Nilmanifolds with a calibrated G_2-structure

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Abstract

We introduce obstructions to the existence of a calibrated G_2-structure on a Lie algebra $\mathfrak{g}$ of dimension seven, not necessarily nilpotent. In particular, we prove that if there is a Lie algebra epimorphism from $\mathfrak{g}$ to a six-dimensional Lie algebra $\mathfrak{h}$ with kernel contained in the center of $\mathfrak{g}$, then $\mathfrak{h}$ has a symplectic form. As a consequence, we obtain a classification of the nilpotent Lie algebras that admit a calibrated G_2-structure.

1   Introduction

A Riemannian 7-manifold with holonomy contained in G_2 can be characterized by the existence of an associated parallel 3-form. The first examples of complete metrics with holonomy G_2 were given by Bryant and Salamon in [4], and the first examples of compact manifolds with such a metric were constructed by Joyce in [16]. Explicit examples on solvable Lie groups were constructed in [5]; examples on nilpotent Lie groups can be obtained by taking a nilpotent six-dimensional group with a half-flat structure and solving Hitchin's evolution equations (see [9, 15]). More generally, one can consider G_2-structures where the associated 3-form φ is closed: then φ defines a calibration ([14]), and the G_2-structure is said to be *calibrated*. An equivalent condition is that the intrinsic torsion lies in the component $\mathcal{X}_2 \cong \mathfrak{g}_2$ ([10]).

Compact calibrated G_2 manifolds have interesting curvature properties. It is well known that a G_2 holonomy manifold is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated G_2 manifold, both the Einstein condition ([6]) and scalar-flatness ([3]) are equivalent to the holonomy being contained in G_2. In fact, [3] shows that the scalar curvature is always non-positive.

Constructing examples is not a straightforward task. For instance, [7] classifies calibrated G_2-manifolds on which a simple groups acts with cohomogeneity one, and no compact manifold occurs in this list. On the other hand, the second author exhibited the first example of a compact calibrated G_2-manifold that

does not have holonomy G_2 [11]. This example is given in terms of a nilpotent Lie algebra $\mathfrak{g}$ and an element of $\Lambda^3\mathfrak{g}^*$ that corresponds to a closed left-invariant 3-form on the associated simply-connected Lie group. Since the structure constants are rational, there exists a uniform discrete subgroup [18]; the quotient, called a nilmanifold, has an induced calibrated G_2-structure.

In this paper we pursue this approach, and classify the nilpotent 7-dimensional Lie algebras that admit a calibrated G_2-structure. Since the structure constants turn out to be rational, each Lie algebra determines a nilmanifold. So, we obtain 12 compact calibrated G_2 nilmanifolds (see Theorem 4, Lemma 5 and Lemma 6). Three of them are reducible: they are the product of a circle with a symplectic half-flat nilmanifold, the latter being classified in [8]. The remaining nine are new.

The proof is based on two necessary conditions that a Lie algebra must satisfy for a calibrated G_2-structure to exist (see Proposition 1 and Lemma 3).

Our first obstruction is related to a construction of [1]. Suppose that M is a 7-manifold with a calibrated G_2 form φ, and X is a unit Killing field, i.e. $\mathcal{L}_X\varphi = 0$. Then if $\eta = X^\flat$, we can write

$$\varphi = \omega \wedge \eta + \psi^+,$$

where ω, $\psi+$ and $d\eta$ are basic forms with respect to the 1-dimensional foliation defined by X. Suppose in addition that X is the fundamental vector field of a free S^1 action; then basic forms can be identified with forms on M/S^1. By

$$0 = \mathcal{L}_X\varphi = d(X\lrcorner\varphi) = d\omega,$$

ω is a symplectic form on M/S^1. Moreover

$$0 = \omega \wedge d\eta + d\psi^+$$

implies that $[d\eta]$ is in the kernel of the map

$$H^2(M/S^1) \to H^4(M/S^1), \quad [\beta] \to [\beta \wedge \omega].$$

If this map is an isomorphism, then the S^1-bundle is trivial: this puts topological restrictions on M, which translate to algebraic conditions in our setup. A similar method was used in [8].

In principle, these restrictions reduce our problem to the classification of symplectic nilpotent Lie algebras of dimension six for which the map $H^2 \to H^4$ is non-injective (see the remark before Lemma 3). The complexity of the required calculations, however, motivate a different approach. In analogy with [9], we introduce a second obstruction, that requires computing the space of closed 3-forms. It consists in the observation that $(X\lrcorner\varphi)^3$ must be nonzero, whenever X is a nonzero vector and φ a 3-form defining a G_2-structure.

The final ingredient is Gong's classification of 7-dimensional indecomposable nilpotent Lie algebras ([12]). This list contains 140 Lie algebras and 9 one-parameter families; in addition, there are 35 decomposable nilpotent Lie algebras ([17], [19]). Calculations on a case-by-case basis show that our list of 12 examples is complete.

2 Calibrated G_2-structures and obstructions

In this section we show obstructions to the existence of a calibrated G_2 form on a Lie algebra (not necessarily nilpotent). First, we recall some definitions and results about G_2-structures.

Let us consider the space $\mathbb{O}$ of the Cayley numbers, which is a non-associative algebra over $\mathbb{R}$ of dimension 8. Thus, we can identify $\mathbb{R}^7$ with the subspace of $\mathbb{O}$ consisting of pure imaginary Cayley numbers. Then, the product on $\mathbb{O}$ defines on $\mathbb{R}^7$ the 3-form given by

$$e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245} \tag{1}$$

(see [10] and [13] for details), where $\{e^1, \ldots, e^7\}$ is the standard basis of $(\mathbb{R}^7)^*$. Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. The group G_2 is the stabilizer of (1) under the standard action of $\mathrm{GL}(7, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. G_2 is one of the exceptional Lie groups, and it is a compact, connected, simply connected simple Lie subgroup of $SO(7)$ of dimension 14.

A G_2-structure on a 7-dimensional Riemannian manifold (M, g) is a reduction of the structure group $O(7)$ of the frame bundle to G_2. Manifolds admitting a G_2-structure are called G_2 manifolds. The existence of a G_2-structure on (M, g) is determined by a global 3-form φ (the G_2 form) which can be locally written as (1) with respect to some (local) basis $\{e^1, \ldots, e^7\}$ of the (local) 1-forms on M. We say that the G_2 manifold M has a *calibrated G_2-structure* if there is a G_2-structure on M such that the 3-form φ is closed, and so φ defines a calibration [14].

If G is a 7-dimensional Lie group with Lie algebra $\mathfrak{g}$, then a G_2-structure on G is left-invariant if and only if the corresponding 3-form is left-invariant. Thus, a left invariant G_2-structure on G corresponds to an element φ of $\Lambda^3 \mathfrak{g}^*$ that can be written as (1) with respect to some coframe $\{e^1, \ldots, e^7\}$ on $\mathfrak{g}^*$; and we shall say that φ defines a G_2-structure on $\mathfrak{g}$. We say that a G_2-structure on $\mathfrak{g}$ is *calibrated* if φ is closed, i.e.

$$d\varphi = 0,$$

where d denotes the Chevalley-Eilenberg differential on $\mathfrak{g}^*$. If Γ is a discrete subgroup of G, a G_2-structure on $\mathfrak{g}$ induces a G_2-structure on the quotient $\Gamma \backslash G$. Moreover, in [18] it is proved that if $\mathfrak{g}$ is nilpotent with rational structure constants, then the associated simply connected Lie group G admits a uniform discrete subgroup Γ. Therefore, a G_2-structure on $\mathfrak{g}$ determines a G_2-structure on the compact manifold $\Gamma \backslash G$, which is called a compact nilmanifold; and if $\mathfrak{g}$ has a calibrated G_2-structure, the G_2-structure on $\Gamma \backslash G$ is also calibrated.

In order to show obstructions to the existence of a calibrated G_2 form on a Lie algebra $\mathfrak{g}$, let us consider first a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. If φ is a G_2 form on $\mathfrak{g}$, and the decomposition is orthogonal with respect to the underlying metric, then

$$\varphi = \omega \wedge \eta + \psi^+,$$

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where ω, ψ^+ are forms on $\mathfrak{h}$ and η generates the dual of the ideal $\mathbb{R}$. The pair (ω, ψ^+) defines an SU(3)-structure on $\mathfrak{h}$. The condition that φ is closed is equivalent to both ω and ψ^+ being closed; this means that the SU(3)-structure is *symplectic half-flat*. There are exactly three nilpotent Lie algebras of dimension six that admit a symplectic half-flat structure, classified in [8]. So, if we focus our attention on decomposable nilpotent Lie algebras, there are at least three 7-dimensional Lie algebras with a calibrated G_2-structure; we will see that these are all.

More generally, every 7-dimensional nilpotent Lie algebra fibres over a nilpotent Lie algebra of dimension six. In fact if ξ is in the center of $\mathfrak{g}$, then the quotient $\mathfrak{g}/\mathrm{Span}\,\{\xi\}$ has a unique Lie algebra structure that makes the projection map
$$\mathfrak{g} \to \frac{\mathfrak{g}}{\mathrm{Span}\,\{\xi\}}$$
a Lie algebra morphism. Moreover, due to the nilpotency assumption every epimorphism $\mathfrak{g} \to \mathfrak{h}$, with $\mathfrak{h}$ of dimension six, is of this form. Using the pullback, we can identify forms on the quotient with *basic forms* on $\mathfrak{g}$; in this setting, α is *basic* if $\xi \lrcorner \alpha = 0$.

Given a G_2-structure on $\mathfrak{g}$ with associated 3-form φ and a nonzero vector ξ in the center, let $\eta = \xi^\flat$; then we can write
$$\varphi = \omega \wedge \eta + \psi^+, \quad \xi \lrcorner \omega = 0 = \xi \lrcorner \psi^+,$$
and up to a normalization coefficient the forms (ω, ψ^+) define an SU(3)-structure on the six-dimensional quotient (see also [1]). In analogy with the case of a circle bundle, we shall think of η as a connection form, and $d\eta$ as the curvature.

Proposition 1. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a calibrated G_2-structure and a non-trivial center. If $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra epimorphism with kernel contained in the center, and $\mathfrak{h}$ of dimension six, then $\mathfrak{h}$ admits a symplectic form ω, and the curvature form is in the kernel of*
$$H^2(\mathfrak{h}^*) \xrightarrow{\cdot \wedge \omega} H^4(\mathfrak{h}^*). \tag{2}$$

If the curvature form is exact on $\mathfrak{h}$, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}$ as Lie algebras.

Proof. Write
$$\varphi = \pi^*\omega \wedge \eta + \pi^*\psi^+$$
where (ω, ψ^+) are forms on $\mathfrak{h}$. Since d commutes with the pullback,
$$0 = d\varphi = d\pi^*\omega \wedge \eta + \pi^*\omega \wedge d\eta + \pi^*d\psi^+,$$
where $\pi^*d\omega$, $d\eta$ and $\pi^*d\psi^+$ are basic. Thus ω is a symplectic form and $d\eta$ is in the kernel of (2).

Now suppose that $d\eta$ is exact on $\mathfrak{h}$. Then, the epimorphism $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is trivial, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. More precisely, we can choose a different, closed connection form $\tilde{\eta}$, and $\mathfrak{g} = \ker \tilde{\eta} \oplus \ker \pi$ is a direct sum of Lie algebras; by construction, $\ker \tilde{\eta}$ is isomorphic to $\mathfrak{h}$. $\square$

Remark. In the previous Proposition, we must notice that when the curvature form is zero, (ω, ψ^+) is a symplectic half-flat structure on $\mathfrak{h}$. Therefore, if $\mathfrak{h}$ is nilpotent, by [8] $\mathfrak{h}$ is one of

$$(0,0,0,0,0,0), \quad (0,0,0,0,12,13), \quad (0,0,0,12,13,23).$$

With notation from [19],$(0,0,0,0,12,13)$ represents a the Lie algebra with a fixed basis $e^1, \ldots e^6$ of $\mathfrak{g}^*$, satisfying

$$de^1 = 0 = de^3 + de^3 = de^4, \quad de^5 = e^{12}, de^6 = e^{13}.$$

Remark. Another obstruction to the existence of a calibrated G_2-structure on a nilpotent Lie algebra is given by the condition $b_3 > 0$. Indeed, if φ is a closed G_2 form on a nilpotent Lie algebra $\mathfrak{g}$, and X is a nonzero vector in the center of $\mathfrak{g}$, then $\mathcal{L}_X \varphi = 0$, so $X \lrcorner \varphi$ is closed. If φ were exact, say $\varphi = d\beta$, then the 7-form

$$(X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \varphi = d((X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \beta)$$

would also be exact, hence zero, which is absurd. On the other hand, b_3 is always positive on a nilpotent Lie algebra of dimension seven.

Proposition 1 motivates the following definition. We say that a 6-dimensional Lie algebra $\mathfrak{h}$ satisfies the *2-Lefschetz property* if, for every symplectic structure on $\mathfrak{h}$, the map (2) is an isomorphism. This condition holds trivially when $\mathfrak{h}$ has no symplectic structure, namely when $\mathfrak{h}$ is one of

$$(0,0,0,12,23,14+35), \quad (0,0,0,12,23,14-35),$$
$$(0,0,0,12,13,14+35), \quad (0,0,0,0,12,15+34),$$
$$(0,0,0,0,0,12+34), \quad (0,0,12,13,14+23,34+52),$$
$$(0,0,12,13,14,34+52), \quad (0,0,0,12,14,24).$$

It is well known [2] that if $(\mathfrak{h}, \omega)$ is a 6-dimensional, symplectic nilpotent Lie algebra, the map

$$H^1(\mathfrak{h}^*) \xrightarrow{\cdot \wedge \omega^2} H^5(\mathfrak{h}^*).$$

is not surjective. However, in the next proposition, we prove that some of those Lie algebras satisfy the 2-Lefschetz property.

Proposition 2. *Among 6-dimensional nilpotent Lie algebras with a symplectic structure, those that satisfy the 2-Lefschetz property are*

$$(0,0,0,0,0,0); \quad (0,0,12,13,23,14); \quad (0,0,12,13,23,14+25).$$

Proof. In the abelian case, the bilinear map

$$H^2 \otimes H^2 \to H^4$$

induced by the wedge product is non-degenerate, in the sense that for every nonzero $\beta \in H^2$, the induced linear map $\cdot \wedge \beta \colon H^2 \to H^4$ is an isomorphism.

For the second Lie algebra, the cohomology class of a generic symplectic form is represented by

$$\omega = \lambda_1 e^{16} + \lambda_2 (e^{15} + e^{24}) + \lambda_3 e^{25} + \lambda_4 (e^{34} - e^{26});$$

non-degeneracy implies $\lambda_4 \neq 0$. The map $H^2 \to H^4$ of (2) is represented by the matrix

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ \lambda_4 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & -2\lambda_4 \\ 0 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$$

which is invertible by the assumption $\lambda_4 \neq 0$.

Similarly, for the last Lie algebra

$$\omega = \lambda_1 e^{14} + \lambda_2 (e^{15} + e^{24}) - \lambda_3 (e^{26} - e^{34}) + \lambda_4 (e^{16} + e^{35}).$$

The map (2) is represented by

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ -\lambda_4 & 2\lambda_3 & 2\lambda_2 & -\lambda_1 \\ 0 & 0 & -2\lambda_3 & 2\lambda_4 \\ 0 & 0 & -\lambda_4 & -\lambda_3 \end{pmatrix},$$

which is invertible unless $\lambda_4^2 + \lambda_3^2 = 0$, which makes ω degenerate.

For all but three of the remaining Lie algebras, we observe that the bilinear map

$$H^2 \otimes H^2 \to H^4$$

is degenerate in the sense that, for every nonzero $\beta \in H^2$, the map

$$\alpha \to \alpha \wedge \beta, \quad H^2 \to H^4$$

is non-injective. The three exceptions are

$$(0, 0, 12, 13, 23, 14 - 25), \quad (0, 0, 0, 12, 13, 23), \quad (0, 0, 0, 0, 0, 12).$$

However, either Lie algebra has a symplectic form that makes the map (2) non-injective. In fact, on the Lie algebra $\mathfrak{h}$ defined by the equations $(0, 0, 12, 13, 23, 14 - 25)$, consider the symplectic form

$$\omega = -e^{16} + e^{15} + e^{35} + e^{34} + e^{24} - e^{26}.$$

Then one can check that $e^{14} + e^{25} + e^{15} + e^{24}$ defines a non-trivial class in $H^2(\mathfrak{h}^*)$, but

$$(e^{14} + e^{25} + e^{15} + e^{24}) \wedge \omega = 2e^{1245} = 2d(e^{146}).$$

Now, on the Lie algebra $(0, 0, 0, 12, 13, 23)$ we consider the symplectic form $\omega = e^{14} + e^{26} + e^{35}$. Then,

$$(-e^{15} - e^{24} + e^{36}) \wedge \omega = d(e^{456});$$

finally, on the Lie algebra $(0, 0, 0, 0, 0, 12)$,

$$(e^{16} + e^{25} + e^{34}) \wedge e^{13} = -de^{356}. \qquad \square$$

In principle, one could try to classify all pairs $(\mathfrak{h}, \omega)$, with $\mathfrak{h}$ nilpotent of dimension six and ω a symplectic form on $\mathfrak{h}$, for which (2) is non-injective. This means that $\omega \wedge \gamma = d\psi^+$, for some $\psi^+ \in \Lambda^3 \mathfrak{h}^*$ and some closed non-exact 2-form $\gamma \in \Lambda^2 \mathfrak{h}^*$. If in addition, (ω, ψ^+) are compatible in the sense that they define an SU(3)-structure, then declaring $de^7 = \gamma$ one obtains a 7-dimensional Lie algebra $\mathfrak{g}$ with a calibrated G_2-structure. By Proposition 1, all calibrated G_2-structures on indecomposable nilpotent Lie algebras are obtained in this way.

However, these calculations turn out to be difficult (although in one dimension less a similar approach was pursued succesfully in [8]), and for this reason we shall use a different method (see Section 4), starting with Gong's classification of 7-dimensional Lie algebras. In fact, given a Lie algebra, it is straightforward to compute the space of its closed 3-forms. In the spirit of [9], the existence of a calibrated G_2-structure puts restrictions on this space. Whilst straightforward, the following result turns out to give an effective obstruction.

Lemma 3. *Let $\mathfrak{g}$ be a 7-dimensional nilpotent Lie algebra. If there is a nonzero X in $\mathfrak{g}$ such that $(X \lrcorner \phi)^3 = 0$ for every closed 3-form on $\mathfrak{g}$, then $\mathfrak{g}$ has no calibrated G_2-structure.*

Proof. Obvious. $\square$

Remark. When $\mathfrak{g}$ fibers over a non-symplectic Lie algebra $\mathfrak{h}$, this obstruction is satisfied automatically. Indeed, suppose $\pi \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra epimorphism; then any closed 3-form on $\mathfrak{g}$ can be written as

$$\pi^* \omega \wedge \eta + \pi^* \psi^+,$$

as in the proof of Proposition 1. So ω is a closed form on $\mathfrak{h}$; if we assume $\mathfrak{h}$ has no symplectic form, then $\omega^3 = 0$. Then the condition of Lemma 3 is satisfied with X a generator of $\ker \pi$.

3 Decomposable case

In this section we classify the decomposable nilpotent Lie algebras with a calibrated G_2-structure. Indeed, we prove:

Theorem 4. *Among the 35 decomposable nilpotent Lie algebras of dimension 7, those that have a calibrated G_2-structure are*

$$(0,0,0,0,0,0,0), \quad (0,0,0,0,12,13,0), \quad (0,0,0,12,13,23,0).$$

Proof. By the remark at the beginning of Section 2, we know that these three Lie algebras have a calibrated G_2-structure (see [11] where the second of these Lie algebras was considered). In fact, on the non-abelian Lie algebras $(0,0,0,0,12,13)$ and $(0,0,0,12,13,23)$ we can consider the symplectic half flat structure (ω_1, ψ_1^+) and (ω_2, ψ_2^+), respectively, defined by

$$\omega_1 = e^{14} + e^{26} + e^{35}, \quad \psi_1^+ = e^{123} + e^{156} + e^{245} - e^{346},$$

and
$$\omega_2 = e^{16} + 2e^{25} + e^{34}, \quad \psi_2^+ = e^{123} + e^{145} + e^{246} - e^{356}.$$

Using Lemma 3, we can see that the decomposable Lie algebra

$$0, 0, 0, 0, 12, 34, 36$$

has no calibrated G_2-structure. Indeed a basis of the space Z^3 of the closed 3-forms is given by

$$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236},$$
$$e^{237}, e^{245}, e^{246}, -e^{126} + e^{345}, e^{346}, e^{347}, e^{127} + e^{356}, e^{367}, e^{467}.$$

Thus $e_7 \lrcorner Z^3$ is the span of $e^{13}, e^{23}, e^{34}, e^{12}, e^{36}, e^{46}$, which contains only degenerate forms.

Since this is the only decomposable nilpotent Lie algebra of dimension seven which does not have the form $\mathfrak{h} \oplus \mathbb{R}$, it remains to prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has a calibrated G_2 form, then $\mathfrak{g}$ must be as in the statement.

Clearly, if $\mathfrak{h}$ is one of the eight Lie algebras defined by (2), Proposition 1 implies that the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has no calibrated G_2 form.

Also, one can check that none of the five Lie algebras defined by

$$(0, 0, 12, 13, 23, 14, 0), \quad (0, 0, 12, 13, 23, 14 + 25, 0), \quad (0, 0, 12, 13, 23, 14 - 25, 0),$$
$$(0, 0, 0, 0, 13 + 42, 14 + 23, 0), \quad (0, 0, 0, 0, 12, 14 + 23, 0),$$

has a calibrated G_2 form because each of these is a bundle over a non-symplectic Lie algebra of dimension six. Explicitly, the base of the bundle and curvature form are given by

$$\pi \colon (0, 0, 12, 13, 23, 14, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14},$$
$$\pi \colon (0, 0, 12, 13, 23, 14 + 25, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} + e^{25},$$
$$\pi \colon (0, 0, 12, 13, 23, 14 - 25, 0) \to (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} - e^{25},$$
$$\pi \colon (0, 0, 0, 0, 13 + 42, 14 + 23, 0) \to (0, 0, 0, 0, 13 + 42, 0), \quad d\eta = e^{14} + e^{23},$$
$$\pi \colon (0, 0, 0, 0, 12, 14 + 23, 0) \to (0, 0, 0, 0, 14 + 23, 0), \quad d\eta = e^{12}.$$

For each of the remaining 18 Lie algebras of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, listed in Table 1 alongside with a basis of the space of closed 3-forms, one can check that the hypothesis of Lemma 3 is satisfied with $X = e_6$. $\qquad \square$

4 Indecomposable case

In this section we complete the classification of 7-dimensional nilpotent Lie algebras with a calibrated G_2-structure. We have seen that there are exactly three decomposable Lie algebras of this type. In order to discuss the indecomposable

Table 1: Closed 3-forms on decomposable Lie algebras

$(0,0,12,13,14+23,24+15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{136}+e^{145}, e^{146}, e^{147},$ $e^{234}, e^{235}+e^{136}, e^{237}, -e^{236}+e^{245}, e^{157}+e^{247}, e^{167}+e^{257}, -\frac{1}{2}e^{156}+e^{345}-\frac{1}{2}e^{246}, e^{167}+e^{347}$
$(0,0,0,12,14,15+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157},$ $e^{234}, e^{235}-e^{146}, -e^{156}+e^{236}, e^{237}, e^{245}, e^{247}, e^{156}+e^{345}, -e^{167}+e^{347}, -e^{267}+e^{457}$
$(0,0,0,12,14-23,15+34,0)$	$e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{126}+e^{145}, e^{146}, e^{147},$ $e^{234}, e^{235}-e^{126}, e^{237}, e^{245}, e^{247}, e^{236}+e^{345}, e^{347}+e^{157}, e^{357}+e^{167}$
$(0,0,0,12,14,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$ $e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}, e^{247}, e^{345}+e^{236}, -e^{267}+e^{457}$
$(0,0,0,12,13,14+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157},$ $e^{234}, e^{235}, e^{236}-e^{146}, e^{237}, e^{245}+e^{146}, e^{246}, e^{247}, e^{257}+e^{167}, e^{345}+e^{156}, e^{347}-e^{167}, e^{357}$
$(0,0,0,12,13,24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157},$ $e^{234}, e^{235}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{256}+e^{346}, e^{257}+e^{347}, e^{357}$
$(0,0,0,12,14,15+24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157},$ $e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{246}-e^{156}, e^{247}, e^{257}+e^{167}, e^{236}+e^{345}, e^{457}-e^{267}$
$(0,0,0,12,14,15+23+24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{136}+e^{146}, e^{147},$ $e^{157}, e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{236}+e^{246}-e^{156}, e^{247}, e^{236}+e^{345}, e^{347}-e^{257}-e^{167}, e^{457}-e^{267}$
$(0,0,12,13,14,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$ $e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}-e^{236}, e^{347}-e^{257}$
$(0,0,12,13,14,23+15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157},$ $e^{234}, e^{235}-e^{146}, e^{236}-e^{156}, e^{237}, e^{245}-e^{156}, e^{247}+e^{167}, e^{347}-e^{257}$
$(0,0,0,12,13+42,14+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{137}, e^{145}, -e^{135}+e^{146}, e^{147},$ $e^{234}, e^{235}, e^{236}-e^{135}, e^{237}, e^{245}+e^{135}, e^{246}, e^{247}, e^{167}+e^{257}, -e^{157}+e^{267}, -e^{167}+e^{347}$
$(0,0,0,12,14,13+42,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157},$ $e^{234}, e^{235}-e^{136}, e^{236}, e^{237}, e^{245}, e^{246}+e^{136}, e^{247}, e^{257}-e^{167}, e^{267}+e^{347}$
$(0,0,0,12,13+14,24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157},$ $e^{234}, e^{235}+e^{136}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{257}+e^{167}, e^{347}$
$(0,0,0,12,13,14,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$ $e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}+e^{236}, e^{246}, e^{247}, e^{347}+e^{257}, e^{357}$
$(0,0,0,0,0,12,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$ $e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, e^{257}, e^{267}, e^{345}, e^{347}, e^{357}, e^{457}$
$(0,0,0,0,12,14+25,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156}, e^{157},$ $e^{234}, e^{235}, e^{237}, e^{245}, e^{246}, e^{247}, -e^{146}+e^{256}, e^{257}, -e^{236}+e^{345}, e^{347}, e^{457}+e^{267}$
$(0,0,0,0,12,34,0)$	$e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157},$ $e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{257}, e^{345}-e^{126}, e^{346}, e^{347}, e^{367}, e^{467}$
$(0,0,0,0,12,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$ $e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}, e^{247}, e^{256}, e^{257}, e^{347}$

Lie algebras, we refer to Gong's classification in [12]. This list consists of 140 Lie algebras and 9 one-parameter families.

The one-parameter families are the following:

$$147E = \left(0,0,0,e^{12},e^{23},-e^{13},\lambda e^{26}-e^{15}-(-1+\lambda)e^{34}\right), \qquad \lambda \neq 0,1;$$

$$1357M = \left(0,0,e^{12},0,e^{24}+e^{13},e^{14},-(-1+\lambda)e^{34}+e^{15}+e^{26}\lambda\right), \qquad \lambda \neq 0;$$

$$1357N = \left(0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}\lambda\right);$$

$$1357S = \left(0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{16}+e^{15}+\lambda e^{26}\right), \qquad \lambda \neq 1;$$

$$12457N = \left(0,0,e^{12},e^{13},e^{23},e^{24}+e^{15},\lambda e^{25}+e^{26}+e^{34}-e^{35}+e^{16}+e^{14}\right);$$

$$123457I = \left(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24},\lambda e^{25}-(-1+\lambda)e^{34}+e^{16}\right);$$

$$147E1 = \left(0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-e^{16}\lambda+\lambda e^{25}\right), \qquad \lambda > 1;$$

$$1357QRS1 = \left(0,0,e^{12},0,e^{13}+e^{24},e^{14}-e^{23},e^{26}\lambda+e^{15}-e^{34}(-1+\lambda)\right), \qquad \lambda \neq 0;$$

$$12457N2 = \left(0,0,e^{12},e^{13},e^{23},-e^{14}-e^{25},e^{15}-e^{35}+e^{16}+e^{24}+e^{25}\lambda\right), \qquad \lambda \geq 0.$$

Recall that a 3-form of type G_2 has the form (1) with respect to some coframe $e^1,\ldots,e^7$; such a coframe identifies the G_2-structure.

Lemma 5. *Exactly three of the above Lie algebras admit a calibrated G_2-structure. Explicit examples are given in terms of a coframe by*

$$1357N(\lambda=1): \quad \sqrt{3}(2e^1-e^7-e^6-e^5), \sqrt{3}(e^4+e^3-2e^2-e^6+e^5), 2e^3-e^6, 2e^5,$$
$$-e^3+3e^4+e^5-e^6, 2e^3-e^5-e^6+3e^7, -\sqrt{3}e^6;$$

$$1357S(\lambda=-3): \quad \sqrt{7}(2e^1+e^2-e^5+e^6), 7e^2+3e^5+5e^6, \sqrt{7}(e^3+2e^4-\frac{3}{2}e^7),$$
$$3e^3+\frac{7}{2}e^7, -\sqrt{70}e^6, \sqrt{10}(2e^5+e^6), -2\sqrt{10}e^3.$$

$$147E1(\lambda=2): \quad \sqrt{3}(2e^1+e^5-e^2+e^6), 3e^2-e^5+e^6, e^3+2e^4, \sqrt{3}(e^3+e^7),$$
$$\sqrt{2}(e^6-e^5), \sqrt{6}(e^5+e^6), 2\sqrt{2}(e^4-e^3).$$

Proof. It is straightforward to verify that each coframe in the statement determines a calibrated G_2-structure on the corresponding Lie algebra. Conversely, for each Lie algebra $\mathfrak{g}$ the vector e_7 is in the center, and determines an epimorphism on a 6-dimensional Lie algebra $\mathfrak{h}$; we view de^7 as the curvature form on $\mathfrak{h}$, and apply Proposition 1.

In the case of $1357M$, the generic element of $H^2(\mathfrak{h}^*)$ is represented by

$$\omega = \lambda_6 e^{46} + \lambda_3 e^{23} + \lambda_1 e^{13} + \lambda_5(e^{15}+e^{34}) + \lambda_2 e^{16} + \lambda_4(e^{15}+e^{26}).$$

Assume $de^7 \wedge \omega$ is exact. Then λ_3, λ_6 are zero, $\lambda_4 = -\lambda_5\lambda$ and

$$(\lambda-\lambda^2-1)\lambda_5 = 0.$$

Since $\lambda^2-\lambda+1$ has no real zeroes, λ_4 and λ_5 are zero as well, and therefore ω^3 is zero. So there is no symplectic form in the cohomology class of ω. By

Proposition 1, if a calibrated G_2-structure existed, then $\mathfrak{g}$ would have to be decomposable, which is absurd.

The other cases are similar. $\qquad\square$

We now turn to the rest of the list, where no parameters appear.

Lemma 6. *In Gong's list, only six Lie algebras with no parameters in their definition admit a calibrated G_2-structure, which can be expressed in terms of a coframe as follows:*

$0, 0, 12, 0, 0, 13 + 24, 15$	$e^1, e^2, e^5, e^6, e^3, e^7, e^4$
$0, 0, 12, 0, 0, 13, 14 + 25$	$e^1, e^3, e^5, e^7, e^2, e^6, e^4$
$0, 0, 0, 12, 13, 14, 15$	$e^1, e^2, e^4, e^7, e^5, e^6, e^3$
$0, 0, 0, 12, 13, 14 + 23, 15$	$e^2 + e^7, e^3 + e^6, e^7, e^6, e^5, e^4, e^1$
$0, 0, 12, 13, 23, 15 + 24, 16 + 34$	$e^2 + e^4, e^7, e^2, e^5, e^3, e^6, e^1$
$0, 0, 12, 13, 23, 15 + 24, 16 + 25 + 34$	$\sqrt{3}(2e^2 + e^5 + e^7), 2e^4 - 3e^5 - e^7, \sqrt{3}(e^1 - e^3 + 2e^6),$
	$e^1 + 3e^3, \sqrt{6}e^7, \sqrt{2}(2e^4 - e^7), 2\sqrt{2}e^1$

Theorem 7. *Up to isomorphism, there are exactly 12 nilpotent Lie algebras that admit a calibrated G_2-structure, namely those appearing in Theorem 4, Lemma 5 and Lemma 6.*

Proof. We must show that the remaining Lie algebras in Gong's list satisfy one of the two obstructions of Section 2; we do so in the Appendix, where we reproduce Gong's list, and note which obstruction applies to each Lie algebra (as a preference, we try to use Proposition 1 rather than Lemma 3 whenever possible, because the former does not require computing the space of closed 3-forms). $\qquad\square$

Appendix

This appendix contains a list of all indecomposable nilpotent Lie algebras of dimension 7, taken from [12], except the 9 one-parameter families that we listed at the beginning of Section 4. Alongside each Lie algebra $\mathfrak{g}$, we give a chosen vector $\xi \in \mathfrak{g}$ which satisfies the conditions of Proposition 1 (when marked with a (P)) or Lemma 3, and the structure constants of the quotient $\mathfrak{g}/\operatorname{Span}\{\xi\}$. The word "resists" marks instead the six Lie algebras that resist the obstructions. Below each Lie algebra, we give a basis of its space of closed 3-forms, except when Proposition 1 applies.

Table 2: Step 2 nilpotent Lie algebras of dimension 7

$0,0,0,0,12,23,24$	e_7	$[0,0,0,0,e^{12},e^{23}]$

$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{137}+e^{146}, e^{147},$
$e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, e^{257}, e^{267}, e^{345}+e^{137},$
e^{346}, e^{347}

$0,0,0,0,12,23,34$	e_7	$[0,0,0,0,e^{12},e^{23}]$

$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}-e^{127}, e^{147},$
$e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, -e^{127}+e^{345}, e^{346}, e^{347}, e^{367}$

$0,0,0,0,12+34,23,24$	e_5	$[0,0,0,0,e^{23},e^{24}]$

$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}+e^{125}, e^{145}, -e^{125}+e^{146},$
$e^{147}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{267}, -e^{125}+e^{345}, e^{346},$
$e^{347}, -e^{256}+e^{367}, -e^{257}+e^{467}$

$0,0,0,0,12+34,13,24$	e_7	$[0,0,0,0,e^{12}+e^{34},e^{13}]$

$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}+e^{125}, e^{145}, e^{146}, e^{147},$
$e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}+e^{125}, e^{247}, -e^{125}+e^{345}, e^{346}, e^{347}$

$0,0,0,0,0,12,14+35$	$e_6(P)$	$[0,0,0,0,0,e^{14}+e^{35}]$
$0,0,0,0,0,12+34,15+23$	$e_7(P)$	$[0,0,0,0,0,e^{34}+e^{12}]$

$0,0,0,0,0,0,12+34+56$	e_7	$[0,0,0,0,0,0]$

$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235},$
$e^{236}, e^{245}, e^{246}, e^{256}, e^{345}, e^{346}, e^{356}, e^{456}$

$0,0,0,0,12-34,13+24,14$	e_6	$[0,0,0,0,e^{12}-e^{34},e^{14}]$

$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}-e^{125}, e^{137}, e^{145}, e^{146}, e^{147},$
$e^{234}, e^{235}, e^{236}, e^{125}+e^{237}, e^{245}, -e^{125}+e^{246}, e^{247}, e^{125}+e^{345}, e^{346},$
$e^{347}, e^{457}-e^{167}, e^{467}+e^{157}$

$0,0,0,0,12-34,13+24,14-23$	e_7	$[0,0,0,0,e^{12}-e^{34},e^{24}+e^{13}]$

$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}-e^{125}, e^{137}, e^{145}, e^{146}, -e^{125}+e^{147},$
$e^{234}, e^{235}, e^{236}, e^{125}+e^{237}, e^{245}, -e^{125}+e^{246}, e^{247}, e^{125}+e^{345}, e^{346}, e^{347}$

Table 3: Step 3 nilpotent Lie algebras of dimension 7

$0,0,12,0,13,24,14$	e_5	$[0,0,e^{12},0,e^{24},e^{14}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147},e^{157},$$
$$e^{234},e^{235},e^{236},e^{237}+e^{136},e^{245}-e^{136},e^{246},e^{247},e^{345}-e^{156},e^{346}+e^{267},$$
$$e^{347}+e^{167},e^{467}$$

$0,0,12,0,13,23,14$	$e_7(P)$	$[0,0,e^{12},0,e^{13},e^{23}]$
$0,0,12,0,13+24,23,14$	e_6	$[0,0,e^{12},0,e^{24}+e^{13},e^{14}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{137},e^{145},e^{146}+e^{135},e^{147},$$
$$e^{234},e^{235},e^{236},e^{237}+e^{135},-e^{135}+e^{245},e^{246},e^{247},e^{257}+e^{345}+e^{167},$$
$$e^{346}+e^{267},e^{347}+e^{157}$$

$0,0,12,0,0,13+24,15$	resists	

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147},e^{157},$$
$$e^{234},e^{235},e^{236},e^{245},-e^{136}+e^{246},-e^{156}+e^{247},e^{256}+e^{237},e^{257},e^{345}-e^{156},$$
$$-e^{347}-e^{167}+e^{456},e^{457}$$

$0,0,12,0,0,13,14+25$	resists	

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{157},$$
$$e^{234},e^{235},e^{236},e^{245},e^{237}+e^{246},e^{247},e^{256}-e^{137},e^{257}-e^{147},e^{345}+e^{147},$$
$$-e^{167}+e^{356},e^{457}$$

$0,0,12,0,0,13+24,25$	e_7	$[0,0,e^{12},0,0,e^{13}+e^{24}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146},e^{147}+e^{156},e^{157},$$
$$e^{234},e^{235},e^{236},e^{237},e^{245},-e^{136}+e^{246},e^{247},e^{256}-e^{137},e^{257},e^{345}+e^{147},e^{457}$$

$0,0,12,0,0,13+24,14+25$	e_7	$[0,0,e^{12},0,0,e^{13}+e^{24}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146},e^{156}+e^{147},e^{157},$$
$$e^{234},e^{235},e^{236},e^{237}+e^{136},e^{245},e^{246}-e^{136},e^{247},e^{256}-e^{137},e^{257}-e^{147},$$
$$e^{345}+e^{147},e^{457}$$

$0,0,12,0,0,13+45,24$	$e_7(P)$	$[0,0,e^{12},0,0,e^{13}+e^{45}]$
$0,0,12,0,0,13+45,15+24$	$e_7(P)$	$[0,0,e^{12},0,0,e^{13}+e^{45}]$
$0,0,12,0,0,13+24,45$	e_7	$[0,0,e^{12},0,0,e^{13}+e^{24}]$

$$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{146},e^{147},-e^{127}+e^{156},e^{157},$$
$$e^{234},e^{235},e^{236},e^{245},e^{246}-e^{136},e^{247},e^{257},e^{345}-e^{127},e^{456}-e^{137},e^{457}$$

$0,0,12,0,0,13+14,15+23$	e_7	$[0,0,e^{12},0,0,e^{14}+e^{13}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{156},$$
$$e^{234},e^{235},-e^{147}+e^{236},e^{237}-e^{157},e^{245},e^{147}+e^{246},e^{247}+e^{256}+e^{157},$$
$$e^{257},-e^{247}+e^{345}$$

$0,0,12,0,0,13+24,15+23$	e_7	$[0,0,e^{12},0,0,e^{24}+e^{13}]$

$$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{146},e^{147}+e^{136},$$
$$e^{234},e^{235},e^{236},-e^{157}+e^{237},e^{245},e^{246}-e^{136},-e^{156}+e^{247},e^{157}+e^{256},$$
$$e^{257},-e^{156}+e^{345}$$

$0,0,12,0,0,13,23+45$	$e_6(P)$	$[0,0,e^{12},0,0,e^{23}+e^{45}]$
$0,0,12,0,0,13+24,23+45$	$e_6(P)$	$[0,0,e^{12},0,0,e^{23}+e^{45}]$

$0,0,0,12,13,14,15$	resists	

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147},$$
$$e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{245} + e^{236}, e^{246}, e^{345} + e^{237}, e^{346} + e^{256} + e^{247},$$
$$e^{257} + e^{347} + e^{356}, e^{357}$$

$0,0,0,12,13,14,35$	e_7	$[0,0,0,e^{12},e^{13},e^{14}]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{156}, e^{157},$$
$$e^{234}, e^{235}, e^{237}, e^{245} + e^{236}, e^{246}, e^{345} + e^{127}, e^{347} + e^{257}, -e^{147} + e^{356}, e^{357}$$

$0,0,0,12,13,14 + 35,15$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{35} + e^{14}]$
$0,0,0,12,13,14,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{34} + e^{25}]$
$0,0,0,12,13,14 + 15,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{25} + e^{34}]$
$0,0,0,12,13,24 + 35,25 + 34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{24} + e^{35}]$
$0,0,0,12,13,14 + 15 + 24 + 35,25 + 34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{35} + e^{24} + e^{15} + e^{14}]$
$0,0,0,12,13,14 + 24 + 35,25 + 34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{24} + e^{35} + e^{14}]$
$0,0,0,12,13,25 + 34,35$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{25} + e^{34}]$
$0,0,0,12,13,15 + 35,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{34} + e^{25}]$
$0,0,0,12,13,14 + 35,25 + 34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{14} + e^{35}]$
$0,0,0,12,13,14 + 23,15$	resists	

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157},$$
$$e^{234}, e^{235}, -e^{146} + e^{236}, -e^{156} + e^{237}, e^{245} + e^{146}, e^{246}, e^{167} + e^{257}, e^{156} + e^{345},$$
$$e^{346} + e^{256} + e^{247}, -e^{167} + e^{356} + e^{347}, e^{357}$$

$0,0,0,12,13,14 + 23,35$	e_7	$[0,0,0,e^{12},e^{13},e^{23} + e^{14}]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, -e^{127} + e^{156}, e^{157},$$
$$e^{234}, e^{235}, -e^{146} + e^{236}, e^{237}, e^{146} + e^{245}, e^{246}, e^{127} + e^{345}, e^{257} + e^{347},$$
$$-e^{147} + e^{356}, e^{357}$$

$0,0,0,12,13,15 + 24,23$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{24} + e^{15}]$
$0,0,0,12,13,14 + 35,15 + 23$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{35} + e^{14}]$
$0,0,0,12,13,23,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{25} + e^{34}]$
$0,0,0,12,13,14 + 23,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{25} + e^{34}]$
$0,0,0,12,13,14 + 15 + 23,25 + 34$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{25} + e^{34}]$
$0,0,12,0,0,0,13 + 24 + 56$	e_7	$[0,0,e^{12},0,0,0]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235},$$
$$e^{236}, e^{245}, e^{246}, e^{256}, -e^{157} + e^{345}, -e^{167} + e^{346}, e^{356} - e^{127}, e^{456}$$

$0,0,0,12,13,0,16 + 25 + 34$	e_7	$[0,0,0,e^{12},e^{13},0]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235},$$
$$e^{236}, e^{245} + e^{127}, e^{246}, e^{256} - e^{237}, e^{345} - e^{137}, e^{346} + e^{237}, e^{356}$$

$0,0,0,12,13,0,14 + 26 + 35$	e_7	$[0,0,0,e^{12},e^{13},0]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235},$$
$$e^{236}, e^{237} + e^{245}, e^{246}, e^{137} + e^{256}, e^{127} + e^{345}, -e^{137} + e^{346}, e^{356}, e^{157} + e^{456} - e^{367}$$

$0,0,0,12,23,-13,15 + 26 + 16 - 2*34$	e_7	$[0,0,0,e^{12},e^{23},-e^{13}]$

$$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, \tfrac{1}{2}e^{127} + e^{145}, e^{146},$$
$$e^{156} - e^{137}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246} + \tfrac{1}{2}e^{127}, e^{256} - e^{137} - e^{237},$$
$$e^{345} + e^{137} + e^{237}, e^{346} - e^{137}, e^{356}$$

$0,0,0,0,12,34,15 + 36$	e_7	$[0,0,0,0,e^{12},e^{34}]$

$$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236},$$
$$e^{245}, e^{246}, e^{345} - e^{126}, e^{346}, e^{347} - e^{156}, e^{356} + e^{127}, -e^{157} + e^{367}$$

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$0,0,0,0,12,34,15+24+36$	e_7	$[0,0,0,0,e^{12},e^{34}]$
$e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137}+e^{126},e^{145},e^{146},e^{234},e^{235},$ $e^{236},e^{245},e^{246},e^{345}-e^{126},e^{346},e^{347}-e^{156},e^{356}+e^{127}$		
$0,0,0,0,12,14+23,16-35$	e_7	$[0,0,0,0,e^{12},e^{14}+e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145},e^{156}+e^{127},e^{157},$ $e^{234},e^{235},-e^{146}+e^{236},e^{245},e^{246},-e^{146}+e^{345},e^{346},e^{147}+e^{237}+e^{356}$		
$0,0,0,0,12,14+23,16+24-35$	e_7	$[0,0,0,0,e^{12},e^{23}+e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{137}+e^{146},e^{156}+e^{127},$ $e^{234},e^{235},e^{236}+e^{137},e^{245},e^{246},e^{157}+e^{256},e^{345}+e^{137},e^{346},e^{356}+e^{237}+e^{147}$		
$0,0,12,0,0,13+14+25,15+23$	e_7	$[0,0,e^{12},0,0,e^{25}+e^{14}+e^{13}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{156},e^{157}+e^{136},$ $e^{234},e^{235},-e^{147}+e^{236},e^{136}+e^{237},e^{245},e^{147}+e^{246},e^{146}+e^{247},e^{256}-e^{146}-e^{136},$ $e^{257},e^{345}+e^{146}$		
$0,0,0,12,13,14,24+35$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{24}+e^{35}]$
$0,0,0,12,13,24-35,25+34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},-e^{35}+e^{24}]$
$0,0,0,12,13,14+24-35,25+34$	$e_7(P)$	$[0,0,0,e^{12},e^{13},e^{24}-e^{35}+e^{14}]$
$0,0,0,12,13,23,24+35$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{35}+e^{24}]$
$0,0,0,12,13,14+23,24+35$	$e_6(P)$	$[0,0,0,e^{12},e^{13},e^{24}+e^{35}]$
$0,0,0,12,13,0,16+24+35$	e_7	$[0,0,0,e^{12},e^{13},0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},$ $e^{236},e^{245}-e^{137},e^{246},e^{256}-e^{237},e^{127}+e^{345},e^{237}+e^{346},e^{356}$		
$0,0,0,0,13+24,14-23,15+26$	e_7	$[0,0,0,0,e^{13}+e^{24},e^{14}-e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},-e^{135}+e^{146},e^{234},e^{235},$ $e^{135}+e^{236},e^{237}-e^{147}+e^{156},-e^{135}+e^{245},e^{246},e^{137}+e^{247}+e^{256},e^{257}-e^{167},$ e^{345},e^{346}		
$0,0,0,0,13+24,14-23,15+26+24$	e_7	$[0,0,0,0,e^{24}+e^{13},e^{14}-e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},e^{146}-e^{135},e^{234},e^{235},$ $e^{135}+e^{236},e^{156}-e^{147}+e^{237},-e^{135}+e^{245},e^{246},e^{256}+e^{137}-e^{135}+e^{247},$ $e^{257}-e^{167}-e^{147},e^{345},e^{346}$		

Table 4: Step 4 nilpotent Lie algebras of dimension 7

$0,0,12,13,0,14,15$	e_7	$[0,0,e^{12},e^{13},0,e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{146},e^{147},$ $e^{156},e^{157},e^{167},e^{234},e^{235},-e^{237}+e^{245},e^{257},e^{256}+e^{345}$		
$0,0,12,13,0,25,14$	e_7	$[0,0,e^{12},e^{13},0,e^{25}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156},e^{157},$ $e^{234},e^{235},e^{236},e^{136}+e^{245},e^{256},-e^{146}+e^{257},e^{146}+e^{345}$		
$0,0,12,13,0,14+25,15$	e_7	$[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156},e^{157},$ $e^{234},e^{235},e^{136}+e^{237},e^{136}+e^{245},e^{256}-e^{146},e^{257},e^{345}+e^{146},e^{357}+e^{167}$		

$0,0,12,13,0,14+23+25,15$	e_6	$[0,0,e^{12},e^{13},0,e^{15}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},$
$e^{147},e^{136}+e^{156},e^{157},e^{234},e^{235},e^{237}+e^{136},e^{245}+e^{136},$
$e^{256}+e^{236}-e^{146},e^{257},e^{345}-e^{236}+e^{146},e^{247}-e^{236}+e^{357}+e^{167}+e^{146}$

$0,0,12,13,0,23+25,14$	e_7	$[0,0,e^{12},e^{13},0,e^{23}+e^{25}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{137},e^{145},e^{147},e^{156}+e^{136},$
$e^{157},e^{234},e^{235},e^{236},e^{136}+e^{245},e^{256},e^{257}-e^{146}+e^{237},e^{345}+e^{146}-e^{237}$

$0,0,12,13,0,14+23,15$	e_7	$[0,0,e^{12},e^{13},0,e^{23}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{147},e^{157},e^{234},e^{235},$
$-e^{146}+e^{236},-e^{156}+e^{237},e^{245}-e^{156},e^{247}+e^{256}+e^{167},e^{257},e^{345}-e^{247}-e^{167}$

$0,0,12,13,0,15+23,14$	e_7	$[0,0,e^{12},e^{13},0,e^{15}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{137},e^{145},e^{147},e^{157},$
$e^{234},e^{235},-e^{156}+e^{236},-e^{146}+e^{237},-e^{156}+e^{245},e^{167}+e^{247},e^{256},e^{345}+e^{257}$

$0,0,12,13,0,23,14+25$	$e_7(P)$	$[0,0,e^{12},e^{13},0,e^{23}]$

$0,0,12,13,0,14+23,25$	e_7	$[0,0,e^{12},e^{13},0,e^{14}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{137}+e^{156},e^{157},$
$e^{234},e^{235},-e^{146}+e^{236},e^{237},e^{137}+e^{245},-e^{147}+e^{256},e^{257},e^{147}+e^{345}$

$0,0,12,13,0,14+23,23+25$	e_7	$[0,0,e^{12},e^{13},0,e^{23}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{156}+e^{137},e^{157}+e^{137},$
$e^{234},e^{235},-e^{146}+e^{236},e^{237},e^{137}+e^{245},e^{146}-e^{147}+e^{256},e^{257},-e^{146}+e^{147}+e^{345}$

$0,0,12,13,0,15+23,14+25$	e_7	$[0,0,e^{12},e^{13},0,e^{15}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{156}+e^{137},e^{157},e^{234},e^{235},e^{236}+e^{137},$
$-e^{146}+e^{237},e^{245}+e^{137},e^{256},-e^{147}+e^{257},e^{345}+e^{147},e^{356}-e^{167}-e^{247}$

$0,0,12,13,23,14+25,15+24$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{23},e^{14}+e^{25}]$

$0,0,12,13,23,24+15,14$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{23},e^{14}]$

$0,0,0,12,14+23,23,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{127},e^{146}+e^{127},e^{147},e^{234},$
$e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},-e^{156}+e^{346}+e^{237},e^{167}-\frac{1}{2}e^{157}-\frac{1}{2}e^{347}+e^{356}$

$0,0,0,12,14+23,13,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{13}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{127}+e^{145},e^{146},e^{147},$
$e^{234},e^{127}+e^{235},e^{236},e^{245},-e^{127}+e^{246},e^{237}+e^{345},-e^{156}+e^{346},e^{167}+e^{356}$

$0,0,0,12,14+23,24,15-34$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},$
$e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},e^{247}+e^{346}-e^{156}$

$0,0,0,12,14+23,13+24,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{24}+e^{13}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},$
$e^{147},e^{234},e^{127}+e^{235},e^{236},e^{245},e^{246}-e^{127},e^{345}+e^{237},e^{346}-e^{156}+e^{247}$

$0,0,12,13,0,0,14+56$	e_7	$[0,0,e^{12},e^{13},0,0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{157},e^{167},$
$e^{234},e^{235},e^{236},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},e^{456}-e^{137},e^{567}-e^{147}$

$0,0,12,13,0,0,23+14+56$	e_7	$[0,0,e^{12},e^{13},0,0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},e^{236},-e^{157}+e^{245},$
$-e^{167}+e^{246},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},-e^{137}+e^{456}$

$0,0,0,12,14+23,0,15+26-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{127},e^{146},-e^{137}+e^{156},$
$e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{256}+e^{147},e^{345}+e^{237},e^{346}-e^{137},e^{356}+e^{167}$

$0,0,0,12,14+23,0,15+36-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{146},e^{127}+e^{156}+e^{145},-e^{147}+e^{167},$
$e^{234},e^{235}-e^{145},e^{236},e^{245},e^{246},e^{237}+e^{345},e^{346}+e^{127}+e^{145},e^{147}+e^{356}$

$0,0,0,12,14+23,0,15+24+36-34$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{137},e^{146},e^{156}+e^{127}-e^{137},$
$e^{234},e^{235}+e^{137},e^{236},e^{245},e^{246},-e^{167}+e^{147}+e^{256},$
$e^{345}+e^{237},e^{346}+e^{127}-e^{137},e^{147}+e^{356}$

$0,0,12,0,23,24,16+25+34$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},e^{235},e^{236},$
$e^{147}+e^{237}+e^{156},e^{245},e^{246},e^{256},e^{147}+e^{345},-e^{247}+e^{346},e^{456}+e^{267}$

$0,0,12,0,23,24,25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145}+e^{136},e^{146},e^{234},e^{235},e^{236},$
$e^{245},e^{246},e^{247},e^{256},e^{267},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456},e^{467}-e^{257}$

$0,0,12,0,23,24,13+25-46$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{145},e^{146},e^{234},e^{235},e^{236},e^{245},$
$e^{246},e^{247}-e^{136},e^{256},-e^{147}+e^{267}-e^{156},e^{147}+e^{345},e^{127}+e^{346},-e^{237}+e^{456}$

$0,0,12,0,23,14,16+25$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$
$e^{156}+e^{237},e^{245},e^{246},-e^{147}+e^{256},-e^{167}+e^{257},e^{147}+e^{345},e^{346}-e^{247}$

$0,0,12,0,23,14,16+25+26-34$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{127}+e^{145},e^{146},e^{234},e^{235},e^{127}+e^{236},$
$e^{156}+e^{237},e^{245},e^{246},e^{256}-e^{247}-e^{147},e^{345}+e^{247},e^{147}+e^{346}-e^{247}$

$0,0,12,0,23,14,25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$
$e^{245},e^{246},e^{247},e^{256}-e^{147},e^{345}+e^{147},e^{346}-e^{127},e^{237}+e^{456},-e^{257}+e^{467}$

$0,0,12,0,23,14,13+25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},$
$e^{245},e^{246},e^{145}+e^{247},e^{256}-e^{147},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456}$

$0,0,12,0,13+24,14,15+23+1/2*(26+34)$	e_7	$[0,0,e^{12},0,e^{13}+e^{24},e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},2e^{127}+e^{135},e^{136},e^{145},e^{146},4e^{127}-2e^{147}+e^{156},$
$e^{234},e^{235},-2e^{127}+e^{236},2e^{127}+e^{245},e^{246},2e^{137}+e^{247}+e^{256},-e^{247}+e^{345},$
$-4e^{127}+e^{346}+2e^{147},e^{456}+4e^{137}+2e^{167}$

$0,0,12,0,13+24,23,16+25$	e_7	$[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},e^{146}+e^{135},e^{234},e^{235},e^{236},-e^{135}+e^{245},$
$e^{246},e^{156}+e^{137}+e^{247},e^{256}-e^{237},e^{267},e^{345}+e^{147},e^{346}+e^{156}+e^{137}$

$0,0,12,0,13+24,23,15+26+34$	e_7	$[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{127}+e^{135},e^{136},e^{145},e^{146}-e^{127},e^{234},$
$e^{235},e^{236},e^{237}-e^{156}+e^{147},e^{127}+e^{245},e^{246},e^{256}+e^{137},-e^{247}+e^{345},e^{346}+e^{147}$

$0,0,12,0,13,23+24,15+26$	e_7	$[0,0,e^{12},0,e^{13},e^{23}+e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{136}+e^{146},e^{234},e^{235},e^{236},-e^{136}+e^{245},$
$e^{246},e^{247}+e^{237}-e^{156},e^{256}+e^{137},-e^{157}+e^{267},e^{237}-e^{156}+e^{345},e^{147}+e^{346}$

$0,0,12,0,13,23+24,16+25+34$	e_7	$[0,0,e^{12},0,e^{13},e^{24}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{145},-e^{127}+e^{146},e^{137}+e^{147}+e^{156},$
$e^{234},e^{235},e^{236},e^{127}+e^{245},e^{246},e^{256}-e^{237},e^{345}+e^{147},e^{346}-e^{247}$

$0,0,12,13,23,14-25,15+24$	e_7	$[0,0,e^{12},e^{13},e^{23},-e^{25}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145}+e^{137},e^{147},e^{234},e^{235},$
$e^{236}+e^{137},-e^{136}+e^{237},-e^{136}+e^{245},e^{156}+e^{246},e^{247}+e^{157}-2e^{146},$
$e^{256}+e^{146},e^{257},-e^{146}+e^{345}$

$0,0,0,12,14+23,13-24,15-34$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},e^{13}-e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{137},e^{145}+e^{127},e^{146},$
$e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246}-e^{127},e^{345}+e^{237},-e^{247}+e^{346}-e^{156}$

$0,0,12,0,23,24,13+25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145}+e^{136},e^{146},e^{234},e^{235},e^{236},$
$e^{245},e^{246},e^{247}-e^{136},e^{256},e^{267}-e^{147}-e^{156},e^{345}+e^{147},e^{346}-e^{127},e^{237}+e^{456}$

$0,0,12,0,13+24,23,15+34-26$	e_7	$[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135}+e^{127},e^{136},e^{145},e^{146}-e^{127},e^{234},$
$e^{235},e^{236},-e^{156}-e^{147}+e^{237},e^{245}+e^{127},e^{246},e^{256}-e^{137},e^{345}-e^{247},e^{346}-e^{147}$

$0,0,12,0,13,23+24,15-26$	e_7	$[0,0,e^{12},0,e^{13},e^{24}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{145},e^{146}+e^{136},e^{234},e^{235},e^{236},e^{245}-e^{136},$
$e^{246},-e^{156}+e^{237}+e^{247},-e^{137}+e^{256},e^{157}+e^{267},-e^{156}+e^{345}+e^{237},-e^{147}+e^{346}$

Table 5: Step 5 nilpotent Lie algebras of dimension 7

$0,0,12,13,14,15,23$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{14},e^{23}]$
$0,0,12,13,14,25-34,23$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{14},e^{25}-e^{34}]$
$0,0,12,13,14,15,25-34$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{14},e^{25}-e^{34}]$
$0,0,12,13,14,15+23,25-34$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{14},-e^{34}+e^{25}]$
$0,0,12,13,14+23,15+24,23$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{23}+e^{14},e^{23}]$
$0,0,12,13,14+23,25-34,23$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{14}+e^{23},e^{25}-e^{34}]$
$0,0,12,13,14+23,15+24,25-34$	$e_6(P)$	$[0,0,e^{12},e^{13},e^{23}+e^{14},-e^{34}+e^{25}]$

$0,0,12,13,14,0,15+26$	e_7	$[0,0,e^{12},e^{13},e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{167},$
$e^{234},e^{236},-e^{237}+e^{245},e^{246}+e^{137},e^{256}+e^{147},-e^{157}+e^{267},e^{147}+e^{346}$

$0,0,12,13,14,0,15+23+26$	e_7	$[0,0,e^{12},e^{13},e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{137}+e^{167},e^{234},e^{236},$
$-e^{237}+e^{245},e^{137}+e^{246},e^{147}+e^{256}-e^{235},-e^{157}+e^{267}+e^{237},e^{147}-e^{235}+e^{346}$

$0,0,12,13,14,0,16+25-34$	e_7	$[0,0,e^{12},e^{13},e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{127}+e^{235},$
$e^{236},e^{137}+e^{245},-e^{237}+e^{246},e^{147}+e^{345},e^{346}-e^{256}$

$0,0,12,13,14+23,0,15+24+26$	e_7	$[0,0,e^{12},e^{13},e^{14}+e^{23},0]$

$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{146},-e^{137}+e^{156}-e^{145},$
$e^{167}+e^{147},e^{234},e^{235}-e^{145},e^{236},-e^{237}+e^{245},e^{137}+e^{145}+e^{246},e^{147}+e^{256},$
$-\frac{1}{2}e^{157}-\frac{1}{2}e^{247}+\frac{1}{2}e^{267}+e^{345},e^{147}+e^{346}$

$0,0,12,13,14+23,0,16+25-34$	e_7	$[0,0,e^{12},e^{13},e^{23}+e^{14},0]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{127},e^{146},e^{234},e^{127}+e^{235},$
$e^{236},e^{237}+e^{156},e^{137}+e^{245},e^{246}+e^{156},e^{147}+e^{345},e^{346}-e^{256}$

$0,0,12,13,14,23,15+26$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{14},e^{23}]$
$0,0,12,13,14,23,16+24+25-34$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{14},e^{23}]$
$0,0,12,13,14,23,15+25+26-34$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{14},e^{23}]$
$0,0,12,13,0,14+25,16+35$	e_7	$[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},e^{145},e^{156},e^{157},$
$e^{234},e^{235},e^{245}-e^{127},-e^{237}+e^{246},e^{256}-e^{146},$
$e^{345}+e^{146},-e^{257}+e^{356}-e^{147},-\frac{1}{2}e^{167}-\frac{1}{2}e^{357}+e^{456}$

$0,0,12,13,0,14+25,16+25+35$	e_7	$[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{137}+e^{127},e^{145},e^{156},$
$e^{157},e^{234},e^{235},e^{245}-e^{127},e^{246}-e^{237},e^{256}-e^{146},e^{345}+e^{146},$
$-e^{257}+e^{356}+e^{146}-e^{147},\frac{1}{2}e^{257}-\frac{1}{2}e^{357}+e^{456}-\frac{1}{2}e^{167}$

$0,0,12,13,0,14+25,26-34$	e_7	$[0,0,e^{12},e^{13},0,e^{25}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156},e^{234},e^{235},e^{236}+e^{127},$
$e^{237},e^{245}+e^{136},e^{246}+e^{137},e^{256}-e^{146},e^{146}+e^{345},e^{147}+e^{257}+e^{346}$

$0,0,12,13,0,14+25,15+26-34$	e_7	$[0,0,e^{12},e^{13},0,e^{14}+e^{25}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156},e^{234},e^{235},e^{127}+e^{236},$
$e^{136}+e^{237},e^{136}+e^{245},e^{137}+e^{246},e^{256}-e^{146},e^{345}+e^{146},e^{257}+e^{346}+e^{147}$

$0,0,12,13,0,14+23+25,16+24+35$	e_7	$[0,0,e^{12},e^{13},0,e^{14}+e^{25}+e^{23}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145},e^{156}-e^{127},e^{157}+e^{137}+e^{146},$
$e^{234},e^{235},e^{137}+e^{236},e^{245}-e^{127},-e^{237}+e^{246},e^{256}-e^{137}-e^{146},$
$e^{345}+e^{137}+e^{146},-e^{257}-e^{147}+e^{356}$

$0,0,12,13,0,14+23+25,26-34$	e_7	$[0,0,e^{12},e^{13},0,e^{23}+e^{14}+e^{25}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{156}+e^{136},e^{234},e^{235},e^{236}+e^{127},$
$e^{237},e^{245}+e^{136},e^{246}+e^{137},e^{256}-e^{146}-e^{127},e^{345}+e^{146}+e^{127},e^{346}+e^{257}+e^{147}$

$0,0,12,13,0,14+23+25,15+26-34$	e_7	$[0,0,e^{12},e^{13},0,e^{23}+e^{25}+e^{14}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145},e^{136}+e^{156},e^{234},e^{235},e^{127}+e^{236},e^{136}+e^{237},$
$e^{136}+e^{245},e^{137}+e^{246},-e^{127}+e^{256}-e^{146},e^{345}+e^{127}+e^{146},e^{346}+e^{147}+e^{257}$

$0,0,12,13,23,15+24,16+34$	resists	

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},-e^{127}+e^{145},e^{146},$
$e^{147},e^{234},e^{235},e^{245}-e^{236},e^{246}-e^{156}-2e^{237},e^{256},e^{345}-e^{156}-e^{237},$
$e^{346}-e^{247}-e^{157},e^{267}-e^{357}+e^{456}$

$0,0,12,13,23,15+24,16+25+34$	resists	

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},$
$e^{235},e^{236}+e^{137},e^{237}+e^{147}+e^{156},e^{137}+e^{245},2e^{147}+e^{156}+e^{246},e^{256},$
$e^{345}+e^{147},e^{346}-e^{247}-e^{157},e^{456}-e^{357}+e^{267}$

$0,0,12,13,23,15+24,16+14+25+34$	e_7	$[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},$
$e^{235},e^{236}+e^{137},e^{237}+e^{156}+e^{147}-e^{127},e^{245}+e^{137},e^{246}+e^{156}+2e^{147},$
$e^{256},e^{345}+e^{147},e^{346}-e^{247}-e^{157}$

$0,0,12,13,23,15+24,16+14+34$	e_7	$[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},e^{137},-e^{127}+e^{145},e^{146},$
$e^{147},e^{234},e^{235},-e^{236}+e^{245},2e^{127}-2e^{237}-e^{156}+e^{246},$
$e^{256},e^{127}-e^{237}-e^{156}+e^{345},-e^{247}-e^{157}+e^{346}$

$0,0,12,13,23,15+24,16+26+34-35$	e_7	$[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{145},e^{146},e^{234},e^{235},e^{127}+e^{136}+e^{236},$
$e^{137}+e^{237},e^{127}+e^{136}+e^{245},e^{246}+2e^{137}-e^{156},$
$e^{256},e^{345}+e^{137}-e^{156},-e^{346}+e^{356}+e^{247}+e^{157}$

$0,0,0,12,14+23,15-34,16-35$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},-e^{34}+e^{15}]$

$e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{126}+e^{145},e^{146},e^{234},e^{126}+e^{235},e^{127}+e^{236},$
$e^{237}+e^{147}+e^{156},e^{245},-e^{127}+e^{345},-2e^{147}+e^{346}-e^{156},-e^{347}+e^{356}+e^{157}$

$0,0,0,12,14+23,15-34,16+23-35$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{15}-e^{34}]$

$e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{126},e^{146},e^{234},e^{235}+e^{126},$
$e^{236}+e^{127},e^{237}+e^{126}+e^{156}+e^{147},e^{245},e^{345}-e^{127},$
$e^{346}-2e^{126}-e^{156}-2e^{147},e^{356}-e^{347}+e^{127}+e^{157}$

$0,0,0,12,14+23,15-34,16+24-35$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},e^{15}-e^{34}]$

$e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},-e^{126}+e^{137},e^{126}+e^{145},e^{146},$
$e^{234},e^{126}+e^{235},e^{236}+e^{127},e^{156}+e^{147}+e^{237},e^{245},$
$e^{345}-e^{127},-e^{156}+e^{346}-2e^{147},-e^{246}+e^{356}-e^{347}+e^{157}$

$0,0,12,13,23,24+15,16+14-25+34$	e_7	$[0,0,e^{12},e^{13},e^{23},e^{24}+e^{15}]$

$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{127}+e^{136},-e^{127}+e^{145},e^{146},e^{234},$
$e^{235},e^{236}-e^{137},-e^{127}+e^{237}-e^{147}+e^{156},-e^{137}+e^{245},-2e^{147}+e^{156}+e^{246},$
$e^{256},-e^{147}+e^{345},-e^{247}-e^{157}+e^{346}$

$0,0,12,13,23,-14-25,16-35$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{23},-e^{14}-e^{25}]$
$0,0,12,13,23,-14-25,16-35+25$	$e_7(P)$	$[0,0,e^{12},e^{13},e^{23},-e^{25}-e^{14}]$

$0,0,0,12,14+23,15-34,16-23-35$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{15}-e^{34}]$

$e^{123},e^{124},e^{125},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{126},e^{146},e^{234},e^{235}+e^{126},$
$e^{236}+e^{127},e^{237}-e^{126}+e^{156}+e^{147},e^{245},$
$e^{345}-e^{127},e^{346}+2e^{126}-e^{156}-2e^{147},e^{356}-e^{347}-e^{127}+e^{157}$

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