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MANN ITERATIONS WITH POWER MEANS

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Abstract

In this paper we analyze a recurrence $x_{n+1} = f(z_n)$, where z_n is a *weighted power mean* of $x_0, \ldots, x_n$. Such an iteration scheme has been proposed to model a class of non-linear forward-looking economic models (the state today is affected by tomorrow' s expectation) under bounded rationality; the agents employ a recursive learning rule to update beliefs using weighted power means of the past states. A proposition on the convergence of the dynamical system with memory, proven with a general weighted power mean, generalizes some results given in the literature, where only the arithmetic mean is considered. A power weighted mean with exponentially decreasing weights decreasing is proposed to simulate a fading memory. In this case the iteration scheme with memory is reduced to an equivalent two-dimensional autonomous map whose possible kinds of asymptotic behaviors are the same as those of a one-dimensional map. By this general technique it is proved, for a function *f* which maps a compact interval into itself, that the presence of a long memory has a stabilizing effect, in the sense that with a sufficiently strong memory convergence to a steady state is obtained even for an otherwise oscillating, or chaotic, dynamical system. In the appendix is considered an economic example from an overlapping generation models which leads to a harmonic mean.

Keywords: Forward-looking models, Learning, Mann Iterations, Nonautonomous difference equations

1. Introduction

A Mann iteration is an iterative scheme of the form:

(1.1) $$x_{n+1} = f(z_n)$$

where $f : I \to I$, $I = [a,b] \subset \Re_+$, and z_n is the arithmetic mean of all the previous values x_i, $0 \le i \le n$:

(1.2) $$z_n = \sum_{k=0}^{n} a_{nk} x_k$$

with

(1.3) $$\sum_{k=0}^{n} a_{nk} = 1$$

Such an iteration scheme has been used to model economic and social systems with agents who have not perfect foresight, so they learn from the past experiences using all the available information, that is present and past data, in order to calculate the expected values of future states. If n represents discrete time periods and x_n the value of the *state variable* in period n, z_n can be interpreted as the *expected value* (see e.g. Bray, 1983, Lucas, 1986, Balasko and Royer, 1996). Starting from the seminal paper of Mann (1953), iterations (1.1) have been studied by many authors, among others Borwein and Borwein (1991), Rhoades (1974), Aicardi and Invernizzi (1992).

A recurrence of the form (1.1) with z_n given by an uniform arithmetic mean

(1.4) $$z_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k$$

has been proposed by Bray (1983) as a learning mechanism. In this case the Mann iteration coincides with the Cesáro iteration, whose dynamics are very simple since in this case every $x_0 \in I$ generates a sequence $\{x_n\}$ converging to a fixed point of f (Franks and Marzek, 1971). This suggests a strong stabilizing effect of a distributed uniform memory since any kind of dynamics more complex than convergence towards a fixed point of f is excluded, and the only possibility of a non trivial dynamics is the existence of more than one fixed point of f in I, so that different basins of attraction must be considered.

In this paper we propose a generalization of (1.2) expressed by the *power mean*

(1.5) $$z_n = \left(\sum_{k=0}^{n} a_{nk} x_k^p \right)^{\frac{1}{p}}, \quad p \ne 0$$

The arithmetic mean (1.2) is a special case of the power mean (1.5) when $p = 1$, but other commonly used algebraic means can be obtained from (1.5), such as the weighted quadratic mean for $p = 2$ and the weighted harmonic mean for $p = -1$. Furthermore the weighted geometric mean is obtained as a limiting case for $p \to 0$, since

(1.6) $$\lim_{p\to 0}\left(\sum_{k=0}^{n} a_{nk} x_k^p\right)^{\frac{1}{p}} = \prod_{k=0}^{n} x_k^{a_{nk}} .$$

Of course if $p < 0$ the further condition $x_i > 0$ for each i should be verified
This study is motivated by the possibility that some learning mechanism can be expressed by the iteration scheme (1.1) with algebraic means of the form (1.5) with $p \neq 1$ (an example is given in appendix A, and some properties of such means, as well as their applications, can be found in Vajani, 1981, ch.6).
In Section 2 the iteration scheme (1.1) with (1.2) replaced by (1.5) is reduced to a first order nonautonomous recurrence, and some convergence results are given which generalize the results of Mann (1953) and Borwein and Borwein (1991), where only the arithmetic mean (1.2) is considered. In sections 3 the power mean (1.5) is considered with weights decreasing as the terms of a geometric progression. These are often used in applications since they describe, as suggested in Friedman (1973), agents which "form their expectations according to a weighted estimation procedure which exponentially discounts older observations", that is, an *exponentially fading memory*. In this case the assumptions of the propositions of Section 2 do not hold, and more complex asymptotic dynamics can be obtained. The results of this section generalize, to the case of power means, the results given in Bischi and Gardini (1995) and Bischi et al. (1995) on Mann iterations which can be reduced to two-dimensional maps.

2. Convergence of recurrences with power means

In the following we assume that the weights are obtained as

(2.1) $$a_{nk} = \frac{\omega_k^{(n)}}{W_n}, \quad \omega_k \geq 0$$

where, for each $n \geq 0$, the $(n+1)$ dimensional vector of nonnegative weights

(2.2) $$\omega^{(n)} = \left\{\omega_0^{(n)}, \omega_1^{(n)}, ..., \omega_n^{(n)}\right\}$$

defines the relative influence of each state x_k, k=0, , n, in the computation of the average z_n, and

(2.3) $$W_n = \sum_{k=0}^{n} \omega_k^{(n)}$$

so that (1.3) is satisfied.
In this section we assume, as in Rhoades (1974) and in Borwein and Borwein (1991), that at each n the vector of relative weights is obtained by adding the last component without any change of the previous ones, that is, from $\omega^{(n)} = (\omega_0, \omega_1, ..., \omega_n)$ we obtain $\omega^{(n+1)} = (\omega_0, \omega_1, ..., \omega_n, \omega_{n+1})$. In this case we have

(2.4) $$W_{n+1} = W_n + \omega_{n+1} .$$

The iterative scheme (1.1) with a power mean (1.5) becomes

(2.5) $$x_{n+1} = f(z_n)$$

with

$$z_n = \left(\sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^p \right)^{\frac{1}{p}}, \quad W_n = \sum_{k=0}^{n} \omega_k , \quad p \neq 0,$$

where f has at least one fixed point in I being it a continuous function which maps the compact set I into itself.
Recurrence (2.5) with $p = 1$ is a Mann iteration, for which the following classical result holds:

Theorem (Mann, 1953). *Let $p = 1$ and $W_n \to \infty$. If either of the sequences $\{x_n\}$ and $\{z_n\}$ converges then the other also converges to the same point and their common limit is a fixed point of f.*

In Rhoades (1974) and Borwein and Borwein (1991) a Mann iteration (1.1) is reduced to the following nonautonomous iteration, called *segmenting Mann iteration*

(2.6) $$z_{n+1} = (1 - t_n) z_n + t_n f(z_n)$$

where $z_0 = x_0 \in I$, and

(2.7) $$t_n = \frac{\omega_{n+1}}{W_{n+1}}$$

From $\{z_n\}$ the sequence of states $\{x_n\}$ can be easily obtained as the images of z_n under f:

(2.8) $$x_{n+1} = f(z_n).$$

The following result is proved in Borwein and Borwein (1991):

Theorem (Borwein and Borwein, 1991). *Suppose that $\{t_n\}$ tends to zero. Then the sequence $\{z_n\}$ converges.*

In this section we generalize these theorems to the case of power mean with $p \neq 1$. This can easily be done once the more general iterative scheme (2.5) is put into a recursive form, for the expected variables z_n, similar to (2.6). In fact, even with $p \neq 1$, from (2.5) we get:

$$z_{n+1} = \left(\sum_{k=0}^{n} \frac{\omega_k}{W_{n+1}} x_k^p + \frac{\omega_{n+1}}{W_{n+1}} x_{n+1}^p \right)^{\frac{1}{p}} = \left(\frac{W_n}{W_{n+1}} \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^p + \frac{\omega_{n+1}}{W_{n+1}} x_{n+1}^p \right)^{\frac{1}{p}}$$

from which, by using the definition (2.7) of t_n and the identity (2.4) we obtain what we shall call *generalized segmenting Mann iteration*,

$$z_{n+1} = F(n, z_n) = \left((1-t_n)z_n^p + t_n[f(z_n)]^p\right)^{\frac{1}{p}}. \tag{2.9}$$

Also in this case the iterative process described by the nonautonomous first order difference equation is equivalent to the iterative process (2.5), in the sense that given an initial condition $z_0 = x_0$ the sequence of expected values obtained from (2.9) is the same as that obtained from (2.5) (and the sequence of states is given by (2.8)).
We recall that a fixed point (or stationary state) of the iteration (2.5) is defined as a value $x^* \in \Re$ such that if $x_0 = x^*$ then (2.5) generates the sequence $x_n = x^*$ for each $n \geq 0$. The following results are straightforward:

Proposition 2.1. (i) x^* *is a fixed point of the iteration (2.5) if and only if it is a fixed point of the function f.*
(ii) z^* *is a fixed point of* $F(n,z)$ *if and only if it is a fixed point of f.*

We recall that a fixed point (or stationary state) of the nonautonomous difference equation (2.9) is defined as a value z^* such that $F(n, z^*) = z^*$ for each n.
The following proposition, which is proved in the appendix B, generalizes the theorems quoted above .

Proposition 2.2. (i) *If* $W_n \to \infty$ *then the sequence* $\{x_n\}$ *defined in (2.5) converges if and only if the sequence* $\{z_n\}$ *in (2.9) converges and the two sequences converge to a common limit which is a fixed point of f.* (ii) *If in (2.9)* $\{t_n\}$ *is a positive sequence which tends to zero then the sequence* $\{z_n\}$ *is convergent.*

Of course if f has a unique fixed point $x^* \in I$, then it is globally attracting in I, i.e. $x_n \to x^*$ for each $x_0 \in I$.
A typical example in which these propositions can be applied is that of a uniform power mean, that is with equal weights $\omega_k = \omega$ for any k. In fact in this case we have $t_n = 1/(n+1) \to 0$ and $W_n \to \infty$.This constitutes a generalization of the result of Franks and Marzek (1971) on the Cesàro iteration, since it includes the uniform arithmetic mean for $p = 1$, the uniform harmonic mean for $p = -1$, the uniform geometric mean for $p \to 0$, and so on.

3. Asymptotic dynamics with exponentially decreasing weights

Another method for defining, at each n, the vector of relative weights, is that of assigning a fixed value to the weight of the last state, say $\omega_n^{(n)} = \omega_0^{(0)} = 1$, and the values of the previous ones are obtained so that the ratio between two successive weights is fixed, say $\omega_k^{(n)} / \omega_{k+1}^n = \rho$, i.e. from $\omega^{(n)} = (\rho^n, \rho^{n-1}, ..., \rho, 1)$ we obtain $\omega^{(n+1)} = (\rho^{n+1}, \rho^n, ..., \rho, 1)$, or, more concisely, (3.1) $\omega_k^{(n)} = \rho^{n-k}, \quad 0 \le k \le n$.

With these weights the following relation holds:

(3.2) $W_{n+1} = 1 + \rho W_n$,

and the recurrence with fading memory becomes

(3.3) $x_{n+1} = f(z_n)$

with

$$z_n = \left(\sum_{k=0}^{n} \frac{\rho^{n-k}}{W_n} x_k^p \right)^{\frac{1}{p}}, \qquad W_n = \sum_{k=0}^{n} \rho^{n-k} = \frac{1-\rho^{n+1}}{1-\rho}, \qquad p \neq 0.$$

As already stressed in Section 1 these weights are often used in economic modelling (see Ganfolfo et al. 1991, Aicardi and Invernizzi, 1992) since, with a *memory ratio* $\rho \in (0,1)$, they represent the realistic assumption of an exponentially fading memory (see Friedman, 1979, Radner, 1983). Lt us first show that the relation (3.2) allows us to obtain, also in this case, a generalized segmenting Mann iteration. In fact we have

$$z_{n+1} = \left(\frac{1}{W_{n+1}} \left(\sum_{K=0}^{n} \rho \rho^{n-k} x_k^p + x_{n+1}^p \right) \right)^{\frac{1}{p}} = \left(\frac{\rho W_n}{W_{n+1}} z_n^p + \frac{1}{W_{n+1}} [f(z_n)]^p \right)^{\frac{1}{p}},$$

and defining

$$t_n = \frac{1}{W_{n+1}} = \frac{1-\rho}{1-\rho^{n+1}}$$

and making use of the identity (3.2), we get the required nonautonomous difference equation:

(3.5) $z_{n+1} = F(n, z_n) = \left((1-t_n) z_n^p + t_n [f(z_n)]^p \right)^{\frac{1}{p}}$.

When $\rho \ge 1$ (non decreasing memory) the main results of section 2 can be applied, without substantial changes, also to the case of geometric weights. In the following we shall consider the more realistic case of memory ratio $\rho \in (0,1)$ (exponentially fading memory). In this case the propositions of Section 2 do not apply, because the sequence of partial sums W_n converges to the value $W^* = 1/(1-\rho)$ and the sequence t_n, defined in (3.4), is not convergent to zero, being $t_n \to (1-\rho)$. For $\rho = 0$ (no memory of the past), the problem reduces to the study of the dynamics of an ordinary one-dimensional map $x_{n+1} = f(x_n)$. Since, as it is well known, the asymptotic

dynamics of this iteration may be periodic of period $k \geq 1$, or aperiodic (i.e. chaotic) depending on the shape of the function f, we can expect complex dynamics also for $\rho > 0$.
We have seen in Section 2 that the only possible fixed points of the generalized Mann iteration are the fixed points of the function f. One may ask if also different asymptotic states, as k-cycles, $k \geq 2$, are related to k-cycles of the map f. The answer is no. If $0 < \rho < 1$, and a k-cycle of (3.5) exist, then in general it is not a k-cycle of the map f. However such cycles are related to those of another one-dimensional (autonomous) map. This can be intuitively justified on the basis of the observation that the sequences of the time-dependent coefficients in the right hand side of (3.4) are convergent, since $t_n \to (1-\rho)$, so that the right hand side of (1.16) possesses an autonomous *limiting form:*

(3.6) $$z_{n+1} = g_\rho(z_n), \qquad \text{with } g_\rho(z) = \left(\rho z^p + (1-\rho)[f(z)]^p\right)^{\frac{1}{p}}.$$

It comes natural to conjecture that the asymptotic behavior of (3.5) is related to that of the map $g_\rho(z)$. That this is the case can be rigorously proved by making use of a two-dimensional map. Let us note, in fact, that the sequence of the partial sums W_n of the geometric weights can be defined recursively by (3.2) and this allows us to obtain a two dimensional map $(z_{n+1}, W_{n+1}) = T(z_n, W_n)$ defined as

(3.7) $$T: \begin{cases} z_{n+1} = \left(\dfrac{\rho W_n}{1+\rho W_n} z_n^p + \dfrac{1}{1+\rho W_n}[f(z_n)]^p\right)^{\frac{1}{p}} \\ W_{n+1} = 1 + \rho W_n. \end{cases}$$

This map is equivalent to (3.5) if the initial condition is taken with $W_0 = 1$, i.e.

(3.8) $$(z_0, W_0) = (x_0, 1), \qquad x_0 \in I.$$

In fact, in such a case, the sequence $\{z_n\}$ given by (3.7) coincides with the sequence obtained from the generalized segmenting Mann iteration (3.5) related to the same initial condition z_0. In other words, the projection on the z-axis of an orbit of the map T (with initial condition as in (3.8)) is the orbit of the nonautonomous iterative process (3.5).
The map (3.7) is a triangular map, that is a map with the structure $T(z,W) = (T_1(z,W), T_2(W))$. We notice that the map T is not defined on the points of the line of equation $W = -\frac{1}{\rho}$, but, since the initial conditions are to be taken on the line $W = 1$, we shall consider the restriction of T to the half-plane $W > -\frac{1}{\rho}$. In fact this half-plane is mapped into itself by T because the second difference equation in (3.7) gives an increasing sequence (the partial sums of the geometric series starting from $W = 1$) always converging to the limit

(3.9) $$W^* = \frac{1}{1-\rho}.$$

This also implies that the line $W = W^*$ is mapped into itself by T (i.e. is a trapping set), and is globally attracting for T in the half-space $W > -\frac{1}{\rho}$ (which means that for any point in the domain $W > -\frac{1}{\rho}$, the limit set of its orbit belongs to the trapping line $W = W^*$). In particular, any initial condition (3.8) has an orbit which is bounded in the rectangle $S = I \times J$, with $J = [1, W^*]$, and the limit set of the orbit belongs to the segment of S on the line $W = W^*$. Thus the limit set of any orbit in the domain $W > -\frac{1}{\rho}$ is an invariant set of the restriction of T to the line $W = W^*$, which is the one dimensional map $g_\rho(z)$ given in (3.6). In other words, the limiting map (3.6) can be obtained from the two-dimensional map T by identifying a point (z, W^*) on the line $W = W^*$ with a point "z" on the real line.

The above considerations prove the following proposition.

Proposition 3.1. *Let* $f : I \to I$, $0 < \rho < 1$, g_ρ *defined as in (3.6) and T defined as in (3.7). Then*

(i) *The orbits of the nonautonomous equation (3.5) are in one-to-one correspondence with the orbits of the autonomous two-dimensional map T associated with an initial condition on the line* $W = 1$.

(ii) *The invariant sets of T belong to the line* $W = W^*$.

(iii) *The invariant sets of T and those of* g_ρ *are in one-to-one correspondence.*

(iv) *An invariant set of T is attracting (resp. repelling) if and only if the corresponding invariant set of* g_ρ *is attracting (resp. repelling).*

This proposition is useful in order to define which are the possible asymptotic sets of the recurrence (3.5), which are to be searched among the invariant sets of the limiting map. Now we investigate if the knowledge of stability/instability of the cycles of the map g_ρ, may be useful in order to decide on the "existence" and on the "stability" of cycles for the nonautonomous recurrence (3.5).

An answer to this question can be obtained from an analysis of the global properties of T. In fact, from the properties of the limiting map g_ρ we know the local properties of T near the asymptotic line $W = W^*$ but, since the initial conditions for T must be taken on the line $W = 1$, we need a global study of the map T in order to obtain information on the properties of the nonautonomous equation (3.5). The following proposition gives an answer to this question

Proposition 3.2. *Let A be a k-cycle,* $k \geq 1$, *of the map* $g_\rho(z)$, $0 < \rho < 1$. *Then:*

(i) *if A is attracting, or attracting from one side, for the limiting map* g_ρ *then it is an attracting cycle for the nonautonomous process (3.5), and hence* $f(A)$ *is an attracting set of the iteration (3.3);*

(ii) *the basin of attraction D of the attractor* $f(A)$ *of (3.3) is given by the intersection of the two-dimensional basin, say* $\mathcal{D}$, *of the cycle* $\mathcal{A} = A x \{W^*\}$ *of the map T (located on the trapping line* $W = W^*$*) with the line of initial conditions* $W = 1$: $\mathcal{D} \cap \{W = 1\} = D \times \{1\}$.

In this proposition the term attracting k-cycle, for the process with memory, means that the process generated by (3.3) converges asymptotically to the cycle starting from a set of initial conditions of measure greater than zero. It can be noticed that the attracting sets are not, in general, invariant sets (as usual for the nonautonomous processes). This means that starting from a point of a an attracting k-cycle the sequence $\{x_n\}$ generated by (3.3) may not converge to the k-cycle, that is, the basin of a given attractor may not contain the points of the cycle itself.
The principal idea emerging from these two propositions is that if the process (3.3) is considered with $z_n = x_n$ (no memory case) then its asymptotic behavior is given by the study of the map $f(z)$, whereas if an exponentially fading memory is considered its limit sets must be searched among the invariant sets of another one-dimensional autonomous map, the limiting map g_ρ defined in (3.6), even if their basins of attraction can only be determined through a global study of the two-dimensional map T. As we have already observed, only the fixed points of the map g_ρ coincide with the fixed points of the map f, whereas the other invariant sets, k-cycles or chaotic sets, are in general different.
Of course the shape of the map g_ρ depends on that of f: in fact from the definition (3.6) the function $g_\rho(z)$ is a power mean of z and $f(z)$, so for each $z \in I$

(3.10) $$\min(z, f(z)) \le g_\rho(z) \le \max(z, f(z)).$$

This means that the graph of g_ρ always belongs to the area between the bisectrix and the graph of f, and the graphs of f and g_ρ intersect at the common fixed points.
The derivative of the function g_ρ is:

(3.11) $$g'_\rho(z) = \left(\rho z^p + (1-\rho)[f(z)]^p\right)^{\frac{1-p}{p}} \left(\rho z^{p-1} + (1-\rho)[f(z)]^{p-1} f'(z)\right)$$

and if z^* is a positive fixed point of f it becomes

(3.12) $$g'_\rho(z^*) = \rho + (1-\rho) f'(z^*)$$

which implies

(3.13) $$\min\left(1, f'(z^*)\right) \le g'_\rho(z^*) \le \max\left(1, f'(z^*)\right)$$

If $z^* = 0$, i.e. $f(0) = 0$, $g'_\rho(z^*)$ is not defined. However in this case

$$\lim_{z \to 0^+} g'_\rho(z) = \left(\rho + (1-\rho)[f'(0)]^p\right)^{\frac{1}{p}}$$

so (3.13) holds even for $z^* = 0$.
If $-1 < f'(z^*) < 1$, so that z^* is an attracting fixed point of the map f, then (3.13) implies $-1 < g'_\rho(z^*) < 1$, thus z^* is attracting for the map g_ρ too. If $\left|f'(z^*)\right| > 1$, so that z^* is a repelling

fixed point of f, then z^* may be attracting or repelling for g_ρ. In particular, if $f^{'}(z^*) > 1$ then z^* is repelling also for g_ρ since from (3.13) we have $1 < g_\rho^{'}(z^*) < f^{'}(z^*)$, while $f^{'}(z^*) < -1$ gives $f^{'}(z^*) < g_\rho^{'}(z^*) < 1$ and in this case z^* may be attracting for g_ρ.
More exactly if $f^{'}(z^*) < -1$ let $\tilde{\rho} \in (0,1)$ be defined as

(3.14) $$\tilde{\rho} = \frac{f^{'}(z^*) + 1}{f^{'}(z^*) - 1}.$$

Then the sufficient condition for the stability of the fixed point of the map g_ρ, $\left|g_\rho^{'}(z^*)\right| < 1$, which can be written as $-\frac{1+\rho}{1-\rho} < f^{'}(z^*) < 1$ is satisfied for $\tilde{\rho} < \rho < 1$, i.e. with a sufficiently strong memory. These arguments are summarized in the following proposition, whose part (ii) states the stabilizing effect of a strong memory.

Proposition 3.3. *Let* z^* *be a fixed point of f.*
(i) *If* $\left|f^{'}(z^*)\right| < 1$ *then also* $\left|g_\rho^{'}(z^*)\right| < 1$ *for each* $\rho \in (0,1)$*;*
(ii) *if* $f^{'}(z^*) < -1$ *a value* $\bar{\rho} \in (0,1)$ *exists, given by (4.5), such that* $\left|g_\rho^{'}(z^*)\right| < 1$ *for* $\tilde{\rho} < \rho < 1$;
(iii) *if* $f^{'}(z^*) > 1$ *then also* $g_\rho^{'}(z^*) > 1$.

This proposition allows us to distinguish, among the fixed points of the map f, those which will be attracting for the process with a sufficiently strong memory (in particular with a uniform memory, obtained in the limiting case $\rho \to 1$).

4. Conclusions

In this paper an iterative scheme of the form $x_{n+1} = f(z_n)$, where z_n is a weighted power mean of all the previous state variables $x_0,...,x_n$, has been studied. The results given extend, to a general class of commonly used algebraic means, including arithmetic, quadratic, harmonic, and geometric means with arbitrary weights, some the results existing in the literature for arithmetic mean only.
These iterative schemes can be used to model learning mechanisms in economic and social systems where the agents use all available past data to compute expected values by some averaging method.
A particular distribution of weights, exponentially decreasing like the terms of a convergent geometric series of ratio ρ (called memory ratio), has been used to investigate the effects of a fading memory on the asymptotic properties of the discrete process. This has been obtained through the reduction of the problem to the study of an equivalent two-dimensional triangular map whose asymptotic behavior is governed by a one-dimensional map.
This allows us to state that the presence of a strong memory, that is, with a memory ratio close to 1, has a stabilizing effect.

Appendix A. A learning mechanism which leads to a harmonic mean.

We consider an economic system whose law of motion is expressed in the classical forward-looking form

(A.1) $$x_n = f(x_{n+1}^{(e)})$$

where $x_{n+1}^{(e)}$ represents the expected value of the state variable x for the next time period, and will be identified, in the following, with the expected variable z_n. We suppose that the agents use all the available data to compute the expected value as a linear combination of the values of the past:

(A.2) $$z_n = x_{n+1}^{(e)} = \sum_{k=0}^{n-1} a_{nk} x_k$$

where $a_{nk} = \frac{\omega_k^{(n)}}{W_n}$ and W_n is the sum of the relative weights. We now introduce the further assumption, similar to that proposed in Benassy and Blad (1989), that also the relative weights $\omega_k^{(n)}$ are estimated on the basis of the past observations. Since $\omega_k^{(n)}$ represents the relative influence of the past value x_k on the expected value, we assume that the it is computed on the basis of the observed influence of x_k on the present value x_n, that is from the relation $x_n \cong \omega_k^{(n)} x_k$ the agents compute

(A.3) $$\omega_k^{(n)} = \frac{x_n}{x_k}.$$

All in all, we have

$$z_n = \frac{\sum_{k=0}^{n-1} \frac{x_n}{x_k}}{W_n} x_k = \frac{n}{\sum_{k=0}^{n-1} \frac{1}{x_k}}$$

from which the law of motion (A.1) becomes

$$x_{n+1} = f(z_n) \quad \text{with} \quad \frac{1}{z_n} = \frac{1}{n} \sum_{K=0}^{n-1} \frac{1}{x_k}$$

Appendix B. Proof of proposition 2.2.

(i) First we shall see, under the assumption $W_n \to \infty$, that if x_n is convergent then also z_n converges to the same limit.

Let $x_n \to q > 0$ (the case $q = 0$ will be treated separately). Then, for each p, $x_n^p \to q^p$, i.e. for each $\varepsilon > 0$ an $N > 0$ exists such that

(B.1) $$q^p - \varepsilon < x_n^p < q^p + \varepsilon \qquad \text{for } n > N.$$

Now we shall prove that $(z_n^p - q^p) \to 0$, that is $z_n^p \to q^p$ which implies $z_n \to q$.
For $n > N$ we have

(B.2)

$$(z_n^p - q^p) = \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^p - q^p = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^p + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^p - q^p = \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^p + \frac{W_n - W_N}{W_n} \sum_{k=N+1}^{n} \frac{\omega_k}{W_n - W_N} x_k^p - q^p$$

From the right inequality in (B.1) we have

(B.3) $$(z_n^p - q^p) \le \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^p + \frac{W_n - W_N}{W_n} (q^p + \varepsilon) - q^p$$

since $$\sum_{k=N+1}^{n} \frac{\omega_k}{W_n - W_N} = 1.$$

Analogously, from the left inequality in (B.1) we have

(B.4) $$(z_n^p - q^p) \ge \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^p + \frac{W_n - W_N}{W_n} (q^p - \varepsilon) - q^p$$

Since $W_n \to \infty$ and the ω_k are bounded, from (B.3) follows that

(B.5) $$\lim_{n \to \infty} (z_n^p - q^p) \le \varepsilon$$

and from (B.4)

(B.6) $$\lim_{n \to \infty} (z_n^p - q^p) \ge -\varepsilon$$

Since ε is arbitrarily small (B.5) and (B.6) prove that $\lim_{n \to \infty} (z_n^p - q^p) = 0$.

Consider now the case $x_n \to 0$. If $p > 0$ the arguments above can be applied with no substantial modifications. If $p < 0$, since the x_n are supposed to be positive, we have that $x_n^p \to +\infty$, i.e. for each $M > 0$ an $N > 0$ exists such that

(B.7) $$x_n^p > M \qquad \text{for } n > N.$$

For $n > N$ we have

$$z_n^p = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^p + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^p > M \sum_{k=N+1}^{n} \frac{\omega_k}{W_n}$$

and since M can be arbitrarily large this implies $z_n^p \to +\infty$, from which, since $p < 0$, follows $z_n \to 0$.

To complete this part of the proof it remains to show that the common limit is a fixed point of f. Indeed, since f is continuous, from $z_n \to q$ follows that $f(z_n) \to f(q)$. But $x_{n+1} = f(z_n)$ so that $q = f(q)$.

We assume now that z_n converges and we prove that also x_n converges to the same limit. If $z_n \to r$ then $x_n \to f(r)$ because f is continuous. From the argument above, it must also be $z_n \to f(r)$ which implies $r = f(r)$.

(ii) Since, for $z_0 \in I = [a,b] \subseteq \Re_+$, the whole sequence $\{z_n\}$ is contained in I, it has at least one limit point. We shall see that it must be unique.

From (2.9), rewritten as

(B.8) $$z_{n+1}^p - z_n^p = t_n \left([f(z_n)]^p - z_n^p \right)$$

we deduce that, since $t_n \to 0$, z_n and $f(z_n)$ are bounded, for each $\varepsilon > 0$ a $m > 0$ exists such that

(B.9) $$\left| z_{n+1}^p - z_n^p \right| < \varepsilon \qquad \text{for } n > m$$

Following the argument used by Borwein and Borwein (1991) let us assume, for sake of contradiction, that ξ and η, with $a \le \xi < \eta \le b$, are two distinct limit points. A consequence of this assumption is that $f(z) = z$ for each $z \in (\xi, \eta)$. In fact let c be a point such that $\xi < c < \eta$. If $f(c) > c$ then, by the continuity of f, a $\delta \in (0,c)$ exists such that

(B.10) $$f(z) > z \qquad \text{whenever} \qquad |z - c| < \delta$$

Since η is a limit point for $\{z_n\}$ a $N > m$ exists such that $|z_N - \eta| < (\eta - c)$ which implies $z_N > c$. It follows that $z_n > c$ for each $n > N$. To prove this we separately analyze the cases of positive and of negative p. Consider first $p > 0$. If $c < z_N < c + \delta$, from (B.10) follows $f(z_N) > z_N$ which gives, since $p > 0$, $[f(z_N)]^p > z_N^p$. From (B.8) follows $z_{N+1}^p > z_N^p$ (remember that $t_n > 0$) and this implies $z_{N+1} > z_N$ because $p > 0$. If $z_N \ge c + \delta$ we have $z_N^p \ge (c+\delta)^p$ so that:

(B.11) $$z_{N+1}^p - c^p = z_{N+1}^p - z_N^p + z_N^p - c^p \ge z_{N+1}^p - z_N^p + (c+\delta)^p - c^p > -\varepsilon + (c+\delta)^p - c^p$$

where (B.9) has been used. Since $\delta < c$ from the binomial series we have: $(c+\delta)^p = c^p + p\delta c^{p-1} + \frac{p(p-1)}{2}\delta^2 c^{p-2} + \frac{p(p-1)(p-2)}{3!}\delta^3 c^{p-3} + \ldots$

so that $(c+\delta)^p - c^p > p\delta c^{p-1}$ for $p \geq 1$,

and $(c+\delta)^p - c^p > p\delta c^{p-2}(c - \frac{1-p}{2}\delta)$ for $0 < p < 1$. Thus if for $p \geq 1$ we take $0 < \varepsilon < p\delta c^{p-1}$ or, for $0 < p < 1$, $0 < \varepsilon < p\delta c^{p-2}(c - \frac{1-p}{2}\delta)$, (B.11) gives $z_{N+1}^p - c^p > 0$ which, for $p > 0$, implies $z_{N+1} > c$.

Consider now $p < 0$. If $c < z_N < c+\delta$ from (B.10) follows $f(z_N) > z_N$ which gives, since $p < 0$, $[f(z_N)]^p < z_N^p$. From (B.8) follows $z_{N+1}^p < z_N^p$ which implies $z_{N+1} > z_N$ because $p < 0$. If $z_N \geq c+\delta$ we have $z_N^p \leq (c+\delta)^p$ so that:

(B.12) $$z_{N+1}^p - c^p = z_{N+1}^p - z_N^p + z_N^p - c^p \leq z_{N+1}^p - z_N^p + (c+\delta)^p - c^p < \varepsilon + (c+\delta)^p - c^p$$

where (B.9) has been used. From the binomial series with $p < 0$ we have $(c+\delta)^p - c^p < p\delta c^{p-1}$ so that if we take $0 < \varepsilon < -p\delta c^{p-1}$ (B.12) gives $z_{N+1}^p - c^p < 0$ which, for $p < 0$, implies again $z_{N+1} > c$.

Hence, by induction, $z_n > c$ for $n \geq N$ against the assumption that $\xi < c$ is a limit point of $\{z_n\}$.

If $f(c) < c$ a similar reasoning contradicts the assumption that η is a limit point. Thus $f(c) = c$ for each $\xi < c < \eta$.

Now, if for a given $\bar{n}$ we have $\xi < z_{\bar{n}} < \eta$ then $z_{\bar{n}+1} = z_{\bar{n}}$ and so $z_n = z_{\bar{n}}$ for each $n \geq \bar{n}$ which contradicts the fact that ξ and η are both limit points. If this is not the case, since $\{z_n\}$ cannot oscillate out of the interval (ξ, η) because of (B.9), taking $\varepsilon < (\eta - \xi)$ it remains $z_n > \eta$ or $z_n < \xi$ for each n, and again this excludes the possibility that $\xi < \eta$ be both limit points. Therefore $\{z_n\}$ converges to its unique limit point.

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