : T \times E \times S_1 \times ... \times S_k \to \Re$$

and we require that it is continuous on $E \times S_1 \times ... \times S_k$.

We assume that $\{t \in T | u(t, e_i, s_1, ..., s_k) > u(t, e_j, s_1, ..., s_k)\}$ is measurable for any $s_1, ..., s_k$ and for any $e_i, e_j \in E$.

Definition 9 *A pure strategy profile* $f \in F$ *is a Nash Equilibrium if for almost all* $t \in T$, $u(t, f(t), \int_{T_1} f, ..., \int_{T_k} f) \geq u(t, e_i, \int_{T_1} f, ..., \int_{T_k} f)\ \forall e_i \in E$.

The result of this extension is:

Theorem 10 *The normal form game* $\{T, (\tau_1, ..., \tau_k), E, u\}$ *has a Nash Equilibrium in pure strategies.*

Proof. Define the best reply correspondence $B : T \times S_1 \times ... \times S_k \to E$ as

$$B(t, s_1, ..., s_k) = \{a | u(t, a, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k) \ \forall e_i \in E\}$$

For any $(t, s_1, ..., s_k)$, $B(t, s_1, ..., s_k)$ is non empty because of the finite number of alternatives and because of the continuity of the utility function. For any $t \in T$, $B(t, \cdot)$ has closed graph. Indeed, for any pair of sequences $\{s_1^m, ..., s_k^m\} \to (s_1, ..., s_k)$ and $\{a^m\} \to a$ such that $a^m \in B(t, s_1^m, ..., s_k^m) \ \forall m$ we have that $u(t, a^m, s_1^m, ..., s_k^m) \geq u(t, e_i, s_1^m, ..., s_k^m) \ \forall e_i \in E$. Since $u(t, \cdot)$ is continuous on $E \times S_1 \times ... \times S_k$ at the limit we have that $u(t, a, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k) \ \forall e_i \in E$ and thus the graph is closed.

Let us define the correspondence $\Gamma : S_1 \times ... \times S_k \to S_1 \times ... \times S_k$ as

$$\Gamma(s_1, ..., s_k) = \prod_{h=1}^{k} \left[\int_{T_h} B(t, s_1, ..., s_k) \right]$$

- Γ is non empty for all $s_1, ..., s_k$.

  Fix a profile $(s_1, ..., s_k) \in S_1 \times ... \times S_k$. For any $e_i, e_j \in E$ define the set $V_{ij} = \{t \in T | u(t, e_i, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k)\}$ or the set of all those players that prefer e_i to e_j when facing the profile $(s_1, ..., s_k)$. Because of the assumption of measurability, such a set is measurable. Now we construct a partition of T starting from the family of set V_{ij}. $V_i = \bigcap_{j \neq i} V_{ij}$ is the set of players that prefer e_i to any other alternative. Such a set is measurable. Let $V_1' = V_1$ and $V_i' = V_i \cap (\cup_{j<i} V_j')^c$ for $i = 2, ..., n$. By construction $\{V_1', ..., V_n'\}$ is a partition of T of measurable subsets.

  Let us define the function $g : T \to E$ as $g(t) = e_i$ if $t \in V_i'$. Therefore $g(\cdot)$ is measurable by definition and $g(t) \in B(t, s_1, ..., s_k)$ for all $t \in T$, since $g(t)$ represents the best response for t, by construction. Therefore for any $(s_1, ..., s_k)$ there exists a measurable selection of $B(\cdot, s_1, ..., s_k)$ represented by $g(t)$. Thus $\Gamma(s_1, ..., s_k)$ is non empty for any $(s_1, ..., s_k)$.

- $\Gamma(\cdot)$ is convex valued.

  Since λ is atomless $\Gamma(\cdot)$ is convex valued (this comes from the definition of Lebesgue integral of a correspondence)

- $\Gamma(\cdot)$ has closed graph (and therefore it is upper hemicontinuous since the image set is compact).

Let the function $H : T \to \Re^n$ defined as $h(t) = (1, ..., 1) = e \; \forall t \in T$. Clearly $h(\cdot)$ is bounded as well as $\int_T h(t)d\lambda$. Therefore the strategy profile $f(t) \leq h(t) \; \forall t \in T$ and $\forall f \in F$, since $f(t)$ is a unit vector for all t. Given that we showed $B(t, \cdot)$ having closed graph, Γ has closed graph since any possible selection is bounded (Aumann's theorem (1976) or "integration preserves upper hemicontinuity").

By the Kakutani's fixed point theorem, Γ has a fixed point $(s_1^*, ..., s_k^*)$. Therefore it exists a pure strategy profile $f^* \in F$ such that $\prod_{h=1}^{k} \int_{T_h} f^* d\lambda = (s_1^*, ..., s_k^*) \in \Gamma(s_1^*, ..., s_k^*)$ and thus $f^* \in B(t, s_1^*, ..., s_k^*)$ for almost all $t \in T$ or, equivalently, $f^*(t)$ is a selection of $B(t, s_1^*, ..., s_k^*)$ for almost all $t \in T$, or, again, f^* is a pure strategy Nash Equilibrium. ■

This proof closely follows the procedure found by Rath. The main extension is represented by the definition of the Γ correspondence as a Cartesian product of Lebesgue integrals, rather than a simple Lebesgue integral of the best reply correspondence.

4 Three applications.

As mentioned in the introductory section, the following three applications concern about how the researchers distribute themselves over the possible alternative studies. Many ideas contained here are usual in the economics of science, as intended by P.E. Stephan (1996) and by P. Dasgupta and P.A. David (1994). In each subsection I briefly describe the "economics of science" phenomenon underlying the game structure. For more details you can refer to L. Rocco (2000).

4.1 Xenophobia in research.

We consider two groups of researchers, the stars and the "common" researchers. Empirical studies showed that the 6% of all researchers produces half of the papers (the so called "Lotka law"). We suppose that the researchers compete between them to maximize an index of reputation, say the index of citation. Since this index is an index of impact of the individual scientific production, the researchers are interested in exercising a considerable effort in "marketing activities" to make their work known. Such activities can be workshop organization or on-line publication of the papers, or participation to several and different meetings. Such activity increases the natural positive externalities between researchers which work on the same topic, making research and so publication easier and more probable respectively. Publication is more probable because a topic dealt by a large mass of researchers has also a large audience. Also the possibility of coauthorship is increased, being coauthorship either a way to formalize a cooperation or a good strategy to increase the individual scientific production.

However it is apparent in many fields of research that there is a clusterization in the research community: it is easier to benefit from the externalities or from the cooperation of members belonging to the same group. Indeed, stars cooperate with other stars or participate to meetings where other stars are present more often than they do with "common" researchers. In a sense, stars fear the possibility that "common" researchers act as free riders on them and benefit almost for free of the star's reputation. Being a star or not is a tag on which basis the other stars discriminate you when they have to decide with whom to cooperate.

Let us now try to formalize heuristically these ideas.

The framework is that considered in the section 3 and the notation is the same.

Consider a set of players $T = [0, \gamma] \cup]\gamma, 1]$ where the former interval represent the "common" researchers (group 1) and the latter the stars (group 2).

The set of topics is binary or $E = \{e_1, e_2\}$ defined as usual by the set of the unit vectors.

The payoff functions are $u_1(e_i, s_1, s_2) = \alpha_1 s_{1i} + \beta_1 s_{2i}$ for the "common" researchers and $u_2(e_i, s_1, s_2) = \alpha_2 s_{1i} + \beta_2 s_{2i}$ for the stars, where $\alpha_i, \beta_i \in [0, 1]$ and $\alpha_1 > \beta_1$ and $\alpha_2 < \beta_2$. Such a formulation capture the idea that working on the same topic of others is more valuable and that such externalities are more easily enjoyable if they come from the same group.

The best reply function has the following form for both groups $j = \{1, 2\}$

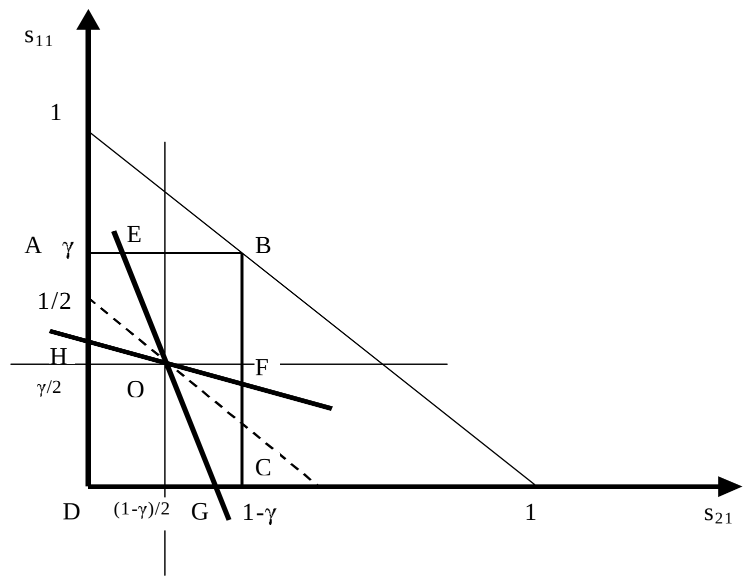
$$B_j(s_1, s_2) = \begin{cases} e_1 & \text{if } \alpha_j s_{11} + \beta_j s_{21} \geq \alpha_j s_{12} + \beta_j s_{22} \\ e_2 & \text{otherwise} \end{cases}$$

In other words, group j prefers e_1 for all the pairs (s_1, s_2) above the straight line

$$s_{11} = -\frac{\beta_j}{\alpha_j} s_{21} + \frac{\gamma(\alpha_j - \beta_j) + \beta_j}{2\alpha_j}$$

This is a family of lines centered on $(\frac{1-\gamma}{2}, \frac{\gamma}{2})$ and with negative slope belonging to the interval $[0, -\infty)$, given the possible values of α_j and β_j. It is clear that all the lines associated with the group 2 are more sloped, in absolute value, than those associated with the group 1.

Consider the following picture and notice that each point can be thought as a 4-tuple $(s_{11}, s_{12}, s_{21}, s_{22})$ where $s_{12} = \gamma - s_{11}$ and $s_{22} = (1-\gamma) - s_{21}$:



All the pairs (s_1, s_2) in the area EOFB are such that both groups prefer the alternative e_1. Therefore the unique fixed point for this region is the point B. Indeed the best response to the point B is B.

Symmetrically, all the pairs (s_1, s_2) in the area HOGD lead both groups to chose the alternative e_2 and the unique fixed point is D.

In the region HOEA, group 1 prefers e_1 but group 2 prefers e_2. The only possible fixed point is A and this represent the separation equilibrium. Nonetheless, such equilibrium exists only under some conditions that I wish discuss briefly below.

The last region is GOFC and here the unique fixed point is C that also exists under the same condition for A.

The existence of the fixed points C and A depends on the parameters. If $\gamma \geq \frac{1}{2}$ we need that $\gamma(\alpha_2 + \beta_2) \leq \beta_2$ or that for a star is much more valuable working with other stars that with common researchers ($\beta_2 >> \alpha_2$). Symmetrically, if $\gamma < \frac{1}{2}$, we need that $\gamma(\alpha_1 + \beta_1) \geq \beta_1$ or that for a common researcher is much more valuable working with others common researches ($\alpha_1 >> \beta_1$). Thus the separation equilibria exist if the stars' "xenophobia" is high enough.

Obviously, this conclusion is very weak, since the pooling equilibria always exist, independently of the degree of "xenophobia". However a sufficiently high degree of "xenophobia" may lead to segregation equilibria.

4.2 Willingness of separation.

Quite usually, one may observe that there are communities of researchers that avoid the comparison with others, especially with the stars and organize an autoreferential circle, impermeable enough, that allows them to make research and publish on topics that we may say reserved. Such a practice allows a researcher to get a certain reputation (an so a certain utility) recognized inside the circle, also if in absolute value the quality or the interest is low. The circle may also edit a journal that formalize its activities.

Such a system may survive if the stars do not write on the reserved topic and do not publish on the circle's journal.

Let us formalize such ideas.

The set of players is $T = [0, \gamma] \cup]\gamma, 1]$, with the same interpretation of the previous subsection, the alternatives are $E = \{e_1, e_2\}$ and the payoff function are $u_j(e_i, s_1, s_2) = \pi_{ji}(s_{11}, s_{22})P_i(s_{1i}, s_{2i})$ where the subscript j is for the groups and the subscript i for the alternatives. $\pi_{ji}(\cdot)$ represents the probability to publish on the journal specialized on the topic i, for the group j. $P_i(\cdot)$ represents the reputation of the journal i (and so the reputation a researcher obtains by publishing on it). The journal reputation is a function of the number of common researchers and stars that publish on its pages. We make the following specifications:

- $\pi_{11}(s_{11}, s_{22}) = \pi_{11}(s_{11})$ increasing on s_{11}
- $\pi_{12}(s_{11}, s_{22}) = \pi_{12}(s_{22})$ decreasing on s_{22}
- $\pi_{2i}(s_{11}, s_{22}) = \pi_{2i}$ constant
- $\pi_{11}(s_{11}) > \pi_{12}(s_{22})\ \forall s_{11}, s_{22}$ and $\pi_{12}(1-\gamma) = 0$
- $\pi_{21} > \pi_{22}$
- $\pi_{21} > \pi_{11}(s_{11})\ \forall s_{11}$ and $\pi_{22} > \pi_{12}(s_{22})\ \forall s_{22}$

- $P_1(0,0) = P_2(0,0)$
- $P_i(s_{1i}, s_{2i})$ decreases on s_{1i} and increases on s_{2i}
- $P_i(s_{1i}, s_{2i}) > 0 \ \forall i, s_{1i}, s_{2i}$

The meaning of all these assumptions is that the probability to publish is always higher for the stars than for the common researchers. The probability to publish a paper on the topic 1 is higher than the probability to publish on topic 2 for both groups, or topic 1 is "easier" than topic 2. However, group 1 faces a probability to publish on topic 1 increasing in the number of members of the same group that writes on topic 1, while the probability to publish on topic 2 is decreasing on the number of members of group 2 that writes on topic 2. There are two journals that publish only one topic. The reputation of the journal depends negatively on the number of common researchers that publish on that topic and depends positively on the number of stars that publish on that topic. The payoff of each researcher depends on the expected reputation he receives from publishing on a journal.

Let us now find the equilibrium set of the game.

First we simply show that there not exist equilibria where both groups are distributed on the two topics. Indeed the condition would be

$$\begin{cases} \pi_{11}(s_{11})P_1(s_{11}, s_{21}) = \pi_{12}(s_{22})P_2(s_{12}, s_{22}) \\ \pi_{21}P_1(s_{11}, s_{21}) = \pi_{22}P_2(s_{12}, s_{22}) \end{cases}$$

Given the conditions on π_{ji} the two equalities above can never be verified simultaneously. Neither it can be possible that one group is distributed on the two topics in equilibrium.

The unique possibilities are the separation equilibria. Only the case where all commons researchers write on topic 1 (the easiest) and all the stars write on topic 2 is possible. Indeed the corresponding condition

$$\begin{cases} \pi_{11}(\gamma)P_1(\gamma, 0) \geq \pi_{12}(1-\gamma)P_2(0, 1-\gamma) \\ \pi_{21}P_1(\gamma, 0) \leq \pi_{22}P_2(0, 1-\gamma) \end{cases}$$

is verified.

Thus the common researchers close themselves into a circle that deals only with topic 1 and publish on the journal 1. Stars find not convenient to deal with topic 1, even if easier, because the low standing of journal 1.

4.3 The planet of the Gods.

Here we consider a setting where the payoff of the group 1 depends negatively on the number of common researchers that write on the same topic and positively on the number of the stars. This stays for the fact that an higher competition reduces the possibility of publishing, but the presence of stars on a topic increases its interest. On the other hand, the stars are interested only on their own distribution and do not care on the common researchers' decision. Indeed the stars influence but they are not touched by the common researchers' decision. We also imagine that the alternatives are ordered by their degree of difficulty and interest represented by the probability to discover something (a low probability stays for an higher interest of the discovery).

The formal setting is:

$T = [0, \gamma] \cup]\gamma, 1]$ is the set of the players

$E = \{e_1, ..., e_n\}$ is the set of the alternatives or of the topics.

The payoff functions are $u_1(e_i, s_1, s_2) = \pi_{1i} P(s_{1i}, s_{2i})$ $\forall t \in [0, \gamma]$ and $u_2(e_i, s_1, s_2) = \pi_{2i} R(s_{2i})$ $\forall t \in]\gamma, 1]$. We assume that $P_{s_{1i}} < 0$, $P_{s_{2i}} > 0$ and $R_{s_2} > 0$. The probabilities π_{ji} are such that $\pi_{11} < ... < \pi_{1n}$ and $\pi_{21} < ... < \pi_{2n}$ and finally $\pi_{1i} < \pi_{2i}$ $\forall i \in \{1, ..., n\}$. Indeed, as in the previous example, the group 2 has an advantage in terms of probability to discover something or it has more talent.

Moreover we assume that $\pi_{1i} P(0, s_{2i}) \geq \pi_{1j} P(s_{1j}, s_{2j})$ $\forall j, s_{1j}, s_{2j}$. This assumption means that for a common researcher is always preferable a topic where no other common researchers work.

Therefore the only equilibrium distribution is a distribution where all topics are covered. Indeed, the equilibrium condition is

$$\pi_{11} P(s_{11}, s_{21}) = ... = \pi_{1n} P(s_{1n}, s_{2n})$$

On the other hand, the group of stars is unaffected by group 1 and a necessary condition for an equilibrium is either $s_{2i} > s_{2j}$ for any $j > i$ or $s_{2i} = 0$.

Let us now specify the functions to find a closed form equilibrium.

Let $P(s_{1i}, s_{2i}) = \begin{cases} \Pi & \text{if } s_{1i}=0 \\ \alpha s_{2i} - \beta s_{1i} & \text{otherwise} \end{cases}$ and $R(s_{2i}) = s_{2i}$. Π is large enough to satisfy the condition on $\pi_{1i} P(0, s_{2i})$. Moreover we impose that $\pi_{2i} = (1+\delta)\pi_{1i}$ with $\delta > 0$. We consider only the equilibria were the stars' group allocates itself over the first m topics and, following the necessary condition, in a decreasing way. Notice that, contrary to the requirements of the theorem of existence, $P(\cdot,\cdot)$ is not always continuous. However, the considered discontinuity is very special and it does not prejudge the equilibrium existence.

The equilibrium condition, given the distribution of the stars, is

$$\pi_{11}(\alpha s_{21}-\beta s_{11}) = ... = \pi_{1m}(\alpha s_{2m}-\beta s_{1m}) = \pi_{1m+1}(-\beta s_{1m+1}) = ... = \pi_{1n}(-\beta s_{1n})$$

Rearranging, for $i \in \{1, m-1\}$ the chain of equalities gives

$$s_{1i+1} = \frac{\pi_{1i}}{\pi_{1i+i}} s_{1i} - \frac{\alpha}{\beta}\left(\frac{\pi_{1i}}{\pi_{1i+i}} s_{2i} - s_{2i+1}\right)$$

For $i \in \{m+1, n\}$ we have:

$$s_{1i+1} = \frac{\pi_{1i}}{\pi_{1i+i}} s_{1i}$$

and for $i = m+1$ we have:

$$s_{1i} = \frac{\pi_{1m}}{\pi_{1i}} s_{1m} - \frac{\alpha}{\beta}\frac{\pi_{1m}}{\pi_{1i}} s_{2m}$$

Now we compute the distribution of the group 2, the stars.

The equilibrium condition is

$$\pi_{1i}(1+\delta)s_{2i} = \pi_{1i+1}(1+\delta)s_{2i+1} \text{ for all } i \in \{1, m-1\}$$

By the chain of equalities we obtain simply

$$s_{2i} = \frac{\pi_{11}}{\pi_{1i}} s_{21} \text{ for } i \in \{1, m\}$$

and

$$s_{2i} = 0 \text{ for } i \in \{m+1, n\}$$

This distribution is a part of an equilibrium if $s_{2i} \geq 0$ and if $\sum_{i=1}^{n} s_{2i} = 1-\gamma$. Therefore we obtain that

$$s_{21} = \frac{1-\gamma}{\pi_{11}A} > 0$$

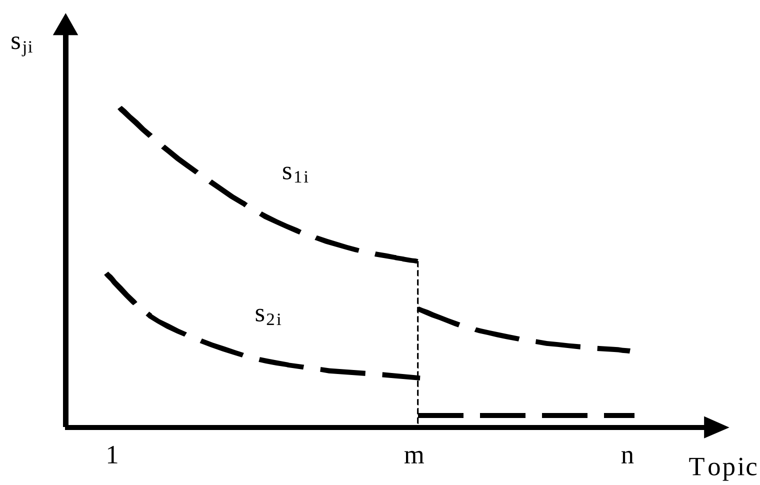
where $A = \sum_{i=1}^{m} \frac{1}{\pi_{1i}}$ and all conditions are verified.

Now, come back to the group 1 distribution.

Substituting for s_{2i},summing all the different s_{1i} and imposing the equality to γ, we obtain

$$s_{11} = \frac{\gamma + \frac{\alpha}{\beta}(1-\gamma)\frac{B}{A}}{\pi_{11}(A+B)} > 0$$

where $B = \sum_{i=m+1}^{n} \frac{1}{\pi_{1i}}$. We have only to check that $s_{1m+1} \geq 0$, i.e., the number of researchers in the first alternative without stars is positive: this is sufficient to guarantee the positivity of all the s_{1i} for $i \in \{m+2, n\}$. Such condition is verified for $\gamma > \frac{\alpha}{\alpha+\beta}$ or if the dimension of the group 1 is large enough. To conclude, the equilibrium of this game that we have considered can be represented as in the following picture.



The distribution of both groups over the alternatives is decreasing, with a higher concentration over the more difficult or interesting topics. This feature is completely explained by the stars' distribution effect that more than compensates the competition effect, summarized in the condition $P_{s_{1i}} < 0$.

5 Conclusions.

This paper mainly analyzes the existence of a Nash Equilibrium in pure strategies of the class of normal form nonatomic games. It presents three different approaches to the problem and the corresponding proofs of existence. Following the Rath's approach, this paper proves the existence of a Nash Equilibrium in pure strategies for the class of nonatomic games whose payoff functions depend over the distribution, or the Lebesgue integral, of a finite number of players' subsets. This extension allows to model a much larger set of problems, given the fact that it allows to differentiate the players in groups, avoiding the limitation implied by a complete anonymity (or symmetry) among players.

Three application based on the economics of science findings are presented as an illustration of some possibilities opened by the nonatomic games with limited anonymity.

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