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Reliability studies of Dagum distribution

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Abstract

In this work, it is proposed to use the Dagum model (Dagum, 1977) in the reliability theory. The main motivation is due to the fact that the hazard rate of this model is very flexible; in fact, it is proved (Domma, 2002) that, according to the values of the parameters, the hazard rate of the Dagum distribution has a decreasing, or a Upside-down Bathtub, or Bathtub and then Upside-down Bathtub failure rate. In this work we study some features of the Dagum distribution as the reversed hazard rate, the mean and variance of the random variables residual life and reversed residual life and their monotonic properties. One published data set has been analyzed for illustrative purpose.

Key words: Reversed Hazard Function, Mean Residual Life, Mean Waiting Time, Variance of Residual Life, Variance of Reversed Residual Life.

1 Introduction

The Dagum distribution was introduced by Dagum in 1977 and it is appeared as the example III of the Burr system of distributions (Burr, 1942). In this way the Dagum distribution is closely related to the Burr XII distribution. In actuarial literature, in fact, it is also called the inverse Burr. The Dagum model has been used in studies of income and wage distribution as well as wealth distribution (see, for example, Kleiber and Kotz (2003) and Kleiber (2007) for excellent survey on the genesis and on the empirical applications of the Dagum model). In this context, its features have been extensively analyzed by many authors. Recently, Quintano and D'Agostino (2006) proposed to model the income distribution in terms of individual characteristics using Dagum distributions with heterogeneous model parameters. Domma (2007) studied the asymptotic distribution of the maximum likelihood estimators of the parameters of the right-truncated Dagum model. Nevertheless, Kleiber and Kotz (2003, p. 215) highlighted that ".... *the hazard function and mean excess function of the Dagum distribution have not been investigated in the statistical literature* ". In this paper, we attempt to fill this gap in literature studying the Dagum distribution from a reliability point of view. In particular, we will study the reversed hazard rate, the mean residual life, the mean waiting time function, the variance of random variables residual life and reversed residual life and their monotonic properties.

The organization of this paper is as follows: in *Section 2*, the Dagum model is described. *Section 3* contains some definitions and background of reliability functions. The hazard rate, the reversed hazard rate, the mean residual life, the mean waiting time function, the variance of residual life and reversed residual life of the Dagum distribution and their monotonicity are discussed in *Section 4*. In *Section 5* an illustrative example is proposed.

2 The Dagum distribution

It is well-known that the probability density function (*pdf*) of the random variable of Dagum (1977, 1980) has the following form:

$$f_T(t, \boldsymbol{\theta}) = \beta\lambda\delta t^{-\delta-1}\left(1+\lambda t^{-\delta}\right)^{-\beta-1} \tag{1}$$

with $t > 0$, $\boldsymbol{\theta} = (\beta, \lambda, \delta)$, and $\beta > 0$, $\lambda > 0$ and $\delta > 0$ and cumulative distribution function (*cdf*):

$$F_T(t, \boldsymbol{\theta}) = \left(1+\lambda t^{-\delta}\right)^{-\beta}. \tag{2}$$

The parameter λ is a scale parameter, while β and δ are shape parameters. Through all this paper, the Dagum distribution with parameters β, λ and δ will be denoted by $Da(\beta, \lambda, \delta)$.

The Dagum random variable with distribution function $F_T(t; \beta, \lambda, \delta) = [H_T(t; \lambda, \delta)]^{\beta}$ can be thought as an exponentiated random variable (see, for example, Gupta et al., 1998; Sarabia and Castillo, 2005) with baseline distribution $H_T(t; \lambda, \delta) = \left(1+\lambda t^{-\delta}\right)^{-1}$. In fact, for $\beta = 1$, $F_T(t; 1, \lambda, \delta)$ is again a distribution function, called Fisk distribution (or log-logistic distribution) (Kleiber and Kotz, 2003).

The Dagum distribution has positive asymmetry, it is unimodal for $\beta\delta > 1$ and zero-modal for $\beta\delta \leq 1$. Moreover, the *q-th* quantile of the Dagum distribution is

$$t(q) = \lambda^{\frac{1}{\delta}}(q^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\delta}}$$

and the *r-th* moment is given by:

$$E(T^r) = \beta\lambda^{\frac{r}{\delta}} B\left(\beta + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right) \tag{3}$$

for $\delta > r$, where $B(.,.)$ is the mathematical function Beta.

3 Some definitions and background of reliability functions

Let T be a non-negative random variable, usually representing the time to failure of a unit, with *cdf* $F_T(t; \boldsymbol{\xi})$ and *pdf* $f_T(t; \boldsymbol{\xi})$, where $\boldsymbol{\xi} \in \Xi \subset \mathbb{R}^p$ with $p \geq 1$ and $p \in \mathbb{N}$. Moreover let $f_T(t; \boldsymbol{\xi})$ be continuous and twice differentiable on $(0, \infty)$. It is well-known that an important measure of ageing is the hazard rate (HR), defined as

$$h_T(t; \boldsymbol{\xi}) = \lim_{\Delta t \to 0} \frac{P\left[T < t + \Delta t | T > t\right]}{\Delta t} = \frac{f_T(t; \boldsymbol{\xi})}{S_T(t; \boldsymbol{\xi})}, \tag{4}$$

where $S_T(t; \boldsymbol{\xi}) = 1 - F_T(t; \boldsymbol{\xi})$ is the survival function of T. That is, $h_T(t; \boldsymbol{\xi}) \cdot \Delta t$ represents the conditional probability that a unit of age t will fail within the interval $(t, t + \Delta t)$.
Similarly, for all $t > 0$ and for all $\boldsymbol{\xi} \in \Xi$ such as $F_T(t; \boldsymbol{\xi}) > 0$, the reversed hazard rate (RHR) is

$$r_T(t; \boldsymbol{\xi}) = \lim_{\Delta t \to 0} \frac{P\left[T > t - \Delta t | T \leq t\right]}{\Delta t} = \frac{f_T(t; \boldsymbol{\xi})}{F_T(t; \boldsymbol{\xi})}. \tag{5}$$

One interpretation of the reversed hazard rate at time t is the following. The $r_T(t; \boldsymbol{\xi}) \cdot \Delta t$ represents the conditional probability that a unit has failed in $(t - \Delta t, t)$, given that the unit has already failed at time t. The RHR function of a random variable plays an important role in estimating the survival function for left-censored lifetimes (Kalbfleisch and Lawless, 1989).
In reliability theory the random variable residual life, R_t, and the random variable reversed residual life, W_t, play an important role because the hazard function is related to *r.v.* R_t; instead, the reversed hazard function is related to random variable W_t. These random variables are defined as follows. Given that a unit or a system is of age t, the remaining lifetime after t is a *r.v.* referred as the residual life (or remaining lifetime) and it is denoted by $R_t = (T - t)|(T \geq t)$. Likewise, the *r.v.* $W_t = (t - T)|(T \leq t)$ denotes the time elapsed after failure till time t, given that the unit has already failed by time t. The *r.v.* W_t is called reversed residual life (or time since failure).
Another ageing measure widely used in reliability analysis and closely related to the *r.v.* R_t, is the mean residual life (MRL), defined as the expectation of the R_t, that is:

$$\begin{aligned} \mu(t, \boldsymbol{\xi}) = E[R_t; \boldsymbol{\xi}] &= \frac{1}{S_T(t; \boldsymbol{\xi})} \int_t^{\infty} (u - t) f_T(u; \boldsymbol{\xi}) du = \\ &= \frac{1}{S_T(t; \boldsymbol{\xi})} \{E(T; \boldsymbol{\xi}) - E_{T \leq t}(T; \boldsymbol{\xi}) - t \cdot S_T(t; \boldsymbol{\xi})\} \end{aligned} \tag{6}$$

where $E_{T \leq t}(T; \boldsymbol{\xi}) = \int_0^t u f_T(u; \boldsymbol{\xi}) du$. That is, (6) is the mean of random variable residual life, R_t. For a unit having already survived up to time t, $\mu(t; \boldsymbol{\xi})$ measures its expected remaining lifetime.
The MRL has a mirror image, called mean waiting time (MWT), denoted in the literature also as the expected inactivity time (Ghitany et al., 2005) or mean advantage over inferiors function (Bagnoli and Bergstrom, 2005). The MWT of a non negative continuous random variable T is defined as:

$$\bar{\mu}(t, \boldsymbol{\xi}) = E[W_t; \boldsymbol{\xi}] = t - \int_0^t u \frac{f_T(u; \boldsymbol{\xi})}{F_T(t; \boldsymbol{\xi})} du = t - \frac{E_{T \leq t}(T; \boldsymbol{\xi})}{F_T(t; \boldsymbol{\xi})}. \tag{7}$$

That is, MWT is the mean of random variable reversed residual life, W_t. In other words, $\bar{\mu}(t, \boldsymbol{\xi})$ defines the mean waiting time elapsed for a unit that failed in a interval $[0, t]$ (Finkelstein, 2002). Other functions that have generated interest in the recent years are the variance of residual life (VRL) and the variance of reversed residual life (VRRL) given, respectively, by

$$\sigma^2(t, \boldsymbol{\xi}) = V[R_t; \boldsymbol{\xi}] = E[T^2|T \geq t; \boldsymbol{\xi}] - [E\,(T|T \geq t; \boldsymbol{\xi})]^2 \tag{8}$$

and

$$\bar{\sigma}^2(t, \boldsymbol{\xi}) = V[W_t; \boldsymbol{\xi}] = E[T^2|T \leq t; \boldsymbol{\xi}] - [E\,(T|T \leq t; \boldsymbol{\xi})]^2\,, \tag{9}$$

see Gupta and Kirmani (2000) and Gupta (2006). Obviously, all functions defined above highlight different aspects of residual life (HR, MRL and VRL) and of reversed residual life (RHR, MWT and VRRL). In order to study the behavior of the HR, the RHR, the MRL, the MWT, VRL and VRRL we consider the following definitions. Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real valuated differentiable function. Then $\chi(t)$ is said to be

1. Increasing if $\chi^{'}(t) > 0$ for all t and is denoted by I.

2. Decreasing if $\chi^{'}(t) < 0$ for all t and is denoted by D.

3. Bathtub shaped if $\chi^{'}(t) < 0$ for $t \in (0, t_*)$, $\chi^{'}(t_*) = 0$, $\chi^{'}(t) > 0$ for $t > t_*$ and is denoted by B.

4. Upside down bathtub shaped if $\chi^{'}(t) > 0$ for $t \in (0, t^*)$, $\chi^{'}(t^*) = 0$, $\chi^{'}(t) < 0$ for $t > t^*$ and is denoted by U.

5. Upside down bathtub and then bathtub if there exists t_1 and t_2 such that $\chi^{'}(t) > 0$ for $t \in (0, t_1)$, $\chi^{'}(t_1) = 0$, $\chi^{'}(t) < 0$ for $t \in (t_1, t_2)$, $\chi^{'}(t_2) = 0$, $\chi^{'}(t) > 0$ for $t > t_2$ and is denoted by UB.

6. Bathtub and then upside down bathtub if there exists t_1 and t_2 such that $\chi^{'}(t) < 0$ for $t \in (0, t_1)$, $\chi^{'}(t_1) = 0$, $\chi^{'}(t) > 0$ for $t \in (t_1, t_2)$, $\chi^{'}(t_2) = 0$, $\chi^{'}(t) < 0$ for $t > t_2$ and is denoted by BU.

(Barlow and Proschan, 1975).

Let $T_{i:n}$ be the *ith* order statistic from the random sample independent and identically distributed (*iid*) $T_1, T_2, ..., T_n$ from $F_T(t; \xi)$. Denote the *pdf*, *cdf*, hazard rate and reversed hazard rate of $T_{i:n}$ by $f_{i:n}(t; \boldsymbol{\xi})$, $F_{i:n}(t; \boldsymbol{\xi})$, $h_{i:n}(t; \boldsymbol{\xi})$ and $r_{i:n}(t; \boldsymbol{\xi})$, respectively. It is well known that the distribution of the maximum (minimum) of n random variables plays an important role in various statistical applications. In particular, in reliability studies $T_{1:n} = min\,\{T_1, ..., T_n\}$ is observed time if the components are arranged in a series system and $T_{n:n} = max\,\{T_1, ..., T_n\}$ is observed if the components are arranged in a parallel system. It is simple to verify that the hazard rate of $T_{1:n}$ is given by

$$h_{1:n}(t;\boldsymbol{\xi}) = \frac{f_{1:n}(t;\boldsymbol{\xi})}{S_{1:n}(t;\boldsymbol{\xi})} = nh_T(t;\boldsymbol{\xi})$$

thus the hazard rate of the minimum $T_{1:n}$ is n times that of T; similarly, the reversed hazard rate of maximum is

$$r_{n:n}(t;\boldsymbol{\xi}) = \frac{f_{n:n}(t;\boldsymbol{\xi})}{F_{n:n}(t;\boldsymbol{\xi})} = nr_T(t;\boldsymbol{\xi}).$$

4 Properties of the Dagum distribution in terms of reliability analysis

In this section we describe some properties of Dagum distribution useful in the reliability analysis. In particular, we refer to the behavior of the hazard rate described in Domma (2002) and we find the reversed hazard rate, the mean residual life, the mean waiting time, the variance of residual life and reversed residual life and we study their monotonic properties.

4.1 The Hazard Rate and Reversed Hazard Rate

The **hazard rate** of the Dagum distribution is:

$$h_T(t,\boldsymbol{\theta}) = \beta\lambda\delta\{t(t^{\delta}+\lambda)[(1+\lambda t^{-\delta})^{\beta}-1]\}^{-1}. \tag{10}$$

Using the Glaser's method (Glaser, 1980), Domma (2002) proved the following *Proposition*.

Proposition 1. *Let* $\beta^* = \frac{2}{\delta} - 1$, $\overline{\beta} = \frac{1}{\delta}$ *and* $\beta_2 = \frac{3-\delta}{\delta+1}$.
The hazard rate of the Dagum distribution is U if: a) $\beta\delta > 1$ *and* $\beta \neq \beta^*$*; b)* $\beta\delta = 1, \beta > \beta^*$ *and* $\delta > 1$.
It is D if: a) $\beta = \beta^*$ *and* $\delta < 2$*; b)*$\beta\delta = 1$, $\beta < \beta^*$ *and* $\delta < 1$*; c)* $\delta < 1$ *and* $\beta \in (\beta_2, \overline{\beta})$.
It is BU if: a) $\delta \in (1,3)$ *and* $\beta \in (\beta_2, \overline{\beta})$*; b)* $\delta \geq 3$ *and* $\beta \in (0, \overline{\beta})$.

In *Figure 1* the behavior of hazard rate for three Dagum distributions is illustrated.

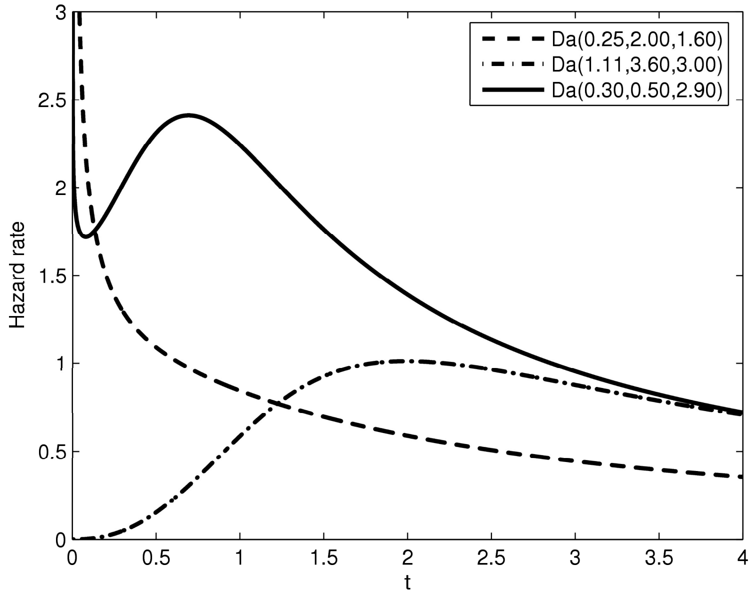


Figure 1: The hazard rate for three Dagum distributions.

It can be proved that the reversed hazard rate of $Da(\beta, \lambda, \delta)$ distribution is D. In fact, by (1), (2) and (5), we have

$$r_T(t; \boldsymbol{\theta}) = \frac{\beta\lambda\delta}{t\,(t^\delta + \lambda)}, \tag{11}$$

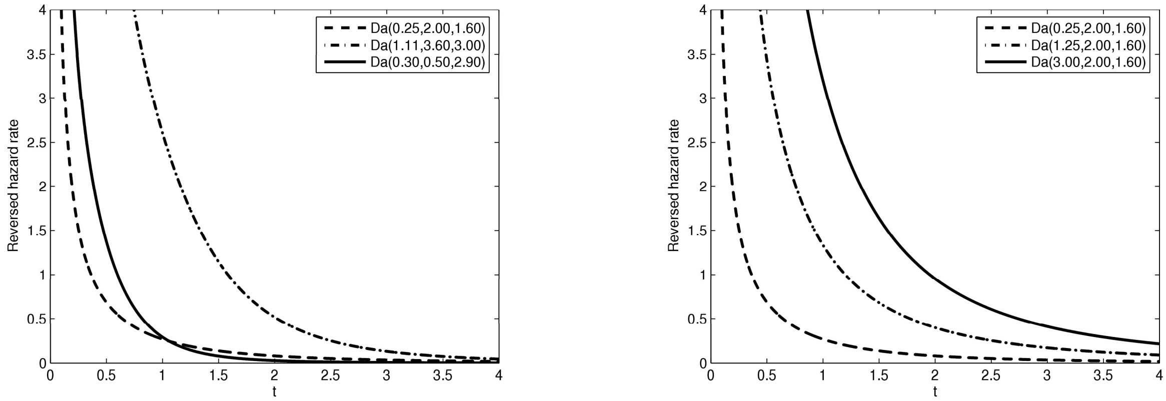
with $\lim_{t\to 0^+} r_T(t; \boldsymbol{\theta}) = +\infty$ and $\lim_{t\to +\infty} r_T(t; \boldsymbol{\theta}) = 0$. Moreover,

$$\frac{\partial r_T(t; \boldsymbol{\theta})}{\partial t} = \frac{-\beta\lambda\delta\left\{(\delta+1)t^\delta + \lambda\right\}}{t^2\,(t^\delta + \lambda)^2}$$

is always negative for $t > 0$ and $\beta > 0$, $\lambda > 0$ and $\delta > 0$.

It is worthwhile to point out that the distribution function of the Dagum random variable, $F_T(t; \boldsymbol{\theta})$, is log-concave because the reversed hazard rate is D. This property will be used subsequently in studying the monotonicity of MWT.

Now, let T_1 and T_2 be two Dagum random variables having distribution function $F_{T_i}(t; \beta_i, \lambda, \delta) = [H_{T_i}(t; \lambda, \delta)]^{\beta_i}$ for $i = 1, 2$. If the baseline distribution, $H_{T_i}(t; \lambda, \delta)$ is the same, then there is a reversed hazard rate order between T_1 and T_2 if $\beta_1 \neq \beta_2$; in particular, if $\beta_1 \leq \beta_2$ then $r_{T_1}(t; \beta_1, \lambda, \delta) \leq r_{T_2}(t; \beta_2, \lambda, \delta)$, in other words, T_1 is smaller than T_2 in the reversed hazard rate order (denoted by $T_1 \leq_{rh} T_2$). *Figure 2* reports the RHR for several Dagum distributions.



(a) The reversed hazard rate for the Dagum distributions with different values of the parameters (no *reversed hazard rate* order.)

(b) The reversed hazard rate for the Dagum distributions with same values for λ and δ, but different values of β (*reversed hazard rate* order.)

Figure 2: The reversed hazard rate for several Dagum distributions.

4.2 The Mean Residual Life and the Mean Waiting Time

In this sub-section, we study the mean of random variables residual life and reversed residual life and their monotonic properties for the Dagum distribution. To such aim, we calculate the $r-th$ incomplete moments

$$E_{T\leq t}\left[T^r;\boldsymbol{\theta}\right] = \beta\lambda^{\frac{r}{\delta}} B\left(z^*;\beta+\frac{r}{\delta},1-\frac{r}{\delta}\right) \tag{12}$$

for $\delta > r$, where $B(z^*;p,q) = \int_0^{z^*} u^{p-1}(1-u)^{q-1}du$ with $z^* = (1+\lambda t^{-\delta})^{-1} < 1$. Using the relations in (6) and (12), the mean residual life for Dagum distribution is:

$$\mu(t,\boldsymbol{\theta}) = \frac{\beta\lambda^{\frac{1}{\delta}}\left[B\left(\beta+\frac{1}{\delta},1-\frac{1}{\delta}\right) - B\left(z^*;\beta+\frac{1}{\delta},1-\frac{1}{\delta}\right)\right]}{S(T;\boldsymbol{\theta})} - t \tag{13}$$

clearly $\mu(0;\boldsymbol{\theta}) = E(T;\boldsymbol{\theta})$.

In order to study the monotonicity of MRL, it is useful to introduce the following theorem (Gupta and Akman, 1995).

Theorem 2.

1. *If $h_T(t;\boldsymbol{\theta})$ is I, then $\mu(t,\boldsymbol{\theta})$ is D.*

2. *If $h_T(t;\boldsymbol{\theta})$ is D, then $\mu(t,\boldsymbol{\theta})$ is I.*

3. *Suppose that $h_T(t;\boldsymbol{\theta})$ is B and $\mu = \mu\left(0,\boldsymbol{\theta}\right)$. Then*

(a) *if* $\mu \times h_T(0;\boldsymbol{\theta}) \leq 1$, *then* $\mu(t,\boldsymbol{\theta})$ *is* D;

(b) *if* $\mu \times h_T(0;\boldsymbol{\theta}) > 1$, *then* $\mu(t,\boldsymbol{\theta})$ *is* U.

4. *Suppose that* $h_T(t;\boldsymbol{\theta})$ *is* U *and* $\mu = \mu(0,\boldsymbol{\theta})$. *Then*

(a) *if* $\mu \times h_T(0;\boldsymbol{\theta}) \geq 1$, *then* $\mu(t,\boldsymbol{\theta})$ *is* I;

(b) *if* $\mu \times h_T(0;\boldsymbol{\theta}) \leq 1$, *then* $\mu(t,\boldsymbol{\theta})$ *is* B.

By using jointly the *Proposition 1* and the previous Gupta and Akman's *Theorem*, we prove the following:

Theorem 3. *Let* $T \sim Da(\beta,\lambda,\delta)$, *then:*

1. *If*
   (a) $\beta = \beta^*$ *and* $\delta < 2$ *or*
   (b) $\beta\delta = 1$, $\beta < \beta^*$ *and* $\delta < 1$ *or*
   (c) $\delta < 1$ *and* $\beta \in (\beta_2, \overline{\beta})$

   then $\mu(t,\boldsymbol{\theta})$ *is* I.

2. *If* $\beta\delta = 1$, $\beta > \beta^*$ *and* $\delta \in (1,2]$, *then* $\mu(t,\boldsymbol{\theta})$ *is* I.

3. *If*
   (a) $\beta\delta > 1$ *and* $\beta \neq \beta^*$ *or*
   (b) $\beta\delta = 1$, $\beta > \beta^*$ *and* $\delta > 2$

   then $\mu(t,\boldsymbol{\theta})$ *is* B.

Proof.

1. For these combinations of the parameter values the hazard rate is D. Then for the point 1. of the *Theorem 2*, the MRL is I.

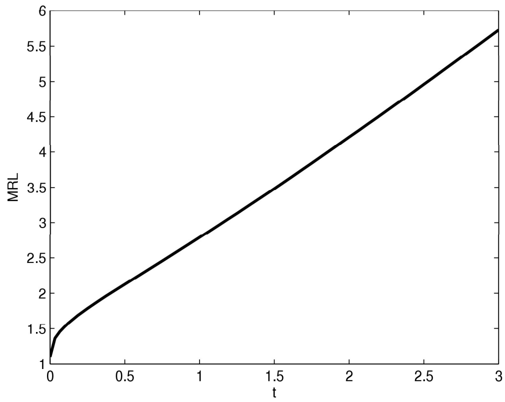
2. For $\beta\delta = 1$, $\beta > \beta^*$ and $\delta \in (1,2]$ the hazard rate is U. Given that, in this case, $f_T(0;\boldsymbol{\theta}) = \lambda^{-\beta}$, $\mu = \frac{\lambda^{\frac{1}{\delta}}}{\delta} B\left(\frac{2}{\delta}, 1-\frac{1}{\delta}\right)$ and $\mu \times h_T(0;\boldsymbol{\theta}) \geq 1$ then, for the point 4(a) of the *Theorem 2*, the MRL is I.

3. (a) For $\beta\delta > 1$ and $\beta \neq \beta^*$ the hazard rate is U. Given that, under this constraint, $\lim_{t\to 0^+} f_T(t;\boldsymbol{\theta}) = 0$ and $\mu \times h_T(0;\boldsymbol{\theta}) < 1$, for the point 4(b) of the *Theorem 2*, the MRL is B.

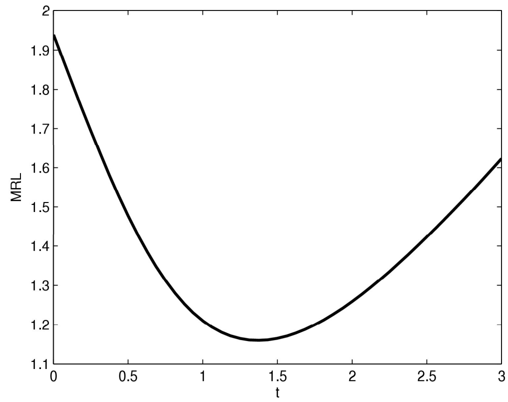
   (b) For $\beta\delta = 1$, $\beta > \beta^*$ and $\delta > 2$ the hazard rate is U. Given that, in this case, $f_T(0;\boldsymbol{\theta}) = \lambda^{-\beta}$, $\mu = \frac{\lambda^{\frac{1}{\delta}}}{\delta} B\left(\frac{2}{\delta}, 1-\frac{1}{\delta}\right)$ and $\mu \times h_T(0;\boldsymbol{\theta}) < 1$ then, for the point 4(b) of the *Theorem 2*, the MRL is B.

□

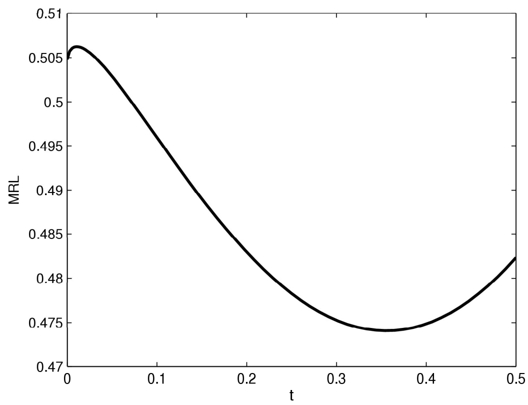
Figure 3 illustrates the behavior of the MRL function for three Dagum distributions.



(a) $Da(0.25, 2.00, 1.60)$



(b) $Da(1.11, 3.60, 3.00)$



(c) $Da(0.30, 0.5, 2.90)$

Figure 3: The Mean Residual Life Function for three Dagum distributions.

It is given as a conjecture that if the hazard rate is BU, then the MRL is UB, as shown in *Figure 3(c)*.

Using relations (12) and (7), it is possible to verify that the MWT for Dagum distribution is given by:

$$\bar{\mu}(t, \boldsymbol{\theta}) = t - \frac{\beta \lambda^{\frac{1}{\delta}} B\left(z^*; \beta + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right)}{\left(1 + \lambda t^{-\delta}\right)^{-\beta}}. \tag{14}$$

By *theorem 5* of Bagnoli and Bergstrom (2005), we can say that $\bar{\mu}(t; \boldsymbol{\theta})$ is monotone increasing because $F_T(t; \boldsymbol{\theta})$ is log-concave.

4.3 The Variance of Residual Life and of Reversed Residual Life

In this sub-section, we study the variance of *r.v.*'s R_t and W_t and their monotonic properties. In order to determine the variance of residual life, we observe that

$$E(T^r|T>t)=\int_t^{+\infty}u^r\frac{f_T(u;\beta,\lambda,\delta)}{S_T(t;\beta,\lambda,\delta)}du=\frac{E\left(T^r\right)-F_T(t;\beta,\lambda,\delta)\times E\left(T^r|T<t\right)}{S_T(t;\beta,\lambda,\delta)} \tag{15}$$

because $E(T^r)=F_T(t;\beta,\lambda,\delta)\times E\left(T^r|T<t\right)+S_T(t;\beta,\lambda,\delta)\times E\left(T^r|T>t\right).$ Now, by (12) and given that $F_T(t)\times E\left(T^r|T<t\right)=E_{T\leq t}\left(T^r\right)$, for $\delta>2$, we have

$$\begin{aligned}V[R_t;\boldsymbol{\theta}] &= E[T^2|T\geq t]-\left[E\left(T|T\geq t\right)\right]^2= \\ &= \frac{\beta\lambda^{\frac{2}{\delta}}\left[B\left(\beta+\frac{2}{\delta},1-\frac{2}{\delta}\right)-B\left(z^*;\beta+\frac{2}{\delta},1-\frac{2}{\delta}\right)\right]}{S_T(t;\beta,\lambda,\delta)}- \\ &\quad \frac{\beta^2\lambda^{\frac{2}{\delta}}\left[B\left(\beta+\frac{1}{\delta},1-\frac{1}{\delta}\right)-B\left(z^*;\beta+\frac{1}{\delta},1-\frac{1}{\delta}\right)\right]^2}{S_T(t;\beta,\lambda,\delta)^2},\end{aligned}$$

where $z^*=\left(1+\lambda t^{-\delta}\right)^{-1}<1$. Moreover, the variance of reversed residual life for the Dagum distribution, for $\delta>2$, is given by

$$\begin{aligned}V[W_t;\boldsymbol{\theta}] &= E[T^2|T<t]-\left[E\left(T|T<t\right)\right]^2= \\ &= \frac{\beta\lambda^{\frac{2}{\delta}}B\left(z^*;\beta+\frac{2}{\delta},1-\frac{2}{\delta}\right)}{F_T(t;\beta,\lambda,\delta)}-\frac{\beta^2\lambda^{\frac{2}{\delta}}\left[B\left(z^*;\beta+\frac{1}{\delta},1-\frac{1}{\delta}\right)\right]^2}{F_T(t;\beta,\lambda,\delta)^2}.\end{aligned}$$

In order to study the monotonic properties of $V(R_t)$ and $V(W_t)$ for the Dagum distribution, it is worthwhile to point out that

$$V(R_t;\boldsymbol{\theta})-\mu(t,\boldsymbol{\theta})^2=\frac{2}{S_T(t;\boldsymbol{\theta})}\int_t^{+\infty}S_T(t;\boldsymbol{\theta})\left[\mu(u;\boldsymbol{\theta})-\mu(x;\boldsymbol{\theta})\right]du \tag{16}$$

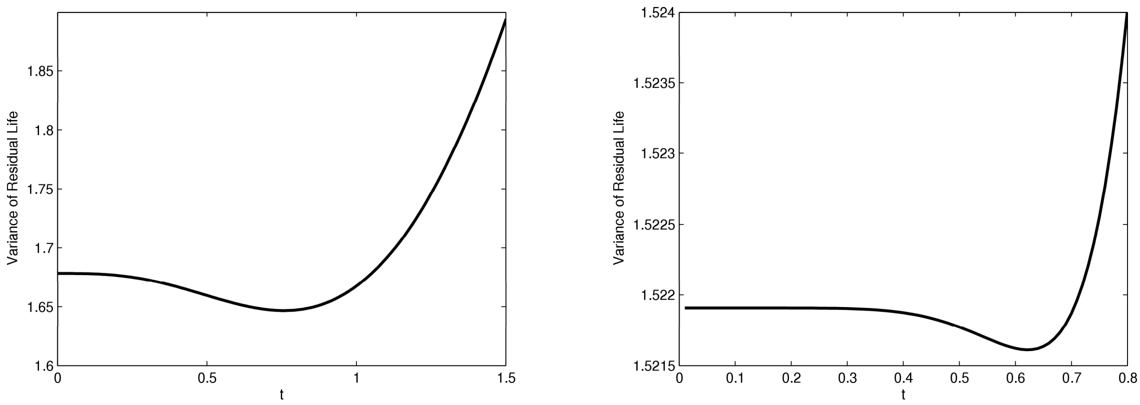
(Gupta, 2006), and that

$$V(W_t;\boldsymbol{\theta})-\bar{\mu}(t;\boldsymbol{\theta})^2=\frac{2}{F_T(t;\boldsymbol{\theta})}\int_0^{x}F_X(x;\boldsymbol{\theta})\left[\bar{\mu}(u;\boldsymbol{\theta})-\bar{\mu}(t;\boldsymbol{\theta})\right]du \tag{17}$$

(Nanda et al., 2003). In the following the behavior of variance of residual life for the Dagum distribution is established. To such aim, using the following result

$$\frac{\partial V(R_t;\boldsymbol{\theta})}{\partial t}=h_T(t;\boldsymbol{\theta})\times\mu(t;\boldsymbol{\theta})^2\left[\frac{V(R_t;\boldsymbol{\theta})}{\mu(t;\boldsymbol{\theta})^2}-1\right] \tag{18}$$

(see Gupta and Kirmani, 2000), it is clear that $V(R_t;\boldsymbol{\theta})$ is increasing if $V(R_t;\boldsymbol{\theta})>\mu(t;\boldsymbol{\theta})^2$; moreover, from (16) $V(R_t;\boldsymbol{\theta})>\mu(t;\boldsymbol{\theta})^2$ if and only if $\mu(t;\boldsymbol{\theta})$ is increasing. Now, given the constraint $\delta>2$, according to the values of parameters of the point 3. of the *Theorem 3*, from (18) and (16), it is simple to verify that, for the Dagum distribution, $V(R_t;\boldsymbol{\theta})$ is Bathtub. In *Figure 4* the behavior of the variance of residual life function for two Dagum distributions is illustrated.



(a) $Da(1.11, 3.60, 3.00)$ (b) $Da(2.1, 2.8, 4.0)$

Figure 4: The Variance of Residual Life Function for two Dagum distributions

Likewise, using the relation

$$\frac{\partial V(W_t;\boldsymbol{\theta})}{\partial t} = r_T(t;\boldsymbol{\theta}) \times \bar{\mu}(t;\boldsymbol{\theta})^2 \left[1 - \frac{V(W_t;\boldsymbol{\theta})}{\bar{\mu}(t;\boldsymbol{\theta})^2}\right] \tag{19}$$

(see Nanda et al. 2003), we can say that, for the Dagum distribution, $V(W_t;\boldsymbol{\theta})$ is always increasing because, from (17), $V(W_t;\boldsymbol{\theta}) - \bar{\mu}(t;\boldsymbol{\theta})^2 < 0$ given that the MWT for the Dagum random variable is monotonic increasing. *Figure 5* reports the variance of reversed residual life for two Dagum distributions.

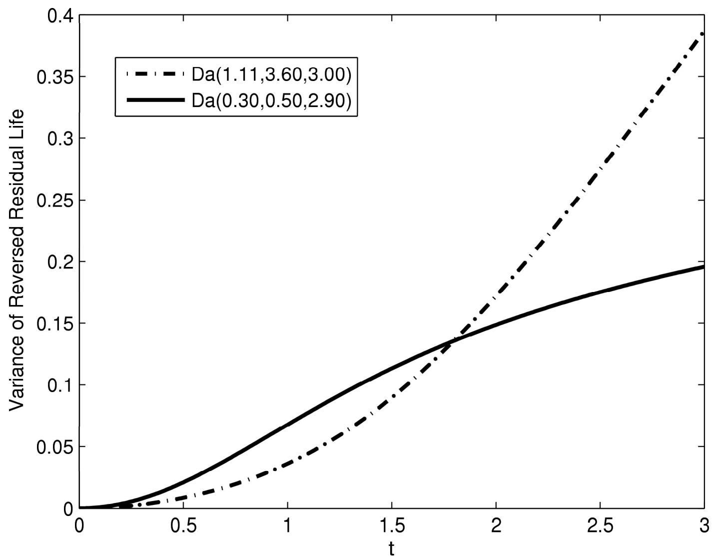


Figure 5: The Variance of Reversed Residual Life for two Dagum distributions.

4.4 The Hazard Rate of the maximum and the Reversed Hazard Rate of the minimum

In this sub-section, we prove that if T_1 and T_2 are *iid*, with $T_i \sim Da(\beta, \lambda, \delta)$ for $i = 1, 2$ then the behavior is preserved *not only* for the hazard rate of the minimum and for the reversed hazard rate of the maximum, but also for the hazard rate of the maximum. Let $M = max\,(T_1, T_2)$ and $m = min\,(T_1, T_2)$ be the maximum and minimum in a random sample *iid* of size two, respectively, from (2). Then, the distribution function of M and of m, respectively, are

$$G_M(t;\beta,\lambda,\delta) = F_T(t;2\beta,\lambda,\delta) \tag{20}$$

and

$$G_m(t;\beta,\lambda,\delta) = 2\left\{F_T(t;\beta,\lambda,\delta) - \frac{F_T(t;2\beta,\lambda,\delta)}{2}\right\}. \tag{21}$$

Under *iid* hypothesis, we have

$$r_M(t;\beta,\lambda,\delta) = 2r_T(t;\beta,\lambda,\delta)$$

and

$$h_m(t;\beta,\lambda,\delta) = 2h_T(t;\beta,\lambda,\delta).$$

Moreover, from (20) it follows that

$$h_M(t;\beta,\lambda,\delta) = h_T(t;2\beta,\lambda,\delta).$$

Now, it can be shown that the behavior of reversed hazard rate of the minimum is smaller of the reversed hazard of T, i.e. $r_m(t;\beta,\lambda,\delta) \leq r_T(t;\beta,\lambda,\delta)$. Indeed, let $g_m(t;\beta,\lambda,\delta)$ be the density function of minimum m, we have

$$\begin{aligned}
r_m(t;\beta,\lambda,\delta) &= \frac{g_m(t;\beta,\lambda,\delta)}{G_m(t;\beta,\lambda,\delta)} = \\
&= \frac{\beta\lambda\delta t^{-\delta-1}\left(1+\lambda t^{-\delta}\right)^{-\beta-1}}{\left(1+\lambda t^{-\delta}\right)^{-\beta} - \frac{1}{2}\left(1+\lambda t^{-\delta}\right)^{-2\beta}} - \frac{\beta\lambda\delta t^{-\delta-1}\left(1+\lambda t^{-\delta}\right)^{-2\beta-1}}{\left(1+\lambda t^{-\delta}\right)^{-\beta} - \frac{1}{2}\left(1+\lambda t^{-\delta}\right)^{-2\beta}} = \\
&= \frac{\beta\lambda\delta t^{-\delta-1}}{\left(1+\lambda t^{-\delta}\right)}\left\{\frac{1}{\left(1-\frac{F_T(t;\beta\lambda\delta)}{2}\right)} - \frac{2}{\left(\frac{2}{F_T(t;\beta,\lambda,\delta)}-1\right)}\right\} = \\
&= r_T(t;\beta,\lambda,\delta)\frac{2\left(1-F_T(t;\beta,\lambda,\delta)\right)}{2-F_T(t;\beta,\lambda,\delta)}
\end{aligned}$$

and, therefore, $r_m(t;\beta,\lambda,\delta) \leq r_T(t;\beta,\lambda,\delta)$ because $\frac{2(1-F_T(t;\beta,\lambda,\delta))}{2-F_T(t;\beta,\lambda,\delta)} \leq 1$.

5 Illustrative Example

We analyze the Traffic data set reported by Bain and Engelhardt (1980). The data set represents 128 observations on times, in seconds, between the arrival of vehicles at a particular location on the road.

Table 1: Traffic data set

0.2, 0.5, 0.8, 0.8, 0.8, 1.0, 1.1, 1.2, 1.2, 1.2, 1.2, 1.2, 1.3, 1.4, 1.5, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8,
1.8, 1.8, 1.8, 1.8, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 2.0, 2.1, 2.1, 2.2, 2.3, 2.3, 2.4, 2.4, 2.5, 2.5,
2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3.0, 3.0, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5.0, 5.1, 5.6,
5.7, 6.0, 6.0, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8.0, 8.3, 8.8, 8.8, 9.3, 9.4,9.5, 10.1, 11.0, 11.3,
11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6,
21.4, 22.8, 23.7, 24.7, 29.7, 30.6, 31.0, 33.7, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8,
51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8, 125.3.

The mean and the standard deviation are respectively 15.80859 and 23.69798. The data are positively skewed, in fact the coefficient of asymmetry is 2.505404. It is fitted the Dagum distribution for the Traffic data with the MLE method, the results are $\hat{\beta} = 4.648562$, $\hat{\lambda} = 1.285682$ and $\hat{\delta} = 0.9445047$. The Log-Likelihood value is -458.9415. *Figure 6* reports the empirical distribution function and the fitted Dagum distribution. It is tested the null hypothesis that the data comes from a $Da(4.648562, 1.285682, 0.9445047)$. The Kolmogorov-Smirnov test for goodness of fit with $\alpha = 0.01$ shows that the fitted Dagum distribution is acceptable for the given data (the value of statistic test is 0.1016 with a *p-value 0.5239*). Using the AIC method, it is possible to assert that the Dagum distribution fits better the data than a Fisk distribution, a Gamma distribution, a Weibull distribution. *Table 2* reports the AIC values.

Table 2: Comparison among the variables: the AIC values.

	Dagum	Fisk	Gamma	Weibull
AIC	923.883	930.0072	951.13	943.3848

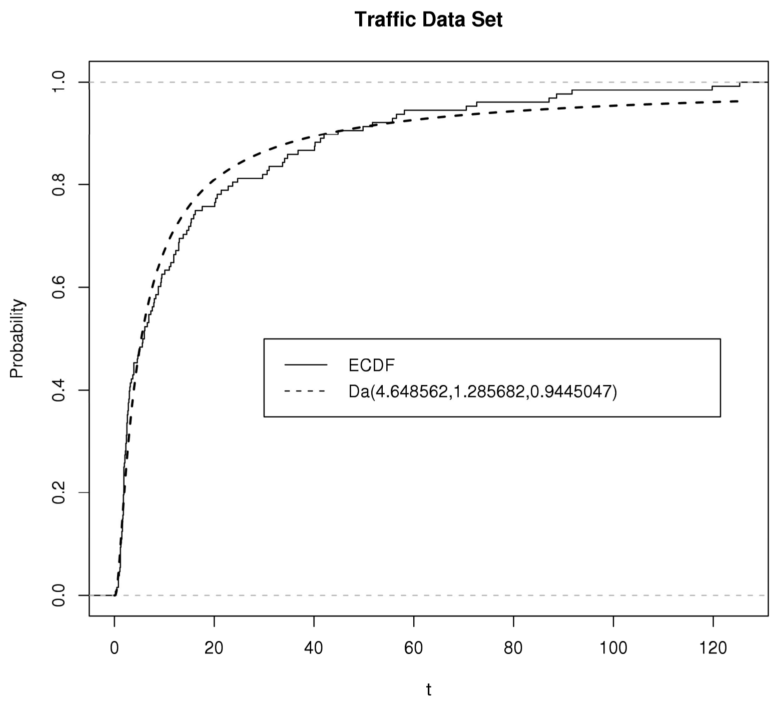


Figure 6: Empirical distribution function versus fitted Dagum distribution.

Using the *Proposition 1* is possible to assert that the hazard function of the estimated Dagum distribution is U. In fact, $\beta \cdot \delta = 4.390589 > 1$ and $\beta^* = 0.555595 \neq \beta = 4.648562$. In *Figure 7* the failure rate of data is compared to the fitted Dagum distribution. For the shape of the MRL distribution, it is easy to prove that it is B, because of the point 3 of the *Theorem 3*.

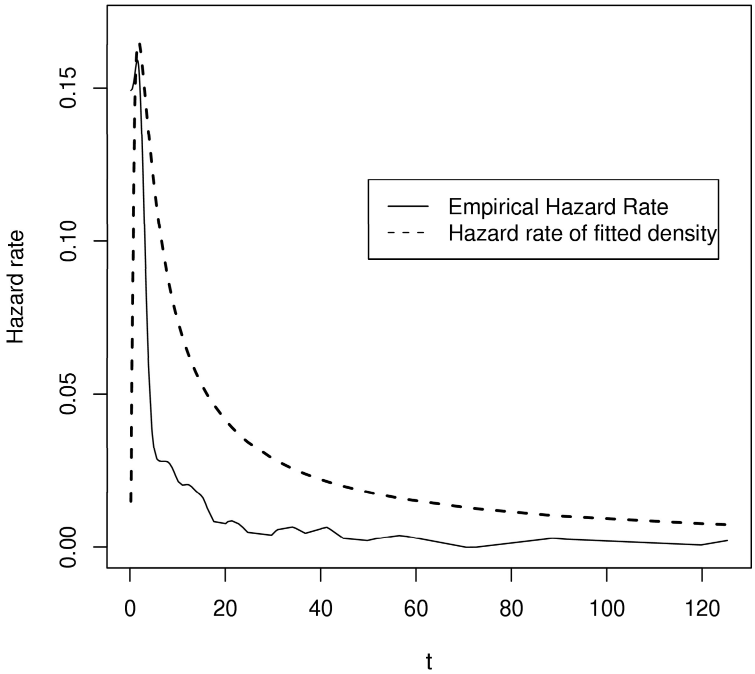


Figure 7: Empirical hazard function versus fitted hazard function.

6 Conclusions

In this work we have studied the Dagum distribution from a reliability point of view. Analyzing the hazard rate, the reversed hazard rate, the mean residual life and the mean waiting time, the variance of random variable residual life and reversed residual life and their monotonic properties, we can say that this model seems to have attractive reliability properties.

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