CLASSIFICATION OF LOCAL ASYMPTOTICS FOR SOLUTIONS TO HEAT EQUATIONS WITH INVERSE-SQUARE POTENTIALS

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Abstract. Asymptotic behavior of solutions to heat equations with spatially singular inverse-square potentials is studied. By combining a parabolic Almgren type monotonicity formula with blow-up methods, we evaluate the exact behavior near the singularity of solutions to linear and subcritical semilinear parabolic equations with Hardy type potentials. As a remarkable byproduct, a unique continuation property is obtained.

1. Introduction and statement of the main results

We aim to describe the asymptotic behavior near the singularity of solutions to backward evolution equations with inverse square singular potentials of the form

$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2}\, u + f(x,t,u(x,t)) = 0, \tag{1}$$

in $\mathbb{R}^N \times (0,T)$, where $T > 0$, $N \geqslant 3$, $a \in L^\infty(\mathbb{S}^{N-1})$ and $f : \mathbb{R}^N \times (0,T) \times \mathbb{R} \to \mathbb{R}$. Inverse square potentials are related to the well-known classical Hardy's inequality

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx, \quad \text{for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \quad N \geqslant 3,$$

see e.g. [18, 20]. Parabolic problems with singular inverse square Hardy potentials arise in the linearization of standard combustion models, see [24]. The properties of the heat operator are strongly affected by the presence of the singular inverse square potential, which, having the same order of homogeneity as the laplacian and failing to belong to the Kato class, cannot be regarded as a lower order term. Hence, singular problems with inverse square potentials represent a borderline case with respect to the classical theory of parabolic equations. Such a criticality makes parabolic equations of type (1) and their elliptic versions quite challenging from the mathematical point of view, thus motivating a large literature which, starting from the pioneering paper by [6], has been devoted to their analysis, see e.g. [18, 32] for the parabolic case and [1, 29, 31] for the elliptic counterpart. In particular, the influence of the Hardy potential in semilinear parabolic problems has been studied in [2], in the case $f(x,t,s) = s^p$, $p > 1$, and for $a(x/|x|) = \lambda$, $\lambda > 0$; the analysis carried out in [2] highlighted a deep difference with respect to the classical heat equation ($\lambda = 0$),

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showing that, if $\lambda > 0$, there exists a critical exponent $p_+(\lambda)$ such that for $p \geqslant p_+(\lambda)$, there is no solution even in the weakest sense for any nontrivial initial datum.

The present paper is addressed to the problem of describing the behavior of solutions along the directions $(\lambda x, \lambda^2 t)$ naturally related to the heat operator. Indeed, the unperturbed operator $u_t + \Delta u + \frac{a(x/|x|)}{|x|^2} u$ is invariant under the action $(x,t) \mapsto (\lambda x, \lambda^2 t)$. Then we are interested in evaluating the asymptotics of

$$u(\sqrt{t}x, t) \quad \text{as } t \to 0^+$$

for solutions to (1). Our analysis will show that $u(\sqrt{t}x, t)$ behaves as a singular self-similar eigenfunction of the Ornstein-Uhlenbeck operator with inverse square potential, multiplied by a power of t related to the corresponding eigenvalue, which can be selected by the limit of a frequency type function associated to the problem.

We consider both linear and subcritical semilinear parabolic equations of type (1). More precisely, we deal with the case $f(x,t,s) = h(x,t)s$ corresponding to the linear problem

$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2} u + h(x,t)u = 0, \quad \text{in } \mathbb{R}^N \times (0,T), \tag{2}$$

with a perturbing potential h satisfying

$$h, h_t \in L^r\big((0,T), L^{N/2}(\mathbb{R}^N)\big) \quad \text{for some } r > 1, \quad h_t \in L^\infty_{\rm loc}\big((0,T), L^{N/2}(\mathbb{R}^N)\big), \tag{3}$$

and negligible with respect to the inverse square potential $|x|^{-2}$ near the singularity in the sense that there exists $C_h > 0$ such that

$$|h(x,t)| \leqslant C_h(1 + |x|^{-2+\varepsilon}) \quad \text{for all } t \in (0,T), \text{ a.e. } x \in \mathbb{R}^N, \text{ and for some } \varepsilon \in (0,2). \tag{4}$$

We also treat the semilinear case $f(x,t,s) = \varphi(x,t,s)$, with a nonlinearity $\varphi \in C^1(\mathbb{R}^N \times (0,T) \times \mathbb{R})$ satisfying the following growth condition

$$\begin{cases} \dfrac{|\varphi(x,t,s)| + |x \cdot \nabla_x \varphi(x,t,s)| + |t \frac{\partial \varphi}{\partial t}(x,t,s)|}{|s|} \leqslant C_\varphi (1 + |s|^{p-1}) \\ \big|\varphi(x,t,s) - s \frac{\partial \varphi}{\partial s}(x,t,s)\big| \leqslant C_\varphi |s|^q \end{cases} \tag{5}$$

for all $(x,t,s) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}$ and some $1 < p < 2^* - 1$ and $2 \leqslant q < p+1$, where $2^* = \frac{2N}{N-2}$ is the critical exponent for Sobolev's embedding and $C_\varphi > 0$ is independent of $x \in \mathbb{R}^N$, $t \in (0,T)$, and $s \in \mathbb{R}$. In particular, we are going to classify the behavior of solutions to the semilinear parabolic problem

$$u_t + \Delta u + \frac{a(x/|x|)}{|x|^2} u + \varphi(x,t,u(x,t)) = 0, \quad \text{in } \mathbb{R}^N \times (0,T), \tag{6}$$

satisfying

$$u \in L^\infty(0,T, L^{p+1}(\mathbb{R}^N)) \tag{7}$$

and

$$t u_t \in L^\infty(0,T, L^{\frac{p+1}{p+1-q}}(\mathbb{R}^N)) \text{ and } \sup_{t \in (0,T)} t^{N/2} \int_{\mathbb{R}^N} |x|^{\frac{2(p+1)}{p-1}} |u(\sqrt{t}x, t)|^{p+1}\, dx < \infty. \tag{8}$$

In order to introduce a suitable notion of solution to (1), for every $t > 0$ let us define the space $\mathcal{H}_t$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to

$$\|u\|_{\mathcal{H}_t} = \bigg(\int_{\mathbb{R}^N} \big(t|\nabla u(x)|^2 + |u(x)|^2\big)G(x,t)\,dx\bigg)^{1/2},$$

where

$$G(x,t) = t^{-N/2}\exp\Big(-\frac{|x|^2}{4t}\Big)$$

is the heat kernel satisfying

$$G_t - \Delta G = 0 \quad \text{and} \quad \nabla G(x,t) = -\frac{x}{2t}\,G(x,t) \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{9}$$

We denote as $\big(\mathcal{H}_t\big)^\star$ the dual space of $\mathcal{H}_t$ and by ${}_{(\mathcal{H}_t)^\star}\langle\cdot,\cdot\rangle_{\mathcal{H}_t}$ the corresponding duality product. For every $t > 0$, we also define the space $\mathcal{L}_t$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to

$$\|u\|_{\mathcal{L}_t} = \bigg(\int_{R^N} |u(x)|^2 G(x,t)\,dx\bigg)^{1/2}.$$

Definition 1.1. We say that $u \in L^1_{\rm loc}(\mathbb{R}^N \times (0,T))$ is a weak solution to (1) in $\mathbb{R}^N \times (0,T)$ if

$$\int_\tau^T \|u(\cdot,t)\|^2_{\mathcal{H}_t}\,dt < +\infty, \qquad \int_\tau^T \Big\|u_t + \frac{\nabla u\cdot x}{2t}\Big\|^2_{(\mathcal{H}_t)^\star} dt < +\infty \text{ for all } \tau \in (0,T), \tag{10}$$

$$\begin{aligned} &{}_{\mathcal{H}_t^\star}\Big\langle u_t + \frac{\nabla u\cdot x}{2t}, \phi\Big\rangle_{\mathcal{H}_t} \\ &\quad = \int_{\mathbb{R}^N}\bigg(\nabla u(x,t)\cdot\nabla\phi(x) - \frac{a(x/|x|)}{|x|^2}\,u(x,t)\phi(x) - f(x,t,u(x,t))\phi(x)\bigg)G(x,t)\,dx \end{aligned} \tag{11}$$

for a.e. $t \in (0,T)$ and for each $\phi \in \mathcal{H}_t$.

It will be clear from the parabolic Hardy type inequality of Lemma 2.1 and the Sobolev weighted inequality of Corollary 2.8, that the integral $\int_{\mathbb{R}^N} f(x,t,u(x,t))\phi(x)G(x,t)dx$ in the above definition is finite for a.e. $t \in (0,T)$, both in the linear case $f(x,t,s) = h(x,t)s$ under assumptions (3–4) and in the semilinear case $f(x,t,s) = \varphi(x,t,s)$ under condition (5) and for u satisfying (7).

Remark 1.2. If $u \in L^1_{\rm loc}(\mathbb{R}^N \times (0,T))$ satisfies (10), then the function

$$v(x,t) := u(\sqrt{t}x,t)$$

satisfies

$$v \in L^2(\tau,T;\mathcal{H}) \quad \text{and} \quad v_t \in L^2(\tau,T;(\mathcal{H})^\star) \quad \text{for all } \tau \in (0,T), \tag{12}$$

where we have set

$$\mathcal{H} := \mathcal{H}_1,$$

i.e. $\mathcal{H}$ is the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to

$$\|v\|_{\mathcal{H}} = \bigg(\int_{\mathbb{R}^N}\big(|\nabla v(x)|^2 + |v(x)|^2\big)e^{-|x|^2/4}\,dx\bigg)^{1/2}.$$

We notice that from (12) it follows that

$$v \in C^0([\tau,T],\mathcal{L}),$$

see e.g. [27, Theorem 1.2], where $\mathcal{L} := \mathcal{L}_1$ is the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to the norm $\|v\|_{\mathcal{L}} = \big(\int_{\mathbb{R}^N} |v(x)|^2 e^{-|x|^2/4}\,dx\big)^{1/2}$. Moreover the function

$$t \in [\tau, T] \mapsto \|v(t)\|^2_{\mathcal{L}} = \int_{\mathbb{R}^N} u^2(x,t) G(x,t)\,dx$$

is absolutely continuous and

$$\frac{1}{2}\frac{1}{dt}\int_{\mathbb{R}^N} u^2(x,t)G(x,t) = \frac{1}{2}\frac{1}{dt}\|v(t)\|^2_{\mathcal{L}} = {}_{\mathcal{H}^\star}\langle v_t(\cdot,t), v(\cdot,t)\rangle_{\mathcal{H}} = {}_{\mathcal{H}_t^\star}\Big\langle u_t + \frac{\nabla u \cdot x}{2t}, u(\cdot,t)\Big\rangle_{\mathcal{H}_t}$$

for a.e. $t \in (0,T)$.

Remark 1.3. If u is a weak solution to (1) in the sense of definition 1.1, then the function $v(x,t) := u(\sqrt{t}x, t)$ defined in Remark 1.2 is a weak solution to

$$v_t + \frac{1}{t}\bigg(\Delta v - \frac{x}{2}\cdot\nabla v + \frac{a(x/|x|)}{|x|^2}\,v + t f(\sqrt{t}x, t, v(x,t))\bigg) = 0,$$

in the sense that, for every $\phi \in \mathcal{H}$,

$$\begin{aligned}(13)\qquad & {}_{\mathcal{H}^\star}\langle v_t, \phi\rangle_{\mathcal{H}} \\ &= \frac{1}{t}\int_{\mathbb{R}^N}\bigg(\nabla v(x,t)\cdot\nabla\phi(x) - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\,v(x,t)\phi(x) - t\,f(\sqrt{t}x, t, v(x,t))\phi(x)\bigg)G(x,1)\,dx.\end{aligned}$$

In particular, if u is a weak solution to (2), then $v(x,t) := u(\sqrt{t}x,t)$ weakly solves

$$v_t + \frac{1}{t}\bigg(\Delta v - \frac{x}{2}\cdot\nabla v + \frac{a(x/|x|)}{|x|^2}\,v + t h(\sqrt{t}x,t) v\bigg) = 0,$$

whereas, if u is a weak solution to (6), then $v(x,t) := u(\sqrt{t}x,t)$ weakly solves

$$v_t + \frac{1}{t}\bigg(\Delta v - \frac{x}{2}\cdot\nabla v + \frac{a(x/|x|)}{|x|^2}\,v + t\varphi(\sqrt{t}x,t,v)\bigg) = 0.$$

We give a precise description of the asymptotic behavior at the singularity of solutions to (2) and (6) in terms of the eigenvalues and eigenfunctions of the Ornstein-Uhlenbeck operator with singular inverse square potential

$$(14)\qquad L : \mathcal{H} \to (\mathcal{H})^\star, \quad L = -\Delta + \frac{x}{2}\cdot\nabla - \frac{a(x/|x|)}{|x|^2},$$

acting as

$${}_{\mathcal{H}^\star}\langle Lv, w\rangle_{\mathcal{H}} = \int_{\mathbb{R}^N}\bigg(\nabla v(x)\cdot\nabla w(x) - \frac{a(x/|x|)}{|x|^2}\,v(x)w(x)\bigg)G(x,1)\,dx, \quad \text{for all } v, w \in \mathcal{H}.$$

In order to describe the spectrum of L, we consider the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$. For any $a \in L^\infty\big(\mathbb{S}^{N-1}\big)$, $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ admits a diverging sequence of eigenvalues

$$\mu_1(a) < \mu_2(a) \leqslant \cdots \leqslant \mu_k(a) \leqslant \cdots,$$

the first of which is simple and can be characterized as

$$(15)\qquad \mu_1(a) = \min_{\psi \in H^1(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS(\theta) - \int_{\mathbb{S}^{N-1}} a(\theta)\psi^2(\theta)\,dS(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta)\,dS(\theta)},$$

see [16]. Moreover the quadratic form associated to $-\Delta - \frac{a(x/|x|)}{|x|^2}$ is positive definite if and only if

$$\mu_1(a) > -\frac{(N-2)^2}{4}, \tag{16}$$

see [16, Lemma 2.5]. To each $k \in \mathbb{N}$, $k \geqslant 1$, we associate a $L^2(\mathbb{S}^{N-1})$-normalized eigenfunction ψ_k of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ corresponding to the k-th eigenvalue $\mu_k(a)$, i.e. satisfying

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}}\psi_k(\theta) - a(\theta)\psi_k(\theta) = \mu_k(a)\,\psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\, dS(\theta) = 1. \end{cases} \tag{17}$$

In the enumeration $\mu_1(a) < \mu_2(a) \leqslant \cdots \leqslant \mu_k(a) \leqslant \cdots$ we repeat each eigenvalue as many times as its multiplicity; thus exactly one eigenfunction ψ_k corresponds to each index $k \in \mathbb{N}$. We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1})$.

The following proposition describes completely the spectrum of the operator L, thus extending to the anisotropic case the spectral analysis performed in [32, §9.3] in the isotropic case $a(\theta) \equiv \lambda$; see also [8, §4.2] and [14, §2] for the non singular case.

Proposition 1.4. *The set of the eigenvalues of the operator L is*

$$\{\gamma_{m,k} : k, m \in \mathbb{N}, k \geqslant 1\}$$

where

$$\gamma_{m,k} = m - \frac{\alpha_k}{2}, \quad \alpha_k = \frac{N-2}{2} - \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \mu_k(a)}, \tag{18}$$

and $\mu_k(a)$ is the k-th eigenvalue of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the sphere $\mathbb{S}^{N-1}$. Each eigenvalue $\gamma_{m,k}$ has finite multiplicity equal to

$$\#\Big\{j \in \mathbb{N}, j \geqslant 1 : \gamma_{m,k} + \frac{\alpha_j}{2} \in \mathbb{N}\Big\}$$

and a basis of the corresponding eigenspace is

$$\Big\{V_{n,j} : j, n \in \mathbb{N}, j \geqslant 1, \gamma_{m,k} = n - \frac{\alpha_j}{2}\Big\},$$

where

$$V_{n,j}(x) = |x|^{-\alpha_j} P_{j,n}\Big(\frac{|x|^2}{4}\Big)\psi_j\Big(\frac{x}{|x|}\Big), \tag{19}$$

ψ_j is an eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the sphere $\mathbb{S}^{N-1}$ associated to the j-th eigenvalue $\mu_j(a)$ as in (17), and $P_{j,n}$ is the polynomial of degree n given by

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\big(\frac{N}{2} - \alpha_j\big)_i} \frac{t^i}{i!},$$

denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol $(s)_i = \prod_{j=0}^{i-1}(s+j)$, $(s)_0 = 1$.

The following theorems provide a classification of singularity rating of any solution u to (1) based on the limit as $t \to 0^+$ of the *Almgren type frequency function* (see [5, 25]),

$$\mathcal{N}_{f,u}(t) = \frac{t \int_{\mathbb{R}^N} \left(|\nabla u(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} u^2(x,t) - f(x,t,u(x,t))u(x,t)\right) G(x,t)\, dx}{\int_{\mathbb{R}^N} u^2(x,t)\, G(x,t)\, dx}. \tag{20}$$

In the linear case $f(x,t,u) = h(x,t)u$, the behavior of weak solutions to (2) is described by the following theorem.

Theorem 1.5. *Let $u \not\equiv 0$ be a weak solution to (2) in the sense of Definition 1.1, with h satisfying (3) and (4) and $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying (16). Then there exist $m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\lim_{t \to 0^+} \mathcal{N}_{hu,u}(t) = \gamma_{m_0,k_0}, \tag{21}$$

where $\mathcal{N}_{hu,u}$ is defined in (20) and γ_{m_0,k_0} is as in (18). Furthermore, denoting as J_0 the finite set of indices

$$J_0 = \{(m,k) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\}) : m - \frac{\alpha_k}{2} = \gamma_{m_0,k_0}\}, \tag{22}$$

for all $\tau \in (0,1)$ there holds

$$\lim_{\lambda \to 0^+} \int_\tau^1 \left\| \lambda^{-2\gamma_{m_0,k_0}} u(\lambda x, \lambda^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(m,k) \in J_0} \beta_{m,k} \widetilde{V}_{m,k}(x/\sqrt{t}) \right\|_{\mathcal{H}_t}^2 dt = 0 \tag{23}$$

and

$$\lim_{\lambda \to 0^+} \sup_{t \in [\tau,1]} \left\| \lambda^{-2\gamma_{m_0,k_0}} u(\lambda x, \lambda^2 t) - t^{\gamma_{m_0,k_0}} \sum_{(m,k) \in J_0} \beta_{m,k} \widetilde{V}_{m,k}(x/\sqrt{t}) \right\|_{\mathcal{L}_t} = 0, \tag{24}$$

where $\widetilde{V}_{m,k} = V_{m,k}/\|V_{m,k}\|_{\mathcal{L}}$, $V_{m,k}$ are as in (19),

$$\begin{aligned} \beta_{m,k} = {} & \Lambda^{-2\gamma_{m_0,k_0}} \int_{\mathbb{R}^N} u(\Lambda x, \Lambda^2) \widetilde{V}_{m,k}(x) G(x,1)\, dx \\ & + 2 \int_0^\Lambda s^{1-2\gamma_{m_0,k_0}} \left(\int_{\mathbb{R}^N} h(sx, s^2) u(sx, s^2) \widetilde{V}_{m,k}(x) G(x,1)\, dx \right) ds \end{aligned} \tag{25}$$

for all $\Lambda \in (0, \Lambda_0)$ and for some $\Lambda_0 \in (0, \sqrt{T})$, and $\beta_{m,k} \neq 0$ for some $(m,k) \in J_0$.

An analogous result holds in the semilinear case for solutions to (6) satisfying the further conditions (7) and (8).

Theorem 1.6. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ satisfy (16) and $\varphi \in C^1(\mathbb{R}^N \times (0,T) \times \mathbb{R})$ such that (5) holds. If $u \not\equiv 0$ satisfies (7–8) and is a weak solution to (6) in the sense of Definition 1.1, then there exist $m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\lim_{t \to 0^+} \mathcal{N}_{\varphi,u}(t) = \gamma_{m_0,k_0}, \tag{26}$$

where $\mathcal{N}_{\varphi,u}$ is defined in (20) and γ_{m_0,k_0} is as in (18). Furthermore, letting J_0 the finite set of indices defined in (22), for all $\tau \in (0,1)$ convergences (23) and (24) hold with

$$\beta_{m,k} = \Lambda^{-2\gamma_{m_0,k_0}} \int_{\mathbb{R}^N} u(\Lambda x, \Lambda^2)\widetilde{V}_{m,k}(x)G(x,1)\,dx + 2\int_0^\Lambda s^{1-2\gamma_{m_0,k_0}}\left(\int_{\mathbb{R}^N} \varphi(sx, s^2, u(sx,s^2))\widetilde{V}_{m,k}(x)G(x,1)\,dx\right)ds \tag{27}$$

for all $\Lambda \in (0,\Lambda_0)$ and for some $\Lambda_0 \in (0,\sqrt{T})$, and $\beta_{m,k} \neq 0$ for some $(m,k) \in J_0$.

(25) and (27) can be seen as Cauchy's integral type formulas for solutions to problems (2) and (6), since they allow reconstructing, up to the perturbation, the solution at the singularity by the values it takes at any positive time.

The proofs of theorems 1.5 and 1.6 are based on a parabolic Almgren type monotonicity formula combined with blow-up methods. Almgren type frequency functions associated to parabolic equations were first introduced by C.-C. Poon in [25], where unique continuation properties are derived by proving a monotonicity result which is the parabolic counterpart of the monotonicity formula introduced by Almgren in [5] and extended by Garofalo and Lin in [19] to elliptic operators with variable coefficients. A further development in the use of Almgren monotonicity methods to study regularity of solutions to parabolic problems is due to the recent paper [7]. We also mention that an Almgren type monotonicity method combined with blow-up was used in [15] in an elliptic context to study the behavior of solutions to stationary Schrödinger equations with singular electromagnetic potentials.

Theorem 1.5 and Theorem 1.6 imply *a strong unique continuation property* at the singularity, as the following corollary states.

Corollary 1.7. *Suppose that either u is a weak solution to (2) under the assumptions of Theorem 1.5 or u satisfies (7–8) and weakly solves (6) under the assumptions of Theorem 1.6. If*

$$u(x,t) = O\big((|x|^2+t)^k\big) \quad \text{as } (x,t)\to(0,0) \quad \text{for all } k\in\mathbb{N}, \tag{28}$$

then $u \equiv 0$ in $\mathbb{R}^N \times (0,T)$.

As a byproduct of the proof of Theorems 1.5 and 1.6, we also obtain the following result, which can be regarded as a *unique continuation property* with respect to time.

Proposition 1.8. *Suppose that either u is a weak solution to (2) under the assumptions of Theorem 1.5 or u satisfies (7–8) and weakly solves (6) under the assumptions of Theorem 1.6. If there exists $t_0 \in (0,T)$ such that*

$$u(x,t_0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N,$$

then $u \equiv 0$ in $\mathbb{R}^N \times (0,T)$.

There exists a large literature dealing with strong continuation properties in the parabolic setting. [21] (see too [22]) studies parabolic operators with $L^{\frac{N+1}{2}}$ time-independent coefficients obtaining a unique continuation property at a fixed time t_0: the used technique relies on a representation formula for solutions of parabolic equations in terms of eigenvalues of the corresponding elliptic operator and cannot be applied to more general equations with time-dependant coefficients. [26] and [30] use parabolic variants of the Carleman weighted inequalities to obtain a unique continuation property at fixed time t_0 for parabolic operators with time-dependant coefficients. In this direction, it is worth mentioning the work of Chen [8] which contains not only a unique continuation result but also some local asymptotic analysis of solutions to parabolic inequalities with bounded coefficients; the approach is based in recasting equations in terms of parabolic self-similar variables. We also quote [4, 9, 10, 12, 13, 17] for unique continuation results for parabolic equations with time-dependent potentials by Carleman inequalities and monotonicity methods.

The present paper is organized as follows. In section 2, we state some parabolic Hardy type inequalities and weighted Sobolev embeddings related to equations (2) and (6). In section 3, we completely describe the spectrum of the operator L defined in (14) and prove Proposition 1.4. Section 4 contains an Almgren parabolic monotonicity formula which provides the unique continuation principle stated in Proposition 1.8 and is used in section 5, together with a blow-up method, to prove Theorems 1.5 and 1.6.

Notation. We list below some notation used throughout the paper.

- const denotes some positive constant which may vary from formula to formula.
- dS denotes the volume element on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$.
- ω_{N-1} denotes the volume of $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} dS(\theta)$.
- For all $s \in \mathbb{R}$, $(s)_i$ denotes the Pochhammer's symbol $(s)_i = \prod_{j=0}^{i-1}(s+j)$, $(s)_0 = 1$.

2. Parabolic Hardy type inequalities and Weighted Sobolev embeddings

The following lemma provides a Hardy type inequality for parabolic operators. We refer to [25, Proposition 3.1] for a proof.

Lemma 2.1. *For every $t > 0$ and $u \in \mathcal{H}_t$ there holds*

$$\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, G(x,t)\, dx \leqslant \frac{1}{(N-2)t} \int_{\mathbb{R}^N} u^2(x) G(x,t)\, dx + \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u(x)|^2\, G(x,t)\, dx.$$

In the anisotropic version of the above inequality, a crucial role is played by the first eigenvalue $\mu_1(a)$ of the angular operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the unit sphere $\mathbb{S}^{N-1}$ defined in (15).

Lemma 2.2. *For every $a \in L^\infty(\mathbb{S}^{N-1})$, $t > 0$, and $u \in \mathcal{H}_t$, there holds*

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 - \frac{a(x/|x|)}{|x|^2}\, u^2(x) \right) G(x,t)\, dx + \frac{N-2}{4t} \int_{\mathbb{R}^N} u^2(x) G(x,t)\, dx \geqslant \left(\mu_1(a) + \frac{(N-2)^2}{4} \right) \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, G(x,t)\, dx.$$

Proof. Let $u \in C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\})$. The gradient of u can be written in polar coordinates as

$$\nabla u(x) = \big(\partial_r u(r,\theta)\big)\theta + \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} u(r,\theta), \quad r = |x|, \quad \theta = \frac{x}{|x|},$$

hence

$$|\nabla u(x)|^2 = \big|\partial_r u(r,\theta)\big|^2 + \frac{1}{r^2} \big|\nabla_{\mathbb{S}^{N-1}} u(r,\theta)\big|^2$$

and

$$\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2-\frac{a(x/|x|)}{|x|^2}\,u^2(x)\right)G(x,t)\,dx \tag{29}$$
$$=t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}r^{N-1}e^{-\frac{r^2}{4t}}|\partial_r u(r,\theta)|^2\,dr\right)dS(\theta)$$
$$+\,t^{-\frac{N}{2}}\int_0^{+\infty}\frac{r^{N-1}e^{-\frac{r^2}{4t}}}{r^2}\left(\int_{\mathbb{S}^{N-1}}\left[|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)|^2-a(\theta)|u(r,\theta)|^2\right]dS(\theta)\right)dr.$$

For all $\theta\in\mathbb{S}^{N-1}$, let $\varphi_\theta\in C^\infty_{\rm c}((0,+\infty))$ be defined by $\varphi_\theta(r)=u(r,\theta)$, and $\widetilde\varphi_\theta\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\})$ be the radially symmetric function given by $\widetilde\varphi_\theta(x)=\varphi_\theta(|x|)$. From Lemma 2.1, it follows that

$$t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}r^{N-1}e^{-\frac{r^2}{4t}}|\partial_r u(r,\theta)|^2\,dr\right)dS(\theta) \tag{30}$$
$$=t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}r^{N-1}e^{-\frac{r^2}{4t}}|\varphi_\theta'(r)|^2\,dr\right)dS(\theta)$$
$$=\frac{1}{\omega_{N-1}}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\nabla\widetilde\varphi_\theta(x)|^2G(x,t)\,dx\right)dS(\theta)$$
$$\geqslant\frac{1}{\omega_{N-1}}\frac{(N-2)^2}{4}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}\frac{|\widetilde\varphi_\theta(x)|^2}{|x|^2}G(x,t)\,dx\right)dS(\theta)$$
$$-\frac{1}{\omega_{N-1}}\frac{N-2}{4t}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\widetilde\varphi_\theta(x)|^2G(x,t)\,dx\right)dS(\theta)$$
$$=t^{-\frac{N}{2}}\frac{(N-2)^2}{4}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}\frac{r^{N-1}e^{-\frac{r^2}{4t}}}{r^2}|u(r,\theta)|^2\,dr\right)dS(\theta)$$
$$-\,t^{-\frac{N}{2}}\frac{N-2}{4t}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}r^{N-1}e^{-\frac{r^2}{4t}}|u(r,\theta)|^2\,dr\right)dS(\theta)$$
$$=\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,G(x,t)\,dx-\frac{N-2}{4t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx,$$

where ω_{N-1} denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1}=\int_{\mathbb{S}^{N-1}}dS(\theta)$. On the other hand, from the definition of $\mu_1(a)$ it follows that

$$\int_{\mathbb{S}^{N-1}}\left[|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)|^2-a(\theta)|u(r,\theta)|^2\right]dS(\theta)\geqslant\mu_1(a)\int_{\mathbb{S}^{N-1}}|u(r,\theta)|^2dS(\theta). \tag{31}$$

From (29), (30), and (31), we deduce that

$$\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2-\frac{a(x/|x|)}{|x|^2}\,u^2(x)\right)G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx$$
$$\geqslant\left(\mu_1(a)+\frac{(N-2)^2}{4}\right)\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}\,G(x,t)\,dx,$$

for all $u\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\})$, thus yielding the required inequality by density of $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\})$ in $\mathcal{H}_t$. □

The following corollary provides a norm in $\mathcal{H}_t$ equivalent to $\|\cdot\|_{\mathcal{H}_t}$ and naturally related to the heat operator with the Hardy potential of equation (1).

Corollary 2.3. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying (16). Then, for every $t>0$,*

$$\inf_{u\in\mathcal{H}_t\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2-\frac{a(x/|x|)}{|x|^2}u^2(x)\right)G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx}{\int_{\mathbb{R}^N}|\nabla u(x)|^2G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx}$$

$$=\inf_{v\in\mathcal{H}\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|\nabla v(x)|^2-\frac{a(x/|x|)}{|x|^2}v^2(x)\right)G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}v^2(x)G(x,1)\,dx}{\int_{\mathbb{R}^N}|\nabla v(x)|^2G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}v^2(x)G(x,1)\,dx}>0.$$

Proof. The equality of the two infimum levels follows by the change of variables $u(x)=v(x/\sqrt{t})$. To prove that they are strictly positive, we argue by contradiction and assume that for every $\varepsilon>0$ there exists $v_\varepsilon\in\mathcal{H}\setminus\{0\}$ such that

$$\int_{\mathbb{R}^N}\left(|\nabla v_\varepsilon(x)|^2-\frac{a(x/|x|)}{|x|^2}v_\varepsilon^2(x)\right)G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}v_\varepsilon^2(x)G(x,1)\,dx$$
$$<\varepsilon\left(\int_{\mathbb{R}^N}|\nabla v_\varepsilon(x)|^2G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}v_\varepsilon^2(x)G(x,1)\,dx\right),$$

which, by Lemma 2.2, implies that

$$\left(\mu_1\left(\frac{a}{1-\varepsilon}\right)+\frac{(N-2)^2}{4}\right)\int_{\mathbb{R}^N}\frac{v_\varepsilon^2(x)}{|x|^2}G(x,1)\,dx$$
$$\leqslant\int_{\mathbb{R}^N}\left(|\nabla v_\varepsilon(x)|^2-\frac{a(x/|x|)}{(1-\varepsilon)|x|^2}v_\varepsilon^2(x)\right)G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}v_\varepsilon^2(x)G(x,1)\,dx<0$$

and consequently

$$\mu_1\left(\frac{a}{1-\varepsilon}\right)+\frac{(N-2)^2}{4}<0.$$

By continuity of the map $a\mapsto\mu_1(a)$ with respect to the $L^\infty(\mathbb{S}^{N-1})$-norm, letting $\varepsilon\to0$ the above inequality yields $\mu_1(a)+\frac{(N-2)^2}{4}\leqslant0$, giving rise to a contradiction with (16). □

The above results combined with the negligibility assumption (4) on h allow estimating the quadratic form associated to the linearly perturbed equation (2) for small times as follows.

Corollary 2.4. *Let $a\in L^\infty(\mathbb{S}^{N-1})$ satisfy (16) and $h\in L^\infty_{\rm loc}(\mathbb{R}^N\setminus\{0\}\times(0,T))$ satisfy (4). Then there exist $C_1',C_2>0$ and $\overline{T}_1>0$ such that for every $t\in(0,\overline{T}_1)$, $s\in(0,T)$, and $u\in\mathcal{H}_t$ there holds*

$$\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2-\frac{a(x/|x|)}{|x|^2}u^2(x)-h(x,s)u^2(x)\right)G(x,t)\,dx$$
$$\geqslant C_1'\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^2}G(x,t)\,dx-\frac{C_2}{t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx$$

$$\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2-\frac{a(x/|x|)}{|x|^2}u^2(x)-h(x,s)u^2(x)\right)G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx$$
$$\geqslant C_1'\left(\int_{\mathbb{R}^N}|\nabla u(x)|^2G(x,t)\,dx+\frac{1}{t}\int_{\mathbb{R}^N}u^2(x)G(x,t)\,dx\right).$$

Proof. From (4), we have that, for every $u \in \mathcal{H}_t$, there holds

$$
\begin{aligned}
(32)\quad \left|\int_{\mathbb{R}^N} h(x,s)u^2(x)G(x,t)\,dx\right| &\leqslant C_h\bigg(\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx + \int_{\mathbb{R}^N} |x|^{-2+\varepsilon}u^2(x)G(x,t)\,dx\bigg)\\
&\leqslant C_h\bigg(\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx + t^{\varepsilon/2}\int_{|x|\leqslant\sqrt{t}} \frac{u^2(x)}{|x|^2}G(x,t)\,dx + t^{-1+\varepsilon/2}\int_{|x|\geqslant\sqrt{t}} u^2(x)G(x,t)\,dx\bigg)\\
&= \frac{C_h}{t}(t+t^{\varepsilon/2})\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx + C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}G(x,t)\,dx.
\end{aligned}
$$

The stated inequalities follow from (32), Lemma 2.1, Corollary 2.3, and assumption (16). □

In order to estimate the quadratic form associated to the nonlinearly perturbed equation (6), we derive a Sobolev type embedding in spaces $\mathcal{H}_t$. To this purpose, we need the following inequality, whose proof can be found in [11, Lemma 3].

Lemma 2.5. *For every $u \in \mathcal{H}$, $|x|u \in \mathcal{L}$ and*

$$\frac{1}{16}\int_{\mathbb{R}^N} |x|^2u^2(x)G(x,1)\,dx \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^2G(x,1)\,dx + \frac{N}{4}\int_{\mathbb{R}^N} u^2(x)G(x,1)\,dx.$$

The change of variables $u(x) = v(x/\sqrt{t})$ in Lemma 2.5, yields the following inequality in $\mathcal{H}_t$.

Corollary 2.6. *For every $u \in \mathcal{H}_t$, there holds*

$$\frac{1}{16t^2}\int_{\mathbb{R}^N} |x|^2u^2(x)G(x,t)\,dx \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^2G(x,t)\,dx + \frac{N}{4t}\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx.$$

From Lemma 2.5 and classical Sobolev embeddings, we can easily deduce the following weighted Sobolev inequality (see also [14]).

Lemma 2.7. *For all $u \in \mathcal{H}$ and $s \in [2,2^*]$, there holds $u\sqrt{G(\cdot,1)} \in L^s(\mathbb{R}^N)$. Moreover, for every $s \in [2,2^*]$ there exists $C_s > 0$ such that*

$$\bigg(\int_{\mathbb{R}^N} |u(x)|^sG^{\frac{s}{2}}(x,1)\,dx\bigg)^{\frac{2}{s}} \leqslant C_s\bigg(\int_{\mathbb{R}^N} \big(|\nabla u(x)|^2 + u^2(x)\big)G(x,1)\,dx\bigg)$$

for all $u \in \mathcal{H}$.

Proof. From Lemma 2.5, it follows that, if $u \in \mathcal{H}$, then $u\sqrt{G(\cdot,1)} \in H^1(\mathbb{R}^N)$; hence, by classical Sobolev embeddings, $u\sqrt{G(\cdot,1)} \in L^s(\mathbb{R}^N)$ for all $s \in [2,2^*]$. The stated inequality follows from classical Sobolev inequalities and Lemma 2.5. □

The change of variables $u(x) = v(x/\sqrt{t})$ in Lemma 2.7, yields the following inequality in $\mathcal{H}_t$.

Corollary 2.8. *For every $t > 0$, $u \in \mathcal{H}_t$, and $2 \leqslant s \leqslant 2^*$, there holds*

$$\bigg(\int_{\mathbb{R}^N} |u(x)|^sG^{\frac{s}{2}}(x,t)\,dx\bigg)^{\frac{2}{s}} \leqslant C_s t^{-\frac{N}{s}\left(\frac{s-2}{2}\right)}\|u\|^2_{\mathcal{H}_t}.$$

The above Sobolev estimate allows proving the nonlinear counterpart of Corollary 2.4.

Corollary 2.9. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ satisfy (16) and $\varphi \in C^1(\mathbb{R}^N \times (0,T) \times \mathbb{R})$ satisfy (5) for some $1 \leqslant p < 2^* - 1$. Then there exist $C_1'' > 0$ and a function $\overline{T}_2 : (0,+\infty) \to \mathbb{R}$ such that, for every $R > 0$, $t \in (0, \overline{T}_2(R))$, $s \in (0,T)$, and $u \in \{v \in \mathcal{H}_t \cap L^{p+1}(\mathbb{R}^N) : \|v\|_{L^{p+1}(\mathbb{R}^N)} \leqslant R\}$, there holds*

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 - \frac{a(x/|x|)}{|x|^2} u^2(x) - \varphi(x,s,u(x))u(x) \right) G(x,t)\,dx + \frac{N-2}{4t} \int_{\mathbb{R}^N} u^2(x) G(x,t)\,dx \geqslant C_1'' \bigg(\int_{\mathbb{R}^N} |\nabla u(x)|^2 G(x,t)\,dx + \frac{1}{t} \int_{\mathbb{R}^N} u^2(x) G(x,t)\,dx \bigg).$$

Proof. From (5), Hölder's inequality, and Corollary 2.8, we have that, for all $u \in \mathcal{H}_t \cap L^{p+1}(\mathbb{R}^N)$, there holds

$$\begin{aligned} (33) \qquad & \left| \int_{\mathbb{R}^N} \varphi(x,s,u(x))u(x)G(x,t)\,dx \right| \\ & \leqslant C_\varphi \bigg(\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx + \int_{\mathbb{R}^N} u^2(x)|u(x)|^{p-1} G(x,t)\,dx \bigg) \\ & \leqslant C_\varphi \bigg(\int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx + \bigg(\int_{\mathbb{R}^N} |u(x)|^{p+1} G^{\frac{p+1}{2}}(x,t)\,dx \bigg)^{\frac{2}{p+1}} \|u\|^{p-1}_{L^{p+1}(\mathbb{R}^N)} \bigg) \\ & \leqslant C_\varphi \bigg(C_{p+1} t^{\frac{(N+2)-p(N-2)}{2(p+1)}} \|u\|^{p-1}_{L^{p+1}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla u(x)|^2 G(x,t)\,dx \\ & \qquad + \Big(t + C_{p+1} t^{\frac{(N+2)-p(N-2)}{2(p+1)}} \|u\|^{p-1}_{L^{p+1}(\mathbb{R}^N)} \Big) \frac{1}{t} \int_{\mathbb{R}^N} u^2(x)G(x,t)\,dx \bigg) \end{aligned}$$

with C_{p+1} as in Corollary 2.8. The stated inequality follows from Corollary 2.3 and (33) by choosing t sufficiently small depending on $\|u\|_{L^{p+1}(\mathbb{R}^N)}$. □

3. Spectrum of Ornstein-Uhlenbeck type operators with inverse square potentials

In this section we describe the spectral properties of the operator L defined in (14), extending to anisotropic singular potentials the analysis carried out in [32] for $a \equiv \lambda$ constant. Following [14], we first prove the following compact embedding.

Lemma 3.1. *The space $\mathcal{H}$ is compactly embedded in $\mathcal{L}$.*

Proof. Let us assume that $u_k \rightharpoonup u$ weakly in $\mathcal{H}$. From Rellich's theorem $u_k \to u$ in $L^2_{\rm loc}(\mathbb{R}^N)$. For every $R > 0$ and $k \in \mathbb{N}$, we have

$$(34) \qquad \int_{\mathbb{R}^N} |u_k - u|^2 G(x,1)\,dx = A_k(R) + B_k(R)$$

where

$$(35) \qquad A_k(R) = \int_{\{|x| \leqslant R\}} |u_k(x) - u(x)|^2 e^{-|x|^2/4}\,dx \to 0 \quad \text{as } k \to +\infty, \quad \text{for every } R > 0$$

and

$$B_k(R) = \int_{\{|x|>R\}} |u_k(x) - u(x)|^2 G(x,1)\,dx.$$

From Lemma 2.5 and boundedness of u_k in $\mathcal{H}$, we deduce that

$$B_k(R) \leqslant R^{-2} \int_{\{|x|>R\}} |x|^2 |u_k(x) - u(x)|^2 G(x,1)\, dx \tag{36}$$

$$\leqslant \frac{1}{R^2} \bigg(16 \int_{\mathbb{R}^N} |\nabla(u_k - u)(x)|^2 G(x,1)\, dx + 4N \int_{\mathbb{R}^N} |u_k(x) - u(x)|^2 G(x,1)\, dx \bigg) \leqslant \frac{\text{const}}{R^2}.$$

Combining (34), (35), and (36), we obtain that $u_k \to u$ strongly in $\mathcal{L}$. □

From classical spectral theory we deduce the following abstract description of the spectrum of L.

Lemma 3.2. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ such that (16) holds. Then the spectrum of the operator L defined in (14) consists of a diverging sequence of real eigenvalues with finite multiplicity. Moreover, there exists an orthonormal basis of $\mathcal{L}$ whose elements belong to $\mathcal{H}$ and are eigenfunctions of L.*

Proof. By Corollary 2.3 and the Lax-Milgram Theorem, the bounded linear self-adjoint operator

$$T : \mathcal{L} \to \mathcal{L}, \quad T = \Big(L + \frac{N-2}{4}\,\mathrm{Id} \Big)^{-1}$$

is well defined. Moreover, by Lemma 3.1, T is compact. The result then follows from the Spectral Theorem. □

Let us now compute explicitly the eigenvalues of L with the corresponding multiplicities and eigenfunctions by proving Proposition 1.4.

Proof of Proposition 1.4. Assume that γ is an eigenvalue of L and $g \in \mathcal{H}\setminus\{0\}$ is a corresponding eigenfunction, so that

$$-\Delta g(x) + \frac{\nabla g(x)\cdot x}{2} - \frac{a(x/|x|)}{|x|^2}\, g(x) = \gamma\, g(x) \tag{37}$$

in a weak $\mathcal{H}$-sense. From classical regularity theory for elliptic equations, $g \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N \setminus \{0\})$. Hence g can be expanded as

$$g(x) = g(r\theta) = \sum_{k=1}^{\infty} \phi_k(r)\psi_k(\theta) \quad \text{in } L^2(\mathbb{S}^{N-1}),$$

where $r = |x| \in (0,+\infty)$, $\theta = x/|x| \in \mathbb{S}^{N-1}$, and

$$\phi_k(r) = \int_{\mathbb{S}^{N-1}} g(r\theta)\psi_k(\theta)\, dS(\theta).$$

Equations (17) and (37) imply that, for every k,

$$\phi_k'' + \Big(\frac{N-1}{r} - \frac{r}{2} \Big)\phi_k' + \Big(\gamma - \frac{\mu_k}{r^2} \Big)\phi_k = 0 \quad \text{in } (0,+\infty). \tag{38}$$

Since $g \in \mathcal{H}$, we have that

(39)

$$\infty > \int_{\mathbb{R}^N} g^2(x) G(x,1)\, dx = \int_0^\infty \bigg(\int_{\mathbb{S}^{N-1}} g^2(r\theta)\, dS(\theta) \bigg) r^{N-1} e^{-\frac{r^2}{4}}\, dr \geqslant \int_0^\infty r^{N-1} e^{-\frac{r^2}{4}} \phi_k^2(r)\, dr$$

and, by the Hardy type inequality of Lemma 2.1,

$$\infty > \int_{\mathbb{R}^N} \frac{g^2(x)}{|x|^2} G(x,1)\, dx \geqslant \int_0^\infty r^{N-3} e^{-\frac{r^2}{4}} \phi_k^2(r)\, dr. \tag{40}$$

For all $k = 1, 2, \dots$ and $t > 0$, we define $w_k(t) = (4t)^{\frac{\alpha_k}{2}} \phi_k(\sqrt{4t})$, with $\alpha_k = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(a)}$. From (38), w_k satisfies

$$tw_k''(t) + \left(\frac{N}{2} - \alpha_k - t\right) w_k'(t) + \left(\frac{\alpha_k}{2} + \gamma\right) w_k(t) = 0 \quad \text{in } (0, +\infty).$$

Therefore, w_k is a solution of the well known Kummer Confluent Hypergeometric Equation (see [3] and [23]). Then there exist $A_k, B_k \in \mathbb{R}$ such that

$$w_k(t) = A_k M\Big(-\frac{\alpha_k}{2} - \gamma, \frac{N}{2} - \alpha_k, t\Big) + B_k U\Big(-\frac{\alpha_k}{2} - \gamma, \frac{N}{2} - \alpha_k, t\Big), \quad t \in (0, +\infty).$$

Here $M(c, b, t)$ and, respectively, $U(c, b, t)$ denote the Kummer function (or confluent hypergeometric function) and, respectively, the Tricomi function (or confluent hypergeometric function of the second kind); $M(c, b, t)$ and $U(c, b, t)$ are two linearly independent solutions to the Kummer Confluent Hypergeometric Equation

$$tw''(t) + (b - t)w'(t) - ct = 0, \quad t \in (0, +\infty).$$

Since $\left(\frac{N}{2} - \alpha_k\right) > 1$, from the well-known asymptotics of U at 0 (see e.g. [3]), we have that

$$U\Big(-\frac{\alpha_k}{2} - \gamma, \frac{N}{2} - \alpha_k, t\Big) \sim \text{const}\, t^{1-\frac{N}{2}+\alpha_k} \quad \text{as } t \to 0^+,$$

for some const $\neq 0$ depending only on N, γ, and α_k. On the other hand, M is the sum of the series

$$M(c, b, t) = \sum_{n=0}^{\infty} \frac{(c)_n}{(b)_n} \frac{t^n}{n!}.$$

We notice that M has a finite limit at 0^+, while its behavior at ∞ is singular and depends on the value $-c = \frac{\alpha_k}{2} + \gamma$. If $\frac{\alpha_k}{2} + \gamma = m \in \mathbb{N} = \{0, 1, 2, \cdots\}$, then $M\big(-\frac{\alpha_k}{2} - \gamma, \frac{N}{2} - \alpha_k, t\big)$ is a polynomial of degree m in t, which we will denote as $P_{k,m}$, i.e.,

$$P_{k,m}(t) = M\Big(-m, \tfrac{N}{2} - \alpha_k, t\Big) = \sum_{n=0}^{m} \frac{(-m)_n}{\left(\frac{N}{2} - \alpha_k\right)_n} \frac{t^n}{n!}.$$

If $\left(\frac{\alpha_k}{2} + \gamma\right) \notin \mathbb{N}$, then from the well-known asymptotics of M at ∞ (see e.g. [3]) we have that

$$M\Big(-\frac{\alpha_k}{2} - \gamma, \frac{N}{2} - \alpha_k, t\Big) \sim \text{const}\, e^t t^{-\frac{N}{2}+\frac{\alpha_k}{2}-\gamma} \quad \text{as } t \to +\infty,$$

for some const $\neq 0$ depending only on N, γ, and α_k.

Now, let us fix $k \in \mathbb{N}$, $k \geqslant 1$. From the above description, we have that

$$w_k(t) \sim \text{const}\, B_k t^{1-\frac{N}{2}+\alpha_k} \quad \text{as } t \to 0^+,$$

for some const $\neq 0$, and hence

$$\phi_k(r) = r^{-\alpha_k} w_k\Big(\frac{r^2}{4}\Big) \sim \text{const}\, B_k r^{2-N+\alpha_k} \quad \text{as } r \to 0^+,$$

for some const $\neq 0$. Therefore, condition (40) can be satisfied only for $B_k = 0$. If $\frac{\alpha_k}{2} + \gamma \notin \mathbb{N}$, then

$$w_k(t) \sim \text{const}\, A_k e^t t^{-\frac{N}{2}+\frac{\alpha_k}{2}-\gamma} \quad \text{as } t \to +\infty,$$

for some const $\neq 0$, and hence

$$\phi_k(r) = r^{-\alpha_k} w_k\Big(\frac{r^2}{4}\Big) \sim \text{const}\, A_k r^{-N-2\gamma} e^{r^2/4} \quad \text{as } r \to +\infty,$$

for some const $\neq 0$. Therefore, condition (39) can be satisfied only for $A_k = 0$. If $\frac{\alpha_k}{2} + \gamma = m \in \mathbb{N}$, then $r^{-\alpha_k} P_{k,m}\big(\frac{r^2}{4}\big)$ solves (38); moreover the function

$$|x|^{-\alpha_k} P_{k,m}\Big(\frac{|x|^2}{4}\Big)\psi_k\Big(\frac{x}{|x|}\Big)$$

belongs to $\mathcal{H}$, thus providing an eigenfunction of L.

We can conclude from the above discussion that if $\frac{\alpha_k}{2} + \gamma \notin \mathbb{N}$ for all $k \in \mathbb{N}$, $k \geqslant 1$, then γ is not an eigenvalue of L. On the other hand, if there exist $k_0, m_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that

$$\gamma = \gamma_{m_0,k_0} = m_0 - \frac{\alpha_{k_0}}{2}$$

then γ is an eigenvalue of L with multiplicity

$$m(\gamma) = m(\gamma_{m_0,k_0}) = \#\Big\{ j \in \mathbb{N}, j \geqslant 1 : \gamma_{m_0,k_0} + \frac{\alpha_j}{2} \in \mathbb{N}\Big\} < +\infty \tag{41}$$

and a basis of the corresponding eigenspace is

$$\bigg\{ |x|^{-\alpha_j} P_{j,\gamma_{m_0,k_0}+\alpha_j/2}\Big(\frac{|x|^2}{4}\Big)\psi_j\Big(\frac{x}{|x|}\Big) : j \in \mathbb{N}, j \geqslant 1, \gamma_{m_0,k_0} + \frac{\alpha_j}{2} \in \mathbb{N}\bigg\}.$$

The proof is thereby complete. □

Remark 3.3. If $a(\theta) \equiv 0$, then $\mu_k(0) = k(N + k - 2)$, so that $\alpha_k = \frac{(N-2)}{2} - \sqrt{\big(\frac{N-2}{2} + k\big)^2} = -k$, and $\gamma_{m,k} = \frac{k}{2} + m$. Hence, in this case we recover the well known fact (see e.g. [8] and [14]) that the eigenvalues of the Ornstein-Uhlenbeck operator $-\Delta + \frac{x}{2}\cdot\nabla$ are the positive half-integer numbers.

Remark 3.4. Due to orthogonality of eigenfunctions $\{\psi_k\}_k$ in $L^2(\mathbb{S}^{N-1})$, it is easy to verify that

$$\text{if } (m_1, k_1) \neq (m_2, k_2) \quad \text{then} \quad V_{m_1,k_1} \text{ and } V_{m_2,k_2} \text{ are orthogonal in } \mathcal{L}.$$

By Lemma 3.2, it follows that

$$\Big\{ \widetilde{V}_{n,j} = \frac{V_{n,j}}{\|V_{n,j}\|_{\mathcal{L}}} : j, n \in \mathbb{N}, j \geqslant 1\Big\}$$

is an orthonormal basis of $\mathcal{L}$.

4. The parabolic Almgren monotonicity formula

Throughout this section, we will assume that $a \in L^\infty(\mathbb{S}^{N-1})$ satisfies (16) and *either*

(I) u is a weak solution to (2) with h satisfying (3) and (4)

or

(II) u satisfies (7–8) and weakly solves (6) for some $\varphi \in C^1(\mathbb{R}^N \times (0,T) \times \mathbb{R})$ satisfying (5).

We denote as

$$f(x,t,s)=\begin{cases} h(x,t)s, & \text{in case } \textbf{(I)},\\ \varphi(x,t,s), & \text{in case } \textbf{(II)},\end{cases}$$

so that, in both cases, u is a weak solution to (1) in $\mathbb{R}^N\times(0,T)$ in the sense of Definition 1.1. Let

$$\overline{T}=\begin{cases} \overline{T}_1, & \text{in case } \textbf{(I)},\\ \overline{T}_2(R_0), & \text{in case } \textbf{(II)},\end{cases} \quad\text{and}\quad C_1=\begin{cases} C_1', & \text{in case } \textbf{(I)},\\ C_1'', & \text{in case } \textbf{(II)},\end{cases} \tag{42}$$

being $C_1',\overline{T}_1$ as in Corollary 2.4 and $C_1'',\overline{T}_2(R_0)$ as in Corollary 2.9 with

$$R_0=\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^{p+1}(\mathbb{R}^N)}$$

(notice that R_0 is finite by assumption (7)). We denote

$$\alpha=\frac{T}{2\big(\lfloor T/\overline{T}\rfloor+1\big)},$$

where $\lfloor\cdot\rfloor$ denotes the floor function, i.e. $\lfloor x\rfloor:=\max\{j\in\mathbb{Z}:\ j\leqslant x\}$. Then

$$(0,T)=\bigcup_{i=1}^{k}(a_i,b_i)$$

where

$$k=2\big(\lfloor T/\overline{T}\rfloor+1\big)-1,\quad a_i=(i-1)\alpha,\quad\text{and}\quad b_i=(i+1)\alpha.$$

We notice that $0<2\alpha<\overline{T}$ and $(a_i,b_i)\cap(a_{i+1},b_{i+1})=(i\alpha,(i+1)\alpha)\neq\emptyset$. For every i, $1\leqslant i\leqslant k$, we define

$$u_i(x,t)=u(x,t+a_i),\quad x\in\mathbb{R}^N,\ t\in(0,2\alpha). \tag{43}$$

Lemma 4.1. *For every $i=1,\dots,k$, the function u_i defined in (43) is a weak solution to*

$$(u_i)_t+\Delta u_i+\frac{a(x/|x|)}{|x|^2}\,u_i+f(x,t+a_i,u_i(x,t))=0 \tag{44}$$

in $\mathbb{R}^N\times(0,2\alpha)$ in the sense of Definition 1.1. Furthermore, the function $v_i(x,t):=u_i(\sqrt{t}x,t)$ is a weak solution to

$$(v_i)_t+\frac{1}{t}\bigg(\Delta v_i-\frac{x}{2}\cdot\nabla v_i+\frac{a(x/|x|)}{|x|^2}\,v_i+tf\big(\sqrt{t}x,t+a_i,v_i(x,t)\big)\bigg)=0 \tag{45}$$

in $\mathbb{R}^N\times(0,2\alpha)$ in the sense of Remark 1.3.

Proof. If $i=1$, then $a_1=0$, $u_1(x,t)=u(x,t)$ in $\mathbb{R}^N\times(0,2\alpha)$, and we immediately conclude. For every $1<i\leqslant k$, $a_i\neq0$, and, being $G(x,t)$ as in (9), the following properties hold for all

$t \in (a_i, b_i)$:

(i) $G\big(x, \frac{t(t-a_i)}{a_i}\big)G(x,t) = \big(\frac{t^2}{a_i}\big)^{-N/2} G(x, t-a_i)$;

(ii) if $\phi \in \mathcal{H}_{t-a_i}$, then $\phi\, G\big(\cdot, \frac{t(t-a_i)}{a_i}\big) \in \mathcal{H}_t$;

(iii) if $\psi \in (\mathcal{H}_t)^\star$, then $\psi \in (\mathcal{H}_{t-a_i})^\star$ and

$$ {}_{\mathcal{H}^\star_{t-a_i}}\big\langle \psi, \phi \big\rangle_{\mathcal{H}_{t-a_i}} = \left(\frac{t^2}{a_i}\right)^{\frac{N}{2}} {}_{\mathcal{H}^\star_t}\Big\langle \psi, \phi\, G\big(\cdot, \tfrac{t(t-a_i)}{a_i}\big)\Big\rangle_{\mathcal{H}_t}, \quad \text{for all } \phi \in \mathcal{H}_{t-a_i}. $$

Let $1 < i \leqslant k$ and $\phi \in \mathcal{H}_{t-a_i}$. Due to (ii), $\phi\, G\big(\cdot, \frac{t(t-a_i)}{a_i}\big) \in \mathcal{H}_t$ and then, since u is a solution to (1) in the sense of of Definition 1.1, for a.e. $t \in (a_i, b_i)$ we have

$$ (46) \qquad {}_{\mathcal{H}^\star_t}\Big\langle u_t + \frac{\nabla u \cdot x}{2t}, \phi\, G\big(x, \tfrac{t(t-a_i)}{a_i}\big)\Big\rangle_{\mathcal{H}_t} $$
$$ = \int_{\mathbb{R}^N} \nabla u(x,t) \cdot \nabla \phi(x)\, G\big(x, \tfrac{t(t-a_i)}{a_i}\big) G(x,t)\, dx - \int_{\mathbb{R}^N} \phi(x) \frac{a_i x \cdot \nabla u(x,t)}{2(t-a_i)t} G\big(x, \tfrac{t(t-a_i)}{a_i}\big) G(x,t)\, dx $$
$$ - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2}\, u(x,t)\phi(x) G\big(x, \tfrac{t(t-a_i)}{a_i}\big) G(x,t)\, dx - \int_{\mathbb{R}^N} f(x,t,u(x,t))\phi(x) G\big(x, \tfrac{t(t-a_i)}{a_i}\big) G(x,t)\, dx. $$

Therefore, thanks to (i) and (iii), we obtain

$$ {}_{\mathcal{H}^\star_{t-a_i}}\Big\langle u_t + \frac{\nabla u(x,t)\cdot x}{2(t-a_i)}, \phi \Big\rangle_{\mathcal{H}_{t-a_i}} = \int_{\mathbb{R}^N} \Big(\nabla u(x,t) \cdot \nabla \phi(x) - \frac{a(x/|x|)}{|x|^2}\, u(x,t)\phi(x)\Big) G(x, t-a_i)\, dx $$
$$ - \int_{\mathbb{R}^N} f(x,t,u(x,t))\phi(x) G(x, t-a_i)\, dx. $$

By the change of variables $s = t - a_i$, we conclude that $u_i(x,t) = u(x, t+a_i)$ is a weak solution to (44) in $\mathbb{R}^N \times (0, 2\alpha)$ in the sense of Definition 1.1. By a further change of variables, we easily obtain that $v_i(x,t) := u_i(\sqrt{t}x, t)$ is a weak solution to (45) in $\mathbb{R}^N \times (0, 2\alpha)$ in the sense of Remark 1.3. □

For every $i = 1, \dots, k$, we define

$$ (47) \qquad H_i(t) = \int_{\mathbb{R}^N} u_i^2(x,t)\, G(x,t)\, dx, \quad \text{for every } t \in (0, 2\alpha), $$

and

$$ (48) \qquad D_i(t) = \int_{\mathbb{R}^N} \bigg(|\nabla u_i(x,t)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} u_i^2(x,t) - f(x, t+a_i, u_i(x,t)) u_i(x,t) \bigg) G(x,t)\, dx $$

for a.e. $t \in (0, 2\alpha)$.

Lemma 4.2. *For every $1 \leqslant i \leqslant k$, $H_i \in W^{1,1}_{\rm loc}(0, 2\alpha)$ and*

$$ (49) \qquad H_i'(t) = 2\, {}_{\mathcal{H}^\star_t}\Big\langle (u_i)_t + \frac{\nabla u_i \cdot x}{2t}, u_i(\cdot, t) \Big\rangle_{\mathcal{H}_t} = 2 D_i(t) \quad \textit{for a.e. } t \in (0, 2\alpha). $$

Proof. It follows from Lemma 4.1 and Remark 1.2. □

Lemma 4.3. *If C_1 is as in (42), then, for every $i = 1, \dots, k$, the function*

$$t \mapsto t^{-2C_1+\frac{N-2}{2}} H_i(t)$$

is nondecreasing in $(0, 2\alpha)$.

Proof. From Lemma 4.2 and Corollaries 2.4 and 2.9, taking into account that $2\alpha < \overline{T}$, we have that, for all $t \in (0, 2\alpha)$,

$$H_i'(t) \geqslant \frac{1}{t}\left(2C_1 - \frac{N-2}{2}\right) H_i(t),$$

which implies

$$\frac{d}{dt}\left(t^{-2C_1+\frac{N-2}{2}} H_i(t)\right) \geqslant 0.$$

Hence the function $t \mapsto t^{-2C_1+\frac{N-2}{2}} H_i(t)$ is nondecreasing in $(0, 2\alpha)$. □

Lemma 4.4. *If $1 \leqslant i \leqslant k$ and $H_i(\bar{t}) = 0$ for some $\bar{t} \in (0, 2\alpha)$, then $H_i(t) = 0$ for all $t \in (0, \bar{t}\,]$.*

Proof. From Lemma 4.3, the function $t \mapsto t^{-2C_1+\frac{N-2}{2}} H_i(t)$ is nondecreasing in $(0, 2\alpha)$, nonnegative, and vanishing at $\bar{t}$. It follows that $H_i(t) = 0$ for all $t \in (0, \bar{t}]$. □

The regularity of D_i in $(0, 2\alpha)$ is analyzed in the following lemma.

Lemma 4.5. *If $1 \leqslant i \leqslant k$ and $T_i \in (0, 2\alpha)$ is such that $u_i(\cdot, T_i) \in \mathcal{H}_{T_i}$, then*

(i) $\displaystyle\int_\tau^{T_i} \int_{\mathbb{R}^N} \left(\left| (u_i)_t(x,t) + \frac{\nabla u_i(x,t)\cdot x}{2t} \right|^2 G(x,t)\, dx \right) dt < +\infty$ *for all $\tau \in (0, T_i)$;*

(ii) *the function*

$$t \mapsto tD_i(t)$$

belongs to $W^{1,1}_{\rm loc}(0, T_i)$ and its weak derivative is, for a.e. $t \in (0, T_i)$, as follows:

in case **(I)**

$$\begin{aligned}\frac{d}{dt}\big(tD_i(t)\big) &= 2t \int_{\mathbb{R}^N} \left| (u_i)_t(x,t) + \frac{\nabla u_i(x,t)\cdot x}{2t} \right|^2 G(x,t)\, dx \\ &\quad + \int_{\mathbb{R}^N} h(x, t+a_i) \left(\frac{N-2}{2} u_i^2(x,t) + (\nabla u_i(x,t)\cdot x) u_i(x,t) - \frac{|x|^2}{4t} u_i^2(x,t) \right) G(x,t)\, dx \\ &\quad - t \int_{\mathbb{R}^N} h_t(x, t+a_i) u_i^2(x,t) G(x,t)\, dx;\end{aligned}$$

in case **(II)**

$$\frac{d}{dt}\big(tD_i(t)\big) = 2t\int_{\mathbb{R}^N}\left|(u_i)_t(x,t)+\frac{\nabla u_i(x,t)\cdot x}{2t}\right|^2 G(x,t)\,dx$$
$$+\,t\int_{\mathbb{R}^N}\left(\varphi(x,t+a_i,u_i(x,t))-\frac{\partial\varphi}{\partial u_i}(x,t+a_i,u_i(x,t))u_i(x,t)\right)(u_i)_t(x,t)G(x,t)\,dx$$
$$+\int_{\mathbb{R}^N}\left(\frac{N-2}{2}\varphi(x,t+a_i,u_i(x,t))u_i(x,t)-t\frac{\partial\varphi}{\partial t}(x,t+a_i,u_i(x,t))u_i(x,t)\right.$$
$$\left.-N\Phi(x,t+a_i,u_i(x,t))-\nabla_x\Phi(x,t+a_i,u_i(x,t))\cdot x\right)G(x,t)\,dx$$
$$+\int_{\mathbb{R}^N}\frac{|x|^2}{4t}\left(2\Phi(x,t+a_i,u_i(x,t))-\varphi(x,t+a_i,u_i(x,t))u_i(x,t)\right)G(x,t)\,dx$$

where

$$\Phi(x,t,s)=\int_0^s\varphi(x,t,\xi)\,d\xi.$$

Proof. Let us first consider case **(I)**, i.e. $f(x,t,u)=h(x,t)u$, with $h(x,t)$ under conditions (3–4). We test equation (45) with $(v_i)_t$; we notice that this is not an admissible test function for equation (45) since a priori $(v_i)_t$ does not take values in $\mathcal{H}$. However the formal testing procedure can be made rigorous by a suitable approximation. Such a test combined with Corollary 2.4 yields, for all $t\in(0,T_i)$,

$$\int_t^{T_i} s\left(\int_{\mathbb{R}^N}(v_i)_t^2(x,s)G(x,1)\,dx\right)ds \leqslant \text{const}\left(\|u_i(\sqrt{T_i}\,\cdot,T_i)\|^2_{\mathcal{H}}+\int_{\mathbb{R}^N}v_i^2(x,t)G(x,1)\,dx\right.$$
$$+\int_t^{T_i}\left(\int_{\mathbb{R}^N}h(\sqrt{s}x,s+a_i)\left(\frac{|x|^2}{8}v_i^2(x,s)-\frac{\nabla v_i(x,s)\cdot x}{2}v_i(x,s)-\frac{N-2}{4}v_i^2(x,s)\right)G(x,1)\,dx\right)ds$$
$$\left.+\frac{1}{2}\int_t^{T_i}s\left(\int_{\mathbb{R}^N}h_s(\sqrt{s}x,s+a_i)v_i^2(x,s)G(x,1)\,dx\right)ds\right).$$

Since, in view of (3–4) and Lemmas 2.1 and 2.5, the integrals in the last two lines of the previous formula are finite for every $t\in(0,T_i)$, we conclude that

$$(v_i)_t\in L^2(\tau,T_i;\mathcal{L})\quad\text{for all }\tau\in(0,T_i).$$

Testing (45) with $(v_i)_t$ also yields

$$\begin{aligned}
&\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}(v_i)_t^2(x,s)G(x,1)\,dx\bigg)\,ds\\
&+\frac12\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-th(\sqrt t x,t+a_i)v_i^2(x,t)\bigg)G(x,1)\,dx\\
&=\frac12\int_{\mathbb{R}^N}\bigg(|\nabla v_{0,i}(x)|^2-\frac{a(x/|x|)}{|x|^2}\,v_{0,i}^2(x)-T_ih(\sqrt{T_i}x,T_i+a_i)v_{0,i}^2(x)\bigg)G(x,1)\,dx\\
&+\int_t^{T_i}\bigg(\int_{\mathbb{R}^N}h(\sqrt s x,s+a_i)\bigg(\frac{|x|^2}{8}v_i^2(x,s)-\frac{\nabla v_i(x,s)\cdot x}{2}v_i(x,s)-\frac{N-2}{4}v_i^2(x,s)\bigg)G(x,1)\,dx\bigg)\,ds\\
&+\frac12\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}h_s(\sqrt s x,s+a_i)v_i^2(x,s)G(x,1)\,dx\bigg)\,ds,
\end{aligned}$$

for all $t\in(0,T_i)$, where $v_{0,i}(x):=u_i(\sqrt{T_i}x,T_i)\in\mathcal H$. Therefore the function

$$t\mapsto\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-th(\sqrt t x,t+a_i)v_i^2(x,t)\bigg)G(x,1)\,dx$$

is absolutely continuous in (τ,T_i) for all $\tau\in(0,T_i)$ and

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-th(\sqrt t x,t+a_i)v_i^2(x,t)\bigg)G(x,1)\,dx\\
&\quad=2t\int_{\mathbb{R}^N}(v_i)_t^2(x,t)G(x,1)\,dx\\
&\quad-\int_{\mathbb{R}^N}h(\sqrt t x,t+a_i)\bigg(\frac{|x|^2}{4}v_i^2(x,t)-(\nabla v_i(x,t)\cdot x)v_i(x,t)-\frac{N-2}{2}v_i^2(x,t)\bigg)G(x,1)\,dx\\
&\quad-t\int_{\mathbb{R}^N}h_s(\sqrt s x,s+a_i)v_i^2(x,s)G(x,1)\,dx.
\end{aligned}$$

The change of variables $u_i(x,t)=v_i(x/\sqrt t,t)$ leads to the conclusion in case **(I)**.

Let us now consider case **(II)**, i.e. $f(x,t,u)=\varphi(x,t,u)$ with φ satisfying (5) and u satisfying (7–8). We test equation (45) with $(v_i)_t$ (passing through a suitable approximation) and, by Corollary 2.9, we obtain, for all $t\in(0,T_i)$,

$$\begin{aligned}
\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}(v_i)_t^2(x,s)G(x,1)\,dx\bigg)\,ds&\leqslant \text{const}\,\bigg(\|u_i(\sqrt{T_i}\,\cdot,T_i)\|^2_{\mathcal H}+\int_{\mathbb{R}^N}v_i^2(x,t)G(x,1)\,dx\bigg)\\
&-\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}\varphi(\sqrt s x,s+a_i,v_i(x,t))(v_i)_t(x,t)G(x,1)\,dx\bigg)\,ds\\
&+\frac12\int_t^{T_i}\frac{d}{ds}\bigg(s\int_{\mathbb{R}^N}\varphi(\sqrt s x,s+a_i,v_i(x,t))v_i(x,s)G(x,1)\,dx\bigg)\,ds.
\end{aligned}$$

Since in view of hypothesis (5) on φ, conditions (7) and (8) on u, and Lemma 2.7 the integrals at the right hand side lines of the previous formula are finite for every $t\in(0,T_i)$, we conclude that

$$(v_i)_t\in L^2(\tau,T_i;\mathcal L)\quad\text{for all }\tau\in(0,T_i).$$

Testing (45) for v_i with $(v_i)_t$ also yields

$$
\begin{aligned}
&\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N} (v_i)_t^2(x,s)G(x,1)\,dx\bigg)\,ds\\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-t\varphi(\sqrt{t}x,t+a_i,v_i(x,t))v_i(x,t)\bigg)G(x,1)\,dx\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^N}\bigg(|\nabla v_{0,i}(x)|^2-\frac{a(x/|x|)}{|x|^2}\,v_{0,i}^2(x)-T_i\varphi(\sqrt{T_i}x,T+a_i,v_{0,i})v_{0,i}(x)\bigg)G(x,1)\,dx\\
&\quad-\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}\varphi(\sqrt{s}x,s+a_i,v_i(x,t))(v_i)_t(x,t)G(x,1)\,dx\bigg)\,ds\\
&\quad+\frac{1}{2}\int_t^{T_i}\frac{d}{ds}\bigg(s\int_{\mathbb{R}^N}\varphi(\sqrt{s}x,s+a_i,v_i(x,s))v_i(x,s)G(x,1)\,dx\bigg)\,ds,
\end{aligned}
$$

for a.e. $t\in(0,T_i)$, where $v_{0,i}(x):=u_i(\sqrt{T_i}x,T_i)\in\mathcal{H}$. Therefore the function

$$
t\mapsto\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-t\varphi(\sqrt{t}x,t+a_i,v_i(x,t))v_i(x,t)\bigg)G(x,1)\,dx
$$

is absolutely continuous in $(0,\tau)$ for all $\tau\in(0,T_i)$ and

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\bigg(|\nabla v_i(x,t)|^2-\frac{a(x/|x|)}{|x|^2}\,v_i^2(x,t)-t\varphi(\sqrt{t}x,t+a_i,v_i(x,t))v_i(x,t)\bigg)G(x,1)\,dx\\
&\quad=2t\int_{\mathbb{R}^N}(v_i)_t^2(x,t)G(x,1)\,dx+2t\int_{\mathbb{R}^N}\varphi(\sqrt{t}x,t+a_i,v_i(x,t))(v_i)_t(x,t)G(x,1)\,dx\\
&\qquad-\frac{d}{dt}\bigg(t\int_{\mathbb{R}^N}\varphi(\sqrt{t}x,t+a_i,v_i(x,t))v_i(x,t)G(x,1)\,dx\bigg).
\end{aligned}
$$

The change of variables $u_i(x,t)=v_i(x/\sqrt{t},t)$ leads to

$$
\begin{gathered}
\frac{d}{dt}(tD_i(t))=2t\int_{\mathbb{R}^N}\bigg|(u_i)_t(x,t)+\frac{\nabla u_i(x,t)\cdot x}{2t}\bigg|^2 G(x,t)\,dx\\
+2t\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))(u_i)_t(x,t)G(x,t)\,dx+\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))\nabla u_i(x,t)\cdot xG(x,t)\,dx\\
-\frac{d}{dt}\bigg(t\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))u_i(x,t)G(x,t)\,dx\bigg)
\end{gathered}
$$

and hence

$$\frac{d}{dt}(tD_i(t)) = 2t\int_{\mathbb{R}^N}\left|(u_i)_t(x,t)+\frac{\nabla u_i(x,t)\cdot x}{2t}\right|^2 G(x,t)\,dx$$
$$+2t\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))(u_i)_t(x,t)G(x,t)\,dx+\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))\nabla u_i(x,t)\cdot x\,G(x,t)\,dx$$
$$+\frac{N-2}{2}\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))u_i(x,t)G(x,t)\,dx-t\int_{\mathbb{R}^N}\frac{\partial\varphi}{\partial t}(x,t+a_i,u_i(x,t))u_i(x,t)G(x,t)\,dx$$
$$-t\int_{\mathbb{R}^N}\left(\frac{\partial\varphi}{\partial u_i}(x,t+a_i,u_i(x,t))u_i(x,t)+\varphi(x,t+a_i,u_i(x,t))\right)(u_i)_t(x,t)G(x,t)\,dx$$
$$-\int_{\mathbb{R}^N}\frac{|x|^2}{4t}\varphi(x,t+a_i,u_i(x,t))u_i(x,t)G(x,t)\,dx.$$

Integration by parts yields (these formal computations can be made rigorous through a suitable approximation)

$$\int_{\mathbb{R}^N}\varphi(x,t+a_i,u_i(x,t))\nabla u_i(x,t)\cdot x\,G(x,t)\,dx = -N\int_{\mathbb{R}^N}\Phi(x,t+a_i,u_i(x,t))G(x,t)\,dx$$
$$+\int_{\mathbb{R}^N}\frac{|x|^2}{2t}\Phi(x,t+a_i,u_i(x,t))G(x,t)\,dx-\int_{\mathbb{R}^N}\nabla_x\Phi(x,t+a_i,u_i(x,t))\cdot xG(x,t)\,dx$$

thus yielding the conclusion in case **(II)**. □

For all $i=1,\dots,k$, let us introduce the *Almgren type frequency function* associated to u_i

(50) $$N_i:(0,2\alpha)\to\mathbb{R}\cup\{-\infty,+\infty\},\quad N_i(t):=\frac{tD_i(t)}{H_i(t)}.$$

Frequency functions associated to unperturbed parabolic equations of type (1) (i.e. in the case $f(x,t,s)\equiv 0$) were first studied by C.-C. Poon in [25], where unique continuation properties are derived by proving monotonicity of the quotient in (50). Due to the presence of the perturbing function $f(x,t+a_i,u(x,t))$, the functions N_i will not be nondecreasing as in the case treated by Poon; however in both cases **(I)** and **(II)**, we can prove that their derivatives are integrable perturbations of nonnegative functions wherever the N_i's assume finite values. Moreover our analysis will show that actually the N_i's assume finite values all over $(0,2\alpha)$.

Lemma 4.6. *Let $i\in\{1,\dots,k\}$. If there exist $\beta_i,T_i\in(0,2\alpha)$ such that*

(51) $$\beta_i<T_i,\quad H_i(t)>0\ \text{for all}\ t\in(\beta_i,T_i),\quad\text{and}\quad u_i(\cdot,T_i)\in\mathcal{H}_{T_i},$$

then the function N_i defined in (50) *belongs to $W^{1,1}_{\rm loc}(\beta_i,T_i)$ and*

$$N_i'(t)=\nu_{1i}(t)+\nu_{2i}(t)$$

in a distributional sense and a.e. in (β_i,T_i) where

$$\nu_{1i}(t)=\frac{2t}{H_i^2(t)}\Bigg(\bigg(\int_{\mathbb{R}^N}\left|(u_i)_t(x,t)+\frac{\nabla u_i(x,t)\cdot x}{2t}\right|^2 G(x,t)\,dx\bigg)\bigg(\int_{\mathbb{R}^N}u_i^2(x,t)\,G(x,t)\,dx\bigg)$$
$$-\bigg(\int_{\mathbb{R}^N}\Big((u_i)_t(x,t)+\frac{\nabla u_i(x,t)\cdot x}{2t}\Big)u_i(x,t)G(x,t)\,dx\bigg)^2\Bigg)$$

and ν_{2i} is as follows:

in case **(I)**

$$\nu_{2i}(t) = \frac{1}{H_i(t)} \int_{\mathbb{R}^N} h(x,t+a_i) \left(\frac{N-2}{2} u_i^2(x,t) + (\nabla u_i(x,t) \cdot x) u_i(x,t) - \frac{|x|^2}{4t} u_i^2(x,t) \right) G(x,t)\, dx$$
$$- \frac{t}{H_i(t)} \left(\int_{\mathbb{R}^N} h_t(x,t+a_i) u_i^2(x,t) G(x,t)\, dx \right),$$

in case **(II)**

$$\begin{aligned}\nu_{2i}(t) = \frac{1}{H_i(t)} \bigg(t \int_{\mathbb{R}^N} \Big(\varphi(x,t+a_i,u_i(x,t)) - \frac{\partial \varphi}{\partial u_i}(x,t+a_i,u_i(x,t)) u_i(x,t) \Big) (u_i)_t(x,t) G(x,t)\, dx \\ + \int_{\mathbb{R}^N} \Big(\frac{N-2}{2} \varphi(x,t+a_i,u_i(x,t)) u_i(x,t) - t \frac{\partial \varphi}{\partial t}(x,t+a_i,u_i(x,t)) u_i(x,t) \\ - N\Phi(x,t+a_i,u_i(x,t)) - \nabla_x \Phi(x,t+a_i,u_i(x,t)) \cdot x \Big) G(x,t)\, dx \\ + \int_{\mathbb{R}^N} \frac{|x|^2}{4t} \Big(2\Phi(x,t+a_i,u_i(x,t)) - \varphi(x,t+a_i,u_i(x,t)) u_i(x,t) \Big) G(x,t)\, dx \bigg).\end{aligned}$$

Proof. From Lemma 4.2 and 4.5, it follows that $N_i \in W^{1,1}_{\rm loc}(\beta_i, T_i)$. From (49) we deduce that

$$N_i'(t) = \frac{(tD_i(t))' H_i(t) - tD_i(t) H_i'(t)}{H_i^2(t)} = \frac{(tD_i(t))' H_i(t) - 2tD_i^2(t)}{H_i^2(t)},$$

which yields the conclusion in view of (47), (48), and Lemma 4.5. □

The term ν_{2i} can estimated as follows.

Lemma 4.7. *There exists $C_3 > 0$ such that, if $i \in \{1, \dots, k\}$ and $\beta_i, T_i \in (0, 2\alpha)$ satisfy (51), then*

$$\big|\nu_{2i}(t)\big| \leqslant \begin{cases} C_3 \big(N_i(t) + \frac{N-2}{4}\big) \big(t^{-1+\varepsilon/2} + \|h_t(\cdot, t+a_i)\|_{L^{N/2}(\mathbb{R}^N)}\big), & \text{in case (I)}, \\ C_3 \big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{in case (II) if } i = 1, \\ C_3 \beta_i^{-1} \big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{in case (II) if } i > 1, \end{cases}$$

for a.e. $t \in (\beta_i, T_i)$, where ν_{2i} is as in Lemma 4.6.

Proof. Let us first consider case **(I)**, i.e. $f(x,t,u) = h(x,t)u$, with $h(x,t)$ under conditions (3–4). In order to estimate ν_{2i} we observe that, from (4),

$$
\begin{aligned}
(52)\quad & \left| \int_{\mathbb{R}^N} h(x,t+a_i)(\nabla u_i(x,t)\cdot x)u_i(x,t)G(x,t)\,dx \right| \\
& \leqslant C_h \int_{\mathbb{R}^N} (1+|x|^{-2+\varepsilon})|\nabla u_i(x,t)||x||u_i(x,t)|G(x,t)\,dx \\
& \leqslant C_h t \int_{\mathbb{R}^N} |\nabla u_i(x,t)| \frac{|x|}{t}|u_i(x,t)|G(x,t)\,dx + C_h t^{\varepsilon/2} \int_{\{|x|\leqslant\sqrt{t}\}} |\nabla u_i(x,t)| \frac{|u_i(x,t)|}{|x|} G(x,t)\,dx \\
& \qquad + C_h t^{\varepsilon/2} \int_{\{|x|\geqslant\sqrt{t}\}} |\nabla u_i(x,t)| \frac{|x|}{t}|u_i(x,t)|G(x,t)\,dx \\
& \leqslant \frac{1}{2}C_h(t+t^{\varepsilon/2}) \int_{\mathbb{R}^N} |\nabla u_i(x,t)|^2 G(x,t)\,dx + \frac{1}{2}C_h(t+t^{\varepsilon/2}) \int_{\mathbb{R}^N} \frac{|x|^2}{t^2} u_i^2(x,t)G(x,t)\,dx \\
& \qquad + \frac{1}{2}C_h t^{\varepsilon/2} \int_{\mathbb{R}^N} |\nabla u_i(x,t)|^2 G(x,t)\,dx + \frac{1}{2}C_h t^{\varepsilon/2} \int_{\mathbb{R}^N} \frac{u_i^2(x,t)}{|x|^2} G(x,t)\,dx \\
& \leqslant \frac{1}{2}C_h t^{\varepsilon/2}(2+\overline{T}^{1-\varepsilon/2}) \int_{\mathbb{R}^N} |\nabla u_i(x,t)|^2 G(x,t)\,dx \\
& \quad + \frac{1}{2}C_h t^{\varepsilon/2}(1+\overline{T}^{1-\varepsilon/2}) \int_{\mathbb{R}^N} \frac{|x|^2}{t^2} u_i^2(x,t)G(x,t)\,dx + \frac{1}{2}C_h t^{\varepsilon/2} \int_{\mathbb{R}^N} \frac{u_i^2(x,t)}{|x|^2} G(x,t)\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
(53)\quad & \int_{\mathbb{R}^N} |h(x,t+a_i)||x|^2 u_i^2(x,t)G(x,t)\,dx \leqslant C_h \int_{\mathbb{R}^N} |x|^2 u_i^2(x,t)G(x,t)\,dx \\
& \qquad\qquad + C_h \int_{\mathbb{R}^N} |x|^{-2+\varepsilon}|x|^2 u_i^2(x,t)G(x,t)\,dx \\
& \leqslant C_h \int_{\mathbb{R}^N} |x|^2 u_i^2(x,t)G(x,t)\,dx + C_h t^{\varepsilon/2} \int_{\{|x|\leqslant\sqrt{t}\}} u_i^2(x,t)G(x,t)\,dx \\
& \qquad\qquad + C_h t^{-1+\varepsilon/2} \int_{\{|x|\geqslant\sqrt{t}\}} |x|^2 u_i^2(x,t)G(x,t)\,dx \\
& \leqslant C_h t^{-1+\varepsilon/2}(1+\overline{T}^{1-\varepsilon/2}) \int_{\mathbb{R}^N} |x|^2 u_i^2(x,t)G(x,t)\,dx + C_h t^{\varepsilon/2} \int_{\mathbb{R}^N} u_i^2(x,t)G(x,t)\,dx,
\end{aligned}
$$

for a.e. $t \in (\beta_i, T_i)$. Moreover, by Hölder's inequality and Corollary 2.8,

$$
(54)\qquad \int_{\mathbb{R}^N} |h_t(x,t+a_i)|u_i^2(x,t)G(x,t)\,dx \leqslant C_{2^*} t^{-1} \|u_i\|^2_{\mathcal{H}_t} \|h_t(\cdot,t+a_i)\|_{L^{N/2}(\mathbb{R}^N)}
$$

for a.e. $t \in (\beta_i, T_i)$. Collecting (32), (52), (53) and (54), we obtain that

$$(55)\quad \Big|\nu_{2i}(t)\Big| \leqslant \frac{\mathrm{const}\, t^{\varepsilon/2}}{H_i(t)}\bigg(\frac{1}{t}\int_{\mathbb{R}^N} u_i^2(x,t)G(x,t)\,dx + \int_{\mathbb{R}^N}\frac{u_i^2(x,t)}{|x|^2}G(x,t)\,dx + \int_{\mathbb{R}^N}|\nabla u_i(x,t)|^2 G(x,t)\,dx + \frac{1}{t^2}\int_{\mathbb{R}^N}|x|^2 u_i^2(x,t)G(x,t)\,dx\bigg) + \frac{C_{2^*}}{H_i(t)}\|u_i\|^2_{\mathcal{H}_t}\|h_t(\cdot,t+a_i)\|_{L^{N/2}(\mathbb{R}^N)}.$$

From inequality (55), Lemma 2.1, Corollary 2.4, and Corollary 2.6, we deduce that there exists $C_3 > 0$ depending only on C_h, $\overline{T}$, and N, such that, for a.e. $t \in (\beta_i, T_i)$,

$$\Big|\nu_{2i}(t)\Big| \leqslant \frac{C_3}{H_i(t)}\Big(tD_i(t) + \tfrac{N-2}{4}H_i(t)\Big)\Big(t^{-1+\varepsilon/2} + \|h_t(\cdot,t+a_i)\|_{L^{N/2}(\mathbb{R}^N)}\Big) = C_3\big(N_i(t) + \tfrac{N-2}{4}\big)\Big(t^{-1+\varepsilon/2} + \|h_t(\cdot,t+a_i)\|_{L^{N/2}(\mathbb{R}^N)}\Big)$$

thus completing the proof in case **(I)**.

Let us now consider case **(II)**, i.e. $f(x,t,s) = \varphi(x,t,s)$ with φ under condition (5) and u satisfying (7) and (8). From (5), we have that

$$(56)\quad \Big|\nu_{2i}(t)\Big| \leqslant \frac{\mathrm{const}}{H_i(t)}\bigg(t\int_{\mathbb{R}^N}|u_i(x,t)|^q|(u_i)_t(x,t)|G(x,t)\,dx + \int_{\mathbb{R}^N}\big(|u_i(x,t)|^2 + |u_i(x,t)|^{p+1}\big)G(x,t)\,dx + \int_{\mathbb{R}^N}\frac{|x|^2}{t}\big(|u_i(x,t)|^2 + |u_i(x,t)|^{p+1}\big)G(x,t)\,dx\bigg).$$

From Hölder's inequality, Corollary 2.8, and assumptions (7–8), it follows that

$$(57)\quad t\int_{\mathbb{R}^N}|u_i(x,t)|^q|(u_i)_t(x,t)|G(x,t)\,dx \leqslant t\bigg(\int_{\mathbb{R}^N}|u_i(x,t)|^{p+1}G^{\frac{p+1}{2}}(x,t)\,dx\bigg)^{\frac{2}{p+1}}\|u(\cdot,t+a_i)\|^{q-2}_{L^{p+1}(\mathbb{R}^N)}\|u_t(\cdot,t+a_i)\|_{L^{\frac{p+1}{p+1-q}}(\mathbb{R}^N)} \leqslant \mathrm{const}\, t^{-\frac{N}{p+1}\frac{p-1}{2}}\|u_i\|^2_{\mathcal{H}_t}$$

and, taking into account also Corollary 2.6,

$$(58)\quad \int_{\mathbb{R}^N}\frac{|x|^2}{t}\big(|u_i(x,t)|^2 + |u_i(x,t)|^{p+1}\big)G(x,t)\,dx \leqslant \int_{\mathbb{R}^N}\frac{|x|^2}{t}|u_i(x,t)|^2G(x,t)\,dx + \frac{t+a_i}{t}\bigg(\int_{\mathbb{R}^N}|u_i(x,t)|^{p+1}G^{\frac{p+1}{2}}(x,t)\,dx\bigg)^{\frac{2}{p+1}}\bigg(\int_{\mathbb{R}^N}\bigg(\frac{|x|^2}{t+a_i}\bigg)^{\frac{p+1}{p-1}}|u(x,t+a_i)|^{p+1}\,dx\bigg)^{\frac{p-1}{p+1}}$$
$$\leqslant \begin{cases}\mathrm{const}\, t^{-\frac{N}{p+1}\frac{p-1}{2}}\|u_i\|^2_{\mathcal{H}_t}, & \text{if } i=1,\\[1ex] \mathrm{const}\, b_i\beta_i^{-1}t^{-\frac{N}{p+1}\frac{p-1}{2}}\|u_i\|^2_{\mathcal{H}_t}, & \text{if } i>1.\end{cases}$$

As in (33) we can estimate

$$(59)\quad \int_{\mathbb{R}^N}\big(|u_i(x,t)|^2 + |u_i(x,t)|^{p+1}\big)G(x,t)\,dx \leqslant \mathrm{const}\, t^{-\frac{N}{p+1}\frac{p-1}{2}}\|u_i\|^2_{\mathcal{H}_t}.$$

Collecting (56), (57), (58), and (59), and using Corollary 2.9, we obtain that there exists some positive constant C_3 such that, for a.e. $t \in (\beta_i, T_i)$,

$$\left|\nu_{2i}(t)\right| \leqslant \begin{cases} \frac{C_3}{H_i(t)}\, t^{-\frac{N}{p+1}\frac{p-1}{2}}\big(tD_i(t) + \frac{N-2}{4}H_i(t)\big) = C_3\big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{if } i = 1,\\ \frac{C_3\beta_i^{-1}}{H_i(t)}\, t^{-\frac{N}{p+1}\frac{p-1}{2}}\big(tD_i(t) + \frac{N-2}{4}H_i(t)\big) = C_3\beta_i^{-1}\big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{if } i > 1, \end{cases}$$

thus completing the proof in case **(II)**. □

Lemma 4.8. *There exists $C_4 > 0$ such that, if $i \in \{1, \dots, k\}$ and $\beta_i, T_i \in (0, 2\alpha)$ satisfy (51), then, for every $t \in (\beta_i, T_i)$,*

$$N_i(t) \leqslant \begin{cases} -\frac{N-2}{4} + C_4\big(N_i(T_i) + \frac{N-2}{4}\big), & \textit{in case } \textbf{(I)} \textit{ and in case } \textbf{(II)} \textit{ if } i = 1,\\ -\frac{N-2}{4} + C_4^{1/\beta_i}\big(N_i(T_i) + \frac{N-2}{4}\big), & \textit{in case } \textbf{(II)} \textit{ if } i > 1. \end{cases}$$

Proof. Let ν_{1i} and ν_{2i} as in Lemma 4.6. By Schwarz's inequality,

$$\nu_{1i} \geqslant 0 \quad \text{a.e. in } (\beta_i, T_i). \tag{60}$$

From Lemma 4.6, (60), and Lemma 4.7, we deduce that

$$\frac{d}{dt}N_i(t) \geqslant \begin{cases} -C_3\big(N_i(t) + \frac{N-2}{4}\big)\Big(t^{-1+\varepsilon/2} + \|h_t(\cdot, t + a_i)\|_{L^{N/2}(\mathbb{R}^N)}\Big), & \text{in case } \textbf{(I)},\\ -C_3\big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{in case } \textbf{(II)} \text{ if } i = 1,\\ -C_3\beta_i^{-1}\big(N_i(t) + \frac{N-2}{4}\big)\, t^{-1+\frac{N+2-p(N-2)}{2(p+1)}}, & \text{in case } \textbf{(II)} \text{ if } i > 1, \end{cases}$$

for a.e. $t \in (\beta_i, T_i)$. After integration, it follows that

$$N_i(t) \leqslant \begin{cases} -\frac{N-2}{4} + \Big(N_i(T_i) + \frac{N-2}{4}\Big)\exp\Big(\frac{2C_3}{\varepsilon}T_i^{\varepsilon/2} + C_3\|h_t\|_{L^1((0,T),L^{N/2}(\mathbb{R}^N))}\Big), & \text{in case } \textbf{(I)},\\ -\frac{N-2}{4} + \Big(N_i(T_i) + \frac{N-2}{4}\Big)\exp\Big(\frac{2(p+1)C_3}{N+2-p(N-2)}T_i^{\frac{N+2-p(N-2)}{2(p+1)}}\Big), & \text{in case } \textbf{(II)},\ i = 1,\\ -\frac{N-2}{4} + \Big(N_i(T_i) + \frac{N-2}{4}\Big)\exp\Big(\frac{2(p+1)C_3\beta_i^{-1}}{N+2-p(N-2)}T_i^{\frac{N+2-p(N-2)}{2(p+1)}}\Big), & \text{in case } \textbf{(II)},\ i > 1, \end{cases}$$

for any $t \in (\beta_i, T_i)$, thus yielding the conclusion. □

Lemma 4.9. *Let $i \in \{1, \dots, k\}$. If $H_i \not\equiv 0$, then*

$$H_i(t) > 0 \quad \textit{for all } t \in (0, 2\alpha).$$

Proof. From continuity of H_i, the assumption $H_i \not\equiv 0$, and the fact that $u_i(\cdot, t) \in \mathcal{H}_t$ for a.e. $t \in (0, 2\alpha)$, we deduce that there exists $T_i \in (0, 2\alpha)$ such that

$$H_i(T_i) > 0 \quad \text{and} \quad u_i(\cdot, T_i) \in \mathcal{H}_{T_i}. \tag{61}$$

Lemma 4.4 implies that $H_i(t) > 0$ for all $t \in [T_i, 2\alpha)$. We consider

$$t_i := \inf\{s \in (0, T_i) : H_i(t) > 0 \text{ for all } t \in (s, 2\alpha)\}.$$

Due to Lemma 4.4, either

$$t_i = 0 \text{ and } H_i(t) > 0 \text{ for all } t \in (0, 2\alpha) \tag{62}$$

or

$$0 < t_i < T_i \text{ and } \begin{cases} H_i(t) = 0 & \text{if } t \in (0, t_i] \\ H_i(t) > 0 & \text{if } t \in (t_i, 2\alpha) \end{cases}. \tag{63}$$

The argument below will exclude alternative (63). Assume by contradiction that (63) holds. From Lemma 4.8 and (49), it follows

$$\frac{t}{2} H_i'(t) \leqslant c_i H_i(t)$$

where

$$c_i = \begin{cases} -\frac{N-2}{4} + C_4\big(N_i(T_i) + \frac{N-2}{4}\big) & \text{in case } \textbf{(I)} \text{ and in case } \textbf{(II)} \text{ if } i = 1, \\ -\frac{N-2}{4} + C_4^{1/t_i}\big(N_i(T_i) + \frac{N-2}{4}\big) & \text{in case } \textbf{(II)} \text{ if } i > 1, \end{cases}$$

for a.e. $t \in (t_i, T_i)$. By integration, it follows that

$$H_i(t) \geqslant \frac{H_i(T_i)}{T_i^{2c_i}} t^{2c_i} \quad \text{for all } t \in [t_i, T_i). \tag{64}$$

By (63) $H_i(t_i) = 0$, giving rise to contradiction with (64) because of (61). Therefore, we exclude (63) and conclude that (62) holds. □

Lemma 4.10. *Let $i \in \{1, \ldots, k\}$. Then*

$$H_i(t) \equiv 0 \textit{ in } (0, 2\alpha) \textit{ if and only if } H_{i+1}(t) \equiv 0 \textit{ in } (0, 2\alpha).$$

Proof. First, we prove that $H_i(t) \equiv 0$ in $(0, 2\alpha)$ implies $H_{i+1}(t) \equiv 0$ in $(0, 2\alpha)$. Let's suppose by contradiction that $H_{i+1}(t) \not\equiv 0$. By Lemma 4.9, we conclude that $H_{i+1}(t) > 0$ for all $t \in (0, 2\alpha)$. It follows that $u_{i+1}(\cdot, t) \not\equiv 0$ for all $t \in (0, 2\alpha)$ and $u(\cdot, t) \not\equiv 0$, for all $t \in (i\alpha, (i+1)\alpha)$. Hence, $u_i(\cdot, t) \not\equiv 0$, for all $t \in (\alpha, 2\alpha)$ and thus $H_i \not\equiv 0$ in $(0, 2\alpha)$, a contradiction.

Let us now prove that $H_{i+1}(t) \equiv 0$ in $(0, 2\alpha)$ implies $H_i(t) \equiv 0$ in $(0, 2\alpha)$. Let's suppose by contradiction that $H_i(t) \not\equiv 0$, then, by Lemma 4.4, $H_i(t) > 0$ in $(\bar{t}, 2\alpha)$ for some $\bar{t} \in (\alpha, 2\alpha)$. Hence, $u_i(\cdot, t) \not\equiv 0$ in $(\bar{t}, 2\alpha)$ and then $u_{i+1}(\cdot, t) \not\equiv 0$ in $(\bar{t} - \alpha, \alpha)$, thus implying $H_{i+1}(t) \not\equiv 0$, a contradiction. □

Corollary 4.11. *If $u \not\equiv 0$ in $\mathbb{R}^N \times (0, T)$, then*

$$H_i(t) > 0$$

for all $t \in (0, 2\alpha)$ and $i = 1, \ldots, k$. In particular,

$$\int_{\mathbb{R}^N} u^2(x,t) G(x,t)\, dx > 0 \quad \textit{for all } t \in (0, T). \tag{65}$$

Proof. If $u \not\equiv 0$, then there exists some $i_0 \in \{1, \ldots, k\}$ such that $u_{i_0} \not\equiv 0$ in $(0, 2\alpha)$. Hence, $H_{i_0}(t) \not\equiv 0$ in $(0, 2\alpha)$ and, thanks to lemma 4.10, $H_i(t) \not\equiv 0$ in $(0, 2\alpha)$ for all $i = 1, \ldots, k$. Applying Lemma 4.9, we conclude that, for all $i = 1, \ldots, k$, $H_i(t) > 0$ in $(0, 2\alpha)$, thus implying (65). □

Proof of Proposition 1.8. It follows immediately from Corollary 4.11. □

Henceforward, we assume $u \not\equiv 0$ and we denote, for all $t \in (0, 2\alpha)$,

$$H(t) = H_1(t) = \int_{\mathbb{R}^N} u^2(x,t)\, G(x,t)\, dx,$$

$$D(t) = D_1(t) = \int_{\mathbb{R}^N} \bigg(|\nabla u(x,t)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} u^2(x,t) - f(x,t,u(x,t))u(x,t) \bigg) G(x,t)\, dx.$$

Corollary 4.11 ensures that, if $u \not\equiv 0$ in $\mathbb{R}^N \times (0,T)$, $H(t) > 0$ for all $t \in (0, 2\alpha)$ and hence the *Almgren type frequency function*

$$\mathcal{N}(t) = \mathcal{N}_{f,u}(t) = N_1(t) = \frac{tD(t)}{H(t)}$$

is well defined over all $(0, 2\alpha)$. Moreover, by Lemma 4.6, $\mathcal{N} \in W^{1,1}_{\rm loc}(0, 2\alpha)$ and

$$\mathcal{N}'(t) = \nu_1(t) + \nu_2(t) \quad \text{for a.e. } t \in (0, 2\alpha),$$

where

$$\nu_1(t) = \nu_{11}(t) \quad \text{and} \quad \nu_2(t) = \nu_{21}(t), \tag{66}$$

with ν_{11}, ν_{21} as in Lemma 4.6. Since, by (10), $u(\cdot, t) \in \mathcal{H}_t$ for a.e. $t \in (0,T)$, we can fix T_0 such that

$$T_0 \in (0, 2\alpha) \quad \text{and} \quad u(\cdot, T_0) \in \mathcal{H}_{T_0}. \tag{67}$$

The following result clarifies the behavior of $\mathcal{N}(t)$ as $t \to 0^+$.

Lemma 4.12. *The limit*

$$\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$$

exists and it is finite.

Proof. We first observe that $\mathcal{N}(t)$ is bounded from below in $(0, 2\alpha)$. Indeed from Corollaries 2.4 and 2.9, we obtain that, for all $t \in (0, 2\alpha)$,

$$tD(t) \geqslant \Big(C_1 - \frac{N-2}{4} \Big) H(t),$$

and hence

$$\mathcal{N}(t) \geqslant C_1 - \frac{N-2}{4}. \tag{68}$$

Let T_0 as in (67). By Schwarz's inequality, $\nu_1(t) \geqslant 0$ for a.e. $t \in (0, T_0)$. Furthermore, from Lemmas 4.7 and 4.8, ν_2 belongs to $L^1(0, T_0)$. In particular, $\mathcal{N}'(t)$ turns out to be the sum of a nonnegative function and of a L^1 function over $(0, T_0)$. Therefore,

$$\mathcal{N}(t) = \mathcal{N}(T_0) - \int_t^{T_0} \mathcal{N}'(s)\, ds$$

admits a limit as $t \to 0^+$ which is finite in view of (68) and Lemma 4.8. □

Lemma 4.13. *Let $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ be as in Lemma 4.12. Then there exists a constant $K_1 > 0$ such that*

$$H(t) \leqslant K_1 t^{2\gamma} \quad \text{for all } t \in (0, T_0). \tag{69}$$

Furthermore, for any $\sigma > 0$, there exists a constant $K_2(\sigma) > 0$ depending on σ such that

$$H(t) \geqslant K_2(\sigma)\, t^{2\gamma+\sigma} \quad \text{for all } t \in (0, T_0). \tag{70}$$

Proof. From Lemma 4.6, (60), Lemma 4.7, and Lemma 4.8, we infer that

$$\begin{aligned}
\mathcal{N}(t) - \gamma &= \int_0^t (\nu_1(s) + \nu_2(s))ds \geqslant \int_0^t \nu_2(s)ds \\
&\geqslant \begin{cases} -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big) \int_0^t \Big(s^{-1+\varepsilon/2} + \|h_t(\cdot,s)\|_{L^{N/2}(\mathbb{R}^N)}\Big)\, ds, & \text{in case } \textbf{(I)}, \\ -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big) \int_0^t s^{-1+\frac{N+2-p(N-2)}{2(p+1)}}\, ds, & \text{in case } \textbf{(II)}, \end{cases} \\
&\geqslant \begin{cases} -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big)\Big(\frac{2}{\varepsilon} t^{\varepsilon/2} + \|h_t\|_{L^r((0,T),L^{N/2}(\mathbb{R}^N))} t^{1-1/r}\Big), & \text{in case } \textbf{(I)}, \\ -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big) \frac{2(p+1)}{N+2-p(N-2)} t^{\frac{N+2-p(N-2)}{2(p+1)}}, & \text{in case } \textbf{(II)} \end{cases} \\
&\geqslant -C_5 t^{\delta}
\end{aligned}$$

with

$$\delta = \begin{cases} \min\{\varepsilon/2, 1 - 1/r\}, & \text{in case } \textbf{(I)}, \\ \frac{N+2-p(N-2)}{2(p+1)}, & \text{in case } \textbf{(II)}, \end{cases} \tag{71}$$

for some constant $C_5 > 0$ and for all $t \in (0, T_0)$. From above and (49), we deduce that

$$(\log H(t))' = \frac{H'(t)}{H(t)} = \frac{2}{t}\mathcal{N}(t) \geqslant \frac{2}{t}\gamma - 2C_5 t^{-1+\delta}.$$

Integrating over (t, T_0) we obtain

$$H(t) \leqslant \frac{H(T_0)}{T_0^{2\gamma}} e^{2C_5 T_0^{\delta}} t^{2\gamma}$$

for all $t \in (0, T_0)$, thus proving (69).

Let us prove (70). Since $\gamma = \lim_{t\to 0^+} \mathcal{N}(t)$, for any $\sigma > 0$ there exists $t_\sigma > 0$ such that $\mathcal{N}(t) < \gamma + \sigma/2$ for any $t \in (0, t_\sigma)$ and hence

$$\frac{H'(t)}{H(t)} = \frac{2\,\mathcal{N}(t)}{t} < \frac{2\gamma + \sigma}{t}.$$

Integrating over the interval (t, t_σ) and by continuity of H outside 0, we obtain (70) for some constant $K_2(\sigma)$ depending on σ. □

5. The blow-up analysis

If u is a weak solution to (1) in the sense of Definition 1.1, then, for every $\lambda > 0$, the function

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$$

is a weak solution to

$$(u_\lambda)_t + \Delta u_\lambda + \frac{a(x/|x|)}{|x|^2} u_\lambda + \lambda^2 f(\lambda x, \lambda^2 t, u_\lambda) = 0 \quad \text{in } \mathbb{R}^N \times (0, T/\lambda^2), \tag{72}$$

in the sense that

$$\int_\tau^{\frac{T}{\lambda^2}} \|u_\lambda(\cdot,t)\|^2_{\mathcal{H}_t}\, dt < +\infty, \quad \int_\tau^{\frac{T}{\lambda^2}} \left\|(u_\lambda)_t + \frac{\nabla u_\lambda \cdot x}{2t}\right\|^2_{(\mathcal{H}_t)^\star} < +\infty \text{ for all } \tau \in \Big(0, \frac{T}{\lambda^2}\Big),$$

$$\begin{aligned} &{}_{\mathcal{H}_t^\star}\Big\langle (u_\lambda)_t + \frac{\nabla u_\lambda \cdot x}{2t}, w \Big\rangle_{\mathcal{H}_t} \\ &\quad = \int_{\mathbb{R}^N} \Big(\nabla u_\lambda(x,t) \cdot \nabla w(x) - \frac{a(x/|x|)}{|x|^2} u_\lambda(x,t) w(x) - \lambda^2 f(\lambda x, \lambda^2 t, u_\lambda(x,t)) w(x) \Big) G(x,t)\, dx \end{aligned}$$

for a.e. $t \in \big(0, \frac{T}{\lambda^2}\big)$ and for each $w \in \mathcal{H}_t$. The frequency function associated to the scaled equation (72) is

$$\mathcal{N}_\lambda(t) = \frac{t\, D_\lambda(t)}{H_\lambda(t)}, \tag{73}$$

where

$$\begin{aligned} D_\lambda(t) &= \int_{\mathbb{R}^N} \Big(|\nabla u_\lambda(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} u_\lambda^2(x,t) - \lambda^2 f(\lambda x, \lambda^2 t, u_\lambda(x,t)) u_\lambda(x,t) \Big) G(x,t)\, dx, \\ H_\lambda(t) &= \int_{\mathbb{R}^N} u_\lambda^2(x,t) G(x,t)\, dx. \end{aligned}$$

The scaling properties of the operator combined with a suitable change of variables easily imply that

$$D_\lambda(t) = \lambda^2 D(\lambda^2 t) \quad \text{and} \quad H_\lambda(t) = H(\lambda^2 t), \tag{74}$$

and consequently

$$\mathcal{N}_\lambda(t) = \mathcal{N}(\lambda^2 t) \quad \text{for all } t \in \Big(0, \frac{2\alpha}{\lambda^2}\Big). \tag{75}$$

Lemma 5.1. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ satisfy (16) and $u \not\equiv 0$ be, in the sense of Definition 1.1, either a weak solution to (2), with h satisfying (3) and (4), or a weak solution to (6) satisfying (7–8) with $\varphi \in C^1(\mathbb{R}^N \times (0,T) \times \mathbb{R})$ under assumption (5). Let $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ as in Lemma 4.12. Then*

(i) *γ is an eigenvalue of the operator L defined in (14);*

(ii) *for every sequence $\lambda_n \to 0^+$, there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction g of the operator L associated to γ such that, for all $\tau \in (0,1)$,*

$$\lim_{k\to+\infty} \int_\tau^1 \left\| \frac{u(\lambda_{n_k} x, \lambda_{n_k}^2 t)}{\sqrt{H(\lambda_{n_k}^2)}} - t^\gamma g(x/\sqrt{t}) \right\|^2_{\mathcal{H}_t} dt = 0$$

and

$$\lim_{k\to+\infty}\sup_{t\in[\tau,1]}\left\|\frac{u(\lambda_{n_k}x,\lambda_{n_k}^2t)}{\sqrt{H(\lambda_{n_k}^2)}}-t^{\gamma}g(x/\sqrt{t})\right\|_{\mathcal{L}_t}=0.$$

Proof. Let

$$w_\lambda(x,t):=\frac{u_\lambda(x,t)}{\sqrt{H(\lambda^2)}}, \tag{76}$$

with $\lambda\in(0,\sqrt{T_0})$, so that $1<T_0/\lambda^2$. From Lemma 4.3 we obtain that, for all $t\in(0,1)$,

$$\int_{\mathbb{R}^N}w_\lambda^2(x,t)G(x,t)\,dx=\frac{H(\lambda^2t)}{H(\lambda^2)}\leqslant t^{2C_1-\frac{N-2}{2}}, \tag{77}$$

with C_1 as in (42). Lemma 4.8, Corollaries 2.4 and 2.9, and (74) imply that

$$\begin{aligned}\frac{1}{t}\bigg(-\frac{N-2}{4}+C_4\bigg(\mathcal{N}(T_0)+\frac{N-2}{4}\bigg)\bigg)H_\lambda(t)&\geqslant\lambda^2D(\lambda^2t)\\&\geqslant\frac{1}{t}\bigg(C_1-\frac{N-2}{4}\bigg)H_\lambda(t)+C_1\int_{\mathbb{R}^N}|\nabla u_\lambda(x,t)|^2G(x,t)\,dx\end{aligned}$$

and hence, in view of (77),

$$\begin{aligned}t\int_{\mathbb{R}^N}|\nabla w_\lambda(x,t)|^2G(x,t)\,dx&\leqslant C_1^{-1}\big(C_4\big(\mathcal{N}(T_0)+\tfrac{N-2}{4}\big)-C_1\big)\int_{\mathbb{R}^N}w_\lambda^2(x,t)G(x,t)\,dx\\&\leqslant C_1^{-1}\big(C_4\big(\mathcal{N}(T_0)+\tfrac{N-2}{4}\big)-C_1\big)t^{2C_1-\frac{N-2}{2}},\end{aligned} \tag{78}$$

for a.e. $t\in(0,1)$. Let us consider the family of functions

$$\widetilde{w}_\lambda(x,t)=w_\lambda(\sqrt{t}x,t)=\frac{u(\lambda\sqrt{t}x,\lambda^2t)}{\sqrt{H(\lambda^2)}},$$

which, by scaling, satisfy

$$\int_{\mathbb{R}^N}\widetilde{w}_\lambda^2(x,t)G(x,1)\,dx=\int_{\mathbb{R}^N}w_\lambda^2(x,t)G(x,t)\,dx \tag{79}$$

and

$$\int_{\mathbb{R}^N}|\nabla\widetilde{w}_\lambda(x,t)|^2G(x,1)\,dx=t\int_{\mathbb{R}^N}|\nabla w_\lambda(x,t)|^2G(x,t)\,dx. \tag{80}$$

From (77), (78), (79), and (80), we deduce that, for all $\tau\in(0,1)$,

$$\big\{\widetilde{w}_\lambda\big\}_{\lambda\in(0,\sqrt{T_0})}\text{ is bounded in }L^\infty(\tau,1;\mathcal{H}) \tag{81}$$

uniformly with respect to $\lambda\in(0,\sqrt{T_0})$. Since

$$\widetilde{w}_\lambda(x,t)=\frac{v(x,\lambda^2t)}{\sqrt{H(\lambda^2)}}\quad\text{and}\quad(\widetilde{w}_\lambda)_t(x,t)=\frac{\lambda^2}{\sqrt{H(\lambda^2)}}\,v_t(x,\lambda^2t)$$

with v as in Remark 1.2, from (13) we deduce that, for all $\phi \in \mathcal{H}$,

$$
{}_{\mathcal{H}^\star}\big\langle(\widetilde{w}_\lambda)_t,\phi\big\rangle_{\mathcal{H}} = \frac{1}{t}\int_{\mathbb{R}^N}\bigg(\nabla\widetilde{w}_\lambda(x,t)\cdot\nabla\phi(x) - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\,\widetilde{w}_\lambda(x,t)\phi(x) - \frac{\lambda^2 t}{\sqrt{H(\lambda^2)}}f\Big(\lambda\sqrt{t}x,\lambda^2 t,\sqrt{H(\lambda^2)}\widetilde{w}_\lambda(x,t)\Big)\phi(x)\bigg)G(x,1)\,dx. \tag{82}
$$

In case **(I)**, from (4) and Lemma 2.1, we can estimate the last term in the above integral as

$$
\begin{aligned}
&\lambda^2\left|\int_{\mathbb{R}^N} h(\lambda\sqrt{t}x,\lambda^2 t)\widetilde{w}_\lambda(x,t)\phi(x)G(x,1)\,dx\right| \\
&\leqslant C_h\lambda^2\int_{\mathbb{R}^N}|\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx + C_h\frac{\lambda^\varepsilon}{t}\int_{\mathbb{R}^N}|x|^{-2+\varepsilon}|\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx \\
&\leqslant C_h\lambda^2\|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}} + C_h\frac{\lambda^\varepsilon}{t}\int_{|x|\leqslant 1}\frac{|\widetilde{w}_\lambda(x,t)||\phi(x)|}{|x|^2}G(x,1)\,dx \\
&\qquad + C_h\frac{\lambda^\varepsilon}{t}\int_{|x|\geqslant 1}|\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx \\
&\leqslant C_h\frac{\lambda^\varepsilon}{t}\bigg(t\,\lambda^{2-\varepsilon} + \frac{\max\{4,N-2\}}{(N-2)^2} + 1\bigg)\|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}
\end{aligned} \tag{83}
$$

for all $\lambda \in (0,\sqrt{T_0})$ and a.e. $t \in (0,1)$. From (82), (83), and Lemma 2.1 it follows that, for all $\lambda \in (0,\sqrt{T_0})$ and a.e. $t \in (0,1)$,

$$
\begin{aligned}
&\big|{}_{\mathcal{H}^\star}\big\langle(\widetilde{w}_\lambda)_t,\phi\big\rangle_{\mathcal{H}}\big| \\
&\quad\leqslant \Big(1 + \tfrac{\max\{4,N-2\}}{(N-2)^2}\|a\|_{L^\infty(\mathbb{S}^{N-1})} + C_h T_0^{\varepsilon/2}\Big(T_0^{1-\varepsilon/2} + \tfrac{\max\{4,N-2\}}{(N-2)^2} + 1\Big)\Big)\frac{\|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}}{t}
\end{aligned}
$$

and hence

$$
\|(\widetilde{w}_\lambda)_t(\cdot,t)\|_{\mathcal{H}^\star} \leqslant \frac{\mathrm{const}}{t}\|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}}. \tag{84}
$$

In case **(II)**, from (5), Hölder's inequality, and Lemma 2.7, we obtain

$$
\begin{aligned}
&\left|\frac{\lambda^2}{\sqrt{H(\lambda^2)}}\int_{\mathbb{R}^N}\varphi\Big(\lambda\sqrt{t}x,\lambda^2 t,\sqrt{H(\lambda^2)}\widetilde{w}_\lambda(x,t)\Big)\phi(x)G(x,1)\,dx\right| \\
&\leqslant C_\varphi\frac{\lambda^2}{\sqrt{H(\lambda^2)}}\int_{\mathbb{R}^N}\Big(\sqrt{H(\lambda^2)}|\widetilde{w}_\lambda(x,t)| + (\sqrt{H(\lambda^2)})^p|\widetilde{w}_\lambda(x,t)|^p\Big)|\phi(x)|G(x,1)\,dx \\
&\leqslant C_\varphi\lambda^2\int_{\mathbb{R}^N}|\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx + C_\varphi\lambda^2(H(\lambda^2))^{\frac{p-1}{2}}\int_{\mathbb{R}^N}|\widetilde{w}_\lambda(x,t)|^p|\phi(x)|G(x,1)\,dx \\
&\leqslant \|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}\frac{\lambda^{\frac{N+2-p(N-2)}{p+1}}}{t}\bigg(C_\varphi t\lambda^{\frac{N(p-1)}{p+1}} + C_\varphi C_{p+1}t^{\frac{N+2-p(N-2)}{2(p+1)}}\bigg(\int_{\mathbb{R}^N}|u(x,\lambda^2 t)|^{p+1}\,dx\bigg)^{\frac{p-1}{p+1}}\bigg).
\end{aligned} \tag{85}
$$

From (82), (85), Lemma 2.1, the fact that $p < 2^* - 1$, and (7), it follows that, for all $\lambda \in (0,\sqrt{T_0})$ and a.e. $t \in (0,1)$, estimate (84) holds also in case **(II)**. Then, in view of (81), estimate (84) yields, for all $\tau \in (0,1)$,

$$\big\{(\widetilde{w}_\lambda)_t\big\}_{\lambda\in(0,\sqrt{T_0})} \text{ is bounded in } L^\infty(\tau,1;\mathcal{H}^\star) \tag{86}$$

uniformly with respect to $\lambda \in (0,\sqrt{T_0})$. From (81), (86), and [28, Corollary 8], we deduce that $\big\{\widetilde{w}_\lambda\big\}_{\lambda\in(0,\sqrt{T_0})}$ is relatively compact in $C^0([\tau,1],\mathcal{L})$ for all $\tau \in (0,1)$. Therefore, for any given sequence $\lambda_n \to 0^+$, there exists a subsequence $\lambda_{n_k} \to 0^+$ such that

$$\widetilde{w}_{\lambda_{n_k}} \to \widetilde{w} \quad \text{in} \quad C^0([\tau,1],\mathcal{L}) \tag{87}$$

for all $\tau \in (0,1)$ and for some $\widetilde{w} \in \bigcap_{\tau\in(0,1)} C^0([\tau,1],\mathcal{L})$. We notice that a diagonal procedure allows subtracting a subsequence which does not depend on τ. Since

$$1 = \|\widetilde{w}_{\lambda_{n_k}}(\cdot,1)\|_{\mathcal{L}},$$

the convergence (87) ensures that

$$\|\widetilde{w}(\cdot,1)\|_{\mathcal{L}} = 1. \tag{88}$$

In particular $\widetilde{w}$ is nontrivial. Furthermore, by (81) and (86), the subsequence can be chosen in such a way that also

$$\widetilde{w}_{\lambda_{n_k}} \rightharpoonup \widetilde{w} \quad \text{weakly in } L^2(\tau,1;\mathcal{H}) \quad \text{and} \quad (\widetilde{w}_{\lambda_{n_k}})_t \rightharpoonup \widetilde{w}_t \quad \text{weakly in } L^2(\tau,1;\mathcal{H}^\star) \tag{89}$$

for all $\tau \in (0,1)$; in particular $\widetilde{w} \in \bigcap_{\tau\in(0,1)} L^2(\tau,1;\mathcal{H})$ and $\widetilde{w}_t \in \bigcap_{\tau\in(0,1)} L^2(\tau,1;\mathcal{H}^\star)$. We now claim that

$$\widetilde{w}_{\lambda_{n_k}} \to \widetilde{w} \quad \text{strongly in} \quad L^2(\tau,1;\mathcal{H}) \quad \text{for all } \tau \in (0,1). \tag{90}$$

To prove the claim, we notice that (89) allows passing to the limit in (82). Therefore, in view of (83) and (84) which ensure the vanishing at the limit of the perturbation term,

$${}_{\mathcal{H}^\star}\langle \widetilde{w}_t,\phi\rangle_{\mathcal{H}} = \frac{1}{t}\int_{\mathbb{R}^N}\bigg(\nabla\widetilde{w}(x,t)\cdot\nabla\phi(x) - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\,\widetilde{w}(x,t)\phi(x)\bigg)G(x,1)\,dx \tag{91}$$

for all $\phi \in \mathcal{H}$ and a.e. $t \in (0,1)$, i.e. $\widetilde{w}$ is a weak solution to

$$\widetilde{w}_t + \frac{1}{t}\bigg(\Delta\widetilde{w} - \frac{x}{2}\cdot\nabla\widetilde{w} + \frac{a(x/|x|)}{|x|^2}\,\widetilde{w}\bigg) = 0.$$

Testing the difference between (82) and (91) with $(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})$ and integrating with respect to t between τ and 1, we obtain

$$\begin{aligned}
&\int_\tau^1\bigg(\int_{\mathbb{R}^N}\bigg(|\nabla(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}\,|(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)|^2\bigg)\,G(x,1)\,dx\bigg)dt\\
&= \frac{1}{2}\|\widetilde{w}_{\lambda_{n_k}}(1)-\widetilde{w}(1)\|_{\mathcal{L}}^2 - \frac{\tau}{2}\|\widetilde{w}_{\lambda_{n_k}}(\tau)-\widetilde{w}(\tau)\|_{\mathcal{L}}^2 - \int_\tau^1\bigg(\int_{\mathbb{R}^N}|(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)|^2\,G(x,1)\,dx\bigg)dt\\
&+ \frac{\lambda_{n_k}^2}{\sqrt{H(\lambda_{n_k}^2)}}\int_\tau^1\bigg(\int_{\mathbb{R}^N} tf\Big(\lambda_{n_k}\sqrt{t}x,\lambda_{n_k}^2 t,\sqrt{H(\lambda_{n_k}^2)}\widetilde{w}_{\lambda_{n_k}}(x,t)\Big)(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)G(x,1)\,dx\bigg)dt.
\end{aligned}$$

Then, from (83), (85), and (87), we obtain that, for all $\tau \in (0,1)$,

$$\lim_{k\to+\infty}\int_\tau^1\bigg(\int_{\mathbb{R}^N}\bigg(|\nabla(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}\,|(\widetilde{w}_{\lambda_{n_k}}-\widetilde{w})(x,t)|^2\bigg)\,G(x,1)\,dx\bigg)dt = 0,$$

which, by Corollary 2.3 and (87), implies the convergence claimed in (90). Thus, we have obtained that, for all $\tau \in (0,1)$,

$$\lim_{k\to+\infty}\int_\tau^1 \|w_{\lambda_{n_k}}(\cdot,t)-w(\cdot,t)\|^2_{\mathcal{H}_t}dt=0 \tag{92}$$

and

$$\lim_{k\to+\infty}\sup_{t\in[\tau,1]}\|w_{\lambda_{n_k}}(\cdot,t)-w(\cdot,t)\|_{\mathcal{L}_t}=0,$$

where

$$w(x,t):=\widetilde{w}\Big(\frac{x}{\sqrt{t}},t\Big)$$

is a weak solution (in the sense of Definition 1.1) of

$$w_t+\Delta w+\frac{a(x/|x|)}{|x|^2}\,w=0. \tag{93}$$

We notice that, by (73) and (76),

$$\mathcal{N}_\lambda(t) = \frac{t\int_{\mathbb{R}^N}\Big(|\nabla w_\lambda(x,t)|^2-\frac{a(x/|x|)}{|x|^2}w_\lambda^2(x,t)-\frac{\lambda^2}{\sqrt{H(\lambda^2)}}f\big(\lambda x,\lambda^2 t,\sqrt{H(\lambda^2)}w_\lambda(x,t)\big)w_\lambda(x,t)\Big)G(x,t)\,dx}{\int_{\mathbb{R}^N}w_\lambda^2(x,t)G(x,t)\,dx}$$

for all $t\in(0,1)$. Since, by (92), $w_{\lambda_{n_k}}(\cdot,t)\to w(\cdot,t)$ in $\mathcal{H}_t$ for a.e. $t\in(0,1)$, and, by (83) and (85),

$$\frac{t\lambda_{n_k}^2}{\sqrt{H(\lambda_{n_k}^2)}}\int_{\mathbb{R}^N}f\Big(\lambda_{n_k}x,\lambda_{n_k}^2t,\sqrt{H(\lambda_{n_k}^2)}w_{\lambda_{n_k}}(x,t)\Big)w_{\lambda_{n_k}}(x,t)G(x,t)\,dx\to 0$$

for a.e. $t\in(0,1)$, we obtain that

$$\int_{\mathbb{R}^N}\bigg(|\nabla w_{\lambda_{n_k}}(x,t)|^2-\frac{a(x/|x|)}{|x|^2}w_{\lambda_{n_k}}^2(x,t)- \frac{\lambda_{n_k}^2}{\sqrt{H(\lambda_{n_k}^2)}}f\Big(\lambda_{n_k}x,\lambda^2t,\sqrt{H(\lambda_{n_k}^2)}w_{\lambda_{n_k}}(x,t)\Big)w_{\lambda_{n_k}}(x,t)\bigg)G(x,t)\,dx\to D_w(t) \tag{94}$$

and

$$\int_{\mathbb{R}^N}w_{\lambda_{n_k}}^2(x,t)G(x,t)\,dx\to H_w(t) \tag{95}$$

for a.e. $t\in(0,1)$, where

$$D_w(t)=\int_{\mathbb{R}^N}\bigg(|\nabla w(x,t)|^2-\frac{a(x/|x|)}{|x|^2}w^2(x,t)\bigg)G(x,t)\,dx \quad\text{and}\quad H_w(t)=\int_{\mathbb{R}^N}w^2(x,t)G(x,t)\,dx.$$

We point out that

$$H_w(t)>0 \quad\text{for all}\quad t\in(0,1); \tag{96}$$

indeed, (88) yields

$$\int_{\mathbb{R}^N}w^2(x,1)G(x,1)\,dx=1, \tag{97}$$

which, arguing as in Lemma 4.9 or applying directly the Unique Continuation Principle proved by [25, Theorem 1.2] to equation (93), implies that $\int_{\mathbb{R}^N} w^2(x,t)G(x,t)\,dx > 0$ for all $t \in (0,1)$. From (94) and (95), it follows that

$$\mathcal{N}_{\lambda_{n_k}}(t) \to \mathcal{N}_w(t) \quad \text{for a.e. } t \in (0,1), \tag{98}$$

where $\mathcal{N}_w$ is the frequency function associated to the limit equation (93), i.e.

$$\mathcal{N}_w(t) = \frac{tD_w(t)}{H_w(t)}, \tag{99}$$

which is well defined on $(0,1)$ by (96).

On the other hand, (75) implies that $\mathcal{N}_{\lambda_{n_k}}(t) = \mathcal{N}(\lambda_{n_k}^2 t)$ for all $t \in (0,1)$ and $k \in \mathbb{N}$. Fixing $t \in (0,1)$ and passing to the limit as $k \to +\infty$, from Lemma 4.12 we obtain

$$\mathcal{N}_{\lambda_{n_k}}(t) \to \gamma \quad \text{for all } t \in (0,1). \tag{100}$$

Combining (98) and (100), we deduce that

$$\mathcal{N}_w(t) = \gamma \quad \text{for all } t \in (0,1). \tag{101}$$

Therefore $\mathcal{N}_w$ is constant in $(0,1)$ and hence $\mathcal{N}_w'(t) = 0$ for any $t \in (0,1)$. By (93) and Lemma 4.6 with $f \equiv 0$, we obtain

$$\bigg(\int_{\mathbb{R}^N} \Big|w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t}\Big|^2 G(x,t)\,dx\bigg)\bigg(\int_{\mathbb{R}^N} w^2(x,t)\,G(x,t)\,dx\bigg) - \bigg(\int_{\mathbb{R}^N} \Big(w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t}\Big) w(x,t)G(x,t)\,dx\bigg)^2 = 0 \quad \text{for all } t \in (0,1),$$

i.e.

$$\bigg(w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}, w(\cdot,t)\bigg)_{\mathcal{L}_t}^2 = \bigg\|w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}\bigg\|_{\mathcal{L}_t}^2 \|w(\cdot,t)\|_{\mathcal{L}_t}^2,$$

where $(\cdot,\cdot)_{\mathcal{L}_t}$ denotes the scalar product in $\mathcal{L}_t$. This shows that, for all $t \in (0,1)$, $w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}$ and $w(\cdot,t)$ have the same direction as vectors in $\mathcal{L}_t$ and hence there exists a function $\beta : (0,1) \to \mathbb{R}$ such that

$$w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t} = \beta(t)w(x,t) \quad \text{for a.e. } t \in (0,1) \text{ and a.e. } x \in \mathbb{R}^N. \tag{102}$$

Testing (93) with $\phi = w(\cdot,t)$ in the sense of (11) and taking into account (102), we find that

$$D_w(t) = {}_{\mathcal{H}_t^\star}\bigg\langle w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}, w(\cdot,t)\bigg\rangle_{\mathcal{H}_t} = \beta(t)H_w(t),$$

which, by (99) and (101), implies that

$$\beta(t) = \frac{\gamma}{t} \quad \text{for a.e. } t \in (0,1).$$

Hence (102) becomes

$$w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t} = \frac{\gamma}{t}\,w(x,t) \text{ for a.e. } (x,t) \in \mathbb{R}^N \times (0,1) \text{ and in a distributional sense.} \tag{103}$$

Combining (103) with (93), we obtain

$$\Delta w + \frac{a(x/|x|)}{|x|^2}\, w - \frac{\nabla w(x,t)\cdot x}{2t} + \frac{\gamma}{t}\, w(x,t) = 0 \tag{104}$$

for a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$ and in a weak sense. From (103), it follows that, letting, for all $\eta > 0$ and a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$, $w^\eta(x,t) := w(\eta x, \eta^2 t)$, there holds

$$\frac{dw^\eta}{d\eta} = \frac{2\gamma}{\eta} w^\eta$$

a.e. and in a distributional sense. By integration, we obtain that

$$w^\eta(x,t) = w(\eta x, \eta^2 t) = \eta^{2\gamma} w(x,t) \quad \text{for all } \eta > 0 \text{ and a.e. } (x,t) \in \mathbb{R}^N \times (0,1). \tag{105}$$

Let

$$g(x) = w(x,1);$$

from (97), we have that $g \in \mathcal{L}$, $\|g\|_{\mathcal{L}} = 1$, and, from (105),

$$w(x,t) = w^{\sqrt{t}}\Big(\frac{x}{\sqrt{t}}, 1\Big) = t^\gamma w\Big(\frac{x}{\sqrt{t}}, 1\Big) = t^\gamma g\Big(\frac{x}{\sqrt{t}}\Big) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times (0,1). \tag{106}$$

In particular, from (106), $g\big(\cdot/\sqrt{t}\big) \in \mathcal{H}_t$ for a.e. $t \in (0,1)$ and hence, by scaling, $g \in \mathcal{H}$. From (104) and (106), we obtain that $g \in \mathcal{H} \setminus \{0\}$ weakly solves

$$-\Delta g(x) + \frac{\nabla g(x)\cdot x}{2} - \frac{a(x/|x|)}{|x|^2}\, g(x) = \gamma\, g(x),$$

i.e. γ is an eigenvalue of the operator L defined in (14) and g is an eigenfunction of L associated to γ. The proof is now complete. □

Let us now describe the behavior of $H(t)$ as $t \to 0^+$.

Lemma 5.2. *Under the same assumptions as in Lemma 5.1, let $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ be as in Lemma 4.12. Then the limit*

$$\lim_{t\to 0^+} t^{-2\gamma} H(t)$$

exists and it is finite.

Proof. In view of (69), it is sufficient to prove that the limit exists. By (49), Lemma 4.12, and Lemma 4.6, we have, for all $t \in (0, T_0)$,

$$\begin{aligned}\frac{d}{dt}\frac{H(t)}{t^{2\gamma}} &= -2\gamma t^{-2\gamma-1} H(t) + t^{-2\gamma} H'(t) = 2t^{-2\gamma-1}(tD(t) - \gamma H(t)) \\ &= 2t^{-2\gamma-1} H(t) \int_0^t (\nu_1(s) + \nu_2(s))\, ds,\end{aligned}$$

with ν_1, ν_2 as in (66). After integration over (t, T_0),

$$\frac{H(T_0)}{T_0^{2\gamma}} - \frac{H(t)}{t^{2\gamma}} = \int_t^{T_0} 2s^{-2\gamma-1} H(s) \left(\int_0^s \nu_1(r)dr\right) ds + \int_t^{T_0} 2s^{-2\gamma-1} H(s) \left(\int_0^s \nu_2(r)dr\right) ds. \tag{107}$$

By (60), $\nu_1(t) \geqslant 0$ and hence

$$\lim_{t\to 0^+} \int_t^{T_0} 2s^{-2\gamma-1} H(s) \left(\int_0^s \nu_1(r)dr\right) ds$$

exists. On the other hand, by Lemmas 4.7 and 4.8 we have that $s^{-\delta}\int_0^s |\nu_2(r)|dr$ is bounded in $(0,T_0)$ with δ defined in (71), while, from Lemma 4.13, we deduce that $t^{-2\gamma}H(t)$ is bounded in $(0,T_0)$. Therefore, for some const > 0, there holds

$$\left|2s^{-2\gamma-1}H(s)\left(\int_0^s \nu_2(r)dr\right)\right| \leqslant \text{const}\, s^{-1+\delta}$$

for all $s \in (0,T_0)$, which proves that $s^{-2\gamma-1}H(s)\left(\int_0^s \nu_2(r)dr\right) \in L^1(0,T_0)$. We conclude that both terms at the right hand side of (107) admit a limit as $t \to 0^+$ thus completing the proof. □

In the following lemma, we prove that $\lim_{t\to 0^+} t^{-2\gamma}H(t)$ is indeed strictly positive.

Lemma 5.3. *Under the same assumptions as in Lemma 5.1 and letting* $\gamma := \lim_{t\to 0^+}\mathcal{N}(t)$ *be as in Lemma 4.12, there holds*

$$\lim_{t\to 0^+} t^{-2\gamma}H(t) > 0.$$

Proof. Let us assume by contradiction that $\lim_{t\to 0^+} t^{-2\gamma}H(t) = 0$ and let $\{\widetilde{V}_{n,j} : j,n \in \mathbb{N}, j \geqslant 1\}$ be the orthonormal basis of $\mathcal{L}$ introduced in Remark 3.4. Since $u_\lambda(x,1) = u(\lambda x, \lambda^2) \in \mathcal{L}$ for all $\lambda \in (0,\sqrt{T_0})$, $u_\lambda(x,1) \in \mathcal{H}$ for a.e. $\lambda \in (0,\sqrt{T_0})$, and $f(\lambda x, \lambda^2, u_\lambda(x,1)) \in \mathcal{H}^\star$ for a.e. $\lambda \in (0,\sqrt{T_0})$, we can expand them as

$$u_\lambda(x,1) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} u_{m,k}(\lambda)\widetilde{V}_{m,k}(x) \quad \text{in } \mathcal{L}, \tag{108}$$

$$f(\lambda x,\lambda^2,u_\lambda(x,1)) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \xi_{m,k}(\lambda)\widetilde{V}_{m,k}(x) \quad \text{in } \mathcal{H}^\star,$$

where

$$u_{m,k}(\lambda) = \int_{\mathbb{R}^N} u_\lambda(x,1)\widetilde{V}_{m,k}(x)G(x,1)\,dx \tag{109}$$

and

$$\xi_{m,k}(\lambda) = {}_{\mathcal{H}^\star}\left\langle f(\lambda\cdot,\lambda^2,u_\lambda(\cdot,1)),\widetilde{V}_{m,k}\right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^N} f(\lambda x,\lambda^2,u_\lambda(x,1))\widetilde{V}_{m,k}(x)G(x,1)\,dx. \tag{110}$$

By orthogonality of the $\widetilde{V}_{m,k}$'s in $\mathcal{L}$, we have that

$$H(\lambda^2) = \sum_{\substack{n,j\in\mathbb{N}\\ j\geqslant 1}} (u_{n,j}(\lambda))^2 \geqslant (u_{m,k}(\lambda))^2 \quad \text{for all } \lambda \in (0,\sqrt{T_0}) \text{ and } m,k\in\mathbb{N},\ k\geqslant 1.$$

Hence, $\lim_{t\to 0^+} t^{-2\gamma}H(t) = 0$ implies that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma}u_{m,k}(\lambda) = 0 \quad \text{for all } m,k\in\mathbb{N},\ k\geqslant 1. \tag{111}$$

Moreover, we can show that the function $\lambda \mapsto u_{m,k}(\lambda)$ is absolutely continuous in $(0,\sqrt{T_0})$ and $u'_{m,k}(\lambda) = {}_{\mathcal{H}^\star}\langle \frac{d}{d\lambda}u_\lambda(x,1), \widetilde{V}_{m,k}(x)\rangle_{\mathcal{H}}$. Hence

$$\frac{d}{d\lambda}u_\lambda(x,1) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} u'_{m,k}(\lambda)\widetilde{V}_{m,k}(x) \quad \text{in } \mathcal{H}^\star.$$

Furthermore,

$$\Delta u_\lambda(x,1) = \lambda^2 \Delta u(\lambda x, \lambda^2) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} u_{m,k}(\lambda) \Delta \widetilde{V}_{m,k}(x) \quad \text{in } \mathcal{H}^\star.$$

From (1) and the fact that $\widetilde{V}_{m,k}(x)$ is an eigenfuntion of the operator L associated to the eigenvalue $\gamma_{m,k}$ defined in (18), it follows that

$$\begin{aligned}
\frac{d}{d\lambda} u_\lambda(x,1) &= 2\lambda u_t(\lambda x, \lambda^2) + \nabla u(\lambda x, \lambda^2)\cdot x \\
&= 2\lambda\bigg(- \Delta u(\lambda x, \lambda^2) - \frac{a(x/|x|)}{\lambda^2 |x|^2} u(\lambda x, \lambda^2) - f(\lambda x, \lambda^2, u(\lambda x, \lambda^2))\bigg) + \nabla u(\lambda x, \lambda^2)\cdot x \\
&= \frac{2}{\lambda} \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} u_{m,k}(\lambda)\bigg(- \Delta \widetilde{V}_{m,k}(x) - \frac{a(x/|x|)}{|x|^2}\widetilde{V}_{m,k}(x) + \frac{\nabla \widetilde{V}_{m,k}\cdot x}{2}\bigg) - 2\lambda \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \xi_{m,k}(\lambda) \widetilde{V}_{m,k}(x) \\
&= \frac{2}{\lambda} \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \gamma_{m,k} u_{m,k}(\lambda) \widetilde{V}_{m,k}(x) - 2\lambda \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \xi_{m,k}(\lambda) \widetilde{V}_{m,k}(x).
\end{aligned}$$

Therefore, we have that

$$u'_{m,k}(\lambda) = \frac{2}{\lambda}\gamma_{m,k} u_{m,k}(\lambda) - 2\lambda \xi_{m,k}(\lambda) \quad \text{for all } m,k\in\mathbb{N},\ k\geqslant 1,$$

a.e. and distributionally in $(0,\sqrt{T_0})$. By integration, we obtain, for all $\lambda, \bar\lambda \in (0,\sqrt{T_0})$,

$$u_{m,k}(\bar\lambda) = \bar\lambda^{2\gamma_{m,k}} \left(\lambda^{-2\gamma_{m,k}} u_{m,k}(\lambda) + 2\int_{\bar\lambda}^{\lambda} s^{1-2\gamma_{m,k}} \xi_{m,k}(s)\, ds\right). \tag{112}$$

From Lemma 5.1, γ is an eigenvalue of the operator L, hence, by Proposition 1.4, there exist $m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that $\gamma = \gamma_{m_0,k_0} = m_0 - \frac{\alpha_{k_0}}{2}$. Let us denote as E_0 the associated eigenspace and by J_0 the finite set of indices $\{(m,k)\in\mathbb{N}\times(\mathbb{N}\setminus\{0\}) : \gamma = m - \frac{\alpha_k}{2}\}$, so that $\#J_0 = m(\gamma)$, with $m(\gamma)$ as in (41), and an orthonormal basis of E_0 is given by $\{\widetilde{V}_{m,k} : (m,k)\in J_0\}$. In order to estimate $\xi_{m,k}$, we distinguish between case **(I)** and case **(II)**.

Case (I): From (4), for all $(m,k)\in J_0$, we can estimate $\xi_{m,k}$ as

$$\begin{aligned}
|\xi_{m,k}(\lambda)| &\leqslant C_h \int_{\mathbb{R}^N} (1+\lambda^{-2+\varepsilon}|x|^{-2+\varepsilon}) |u(\lambda x, \lambda^2)| |\widetilde{V}_{m,k}(x)| G(x,1)\, dx \\
&\leqslant C_h \bigg(\int_{\mathbb{R}^N} u^2(\lambda x,\lambda^2) G(x,1)\,dx\bigg)^{1/2} \bigg(\int_{\mathbb{R}^N} \widetilde{V}^2_{m,k}(x) G(x,1)\,dx\bigg)^{1/2} \\
&\quad + C_h \lambda^{-2+\varepsilon/2} \int_{|x|\leqslant \lambda^{-1/2}} \frac{|u(\lambda x,\lambda^2)||\widetilde{V}_{m,k}(x)|}{|x|^2} G(x,1)\,dx \\
&\quad + C_h \lambda^{-1+\varepsilon/2} \int_{|x|\geqslant \lambda^{-1/2}} |u(\lambda x,\lambda^2)||\widetilde{V}_{m,k}(x)| G(x,1)\,dx \\
&\leqslant C_h(1+\lambda^{-1+\frac{\varepsilon}{2}})\sqrt{H(\lambda^2)} + C_h \lambda^{-2+\frac{\varepsilon}{2}} \bigg(\int_{\mathbb{R}^N} \tfrac{u^2(\lambda x,\lambda^2)}{|x|^2} G(x,1)\,dx\bigg)^{\frac12} \bigg(\int_{\mathbb{R}^N} \tfrac{\widetilde{V}^2_{m,k}(x)}{|x|^2} G(x,1)\,dx\bigg)^{\frac12}.
\end{aligned} \tag{113}$$

From Corollary 2.4 and Lemma 4.8, it follows that

$$\int_{\mathbb{R}^N} \frac{u^2(\lambda x,\lambda^2)}{|x|^2} G(x,1)\,dx = \lambda^2 \int_{\mathbb{R}^N} \frac{u^2(y,\lambda^2)}{|y|^2} G(y,\lambda^2)\,dy \leqslant \frac{\lambda^2}{C_1'}\Big(D(\lambda^2) + \frac{C_2}{\lambda^2} H(\lambda^2)\Big) = \frac{H(\lambda^2)}{C_1'}\big(\mathcal{N}(\lambda^2) + C_2\big) \leqslant \frac{C_2 - \frac{N-2}{4} + C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big)}{C_1'} H(\lambda^2), \tag{114}$$

while, from Lemma 2.2, for all $(m,k)\in J_0$,

$$\int_{\mathbb{R}^N} \frac{\widetilde{V}^2_{m,k}(x)}{|x|^2} G(x,1)\,dx \leqslant \Big(\mu_1(a) + \frac{(N-2)^2}{4}\Big)^{-1}\Big(\gamma + \frac{N-2}{4}\Big). \tag{115}$$

From (113), (114), (115), and Lemma 4.13, we deduce that

$$|\xi_{m,k}(\lambda)| \leqslant C_6 \lambda^{-2+\frac{\varepsilon}{2}+2\gamma}, \quad \text{for all } \lambda\in(0,\sqrt{T_0}) \tag{116}$$

and for some positive constant C_6 depending on $a, N, \gamma, h, T_0, K_1, \varepsilon$ but independent of λ and $(m,k)\in J_0$.

Case (II): From (5) and Lemma 2.7, for all $(m,k)\in J_0$, we can estimate $\xi_{m,k}$ as

$$\begin{aligned}
|\xi_{m,k}(\lambda)| &\leqslant C_\varphi \int_{\mathbb{R}^N} \big(|u(\lambda x,\lambda^2)| + |u(\lambda x,\lambda^2)|^p\big)|\widetilde{V}_{m,k}(x)| G(x,1)\,dx \\
&\leqslant C_\varphi \Big(\int_{\mathbb{R}^N} u^2(\lambda x,\lambda^2) G(x,1)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} \widetilde{V}^2_{m,k}(x) G(x,1)\,dx\Big)^{1/2} \\
&\quad + C_\varphi \Big(\int_{\mathbb{R}^N} |u(\lambda x,\lambda^2)|^{p+1} |G(x,1)|^{\frac{p+1}{2}}\,dx\Big)^{\frac{1}{p+1}}\Big(\int_{\mathbb{R}^N} |\widetilde{V}_{m,k}(x)|^{p+1}|G(x,1)|^{\frac{p+1}{2}}\,dx\Big)^{\frac{1}{p+1}} \\
&\qquad \times \Big(\int_{\mathbb{R}^N} |u(\lambda x,\lambda^2)|^{p+1}\,dx\Big)^{\frac{p-1}{p+1}} \\
&\leqslant C_\varphi\sqrt{H(\lambda^2)} + C_\varphi C_{p+1}\lambda^{-N\frac{p-1}{p+1}} \|u_\lambda(\cdot,1)\|_{\mathcal{H}} \|\widetilde{V}_{m,k}\|_{\mathcal{H}} \Big(\int_{\mathbb{R}^N} |u(y,\lambda^2)|^{p+1}\,dy\Big)^{\frac{p-1}{p+1}}.
\end{aligned} \tag{117}$$

From Corollary 2.9 and Lemma 4.8, it follows that

$$\|u_\lambda(\cdot,1)\|^2_{\mathcal{H}} = \|u(\cdot,\lambda^2)\|^2_{\mathcal{H}_{\lambda^2}} \leqslant \frac{\lambda^2}{C_1''}\Big(D(\lambda^2) + \frac{N-2}{4\lambda^2} H(\lambda^2)\Big) = \frac{H(\lambda^2)}{C_1''}\Big(\mathcal{N}(\lambda^2) + \frac{N-2}{4}\Big) \leqslant \frac{C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big)}{C_1''} H(\lambda^2), \tag{118}$$

while, from Corollary 2.3, for all $(m,k)\in J_0$,

$$\|\widetilde{V}_{m,k}\|_{\mathcal{H}} \leqslant \text{const}\Big(\gamma + \frac{N-2}{4}\Big). \tag{119}$$

From (117), (118), (119), and Lemma 4.13, we deduce that

$$|\xi_{m,k}(\lambda)| \leqslant C_7 \lambda^{-2+\frac{N+2-p(N-2)}{p+1}+2\gamma}, \quad \text{for all } \lambda\in(0,\sqrt{T_0}) \tag{120}$$

and for some positive constant C_7 depending on $\|u\|_{L^\infty(0,T,L^{p+1}(\mathbb{R}^N))}$, a, N, γ, φ, T_0, K_1, p, but independent of λ and $(m,k)\in J_0$.

Collecting (116) and (120), we have that

$$|\xi_{m,k}(\lambda)| \leqslant C_8 \lambda^{-2+\tilde{\delta}+2\gamma}, \quad \text{for all } \lambda \in (0, \sqrt{T_0}) \tag{121}$$

for some $C_8 > 0$ which is independent of λ and $(m, k) \in J_0$ and

$$\tilde{\delta} = \begin{cases} \varepsilon/2, & \text{in case } \textbf{(I)}, \\ \frac{N+2-p(N-2)}{p+1}, & \text{in case } \textbf{(II)}. \end{cases}$$

Estimate (121) implies that the function $s \mapsto s^{1-2\gamma}\xi_{m,k}(s)$ belongs to $L^1(0, \sqrt{T_0})$. Therefore, letting $\bar{\lambda} \to 0^+$ in (112) and using (111), we deduce that, for all $\lambda \in (0, \sqrt{T_0})$,

$$u_{m,k}(\lambda) = -2\lambda^{2\gamma} \int_0^\lambda s^{1-2\gamma}\xi_{m,k}(s)\, ds. \tag{122}$$

From (121) and (122), we obtain that, for all $(m, k) \in J_0$ and $\lambda \in (0, \sqrt{T_0})$,

$$|u_{m,k}(\lambda)| \leqslant \frac{2C_8}{\tilde{\delta}} \lambda^{2\gamma+\tilde{\delta}}. \tag{123}$$

Let us fix $\sigma \in (0, \tilde{\delta})$; by Lemma 4.13, there exists $K_2(\sigma)$ such that

$$H(\lambda^2) \geqslant K_2(\sigma)\lambda^{2(2\gamma+\sigma)} \quad \text{for } \lambda \in (0, \sqrt{T_0}).$$

Therefore, in view of (123), for all $(m, k) \in J_0$ and $\lambda \in (0, \sqrt{T_0})$,

$$\frac{|u_{m,k}(\lambda)|}{\sqrt{H(\lambda^2)}} \leqslant \frac{2C_8}{\tilde{\delta}\sqrt{K_2(\sigma)}} \lambda^{\tilde{\delta}-\sigma}$$

and hence

$$\frac{u_{m,k}(\lambda)}{\sqrt{H(\lambda^2)}} \to 0 \quad \text{as } \lambda \to 0^+. \tag{124}$$

On the other hand, by Lemma 5.1, for every sequence $\lambda_n \to 0^+$, there exists a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$ and an eigenfunction $g \in E_0 \setminus \{0\}$ of the operator L associated to γ such that

$$\frac{u_{\lambda_{n_j}}(x,1)}{\sqrt{H(\lambda_{n_j}^2)}} \to g \quad \text{in } \mathcal{L} \quad \text{as } j \to +\infty,$$

thus implying, for all $(m, k) \in J_0$,

$$\frac{u_{m,k}(\lambda_{n_j})}{\sqrt{H(\lambda_{n_j}^2)}} = \left(\frac{u_{\lambda_{n_j}}(x,1)}{\sqrt{H(\lambda_{n_j}^2)}}, \widetilde{V}_{m,k} \right)_{\mathcal{L}} \to (g, \widetilde{V}_{m,k})_{\mathcal{L}} \quad \text{as } j \to +\infty. \tag{125}$$

From (124) and (125), we deduce that $(g, \widetilde{V}_{m,k})_{\mathcal{L}} = 0$ for all $(m, k) \in J_0$. Since $g \in E_0$ and $\{\widetilde{V}_{m,k} : (m, k) \in J_0\}$ is an orthonormal basis of E_0, this implies that $g = 0$, a contradiction. □

We now complete the description of the asymptotics of solutions by combining Lemmas 5.1 and 5.3 and obtaining some convergence of the blowed-up solution continuously as $\lambda \to 0^+$ and not only along subsequences, thus proving Theorems 1.5 and 1.6.

Proof of Theorems 1.5 and 1.6. Identities (21) and (26) follow from part (i) of Lemma 5.1 and Proposition 1.4, which imply that there exists an eigenvalue $\gamma_{m_0,k_0} = m_0 - \frac{\alpha_{k_0}}{2}$ of L, $m_0, k_0 \in \mathbb{N}$,

$k_0 \geqslant 1$, such that $\gamma = \lim_{t\to 0^+} \mathcal{N}(t) = \gamma_{m_0,k_0}$. Let E_0 be the associated eigenspace and J_0 the finite set of indices $\{(m,k) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\}) : \gamma_{m_0,k_0} = m - \frac{\alpha_k}{2}\}$, so that $\{\widetilde{V}_{m,k} : (m,k) \in J_0\}$, with the $\widetilde{V}_{m,k}$'s as in Remark 3.4, is an orthonormal basis of E_0. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ such that $\lim_{n\to+\infty} \lambda_n = 0$. Then, from part (ii) of Lemma 5.1 and Lemmas 5.2 and 5.3, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and real numbers $\{\beta_{n,j} : (n,j) \in J_0\}$ such that $\beta_{n,j} \neq 0$ for some $(n,j) \in J_0$ and, for any $\tau \in (0,1)$,

$$\lim_{k\to+\infty} \int_\tau^1 \left\| \lambda_{n_k}^{-2\gamma} u(\lambda_{n_k} x, \lambda_{n_k}^2 t) - t^\gamma \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{V}_{n,j}(x/\sqrt{t}) \right\|_{\mathcal{H}_t}^2 dt = 0 \tag{126}$$

and

$$\lim_{k\to+\infty} \sup_{t\in[\tau,1]} \left\| \lambda_{n_k}^{-2\gamma} u(\lambda_{n_k} x, \lambda_{n_k}^2 t) - t^\gamma \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{V}_{n,j}(x/\sqrt{t}) \right\|_{\mathcal{L}_t} = 0. \tag{127}$$

In particular,

$$\lambda_{n_k}^{-2\gamma} u(\lambda_{n_k} x, \lambda_{n_k}^2) \underset{k\to+\infty}{\longrightarrow} \sum_{(n,j)\in J_0} \beta_{n,j} \widetilde{V}_{n,j}(x) \quad \text{in } \mathcal{L}. \tag{128}$$

We now prove that the $\beta_{n,j}$'s depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Let us fix $\Lambda \in (0, \sqrt{T_0})$ and define $u_{m,i}$ and $\xi_{m,i}$ as in (109-110). By expanding $u_\lambda(x,1) = u(\lambda x, \lambda^2) \in \mathcal{L}$ in Fourier series as in (108), from (128) it follows that, for any $(m,i) \in J_0$,

$$\lambda_{n_k}^{-2\gamma} u_{m,i}(\lambda_{n_k}) \to \sum_{(n,j)\in J_0} \beta_{n,j} \int_{\mathbb{R}^N} \widetilde{V}_{n,j}(x) \widetilde{V}_{m,i}(x) G(x,1)\, dx = \beta_{m,i} \tag{129}$$

as $k \to +\infty$. As deduced in the proof of Lemma 5.3 (see (112)), for any $(m,i) \in J_0$ and $\lambda \in (0,\Lambda)$ there holds

$$u_{m,i}(\lambda) = \lambda^{2\gamma} \left(\Lambda^{-2\gamma} u_{m,i}(\Lambda) + 2 \int_\lambda^\Lambda s^{1-2\gamma} \xi_{m,i}(s)\, ds \right). \tag{130}$$

Furthermore, arguing again as in Lemma 5.3 (see (121)), $s \mapsto s^{1-2\gamma}\xi_{m,i}(s)$ belongs to $L^1(0,\sqrt{T_0})$. Hence, combining (129) and (130), we obtain, for every $(m,i) \in J_0$,

$$\begin{aligned}
\beta_{m,i} &= \Lambda^{-2\gamma} u_{m,i}(\Lambda) + 2 \int_0^\Lambda s^{1-2\gamma} \xi_{m,i}(s)\, ds \\
&= \Lambda^{-2\gamma} \int_{\mathbb{R}^N} u(\Lambda x, \Lambda^2) \widetilde{V}_{m,i}(x) G(x,1)\, dx \\
&\qquad + 2 \int_0^\Lambda s^{1-2\gamma} \left(\int_{\mathbb{R}^N} f(sx, s^2, u(sx,s^2)) \widetilde{V}_{m,i}(x) G(x,1)\, dx \right) ds.
\end{aligned}$$

In particular the $\beta_{m,i}$'s depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergences in (126) and (127) actually hold as $\lambda \to 0^+$ and proving the theorems. □

The strong unique continuation property is a direct consequence of Theorems 1.5 and 1.6.

Proof of Corollary 1.7. Let us assume by contradiction that $u \not\equiv 0$ in $\mathbb{R}^N \times (0,T)$ and fix $k \in \mathbb{N}$ such that $k > \gamma$, with $\gamma = \gamma_{m_0,k_0}$ as in Theorems 1.5 and 1.6. From assumption (28), it follows that, for a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$,

$$\lim_{\lambda \to 0^+} |\lambda^{-2\gamma} t^{-\gamma} u(\lambda x, \lambda^2 t)| = 0. \tag{131}$$

On the other hand, from Theorems 1.5 and 1.6, it follows that there exists $g \in \mathcal{H} \setminus \{0\}$ such that g is an eigenfunction of the operator L associated to γ and, for all $t \in (0,1)$ and a.e. $x \in \mathbb{R}^N$,

$$\lambda^{-2\gamma} t^{-\gamma} u(\lambda x, \lambda^2 t) \to g(x/\sqrt{t}),$$

which, in view of (131), implies $g \equiv 0$, a contradiction. □

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