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Multiplicity of solutions to elliptic equations

The case of singular potentials in second order problems and Morse theory in a fourth order problem

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To my family,
and in particular to Marco.

To our friends, who always help us in growing up.

Contents

Introduction

In the context of partial differential equations, a *stationary* Schrödinger operator takes the form

$$-\Delta + q(x).$$

The absence of an explicit dependence on the time variable makes it different from the usual Schrödinger operator. One can look at solutions to stationary Schrödinger equations as solutions to Schrödinger equations which do not depend on time: the so-called *stationary solutions*. Here Δ stands for the Laplace operator and $q(x)$ is a real-valued function. According to some possible integrability properties of q, this operator will share some particular spectral properties with $-\Delta$ as it will be briefly discussed later.

We will be interested in solutions to stationary Schrödinger equations in the whole space $\mathbb{R}^N$, with critical growth. This choice presents several different features from the bounded and subcritical case: most of all, the lack of compactness. As it is wellknown, the bounded case enjoys several compactness properties, namely the Sobolev embeddings, which hold true up to the critical threshold (not included) and may help in proving the existence of solutions via variational methods. On the contrary, here we have at least two reasons for losing compactness: the unbounded domain and the critical growth.

Nevertheless, in both these cases, something can be made. A widespread strategy to regain compactness is to exploit symmetries the equation is invariant for. For instance, we recall the wellknown result by A. Strauss:

Theorem (Strauss) *$H^1_{rad}(\mathbb{R}^N)$ is compactly embedded in $L^{p+1}(\mathbb{R}^N)$ for all $p \in \big(1, (N+2)/(N-2)\big)$, being H^1_{rad} the set of all H^1 functions which are radially symmetric.*

Together with the symmetry strategy, one can develop concentration-compactness arguments, both in the form due to Willem and Lions, which involves measure convergence, and in the special form due to Solimini ([58]). We will take advantage from the second statement, which provides a sort of decomposition of minimizing sequences in many spread away profiles.

Investigating possible existence of solution to equations, one of the main issues is what kind of solution one is interested in, which is strictly related to the question of what functions spaces are involved. To overcome an understandable initial confusion, one can consider the space of compactly supported functions $C^\infty_C(\Omega)$, being Ω bounded or not, and close this space under the norm the equation suggests. In case of $\mathbb{R}^N$ and Schrödinger equation, the final space turns out to be the so-called $D^{1,2}(\mathbb{R}^N)$, which may be also characterized as the space of all $L^{2^*}(\mathbb{R}^N)$ functions which have the gradient square-integrable. By definition it follows that the space $D^{1,2}(\mathbb{R}^N)$ is continuosly embedded in $L^{2^*}(\mathbb{R}^N)$, as well as it happens for bounded domains. Of course, even in this case the embedding is not compact.

Nevertheless, the best Sobolev constant's definition is well-posed. Moreover, some remarkable facts hold about it. First, the L^{2^*} Sobolev constant is independent from the domain, but it is achieved if and only if the domain is the whole $\mathbb{R}^N$. When investigating joint problems of critical growth on unbounded domains, these facts turn out to be crucial. Indeed, they make possible certain comparisons between critical levels of the functional, and pose some questions about what symmetries the solutions enjoy, as we shall have chance to see. As a hint, we recall that the best Sobolev constant is achieved by an entire family of functions, the so-called Talenti's functions, which are of the form

$$\frac{\big(N(N-2)\big)^{(N-2)/4}}{\big(1+|x|^2\big)^{(N-2)/2}}$$

and which clearly enjoy important symmetries.

A further step on the involved operator

We are interested in *magnetic Schrödinger operators*, where the *magnetic Laplacian* takes the place of the usual Laplacian. It is defined as $(i\nabla - A)^2$ where $A:\mathbb{R}^N \to \mathbb{R}^N$ stands for a vector field. Such operators occur in quantum mechanics when the dimension is restricted to $N=3$ to describe the Hamiltonian associated to a charged particle in an electromagnetic field. More precisely, the Hamiltonian is given by

$$(i\nabla - A)^2 + V$$

where $V:\mathbb{R}^N \to \mathbb{R}$ is the electric potential. In this context a magnetic field $B:\mathbb{R}^3 \to \mathbb{R}^3$ is usually introduced, so that A is the vector magnetic potential which satisfies

$$\mathrm{curl} A = B.$$

In higher dimensions, B will be intended as the differential 2–form $B = da$, a being the 1–form canonically associated to the vector field A.

At the very beginning, no regularity is needed for the magnetic potential. In general A is not even a bounded vector field: for example, if B is the constant vector field $(0,0,1)$, then a suitable vector potential A is given by $(-x_2,0,0)$. A is not needed to be smooth either, since we could add an arbitrary gradient to A and still get the same magnetic field. This important property is called the *gauge invariance*. As a consequence the potential A could be a wild function even if the magnetic field has a nice behavior. For these reasons it is usual to fix the regularity grade as $V, A_j \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, in order to make distributional sense to $(i\nabla - A)^2 + V$ when acting on $L^2_{\text{loc}}(\mathbb{R}^N)$ functions. Indeed, with this choice, $(i\nabla - A)f + Vf$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N)$ so that it is a distribution for every $f \in L^2_{\text{loc}}(\mathbb{R}^N)$.

By the way, we are taking into consideration "very singular" potential, such as $A(\theta)/|x|$ for magnetic potential and $a(\theta)/|x|^2$ for the electric one, being $\theta \in \mathbb{S}^N$ the angular component of x and $|x|$ the radial one. Such magnetic potentials appear in a physical context as limits of thin solenoids, when the circulation remains constant as the sequence of solenoids' radii tends to zero. The limiting vector field is then a singular measure supported in a lower dimensional set. Though the resulting magnetic field vanishes almost everywhere, its presence still affects the spectrum of the operator, giving rise to the so-called "Aharonov-Bohm effect". From the mathematical point of view, this class of operators is worth being investigated, mainly because of their critical behaviour. Indeed they share with the Laplacian the same degree of homogeneity and invariance under the Kelvin transform, and therefore they cannot be regarded as lower order perturbations of the Laplace operator; in other words, they do not belong to the *Kato class*:

Definition *A potential $q(x)$ is said to belong to the* Kato class *if and only if the function*

$$M_q(x;r) := \int_{|x-y|<r} \frac{|q(y)|}{|x-y|}\, dy$$

converges to zero as $r \to 0$ uniformly with respect to $x \in \mathbb{R}^N$.

Potentials in the *Kato class* may bring additional spectral properties to the operator with respect to the general case, since they make the Schrödinger operator $-\Delta$-bounded with relative bound less then 1, so that it inherits peculiar spectral properties as essential selfadjointness:

Theorem (Kato-Rellich) *Suppose the operator S is self-adjoint, T is symmetric, and T is S-bounded with relative bound $s < 1$. Then $S + T$ is self-adjoint on the domain $D(S)$ of S and essentially self-adjoint on any core of S, that is any submanifold D of $D(S)$ such that the set $\{(u, Su) : u \in D\}$ is dense in the graph of S.*

On the other hand, in [33] some spectral properties are established for Schrödinger operators $-\Delta + q(x)$ with this kind of singular perturbations, namely

Theorem *Let $q(x) = \beta/|x|^2$, $N \geq 5$. Then $-\Delta + q(x)$ is essentially selfadjoint if and only if $\beta \geq \beta_0 := 1 - (\frac{N-2}{2})^2$. It is bounded from below if and only if $\beta \geq -(\frac{N-2}{2})^2$.*

However, our main concern is not operators' spectral properties, since it is a rather physical approach. We are interested in more "analytic" issues, like existence of solutions, their possible symmetry properties, and so on.

We are considering in particular magnetic Schrödinger equations with this kind of singular potentials together with a critical nonlinearity, so that our reference equation will be

$$\left(i\nabla - \frac{A(\theta)}{|x|}\right)^2 u - \frac{a(\theta)}{|x|^2} u = |u|^{2^*-2} u \qquad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1}$$

As already explained, we begin our analysis choosing a suitable space of functions in which the seeking for solutions makes sense. In Appendix B of this thesis the reader may find some basic facts about this kind of operators; among all, the suitable function space turns out again to be $D^{1,2}(\mathbb{R}^N)$.

Together with this sort of singularities, we are taking into account the "Aharonov-Bohm" type potentials. They present a singularity of the same degree as well, but on a subspace which has a higher codimension. Indeed, in $\mathbb{R}^2$ a vector potential associated to the Aharonov-Bohm magnetic field has the form

$$\mathcal{A}(x_1, x_2) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

where $\alpha \in \mathbb{R}$ stands for the circulation of $\mathcal{A}$ around the thin solenoid. Here we are considering the analogue of these potentials in $\mathbb{R}^N$ for $N \geq 4$, that is

$$\mathcal{A}(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0 \right) \qquad (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{R}^{N-2}.$$

In order to investigate the solutions' qualitative properties (mainly symmetries they enjoy), we found useful to restrict our analysis to the minima of the Sobolev quotient, which is now well-defined; moreover, when it will be needed, we will seek our solutions among functions which enjoy certain symmetry properties. By the mean of this choice, we will exploit the basic facts about best Sobolev constants previously mentioned.

Our main result on this theme can be stated as follows:

Theorem A *Assume $N \geq 4$, $a(\theta) \equiv a \in \mathbb{R}^-$ and $A(\theta)$ is equivariant for the group action of $SO(2) \times SO(N-2)$, that is $A(g\theta) = gA(\theta)$ for all $g \in SO(2) \times SO(N-2)$ and for all $\theta \in \mathbb{S}^N$. Then there exist $a^* < 0$ such that, when $a < a^*$, the equation (1.2) admits at least two distinct solutions in $D^{1,2}(\mathbb{R}^N)$: one is $SO(2) \times SO(N-2)$-invariant, while the second one in only $\mathbb{Z}_k \times SO(N-2)$-invariant for some integer k.*

A similar result holds for Aharonov-Bohm type potentials.

Further, such a symmetry breaking occurs for solutions which have a non-zero angular momenta in $\mathbb{R}^2$.

As far as we know this is the first result in literature regarding multiplicity of solutions to Schrödinger equations with such singular potentials, except for those contained in [21], which exhibit the existence of at least a so–called "biradial" solution for Aharonov–Bohm potentials when the dimension is restricted to $N = 3$. Nevertheless we imagine their argument might be extended even in further dimensions.

The proof of Theorem A relies on a preliminary self-sufficient result, which is worth being stated on its own:

Theorem B *Let $u \in D^{1,2}(\mathbb{R}^N)$ be a biradial solution (i.e. invariant under a toric group of rotations) to*

$$-\Delta u = \frac{a}{|x|^2} u + f(|x|, u)$$

with $a < \left(\frac{N-2}{2}\right)^2$ *and* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *being a Carathéodory function,* C^1 *with respect to* z*, such that it satisfies the growth restriction*

$$\left|f'_y(|x|\,, y)\right| \leq C(1 + |y|^{2^*-2})$$

for a.e. $x \in \mathbb{R}^N$ *and for all* $y \in \mathbb{C}$*.*

If the solution u *has biradial Morse index* $m(u) \leq 1$*, then* u *is radially symmetric.*

The two proofs are quite different. The first one relies on classical symmetry breaking methods, the second one is rather a "geometric" proof.

Symmetry breaking methods are based on comparisons between different levels of the funtional. For instance, assume that these levels are always achieved, and the functional level over the whole space H^1 is strictly less than the level over H^1_{rad}, then there will exist at least two solutions: one will be radially symmetric, and the other one not. Many issues are hidden in this apparently simple statement. First of all, the possible achievement of the Sobolev constants. What conditions can assure it? When considering the Rayleigh quotient on $D^{1,2}(\mathbb{R}^N)$, a sufficient condition which guarantees existence of solution is merely that the quotient is strictly less than the best Sobolev constant. To prove it one can use a concentration–compactness argument. Here we propose a similar argument for our "electromagnetic infima" taking into account Solimini's statement. As already mentioned, it states for any bounded sequence u_n the existence of a sequence of profiles ϕ_i and a sequence of mutually divergent rescalings of the form $\rho^i_n(u) := (\lambda^i_n)^{(N-2)/2}u(x_n + \lambda^i_n(x - x_n))$ such that - up to subsequences - u_n is $\sum_i \rho^i_n \phi_i$ in L^{2^*} up to $o(1)$. Thanks to the quotient's invariance under Solimini's rescaling, we are able immediately to exclude concentration and vanishing of the solution, and the aforementioned condition will be sufficient to avoid the translation divergence. When the discrete group of symmetries $\mathbb{Z}_k \times SO(N-2)$ is introduced, defined as

$$u(z, y) \;\longmapsto\; u(\mathrm{e}^{i\frac{2\pi}{k}}z, Ry) \quad \text{for } (z, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2},\; k \in \mathbb{N},\; R \in SO(N-2),$$

the corresponding condition will be likely related to the group order k. More precisely, the threshold which guarantees the achievement of the best constant is increasing proportionally to the best Sobolev constant of a factor $k^{2/N}$. This new upper bound can be deduced from some involved integral estimates and it is quite challenging to prove, especially in case of Aharonov–Bohm potentials. Secondly, suppose we are restricting the quotient under symmetry constraints: are the minima solutions to the unconstrained equation? The classical *Symmetric Criticality Principle* supplies a satisfying criterion: the compactness of the symmetry group is sufficient, and here this is clearly fulfilled. Thirdly, the estimation of the Rayleigh quotient. For what concerns this last point, the main part has been already done: we exploit the previous sufficient conditions to attain the infima and their corresponding thresholds. Only one point is left over: the comparison between the radial and biradial levels. The idea comes from the possibility to increase arbitrarily the group order k, eventually letting it to infinity. In the limit, one is expecting to find the so–called *biradial* functions, that are invariant by definition under the group action $SO(2) \times SO(N-2)$.

The natural question arises if biradial solutions are distinct from radial solutions. Here the second theorem plays a role.

As already mentioned, the proof of Theorem B has different features from the previous one: we have already said it is rather geometric. Although it works on an auxiliary linear equation, it relies on the simple idea of seeking for directions on the sphere which the solution is invariant along. Such directions will be identified by suitable vector fields on the tangent space. To do this, an information about the possible negativity of the associated quadratic form will be needed. We will obtain a minimum number of directions which the quadratic form is negative on, in order to compare it with the solution's Morse index. Moreover, to gain this information, some asymptotics of solutions turn out very useful. On the other hand, a similar result is proved on the sphere via conformal equivalence between manifolds and equations. Here a multiplicity of solutions is found in a paper by Ding [24]: they can be characterized by their Morse index and this allows us to prove the optimality of the condition on the Morse index.

One more little remark can be done about the hypothesis $m(u) \leq 1$. Recent literature indicates that, for general semilinear equations, solutions having low Morse index do likely possess extra symmetries. Moreover, this low Morse index is usually somehow related to the space dimension. About this, we cite [32, 50, 51]. Our result may be read in the spirit of this branch of research.

Morse index of solutions

Taking into account the minima of the Sobolev quotient, one is led to consider the corresponding Morse index. We recall it is defined as follows:

Definition *The* Morse index *of a solution u is the dimension of the maximal subspace of the space of all functions of $\mathcal{C}_0^\infty(\mathbb{R}^N \setminus \{0\})$ on which the quadratic form associated to the linearized equation at u is negative definite.*

For instance, usually the minima's Morse index is 1, since they are mountain pass solutions. When restricting to symmetric functions' subspaces, one can even consider the "symmetric" Morse index, which means that the previous definition is checked over the test functions which share a given symmetry.

If we look at the general Morse theory in finite dimension, we see it is developed by variational arguments: its main point is connecting the number of the negative eigenvalue of the hessian of the functional (in case this last is regular enough) with some topological properties of its sublevels. More precisely, if $[a, b] \subset \mathbb{R}$ does not contain critical values for the functional F, then the sublevels $\{x : F(x) \leq b\}$ and $\{x : F(x) \leq a\}$ are homotopically equivalent; on the other hand, if there is a critical value $c \in [a, b]$ such that its critical point has Morse index k, for any $\varepsilon > 0$ the sublevel $\{x : F(x) \leq c+\varepsilon\}$ is homotopically equivalent to $\{x : F(x) \leq c - \varepsilon\} \cup D^k$, where D^k stands for a k–cell. Similar arguments may be developed even in infinite dimension with the suitable modifications.

With the aim of getting more familiar with these kind of tecniques, we got interested in a problem of a fourth order elliptic equation with exponential nonlinearity.

Exponential growth appears as a natural nonlinearity when investigating elliptic equations of second order in domains in $\mathbb{R}^2$, as the Moser-Trudinger inequality shows. This kind of problems excited much interest because of their physical meaning, describing a mean field equation of Euler flows in mathematical physics or a self-dual condensates of some Chern–Simons–Higgs model in physics. On the other hand, from a mathematical point of view, these problems arise in general conformal geometry, in particular in prescribed mean curvature equations and conformally covariant operators. We recall a conformal change of metric on a compact Riemannian manifold of dimension n (M^n, g) is

$$g_w = \mathrm{e}^{2w}\, g.$$

Conformal geometry aims at studying conformal invariant operators and their associated invariants. A conformally covariant operator of bidegree (a, b) is an operator A if under $g_w = \mathrm{e}^{2w}\, g$ it holds true

$$A_{g_w}(\Phi) = \mathrm{e}^{-bw} A(\mathrm{e}^{aw}\Phi) \quad \forall \Phi \in C^\infty_C(M^n).$$

For instance, in dimension 2 the usual Laplace–Beltrami operator is conformally invariant, meaning conformally covariant of bidegree $(0, 2)$, whereas in higher dimensions one can define the so-called *conformal Laplacian*, defined as a suitable multiple of the Laplace–Beltrami plus the scalar curvature of the manifold, which is in fact conformally covariant of bidegree $\left(\frac{N-2}{2}, \frac{N+2}{2}\right)$. In dimension 4, we can also consider the *Paneitz operator*, which is defined as

$$P\varphi := \Delta^2\varphi + \operatorname{div}\left\{\frac{2}{3}Rg - 2Ric\right\} d\varphi \quad \forall \varphi \in C^\infty_C(M^4),$$

where Δ is again the Laplace–Beltrami operator, Ric is the Ricci tensor and d is the deRham differential. It is possible to check that this operator is conformally covariant of bidegree $(0, 4)$, that is

$$P_{g_w}(\Phi) = \mathrm{e}^{-4w} P(\Phi) \quad \forall \Phi \in C^\infty_C(M^4).$$

We recall in 2-dimensional case the Laplace-Beltrami operator is strictly related to the Gauss curvature K_g of the manifold (the function which maps every point onto the sectional curvature of the tangent plane in that point), in particular throughout the prescribed Gauss curvature equation

$$-\Delta_g w + K_g = K_{g_w}\mathrm{e}^{2w}.$$

As well as it happens in dimension 2for $-\Delta_g$, in the 4-dimensional case the Paneitz operator is related to the so-called *Q-curvature*, which is defined by means of Laplace–Beltrami, scalar curvature of the manifold, and modulus of the Ricci tensor. We skip some details, but the corresponding equation which shows this connection is the following:

$$P_g w + 2Q_g = 2Q_{g_w}\mathrm{e}^{4w}.$$

We found very interesting a tecnique developed in [48] by Malchiodi, which allows to connect Leray-Schauder degree counting formulas for such equations with direct methods of min-max principles via Morse theory and deformation lemmas. Here the argument is restricted to compact 4-dimensional manifolds without any boundary. In order to cover even the case with boundary, we started from the simplest case: a regular connected bounded domain Ω in $\mathbb{R}^4$. We gathered that the same tecnique as in [48] essentially applies in this case providing very slight modifications. Restricting our interest to analytic aspects of the equation, we skip any geometrical meaning of the equation and of the quantities involved; moreover, in our case the Paneitz operator reduces to Δ^2, the Q-curvature is a constant and the corresponding equation may become

$$\begin{cases} \Delta^2 u = \tau \dfrac{h(x)\mathrm{e}^u}{\int_\Omega h(x)\mathrm{e}^u} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where h is a $C^{2,\alpha}$ positive function for $\alpha \in (0,1)$ and τ a real positive parameter. We observe the problem has a variational structure and solutions may be found as critical points of the functional

$$I_\tau(u) := \frac{1}{2}\|u\|^2 - \tau \log\left(\frac{1}{|\Omega|}\int_\Omega h(x)\mathrm{e}^u\,dx\right),$$

where $\|\cdot\|$ denotes an equivalent norm over the space $\mathcal{H} := H^2(\Omega) \cap H_0^1(\Omega)$. The main result we reached can be stated as follows

Theorem C *If $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, the Leray–Schauder degree of the problem is given by*

$$\deg_{LS} = \binom{k - \chi(\Omega)}{k}.$$

We provide some preliminary remarks about the τ parameter. We avoid the case when τ is an integer multiple of $64\pi^2$ since by virtue of [40] some compactness properties hold true in this case, as for instance equiboundedness of solutions near the boundary. When $\tau < 64\pi^2$ it is quite simple to prove the functional is coercive, then the direct calculus of variations applies. On the contrary, when $\tau > 64\pi^2$ the functional in not bounded from below and its sublevels are not bounded. Moreover, via an improved Moser–Trudinger inequality it is possible to show that very low sublevels carry a sort of concentration of $\int_\Omega \mathrm{e}^u$ essentially in k points depending on u. These facts provoke several difficulties to face. In the direct calculus of variation, one of the main points is indeed the *Palais–Smale condition*, which ensures the convergence of each bounded Palais–Smale sequence. But here the boundedness of PS sequences fails. Another trouble source is the Poincaré–Hopf theorem. It provides a degree counting formula in terms of the Euler characteristic of the couple of the sublevels once the critical points in the interval are non degenerate. Here we can not assure it, so we will need to work with a suitable perturbation of the functional which will have the same LS degree. Another point is the counting of the characteristic $\chi(\mathcal{H}^b, \mathcal{H}^a)$. To overcome this last challenge, the strategy suggested in [48] is based once more on homotopic equivalences. The level b will be chosen so large that the sublevel $\mathcal{H}^b$ will be homotopic to the whole space $\mathcal{H}$; whereas a will be chosen so small that a certain homotopic equivalence can be established with the so-called *formal set of baricenters*, which is strictly related to the concentration of the exponential density mentioned above.

The results presented here form the core of three papers:

- L.A., S. Terracini, *Solutions to nonlinear Schrödinger equations with singular electromagnetic potential and critical exponent*, to appear in Journal of Fixed Point Theory and Applications.
- L.A., S. Terracini, *A note on the complete rotational invariance of biradial solutions to semilinear elliptic equations*, to appear in Advanced Nonlinear Studies (May, 2011).
- L.A., A. Portaluri, *Morse theory for a fourth order elliptic equation with exponential nonlinearity*, to appear in Nonlinear Differential Equations and Applications.

Chapter 1

Solutions to nonlinear Schrödinger equations with singular electromagnetic potential and critical exponent

1.1 Introduction

In norelativistic quantum mechanics, the Hamiltonian associated with a charged particle in an electromagnetic field is given by $(i\nabla - A)^2 + V$ where $A: \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential and $V: \mathbb{R}^N \to \mathbb{R}$ is the electric one. The vector field $B = \text{curl}A$ has to be intended as the differential 2–form $B = da$, a being the 1–form canonically associated with the vector field A. Only in three dimensions, by duality, B is represented by another vector field.

In this paper we are concerned with differential operators of the form

$$\left(i\nabla - \frac{A(\theta)}{|x|}\right)^2 - \frac{a(\theta)}{|x|^2}$$

where $A(\theta) \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a(\theta) \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$. Notice the presence of homogeneous (fuchsian) singularities at the origin. In some situations the potentials may also have singularities on the sphere.

This kind of magnetic potentials appear as limits of thin solenoids, when the circulation remains constant as the sequence of solenoids' radii tends to zero, The limiting vector field is then a singular measure supported in a lower dimensional set. Though the resulting magnetic field vanishes almost everywhere, its presence still affects the spectrum of the operator, giving rise to the so-called "Aharonov-Bohm effect".

Also from the mathematical point of view this class of operators is worty being investigated, mainly because of their critical behaviour. Indeed, they share with the Laplacian the same degree of homogeneity and invariance under the Kelvin transform. Therefore they cannot be regarded as lower order perturbations of the Laplace operator (they do not belong to the Kato's class: see fo instance [28], [30] and references therein).

Here we shall always assume $N \geq 3$, otherwise specified. A quadratic form is associated with the differential operator, that is

$$\int_{\mathbb{R}^N} \left| \left(i\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} \, u^2. \tag{1.1}$$

As its natural domain we shall take the closure of compactly supported functions $C_C^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to the quadratic form itself. Thanks to Hardy type inequalities, when $N \geq 3$, this space turns out to be the same $D^{1,2}(\mathbb{R}^N)$, provided a is suitably bounded ([28]), while, when $N = 2$, this is a smaller space of functions vanishing at the pole of the magnetic potential. Throughout the paper we shall always assume positivity of (1.1).

We are interested in solutions to the critial semilinear differential equations

$$\left(i\nabla - \frac{A(\theta)}{|x|} \right)^2 u - \frac{a(\theta)}{|x|^2} u = |u|^{2^*-2} \, u \qquad \text{in } \mathbb{R}^N \setminus \{0\} \tag{1.2}$$

and in particular in their symmetry properties. The critical exponent appears as the natural one whenever seeking finite energy solutions: indeed, Pohozaev type identitities prevent the existence of entire solutions for power nonlinearities of different degrees.

The first existence results for equations of type (1.2) are given in [26] for subcritical nonlinearities. In addition, existence and multiplicity of solutions are investigated for instance in [16, 20, 36, 57] mainly via variational methods and concentration-compactness arguments. Some results involving critical nonlinearities are present in [4, 14]. Concerning results on semiclassical solutions we quote [18, 19]. As far as we know, not many papers are concerned when electromagnetic potentials which are singular, except those in [35], where anyway several integrability hypotheses are assumed on them, and, much more related with ours, the paper [21] that we discuss later on.

We are interested in the existence of solutions to Equation (1.2) distinct by symmetry properties, as it happens in [62] for Schrödinger operators when magnetic vector potential is not present. To investigate these questions, we aim to extend some of the results contained in [62] when a singular electromagnetic potential is present.

To do this, we refer to solutions which minimize the Rayleigh quotient

$$\frac{\displaystyle\int_{\mathbb{R}^N} \left| \left(i\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} \, u^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}}.$$

We find useful to stress that, although, in general, ground states in $D^{1,2}(\mathbb{R}^N)$ to equation (1.2) do not exist (see Section 3), the existence of minimizers con be granted in suitable subspaces of symmetric functions.

We are concerned with Aharonov-Bohm type potentials too. In $\mathbb{R}^2$ a vector potential associated to the Aharonov-Bohm magnetic field has the form

$$\mathcal{A}(x_1, x_2) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

where $\alpha \in \mathbb{R}$ stands for the circulation of $\mathcal{A}$ around the thin solenoid. Here we consider the analogous of these potentials in $\mathbb{R}^N$ for $N \geq 4$, that is

$$\mathcal{A}(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0 \right) \qquad (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{R}^{N-2}.$$

Our main result can be stated as follows:

Theorem 1.1.1 *Assume $N \geq 4$, $a(\theta) \equiv a \in \mathbb{R}^-$ and $A(\theta)$ is equivariant for the group action of $SO(2) \times SO(N-2)$, that is $A(g\theta) = gA(\theta)$ for all $g \in SO(2) \times SO(N-2)$ and for all $\theta \in \mathbb{S}^N$. Then there exist $a^* < 0$ such that, when $a < a^*$, the equation (1.2) admits at least two distinct solutions in $D^{1,2}(\mathbb{R}^N)$: one is $SO(2) \times SO(N-2)$-invariant, while the second one in only $\mathbb{Z}_k \times SO(N-2)$-invariant for some integer k.*

A similar result holds for Aharonov-Bohm type potentials.

Further, such a symmetry breaking occurs for solutions which have a non-zero angular momenta in $\mathbb{R}^2$.

We recall a function u is said to be G-invariant if $u(gx) = u(x)$ for every $g \in G$ and $x \in \mathbb{R}^N$.

We point out hypothesis on the dimension is purely technical here. By the way, in dimension $N = 3$ and in case of Aharonov-Bohm potentials, Clapp and Szulkin proved in [21] the existence of at least a solution which enjoys the so-called *biradial* symmetry. However, their argument may be adapted even in further dimensions, provided a cylindrical symmetry is asked to functions with respect to the second set of variables in $\mathbb{R}^{N-2}$.

The proof of our main result is based on a comparison between the different levels of the Rayleigh quotient's infima taken over different spaces of functions which enjoy certain symmetry properties. In particular, we will focus our attention on three different kinds of symmetries:

1. functions which are equivariant under the $\mathbb{Z}_k \times SO(N-2)$ action for $k \in \mathbb{N}$, $m \in \mathbb{Z}$ defined as
$$u(z, y) \mapsto \mathrm{e}^{-i\frac{2\pi}{k}m} u(\mathrm{e}^{i\frac{2\pi}{k}} z, Ry) \qquad \text{for } z \in \mathbb{R}^2 \text{ and } y \in \mathbb{R}^{N-2},\ R \in SO(N-2),$$
$D^{1,2}_{k,m}(\mathbb{R}^N)$ will denote their vector space;
2. functions which we will call "biradial", i.e.
$$D^{1,2}_{\mathrm{birad},m}(\mathbb{R}^N) := \{u \in D^{1,2}(\mathbb{R}^N) : \ u(Sz, Ty) = S^m u(z, y), \\ \forall\ (S, T) \in SO(2) \times SO(N-2)\},$$
so that they have the form $u(z, y) = \rho(|z|, |y|)\mathrm{e}^{im\theta(z)}$ where $\theta(z) = \arg(z)$;

3. functions which are radial, $D^{1,2}_{rad}$ will be their vector space.

We fix the notation we will use throughout the paper:

Definition 1.1.2 $S^{birad,m}_{A,a}$ *is the minimum of the Rayleigh quotient related to the magnetic Laplacian over all the biradial functions in* $D^{1,2}_{birad,m}(\mathbb{R}^N)$*;*

$S^{birad,m}_{0,a}$ *is the minimum of the Rayleigh quotient related to the usual Laplacian over all the biradial functions in* $D^{1,2}(\mathbb{R}^N)$*;*

$S^{rad}_{0,a}$ *is the minimum of the Rayleigh quotient related to the usual Laplacian over all the radial functions in* $D^{1,2}(\mathbb{R}^N)$*;*

$S^{k,m}_{0,a}$ *is the minimum of the Rayleigh quotient related to the usual Laplacian over all the functions in* $D^{1,2}_{k,m}(\mathbb{R}^N)$*;*

$S^{k,m}_{A,a}$ *is the minimum of the Rayleigh quotient related to the magnetic Laplacian over all the functions in* $D^{1,2}_{k,m}(\mathbb{R}^N)$*;*

S *is the usual Sobolev constant for the immersion* $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$*.*

In order to prove these quantities are achieved, we use concentration-compactness arguments, in a special form due to Solimini in [58]. Unfortunately, we are not able to compute the precise values of the abovementioned infima, but only to provide estimates in terms of the Sobolev constant S; nevertheless this is enough to our aims. By the way, it is worth being noticed in [62] a characterization is given for the radial case $S^{rad}_{0,a}$: it is proved $S^{rad}_{0,a}$ is achieved and the author is able to compute its precise value. This will turn out basic when we compare it with the other infimum values in order to deduce some results about symmetry properties.

Both in case of $\frac{A(\theta)}{|x|}$ type potentials and Aharonov-Bohm type potentials, we follow the same outline. We organize the paper as follows: first of all in Section 2 we state the variational framework for our problem; secondly in Section 3 we provide some sufficient conditions to have the infimum of the Rayleigh quotients achieved, beginning from some simple particular cases; in Section 4 we investigate the potential symmetry of solutions; finally in Section 6 we deduce our main result. On the other hand, Section 5 is devoted to the study of Aharonov-Bohm type potentials.

1.2 Variational setting

As initial domain for the quadratic form (1.1) we take the space of compactly supported functions in $\mathbb{R}^N \setminus \{0\}$: we denote it $C^\infty_C(\mathbb{R}^N \setminus \{0\})$. Actually, as a consequence of the following lemmas, one can consider the space $D^{1,2}(\mathbb{R}^N)$ as the maximal domain for the quadratic form. We recall that by definition $D^{1,2}(\mathbb{R}^N) = \overline{C^\infty_C(\mathbb{R}^N)}^{(\int_{\mathbb{R}^N}|\nabla u|^2)^{1/2}}$, i.e. the completion of the compact supported functions on $\mathbb{R}^N$ under the so-called Dirichlet norm.

The main tools for this are the following basic inequalities:

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx &\leq \frac{4}{(N-2)^2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad \text{(Hardy inequality)}\\ \int_{\mathbb{R}^N} |\nabla |u||^2\,dx &\leq \int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A}{|x|}\right)u\right|^2 dx \quad \text{(diamagnetic inequality)}\end{aligned}$$

both with the following lemmas

Lemma 1.2.1 *The completion of $C_C^\infty(\mathbb{R}^N\setminus\{0\})$ under the Dirichlet norm coincide with the space $D^{1,2}(\mathbb{R}^N)$.*

Lemma 1.2.2 *If $A \in L^\infty(\mathbb{S}^{N-1})$ then the norm $\left(\int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2\right)^{1/2}$ is equivalent to the Dirichlet norm on $C_C^\infty(\mathbb{R}^N\setminus\{0\})$.*

Lemma 1.2.3 *The quadratic form (1.1) is equivalent to $Q_A(u) = \int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2$ on its maximal domain $D^{1,2}(\mathbb{R}^N)$ provided $\|a\|_\infty < (N-2)^2/4$. Moreover, it is positive definite.*

We refer to [28] for a deeper analysis of these questions.

We set the following variational problem

$$S_{A,a} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2 - \int_{\mathbb{R}^N}\frac{a(\theta)}{|x|^2}\,u^2}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}}. \tag{1.3}$$

Of course, $S_{A,a}$ is strictly positive since the quadratic form (1.1) is positive definite.

We are now proposing a lemma which will be useful later.

Lemma 1.2.4 *Let $\{x_n\}$ be a sequence of points such that $|x_n|\to\infty$ as $n\to\infty$. Then for any $u \in D^{1,2}(\mathbb{R}^N)$ as $n\to\infty$ we have*

$$\frac{\int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u(\cdot+x_n)\right|^2 - \int_{\mathbb{R}^N}\frac{a(\theta)}{|x|^2}\,|u(\cdot+x_n)|^2}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}} \to \frac{\int_{\mathbb{R}^N}|\nabla u|^2}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}}.$$

Proof. It is sufficient to prove for all $\varepsilon > 0$ there exists a $\overline{n}$ such that $\int_{\mathbb{R}^N} \frac{|u(x+x_n)|^2}{|x|^2} dx < 2\varepsilon$ for $n \geq \overline{n}$. Let us consider $R > 0$ big enough to have

$$\int_{\mathbb{R}^N \setminus B_R(x_n)} \frac{|u(x+x_n)|^2}{|x|^2} dx < \varepsilon$$

for every $n \in \mathbb{N}$. On the other hand, when $x \in B_R(x_n)$ we have $|x| \geq |x_n| - |x - x_n| \geq |x_n| - R$ which is a positive quantity for n big enough. In this way

$$\int_{B_R(x_n)} \frac{|u(x+x_n)|^2}{|x|^2} dx \leq \frac{1}{(|x_n| - R)^2} \int_{B_R(x_n)} |u(x+x_n)|^2 \, dx < \varepsilon$$

for n big enough. □

Exploiting this lemma, we can state the following property holding for $S_{A,a}$:

Proposition 1.2.5 *If S denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$, i.e.*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} \quad , \tag{1.4}$$

it holds $S_{A,a} \leq S$.

Proof. Lemma (1.2.4) shows immediately for all $u \in D^{1,2}(\mathbb{R}^N)$

$$S_{A,a} \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} + o(1) \ .$$

If we choose a minimizing sequence for (1.4) in the line above, we see immediately $S_{A,a} \leq S$. □

1.3 Attaining the infimum

Given the results in [13] due to Brezis and Nirenberg, one could expect that ,if $S_{A,a}$ is strictly less than S, then it is attained. Here we pursue this idea with concentration-compactness arguments, in the special version due to Solimini in [58]. Before proceeding, we find useful to recall some definitions about the so-called *Lorentz spaces*.

Definition 1.3.1 *[58] A* Lorentz space *$L^{p,q}(\mathbb{R}^N)$ is a space of measurable functions affected by two indexes p and q which are two positive real numbers, $1 \leq p\,, q \leq +\infty$, like the indexes* *which determine the usual L^p spaces. The index p is called* principal index *and the index q is called* secundary index. *A monotonicity property holds with respect to the secundary index: if $q_1 < q_2$ then $L^{p,q_1} \subset L^{p,q_2}$. So the strongest case of a Lorentz space with principal index p is $L^{p,1}$; while the weakest case is $L^{p,\infty}$, which is equivalent to the so-called* weak L^p space, *or Marcinkiewicz space. Anyway, the most familiar case of Lorentz space is the intermediate case given by $q = p$, since the space $L^{p,p}$ is equivalent to the classical L^p space.*

Properties 1.3.2 *[58] A basic property about the Lorentz spaces is an appropriate case of the Hölder inequality, which states that the duality product of two functions is bounded by a constant times the product of the norms of the two functions in two respective conjugate Lorentz spaces L^{p_1,q_1} and L^{p_2,q_2} where the two pairs of indexes satisfy the relations $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$.*

Moreover, if we consider the Sobolev space $H^{1,p}(\mathbb{R}^N)$, it is wellknown it is embedded in the Lebesgue space $L^{p^*}(\mathbb{R}^N)$. But this embedding is not optimal: it holds that the space $H^{1,p}(\mathbb{R}^N)$ is embedded in the Lorentz space $L^{p^*,p}$, which is strictly stronger than $L^{p^*} = L^{p^*,p^*}$.

Theorem 1.3.3 ***(Solimini)****([58]) Let $(u_n)_{n\in\mathbb{N}}$ be a given bounded sequence of functions in $H^{1,p}(\mathbb{R}^N)$, with the index p satisfing $1 < p < N$. Then, replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, we can find a sequence of functions $(\phi_i)_{i\in\mathbb{N}}$ belonging to $H^{1,p}(\mathbb{R}^N)$ and, in correspondence of any index n, we can find a sequence of rescalings $(\rho^i_n)_{i\in\mathbb{N}}$ in such a way that the sequence $(\rho^i_n(\phi_i))_{i\in\mathbb{N}}$ is summable in $H^{1,p}(\mathbb{R}^N)$, uniformly with respect to n, and that the sequence $(u_n - \sum_{i\in\mathbb{N}} \rho^i_n(\phi_i))_{n\in\mathbb{N}}$ converges to zero in $L(p^*, q)$ for every index $q > p$.*

Moreover we have that, for any pair of indexes i and j, the two corresponding sequences of rescalings $(\rho^i_n)_{n\in\mathbb{N}}$ and $(\rho^j_n)_{n\in\mathbb{N}}$ are mutually diverging, that

$$\sum_{i=1}^{+\infty} \|\phi_i\|^p_{1,p} \leq M \ , \tag{1.5}$$

where M is the limit of $(\|u_n\|^p_{1,p})_{n\in\mathbb{N}}$, and that the sequence $(u_n - \sum_{i\in\mathbb{N}} \rho^i_n(\phi_i))_{n\in\mathbb{N}}$ converges to zero in $H^{1,p}(\mathbb{R}^N)$ if and only if (1.5) is an equality.

Now we can state the result

Theorem 1.3.4 *If $S_{A,a} < S$ then $S_{A,a}$ is attained.*

Proof. Let us consider a minimizing sequence $u_n \in D^{1,2}(\mathbb{R}^N)$ to $S_{A,a}$. In particular, it is bounded in $D^{1,2}(\mathbb{R}^N)$. By Solimini's theorem (1.3.3), up to subsequences, there will exist a sequence $\phi_i \in D^{1,2}(\mathbb{R}^N)$ and a sequence of mutually divergent rescalings ρ^i_n defined as $\rho^i_n(u) = (\lambda^i_n)^{\frac{N-2}{2}} u(\lambda^i_n x + y^i_n)$, such that $\sum_i \rho^i_n \phi_i \in D^{1,2}(\mathbb{R}^N)$ and $u_n - \sum_i \rho^i_n \phi_i \to 0$ in L^{2^*}. In general the rescalings may be mutually divergent by dilation (concentration or vanishing) or by translation. We divide the proof in two different cases.

1. Suppose there exists at least an index $\bar{j}$ such that the sequence of the corresponding translation remains bounded: $\left|y_n^{\bar{j}}\right| \leq const$ for all n. Then we consider $\widetilde{u}_n := (\rho_n^{\bar{j}})^{-1}(u_n)$, which is again a minimizing sequence. The following convergence can be stated
$$\widetilde{u}_n - \phi_{\bar{j}} + \sum_{j \neq \bar{j}} (\rho_n^{\bar{j}})^{-1} \rho_n^j \phi_j \longrightarrow 0 \qquad \text{in } L^{2^*}.$$
If we call for a moment $v_n = \sum_{j \neq \bar{j}} (\rho_n^{\bar{j}})^{-1} \rho_n^j \phi_j$, we have that $v_n \to 0$ a.e. in $\mathbb{R}^N$ because of the mutual rescalings' divergence. Then $\widetilde{u}_n \to \phi_{\bar{j}}$ a.e. in $\mathbb{R}^N$. If we assume the sequence $\widetilde{u}_n$ is normalized in L^{2^*}, the famous Brezis–Lieb lemma applies and we immediately obtain the relation
$$\|\widetilde{u}_n\|_{2^*} = \left\|\phi_{\bar{j}}\right\|_{2^*} + \|v_n\|_{2^*} + o(1) \qquad \text{as } n \to \infty.$$
At the same time even
$$\|\widetilde{u}_n\|_{D^{1,2}(\mathbb{R}^N)} = \left\|\phi_{\bar{j}}\right\|_{D^{1,2}(\mathbb{R}^N)} + \|v_n\|_{D^{1,2}(\mathbb{R}^N)} + o(1) \qquad \text{as } n \to \infty.$$
So that
$$\begin{aligned} S_{A,a} \quad &\leftarrow \quad \frac{\int_{\mathbb{R}^N} \left|\nabla_A \phi_{\bar{j}}\right|^2 + \int_{\mathbb{R}^N} |\nabla_A v_n|^2 + o(1)}{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*} + \int_{\mathbb{R}^N} |v_n|^{2^*} + o(1)\right)^{2/2^*}} \\ &\geq \quad S_{A,a} \frac{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*}\right)^{2/2^*} + \left(\int_{\mathbb{R}^N} |v_n|^{2^*}\right)^{2/2^*} + o(1)}{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*} + \int_{\mathbb{R}^N} |v_n|^{2^*} + o(1)\right)^{2/2^*}}, \end{aligned}$$
and in order not to fall in contradiction the previous coefficient must tend to zero, and then $\int_{\mathbb{R}^N} |v_n|^{2^*} \to 0$, and also $\|v_n\|_{D^{1,2}(\mathbb{R}^N)} \to 0$, which in particular implies that $\phi_{\bar{j}}$ is a nontrivial function. In conclusion, we have the strong $D^{1,2}(\mathbb{R}^N)$ convergence $u_n(\cdot) - \phi_{\bar{j}} \to 0$, since we have an equality in (1.5) in Theorem (1.3.3).

2. On the other hand, if for all j $|y_n^j| \to \infty$ as $n \to \infty$, then we argue in the following way: let us fix $m \in \mathbb{N}$ and evaluete the quadratic form over the difference $u_n - \sum_{j=1}^m \rho_n^j \phi_j$. Since it is equivalent to the $D^{1,2}(\mathbb{R}^N)$-norm, it will be greater or equal to zero. So

that

$$\begin{aligned}
0 &\leq Q_{A,a}\left(u_n - \sum_{j=1}^{m} \rho_n^j \phi_j\right) \\
&= Q_{A,a}(u_n) + Q_{A,a}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \nabla_A u_n \cdot \nabla_A \rho_n^j \phi_j - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{a}{|x|^2} u_n \, \rho_n^j \phi_j \\
&= Q_{A,a}(u_n) + Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \nabla_A u_n \cdot \nabla_A \rho_n^j \phi_j + o(1) \\
&= Q_{A,a}(u_n) + Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) + o(1) \\
&= Q_{A,a}(u_n) - Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) + o(1) \qquad \text{for any } m,
\end{aligned}$$

thanks to the mutual divergence of the rescalings (see also [63]). Then we have

$$\frac{Q_{A,a}(u_n)}{\|u_n\|_{2^*}^{2/2^*}} \geq \frac{Q_{A,0}(\sum_{j=1}^{\infty} \rho_n^j \phi_j)}{\left\|\sum_{j=1}^{\infty} \rho_n^j \phi_j\right\|_{2^*}^{2/2^*}} = o(1) \geq S + o(1),$$

a contradiction.

□

1.3.1 The case $a \leq 0$

In order to investigate when the infimum is attained depending on the magnetic vector potential A and the electric potential a, we start from the simplest cases. The first of them is the case $a \leq 0$.

Proposition 1.3.5 *If $a \leq 0$ but not identically zero, $S_{A,a}$ is not achieved.*

Proof. First of all, in this case we have $S_{A,a} = S$. Indeed, by diamagnetic inequality, we have

$$\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2 \geq \int_{\mathbb{R}^N} |\nabla |u||^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2 \geq \int_{\mathbb{R}^N} |\nabla |u||^2$$

from which we have $S_{A,a} \geq S$.

Suppose by contradiction $S_{A,a}$ is achieved on a function ϕ. Following the previous argument by Solimini's theorem, according to the negativity of the electric potential, we get $S_{A,a} \geq S + c$, where c is a positive constant due to the convergence of the term $\frac{\int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} \phi(\cdot + x_n)^2}{(\int_{\mathbb{R}^N} |\phi(\cdot + x_n)|^{2^*})^{2/2^*}}$. So we get $S_{A,a} > S$, a contradiction.

Note here we used the considerable fact that

$$\inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla |u||^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} = S \ .$$

Its proof is based on the idea that S is achieved over a radial function. □

1.3.2 The case $a = 0$

In this case we expect in general the infimum is not achieved. Indeed, first of all we have $S_{A,a} = S$, because we have already seen in general $S_{A,a} \leq S$, and in this case the diamagnetic inequality gives the reverse inequality. There is a simple case in which we can immediately deduce a result.

Remark 1.3.6 *If the vector potential* $\dfrac{A}{|x|}$ *is a gradient of a function* $\Theta \in L^1_{loc}(\mathbb{R}^N)$ *such that* $\nabla\Theta \in L^{N,\infty}(\mathbb{R}^N)$*, then* $S_{A,a}$ *is achieved.*
Indeed, suppose $\dfrac{A}{|x|} = \nabla\Theta$ *for a function* $\Theta \in L^1_{loc}(\mathbb{R}^N)$ *such that its gradient has the regularity mentioned above. The change of gauge* $u \mapsto e^{+i\Theta}u$ *makes the problem (1.3) equivalent to (1.4), so that the infimum is necessarly achieved.*
Just a few words about the regularity asked to $\nabla\Theta$*. In order to have the minimum problem wellposed, it would be sufficient* $\nabla\Theta \in L^2$*. But if we require the function* $e^{-i\Theta}u \in D^{1,2}(\mathbb{R}^N)$ *for any* $u \in D^{1,2}(\mathbb{R}^N)$*, this regularity is not sufficient any more. Rather, everything works if* $\nabla\Theta \in L^{N,\infty}(\mathbb{R}^N)$*.*

Now, suppose the infimum $S_{A,a} = S$ is achieved on a function $u \in D^{1,2}(\mathbb{R}^N)$. Then we have

$$S = \frac{\displaystyle\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \geq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla |u||^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \geq S \ .$$

So it is clear the equality must hold in the diamagnetic inequality in order not to fall into a contradiction. We have the following chain of relations:

$$|\nabla |u|| = \left|Re\Big(\frac{\overline{u}}{|u|}\nabla u\Big)\right| = \left|Im\Big(i\frac{\overline{u}}{|u|}\nabla u\Big)\right| = \left|Im\Big(i\nabla u - \frac{A}{|x|}u\Big)\frac{\overline{u}}{|u|}\right| \leq \left|\Big(i\nabla u - \frac{A}{|x|}u\Big)\frac{\overline{u}}{|u|}\right| \ .$$

In order that the equality holds in the last line $Re\Big\{\Big(i\nabla u - \frac{A}{|x|}u\Big)\overline{u}\Big\}$ must vanish. Expanding the expression one finds the equivalent condition is $\frac{A}{|x|} = Re\Big(i\frac{\nabla u}{u}\Big)$. We can rewrite $i\frac{\nabla u}{u} = i\frac{\nabla u}{|u|^2}\overline{u}$ and see

$$Re\Big(i\frac{\nabla u}{u}\Big) = \frac{-Re(u)\nabla\left(Im(u)\right) + Im(u)\nabla\left(Re(u)\right)}{|u|^2} = -\nabla\left(\arctan\frac{Im(u)}{Re(u)}\right)$$

which is equivalent to $-\frac{A}{|x|} = \nabla\Theta$ where Θ is the phase of u.

In conclusion, we can resume our first remark both with this argument to state the following

Proposition 1.3.7 *If the electric potential $a = 0$, the infimum $S_{A,a}$ is achieved if and only if $\dfrac{A}{|x|} = \nabla\Theta$. In this case Θ is the phase of the minimizing function.*

1.3.3 The general case: sufficient conditions

In Theorem (1.3.4) we proved that a sufficient condition for the infimum achieved is $S_{A,a} < S$. In this section we look for the hypotheses on A or a which guarantee this condition.

Proposition 1.3.8 *Suppose there exist a small ball $B_\delta(x_0)$ centered in $x_0 \in \mathbb{S}^{N-1}$ in which*

$$a(x) - |A(x)|^2 \geq \lambda > 0 \quad a.e.\ x \in B_\delta(x_0).$$

Then $S_{A,a} < S$ and so $S_{A,a}$ is achieved.

Proof. We define

$$\mathcal{H}_A(\Omega) = \overline{C_C^\infty(\Omega)}^{(\int_\Omega |\nabla_A u|^2)^{1/2}}$$

the closure of compact supported functions with respect to the norm associated to the quadratic form. We have the following chain of relations:

$$\begin{aligned} S_{A,a} &\leq \inf_{u\in\mathcal{H}_A(B_\delta(x_0))\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla_A u|^2 - \int_{\mathbb{R}^N} \frac{a}{|x|^2}|u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \\ &\leq \inf_{u\in\mathcal{H}_A(B_\delta(x_0),\mathbb{R})\setminus\{0\}} \frac{\displaystyle\int_{B_\delta(x_0)} |\nabla_A u|^2 - \int_{B_\delta(x_0)} \frac{a}{|x|^2}u^2}{\left(\displaystyle\int_{B_\delta(x_0)} |u|^{2^*}\right)^{2/2^*}} \end{aligned}$$

since the quotient is invariant under Solimini's rescalings and we restrict to a proper subset of functions. When we check the quotient over a real function, it reduces to

$$\frac{\displaystyle\int_{B_\delta(x_0)} |\nabla u|^2 + \int_{B_\delta(x_0)} \frac{|A|^2 - a}{|x|^2} u^2}{\left(\displaystyle\int_{B_\delta(x_0)} |u|^{2^*}\right)^{2/2^*}},$$

so the thesis follows from [13], Lemma (1.1). □

Remark 1.3.9 *We can resume the results reached until now: in case the magnetic vector potential $\frac{A}{|x|}$ is a gradient, the infimum $S_{A,a}$ is achieved if $a \equiv 0$ or if its essential infimum is positive and sufficiently small in a neighborhood far from the origin (we mean $\|a\|_\infty \leq (N-2)^2/4$ in order to keep the quadratic form positive definite); while it is never achieved provided $a \leq 0$, neither in case the magnetic potential is a gradient, nor in case it is not. On the other hand, in order to have $S_{A,a}$ achieved, if the magnetic vector potential is not a gradient we need to assume it has a suitably low essential supremum somewhere in a ball far from the origin in relation to the electric potential a (see Proposition (1.3.8)).*

Anyway, it seems reasonable what is important here is not the essential supremum of $\frac{A}{|x|}$ (or A, since we play far from the origin), but "the distance" between the magnetic vector potential and the set of gradients. Pursuing this idea, it seems possible to interpretate a suitable (to be specified) norm of $\mathrm{curl}\frac{A}{|x|}$ as a measure of this distance. In order to specify these ideas we refer to [38] and [11]. We recall the following

Definition 1.3.10 *[38] Let Ω be a open set of $\mathbb{R}^N$ and $\overrightarrow{a}, \overrightarrow{b} \in L^1_{loc}(\Omega)$. We say that $\overrightarrow{a}$ and $\overrightarrow{b}$ are related by a gauge transformation, $\overrightarrow{a} \sim_\Omega \overrightarrow{b}$, if there is a distribution $\lambda \in D'(\Omega)$ satisfying $\overrightarrow{b} = \overrightarrow{a} + \nabla\lambda$.*

By $\mathrm{curl}\overrightarrow{a}$ we denote the skew-symmetric, matrix-valued distribution having $\partial_i \overrightarrow{a}_j - \partial_j \overrightarrow{a}_i \in D'(\Omega)$ as matrix elements.

Lemma 1.3.11 *[38] Let Ω be any open subset of $\mathbb{R}^N$, $1 \leq p < +\infty$ and $\overrightarrow{a}, \overrightarrow{b} \in L^p_{loc}(\Omega)$. Then every λ satisfying $\overrightarrow{b} = \overrightarrow{a} + \nabla\lambda$ belongs to $W^{1,p}(\Omega)$. If Ω is simply-connected then*

$$\overrightarrow{a} \sim_\Omega \overrightarrow{b} \iff \mathrm{curl}\overrightarrow{a} = \mathrm{curl}\overrightarrow{b}\ .$$

Theorem 1.3.12 *[11] Let $M = (0,1)^N$ be the N-dimensional cube of $\mathbb{R}^N$ with $N \geq 2$ and $1 \leq l \leq N-1$. Given any X a l-form with coefficients in $W^{1,N}(M)$ there exists some Y a l-form with coefficients in $W^{1,N} \cap L^\infty(M)$ such that*

$$dY = dX$$

and

$$\|\nabla Y\|_N + \|Y\|_\infty \leq C\,\|dX\|_N\ .$$

The Theorem (1.3.12) will be very useful in our case choosing $l = 1$, so that the external derivative is the curl of the vector field which represents the given 1-form.

Suppose $\frac{A}{|x|} \in W^{1,N}(B_\delta(x_0))$ in a ball far from the origin. Then for Theorem (1.3.12) there exists a vector field $Y \in L^\infty \cap W^{1,N}(B_\delta(x_0))$ such that $\mathrm{curl}\frac{A}{|x|} = \mathrm{curl}Y$ and $\|Y\|_\infty \leq C\left\|\mathrm{curl}\frac{A}{|x|}\right\|_N$. By Lemma (1.3.11), Y is related to $\frac{A}{|x|}$ by a gauge transformation, so, in the spirit of Theorem (1.3.8), it is sufficient $\left\|\mathrm{curl}\frac{A}{|x|}\right\|_N$ is not too large in order to have $S_{A,a} < S$ and hence $S_{A,a}$ achieved.

1.4 Symmetry of solutions

We recall once again in general $S_{A,a} \leq S$. When $S_{A,a} = S$ and $Q_{A,a}(u) > Q(u)$ for any $u \in D^{1,2}(\mathbb{R}^N)$, e.g. when $a \leq 0$ but not identically 0, we lose compactness since clearly $S_{A,a}$ cannot be attained. In this section we follow the idea that introducing symmetry properties to the quadratic form can help in growing the upper bound for $S_{A,a}$, in order to increase the probability for it to be achieved.

We basically follow the ideas in [62], assuming the dimension $N \geq 4$.

Let us write $\mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and denote $x = (z, y)$. Let us fix $k \in \mathbb{N}$, and suppose there is a $\mathbb{Z}_k \times SO(N-2)$ group-action on $D^{1,2}(\mathbb{R}^N)$, denoting

$$D^{1,2}_{k,m}(\mathbb{R}^N) = \{u(z,y) \in D^{1,2}(\mathbb{R}^N) \text{ s.t. } u(\mathrm{e}^{i\frac{2\pi}{k}}z, Ry) = \mathrm{e}^{i\frac{2\pi}{k}m}u(z,y) \text{ for any } R \in SO(N-2)\}$$

the fixed point space. In order to have the quadratic form invariant under this action, let us suppose that $a(\theta) \equiv a \in \mathbb{R}^-$ and

$$A\left(\frac{\mathrm{e}^{i\frac{2\pi}{k}}z, Ry}{|(z,y)|}\right) = \left(\mathrm{e}^{i\frac{2\pi}{k}}(A_1, A_2), RA_3\right)\left(\frac{z,y}{|(z,y)|}\right) \qquad \text{for any } R \in SO(N-2). \tag{1.6}$$

These two conditions allow us to apply the *Symmetric Criticality Principle*, so that the minima of the problem

$$S^{k,m}_{A,a} := \inf_{u \in D^{1,2}_{k,m}(\mathbb{R}^N)} \frac{Q_{A,a}(u)}{\|u\|^2_{2^*}} . \tag{1.7}$$

are solutions to (1.2).

Theorem 1.4.1 *If $S^{k,m}_{A,a} < k^{2/N}S$ then it is achieved.*

Proof. Let us consider a minimizing sequence $\{u_n\}$. The space $D^{1,2}_{k,m}(\mathbb{R}^N)$ is a close subspace in $D^{1,2}(\mathbb{R}^N)$, so Solimini's Theorem (1.3.3) holds in $D^{1,2}_{k,m}(\mathbb{R}^N)$. Up to subsequences we can find a sequence $\Phi_i \in D^{1,2}_{k,m}(\mathbb{R}^N)$ and a sequence of mutually diverging rescalings ρ^i_n such that $u_n - \sum_i \rho^i_n(\Phi_i) \to 0$ in L^{2^*}.

We can basically follow the proof of Theorem (1.3.4).

We just stress that the possible function $\phi_{\bar{j}}$ will be in fact of the form $\sum_{l=1}^k \phi(\cdot + x^l)$, meaning that it will enjoy the same $\mathbb{Z}_k \times SO(N-2)$-group symmetry. □

Remark 1.4.2 *The above result is actually a symmetry breaking result for the equation associated to these minimum problems. Indeed, let us consider the equation*

$$-\Delta_A u = \frac{a}{|x|^2} + |u|^{2^*-2} u \qquad \text{in } \mathbb{R}^N, \tag{1.8}$$

where $-\Delta_A$ *denotes the differential operator we have called* magnetic Laplacian. *Then the minima of (1.3) are solutions to (1.8) and so are those of (1.7), thanks to the* Symmetric Criticality Principle *(the quotient is invariant under the* $\mathbb{Z}_k \times SO(N-2)$ *group-action). Thus, when the electric potential is constant and negative, we find a multiplicity of solutions to (1.8) depending on* k *(we would say an infinite number, at least for* k *not multiples to each other), and each of them is invariant under rotations of angle* $2\pi/k$*, respectively.*

Now we want to check whenever the condition $S^{k,m}_{A,a} < k^{2/N} S$ is fulfilled. Let us pick k points in $\mathbb{R}^N \setminus \{0\}$ of the form $x_j = (R\mathrm{e}^{\frac{2\pi i}{k} j}\xi_0, 0)$ where $|\xi_0| = 1$, and denote

$$w_j = \mathrm{e}^{\frac{2\pi i}{k} jm} \frac{\big(N(N-2)\big)^{\frac{N-2}{4}}}{\big(1 + |x - x_j|^2\big)^{\frac{N-2}{2}}} \ . \tag{1.9}$$

In this way the sum $\sum_{j=1}^k w_j$ is an element of $D^{1,2}_{k,m}(\mathbb{R}^N)$. Additionally we notice w_j are minimizers of the usual Sobolev quotient, and they satisfy

$$-\Delta w_j = |w_j|^{2^*-2} w_j \qquad \text{in } \mathbb{R}^N. \tag{1.10}$$

It is worth to notice that both

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w_j|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |w_j|^{2^*}\right)^{2/2^*}} = S$$

and (1.10) imply

$$\int_{\mathbb{R}^N} |\nabla w_j|^2 = \int_{\mathbb{R}^N} |w_j|^{2^*} = S^{N/2} \, . \tag{1.11}$$

We state the following

Proposition 1.4.3 *Choosing* R *and* k *large enough, the quotient evaluated over* $\sum_{j=1}^k w_j$ *is strictly less than* $k^{2/N} S$*, and so is the infimum* $S^{k,m}_{A,a}$*.*

In order to prove it, we need some technical results, whose proofs are postponed to the next subsection. We basically follow the ideas in [62].

For sake of semplicity, we introduce the following notation:

$$\begin{aligned}\alpha &= \int_{\mathbb{R}^N} Re\Big\{\sum_{i\neq j} |w_i|^{2^*-2}\, w_i\, \overline{w_j}\Big\}\\ \beta &= \int_{\mathbb{R}^N} \frac{|A|^2-a}{|x|^2}\Big|\sum_{j=1}^k w_j\Big|^2\\ \gamma &= Re\Big\{i\int_{\mathbb{R}^N}\frac{A}{|x|}\cdot\sum_{i,j}\nabla w_i\overline{w_j}\Big\}\ .\end{aligned}$$

Lemma 1.4.4 *It holds $\alpha \geq 0$.*

Lemma 1.4.5 *For every positive δ there exists a positive constant K_δ (independent of k) such that if*

$$\frac{|x_i-x_j|^2}{\log|x_i-x_j|} \geq K_\delta\,(k-1)^{2/(N-2)} \quad \forall i\neq j$$

then

$$\int_{\mathbb{R}^N}\Big|\sum_{j=1}^k w_j\Big|^{2^*} \ \geq\ k\,S^{N/2}+2^*(1-\delta)\int_{\mathbb{R}^N} Re\Big\{\sum_{i\neq j}|w_i|^{2^*-2}\,w_i\,\overline{w_j}\Big\}. \tag{1.12}$$

Lemma 1.4.6 *Given Lemma (1.4.5), it is possible to choose R and k in such a way that the quantity*

$$1+\frac{1}{kS^{N/2}}\Big\{\beta-2\gamma+\alpha\Big(-1+\delta+\frac{2-\delta}{kS^{N/2}}(2\gamma-\beta)\Big)\Big\}$$

is positive and strictly less than 1.

Proof of Proposition (1.4.3). Let us evaluate the quotient over $\sum_{j=1}^k w_j$:

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big|\nabla_A\big(\sum_{j=1}^k w_j\big)\Big|^2-\int_{\mathbb{R}^N}\frac{a}{|x|^2}\Big|\sum_{j=1}^k w_j\Big|^2\\ &=\int_{\mathbb{R}^N}\Big\{\sum_{j=1}^k|\nabla w_j|^2+Re\Big\{\sum_{i\neq j}\nabla w_i\cdot\nabla\overline{w_j}\Big\}+\frac{|A|^2-a}{|x|^2}\Big|\sum_{j=1}^k w_j\Big|^2\\ &\qquad-2\,Re\Big\{i\frac{A}{|x|}\cdot\sum_{i,j}\nabla w_i\overline{w_j}\Big\}\Big\}\\ &=kS^{N/2}+\int_{\mathbb{R}^N}\Big\{Re\Big\{\sum_{i\neq j}|w_i|^{2^*-2}\,w_i\,\overline{w_j}\Big\}+\frac{|A|^2-a}{|x|^2}\Big|\sum_{j=1}^k w_j\Big|^2\\ &\qquad-2\,Re\Big\{i\frac{A}{|x|}\cdot\sum_{i,j}\nabla w_i\overline{w_j}\Big\}\Big\}\end{aligned} \tag{1.13}$$

where in the last equality we have used (1.11) and the equation (1.10). Now we use Lemma (1.4.5) which states the lower bound (1.12) for the denominator of our quotient. Thus using (1.13) and (1.12) the quotient is

$$\begin{aligned}
&\frac{Q_{A,a}\big(\sum_{j=1}^{k} w_j\big)}{\Big\|\big(\sum_{j=1}^{k} w_j\big)\Big\|_{2^*}^2} \\
&\leq \left(kS^{N/2} + \int_{\mathbb{R}^N} \left\{Re\Big\{\sum_{i\neq j} |w_i|^{2^*-2}\, w_i\, \overline{w_j}\Big\} + \frac{|A|^2 - a}{|x|^2}\Big|\sum_{j=1}^{k} w_j\Big|^2 \right.\right. \\
&\qquad \left.\left. -2\, Re\Big\{i\frac{A}{|x|}\cdot\sum_{i,j}\nabla w_i \overline{w_j}\Big\}\right\}\right) \\
&\qquad \cdot \left(kS^{N/2} + 2^*(1-\delta/2)\int_{\mathbb{R}^N} Re\Big\{\sum_{i\neq j}|w_i|^{2^*-2}\, w_i\, \overline{w_j}\Big\}\right)^{-2/2^*} \\
&= k^{2/N} S\left(1 + \frac{1}{kS^{N/2}}(\alpha+\beta-2\gamma)\right)\left(1 - \frac{2(1-\delta/2)}{kS^{N/2}}\alpha\right) + o(1)
\end{aligned}$$

where in the last line we have expanded the denominator in Taylor's serie since the argument is very close to zero if R is large. Up to infinitesimal terms of higher order, the coefficient of $k^{2/N}S$ is

$$1 + \frac{1}{kS^{N/2}}\beta - \frac{2}{kS^{N/2}}\gamma + \frac{1}{kS^{N/2}}\alpha\big(-1+\delta+\frac{2-\delta}{kS^{N/2}}(2\gamma-\beta)\big)\ .$$

Now we invoke Lemma (1.4.6) to conclude the proof. □

1.4.1 Proofs of technical lemmas

In order to prove Lemmas (1.4.4), (1.4.5) and (1.4.6) we need supplementary results mainly about asymptotics of the quantities involved.

Lemma 1.4.7 *We have, as* $|x_i - x_j| \to +\infty$ *and* $|x_i| \to +\infty$

$$\int_{\mathbb{R}^N} |w_i|^{2^*-2}\, w_i\, \overline{w_j} = O\Big(\frac{1}{|x_i - x_j|^{N-2}}\Big) \tag{1.14}$$

$$\int_{\mathbb{R}^N} |w_i\, \overline{w_j}|^{2^*/2} = O\Big(\frac{\log|x_i - x_j|}{|x_i - x_j|^{N}}\Big) \tag{1.15}$$

$$\int_{\mathbb{R}^N} \frac{|w_j|^2}{|x|^2} = \begin{cases} O\big(\frac{\log R}{R^2}\big) & \text{if } N = 4 \\ O\big(\frac{1}{R^2}\big) & \text{for } N \geq 5 \end{cases} \tag{1.16}$$

$$\int_{\mathbb{R}^N} \frac{1}{|x|}\cdot|\nabla w_j|\ |w_i| = O\Big(\frac{1}{R\,|x_i - x_j|^{N-3}}\Big). \tag{1.17}$$

Proof. For what concerns (1.14), (1.15) and (1.16) we refer to [62].

About (1.17) we have

$$\int_{B_{R/2}(0)} \frac{1}{|x|} |\nabla w_j| \, |w_i| = O(\frac{1}{R^{N-1}}) \, O(\frac{1}{R^{N-2}}) \, O(R^{N-1}) = O(\frac{1}{R^{N-2}})$$

since in $B_{R/2}(0)$ $|x - x_i| \geq R - |x| \geq R/2$ and the same holds for $|x - x_j|$;

$$\begin{aligned} \int_{B_{|x_i-x_j|/4}(x_i)} \frac{1}{|x|} |\nabla w_j| \, |w_i| &= O(\frac{1}{R}) \, O(\frac{1}{|x_i - x_j|^{N-1}}) \int_{B_{|x_i-x_j|/4}(x_i)} |w_i| \\ &= O(\frac{1}{R \, |x_i - x_j|^{N-3}}) \end{aligned}$$

since in $B_{|x_i-x_j|/4}(x_i)$ $|x| \geq |x_i| - |x - x_i| \geq R/2$ and $|x - x_j| \geq |x_i - x_j| - |x - x_i| \geq \frac{3}{4} |x_i - x_j|$;

$$\begin{aligned} \int_{B_{|x_i-x_j|/4}(x_j)} \frac{1}{|x|} |\nabla w_j| \, |w_i| &= O(\frac{1}{R}) O(\frac{1}{|x_i - x_j|^{N-2}}) \int_{B_{|x_i-x_j|/4}(x_j)} |\nabla w_j| \\ &= O(\frac{1}{R \, |x_i - x_j|^{N-3}}) \end{aligned}$$

since in $B_{|x_i-x_j|/4}(x_j)$ $|x| \geq |x_j| - |x - x_j| \geq R/2$ and $|x - x_i| \geq |x_i - x_j| - |x - x_j| \geq \frac{3}{4} |x_i - x_j|$; while in $\mathbb{R}^N \setminus (B_{R/2}(0) \cup B_{|x_i-x_j|/4}(x_i) \cup B_{|x_i-x_j|/4}(x_j))$ we have $|x| \geq R/2$, and via Hölder inequality

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus (B_{R/2}(0) \cup B_{|x_i-x_j|/4}(x_i) \cup B_{|x_i-x_j|/4}(x_j))} \frac{1}{|x|} |\nabla w_j| \, |w_i| \\ &= \begin{cases} O\big(\frac{1}{R \, |x_i - x_j|^2} \log |x_i - x_j|\big) & \text{if } N = 4 \\ O\big(\frac{1}{R \, |x_i - x_j|^{2N-6}}\big) & \text{if } N \geq 5. \end{cases} \end{aligned}$$

□

Remark 1.4.8 *The above asymptotics in Lemma (1.4.7) come in terms of k and R as we note*

$$\begin{aligned} |x_i - x_j|^2 &= R^2 \sin^2 \frac{2\pi}{k}(i - j) + R^2 \Big(1 - \cos \frac{2\pi}{k}(i - j)\Big)^2 \\ &\sim \begin{cases} \frac{R^2}{k^2} + \frac{R^2}{k^4} = O(\frac{R^2}{k^2}) & \textit{if } |i - j| \ll k \\ R^2 & \textit{otherwise} \end{cases} . \end{aligned}$$

According to the previous asymptotic, we note we have the worst estimates in Lemma (1.4.7) for $|i - j| \ll k$, that is for the centers x_i, x_j quite near to each other.

Lemma 1.4.9 *The following asymptotic behavior holds for $k \to +\infty$ and $R \to +\infty$*

$$\left|\int_{\mathbb{R}^N} Re\Big\{ i\,\frac{A}{|x|}\cdot \sum_{l,j} \nabla w_l\,\overline{w_j}\Big\}\right| \le \begin{cases} O(\frac{k^2}{R^2}) & \text{if } N=4 \\ O(\frac{k^2\log k}{R^3}) & \text{if } N=5 \\ O(\frac{k^{N-3}}{R^{N-2}}) & \text{for } N\ge 6. \end{cases}$$

Proof. First of all we note if $l=j$ the quantity in the statement is zero. Next,

$$\left|\int_{\mathbb{R}^N} Re\Big\{ i\,\frac{A}{|x|}\cdot \sum_{l,j} \nabla w_l\,\overline{w_j}\Big\}\right| = \left|\sum_{l\neq j} \sin\frac{2\pi}{k} m(l-j) \int_{\mathbb{R}^N} \frac{A}{|x|}\cdot\nabla\,|w_l|\,|w_j|\right|$$
$$\le \frac{C}{R^{N-2}}\sum_{l\neq j}\frac{\left|\sin\frac{2\pi}{k}m(l-j)\right|}{\left(1-\cos\frac{2\pi}{k}(l-j)\right)^{\frac{N-3}{2}}} = \frac{C\,k}{R^{N-2}}\sum_{l=1}^{k-1}\frac{\left|\sin\frac{2\pi}{k}ml\right|}{\left(1-\cos\frac{2\pi}{k}l\right)^{\frac{N-3}{2}}}$$
$$\le \frac{C\,m\,k}{R^{N-2}}\sum_{l=1}^{k-1}\frac{l/k}{(l/k)^{N-3}} = \frac{C\,m\,k^{N-3}}{R^{N-2}}\begin{cases} k & \text{if } N=4\\ \log k & \text{if } N=5 \\ O(1) & \text{for } N\ge 6.\end{cases}$$

□

We recall the following result proved in [62]:

Lemma 1.4.10 *Let $s_1,\dots,s_k \ge 0$. For every positive δ there exists a positive constant K_δ (independent of k) such that if*

$$\frac{|x_i-x_j|^2}{\log|x_i-x_j|} \ge K_\delta\,(k-1)^{2/(N-2)} \quad \forall i\neq j$$

then

$$\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k s_i\Big)^{2^*} \ge k\,S^{N/2} + 2^*(1-\delta/2)\int_{\mathbb{R}^N}\sum_{i\neq j} s_i^{2^*-1}s_j \tag{1.18}$$

Proof of Lemma (1.4.4). We split the sum in two contributions: indexes for which $\cos\frac{2\pi}{k}(j-l)\ge 0$ (we will call them *j,l pos*), and indexes for which $\cos\frac{2\pi}{k}(j-l)\le 0$ (we will call them *j,l neg*). We note in the first case, we have $|x_j-x_l|\sim\frac{R}{k}$, whereas in the second case $|x_j-x_l|\sim R$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} Re\Big\{\sum_{l,j\ pos}|w_j|^{2^*-2}\,w_j\,\overline{w_l}\Big\} &\ge \int_{\mathbb{R}^N} Re\Big\{\sum_j |w_j|^{2^*-2}\,w_j\,\overline{w_{j+1}}\Big\} \\ &= k\int_{\mathbb{R}^N} Re\Big\{|w_2|^{2^*-2}\,w_2\,\overline{w_1}\Big\} = O(\frac{k^{N-1}}{R^{N-2}}).\end{aligned} \tag{1.19}$$

On the other hand,

$$\int_{\mathbb{R}^N} Re\Big\{ \sum_{l,j\ neg} |w_j|^{2^*-2} w_j \overline{w_l} \Big\} \le k^2 \int_{\mathbb{R}^N} Re\Big\{ |w_l|^{2^*-2} w_l \, \overline{w_1} \Big\} = O(\frac{k^2}{R^{N-2}});$$

so that for k large enough we have the thesis. □

Proof of Lemma (1.4.5). By convexity of the function $(\cdot)^{2^*/2}$ we have

$$\begin{aligned}
\Big|\sum_{j=1}^k w_j\Big|^{2^*} &= \Bigg(\Big|\sum_{j=1}^k w_j\Big|^2\Bigg)^{2^*/2} = \Bigg(\sum_{i,j=1}^k Re\{w_i\,\overline{w_j}\}\Bigg)^{2^*/2} \\
&= \Bigg(\sum_{i,j} |w_i|\,|w_j| - \sum_{i,j} |w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\Bigg)^{2^*/2} \\
&\ge \Bigg(\sum_{i,j} |w_i|\,|w_j|\Bigg)^{2^*/2} - \frac{2^*}{2}\Bigg(\sum_{i,j} |w_i|\,|w_j|\Bigg)^{2^*/2-1} \sum_{i,j} |w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)
\end{aligned} \tag{1.20}$$

For what concerns the first term $\Big(\sum_{i,j}|w_i|\,|w_j|\Big)^{2^*/2} = \Big(\sum_j^k |w_j|\Big)^{2^*}$, we can apply directly inequality (1.18) with $s_j = |w_j|$ in order to have

$$\int_{\mathbb{R}^N}\Bigg(\sum_{j=1}^k |w_j|\Bigg)^{2^*} \ \ge\ k\,S^{N/2} + 2^*(1-\delta/2)\int_{\mathbb{R}^N}\sum_{i\ne j} |w_i|^{2^*-1}\,|w_j|\ . \tag{1.21}$$

We want to stress that

$$\int_{\mathbb{R}^N}\sum_{i\ne j} |w_i|^{2^*-1}\,|w_j| \ \ge\ \int_{\mathbb{R}^N}\sum_{j=1}^k |w_j|^{2^*-1}\,|w_{j+1}| \ =\ k\int_{\mathbb{R}^N} |w_1|^{2^*-1}\,|w_2|$$

(see also [62], equation (6.22)), so that

$$\int_{\mathbb{R}^N}\sum_{i\ne j} |w_i|^{2^*-1}\,|w_j| \ \ge\ O\big(\frac{k^{N-1}}{R^{N-2}}\big)\,. \tag{1.22}$$

Now we focus our attention on the integral of the second term in (1.20): via Hölder inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Bigg(\sum_{i,j}|w_i|\,|w_j|\Bigg)^{2^*/2-1} \sum_{i,j}|w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big) \\
&\le \Bigg(\int_{\mathbb{R}^N}\Bigg(\sum_{i,j}|w_i|\,|w_j|\Bigg)^{2^*/2}\Bigg)^{\frac{2^*-2}{2^*}} \cdot \Bigg(\int_{\mathbb{R}^N}\Bigg(\sum_{i,j}|w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\Bigg)^{2^*/2}\Bigg)^{2/2^*}
\end{aligned}$$

and

$$\left(\int_{\mathbb{R}^N}\Big(\sum_{i,j}|w_i|\,|w_j|\Big)^{2^*/2}\right)^{\frac{2^*-2}{2^*}}=\left(\int_{\mathbb{R}^N}\Big(\sum_{j}|w_j|\Big)^{2^*}\right)^{\frac{2^*-2}{2^*}}\sim (k\,S^{N/2})^{\frac{2^*-2}{2^*}}$$

thanks to inequality (1.18) and Lemma (1.4.7). On the other hand

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N}\Big(\sum_{i,j}|w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\Big)^{2^*/2}\right)^{2/2^*}\\
&\quad\le\sum_{i,j}\left(\int_{\mathbb{R}^N}\Big(|w_i|\,|w_j|\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\Big)^{2^*/2}\right)^{2/2^*}\\
&\quad=\sum_{i,j}\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\frac{(\log|x_i-x_j|)^{\frac{N-2}{N}}}{|x_i-x_j|^{N-2}}
\end{aligned}$$

according to (1.15). Now, since $|x_i-x_j|\sim R\big(1-\cos\big(\frac{2\pi}{k}(i-j)\big)\big)^{1/2}$, the sum

$$\begin{aligned}
&\sum_{i,j}\Big(1-\cos\big(\frac{2\pi}{k}m(i-j)\big)\Big)\frac{\Big(\log\Big(R\big(1-\cos\big(\frac{2\pi}{k}(i-j)\big)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\Big(1-\cos\big(\frac{2\pi}{k}(i-j)\big)\Big)^{\frac{N-2}{2}}}\\
&\quad\le C(m)\,k\sum_{j}\frac{\Big(\log\Big(R\big(1-\cos\big(\frac{2\pi}{k}j\big)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\Big(1-\cos\big(\frac{2\pi}{k}j\big)\Big)^{N/2-2}}\\
&\quad\sim 2\,C(m)\,k^2\int_0^{1/2}\frac{\Big(\log\Big(R\big(1-\cos\big(2\pi x\big)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\Big(1-\cos(2\pi x)\Big)^{N/2-2}}\,dx\\
&\quad\le\frac{C(m)\,k^2}{R^{N-2}}\begin{cases}O(\log R) & \text{if } N=4\\ O(\log R\log k) & \text{if } N=5\\ O\Big((\log R\log k)^{\frac{N-2}{N}}k^{N-5}\Big) & \text{if } N\ge 6\end{cases}
\end{aligned}$$

so that the second term (1.20) is

$$\begin{aligned}
(1.20) \quad &\le \quad C(m)\, k^{2/N}\frac{k^2}{R^{N-2}}\begin{cases} O(\log R) & \text{if } N=4\\ O(\log R\log k) & \text{if } N=5\\ O\big((\log R\log k)^{\frac{N-2}{N}}k^{N-5}\big) & \text{if } N\ge 6\end{cases}\\
&= \begin{cases} O\big(\dfrac{k^{5/2}\log R}{R^2}\big) & \text{if } N=4\\ O\big(\dfrac{k^{12/5}\log R\log k}{R^3}\big) & \text{if } N=5\\ O\big(\dfrac{k^{N-3+2/N}(\log R\log k)^{\frac{N-2}{N}}}{R^{N-2}}\big) & \text{if } N\ge 6\end{cases} \qquad (1.23)
\end{aligned}$$

which can be made $o\big(\frac{k^{N-1}}{R^{N-2}}\big)$ in every dimension for a suitable choice of the parameters R and k (e.g. $k \sim R^{\alpha}$ with $0<\alpha<1$ since according to the hypothesis of lemma itself we need $k=o(R)$).

Provided the ratio R/k is big enough, from equations (1.21), (1.22) and (1.23) we get

$$\int_{\mathbb{R}^N}\left|\sum_{j=1}^{k} w_j\right|^{2^*} \;\ge\; k\,S^{N/2} + 2^*(1-\delta)\int_{\mathbb{R}^N}\sum_{i\neq j}|w_i|^{2^*-1}|w_j|\ ,$$

which in particular implies the thesis. □

Proof of Lemma (1.4.6) In order to have this quantity (positive) and less than 1, it is sufficient to have

1. α/k, γ/k and β/k small,
2. α arbitrarly greater than β,
3. α arbitrarly greater than γ.

According to Lemma (1.4.7), Lemma (1.4.9) and Remark (1.4.8) we know

$$\begin{aligned}
\beta &= \begin{cases} O\big(k^2\dfrac{\log R}{R^2}\big) & \text{if } N=4\\ O\big(\dfrac{k^2}{R^2}\big) & \text{for } N\ge 5;\end{cases}\\
\gamma &= \begin{cases} O\big(\dfrac{k^2}{R^2}\big) & \text{if } N=4\\ O\big(\dfrac{k^2}{R^3}\log k\big) & \text{for } N=5\\ O\big(\dfrac{k^{N-3}}{R^{N-2}}\big) & \text{for } N\ge 6\end{cases}\\
\alpha &= O\big(k^2\frac{k^{N-2}}{R^{N-2}}\big).
\end{aligned}$$

Let us fix the condition

$$k^{(N-1)/(N-2)} = o(R) \tag{1.24}$$

in order to have α/k small. Consequently we immediately find the request 1 fulfilled. Moreover, we note this does not contradict either the hypothesis of Lemma (1.4.5) (rather, that is a consequence), or the conditions on equation (1.23).

For what concerns request 2 and 3, we recall equation (1.19) states the lower bound $\alpha \gg k^{N-1}/R^{N-2}$.

Thus, we find request 3 satisfied as soon as $k \to \infty$.

About request 2, everything works without any additional hypothesis in dimension 4. In dimension $N \geq 5$, we need $R = o(k^{(N-3)/(N-4)})$: we emphasize this does not contradict condition (1.24) thanks to the order $\frac{N-1}{N-2} < \frac{N-3}{N-4}$. □

As a natural question, letting $k \to \infty$, we wonder if there exists any biradial solution: we mean a function belonging to the space

$$D^{1,2}_{\text{birad},m}(\mathbb{R}^N) = \{u \in D^{1,2}(\mathbb{R}^N) \text{ s.t. } u(R(x_1,x_2), Sx_3) = R^m u((x_1,x_2),x_3) \\ \forall R \in SO(2)\,, \forall S \in SO(N-2)\}\ .$$

As already pointed out, even in this case we need that the magnetic potential A is equivariant with respect to the action of the group $SO(2) \times SO(N-2)$, that is

$$A\left(\frac{Rz, Sy}{|(z,y)|}\right) = (R(A_1,A_2), SA_3)\left(\frac{z,y}{|(z,y)|}\right) \qquad \text{for any } (R,S) \in SO(2) \times SO(N-2),$$

in order that the *Symmetric Criticality Principle* applies. In order to investigate the possible existence of a biradial solution, we set the problem

$$S^{\text{birad},m}_{A,a} = \inf_{u \in D^{1,2}_{\text{birad},m}(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} \left|(i\nabla - \frac{A}{|x|^2})u\right|^2 - \int_{\mathbb{R}^N} \frac{a}{|x|^2}|u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}\,,$$

and we are able to prove

Proposition 1.4.11 *There exists a biradial solution.*

Proof. As we usually do, we consider a minimizing sequence u_n to $S^{\text{birad},m}_{A,a}$ and Solimini's lemma in $D^{1,2}_{\text{birad},m}(\mathbb{R}^N)$, since this is a closed subspace of $D^{1,2}(\mathbb{R}^N)$. As usual, we reconduce ourselves to $u_n - \Phi(\cdot + x_n) \to 0$ in $D^{1,2}_{\text{birad},m}(\mathbb{R}^N)$ with $\Phi \neq 0$ and suppose by contradiction $|x_n| \to +\infty$.

To preserve the symmetry, in Solimini's decomposition we will find all the functions obtained by Φ with a rotation of a $2\pi/k$ angle, for $k \in \mathbb{Z}$ fixed. Thus, we can write $u_n - \sum_{i=1}^k \Phi(\cdot + x_n^i) \to 0$ in $D^{1,2}_{\text{birad},m}(\mathbb{R}^N)$. Now, following the same calculations in Theorem (1.4.1), we obtain $S^{\text{birad},m}_{A,a} \geq S^{k,m}_{A,a} \geq k^{2/N} S$ that leads to $S^{\text{birad},m}_{A,a} = +\infty$ choosing k arbitrary large: a contradiction. □

1.5 Aharonov-Bohm type potentials

In dimension 2, an Aharonov-Bohm magnetic field is a δ-type magnetic field. A vector potential associated to the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ has the form

$$\mathcal{A}(x_1, x_2) = \left(\frac{-\alpha x_2}{|x|^2}, \frac{\alpha x_1}{|x|^2} \right) \qquad (x_1, x_2) \in \mathbb{R}^2$$

where α is the field flux through the origin. In this contest we want to take account of Aharonov-Bohm type potentials in $\mathbb{R}^N$, for $N \geq 4$:

$$\mathcal{A}(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0 \right) \qquad (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{R}^{N-2},$$

paying special attention now the singular set is a whole subspace of $\mathbb{R}^N$ with codimension 2.

1.5.1 Hardy-type inequality

In order to study minimum problems and therefore the quadratic form associated to this kind of potentials, we need a Hardy-type inequality. We know by [37] that a certain Hardy-type inequality holds for Aharonov-Bohm vector potentials in $\mathbb{R}^2$, that is

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{|x|^2} \leq C \int_{\mathbb{R}^2} |(i\nabla - \mathcal{A})\varphi|^2 \qquad \forall \varphi \in C_C^\infty(\mathbb{R}^2 \setminus \{0\}),$$

where the best constant C is

$$H = \left(\min_{k \in \mathbb{Z}} |k - \Phi_{\mathcal{A}}| \right)^2 . \tag{1.25}$$

Here $\Phi_{\mathcal{A}}$ denotes the field flux around the origin

$$\Phi_{\mathcal{A}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt .$$

One can generalize this result and gain a similar inequality to the Aharonov-Bohm potentials in $\mathbb{R}^N$, simply separating the integrals: for all $\varphi \in C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$ one has

$$\int_{\mathbb{R}^N} \frac{|\varphi|^2}{x_1^2 + x_2^2} = \int_{\mathbb{R}^{N-2}} \int_{\mathbb{R}^2} \frac{|\varphi|^2}{x_1^2 + x_2^2}\, dx_1\, dx_2\, dx_3 \leq H \int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})\varphi|^2, \tag{1.26}$$

where H is defined in (1.25). Now a natural question arises: is H the best constant for inequality (1.26)? In other words, is H the infimum of the Rayleigh quotient?

Proposition 1.5.1 *The best constant for the inequality (1.26) is exactly (1.25).*

Proof. To prove this, we consider the approximating sequence u_n to (1.25) in $\mathbb{R}^2$. We can choose this sequence bounded in $L^2(\mathbb{R}^2)$ norm, thanks to the homogeneity of the quotient under dilation.

We claim there exists a sequence of real-valued functions $(\eta_n)_n \subset C_C^\infty(\mathbb{R}^{N-2})$ such that $\int_{\mathbb{R}^{N-2}} |\nabla \eta_n|^2 \longrightarrow 0$ and $\int_{\mathbb{R}^{N-2}} \eta_n^2 \longrightarrow +\infty$ as $n \to +\infty$. We can namely consider a real radial function such that $\eta_n \equiv 1$ in $B_R(0)$ and $\eta_n \equiv 0$ in $\mathbb{R}^{N-2} \setminus B_{R+n^\alpha}(0)$, with $|\nabla \eta_n| \sim \frac{1}{n^\alpha}$, for a suitable $\alpha > 0$ (e.g. $\alpha > \frac{N-2}{2}$).

Now we consider the sequence $v_n(x_1, x_2, x_3) = u_n(x_1, x_2)\eta_n(x_3)$ where x_3 as usual denotes the whole set of variables in $\mathbb{R}^{N-2}$, and test the quotient over this sequence:

$$\frac{\displaystyle\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})v_n|}{\displaystyle\int_{\mathbb{R}^N} \frac{|v_n|^2}{x_1^2 + x_2^2}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v_n|^2 - 2Re \int_{\mathbb{R}^N} \mathcal{A} v_n \cdot \nabla \overline{v}_n + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |v_n|^2}{\displaystyle\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2 + x_2^2}},$$

where the numerator is

$$\int_{\mathbb{R}^N} \eta_n^2 |\nabla u_n|^2 + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |\eta_n|^2 |u_n|^2 - 2Re \int_{\mathbb{R}^N} \eta_n^2 u_n \mathcal{A} \cdot \nabla \overline{u}_n$$
$$+2\, Re \int_{\mathbb{R}^N} u_n \eta_n \nabla \eta_n \cdot \nabla \overline{u}_n + \int_{\mathbb{R}^N} u_n^2 |\nabla \eta_n|^2 - 2Re \int_{\mathbb{R}^N} |u_n|^2 \eta_n \mathcal{A} \cdot \nabla \eta_n\,. \tag{1.27}$$

About the second line (1.27) in the numerator, via Hölder inequality we have

$$\left| \int_{\mathbb{R}^N} u_n \eta_n \nabla \eta_n \cdot \nabla \overline{u}_n \right| \leq \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla \eta_n|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} \eta_n^2 |\nabla u_n|^2 \right)^{1/2}$$
$$= \left(\int_{\mathbb{R}^{N-2}} |\nabla \eta_n|^2 \right)^{1/2} \left(\int_{\mathbb{R}^2} |u_n|^2 \right)^{1/2} \left(\int_{\mathbb{R}^{N-2}} \eta_n^2 \right)^{1/2} \left(\int_{\mathbb{R}^2} |\nabla u_n|^2 \right)^{1/2}$$

and

$$\left| \int_{\mathbb{R}^N} \mathcal{A} u_n \eta_n \cdot \overline{u}_n \nabla \eta_n \right| \leq \left(\int_{\mathbb{R}^N} |\nabla \eta_n|^2 |u_n|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |\mathcal{A}|^2 |\eta_n|^2 |u_n|^2 \right)^{1/2}$$
$$= \left(\int_{\mathbb{R}^{N-2}} |\nabla \eta_n|^2 \int_{\mathbb{R}^2} |u_n|^2 \right)^{1/2} \left(\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} |\mathcal{A}|^2 |u_n|^2 \right)^{1/2}$$

Therefore the Rayleigh quotient is reduced to

$$\frac{\displaystyle\int_{\mathbb{R}^2} |\nabla u_n|^2 + \int_{\mathbb{R}^2} |\mathcal{A}|^2 |u_n|^2 - 2Re \int_{\mathbb{R}^2} \mathcal{A} u_n \cdot \nabla \overline{u}_n}{\displaystyle\int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2 + x_2^2}} +$$
$$+ \frac{\displaystyle 2\, Re \int_{\mathbb{R}^N} u_n \eta_n \nabla \eta_n \cdot \nabla \overline{u}_n}{\displaystyle\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2 + x_2^2}} + \frac{\displaystyle\int_{\mathbb{R}^2} |u_n|^2 \int_{\mathbb{R}^{N-2}} |\nabla \eta_n|^2}{\displaystyle\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2 + x_2^2}} - \frac{\displaystyle 2Re \int_{\mathbb{R}^N} |u_n|^2 \eta_n \mathcal{A} \cdot \nabla \eta_n}{\displaystyle\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2 + x_2^2}}$$
$$= H + o(1)$$

thanks to the properties of the sequence η_n. □

1.5.2 Variational setting

We have seen before the quadratic form associated to $\frac{A}{|x|}$-type potentials is equivalent to the Dirichlet form. On the contrary, we will see in case of Aharonov-Bohm potentials it is stronger than the Dirichlet form, and consequently the function space is a proper subset of $D^{1,2}(\mathbb{R}^N)$.

Indeed, for any $\varphi \in C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$ we have the simple inequality

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\varphi|^2 &= \int_{\mathbb{R}^N} |(i\nabla - \mathcal{A} + \mathcal{A})\varphi|^2 \le C\left(\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})\varphi|^2 + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |\varphi|^2\right)\\ &\le C\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})\varphi|^2\end{aligned}$$

thanks to Hardy-type inequality proved above.

It is immediate to see by this remark

$$\mathcal{H}_\mathcal{A} \doteq \overline{C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})}^{\int_{\mathbb{R}^N}|(i\nabla-\mathcal{A})\varphi|^2} \subseteq D^{1,2}(\mathbb{R}^N)\ .$$

To prove the strict inclusion it is sufficient to show a function lying in $D^{1,2}(\mathbb{R}^N)$ but not in $\mathcal{H}_\mathcal{A}$. One can choose for example $\varphi(x_1, x_2, x_3) = p(x_1, x_2, x_3)\, |x|^{(-N+1)/2}$, where p is a cut-off function which is identically 0 in $B_\varepsilon(0)$ and identically 1 in $\mathbb{R}^N \setminus B_{2\varepsilon}(0)$: we have $|\nabla\varphi|^2 \sim |x|^{-N-1}$ which is integrable in $\mathbb{R}^N \setminus B_\varepsilon(0)$, whereas $\frac{\varphi^2}{x_1^2+x_2^2}$ is not, since φ is far from 0 near the singular set.

Remark 1.5.2 *Of course $\mathcal{H}_\mathcal{A}$ is a closed subspace of $D^{1,2}(\mathbb{R}^N)$. This is a straightforward consequence of the density of $C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$ in $\mathcal{H}_A$ and the relation between the two quadratic forms. Then Solimini's Theorem (1.3.3) holds also in this space.*

Following what we did in the previous case, we state the following

Lemma 1.5.3 *Let x_n a sequence of points such that $|({x_n}^1, {x_n}^2)| \to \infty$ as $n \to \infty$. Then for any $u \in \mathcal{H}_\mathcal{A}$ as $n \to \infty$ we have*

$$\frac{\displaystyle\int_{\mathbb{R}^N} \left|(i\nabla - \mathcal{A})\, u(\cdot + x_n)\right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{x_1^2 + x_2^2}\, |u(\cdot + x_n)|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \to \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}.$$

Proof. We can follow the proof of Lemma (1.2.4) noting here the singularity involves only the first two variables. □

So that we immediately have the following property for $S_{\mathcal{A},a}$:

Proposition 1.5.4 *If the electric potential a is invariant under translations in $\mathbb{R}^{N-2}$ (as the magnetic vector potential actually is), the related minimum problem leads to*

$$S_{\mathcal{A},a} = \inf_{u \in \mathcal{H}_\mathcal{A} \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})u|^2 - \int_{\mathbb{R}^N} \frac{a}{x_1^2 + x_2^2} |u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} \leq S \ .$$

Proof. We follow the proof of Proposition (1.2.5) taking into account Lemma (1.5.3). □

1.5.3 Achieving the Sobolev constant

As in the previous case, we state the following

Proposition 1.5.5 *If $S_{\mathcal{A},a} < S$ then $S_{\mathcal{A},a}$ is achieved.*

Proof. The proof is essentially the same as in Theorem (1.3.4).

1.5.4 Symmetry of solutions

We introduce the space

$$\mathcal{H}_\mathcal{A}^{k,m} = \{u(z,y) \in \mathcal{H}_\mathcal{A} \text{ s.t. } u(\mathrm{e}^{i\frac{2\pi}{k}} z, y) = \mathrm{e}^{i\frac{2\pi}{k}m} u(z, |y|)\} \ ,$$

which is a closed subspace of $\mathcal{H}_\mathcal{A}$, so Solimini's Theorem (1.3.3) holds in it.

We should suppose that the magnetic potential $\mathcal{A}$ is equivariant under the $\mathbb{Z}_k \times SO(N-2)$-group action on $\mathcal{H}_\mathcal{A}$, as in (1.6). But in this case, the magnetic vector potential enjoys this symmetry thanks to its special form. On the other hand, we choose the electric potential a as a negative constant.

Following the same proof as in the previous case, we can state the following

Proposition 1.5.6 *If $S_{\mathcal{A},a}^{k,m} < k^{2/N} S$ then $S_{\mathcal{A},a}^{k,m}$ is achieved.*

Now we look for sufficient conditions to have $S_{\mathcal{A},a}^{k,m} < k^{2/N} S$.

The idea is again to check the quotient over a suitable sequence of test functions. We choose as well $\sum_{i=1}^k w_j$, where w_j are defined in (1.9) and the lines above it. Of course, we need to multiply them by a cut-off function $\varphi(x_1, x_2, x_3) = \varphi(x_1, x_2) = \varphi(\sqrt{x_1^2 + x_2^2}) = \varphi(\rho)$, in order to obtain the necessary integrability near the singular set.

Lemma 1.5.7 *Choosing R and k large enough in (1.9), the quotient evalueted over $\varphi \sum_{i=1}^k w_j$ is strictly less than $k^{2/N} S$, and so is the infimum $S_{\mathcal{A},a}^{k,m}$.*

Proof. Let us check the quotient over $\varphi \sum_{j=1}^k w_j$. In

$$\int_{\mathbb{R}^N} \Big|\nabla(\varphi \sum_{j=1}^k w_j)\Big|^2 + \frac{\alpha^2 - a}{x_1^2 + x_2^2}\varphi^2 \Big|\sum_{j=1}^k w_j\Big|^2 - 2Re\Big\{ i\nabla(\varphi \sum_{j=1}^k w_j) \cdot \mathcal{A}\varphi \sum_{j=1}^k \overline{w}_j \Big\}$$

we study term by term. First of all

$$\Big|\nabla(\varphi \sum_{j=1}^k w_j)\Big|^2 = |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2 + \varphi^2 \Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 + 2\, Re\left\{ \varphi\nabla\varphi \cdot \sum_{j,l} \nabla w_j \overline{w}_l \right\}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} \varphi^2 \Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 &= \int_{\mathbb{R}^N} \Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 - \int_{\mathbb{R}^N} (1-\varphi^2)\Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 \\
&= kS^{N/2} + \int_{\mathbb{R}^N} Re\Big\{ \sum_{j\neq l} |w_j|^{2^*-2} w_j \overline{w}_l \Big\} - \int_{\mathbb{R}^N} (1-\varphi^2)\Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 .
\end{aligned}$$

Secondly

$$\nabla(\varphi \sum_{j=1}^k w_j) \cdot \mathcal{A}\varphi \sum_{j=1}^k \overline{w}_j = \varphi\nabla\varphi \cdot \mathcal{A}\Big|\sum_{j=1}^k w_j\Big|^2 + \varphi^2 \sum_{j,l} \nabla w_j \cdot \mathcal{A}\, \overline{w}_l .$$

So, the quadratic form is the following

$$
\begin{aligned}
&kS^{N/2} + \int_{\mathbb{R}^N} Re\Big\{ \sum_{j\neq l} |w_j|^{2^*-2} w_j \overline{w}_l \} - \int_{\mathbb{R}^N} (1-\varphi^2)\Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 + \int_{\mathbb{R}^N} |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2 \\
&\quad +2\int_{\mathbb{R}^N} Re\Big\{\varphi\nabla\varphi\cdot\sum_{j,l}\nabla w_j\,\overline{w}_l\Big\} + \int_{\mathbb{R}^N} \frac{\alpha^2-a}{x_1^2+x_2^2}\varphi^2\Big|\sum_{i=1}^k w_j\Big|^2 \\
&\quad -2\int_{\mathbb{R}^N} Re\Big\{i\,\varphi\nabla\varphi\cdot\mathcal{A}\Big|\sum_{j=1}^k w_j\Big|^2\Big\} - 2\int_{\mathbb{R}^N} Re\Big\{i\,\varphi^2\sum_{j,l}\nabla w_j\cdot\mathcal{A}\,\overline{w}_l\Big\} \\
&\leq kS^{N/2} + \int_{\mathbb{R}^N} Re\Big\{ \sum_{j\neq l} |w_j|^{2^*-2} w_j \overline{w}_l \} + \int_{\mathbb{R}^N} |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2 \\
&\quad +2\left(\int_{\mathbb{R}^N}\varphi^2\Big|\sum_{j=1}^k\nabla w_j\Big|^2\right)^{1/2}\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2} \\
&\quad +\int_{\mathbb{R}^N} \frac{\alpha^2-a}{x_1^2+x_2^2}\varphi^2\Big|\sum_{j=1}^k w_j\Big|^2 \\
&\quad +2\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2}\left(\int_{\mathbb{R}^N}\varphi^2\,|\mathcal{A}|^2\Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2} \\
&\quad -2\int_{\mathbb{R}^N} Re\Big\{i\,\varphi^2\sum_{j,l}\nabla w_j\cdot\mathcal{A}\,\overline{w}_l\Big\}\ ,
\end{aligned}
$$

whereas the denominator

$$
\begin{aligned}
\int_{\mathbb{R}^N}\varphi^{2^*}\Big|\sum_{j=1}^k w_j\Big|^{2^*} &= \int_{\mathbb{R}^N}\Big|\sum_{j=1}^k w_j\Big|^{2^*} - \int_{\mathbb{R}^N}(1-\varphi^{2^*})\Big|\sum_{j=1}^k w_j\Big|^{2^*} \\
&\geq kS^{N/2} + 2^*(1-\delta/2)\int_{\mathbb{R}^N} Re\sum_{j\neq l}|w_j|^{2^*-2}w_j\,\overline{w}_l \\
&\quad -\int_{\mathbb{R}^N}(1-\varphi^{2^*})\Big|\sum_{j=1}^k w_j\Big|^{2^*}\ . \qquad (1.28)
\end{aligned}
$$

To simplify the notation, we set $R = \sqrt{(x_j{}^1)^2 + (x_j{}^2)^2}$ and we have

$$
\begin{aligned}
\alpha &= \int_{\mathbb{R}^N} Re\Big\{\sum_{j\neq l} |w_j|^{2^*-2}\, w_j\, \overline{w}_l\Big\} \begin{cases} \leq O(\frac{k^N}{R^{N-2}}) \\ \gg \frac{k^{N-1}}{R^{N-2}} \end{cases} && (1.29)\\
\beta &= \int_{\mathbb{R}^N} |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2 \;\leq\; O\Big(\frac{k^2}{R^{2N-4}}\Big)\\
\gamma &= 2\left(\int_{\mathbb{R}^N} \varphi^2 \Big|\nabla(\sum_{j=1}^k w_j)\Big|^2 \int_{\mathbb{R}^N} |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2} \;\leq\; O\Big(\frac{k^{3/2}}{R^{N-2}}\Big)\\
\eta &= \int_{\mathbb{R}^N} \frac{\alpha^2-a}{x_1^2+x_2^2}\varphi^2 \Big|\sum_{j=1}^k w_j\Big|^2 \;\leq\; \begin{cases} O(\frac{k^2}{R^2}\log R) & \text{if } N=4\\ O(\frac{k^2}{R^2}) & \text{if } N\geq 5\end{cases}\\
\xi &= 2\left(\int_{\mathbb{R}^N} |\nabla\varphi|^2 \Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2} \left(\int_{\mathbb{R}^N} \varphi^2\,|\mathcal{A}|^2 \Big|\sum_{j=1}^k w_j\Big|^2\right)^{1/2}\\
&\leq \begin{cases} O(\frac{k^2}{R^3}\log^{1/2} R) & \text{if } N=4\\ O(\frac{k^2}{R^{N-1}}) & \text{if } N\geq 5\end{cases} && (1.30)\\
\psi &= \int_{\mathbb{R}^N} (1-\varphi^{2^*}) \Big|\sum_{j=1}^k w_j\Big|^{2^*} \;\leq\; O(\frac{k^2}{R^{2N}})
\end{aligned}
$$

while for the last term we have

$$
\left|\int_{\mathbb{R}^N} Re\Big\{i\,\varphi^2 \sum_{j,l} \nabla w_j \cdot \mathcal{A}\overline{w}_l\Big\}\right| \leq O(\frac{k^{N-3}}{R^{N-2}}) \tag{1.31}
$$

since Lemma (1.4.9) fits also in this case with the suitable modifications. In (1.29) the symbol $\gg$ stands for α has order strictly greater than k^{N-1}/R^{N-2}.

We note all these quantities α, β, γ, η, ξ, ζ can be chosen small simply taking the quotient k^{N-1}/R^{N-2} small (namely $k^{N-1}/R^{N-2} = \varepsilon$), as we can deduce from (1.29), ..., (1.31).

Moreover, we see $\psi = o(\alpha)$, so that we can improve estimate (1.28) and state

$$
\int_{\mathbb{R}^N} \varphi^{2^*} \Big|\sum_{j=1}^k w_j\Big|^{2^*} \geq kS^{N/2} + 2^*(1-\delta/2)\int_{\mathbb{R}^N} Re \sum_{j\neq l} |w_j|^{2^*-2}\, w_j\, \overline{w}_l
$$

for a different δ from above.

With the simplified notation, the quotient takes the form

$$\frac{kS^{N/2}+\alpha+\beta+\gamma+\eta+\xi+\zeta}{\big(kS^{N/2}+2^*(1-\delta/2)\alpha\big)^{2/2^*}} = k^{2/N}S\,\frac{1+\dfrac{1}{kS^{N/2}}(\alpha+\beta+\gamma+\eta+\xi+\zeta)}{\Big(1+\dfrac{2^*(1-\delta/2)}{kS^{N/2}}\alpha\Big)^{2/2^*}} \,.$$

Expanding the quotient in first order power series, it is asymptotic to

$$\begin{aligned} & k^{2/N}S \quad \Big(1+\frac{1}{kS^{N/2}}(\alpha+\beta+\gamma+\eta+\xi+\zeta)\Big)\Big((1-\frac{2(1-\delta/2)}{kS^{N/2}}\alpha\Big) \\ \sim\ & k^{2/N}S \quad \Big\{1+\frac{1}{kS^{N/2}}\big(\beta+\gamma+\eta+\xi+\zeta\big) \\ & \qquad + \frac{1}{kS^{N/2}}\alpha\Big(-1+\delta+\frac{1}{kS^{N/2}}\big(\beta+\gamma+\eta+\xi+\zeta\big)\Big)\Big\} \end{aligned}$$

Now, in order to have the coefficient of $k^{2/N}S$ strictly less than 1, it is sufficient that β, γ, η, ξ, ζ are $o(k^{N-1}/R^{N-2})$. Taking into account (1.29), ..., (1.30) and (1.31) we see it is sufficient choosing k as in the previous case of $\dfrac{A}{|x|}$-type potentials. □

As we made in the previous section, we wonder if there exists any biradial solution, meaning a function belonging to the space

$$\begin{aligned} \mathcal{H}_{\mathcal{A}}^{\mathrm{birad},m} = \{u \in \mathcal{H}_{\mathcal{A}} \text{ s.t. } \quad & u(R(x_1,x_2),Sx_3) = R^m u((x_1,x_2),x_3) \\ & \forall R \in SO(2)\,, \forall S \in SO(N-2)\}\ . \end{aligned}$$

We note that here the suitable equivariant condition for the magnetic field is already fulfilled thanks to the special form of Aharonov–Bohm potentials, as well as it occurs for the $\mathbb{Z}_k \times SO(N-2)$ action. In order to investigate this question, we set the problem

$$S_{\mathcal{A},a}^{\mathrm{birad},m} = \inf_{u\in\mathcal{H}_{\mathcal{A}}^{\mathrm{birad},m}} \frac{\displaystyle\int_{\mathbb{R}^N}|(i\nabla-A)u|^2 - \int_{\mathbb{R}^N}\frac{a}{x_1^2+x_2^2}|u|^2}{\Big(\displaystyle\int_{\mathbb{R}^N}|u|^{2^*}\Big)^{2/2^*}}\,,$$

and we state

Proposition 1.5.8 *There exists a biradial solution.*

Proof. We follow the proof of Proposition (1.4.11) that fits also in this case with the suitable modifications. □

1.6 Symmetry breaking

In order to proceed in our analysis, we need to recall a result proved in [1]:

Theorem 1.6.1 *([1]) Suppose $u = u(r_1, r_2)$ (where $r_1 = \sqrt{x_1^2 + x_2^2}$ and $r_2 = \sqrt{x_3^2 + \cdots + x_N^2}$) is a $D^{1,2}(\mathbb{R}^N)$ solution to*

$$-\Delta u - \frac{a}{|x|^2} u = f(x, u)$$

with $a \in \mathbb{R}^-$ and $f : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ being a Carathéodory function, C^1 with respect to z, such that it satisfies the growth restriction

$$|f_z'(x, z)| \leq C(1 + |z|^{2^*-2})$$

for a.e. $x \in \mathbb{R}^N$ and for all $z \in \mathbb{C}$.

If the solution u has biradial Morse index $m(u) \leq 1$, then u is a radial solution, that is $u = u(r)$ where $r = \sqrt{x_1^2 + \cdots + x_N^2}$.

We split the argument according to the value of the parameter m.

For $m = 0$, the minimizers for $S_{A,a}^{\mathrm{birad},0}$ can be chosen real-valued and have in fact biradial Morse index exactly 1. Further, if the magnetic potential is not present, we are precisely under the hypothesis of the previous theorem, then the minimizers are in fact completely radial and the two levels of the quotient coincide:

$$S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{\mathrm{rad}} = S\Big(1 - a\frac{4}{(N-2)^2}\Big),$$

where the precise value of $S_{0,a}^{\mathrm{rad}}$ has been stated for instance in [62]. So we can write the following chain of relations:

$$S_{A,a}^{\mathrm{birad},0} \geq S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{rad} = S\Big(1 - a\frac{4}{(N-2)^2}\Big) \geq k^{2/N} S > S_{A,a}^{k,0}$$

where the first inequality holds thanks to diamagnetic inequality; the fact $S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{\mathrm{rad}}$ is a straightforward consequence of the last theorem; the second inequality is proved in [62], Section 6 for sufficiently large values of $|a|$; and the last one is proved in Proposition (1.4.3).

We note that if we assume A is $SO(N)$-equivariant, then $A(\theta) = \lambda\theta$ for some constant λ, then the magnetic potential $A(\theta)/|x|$ is a gradient. Thus the problems $S_{A,a}^{\mathrm{birad},0}$ and $S_{0,a}^{\mathrm{birad},0}$ are in fact equivalent, so there are no biradial solutions distinct from the radial ones.

For $m \neq 0$, the previous argument is sufficient to prove the symmetry breaking as well. Indeed, the functions in $D_{\mathrm{birad},m}^{1,2}(\mathbb{R}^N)$ take the special form $u(z, y) = \rho(|z|, |y|)\mathrm{e}^{im\arg(z)}$, so that

$$|\nabla u|^2 = |\nabla \rho|^2 + m^2 \frac{\rho^2}{|z|^2}.$$

Then we can write the following chain of relations:

$$S_{A,a}^{\mathrm{birad},m} \geq S_{0,a}^{\mathrm{birad},m} > S_{0,m^2-a}^{\mathrm{birad},0} = S_{0,m^2-a}^{\mathrm{rad}} = S\Big(1 + \frac{4(m^2-a)}{(N-2)^2}\Big) \geq k^{2/N} S > S_{A,a}^{k,m},$$

where the first inequality holds again thanks to the diamagnetic inequality; the second one is due to the special form of functions in $D^{1,2}_{\text{birad},m}(\mathbb{R}^N)$ (see also [1]), and the third inequality is a straightforward consequence of its analogue in the case $m = 0$.

Remark 1.6.2 Symmetry breaking for Aharonov-Bohm electromagnetic potentials. *We note the same facts hold also for Aharonov-Bohm electromagnetic fields. Indeed, the diamagnetic inequivalence holds also for them with the* same *best constant, because the Hardy constant is the same (see Section 4.1); moreover,* $\frac{a}{x_1^2+x_2^2} \geq \frac{a}{|x|^2}$ *for* $a > 0$*. So we can rewrite*

$$S^{birad,0}_{\mathcal{A},a} \geq S^{birad,0}_{0,a} = S^{rad}_{0,a} \geq k^{2/N} S > S^{k}_{\mathcal{A},a}$$

where the last inequivalence has been proved in Lemma (1.5.7).

Chapter 2

A note on the complete rotational invariance of biradial solutions to semilinear elliptic equations

2.1 Introduction and statement of the result

Let $x = (\xi, \zeta) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $k, N-k \geq 2$. A function $u\colon \mathbb{R}^N \to \mathbb{R}$ is termed *biradial* if it is invariant under the action of the subgroup $SO(k) \times SO(N-k)$ of the group of rotations, namely, if there exists $\varphi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that $u(\xi, \zeta) = \varphi(|\xi|, |\zeta|)$. Consider the equation

$$-\Delta u = \frac{a}{|x|^2} u + f(|x|, u) \qquad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{2.1}$$

in this paper, we wonder under what circumstances it is possible to assert that a biradial solution to (2.1) is actually radially symmetric.

This problem arises from [62], where the following symmetry breaking result is given for the critical nonlinearity $f(|x|, u) = u^{(N+2)/(N-2)}$: if $a < 0$ and $|a|$ is sufficiently large, there are at least two distinct positive solutions, one being radially symmetric and the second not. These solution are obtained by minimization of the associated Rayleigh quotient over functions possessing either the full radial symmetry or a discrete group of symmetries, namely, for given $k \in \mathbb{Z}$, functions which are invariant under the $\mathbb{Z}_k \times SO(N-2)$-action on $D^{1,2}(\mathbb{R}^N)$ given by

$$u(\xi, \zeta) \mapsto v(\xi, \zeta) = u\big(R\xi, T\zeta\big) ,$$

T being any rotation of $\mathbb{R}^{N-2}$ and R a fixed rotation of order k. Once proved that the infimum taken over the $\mathbb{Z}_k \times SO(N-2)$-invariant functions is achieved, by comparing its value with the infimum taken over the radial functions, one deduces the occurrence of symmetry breaking (see also [2]).

In order to obtain multiplicity of solutions, the first attempt is to increase the order k of the symmetry group and, eventually, to let it diverge to infinity, finding in the limit a minimizer of the Rayleigh quotient over the biradial functions. Now, will all these solutions be distinct and different from the radial one? When examining this question, we need to take into account the construction due to Ding of an infinity of nontrivial biradial solutions to the Lane–Emden equation with critical nonlinearity (cfr [24]). In that case it is well known that there is a unique family of radially symmetric solutions, which are the global minimizers of the Rayleigh quotients, while in Ding's construction the nontrivial biradial solutions have a Morse index larger than 2.

We recall the following definition:

Definition 2.1.1 *The (plain, radial, biradial)* Morse index *of a solution u is the dimension of the maximal subspace of the space of (all, radial, biradial) functions of $\mathcal{C}_0^\infty(\mathbb{R}^N \setminus \{0\})$ on which the quadratic form associated to the linearized equation at u is negative definite.*

We stress it is rather a geometric definition, so it is independent from any spectral theory about the differential operator we are dealing with.

The recent literature indicates that, for general semilinear equations, solutions having low Morse index do likely possess extra symmetries. Following these ideas and questions, we investigated in particular the biradial solutions with a low Morse index, and we are able to prove the following

Theorem 2.1.2 *Let $u \in D^{1,2}(\mathbb{R}^N)$ be a biradial solution to*

$$-\Delta u = \frac{a}{|x|^2} u + f(|x|, u) \tag{2.2}$$

with $a < \left(\frac{N-2}{2}\right)^2$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ being a Carathéodory function, C^1 with respect to z, such that it satisfies the growth restriction

$$\left|f'_y(|x|, y)\right| \le C(1 + |y|^{2^*-2})$$

for a.e. $x \in \mathbb{R}^N$ and for all $y \in \mathbb{C}$.

If the solution u has biradial Morse index $m(u) \le 1$, then u is radially symmetric.

An analogous result also holds for bounded domains having rotational symmetry, and for elliptic equations on the sphere. The following result holds in any dimension $N \ge 3$:

Theorem 2.1.3 *Let $f \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$: if $u \in \mathcal{C}^2(\mathbb{S}^N)$ is a biradial solution to*

$$-\Delta_{\mathbb{S}^N} u = f(u)$$

with $N \ge 3$, and it has biradial Morse index $m(v) \le 1$, then u is constant on the sphere $\mathbb{S}^N$.

The paper is organized as follows: the next section is devoted to introduce the main tools and facts which will play a key role within the proof; in section 3 we present the proofs of Theorems 2.1.2 and 2.1.3 splitting it according to solutions' Morse index. In section 4 we give applications to the estimate of the best constants in some Sobolev type embeddings with symmetries. Finally section 5 is devoted to the discussion of the sharpness of the Theorems with respect to the Morse index.

2.2 Preliminaries

Here we start the proof of Theorem 2.1.2. For the sake of simplicity, we will work in dimension $N = 4$. We devote the last part of the proof to discuss the validity of the result in higher dimensions.

Let us consider the following three orthogonal vector fields in $\mathbb{R}^4$:

$$X_1 = \begin{bmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_4 \\ x_3 \\ -x_2 \\ -x_1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \end{bmatrix}$$

The related derivatives

$$w_i = \nabla u \cdot X_i \ , \qquad i = 1, 2, 3$$

represent the infinitesimal variations of the function u along the flows of the vector fields X_i respectively. As the equation is invariant under the action of such flows, these directional derivatives are solutions to the linearized equation

$$-\Delta w - \frac{a}{|x|^2} w = f'_y(|x|\, , u) w \ . \tag{2.3}$$

We can associate the singular differential operator

$$L_u w = -\Delta w - \frac{a}{|x|^2} w - f'_y(|x|\, , u) w \ . \tag{2.4}$$

Remark 2.2.1 *The vector space of* $\{X_1, X_2, X_3\}$ *generates the whole group of infinitesimal rotations on the sphere of* $\mathbb{R}^4$*, which can be structured as a 3-dimensional manifold. In order to prove Theorem 2.1.2 it will be sufficient to show that every* $w_i \equiv 0$*.*

Obviously, we have $w_1 \equiv 0$ because the vector field X_1 generates the rotations under which the function u is invariant for. Let us fix polar coordinates

$$\begin{cases} x_1 = r_1 \cos\theta_1 \\ x_2 = r_1 \sin\theta_1 \end{cases} \qquad \begin{cases} x_3 = r_2 \cos\theta_2 \\ x_4 = r_2 \sin\theta_2 \ ; \end{cases} \tag{2.5}$$

we have $r_1 = \sqrt{{x_1}^2 + {x_2}^2}$, $r_2 = \sqrt{{x_3}^2 + {x_4}^2}$ and $\theta_1 = \arctan \frac{x_2}{x_1}$, $\theta_2 = \arctan \frac{x_4}{x_3}$.

Therefore, since u is biradial, we have

$$w_i = \nabla u \cdot X_i = w(r_1, r_2) z_i(\theta_1, \theta_2) \ , \qquad i = 2, 3 \ ,$$

where

$$w(r_1, r_2) = \frac{\partial u}{\partial r_1} r_2 - \frac{\partial u}{\partial r_2} r_1, \qquad z_2 = \sin(\theta_1 + \theta_2), \qquad z_3 = -\cos(\theta_1 + \theta_2).$$

Remark 2.2.2 *According to Remark 2.2.1, to our aim it will be sufficient to prove that* $w \equiv 0$.

We now focus our attention on a few fundamental properties of the functions w_i. At first, as the z_i's are spherical harmonics and depend on the angles θ_1 and θ_2 only, we have

$$-\Delta z_i = \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) z_i \qquad \text{for } i = 2, 3. \tag{2.6}$$

Joining this with the linearized equation (2.3) solved by the w_i's, we obtain the equation for w.

Proposition 2.2.3 *The function w is a solution to the following equation*

$$-\Delta w - \frac{a}{|x|^2} w - f_y'(|x|, u)w + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) w = 0. \tag{2.7}$$

Proof. It holds that

$$f_y'(|x|, u)w_i = -\Delta(wz_i) - \frac{a}{|x|^2} wz_i = -\Delta w\, z_i - \nabla w \cdot \nabla z_i - w\Delta z_i - \frac{a}{|x|^2} wz_i.$$

Since $\nabla w \cdot \nabla z_i = 0$, thanks to (2.6), this becomes

$$-\Delta w\, z_i - \frac{a}{|x|^2} wz_i = f_y'(|x|, u)w\, z_i + w\Delta z_i = f_y'(|x|, u)w\, z_i - \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) wz_i$$

that is

$$z_i \left\{ -\Delta w - \frac{a}{|x|^2} w - f_y'(|x|, u)w + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) \right\} = 0.$$

Last, multiplying by z_i and summing for $i = 1, 2$ we obtain the desired equation.

2.3 Proofs

We will split the argument according to the Morse index of solution u: we denote it by $m(u)$.

In order to complete our proof, we need a couple of preliminary results: the first one is about the asymptotics of the solution and is contained in [30].

Lemma 2.3.1 *([30]) Under the assumptions of Theorem 2.1.2, let u be any solution to* (2.1). *Then the following asymptotics hold*

$$u(x) \sim |x|^{\gamma}\, \psi(\frac{x}{|x|}) \qquad \textit{for } |x| \ll 1 \tag{2.8}$$

$$u(x) \sim |x|^{\delta}\, \psi(\frac{x}{|x|}) \qquad \textit{for } |x| \gg 1 \tag{2.9}$$

where $\gamma = \gamma(a,N) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}$, $\delta = \delta(a,N) = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}$ *and* $\mu = \mu(a,N)$ *is one of the eigenvalues of* $-\Delta_{\mathbb{S}^{N-1}} - a$ *on* $\mathbb{S}^{N-1}$, *and* ψ *one of its related eigenfunctions.*

This turns out to be the key for proving the following result.

Lemma 2.3.2 *The function* $\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) w^2$ *is* L^1*-integrable on* $\mathbb{R}^N$*.*

Proof. Since $w(r_1, r_2) = \frac{\partial u}{\partial r_1} r_2 - \frac{\partial u}{\partial r_2} r_1$, we first observe that by regularity of u outside the origin and its radial symmetry, the functions

$$\frac{1}{r_i}\frac{\partial u}{\partial r_i}$$

$i = 1, 2$, are continuous outside the origin. Next we remark that

$$\begin{aligned}\frac{1}{4}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) w^2 &\le \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\left\{\left(\frac{\partial u}{\partial r_1}\right)^2 r_2^2 + \left(\frac{\partial u}{\partial r_2}\right)^2 r_1^2\right\} \\ &= \left(\frac{r_2}{r_1}\right)^2\left(\frac{\partial u}{\partial r_1}\right)^2 + \left(\frac{r_1}{r_2}\right)^2\left(\frac{\partial u}{\partial r_2}\right)^2 + \left(\frac{\partial u}{\partial r_1}\right)^2 + \left(\frac{\partial u}{\partial r_2}\right)^2.\end{aligned}$$

The integrability of the last two terms is a straightforward consequence of $u \in D^{1,2}(\mathbb{R}^N)$. In order to study the other two terms, let us focus our attention in a ball around the origin, namely $B_1(0)$, so that $r_1^2 + r_2^2 \le 1$. Then

$$\left(\frac{\partial u}{\partial r_1}\right)^2\left(\frac{r_2}{r_1}\right)^2 \le \frac{1}{r_1^2}\left(\frac{\partial u}{\partial r_1}\right)^2 - \left(\frac{\partial u}{\partial r_1}\right)^2,$$

so that the question of integrability is restricted to the first term. From Lemma 2.3.1, Equation (2.8) we know $u \sim r^\gamma \psi(r_1, r_2) = (r_1^2 + r_2^2)^{\gamma/2}\psi(r_1, r_2)$, from which

$$\frac{\partial u}{\partial r_1} \sim \psi(r_1, r_2)\gamma(r_1^2 + r_2^2)^{\gamma/2 - 1} r_1 + (r_1^2 + r_2^2)^{\gamma/2}\frac{\partial \psi}{\partial r_1}.$$

So we are lead to consider the integrability of $\int_{B_1(0)} \frac{1}{r_1^2}\left(\frac{\partial \psi}{\partial r_1}\right)^2$. Additionally we know that ψ is the restriction on the sphere of a harmonic polynomial, then it is analytic and its Taylor's expansion is a polynomial whose degree 1 terms vanish, since it is a function of the only variables r_1 and r_2. Then $\left(\frac{\partial \psi}{\partial r_1}\right)^2 \sim r_1^2$, which provides the sought integrability.

For what concerns the integrability at infinity, it is sufficient to show that the terms of type $\frac{r_2^2}{r_1^2}\left(\frac{\partial u}{\partial r_1}\right)^2$ are in $L^1(\mathbb{R}^N)$. We have

$$\frac{r_2^2}{r_1^2}\left(\frac{\partial u}{\partial r_1}\right)^2 \le 2\left\{(r_1^2 + r_2^2)^{\delta - 2} r_2^2 \psi^2 + \frac{r_2^2}{r_1^2}(r_1^2 + r_2^2)^\delta \left(\frac{\partial \psi}{\partial r_1}\right)^2\right\}$$

and exploiting equation (2.9), the expression of the exponent δ provides the sought integrability.

In the following we consider the cut-off function defined as $\eta(r_1, r_2) = \eta_1(r_1)\eta_2(r_2)$ where

$$\eta_1(r_1) = \begin{cases} \frac{1}{\log(R_2/R_1)} \log r_1/R_1 & \text{for } R_1 \le r_1 \le R_2 \\ 1 & \text{for } R_2 \le r_1 \le R_3 \\ 1 - \frac{1}{\log(R_4/R_3)} \log r_1/R_3 & \text{for } R_3 \le r_1 \le R_4 \\ 0 & \text{elsewhere,} \end{cases}$$

η_2 being defined similarly. Given the special form of η, we note $|\nabla\eta|^2 \le |\nabla\eta_1|^2 + |\nabla\eta_2|^2$, that is

$$|\nabla\eta|^2 \le \frac{1}{\log^2 R_2/R_1}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) \qquad \text{for } R_1 \le r_1\,, r_2 \le R_2$$

and analogously for $R_3 \le r_1\,, r_2 \le R_4$. Thus, we have

$$|\nabla\eta|^2 \le 3\left(\frac{1}{\log^2 R_4/R_3} + \frac{1}{\log^2 R_2/R_1}\right)\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right). \tag{2.10}$$

Lemma 2.3.3 *There is a suitable choice of the parameters R_1, R_2, R_3 and R_4 such that the quadratic form associated to the operator* (2.4) *is negative definite both on $\eta\, w^+$ and $\eta\, w^-$.*

Proof. Let us fix $\varepsilon > 0$ small and choose $R_1 = \varepsilon^2$, $R_2 = \varepsilon$ and $R_3 = \varepsilon^{-1}$, $R_4 = \varepsilon^{-2}$. We multiply equation (2.7) by $\eta^2 w^+$ and integrate by parts. We obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\nabla(\eta w^+)\right|^2 - \frac{a}{|x|^2}\eta^2(w^+)^2 - f'_y(|x|\,, u)\eta^2(w^+)^2 \\ &\quad = \int_{\mathbb{R}^N} |\nabla\eta|^2 (w^+)^2 - \int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\eta^2(w^+)^2 . \end{aligned} \tag{2.11}$$

If ε is small enough, the second term in (2.11) is far away from zero, or rather, it is quite close to $\int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(w^+)^2$, say for instance

$$\int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\eta^2(w^+)^2 > \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(w^+)^2.$$

On the other hand, the first term in (2.11) can be made very small with respect to $\int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(w^+)^2$, since from (2.10)

$$\int_{\mathbb{R}^N} |\nabla\eta|^2 (w^+)^2 \le \frac{6}{\log^2 \varepsilon}\int_{\mathbb{R}^N}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)(w^+)^2,$$

so that (2.11) is seen to be negative.

Repeating the same argument multiplying by $\eta^2 w^-$ we reach the same conclusion.

First case: Morse index $m(u) = 0$. In this case Lemma 2.3.3 clearly contradicts the hypothesis $m(u) = 0$, unless $w^+ = w^- \equiv 0$, that is the only stable solution to (2.7) is the trivial one.

Second case: Morse index $m(u) = 1$. *In this case we infer that w has constant sign*, say positive, and therefore $w > 0$ for $r_1 > 0$ and $r_2 > 0$ by the Strong Maximum Principle. Now we show a contradiction. Consider a vector field of the form $\alpha X_2 + \beta X_3$. Along this vector field, choosing $\alpha = \cos\gamma$ and $\beta = \sin\gamma$, the derivative of u is

$$\nabla u \cdot (\alpha X_2 + \beta X_3) = \alpha w_2 + \beta w_3 = w\big(\alpha \sin(\theta_1 + \theta_2) - \beta\cos(\theta_1 + \theta_2)\big) = -w \sin(\theta_1 + \theta_2 - \gamma).$$

Now we turn to the directional derivative of $\theta_1 + \theta_2$ along the vector field $\alpha X_2 + \beta X_3$. Using the polar coordinates (2.5), it results

$$\theta_1 = \arctan\frac{x_2}{x_1}, \qquad \theta_2 = \arctan\frac{x_4}{x_3};$$

so that checking the motion along X_2 we have

$$\begin{aligned}
\nabla\theta_1 \cdot X_2 &= \frac{x_3 x_1 - x_2 x_4}{x_1^2 + x_2^2} = \frac{r_2}{r_1}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) = \frac{r_2}{r_1}\cos(\theta_1 + \theta_2)\\
\nabla\theta_2 \cdot X_2 &= \frac{-x_3 x_1 + x_2 x_4}{x_3^2 + x_4^2} = \frac{r_1}{r_2}(-\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = -\frac{r_1}{r_2}\cos(\theta_1 + \theta_2),
\end{aligned}$$

whereas along X_3

$$\begin{aligned}
\nabla\theta_1 \cdot X_3 &= \frac{x_4 x_1 + x_2 x_3}{x_1^2 + x_2^2} = \frac{r_2}{r_1}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) = \frac{r_2}{r_1}\sin(\theta_1 + \theta_2)\\
\nabla\theta_2 \cdot X_3 &= \frac{-x_2 x_3 - x_1 x_4}{x_3^2 + x_4^2} = \frac{r_1}{r_2}(-\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) = -\frac{r_1}{r_2}\sin(\theta_1 + \theta_2);
\end{aligned}$$

and finally we obtain

$$\begin{aligned}
\nabla(\theta_1 + \theta_2) \cdot (\alpha X_2 + \beta X_3) &= \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)\big(\alpha\cos(\theta_1 + \theta_2) + \beta\sin(\theta_1 + \theta_2)\big)\\
&= \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)\cos(\theta_1 + \theta_2 - \gamma).
\end{aligned}$$

Now we are in good position to conclude. For a given point $\overline{x}$ of the sphere - located by angles $\overline{\theta_1}$ and $\overline{\theta_2}$, we choose $\gamma = \gamma(\overline{x}) = \overline{\theta_1} + \overline{\theta_2} - \pi/2$, so that the quantity $\theta_1 + \theta_2$ is at rest for the associated vector field $\cos\gamma X_2 + \sin\gamma X_3$. With this choice the function u is monotone along the flow $\alpha X_2 + \beta X_3$ since $\dot{u} = -w\sin(\theta_1 + \theta_2 - \gamma) = -w$ and the sign of w is constant by the previous discussion. Since the trajectory of the flow is a circle, we will reach again the initial point in finite time, but with a strictly smaller value of u (if we consider the first eigenfunction w positive). This is clearly a contradiction.

Generalization to higher dimensions. In dimension $N \geq 5$ the argument is very similar. Relabeling we may always assume $u = u(\rho_1, \rho_2)$, where we have fixed the notation $\rho_1 = |\xi|$ and $\rho_2 = |\zeta|$, while $|x| = \sqrt{|\xi|^2 + |\zeta|^2}$, being $x = (\xi, \zeta) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Now we repeat the argument performed in the 4-dimensional space with respect to the variables x_{k-1}, x_k, x_{k+1}, x_{k+2}, considering the vector fields with those same four components as above and the other ones being zero. Hence we define $r_1 = \sqrt{x_{k-1}^2 + x_k^2}$ and $r_2 = \sqrt{x_{k+1}^2 + x_{k+2}^2}$. When discussing the integrability properties, it can be worthwhile noticing that

$$\frac{1}{r_i}\frac{\partial u}{\partial r_i} = \frac{1}{\rho_i}\frac{\partial u}{\partial \rho_i}\,.$$

Arguing as above, we can prove that the solution u is actually radial with respect to those four variables. We can imagine to iterate this proceeding for every hyperplane whose rotations the function u is supposed not to be invariant for. Finally, it follows that u is radial in $\mathbb{R}^N$.

Proof of Theorem 2.1.3. Since now v is a function defined over $\mathbb{S}^N$, recalling the Laplace operator in polar coordinates

$$\Delta_{\mathbb{R}^{N+1}} = \partial_r^2 + \frac{N}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^N},$$

we define $\widetilde{v}(x) = v(y)$ for $x \in (-\varepsilon, \varepsilon) \times \mathbb{S}^N$, so that $\Delta_{\mathbb{S}^N} v = \Delta_{\mathbb{R}^{N+1}}\widetilde{v}$. At first, let us suppose $N = 3$. Obviously, since v is invariant with respect to the group $O(2) \times O(2)$, so is $\widetilde{v}$.

Following the same argument in the proof of Theorem 2.1.2, we wish to prove the vanishing of $\widetilde{w} = \frac{\partial \widetilde{v}}{\partial r_1} r_2 - \frac{\partial \widetilde{v}}{\partial r_2} r_1$. On the other hand, being $\widetilde{v}$ homogenous of degree 0, $\widetilde{w}$ is homogenuos of degree 0 too (it can be proved by differentiating identity $\widetilde{v}(x) = \widetilde{v}(\lambda x)$), then the w associated with v is nothing else that $\widetilde{w}$ restricted on the sphere $\mathbb{S}^N$. Therefore $\Delta_{\mathbb{R}^{N+1}}\widetilde{w} = \Delta_{\mathbb{S}^N} w$, and following the proof of Proposition 2.2.3 we see w is a solution to

$$-\Delta_{\mathbb{S}^N} w - f'(v)w + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) w = 0,$$

analogous to equation (2.7). The rest of the proof fits also in this case. □

2.4 An application to best Sobolev constants with symmetries

Solutions to the critical exponent equation

$$-\Delta u = \frac{a}{|x|^2} u + |u|^{2^*-2} u \tag{2.12}$$

are related to extremals of Sobolev inequalities (cfr [62]). To our purposes, the functions u will be complex-valued and $a \in (-\infty, (N-2)^2/4)$. Then, thanks to Hardy inequality, an equivalent norm on $D^{1,2}(\mathbb{R}^N)$ is

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2 - a\frac{|u|^2}{|x|^2} \right)^{1/2},$$

hence we can seek solutions to (2.12) as extremals of the Sobolev quotient associated with this norm on different symmetric spaces .

The whole group of rotations $SO(2) \times SO(N-2)$ induces the following action on $D^{1,2}(\mathbb{R}^N; \mathbb{C})$:

$$u(\xi, \zeta) \mapsto R^{-m} u(R\xi, T\zeta)$$

for $m \in \mathbb{Z}$ fixed. We denote, as usual, $D^{1,2}_{rad}(\mathbb{R}^N)$ and $D^{1,2}_{\text{birad}}(\mathbb{R}^N)$ the subspaces of real or complex radial and biradial functions. Moreover, let k and m be fixed integers; for a given rotation $R \in SO(2)$ of order k, we consider the space of symmetric functions

$$D^{1,2}_{R,k,m}(\mathbb{R}^N; \mathbb{C}) := \{u \in D^{1,2}(\mathbb{R}^N; \mathbb{C}) : \ u(R\xi, T\zeta) = R^m u(\xi, \zeta), \forall \ T \in SO(N-2)\}.$$

This is of course a proper subspace of

$$D^{1,2}_{\text{birad},m}(\mathbb{R}^N; \mathbb{C}) := \{u \in D^{1,2}(\mathbb{R}^N; \mathbb{C}) : \ u(S\xi, T\zeta) = S^m u(\xi, \zeta), \forall \ (S,T) \in SO(2) \times SO(N-2)\}.$$

Note this last space coincides with the usual space of biradial solution once $m = 0$.

Thanks to its rotational invariance, for any choice of the above spaces $D^{1,2}_*(\mathbb{R}^N; \mathbb{C})$, solutions to the minimization problem

$$\inf_{\substack{u \in D^{1,2}_*(\mathbb{R}^N;\mathbb{C}) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 - a\frac{|u|^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} \tag{2.13}$$

are in fact solutions to equation (2.12).

The minimization of the Sobolev quotient over the space of radial functions follows from a nowadays standard compactness argument; in addition, see for instance [62], we have:

$$\inf_{\substack{u \in D^{1,2}_{rad}(\mathbb{R}^N;\mathbb{C}) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 - a\frac{|u|^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} = S\Big(1 - a\frac{4}{(N-2)^2}\Big)$$

where S denote the best constant for the standard Sobolev embedding. Moreover, generalizing the results in [21] in higher dimensions (see also [2]), one can easily prove existence of minimizers of the Sobolev quotient (2.13) in the spaces $D^{1,2}_{\text{birad},m}(\mathbb{R}^N; \mathbb{C})$, for any choice of the integer m.

At first, let us consider the case $m = 0$. Then it is easily checked that the minimizers can be chosen to be real valued and that the corresponding solution to (2.12) have biradial Morse index exactly one. Hence our Theorem 2.1.2 applies and such biradial solutions are in fact fully radially symmetric, and therefore the infimum on the biradial space equals that on the radial. Now, let us turn to the case $m \neq 0$. We remark that elements of the space $D^{1,2}_{\mathrm{birad},m}(\mathbb{R}^N;\mathbb{C})$ have the form $u((\xi,\zeta) = \rho(|\xi|,|\zeta|)e^{im\theta(\xi)}$, where $\theta(\xi) = \arg(\xi)$, so that

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |m\nabla\theta|^2 = |\nabla\rho|^2 + m^2 \frac{\rho^2}{|\xi|^2}.$$

Then the following chain of inequalities holds:

$$\begin{aligned}
\min_{\substack{u\in D^{1,2}_{\mathrm{birad},m}(\mathbb{R}^N;\mathbb{C})\\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 - a\frac{|u|^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} &= \min_{\substack{\rho\in D^{1,2}_{\mathrm{birad}}(\mathbb{R}^N,\mathbb{R})\\ \rho\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\rho|^2 + m^2\frac{\rho^2}{|\xi|^2} - a\frac{\rho^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} \rho^{2^*}\right)^{2/2^*}} \\
&> \min_{\substack{\rho\in D^{1,2}_{\mathrm{birad}}(\mathbb{R}^N;\mathbb{R})\\ \rho\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\rho|^2 + (m^2-a)\frac{\rho^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} \rho^{2^*}\right)^{2/2^*}} = \min_{\substack{\rho\in D^{1,2}_{rad}(\mathbb{R}^N;\mathbb{R})\\ \rho\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\rho|^2 + (m^2-a)\frac{\rho^2}{|x|^2}}{\left(\displaystyle\int_{\mathbb{R}^N} \rho^{2^*}\right)^{2/2^*}} \\
&= S\left(1 + \frac{4(m^2-a)}{(N-2)^2}\right)
\end{aligned}$$

where we have used $|\xi| \leq |x|$; the intermediate line follows again from Theorem 2.1.2, and the last from [62]. Then, this argument states a very useful lower bound (see [2]) to the minima problems (2.13). Indeed, it allows us to compare the infimum over the space of $D^{1,2}_{R,k,m}(\mathbb{R}^N;\mathbb{C})$ with that on $D^{1,2}_{\mathrm{birad},m}(\mathbb{R}^N;\mathbb{C})$, and to prove the occurrence of symmetry breaking in some circumstances. In fact it has been proven (see [2]) that, for large enough k, the first minimum is achieved and less that $k^{2/N}S$, while the latter increases with $|a|$ and m. Symmetry breaking holds whenever it can be shown that $1 + \frac{4(m^2-a)}{(N-2)^2} > k^{2/N}$ for appopriate choices of the parameters.

2.5 Optimality with respect to the Morse index

We want to stress our results Theorem 2.1.2 and 2.1.3 are sharp with respect to the Morse index. By that, we mean that doubly radial solutions with Morse index greater or equal to 2, need not to be completely radial.

To prove this, we will take advantage from a result proved by Ding in [24] in such a way which will be clear later. The quoted paper by Ding has to do with solutions to a related equation on $\mathbb{S}^N$, for this reason we state first some connections between these two environments.

2.5.1 Conformally equivariant equations

We recall a general fact cited in [24] about elliptic equations on Riemannian manifolds.

Lemma 2.5.1 *Let (M, g) and (N, h) two Riemannian manifolds of dimensions $N \geq 3$. Suppose there is a conformal diffeomorphism $f : M \to N$, that is $f^*h = \varphi^{2^*-2}g$ for some positive $\varphi \in C^\infty(M)$. The scalar curvatures of (M, g) and (N, h) are R_g and R_h respectively. Set the following corresponding equations:*

$$-\Delta_g u + \frac{1}{4}\frac{N-2}{N-1}R_g(x)u = F(x, u) \tag{2.14}$$

$$-\Delta_h v + \frac{1}{4}\frac{N-2}{N-1}R_h(y)v = [(\varphi \circ f^{-1})(y)]^{-\frac{N+2}{N-2}}F(f^{-1}(y), (\varphi \circ f^{-1})(y)v) \tag{2.15}$$

where $F : M \times \mathbb{R} \to \mathbb{R}$ is smooth. Suppose v is a solution of (2.15). Then $u = (v \circ f)\varphi$ is a solution of (2.14) such that $\int_M |u|^{2^*} dV_g = \int_N |v|^{2^*} dV_h$.

We consider the inverse of the stereographic projection $\pi : \mathbb{S}^N \setminus \{p\} \to \mathbb{R}^N$. We denote it by $\Phi = \pi^{-1} : \mathbb{R}^N \to \mathbb{S}^N \setminus \{p\}$, moreover g_0 will denote the standard metric on $\mathbb{S}^N$ and δ the standard one on $\mathbb{R}^N$.

The diffeomorphism Φ is conformal between the two manifolds, since it results

$$g \doteq \Phi^* g_0 = \mu(x)^{\frac{4}{N-2}}\delta ,$$

where

$$\mu(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{N-2}{2}} .$$

In addition, we point out the manifold $(\mathbb{R}^N, g)$ is *the same* as $(\mathbb{S}^N, g_0)$, in terms of diffeomorphic manifolds.

We recall the following

Definition 2.5.2 *We define the* conformal Laplacian *on a differentiable closed manifold (M, g) of dimension N the operator*

$$L_g = -\Delta_g + \frac{N-2}{4(N-1)}R_g$$

where Δ_g denotes the standard Laplace-Beltrami operator on M and R_g the scalar curvature of the manifold.

Moreover, this operator has a simple transformation law under a conformal change of metric, that is

$$\text{if} \quad \widetilde{g} = \mu(x)^{\frac{4}{N-2}}g \qquad \text{then} \quad L_{\widetilde{g}}\,\cdot = \mu(x)^{-\frac{N+2}{N-2}}L_g\big(\mu(x)\cdot\big).$$

In our case we are dealing with the same manifold $\mathbb{R}^N$ endowed with the two metrics δ, the standard one, and $g = \Phi^* g_0$. Thus in our case we have

$$L_\delta = -\Delta \qquad L_g = -\Delta_g + \frac{1}{4}N(N-2)$$

so it is quite easy to check directly the correspondence between the equations stated in Lemma C.0.4 by calculations.

2.5.2 Proof of the optimality of Theorem 2.1.2 with respect to the Morse index

In this section we discuss the optimality of Theorems 2.1.3 with respect to the solutions' Morse index. First of all, we consider the the equation on the sphere $\mathbb{S}^N$ related to (2.2) through the weighted composition with the stereographic projection π as conformal diffeomorphism from $\mathbb{S}^N \setminus \{p\}$ onto $\mathbb{R}^N$: it is immediate to check that it is

$$-\Delta_{\mathbb{S}^N} v(y) + \frac{1}{4}N(N-2)v(y) = f(v(y)) \qquad y \in \mathbb{S}^N.$$

In his paper [24], Ding states the following result:

Lemma 2.5.3 *There exists a sequence $\{v_k\}$ of biradial solutions to the equation*

$$-\Delta_{\mathbb{S}^N} v + \frac{1}{4}N(N-2)v = |v|^{\frac{4}{N-2}}\, v \qquad v \in C^2(\mathbb{S}^N) \tag{2.16}$$

such that $\int_{\mathbb{S}^N} |v_k|^{\frac{2N}{N-2}}\, dV \to \infty$ as $k \to \infty$.

The choice of working in a space of biradial is motivated by the compact embedding of the space of H^1–biradial functions on the sphere into $L^{2N/(N-2)}$. In this way one can overcome the lack of compactness due to the presence of the critical exponent and prove the result as an application of the Ambrosetti-Rabinowitz symmetric Mountain Pass Theorem. We are interested in classifying the solutions according to their Morse index. We can state the following

Lemma 2.5.4 *Among the solutions $\{v_k\}$ in Lemma 2.5.3 there is also a constant one, which is unique and corresponds to the minimum of Sobolev quotient. All the other biradial solutions have biradial Morse index at least 2, and there is at least one non constant biradial solution having Morse index exactly 2.*

Proof. We can check directly there exists a unique constant solution:

$$\frac{1}{4}N(N-2)c = c^{\frac{N+2}{N-2}} \quad \Longrightarrow \quad c = \left(\frac{1}{4}N(N-2)\right)^{\frac{N-2}{4}}.$$

which corresponds to the Talenti functions on the sphere ([61]). We mean it is the image of the function $w(x) = \frac{\left(N(N-2)\right)^{\frac{N-2}{4}}}{\left(1+|x|^2\right)^{\frac{N-2}{2}}} = \mu(x)c$ through the diffeomorphism π^{-1} and

$$L_g c = \mu(x)^{-\frac{N+2}{N-2}} \Delta\left(\mu(x)c\right) .$$

Then it reaches the minimum of Sobolev quotient $\inf_{v \neq 0} \frac{\int_{\mathbb{S}^N} |\nabla v|^2}{\left(\int_{\mathbb{S}^N} |v|^{\frac{2N}{N-2}}\right)^{2/2^*}}$, and therefore it is quite simple to prove it is the mountain pass solution, i.e. its (plain, radial, biradial) Morse index is $m(c) = 1$. Now, thanks to Theorem 2.1.3, every other biradial solution having biradial Morse index at most 1 is constant, hence all the other solutions have biradial Morse index at least 2. Now, it is well known that Talenti's solutions are unique among positive solutions of equation (2.2) on $\mathbb{R}^N$, so we can assert that the only biradial positive solutions of (2.16) are constant. On the other hand, it can be proven for example using Morse Theory in ordered Banach spaces (see [9]), that the equation admits a biradial sign-changing solution having biradial Morse index at most 2. Hence there is a biradial solution of (2.16) with Morse index exactly 2 which is not constant.

Chapter 3

Morse theory for a fourth order elliptic equation with exponential nonlinearity

Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space whose associated norm will be denoted by $\| \cdot \|$. Given an interval Λ of $(0, \infty)$ and K such that

$$K \in C^2(\mathcal{H}, \mathbb{R}), \qquad \text{with} \quad \nabla K : \mathcal{H} \to \mathcal{H} \ \text{ compact}, \tag{3.1}$$

let us consider functionals which are of the form:

$$I(\lambda, u) = \frac{1}{2}\langle u, u \rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}. \tag{3.2}$$

We observe that the conditions (3.1)-(3.2) are not enough to ensure the (PS)-condition which is known to hold only for bounded sequences. (See, [46, Lemma 2.3]). Therefore the classical flow defined by the vector-field $-\nabla_u I(\lambda, u)$ is not suitable to derive a deformation lemma. However, by using a recent deformation result proven by [46, Proposition 1.1], we prove the following result.

Theorem 3.0.5 *Let $I(\lambda, \cdot)$ be a family of functionals satisfying* (3.1)-(3.2) *and fix $\bar{I}(\cdot) := I(\bar{\lambda}, \cdot)$ for some $\bar{\lambda} \in \Lambda$. Given $\varepsilon > 0$, let $\Lambda' := [\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon]$ be a (compact) subset of Λ and consider $a, b \in \mathbb{R}$ $(a < b)$, so that all the critical points $\bar{u}$ of $I(\lambda, \cdot)$ for $\lambda \in \Lambda'$ satisfy $\bar{I}(\bar{u}) \in (a, b)$. If*

$$\bar{I}_a^b := \left\{ u \in \mathcal{H} : a \leq \bar{I}(u) \leq b \right\},$$

and assuming that $\bar{I}$ has no critical points at the levels a, b, we have

$$\deg_{LS}(\nabla \bar{I}, \bar{I}_a^b, 0) = \chi(\bar{I}^b, \bar{I}^a). \tag{3.3}$$

where we denoted by $\deg_{LS}$ *the Leray-Schauder degree of the compact vector field $\nabla \bar{I}$.*

Now, let $\Omega \subset \mathbb{R}^4$ be a bounded smooth domain, and let us consider the following boundary value problem

$$\begin{cases} \Delta^2 u = \tau \dfrac{h(x)e^u}{\int_\Omega h(x)e^u dx} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

where h is a $C^{2,\alpha}$ positive function for $\alpha \in (0,1)$, and $\tau \in \mathbb{R}^+$. In dimension two and in the second order case the problem has been extensively studied by many authors since the importance of this equation is related to its physical meaning. In fact, it arises in mathematical physics as a mean field equation of Euler flows or for the description of self-dual condensates of some Chern-Simons-Higgs model. (See [44, 45, 46, 25, 47], for further details). Moreover semilinear equations involving exponential nonlinearity and fourth order elliptic operators appear naturally in conformal geometry and in particular in prescribing Q-curvature on 4-dimensional Riemannian manifolds. (See, for instance [25]).

We denotes by $\mathscr{H}$ the space of all functions of Sobolev class $H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the equivalent norm $\|u\|_{\mathscr{H}} := \|\Delta u\|_2$, then problem (3.4) has a variational structure and for each fixed constant τ, the (weak) solutions can be found as critical points of the functional

$$I_\tau(u) := \frac{1}{2}\|u\|_{\mathscr{H}}^2 - \tau \log\left(\frac{1}{|\Omega|}\int_\Omega h(x)e^u dx\right) \qquad \forall\, u \in \mathscr{H}, \tag{3.5}$$

where we denoted by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^4$. The key analytic fact which we need in order to compute the Leray-Schauder degree is a version for higher order operators of the Moser-Trudinger inequality. As a direct consequence of this inequality it follows that the functional (3.5) is coercive for $\tau < 64\pi^2$ and thus it is possible to find the solutions of (3.4), by using the direct method of the calculus of variations. If $\tau > 64\pi^2$, the functional I_τ is unbounded both from below and from above and hence the solutions have to be found by other methods, for instance as saddle points, by using some min-max scheme. A general feature of the problem is a compactness property if τ is not integer multiple of $64\pi^2$ as proven by Lin & Wei in [40].

If $\tau < 64\pi^2$ or $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, $k \in \mathbb{N}$, by elliptic regularity and by taking into account the compactness result proven in [41, Theorem 1.2], it is possible to define the Leray-Schauder degree for the boundary value problem (3.4), fixing a large ball $\mathscr{B}_R \subset \mathscr{H}$ centered at 0 and containing all the solutions. In fact, let us consider the family of compact operators $T_\tau : \mathscr{B}_R \to \mathscr{H}$, defined by

$$T_\tau(u) := \tau\, \Delta^{-2} \frac{he^u}{\int_\Omega he^u};$$

then the Leray-Schauder degree

$$d_\tau := \deg_{LS}(I - T_\tau, \mathscr{B}_R, 0)$$

is well-defined for $\tau \neq 64k\pi^2$, $k \in \mathbb{N}$.

Notation 1 *For any two integers* $k_1 \geq k_2$*, we use the notation* $\binom{k_1}{k_2}$ *to denote*

$$\binom{k_1}{k_2} := \begin{cases} \dfrac{k_1(k_1-1)\dots(k_1-k_2+1)}{k_2!} & \text{if } k_2 > 0 \\ 1 & \text{if } k_2 = 0, \end{cases}$$

and $\mathbf{k}$ *to denote the set* $\{1, \dots, k\}$.

By applying Theorem 3.3.1, together with a precise homological properties of the formal set of barycenters obtained in [25] we can reprove the following result.

Theorem 3.0.6 *([42]) For* $\tau \in (64k\pi^2, 64(k+1)\pi^2)$*, and* $k \in \mathbb{N}$*, the Leray-Schauder degree* d_τ *of* (3.4) *is given by*

$$d_\tau = \binom{k - \chi(\Omega)}{k},$$

where $\chi(\Omega)$ *denotes the Euler characteristic of the domain* Ω.

As a direct consequence if $\chi(\Omega) \leq 0$ then the problem (3.4) possesses a solution provided that $\tau \neq 64k\pi^2$, $k \in \mathbb{N}$.

In the rest of the section we briefly describe the method and the main ideas of the proof. As already observed for $\tau > 64\pi^2$, the functional I_τ is unbounded both from above and below due to the so-called *bubbling phenomenon* which often occurs in geometric problems. More precisely, for a given point $x \in \Omega$ and for $\lambda > 0$, we consider the following function

$$\varphi_{\lambda,x}(y) = \log\left(\frac{2\lambda}{1 + \lambda^2 dist(y,x)^2}\right)^4$$

where $dist(\cdot,\cdot)$ denotes the metric distance on Ω. For large λ, one has $e^{\varphi_{\lambda,x}} \rightharpoonup \delta_x$ (the Dirac mass at x) and moreover one can show that $I(\tau, \varphi_{\lambda,x}) \to -\infty$, for $\tau \in (64\pi^2, 128\pi^2)$ as $\lambda \to +\infty$. Similarly, if $\tau \in (64k\pi^2, 64(k+1)\pi^2)$ for $k > 1$, it is possible to construct a function φ of the above form (near at each x_i) with $e^{\varphi_{\lambda,\sigma}} \rightharpoonup \sigma := \sum_{i=1}^k t_i \delta_{x_i}$ and on which I_τ still attains large negative values. A crucial observation, as proven in [25], is that the constant in Moser-Trudinger inequality can be divided by the number of regions where e^u is supported. From this argument we see that one is led naturally to consider the family of elements $\sum_{i=1}^k t_i \delta_{x_i}$ with $(x_i)_i \subset \Omega$ and $\sum_{i=1}^k t_i = 1$, known in literature as the *formal set of barycenyters of* Ω *of order* k and introduced for the first time by Bahri & Coron in [6]. Using the functions $\varphi_{\lambda,x}$, it is indeed possible to map (non-trivially) this set into $\mathscr{H}$ in such a way that the functional I_τ on the image is close to $-\infty$. On the other hand, it is also possible to do the opposite, namely to map appropriate sublevels of I_τ into the formal set of barycenters. The composition of these two maps turns out to be homotopic to the identity on the formal set of barycenters (which is not contractible) and therefore they are both topologically non-trivial. We remark that our method is along the same line of a recent result proven by Malchiodi in [48], for a general Paneitz operator on compact four dimensional Riemannian manifolds without boundary.

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3.1 Preliminaries

The aim of this section is to recall some abstract results from degree theory for α-contractions, Sard's lemma for Fredholm maps and to recall some topological and homological properties of the so-called *formal set of barycenters*. Our main references will be [6, 23, 25, 34, 48].

The Sard-Smale theorem and Kuratowski non-compactness measure. We start this section with the classical Sard-Smale theorem stated in a form suitable for our purposes. See [23, pag.91].

Theorem 3.1.1 *(Sard-Smale) Let Γ be an open subset of a Hilbert space X. Suppose that $\mathscr{G} \in C^1(\Gamma, X)$ is proper when restricted to any closed bounded subset of Γ and that $\nabla\mathscr{G}(x) = \mathrm{Id} - K(x)$ where for every $x \in \Gamma$, $K(x)$ is a compact operator. Then the set of regular values of $\mathscr{G}$ is dense in X.*

We will apply this result to $X = \mathscr{H}$ and $\mathscr{G} = \nabla I_\tau$. Since both the map $\mathscr{G}$ and its Fréchet derivative are of the form $\mathrm{Id} - K$ where K is a compact operator, than the assumptions of theorem 3.1.1 are fulfilled.
Now let Γ be an open subset of X and let $\mathscr{F} : \Gamma \to X$ be a strict α-contraction, meaning that $\alpha(\mathscr{F}(B)) < k\alpha(B)$ for some fixed $k \in [0, 1)$, where $B \subset \Omega$ is a bounded subset and where α denotes the *Kuratowski measure of non-compactness*. If $y \notin (\mathrm{Id} - \mathscr{F})(\partial\Omega)$ and $(\mathrm{Id} - \mathscr{F})^{-1}(\{y\})$ is compact, we can define the *generalized degree* Deg, in such a way that if $\mathrm{Id} - \mathscr{F}$ is a compact vector field and Γ is a bounded subset it enjoys all the properties of the Leray-Schauder degree.

Formal set of barycenters. The aim of this paragraph is to recall some facts about the formal set of barycenters.
Following [6], we let Ω_k denote the family of formal sums

$$\Omega_k := \sum_{i=1}^{k} t_i \delta_{x_i}; \qquad t_i \geq 0, \ \sum_{i=1}^{k} t_i = 1; \quad x_i \in \Omega, \tag{3.6}$$

endowed with the weak topology of distributions. This is known in literature as the *formal set of barycenters* of Ω of order k. We stress the fact that this set is not the family of convex combinations of points in Ω.

In order to give a more topological insight on these spaces, some definitions are in order. We denote by J_k the k-fold join of Ω. We recall that a point $\underline{x} \in J_k$ is specified by:

(i) k real numbers $t_1, \ldots t_k$ satisfying $t_i \geq 0$, $\sum_{i=1}^{k} t_i = 1$, and

(ii) a point $x_i \in \Omega$ for each $i \in \mathbf{k}$ such that $t_i \neq 0$.

Such a point will be denoted by the symbol $\oplus_{i=1}^{k} t_i x_i$, where the elements x_i may be chosen arbitrarily or omitted whenever the corresponding t_i vanishes. Furthermore we will endow this space with the strongest topology such that the coordinate functions are continuous. Now, if Σ^k denotes the symmetric group over k elements, we assume that Σ^k acts on J_k by permuting factors, namely

$$\forall\, \sigma \in \Sigma^k: \quad \sigma(t_1x_1 \oplus \cdots \oplus t_kx_k) := (t_{\sigma(1)}x_{\sigma(1)} \oplus \cdots \oplus t_{\sigma(k)}x_{\sigma(k)}).$$

Thus, the k-th *symmetric join* of Ω, say SJ_k is defined as the quotient of J_k with respect to Σ^k.

Definition 3.1.2 *([34, Definition 5.1]) The k-th barycenter space Ω_k can be defined as the quotient of the symmetric join SJ_k under the equivalence relation $\sim$:*

$$t_1x_1 \oplus t_2x_1 \oplus \cdots \oplus t_kx_k \sim (t_1 + t_2)x_1 \oplus \cdots \oplus t_kx_k.$$

That is a point in Ω_k is a formal abelian sum with the topology that when $t_i = 0$ the entry $0x_i$ is discarded from the sum, and when x_i moves in coincidence with x_j, one identifies $t_ix_i + t_jx_i$ with $(t_i + t_j)x_i$. It is possible to show that we have the embeddings

$$\Omega \hookrightarrow \Omega_2 \hookrightarrow \ldots \hookrightarrow \Omega_{k-1} \hookrightarrow \Omega_k$$

and each factor is contractible in the next one. Let P be the projection on $\mathscr{H}$ (i.e. $P\varphi = \varphi - h$ with $\Delta^2 h = 0$ in Ω and $h = \varphi$ and $\Delta h = \Delta\varphi$ on $\partial\Omega$), $\Sigma \subset \mathscr{H}$ be the unit sphere and finally let

$$R : \mathscr{H}\backslash\{0\} \to \Sigma : u \mapsto R(u) := u/\|u\|_{\mathscr{H}}.$$

Let $g_k : SJ_k \to \Sigma$ be the map defined as: $g_k((x_1, \ldots, x_k), (t_1, \ldots, t_k)) := R\big(\sum_{i=1}^k t_i P\varphi_{\lambda,x_i}\big)$, where $\lambda > 0$ is fixed and φ_{λ,x_i} are given by

$$\varphi_{\lambda,x}(y) = \log\left(\frac{2\lambda}{1 + \lambda^2 dist(y,x)^2}\right)^4. \tag{3.7}$$

We observe that since two elements in SJ_k equivalent for the relation introduced in definition 3.1.2 have the same image through g_k, this implies that g_k is well-defined on the quotient. Denoting by Ω^k the k-fold product of copies of Ω and by Δ_k the *collision set* $\bigcup_{i,j=1}^k \Delta_{i,j}$, where

$$\Delta_{i,j} := \{(x_1, \ldots, x_k) \in \Omega^k |\;\; x_i = x_j, i \neq j, \;\text{ for }\; i, j \in \mathbf{k}\},$$

we define the *configuration space* $\widehat{\mathfrak{X}}_k := \mathfrak{X}_k \backslash \Delta_k$. Let us consider the fibration

$$\mu\colon \widehat{\mathfrak{X}}_k \to \widehat{\mathfrak{X}}_{k-1}, \quad \text{defined by} \quad \mu(x_1, \ldots, x_k) := (x_1, \ldots, x_{k-1}),$$

it is easy to observe that each fiber $\mu^{-1}((x_1, \ldots, x_{k-1})) = \Omega\backslash\{x_1, \ldots, x_{k-1}\}$ is homeomorphic to each other. Thus by using the classical Hopf theorem for fibrations (see, for instance Spanier [59], for further details), the Euler characteristic of $\widehat{\mathfrak{X}}_k$ can be computed through the fiber $\Omega\backslash\{x_1, \ldots, x_{k-1}\}$ and $\widehat{\mathfrak{X}}_{k-1}$. By an easy calculations it follows that

$$\chi(\widehat{\mathfrak{X}}_k) = \chi(\Omega)(\chi(\Omega) - 1) \ldots (\chi(\Omega) - k + 1). \tag{3.8}$$

Lemma 3.1.3 *(well-known) The set $\Omega_k \backslash \Omega_{k-1}$ is an open smooth manifold of dimension $5k-1$ for each $k \in \mathbb{N}$.*

Proof. The case $k = 1$ is trivial, since $\Omega_1 = \Omega$ and Ω is a four dimensional manifold being an open subset of $\mathbb{R}^4$. For $k \geq 2$ the join J_k is a smooth manifold. Since the action of the symmetric group on J_k is free of fixed points than the symmetric join is a smooth manifold. Moreover, since Ω_{k-1} is the boundary of Ω_k, than $\Omega_k \backslash \Omega_{k-1}$ is a smooth open manifold in which the elements in $\Omega_k \backslash \Omega_{k-1}$ are smoothly parameterized by $4k$ coordinates locating the points x_i and $k-1$ coordinates identifying the numbers t_i's. The conclusion immediately follows. □

Lemma 3.1.4 *(well-known) If Ω is not contractible, then for any $k \geq 1$ the set Ω_k is a is non contractible stratified set.*

Proof. (Sketch). It can be proved by arguing as follows. The case $k = 1$ is trivial. For $k \geq 2$ even if the set Ω_{k-2} is not a smooth manifold (actually it is a stratified set) however it is an ENR which implies that there exists a non trivial (mod 2) orientation class with respect to its boundary. However by using the Čech-cohomology, and by taking into account that it is isomorphic to the singular cohomology and over $\mathbb{Z}_2$ to the singular homology, the thesis follows by using the exactness of the pair once it is proven that

$$H_{5k-1}(\Omega_k; \mathbb{Z}_2) \simeq H_{5k-1}(\Omega_k \Omega_{k-1}; \mathbb{Z}_2).$$

(See, for instance, [25, Lemma 3.7], for further details). □

By using the same arguments as in [48, Proposition 5.1], it can be proven the following result.

Lemma 3.1.5 *Let η be positive and let $G \colon (0, +\infty) \to (0, +\infty)$ be the non-increasing function satisfying:*

$$G(t) = \frac{1}{t} \;\; \text{for} \;\; t \in (0, \eta] \qquad G(t) = \frac{1}{2\eta} \;\; \text{for} \;\; t > 2\eta.$$

If $d(x_i, x_j) := dist(x_i, x_j)$ and $F^* : \Omega_k \setminus \Omega_{k-1} \to \mathbb{R}$ as follows

$$F^*(\sum_{i=1}^{k} t_i \delta_{x_i}) := -\sum_{i \neq j} G(d(x_i, x_j)) - \sum_{i=1}^{k} \frac{1}{t_i(1-t_i)}. \tag{3.9}$$

Then we have

$$\sum_{i=1}^{5k-1} c_i = (-1)^{k-1} \frac{\chi(\widehat{\mathfrak{X}}_k)}{k!} \tag{3.10}$$

where c_i denotes the number of critical points of F^* of index i.

The following result will be crucial in order to compute the Leray-Schauder degree of our result.

Proposition 3.1.6 *For any natural number k we have:*

$$\chi(\Omega_k) = 1 - \binom{k-\chi(\Omega)}{k}.$$

Proof. The proof is given by induction over k. The case $k = 1$ is trivial being Ω_1 homeomorphic to Ω. For $k > 1$ we consider the pair (Ω_k, Ω_{k-1}) and we remark that the Euler characteristic is additive. Thus $\chi(\Omega_k) = \chi(\Omega_k, \Omega_{k-1}) + \chi(\Omega_{k-1})$.
Claim 1. The following formula holds for any natural number k

$$\chi(\Omega_k, \Omega_{k-1}) = (-1)^{k-1}\binom{\chi(\Omega)-k}{k}. \tag{3.11}$$

Once this is done the proposition easily follows. By Lemma 3.1.3 the space $\Omega_k \backslash \Omega_{k-1}$ is an open manifold of dimension $5k-1$ with boundary Ω_{k-1} and by the definition of F^*, Palais-Smale condition holds.
Observe that Ω_{k-1} is a deformation retract of the sublevel $F^*_{-L} := \{F^* \le -L\} \cup \Omega_k$ for L sufficiently large and positive (simply by taking the limit for $L \to +\infty$). Thus denoting by $\widehat{F}^*\colon \{F^* \ge -L\} \to \mathbb{R}$ a non-degenerate function C^2-close to the restriction of F^* to the subset $\{F^* \ge -L\}$, by excision of the sublevel $F^*_{-L} := \{F^* < -L\}$ and by the classical Poincaré-Hopf theorem it holds

$$\chi(\Omega_k, \Omega_{k-1}) = \sum_{i=1}^{5k-1} (-1)^i c_i.$$

The thesis follows by formula (3.10) and (3.8). □

Improved Moser-Trudinger inequality. Let $C_c^\infty(\Omega)$ be the set of all smooth functions with compact support in Ω, and let $\mathscr{H}$ be the completion with respect to the norm $\|u\|_{\mathscr{H}} := \|\Delta u\|_2$. The space $\mathscr{H}$ is a Hilbert space with respect to the scalar product $\langle u, v\rangle_{\mathscr{H}} := \int_\Omega \Delta u\, \Delta v dx$ for all $u, v \in \mathscr{H}$, and, by the local regularity theorem and by the Poincaré inequality, it follows that $\mathscr{H}$ agrees with the space of all functions on Ω of Sobolev class $H^2(\Omega) \cap H^1_0(\Omega)$. As immediate consequence of [41, Theorem 1.2] and the Schauder estimates, the following crucial compactness results hold.

Proposition 3.1.7 *([41, Theorem 1.2]) Let $h\colon \Omega \to \mathbb{R}$ be a positive $C^{2,\alpha}$ function and $\tau \neq 64k\pi^2$ for $k \in \mathbb{N}$. Then the solutions of* (3.4) *are bounded in $C^{4,\alpha}(\Omega)$ for any $\alpha \in (0,1)$.*

Proposition 3.1.8 *([40, Lemma 2.1]) Let u be a solution of* (3.4) *with $\tau \le c$, for some constant c. Then there exists a $\delta > 0$ such that*

$$u(x) \le c, \qquad \forall\, x \in \mathcal{U}_\delta(\partial\Omega),$$

where $\mathcal{U}_\delta(\partial\Omega) := \{x \in \Omega : \ d(x, \partial\Omega) \le \delta\}$.

We remark that proposition 3.1.8 excludes boundary bubbles.

Lemma 3.1.9 *There exists a constant C_Ω depending only on Ω such that for all $u \in \mathscr{H}$ one has:*

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar{u})}dx\right) \le C_\Omega + \frac{1}{128\pi^2}\|u\|^2_{\mathscr{H}} \tag{3.12}$$

where $\bar{u} := \frac{1}{|\Omega|}\int_\Omega u dx$ stands for the average of u over Ω.

Proof. In fact by [3, Theorem 1], there exists $C'_\Omega > 0$ depending only on Ω such that for all $u \in C^2_c(\Omega)$ it holds

$$\frac{1}{|\Omega|}\int_\Omega e^{\frac{32\pi^2(u-\bar{u})^2}{\|u\|^2_{\mathscr{H}}}}dx \le C'_\Omega, \qquad \forall\, u \in \mathscr{H}.$$

Since for every $a, b \in \mathbb{R}$, we have $(8\pi a - \frac{1}{8\pi}b)^2 \ge 0$ is $2ab \le \frac{1}{64\pi^2}b^2 + 64\pi^2 a^2$, by setting $a := (u-\bar{u})/\|u\|_{\mathscr{H}}$ and $b = \|u\|_{\mathscr{H}}$, and exponentiating, we have

$$\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar{u})}dx \le e^{\frac{1}{128\pi^2}\|u\|^2_{\mathscr{H}}}\frac{1}{|\Omega|}\int_\Omega e^{\frac{32\pi^2(u-\bar{u})^2}{\|u\|^2_{\mathscr{H}}}}dx \;\le\; e^{\frac{1}{128\pi^2}\|u\|^2_{\mathscr{H}}}C'_\Omega, \qquad \forall\, u \in \mathscr{H}.$$

Taking the logarithm of this last inequality the conclusion follows by setting $C_\Omega := \log C_\Omega$. □

In order to study how the function e^u is *spread* over Ω we need some quantitative results. In fact, we will show that if e^u has integral bounded from below on $(l+1)$-regions, the constant $\frac{1}{128\pi^2}$, can be basically divided by $(l+1)$. The proof of the proposition 3.1.10, is up to minor modifications, an adaptation of the arguments given in [25, Lemma 2.2]; we will reproduce it for the sake of completeness.

Proposition 3.1.10 *For any fixed integer l, let $\Omega_1, \ldots, \Omega_{l+1}$ be subsets of Ω satisfying $dist(\Omega_i, \Omega_j) \ge \delta_0$, for $i \ne j$, when δ_0 be positive real number, and let $\gamma_0 \in (0, \frac{1}{l+1})$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $\widetilde{C} := \widetilde{C}(l, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar{u})}dx\right) \le \frac{1}{128(l+1)\pi^2 - \tilde{\varepsilon}}\|u\|^2_{\mathscr{H}} + \widetilde{C},$$

for all functions $u \in \mathscr{H}$ satisfying

$$\frac{\int_{\Omega_i} e^u\,dx}{\int_\Omega e^u\,dx} \ge \gamma_0 \qquad \forall\, i \in \mathbf{l+1}. \tag{3.13}$$

Proof. We consider $(l+1)$ smooth cut-off functions $g_1, \ldots, g_{l+1}$, satisfying the following properties:

$$\begin{cases} g_i(x) \in [0,1] & \text{for every } \; x \in \Omega; \\ g_i(x) = 1 & \text{for every } \; x \in \Omega_i, i \in \mathbf{l+1}; \\ g_i(x) = 0 & \text{if } \; dist(x, \Omega_i) \ge \frac{\delta_0}{4}; \\ \|g_i\|_{C^4(\Omega)} \le C_{\delta_0}, \end{cases} \tag{3.14}$$

where C_{δ_0} depends only on δ_0. By interpolation, (see, for instance, [43, Prop. 4.1]), for any $\varepsilon > 0$, there exists C_{ε,δ_0}, such that for any $v \in \mathscr{H}$, and for any $i \in \mathbf{l+1}$ there holds

$$\|g_i v\|^2_{\mathscr{H}} := \int_\Omega |\Delta(g_i v)|^2 dx \leq \int_\Omega g_i^2 |\Delta v|^2 dx + \varepsilon \int_\Omega |\Delta v|^2 dx + C_{\varepsilon,\delta_0} \int_\Omega v^2 dx. \tag{3.15}$$

Let $u - \bar{u} = u_1 + u_2$ with $u_1 \in L^\infty(\Omega)$, then from our assumptions we deduce

$$\int_{\Omega_i} e^{u_2} dx \geq e^{-\|u_1\|_{L^\infty(\Omega)}} \int_{\Omega_i} e^{(u-\bar{u})} dx \geq e^{-\|u_1\|_{L^\infty(\Omega)}} \gamma_0 \int_\Omega e^{(u-\bar{u})} dx, \quad i \in \mathbf{l+1}.$$

By invoking inequality (3.12) in lemma 3.1.9, together with the last two inequalities, it follows that

$$\begin{aligned} \log\left(\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar{u})}dx\right) &\leq \log\frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + \log\left(\frac{1}{|\Omega|}\int_\Omega e^{g_i u_2}dx\right) + C_\Omega \\ &\leq \log\frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + \frac{1}{128\pi^2}\|g_i u\|^2_{\mathscr{H}} + C_\Omega. \end{aligned} \tag{3.16}$$

We choose i such that $\int_\Omega |\Delta(g_i u_2)|^2 dx \leq \int_\Omega |\Delta(g_j u_2)|^2 dx$, for each $j \in \mathbf{l+1}$. Since the functions g_j have disjoint supports, the last formula and (3.15), implies that

$$\begin{aligned} \log\left(\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar{u})}dx\right) &\leq \log\frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Omega)} + C_\Omega + \left(\frac{1}{128(l+1)\pi^2} + \varepsilon\right)\|u_2\|^2_{\mathscr{H}} + \\ &+ C_{\varepsilon,\delta_0}\int_\Omega v^2 dx. \end{aligned}$$

Now let $\lambda_{\varepsilon,\delta_0}$ to be an eigenvalue of $-\Delta^2$ such that $\frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} < \varepsilon$, and we set

$$u_1 := P_{V_{\varepsilon,\delta_0}}(u-\bar{u}); \qquad u_2 := P_{V^\perp_{\varepsilon,\delta_0}}(u-\bar{u}).$$

Here V_{ε,δ_0} is the direct sum of the eigenspaces of $-\Delta^2$ with Navier boundary conditions and having eigenvalues less or equal than $\lambda_{\varepsilon,\delta_0}$ and $P_{V_{\varepsilon,\delta_0}}$, $P_{V^\perp_{\varepsilon,\delta_0}}$ the orthogonal projections onto V_{ε,δ_0} and $V^\perp_{\varepsilon,\delta_0}$, respectively. Since V_{ε,δ_0} is finite dimensional, the L^2 norm and L^∞ norm of $u-\bar{u}$ on V_{ε,δ_0} are equivalent; then, by using the Poincaré-Wirtinger inequality (cfr. [12, pag. 308]), there holds:

$$\|u_1\|^2_{L^\infty(\Omega)} \leq \hat{C}_{\varepsilon,\delta_0}\|u_1\|^2_{L^2(\Omega)} \leq \hat{C}_{\varepsilon,\delta_0}\|u_1\|^2_{(H^2\cap H^1_0)(\Omega)} \leq \hat{C}'_{\varepsilon,\delta_0}\|u_1\|^2_{\mathscr{H}},$$

where $C'_{\varepsilon,\delta_0}$ is another constant depending only on ε and δ_0. Furthermore

$$C_{\varepsilon,\delta_0}\int_\Omega u_2^2 dx \leq \frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}}\|u_2\|^2_{H^2(\Omega)\cap H^1_0(\Omega)} \leq \varepsilon\|u_2\|^2_{H^2(\Omega)\cap H^1_0(\Omega)} \leq \varepsilon C'_\Omega\|u_2\|^2_{\mathscr{H}},$$

where C'_Ω is a constant depending only on Ω. Hence the last formulas imply

$$\begin{aligned}\log\left(\frac{1}{|\Omega|}\int_\Omega e^{(u-\bar u)}dx\right) &\leq \log\frac{1}{\gamma_0}+C'_{\varepsilon,\delta_0}\|u_1\|_{\mathscr{H}}+C_\Omega+\left(\frac{1}{128(l+1)\pi^2}+\varepsilon\right)\|u_2\|^2_{\mathscr{H}}+\varepsilon C'_\Omega\|u_2\|_{\mathscr{H}}\\ &\leq \log\frac{1}{\gamma_0}+C_\Omega+\left(\frac{1}{128(l+1)\pi^2}+3\varepsilon\right)\|u\|^2_{\mathscr{H}}+\overline{C}_{\varepsilon,\delta_0}\end{aligned}$$

where $\overline{C}_{\varepsilon,\delta_0}$ depends only on ε and δ_0 (and l which is fixed). This concludes the proof. □

In the next Lemma we show a criterion which implies the situation described in the first condition in (3.13).

Lemma 3.1.11 *([25, Lemma 2.3]) Let l be a given positive integer, and suppose that ε and r are positive numbers. Suppose that for a non-negative function $f\in L^1(\Omega)$ with $\|f\|_1=1$ there holds*

$$\int_{\cup_{i=1}^l B_r(p_i)} f dx < 1-\varepsilon, \qquad \forall\ l-tuple\ \ p_1,\dots,p_l\in\Omega.$$

Then there exists $\bar\varepsilon>0$ and $\bar r>0$, depending on ε, r, l and Ω (but not on f), and $l+1$ points $\bar p_1,\dots,\bar p_{l+1}\in\Omega$ satisfying

$$\int_{B_{\bar r}(\bar p_1)} f dx\geq\bar\varepsilon,\dots,\int_{B_{\bar r}(\bar p_{l+1})} f dx\geq\bar\varepsilon; \qquad B_{2\bar r}(\bar p_i)\cap B_{2\bar r}(\bar p_j)=\emptyset\ \ for\ i\neq j.$$

Lemma 3.1.12 *If $\tau\in(64k\pi^2, 64(k+1)\pi^2)$ with $k\geq 1$, the following property holds. For any $\varepsilon>0$ and any $r>0$ there exists a large positive $L=L(\varepsilon,r)$ such that for every $u\in\mathscr{H}$ with $\frac{1}{|\Omega|}\int_\Omega e^u dx=1$ and $I_\tau(u)\leq -L$ there exist k points $p_{1,u},\dots,p_{k,u}\in\Omega$ such that*

$$\frac{1}{|\Omega|}\int_{\Omega\setminus\cup_{i=1}^k B_r(p_{i,u})} e^u dx<\varepsilon.$$

Proof. To prove the thesis, we argue by contradiction. Thus, there exist $\varepsilon, r>0$ and a sequence $(u_n)_n\in\mathscr{H}$ with $1/|\Omega|\int_\Omega e^{u_n}dx=1$ and $I_\tau(u_n)\to-\infty$ such that for every k-tuple $p_1,\dots,p_k$ in Ω we have $1/|\Omega|\int_{\Omega\setminus\cup_{i=1}^k B_r(p_{i,u})} e^u dx\geq\varepsilon$. Now applying Lemma 3.1.11 with $l=k$, $f=e^{u_n}$ and finally with $\delta_0=2\bar r$, $\Omega_j=B_{\bar r}(\bar p_j)$ and $\bar\gamma_0=\bar\varepsilon$ for $j\in\mathbf{k}$ and where the symbols $\delta_0,\Omega_j,\bar\gamma_0$ were defined in Lemma 3.1.9 and $\bar r, B_{\bar r}(\bar p_j),\bar\varepsilon,(\bar p_j)_j$ were defined in Lemma 3.1.11. By this it follows that, for any given $\tilde\varepsilon>0$ there exists a constant $C>0$ depending on $\varepsilon,\tilde\varepsilon$ and on r such that

$$I_\tau(u_n)\geq\frac{1}{2}\|u_n\|^2_{\mathscr{H}}-C\tau-\frac{\tau}{64(k+1)\pi^2-\tilde\varepsilon}\frac{1}{2}\|u\|^2_{\mathscr{H}}, \tag{3.17}$$

where the constant C does not depends on n. Now since $\tau<64(k+1)\pi^2$, we can choose $\tilde\varepsilon>0$ small enough that the number $1-\frac{\tau}{64(k+1)\pi^2-\tilde\varepsilon}:=\delta'>0$. Therefore the inequality (3.17) reduces to

$$I_\tau(u_n)\geq\frac{\delta'}{2}\|u_n\|^2_{\mathscr{H}}-C\tau\geq -K,$$

where K is a positive constant independent of n. This violates our contradiction assumption, and conclude the proof. □

Given a non-negative L^1 function f on Ω, we define the distance of f from Ω_k as

$$dist(f, \Omega_k) := \sup\left\{\left|\int_\Omega f\psi dx - \langle\sigma, \psi\rangle\right| : \sigma \in \Omega_k, \quad \text{and} \quad \|\psi\|_{C^1(\Omega)} \leq 1\right\},$$

where we denoted by $\langle\cdot,\cdot\rangle$ the usual duality product. We also define the set

$$\mathscr{D}_{\varepsilon,k} = \{f \in L^1(\Omega) \colon f \geq 0, \ \|f\|_{L^1(\Omega)} = 1, \ d(f, \Omega_k) < \varepsilon\}.$$

With this notation in mind, by Lemma 3.1.12 we deduce the following.

Lemma 3.1.13 *Suppose $\tau \in (64k\pi^2, 64(k+1)\pi^2)$ with $k \geq 1$. Then for any $\varepsilon > 0$ there exists a large positive $L = L(\varepsilon)$ such that for all $u \in \mathscr{H}$ with $I(\tau, u) \leq -L$ and $1/|\Omega| \int_\Omega e^u dx = 1$, we have $dist(e^u, \Omega_k) < \varepsilon$.*

We remark that as a direct consequence of [41, Theorem 1.2,(ii)], the blow-up points $p_{j,u}$ at which the local-mass is concentrated cannot lie on the boundary of Ω.

3.2 A topological argument

The aim of this section is to show that an image of the Ω_k can be mapped into very negative sublevels of the Euler functional I_τ. Moreover if Ω is non contractible then this map is non-trivial in the sense that it carries some homology. The goal of this section is to sketch the proof of the following result which is given along the lines of [47].

Proposition 3.2.1 *For any $k \in \mathbb{N}$ and $\tau \in (64k\pi^2, 64(k+1)\pi^2)$, there exists $L > 0$ such that the sublevel $\mathscr{H}^{-L}$ has the same homology as Ω_k.*

The proof of the Proposition 3.2.1 will follows from the homotopy invariance of the homology groups once the following facts will be established.

Mapping Ω_k into very low sublevels of I_τ. To do so, for $\eta > 0$ small enough, consider the smooth non-decreasing cut-off function $\chi_\eta \colon \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

$$\begin{cases} \chi_\eta(t) = t, & \text{for } t \in [0, \eta]; \\ \chi_\eta(t) = 2\eta, & \text{for } t \geq 2\eta; \\ \chi_\eta(t) \in [\eta, 2\eta], & \text{for } t \in [\eta, 2\eta]. \end{cases} \tag{3.18}$$

Then given $\sigma \in \Omega_k$, $\lambda > 0$ and $\delta > 0$ as in proposition 3.1.8, we can define a smooth function $\tilde{\varphi}_{\lambda,\sigma} : \Omega \to \mathbb{R}$ such that in $\Omega \backslash \Omega_\delta \cup \Omega_{\delta/2}$ it is given by:

$$\tilde{\varphi}_{\lambda,\sigma}(y) := \begin{cases} \varphi_{\lambda,\sigma}(y) & \text{for } y \in \Omega \backslash \Omega_\delta \\ 0 & \text{for } y \in \Omega_{\delta/2}, \end{cases} \tag{3.19}$$

for

$$\varphi_{\lambda,\sigma}(y) := \log \sum_{i=1}^{k} t_i \left(\frac{2\lambda}{1+\lambda^2 \chi_\eta^2(d_i(y))} \right)^4$$

with $d_i(y) := d(y, x_i)$.

Proposition 3.2.2 *Let $\tilde{\varphi}_{\lambda,\sigma}$ be defined above. Then as $\lambda \to +\infty$ the following properties hold*

(i) $e^{\varphi_{\lambda,\sigma}} \rightharpoonup \sigma$ weakly in the sense of distributions;

(ii) $I_\tau(\tilde{\varphi}_{\lambda,\sigma}) \to -\infty$ in $\mathscr{H}$ uniformly with respect to $\sigma \in \Omega_k$.

Proof. To prove (i) we first consider the function

$$\bar{\varphi}_{\lambda,x_i}(y) := \left(\frac{2\lambda}{1+\lambda^2 \chi_\eta^2(d_i(y))} \right)^4, \qquad \forall\, y \in \Omega,$$

where x is a fixed point in Ω. It is easy to verify that $\bar{\varphi}_{\lambda,x_i}(y) \to \delta_{x_i}$ for $\lambda \to +\infty$. Then (i) follows from the explicit expression of $\varphi_{\lambda,\sigma}$.
In order to prove (ii), we evaluate separately each term of I_τ, and claim that the following estimates hold

$$\log\left(\frac{1}{|\Omega|} \int_\Omega e^{\tilde{\varphi}_{\lambda,\sigma}} dx \right) = O(1) \qquad \text{as} \quad \lambda \to +\infty. \tag{3.20}$$

$$\frac{1}{2} \|\tilde{\varphi}_{\lambda,\sigma}\|_{\mathscr{H}}^2 \leq (128k\pi^2 + o_\epsilon(1)) \log \lambda + C_\epsilon) \qquad \text{(uniformly in } \sigma \in \Sigma_k), \tag{3.21}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ and where C_ϵ is a constant independent $(x_i)_i$.
The proof of (3.20) is easy and it follows by integrating over Ω. The proof of (3.21) is much more involved and it follows by arguing as in [25, Lemma 4.2]. □

Mapping very low sublevels of I_τ into Ω_k and an homotopy inverse. The main idea is to construct a non-trivial continuous map $\psi \colon \mathscr{H} \to \Omega_k$ from the sublevels of the Euler functional into Ω_k such that the composition $\psi \circ \phi_\lambda$ is homotopic to identity on Ω_k.

Proposition 3.2.3 *Suppose that $\tau \in (64k\pi^2, 64(k+1)\pi^2)$ with $k \geq 1$. Then there exists $L > 0$ and a continuous projection $\psi \colon \mathscr{H}^{-L} \to \Omega_k$ with the following properties.*

(i) If $(u_n)_n \subset \mathscr{H}^{-L}$ is such that $e^{u_n} \rightharpoonup \sigma$, for some $\sigma \in \Omega_k$, then $\psi(u_n) \to \sigma$;

(ii) if $\varphi_{\lambda,\sigma}$ is as in (3.19), then for any λ sufficiently large the map $\sigma \mapsto \psi(\overline{\varphi}_{\lambda,\sigma})$ is homotopic to the identity on Ω_k.

Proof. First of all we observe that item (i) follows directly from item (ii) and Proposition 3.2.2. The non-trivial part is the construction of the global continuous projection map ψ which is a left homotopy inverse has proven in [25, Section 3]. □

We close this section by observing that, up to minor modifications, the above defined map ψ is also a right inverse homotopy as proven in [48, Appendix]. Thus summing up we conclude that

Corollary 3.2.4 *Given L sufficiently large the topological spaces $\mathscr{H}^{-L}$ and Ω_k are equivalent, up to homotopy.*

3.3 A Poincaré-Hopf Theorem without (PS)

The aim of this section is to prove an analogous of the Poincaré-Hopf theorem for a special class of functionals. To do so, let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space whose associated norm will be denoted by $\|\cdot\|$. Given an interval Λ of $(0,\infty)$ and a map K such that

$$K \in C^2(\mathcal{H}, \mathbb{R}), \qquad \text{with} \quad \nabla K : \mathcal{H} \to \mathcal{H} \ \text{ compact}, \tag{3.22}$$

let us consider the functionals which are of the form:

$$I(\lambda, u) = \frac{1}{2}\langle u, u\rangle - \lambda K(u), \quad (\lambda, u) \in \Lambda \times \mathcal{H}. \tag{3.23}$$

It is well-known (see, for instance, [46, Lemma 2.3]) that the conditions (3.22)-(3.23) could not be enough to ensure the (PS)-condition which is known to hold only for *bounded sequences*. Now by using the deformation Lemma proven in [46, Proposition 1.1], we are in position to derive the following result.

Theorem 3.3.1 (A Poincaré-Hopf theorem) *Let $I(\lambda,\cdot)$ be a family of functionals satisfying* (3.22)-(3.23) *and fix $\bar I(\cdot) := I(\bar\lambda,\cdot)$ for some $\bar\lambda \in \Lambda$. Given $\varepsilon > 0$, let $\Lambda' := [\bar\lambda - \varepsilon, \bar\lambda + \varepsilon]$ be a (compact) subset of Λ and consider $a, b \in \mathbb{R}$ ($a < b$), so that all the critical points $\bar u$ of $I(\lambda,\cdot)$ for $\lambda \in \Lambda'$ satisfy $\bar I(\bar u) \in (a,b)$. Assuming that $\bar I$ has no critical points at the levels a, b, we have*

$$\deg_{LS}(\nabla \bar I, \bar I^b_a, 0) = \chi(\bar I^b, \bar I^a). \tag{3.24}$$

The proof of this result will be given into two main steps. In the first step we will assume that all the critical points are non-degenerate; in the second step we will remove this assumption.

Proof. *First step: non-degenerate case.* We let $\mathscr{K}$ denote the set of critical points of $\bar I$ which is compact by hypothesis. By compactness and non-degeneracy assumptions, $\bar I$ has only finitely-many critical levels each of whose consists only of finitely-many critical points. Let R be so large that all the critical points of I_λ for $\lambda \in \Lambda'$ are in $\mathscr{B}_{\frac{R}{2}}(0)$. Then we can define the cut-off function $\theta : \mathcal{H} \to [0,1]$ satisfying

$$\theta(u) = 1 \text{ for } u \in \mathcal{B}_R(0); \qquad \theta(u) = 0 \text{ for } u \in \mathcal{H}\backslash\mathcal{B}_{2R}(0).$$

Following Lucia in [46], let $Z \in C^1(\mathcal{H},\mathcal{H})$ be defined by:

$$Z(u) := -[|\nabla K(u)|\nabla \bar I(u) + |\nabla \bar I(u)|\nabla K(u)],$$

and choose $\omega_\varepsilon \in C^\infty(\mathbb{R})$ such that

$$0 \le \omega_\varepsilon \le 1, \qquad \omega_\varepsilon(\zeta) = 0 \text{ for all } \ \zeta \le \varepsilon, \qquad \omega_\varepsilon(\zeta) = 1 \text{ for all } \ \zeta \ge 2\varepsilon.$$

Finally we can define

$$W(u) := -\omega_\varepsilon \left(\frac{|\nabla \bar{I}(u)|}{|\nabla K(u)|} \right) \nabla \bar{I}(u) + Z(u),$$

where $\omega_\varepsilon \big(|\nabla \bar{I}(u)|/|\nabla K(u)| \big)$ is understood to be equal 1 when $\nabla K(u) = 0$. Given the vector field:

$$\widetilde{W}(u) := -\theta(u) \nabla \bar{I}(u) + (1 - \theta(u)) W(u),$$

we observe that it decreases $\bar{I}$ in the complement of $\mathscr{K}$. We consider the local flow $\eta = \eta(t, u_0)$ defined by the Cauchy problem:

$$\frac{du}{dt} = \widetilde{W}(u), \qquad u(0) = u_0.$$

Claim 1.If $\bar{I}$ has no critical levels inside some interval $[\tilde{a}, \tilde{b}]$, then the sub-level $\bar{I}^{\tilde{a}}$ is a deformation retract of $\bar{I}^{\tilde{b}}$.
To prove this, we argue as follows. Given $u_0 \in \bar{I}^{\tilde{b}}$, we can prove that

$$\bar{I}(\eta(t, u_0)) \le -c^2 t + \bar{I}(u_0).^{1} \tag{3.25}$$

Thus there exists a t such that $\bar{I}(\eta(t, u_0) \le \tilde{a}$. Then we define:

$$t_{\tilde{a}}(u_0) := \begin{cases} \inf\{t \ge 0 : \bar{I}(\eta(t, u_0)) \in \bar{I}^{\tilde{a}}\} & \text{if} \quad \bar{I}(u_0) > \tilde{a} \\ 0 & \text{if} \quad \bar{I}(u_0) \le \tilde{a}. \end{cases}$$

Now the map

$$\widetilde{\eta} : [0, 1] \times \mathcal{H} \to \mathcal{H}, \qquad (s, u_0) \mapsto \eta(s t_{\tilde{a}}(u_0), u_0),$$

is a deformation retraction, as required.
Now let $\bar{c}_i$ be the number of critical points of $\bar{I}$ of index i. By classical Morse-theoretical arguments as in [15, Theorem 3.2, 3.3, pagg. 100-103], by excising $\{\bar{I} < \tilde{a}\}$, it follows that

$$\deg_{LS}(\nabla \bar{I}, \bar{I}^b_a, 0) = \sum_i (-1)^i \bar{c}_i = \chi(\bar{I}^b, \bar{I}^a).$$

This concludes the proof in the non degenerate case.
Second step: degenerate case. We reduce ourselves to the non-degenerate case. To do so, fix a small $\delta > 0$ so that $dist(\mathscr{K}, \bar{I}^b_a) > 4\delta$, and define the set $\mathscr{K}_\delta = \{u \in \mathcal{H} : dist(u, \mathscr{K}) < \delta\}$. We next choose a smooth cut-off function p such that

$$p(u) = 1 \ \text{ for every } u \in \mathscr{K}_\delta; \qquad p(u) = 0 \ \text{ for every } u \in \mathcal{H} \backslash \mathscr{K}_{2\delta}.$$

We can also choose p such that $0 \le p(u) \le 1$ for all $u \in \mathcal{H}$ and having uniformly bounded derivative in $\mathscr{K}_{2\delta}$. Now let $\mathscr{G} := \nabla \bar{I}|_{\mathscr{K}_\delta} : \mathscr{K}_\delta \to \mathcal{H}$. Since the map $\mathscr{G}$ is a compact

[1]The proof of this inequality is the most involved part of this claim and it can be proven up to minor modifications reapeating word by word the arguments given in [46, pagg. 121-122].

perturbation of the identity, by applying the Sard-Smale theorem (see theorem 3.1.1), the set of regular values of $\mathscr{G}$ is dense in $\mathcal{H}$. This implies that we can find an arbitrarily small u_0 such that $\nabla\mathscr{G}(p)$ is non-degenerate for each $p \in \mathscr{G}^{-1}(u_0)$ which is equivalent to say that $\nabla^2 \bar{I}$ is non-degenerate on the set

$$\Gamma(u_0) := \{u \in \mathcal{H} : \nabla I(u) = u_0\} \cap \mathscr{K}_\delta.$$

Moreover we observe that $\|\nabla \bar{I}\| \geq \gamma_\delta > 0$ on $\mathscr{K}_{2\delta} \backslash \mathscr{K}_\delta$ for some constant γ_δ. Now let us consider the function

$$\widetilde{I}(u) := \bar{I}(u) + p(u)\langle u_0, u\rangle.$$

It can be shown that the following facts hold:

(i) $\widetilde{I}$ coincides with $\bar{I}$ in $\mathcal{H} \backslash \mathscr{K}_{2\delta}$;

(ii) $\widetilde{I}$ has the same critical points as $I(\tau, \cdot)$ in $\mathscr{H} \backslash \mathscr{K}_\delta$;

(iii) $\widetilde{I}$ is non-degenerate in $\bar{I}^b_a$.

Item (i) is trivial. To prove (ii) we observe that since $\widetilde{I}$ and $\bar{I}$ coincides out of $\mathscr{K}_{2\delta}$, it is enough to prove the claim for $u \in \overline{\mathscr{K}_{2\delta}} \backslash \mathscr{K}_\delta$. By differentiating, we have

$$\langle \nabla \widetilde{I}(u), v\rangle = \langle \nabla \bar{I}(u) + \nabla p(u)\langle u, u_0\rangle + p(u)u_0, v\rangle, \qquad \forall v \in \mathcal{H}.$$

Thus, by recalling that $u \in \overline{\mathscr{K}_{2\delta}} \backslash \mathscr{K}_\delta$, it follows that

$$\|\nabla \widetilde{I}(u)\| \geq \|\nabla \bar{I}(u)\| - |\langle u, u_0| \|\nabla p(u)\| - p(u)\|u_0\| \geq \gamma_\delta - \|u_0\|(\|\nabla p(u)\|\|u\| + 1) > 0,$$

where the last inequality follows since p has uniformly bounded derivatives and u_0 can be chosen arbitrarily small. To prove (iii) we argue as follows. Since all the critical points of $\widetilde{I}$ are in $\mathscr{K}_\delta$, let us assume by contradiction that $\widetilde{I}$ is degenerate at some critical point $\bar{u}$. Now since $\bar{u} \notin \mathscr{K}$, it follows that $\bar{u} \in \mathscr{K}_\delta \backslash \mathscr{K}$. Moreover $\nabla \widetilde{I}(\bar{u}) = 0$ is equivalent to say that $\nabla \bar{I}(\bar{u}) = u_0$ and therefore $\bar{u} \in \Gamma(u_0)$. But this is contradict the fact that $\nabla^2 \widetilde{I}(p)$ is non-degenerate on $p \in \Gamma(u_0)$.

Now, for $\|u_0\|$ sufficiently small the map $\nabla I - Id$ is a strict α-contraction. (See Section 3.1) and since $(\nabla \widetilde{I})^{-1}(\{u_0\}) = \mathscr{K}$, the generalized degree $\mathrm{Deg}(\nabla \widetilde{I}, \bar{I}^b_a, u_0)$ is well-defined; moreover it coincides with $\mathrm{Deg}(\nabla \widetilde{I}, \bar{I}^b_a, 0)$ since it is locally constant. With the above choice for R and by using the excision property and the homotopy invariance of the generalized degree, (see, for instance, [23] for further details), we have

$$\deg_{LS}(\nabla \widetilde{I}, \mathscr{B}_R, 0) = \mathrm{Deg}(\nabla \widetilde{I}, \mathscr{B}_R, 0).$$

Now choosing a possibly larger R in such a way $\mathscr{K}_{2\delta} \subset \mathscr{B}_{R/2}$, the conclusion readily follows by the first step, simply by replacing $\bar{I}$ with $\widetilde{I}$. □

Corollary 3.3.2 *If $\tau \in (64k\pi^2, 64(k+1)\pi^2)$ for some $k \in \mathbb{N}$ and if b is sufficiently large positive, the sub-level $\mathscr{H}^b$ is a deformation retract of $\mathscr{H}$ and hence it has the homology of a point.*

Proof. This result follows, by using the deformation constructed in the proof of the Poincaré-Hopf theorem. See, for instance [48, Corollary 2.8]. □

Setting

$$J(u) := \log\left(\frac{1}{|\Omega|}\int_\Omega h(x)e^u dx\right) \tag{3.26}$$

the functional (3.5) can be put in the following form: $I_\tau(u) = \frac{1}{2}\|u\|^2_{\mathscr{H}} - \tau J(u)$.

Proof of Theorem 3.0.6. In order to prove theorem 3.0.6, it is enough to apply theorem 3.3.1 to the functional (3.5) for $\lambda = \tau$, $\Lambda = (64k\pi^2, 64(k+1)\pi^2)$ for $k \geq 1$, $\mathcal{H} = \mathscr{H}$ and finally $K(u) = J(u)$ where J was given in (3.26). The only thing it should be noted, is that all the critical points $\bar{u}$ of I_τ for $\tau \in [\bar{\tau} - \varepsilon, \bar{\tau} + \varepsilon] \subset (64k\pi^2, 64(k+1)\pi^2)$ satisfy $\bar{I}(\bar{u}) \in (a, b)$. This is a consequence of proposition 3.1.7 and of the boundedness of J which is consequence of the Moser-Trudinger inequality. Now the conclusion follows choosing $a = -L$ as in proposition 3.2.1 and b as in corollary 3.3.2. In fact by using theorem 3.3.1, we have that

$$d_\tau = \chi(\bar{I}^b, \bar{I}^a) = \chi(\bar{I}^b) - \chi(\bar{I}^a) = \chi(\mathscr{H}) - \chi(\Omega_k) = 1 - \chi(\Omega_k).$$

The conclusion follows by invoking proposition 3.1.6. □

Remark 3.3.3 *We observe that the Leray-Schauder degree in the Sobolev space $\mathscr{H}$ coincides with the degree in every Hölder space $C^{2,\alpha}(\Omega)$, $\alpha \in (0,1)$. See for instance [39, Part I, Appendix B, Theorem B.1-B.2].*

Appendix A

Friedrichs' extension

In this section we are going to consider some remarkable facts about linear operators in Hilbert spaces. Operators will be usually denoted by T, they will be defined on their own domain $\mathcal{D}(T)$, which will be assumed a proper dense subspace of a general Hilbert space $\mathcal{H}$.

We preface the following definitions:

Definition A.0.4 • *The operator* $T : \mathcal{D}(T) \to \mathcal{H}$ *is called to be* symmetric *if*

$$\langle Tu, v\rangle_{\mathcal{H}} = \langle u, Tv\rangle_{\mathcal{H}} \qquad \forall u, v \in \mathcal{D}(T).$$

- *The* adjoint *of an operator* $T : \mathcal{D}(T) \to \mathcal{H}$ *is an operator* T^* *defined on*

$$\mathcal{D}(T^*) : \; = \; \{u \in \mathcal{H} \textit{ s.t. application } \mathcal{D}(T) \ni v \mapsto \langle u, Tv\rangle_{\mathcal{H}} \textit{ can be extended continuosly on whole } \mathcal{H}\}.$$

  Then, via Riesz's Theorem there exists $f_u \in \mathcal{H}$ *such that* $\langle f_u, v\rangle_{\mathcal{H}} = \langle u, Tv\rangle_{\mathcal{H}}$ *for all* $v \in \mathcal{D}(T)$ *and for all* $u \in \mathcal{D}(T^*)$*.* T^* *will be defined as the application* $u \mapsto f_u$*.*

- T *is a* selfadjoint *operator if and only if* $T = T^*$*, that is* $\mathcal{D}(T) = \mathcal{D}(T^*)$ *and* $Tu = T^*u$ *for all* $u \in \mathcal{D}(T)$*.*

- *An symmetric operator* T_0 *with intial domain* $\mathcal{D}(T_0)$ *in a Hilbert space* $\mathcal{H}$ *is called to be* semibounded from below *if there exists a constant* C *such that for all* $u \in \mathcal{D}(T_0)$

$$\langle T_0\, u\,, u\rangle_{\mathcal{H}} \geq -C\, \|u\|^2_{\mathcal{H}}\ .$$

- T *is a* closed *operator if its graph* $G(T) = \{(u, Tu)\}$ *is a closed set with respect to the product topology of* $\mathcal{H} \times \mathcal{H}$*. Equivalently we can say* T *is closed if and only if given a sequence* $u_n \in \mathcal{D}(T)$ *such that* $u_n \to u$ *in* $\mathcal{H}$ *and* Tu_n *is a Cauchy sequence in* $\mathcal{H}$*, then* $u \in \mathcal{D}(T)$ *and* $Tu_n \to Tu$ *in* $\mathcal{H}$*.*

- *T is* closable *if the closure of its graph is a graph, or, equivalently, if and only if given a sequence $u_n \in \mathcal{D}(T)$ such that $u_n \to 0$, then $Tu_n \to v$ implies $v = 0$. $\overline{T}$ is the* closure *of T if $\overline{T}$ is a closed operator and $G(\overline{T}) = \overline{G(T)}$. Equivalently $\overline{T}$ is the continuos extension of T on $\overline{\mathcal{D}(T)}^{\|Tu\|+\|u\|}$.*

- *T will be named* essentially selfadjoint *if its closure is a selfadjoint operator.*

Here there is a list of properties and facts concerning the previous definitions.

Properties A.0.5

- *A symmetric operator is closable.*

  Proof. *Given a sequence $u_n \in \mathcal{D}(T)$ such that $u_n \to 0$ and $Tu_n \to v$, we want to prove $v = 0$. We have the following chain of equalities:*

$$\|v\|^2 = \langle v, v\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle Tu_n, v\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle u_n, Tv\rangle_{\mathcal{H}} = \langle 0, Tv\rangle_{\mathcal{H}} = 0.$$

  Then this implies $v = 0$. □

- *The adjoint operator is closed. Consequently, a selfadjoint operator is closed.*

  Proof. *Let $v_n \in \mathcal{D}(T^*)$ a sequence such that $v_n \to v$ and $T^* v_n \to w^*$. We want to prove (v, w^*) is a point of the graph $G(T^*)$. For any $u \in \mathcal{D}(T)$ we have*

$$\langle Tu, v\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle Tu, v_n\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle u, T^* v_n\rangle_{\mathcal{H}} = \langle u, w^*\rangle_{\mathcal{H}}$$

  and thus $v \in \mathcal{D}(T^*)$ and $T^* v = w^*$. □

- *If a selfadjoint operator T is invertible, then T^{-1} is selfadjoint.*

  Proof. *For any $v \in \mathcal{D}(T)$ the following chain of equalities holds*

$$\langle u, v\rangle_{\mathcal{H}} = \langle TT^{-1}u, v\rangle_{\mathcal{H}} = \langle T^{-1}u, Tv\rangle_{\mathcal{H}} = \langle u, (T^{-1})^* Tv\rangle_{\mathcal{H}}$$

  from which $(T^{-1})^* = T^{-1}$. □

- *If T is selfadjoint, then $T + \lambda\mathbf{1}$ is selfadjoint for any $\lambda \in \mathbb{R}$. (Obvious).*

- *If T is closable, then $\mathcal{D}(T)$ is dense in $\mathcal{H}$ and $T^{**} = \overline{T}$. (The proof is not straightforward, nevertheless we skip it, since it requires several additional propositions).*

- *If T' extends T (in notation $T' \supset T$), then T^* extends $(T')^*$. (The proof is immediate from definition of adjiont operators).*

Theorem A.0.6 ***(Kato-Rellich)*** *Let A be a selfadjoint operator, B be a symmetric operator whose domain contains $\mathcal{D}(A)$. Let us assume the existence of two constants a and b such that $0 \le a < 1$ and $b \ge 0$ such that*

$$\|B\,u\| \le a\,\|A\,u\| + b\,\|u\|$$

for all $u \in \mathcal{D}(A)$. Then $A + B$ is selfadjoint on $\mathcal{D}(A)$. If A is essentially selfadjoint on $\mathcal{D}(A)$, then $A + B$ has the same property.

Theorem A.0.7 ***(Friedrichs' extension)*** *Given a semibounded from below and symmetric operator T_0 with domain $\mathcal{D}(T_0)$ dense in $\mathcal{H}$, there exists a selfadjoint extension.*

Proof. Thanks to Theorem (A.0.6), without loss of generality one can consider the case $\langle T_0\, u\,, u\rangle_{\mathcal{H}} \geq \|u\|^2_{\mathcal{H}}$ for all $u \in \mathcal{D}(T_0)$. Let V be the completion in $\mathcal{H}$ of $\mathcal{D}(T_0)$ with respect to the norm $\sqrt{\langle T_0\, u\,, u\rangle_{\mathcal{H}}}$ and set $u \in V$ if and only if there exists a sequence $u_n \in \mathcal{D}(T_0)$ which is a Cauchy sequence with respect to the aforementioned norm and converges to u in $\mathcal{H}$. As a reasonable norm for V we choose $\|u\|_V = \lim_{n\to+\infty} \sqrt{\langle T_0\, u_n\,, u_n\rangle_{\mathcal{H}}}$, moreover it can be proved this is a good definition, i.e. it does not depend on the Cauchy sequence, and T_0 can be extended continuosly on V. Thus, we have obtained the continuos dense embedding of V into $\mathcal{H}$. Thanks to the density, we can define a scalar product in $\mathcal{H}$ with respect to T_0: for any $u, v \in \mathcal{H}$ let

$$\langle u\,, v\rangle_{T_0} = \lim_{n\to+\infty} \langle T_0\, u_n\,, v_n\rangle_{\mathcal{H}}$$

where $u_n, v_n \in V$ and $u_n \to u$ and $v_n \to v$ in $\mathcal{H}$ as $n \to +\infty$. Now we are in position to conclude via Lax-Milgram's theorem: $\langle u\,, v\rangle_{T_0}$ is a V-elliptic sesquilinear form, so it can be represented by a selfadjoint operator, which is the extension of T_0. □

Remark A.0.8 *For operators, to be essentially selfadjoint is equivalent to admit a unique selfadjoint extension. In this case it will coincide with the Friedrichs' one. Indeed, if T is an essentially selfadjoint operator, its closure is selfadjoint. Suppose S is an extension of T, then S is closed, and thus S is a closed extension of $\overline{T}$, since $\overline{T}$ is by definition the smallest closed extension of T. For it we write $S \supset \overline{T}$. On the other hand, $\overline{T} = T^{**}$, so that $S = S^* \subset \overline{T}^* = (T^{**})^* = T^{**}$ and then $S \subset \overline{T}$.* □

A.1 Examples

To give some examples, we are going to exploit the previous remark. In the following $\widetilde{H}$ will denote the Friedrichs' extension of operator H, whereas $\overline{H}$ will denote the closure of the same.

1. $H = -\Delta + \mathbf{1}$ with domain $C^\infty_C(\mathbb{R}^N)$ is essentially selfadjoint.

   Indeed, we can show its closure coincides with the Friedrichs' extension. To do this, we consider $L^2(\mathbb{R}^N)$ as setting Hilbert space and $C^\infty_C(\mathbb{R}^N)$ as initial domain $\mathcal{D}(H)$. Thus, via integration by parts

$$\langle Hu\,, u\rangle_{L^2} = \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2$$

   and the closure of the initial domain consists in $V = \overline{\mathcal{D}(H)}^{\|\nabla u\|_2^2 + \|u\|_2^2} = H^1(\mathbb{R}^N)$. The sesquilinear form related to the quadratic form is $(u, v) \mapsto \int_{\mathbb{R}^N} \nabla u \nabla \overline{v} + u\overline{v}$ and via

Lax-Milgram's Theorem it will be $\int_{\mathbb{R}^N} \nabla u \nabla \overline{v} + u\overline{v} = \int_{\mathbb{R}^N} fv$ for some $f \in L^2(\mathbb{R}^N)$ and for any $v \in V$. In particular, if $v \in C_C^\infty(\mathbb{R}^N)$, via integration by parts we obtain $-\Delta u = f$ in distributional sense, which implies $-\Delta u \in L^2(\mathbb{R}^N)$. Then the domain of the Friedrichs' extension will be

$$\mathcal{D}(\widetilde{H}) = \{u \in H^1(\mathbb{R}^N) \text{ s.t. } -\Delta u \in L^2(\mathbb{R}^N)\}.$$

On the other hand, the closure $\overline{H}$ is an operator whose domain is by definition

$$\mathcal{D}(\overline{H})^{\|Hu\|_2^2+\|u\|_2^2} = \overline{C_C^\infty(\mathbb{R}^N)}^{\|Hu\|_2^2+\|u\|_2^2}$$

which coincides with $\mathcal{D}(\widetilde{H})$, since we need $u \in H^1(\mathbb{R}^N)$ in order to $-\Delta$ makes sense in distributional sense.

2. $H = -\Delta$ with domain $C_C^\infty(\Omega)$ is not essentially selfadjoint provided Ω is a regular open bounded domain in $\mathbb{R}^N$.

   It is quite simple to see the Friedrichs' extension in this case is $\widetilde{H} = -\Delta$ with domain $\mathcal{D}(\widetilde{H}) = \{u \in H_0^1(\Omega) \text{ s.t. } -\Delta u \in L^2(\Omega)\} = H^2(\Omega) \cap H_0^1(\Omega)$ via standard regularity theorem. On the other hand, the operator $J = -\Delta$ with domain $\mathcal{D}(J) = \{u \in H^2(\Omega) \text{ s.t. } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$ is a selfadjoint extension of H. Indeed, it is a symmetric operator, thanks to the boundary conditions imposed, and it is easy to see the adjoint operator's domain $\mathcal{D}(J^*)$ is the same as $\mathcal{D}(J)$.

Appendix B

Basic facts about stationary Schrödinger operators

We recall the following basic inequalities:

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx &\leq \frac{4}{(N-2)^2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad \text{(Hardy inequality)}\\ \int_{\mathbb{R}^N} |\nabla |u||^2\,dx &\leq \int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A}{|x|}\right)u\right|^2 dx \quad \text{(diamagnetic inequality)}\end{aligned}$$

Lemma B.0.1 *The completion of $C_C^\infty(\mathbb{R}^N \setminus \{0\})$ under the Dirichlet norm coincide with the space $D^{1,2}(\mathbb{R}^N)$.*

Proof. We indeed prove that for all $u \in C_C^\infty(\mathbb{R}^N)$ there exists a sequence $\{v_n\} \subset C_C^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\int_{\mathbb{R}^N} |\nabla(u - v_n)|^2 \to 0$ as $n \to \infty$.

As approximating sequence in the statement we choose $v_\varepsilon = (1-\eta_\varepsilon)u$ for ε small, where η_ε is a cut-off radial function which is identically 1 in $B_\varepsilon(0)$ and identically 0 in $\mathbb{R}^N \setminus B_{\sqrt{\varepsilon}}(0)$, while in the joining region it is $\eta_\varepsilon(r) = \frac{1}{\log\sqrt{\varepsilon}}\log(\frac{r}{\varepsilon}) + 1$. Hence

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\nabla\big((1-\eta_\varepsilon)u\big) - \nabla u\right|^2 &= \int_{\mathbb{R}^N} |\nabla(\eta_\varepsilon u)|^2 = \int_{\mathbb{R}^N} |\eta_\varepsilon \nabla u + u\nabla\eta_\varepsilon|^2 \qquad \text{(B.1)}\\ &\leq C\left(\int_{\mathbb{R}^N} \eta_\varepsilon^2\,|\nabla u|^2 + \int_{\mathbb{R}^N} u^2\,|\nabla\eta_\varepsilon|^2\right).\end{aligned}$$

The first term is $o(1)$ as $\varepsilon \to 0$ since we are integrating over $\operatorname{supp}\eta_\varepsilon$ whose measure tends to zero as $\varepsilon \to 0$; via Hölder inequality the second term is less or equal to

$$C\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{\frac{N-2}{N}} \left(\int_{\mathbb{R}^N} |\nabla\eta_\varepsilon|^N\right)^{2/N}$$

which is asymptotic to

$$\begin{aligned}\int_{\varepsilon}^{\sqrt{\varepsilon}} |\eta_\varepsilon'(r)|^N\, r^{N-1}\, dr &= \int_{\varepsilon}^{\sqrt{\varepsilon}} \left|\frac{1}{r\log(\sqrt{\varepsilon})}\right|^N r^{N-1}\, dr \\ &= \frac{1}{|\log\sqrt{\varepsilon}|^N}\int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{1}{r}\, dr \\ &= -\frac{1}{|\log\sqrt{\varepsilon}|^N}\log\sqrt{\varepsilon} = \frac{1}{|\log\sqrt{\varepsilon}|^{N-1}} \xrightarrow{\varepsilon\to 0} 0\,,\end{aligned}$$

therefore the integral (B.1) tends to zero. □

Lemma B.0.2 *If $A \in L^\infty(\mathbb{S}^{N-1})$ then the norm $\left(\int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2\right)^{1/2}$ is equivalent to the Dirichlet norm on $C_C^\infty(\mathbb{R}^N\setminus\{0\})$.*

Proof. We have to show there exist two positive constants c_1 and c_2 such that $c_1\, Q_A(u) \le \int_{\mathbb{R}^N}|\nabla u|^2$ and $\int_{\mathbb{R}^N}|\nabla u|^2 \le c_2\, Q_A(u)$ for all $u \in C_C^\infty(\mathbb{R}^N\setminus\{0\})$.

The first inequivalence is an immediate consequence of Hardy inequality, using the fact $A \in L^\infty(\mathbb{S}^{N-1})$.

The second one is a consequence of Hardy inequality both with the diamagnetic inequality, noting

$$\int_{\mathbb{R}^N}|\nabla u|^2 = \int_{\mathbb{R}^N}\left|\left(i\,\nabla - \frac{A}{|x|}\right)u + \frac{A}{|x|}u\right|^2 \le C\left(Q_A(u) + \int_{\mathbb{R}^N}\frac{|A|^2}{|x|^2}|u|^2\right)\ ,$$

and

$$\int_{\mathbb{R}^N}\frac{|A|^2}{|x|^2}|u|^2 \le \frac{4\,\|A\|_\infty}{(N-2)^2}\int_{\mathbb{R}^N}|\nabla\,|u||^2 \le \frac{4\,\|A\|_\infty}{(N-2)^2}Q_A(u)\ .$$

□

Lemma B.0.3 *The quadratic form (1.1) is equivalent to $Q_A(u) = \int_{\mathbb{R}^N}\left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2$ on its maximal domain $D^{1,2}(\mathbb{R}^N)$ provided $\|a\|_\infty < (N-2)^2/4$. Moreover, it is positive definite.*

Proof. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\frac{a}{|x|^2}u^2\right| &\le \|a\|_\infty\int_{\mathbb{R}^N}\frac{u^2}{|x|^2} \le \left(\frac{2}{N-2}\right)^2\|a\|_\infty\int_{\mathbb{R}^N}|\nabla\,|u||^2 \\ &\le \left(\frac{2}{N-2}\right)^2\|a\|_\infty\, Q_A(u)\ ,\end{aligned}$$

therefore we immediately have

$$Q_A(u) - \int_{\mathbb{R}^N} \frac{a}{|x|^2}\, u^2 \leq (1+C)\, Q_A(u)$$

where $C = \left(\dfrac{2}{N-2}\right)^2 \|a\|_\infty$. In the same way we get

$$Q_A(u) - \int_{\mathbb{R}^N} \frac{a}{|x|^2}\, u^2 \geq Q_A(u) - \left| \int_{\mathbb{R}^N} \frac{a}{|x|^2}\, u^2 \right| \geq (1-C)\, Q_A(u)$$

and if $\|a\|_\infty < \frac{(N-2)^2}{4}$ it results $(1-C) > 0$ and therefore the quadratic forms (1.1) and Q_A are equivalent and positive definite. □

Appendix C

An examplary case of conformal equivalence between Riemannian manifolds

In Chapter 2, we stated the general theorem

Lemma C.0.4 *Let* (M,g) *and* (N,h) *two Riemannian manifolds of dimensions* $N \geq 3$. *Suppose there is a conformal diffeomorphism* $f : M \to N$, *that is* $f^*h = \varphi^{2^*-2}g$ *for some positive* $\varphi \in C^\infty(M)$. *The scalar curvatures of* (M,g) *and* (N,h) *are* R_g *and* R_h *respectively. Set the following corresponding equations:*

$$-\Delta_g u + \frac{1}{4}\frac{N-2}{N-1}R_g(x)u = F(x,u) \tag{C.1}$$

$$-\Delta_h v + \frac{1}{4}\frac{N-2}{N-1}R_h(y)v = [(\varphi \circ f^{-1})(y)]^{-\frac{N+2}{N-2}}F(f^{-1}(y),(\varphi \circ f^{-1})(y)v) \tag{C.2}$$

where $F : M \times \mathbb{R} \to \mathbb{R}$ *is smooth. Suppose* v *is a solution of* (C.2). *Then* $u = (v \circ f)\varphi$ *is a solution of* (C.1) *such that* $\int_M |u|^{2^*} dV_g = \int_N |v|^{2^*} dV_h$.

We used it in a relatively simple context: we considered the inverse of the stereographic projection $\pi : \mathbb{S}^N \setminus \{p\} \to \mathbb{R}^N$. We denoted it by $\Phi = \pi^{-1} : \mathbb{R}^N \to \mathbb{S}^N \setminus \{p\}$, moreover g_0 will denote the standard metric on $\mathbb{S}^N$ and δ the standard one on $\mathbb{R}^N$, and stated the diffeomorphism Φ is conformal between the two manifolds, since it results

$$g \doteq \Phi^* g_0 = \mu(x)^{\frac{4}{N-2}}\delta \ . \tag{C.3}$$

Thanks to the relative simplicity of the manifolds involved, we can check it directly. We recall

$$\Phi : \quad (x_1,\ldots,x_N) \mapsto \left(\frac{2x_1}{1+|x|^2},\ldots,\frac{2x_N}{1+|x|^2},1-\frac{2}{1+|x|^2}\right)$$

so that

$$\frac{\partial \Phi^h}{\partial x^k} = \begin{cases} \frac{2}{1+|x|^2}\delta_k^h - \frac{4x^k x^h}{\left(1+|x|^2\right)^2} & \text{for } h = 1,\cdots,N \\ \frac{4x^k}{(1+|x|^2)^2} & \text{for } h = N+1. \end{cases}$$

Now we are interested in the new metric $g = \Phi^* g_0$ on $\mathbb{R}^N$. It is defined by its coefficients

$$g_{ij} = g\,(e_i, e_j) = \langle d\Phi\, e_i, d\Phi\, e_j\rangle = \langle \frac{\partial \Phi^h}{\partial x^i}, \frac{\partial \Phi^k}{\partial x^j}\rangle = \left(\frac{2}{1+|x|^2}\right)^2 \delta_{ij}$$

being $\langle \cdot, \cdot \rangle$ the inner product of $\mathbb{R}^{N+1}$. Defining

$$\mu(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2}{2}}$$

we see Φ is conformal according to (C.3).

In addition, we point out the manifold $(\mathbb{R}^N, g)$ is *the same* as $(\mathbb{S}^N, g_0)$, in terms of diffeomorphic manifolds.

Thus, we can check the correspondence of the two equations in Lemma (C.0.4) directly. In general (see [53]) the so-called *conformal laplacian* is defined as follows

Definition C.0.5 *We define the* conformal Laplacian *on a differentiable closed manifold (M, g) of dimension N the operator*

$$L_g = -\Delta_g + \frac{N-2}{4(N-1)} R_g$$

where Δ_g denotes the standard Laplace-Beltrami operator on M and R_g the scalar curvature of the manifold.

Moreover, this operator has a simple transformation law under a conformal change of metric, that is

$$\text{if} \quad \widetilde{g} = \mu(x)^{\frac{4}{N-2}} g \qquad \text{then} \quad L_{\widetilde{g}} \cdot = \mu(x)^{-\frac{N+2}{N-2}} L_g\big(\mu(x)\cdot\big).$$

In our case we are dealing with the same manifold $\mathbb{R}^N$ endowed with the two metrics δ, the standard one, and $g = \Phi^* g_0$. Thus in our case we have

$$L_\delta = -\Delta$$

since the scalar curvature of $\mathbb{R}^N$ is zero, and

$$L_g = -\Delta_g + \frac{1}{4} N(N-2)$$

since the scalar curvature of $\mathbb{S}^N$ is $N(N-1)$.

To our aim, it will be sufficient to compute

$$\Delta\big(\mu(x)v(x)\big) = \sum_{i=1}^{N} \left\{ \frac{\partial^2 \mu}{\partial {x_i}^2} v(x) + 2 \frac{\partial \mu}{\partial x_i} \frac{\partial v}{\partial x_i} + \mu(x) \frac{\partial^2 v}{\partial {x_i}^2} \right\} .$$

It holds

$$\begin{aligned}\frac{\partial\mu}{\partial x_i} &= -\frac{N-2}{2}\mu(x)^{\frac{N}{N-2}}x_i\\ \frac{\partial^2\mu}{\partial {x_i}^2} &= -\frac{N-2}{2}\mu(x)^{\frac{N}{N-2}}+\frac{1}{4}N(N-2)\mu(x)^{\frac{N+2}{N-2}}{x_i}^2\end{aligned}$$

therefore

$$\begin{aligned}\Delta\big(\mu(x)v(x)\big) &= \sum_{i=1}^{N} v(x)\left\{-\frac{N-2}{2}\mu(x)^{\frac{N}{N-2}}+\frac{1}{4}N(N-2)\mu(x)^{\frac{N+2}{N-2}}{x_i}^2\right\}\\ &\quad + 2\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}\left\{-\frac{N-2}{2}\mu(x)^{\frac{N}{N-2}}x_i\right\} \ + \ \mu(x)\Delta v(x)\\ &= -\frac{1}{2}N(N-2)\mu(x)^{\frac{N}{N-2}}v(x) \ + \ \frac{1}{4}N(N-2)\mu(x)^{\frac{N+2}{N-2}}\left|x\right|^2 v(x)\\ &\quad - (N-2)\mu(x)^{\frac{N}{N-2}}\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}x_i \ + \ \mu(x)\Delta v(x)\ ;\end{aligned}$$

and finally we have

$$\begin{aligned}\mu(x)^{-\frac{N+2}{N-2}}\Delta\big(\mu(x)v(x)\big) &= \mu(x)^{-\frac{4}{N-2}}\Delta v(x) \ + \ \frac{1}{4}N(N-2)v(x)\left(\left|x\right|^2-2\mu(x)^{-\frac{2}{N-2}}\right)\\ &\quad - (N-2)\mu(x)^{-\frac{2}{N-2}}\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}x_i\\ &= \mu(x)^{-\frac{4}{N-2}}\Delta v(x) \ - \ \frac{1}{4}N(N-2)v(x)\\ &\quad - (N-2)\mu(x)^{-\frac{2}{N-2}}\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}x_i\ ; \qquad (C.4)\end{aligned}$$

since $\mu(x)^{-\frac{2}{N-2}} = 1/2+\left|x\right|^2/2$. As last step, we need to compute $\Delta_g v(x)$. In local charts we have the form

$$\Delta_g = \frac{1}{\sqrt{g}}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}$$

where $g=\det(g_{ij})$ and $g^{ij}=(g_{ij})^{-1}$. In our case these quantities are

$$\begin{aligned}g_{ij} &= \mu(x)^{\frac{4}{N-2}}\delta_{ij}\\ g &= =\mu(x)^{\frac{4N}{N-2}}\\ g^{ij} &= =\mu(x)^{-\frac{4}{N-2}}\delta^{ij}\ .\end{aligned}$$

Therefore

$$
\begin{aligned}
\Delta_g v(x) &= \mu(x)^{-\frac{2N}{N-2}} \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\mu(x)^2 \frac{\partial v}{\partial x_i}\right) \\
&= \mu(x)^{-\frac{2N}{N-2}} \left(\sum_{i=1}^{N} -(N-2)\mu(x)^{\frac{2N-2}{N-2}} \frac{\partial v}{\partial x_i} x_i + \mu(x)^2 \Delta v(x)\right) \\
&= -(N-2)\mu(x)^{-\frac{2}{N-2}} \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} x_i + \mu(x)^{-\frac{4}{N-2}} \Delta v(x) \ . \qquad \text{(C.5)}
\end{aligned}
$$

Thanks to the equations (C.4) and (C.5) we proved (C.3) in this particular case.

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