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A GEOMETRICAL CHARACTERIZATION OF INTEGRABLE HAMILTONIAN SYSTEMS THROUGH THE THEORY OF POISSON-NIJENHUIS MANIFOLDS

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Riassunto - In questo lavoro si introduce il concetto di varietà di Poisson-Nijenhuis, e si mostra che i sistemi hamiltoniani integrabili sono i "campi fondamentali" di tali varietà. In particolare, si costruisce esplicitamente un semplice modello di varietà PN infinito-dimensionale e si mostra che esso dà luogo alle equazioni di Gel'fand-Dikii. Il principale vantaggio del presente approccio sembra essere la sua semplicità concettuale e la sua sistematicità. Le differenti equazioni hamiltoniane integrabili sono ottenute come differenti riduzioni di una singola struttura PN su sottovarietà individuate dalla struttura PN stessa.

Abstract - In this paper we introduce the concept of Poisson-Nijenhuis manifold and we show that the integrable Hamiltonian systems are the "fundamental fields" of such manifolds. In particular, we explicitly construct a simple model of infinite-dimensional PN manifold, and we show that it gives rise to the Gel'fand-Dikii equations. The many advantages of the present approach seem to be its conceptual simplicity and the property of being systematic. The different equations are obtained as different reductions of a single PN structure, performed over special submanifolds picked out by the geometry of the PN manifold itself.

1. Introduction

The purpose of this paper is to present a new geometrical characterization of the integrable Hamiltonian systems, with both a finite and an infinite number of degrees of freedom. It differs from recent researches on the same subject [1,2] for a marked emphasis on the geometrical foundations of the concept of complete integrability, so as to show that the theory of the integrable Hamiltonian systems is a part of the geometry of a particular class of manifolds, here called Poisson-Nijenhuis manifolds. The origin of this concept will be explained in this introduction.

We recall that two objects are required to define a differentiable dynamical system: a differentiable manifold M, called the phase space, and a vector field φ on M. This field defines a system of equations

$$\dot{m}(t) = \varphi(m(t)) \tag{1.1}$$

which are called the equations of motion of the dynamical system. According to the properties of M and φ, the abstract Eq. (1.1) can give rise to a great variety of evolution equations which may be quite different from one another. For instance, if M is a finite-dimensional manifold and if $\{x^j\}$ are any system of local coordinates on M, (1.1) takes the well-known form

$$\dot{x}^j = \phi^j(x^k) \ . \tag{1.2}$$

Otherwise, if M is the Schwartz space $\mathcal{S}$ of rapidly decreasing C^{∞} functions on R , and if the vector field is defined by a differential operator on $\mathcal{S}$, (1.1) may take the form

$$(1.3) \qquad \partial_t u = \Phi(u, u_x, u_{xx}, \ldots)$$

In principle, (1.1) may represent also a system of integro-differential equations, such as the Benjamin-Ono equation [3,4]

$$(1.4) \qquad \partial_t u = -u u_x - H(u_{xx}) \qquad H(\psi) := \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\psi(x')}{x'-x}\, dx'$$

or equations in more than one space variable (as the Kodomtsev-Petviashvili equation [5]); in any case, these eqqations must be solved with respect to the time derivative and they must be endowed with suitable boundary conditions to be encompassed into the definition of M .

Within the previous class of dynamical systems, the Hamiltonian ones play a quite particular role. To define them, it is necessary to endow M with an additional structure, given by a second-order tensor P : $\mathcal{X}^*(M) \longrightarrow \mathcal{X}(M)$ of type (2,0), skew-symmetric and with a vanishing Schouten bracket (to be defined below) :

$$(1.5) \qquad P + P^* = 0 \qquad [P, P] = 0$$

Such a manifold is called a Poisson manifold. The Poisson tensor P maps the one-forms α into the vector fields φ and, in particular, it maps the closed one-forms into vector fields

$$(1.6) \qquad \varphi_\alpha := P\,\alpha \qquad d\alpha = 0$$

which are said to be (locally) Hamiltonian with respect to P. The property of being Hamiltonian is thus relative to the choice of P, and therefore it may be that a vector field is Hamiltonian with respect to a Poisson tensor but not with respect to another one, that it is not at all Hamiltonian, or that it is Hamiltonian simultaneously with respect to several Poisson tensors (the last property will turn out to be distinctive of the integrable Hamiltonian systems). Again, the abstract form (1.6) gives rise, in the applications, to a large number of particular forms, such as, for finite-dimensional manifolds,

$$\varphi^i = P^{ik}(x)\,\partial_k H \tag{1.7}$$

with

$$P^{ki} + P^{ik} = 0 \;;\; P^{il}\partial_l P^{hk} + P^{hl}\partial_l P^{ki} + P^{kl}\partial_l P^{ih} = 0 \tag{1.8}$$

and

$$\underset{\sim}{\varphi}(x) = \partial_x \underset{\sim}{\alpha} \tag{1.9.1}$$

$$\underset{\sim}{\varphi}(x) = \partial_x \underset{\sim}{\alpha} + [\underset{\sim}{u}, \underset{\sim}{\alpha}] \tag{1.9.2}$$

$$\underset{\sim}{\varphi}(x) = \partial_{xxx} \underset{\sim}{\alpha} - 2(\underset{\sim}{u}\,\underset{\sim}{\alpha})_x - \{\underset{\sim}{u}, \underset{\sim}{\alpha}_x\} + [\underset{\sim}{u}, \underset{\sim}{\alpha}]_x + [\underset{\sim}{u}, \int^x [\underset{\sim}{u}, \underset{\sim}{\alpha}]\,dx] \tag{1.9.3}$$

for the manifold of the n x n matrices $\underset{\sim}{u}$ whose entries are rapidly decreasing C^∞ functions on R. More and more complex examples can be easily found in the literature. However, it can be shown that the class of Poisson tensors and Hamiltonian structures appearing in the literature is not as large as it might seem, since these structures are often strictly related. One of the advantages of the synthetic point of view we are developing in this paper is to show that the whole class of such Poisson tensors may be engendered by a few simple fundamental tensors (see Sec.s 15-16).

Let us now examine the concept of integrable system. No general agreement exists on this concept. The "integrability theories" known in the literature may be classified according to the technique by which the integrability is ascertained or the integrable system is constructed. After all, however, all such theories have to display the existence of a globally defined coordinate system entailing the splitting of the equations of motion (1.2), for finite-dimensional systems, into n equations, each one containing a single coordinate x^j. Among these different techniques, that based on the study of the integrals of the motion seems to be the most popular (and the origin of several variants deriving from it). As is known, one must assume not only that the dynamical system is Hamiltonian, but also that as many integrals of the motion are known, which are independent and in involution, as the number of degrees of freedom. Under these assumptions, this technique allows to obtain the system of coordinates and consequently to solve the equations of motion only by quadratures.

In this paper, we follow a different approach to the integrability problem, which is based directly on the study of the coordinate system without requiring, in principle, that the dynamical system be Hamiltonian. The basic remark, due to the Dutch geometer A. Nijenhuis[6], is that a system of coordinates can be characterized by a special class of tensors of type (1,1), hereafter defined as Nijenhuis tensors. This is easily understood by observing that the distribution of the natural basis associated with any system of coordinates can be seen as the distribution of the eigenvectors of a special tensor N of type (1,1), whose eigenvalues are to be suitably precised. As it was shown by Nijenhuis, in order that the distribution of the eigenvectors be integrable and consequently define a system of coordinates, it is necessary that the torsion of N be vanishing :

$$(1.10) \qquad T(N)(\varphi,\psi) := [N\varphi, N\psi] - N[\varphi, N\psi] - N[N\varphi,\psi] + N^2[\varphi,\psi] = 0$$

For finite-dimensional manifolds, this means that the components N^i_k of N, in any local chart, fulfill the relations

$$(1.11) \qquad \partial_l N^i_j \, N^l_m - \partial_l N^i_m \, N^l_j - N^i_l \, \partial_m N^l_j + N^i_l \, \partial_j N^l_m = 0 .$$

One can remark, however, that although (1.10) has been deduced from the study of tensors on a finite-dimensional manifold, it keeps its meaning also for infinite-dimensional manifolds. For example, the following tensors, defined on the space $\mathcal{S}$ of the Schwartz functions, are Nijenhuis tensors

$$(1.12.1) \qquad \bar{\varphi} = \varphi_{xx} - 4u\varphi - 2u_x \int_{-\infty}^{x} \varphi \, dx$$

$$(1.12.2) \qquad \bar{\utilde{\varphi}} = \utilde{\varphi}_{xx} - 2\Big(\utilde{u} \int_{-\infty}^{x} \utilde{\varphi} \, dx\Big)_x - \{\utilde{u}, \utilde{\varphi}\} + \Big[\utilde{u}, \int_{-\infty}^{x} \utilde{\varphi} \, dx\Big]_x + \Big[\utilde{u}, \int_{-\infty}^{x} [\utilde{u}, \int_{-\infty}^{x} \utilde{\varphi} \, dx] \, dx\Big]$$

$$(1.12.3) \qquad \bar{\utilde{\varphi}} = \Big(\int_{-\infty}^{x} \utilde{u} \, [\utilde{a}, \utilde{u}^{-1} \utilde{\varphi}] \, \utilde{u}^{-1} \, dx\Big) \utilde{u}$$

More in general, Eq.(1.10) is fulfilled by all the "recursion operators" of the equations solvable by the inverse scattering method known in the literature, which turn out to be Nijenhuis tensors.

The relation between a system of coordinates and a suitable tensor of type (1,1) leads to consider the problem of the integrability as the problem of finding the conditions between N and φ in order that the evolution equations related to φ be decoupled when written in the system of coordinates defined by N. It can be shown (Sec.5) that this condition is simply that N be invariant under φ, in the sense that the Lie derivative $L_\varphi(N)$ of N along φ vanishes

(1.13) $$L_\varphi(N) = 0 \; ;$$

summarizing, if φ leaves N invariant, then N decouples φ and makes the equations of motion integrable. The condition (1.13) is clearly analogous to the condition

(1.14) $$L_\varphi(I_j) = 0 \qquad (j=1,2, \quad ,n)$$

which must be fulfilled by the n integrals of the motion I_j. Although the condition (1.13) may seem more stringent than (1.14), one has to recall that (1.14) holds only if the dynamical system is Hamiltonian and the integrals of the motion are in involution; in this case, one can show (Sec.4) that the two conditions (1.13) and (1.14) are equivalent. Moreover, (1.14) becomes meaningless for infinite-dimensional systems, whereas (1.13) keeps its meaning also in this context, and it appears in this case to be the simplest integrability condition at hand.

At last, if the special case of Hamiltonian systems is considered, it becomes meaningful to select a particular class of Nijenhuis tensors, by requiring that the systems of coordinates be canonical with respect to the Hamiltonian structure. To this end, it is necessary that N and P, that is the integrability and the Hamiltonian structures, be suitably coupled. It can be shown that two coupling conditions are required:

$$NP - PN^{*} = 0 \qquad R(P,N) = 0 \tag{1.15}$$

The first one means that $\bar{P} := P \cdot N$ is a skew-symmetric tensor of type (2,0); the second one is expressed by the vanishing of a third order tensor $R(P,N)$ of type (2,1), depending on P, N and their Lie derivatives, which is defined in Sec.2. Its vanishing entails that the Schouten bracket of $\bar{P}$ vanishes, so that (1.15) mean that $\bar{P}$ is itself a Poisson tensor.

Thus we arrive at the main geometric object which is considered in this paper, the PN manifold or Poisson-Nijenhuis manifold: it is a differentiable manifold M endowed with a Hamiltonian structure and with an integrability structure by a Poisson tensor P and a Nijenhuis tensor N fulfilling (1.15). The main thesis of this paper is that the PN manifolds are the natural setting for the theory of integrable Hamiltonian systems. Indeed, it will be shown in Sec.4 that any integrable Hamiltonian system can be seen as a vector field of a PN manifold keeping simultaneously invariant both P and N

$$L_{\varphi}(P) = 0 \qquad L_{\varphi}(N) = 0 \tag{1.16}$$

and that, conversely, any vector field of a (finite-dimensional) PN manifold fulfilling (1.16) is an integrable Hamiltonian vector field (under an additional simple condition of minimal degeneracy on the spectrum of N). Consequently, the study of integrable systems can be replaced by the study of PN manifolds. At first sight, this may seem to be an ambitious program, due to the obvious difficulties in explicitly constructing PN manifolds: notwithstanding, it is worthwhile to carry it out in view of the more synthetic point of view it allows to attain, as it will become clearer from the applications done in Sec.s 15-16. Indeed, from a single PN manifold, different classes of integrable systems can be obtained, by a straightforward application of successive reductions. Moreover, the analysis of the examples strongly suggests that all the equations solvable by the inverse scattering method can be obtained from a unique PN structure. In this paper, however, this statement will be supported only for the class of Gel'fand-Dikii equations.

Having sketched the main ideas inspiring this research, let us examine in detail the contents and results of this paper. It can be divided into four parts:

(i) PN manifolds and integrable systems (Secs.2-5)

(ii) The reduction theory (Secs.6-11)

(iii) The construction of a group-theoretical PN manifold (Secs.12-14)

(iv) The PN structure related with the equations solvable by the inverse scattering method (Secs.15,16).

A remaining part dealing with the PN structure of the non-abelian infinite Toda lattice will be considered in a forthcoming paper.

In the first part the definitions are given and explained of the main structures entering the paper ($P\Omega$ and PN manifolds, fundamental vector fields and one-forms) and the relation between PN manifolds and integrable systems is established. The method is slightly "axiomatic", the concept of PN manifold being taken as primitive (Sec.2), and the theory of integrable systems being deduced from it. For that, in Sec.3 we prove a few general properties of fundamental fields of a PN manifold (i.e., the fields fulfilling (1.16)) holding both for infinite-dimensional and for finite-dimensional manifolds: among them, the iterative properties stated in Proposition 3.2 (Lenard recursion relations) are noteworthy. In Sec.4 we show, for finite-dimensional PN manifolds and under suitable assumptions given in Proposition 4.1 (geometric characterization of integrable systems), that the fundamental fields of PN manifolds coincide with the integrable Hamiltonian fields. At last, Sec.5 is devoted to a brief outline of the integration of the equations of motion by means of the Nijenhuis tensor, mainly to remark a striking analogy of this method with the "inverse scattering method": the discussion of this problem is clearly incomplete, but to make it complete would have taken us too far a field from the main subject of the paper.

The reduction theory discussed in the second part gives the main technical device to deal with PN manifolds in the applications. As is known, the equations solvable by the inverse scattering method arise from (often not clearly justified) "reduction processes", consisting in the choice of particular forms of matrices and so on. The reduction theory of PN manifolds aims to give a theoretical explanation of these techniques, by showing that the arbitrariness in the reduction process can be eliminated by adopting the point of view of the reduction of PN manifolds, and by proving that the reduced structures maintain the properties of the given ones (tensor point of view). Simple as this remark may be from a conceptual point of view, it has a great practical importance, since it allows to avoid a posteriori proofs which turn out to be very cumbersome in most cases (to this regard, it would suffice to look through the literature on the so-called "recursion operators"). This second part consits of six sections. In Secs. 6 and 7 two preliminary reduction Lemmas are proved; Secs.8 and 9 contain the main results of this part, defining the systematic reduction processes for PΩ and PN manifolds (Prop.8.1 and Prop.9.1); Sec.10 gives an example of possible variants of the reduction technique. This technique, finally, is exemplified in Sec.11 in a simple but not trivial case; though the result is the well-known integrability structure of non-abelian KdV equation, the reduction technique which is used has the advantage of not requiring any initial guess, being a systematic application of general methods.

The third part (Secs.12-14) concerns the <u>constructive phase</u> of the theory, that is the setting of practical rules for constructing a PN manifold. To this end, we choose as a base manifold M a Lie group H, which is a particularly structured manifold. The main result (Prop.12.1) is to show that the problem of the construction of a PN structure on this group is reduced to the <u>purely algebraic</u> problem of constructing skew-symmetric tensors $P_e : \mathcal{H}^* \to \mathcal{H}$, on the algebra $\mathcal{H}$, fulfilling the "cocycle condition"

$$(1.17) \qquad \langle \lambda, [P_e\mu, P_e\nu] \rangle + \cdots = 0$$

for any $\lambda,\mu,\nu \in \mathcal{H}^*$. All the following applications are based on this simple result. A detailed study of the group-theoretical PN manifolds is given in Sec.13. Although the analysis is an application of the general scheme of Secs.8,9, in this case one can use the particular structure of the base manifold to obtain a more detailed and precise description of the reduced manifolds. In particular, it results an interesting link between the reduction of group-theoretical PN manifolds and the theory of the momentum mapping (Prop.13.1) : indeed, with any Poisson cocycle P_e one can canonically relate a momentum mapping $J : H \longrightarrow \mathcal{H}^*$ giving a global diffeomorphism betweem the connected subgroup H', whose algebra is $\mathcal{H}' = P_e(\mathcal{H}^*)$, and $\mathcal{H}'^*$. By means of the momentum mapping, the PN structure of H' can be transferred to $\mathcal{H}'^*$, endowing $\mathcal{H}'^*$ with a PN structure (Sec.14). One obtains in this way a simple group-theoretical interpretation of the so-called spectral problem of the inverse scattering theory.

The general methods are straightforwardly applied in the last Part (Secs.15-16) to obtain the Gel'fand-Dikii équations. Here we limit to observe that the whole construction of these equations is carried out in a deductive way from the only remark that if the Lie group H is the group of the non-singular matrices $\underset{\sim}{u}$ whose entries $u^i_k(x)$ are functions rapidly decreasing to δ^i_k for $|x| \to \infty$, then the cocycle condition (1.17) has the simple solution

(1.18) $$P_0^{-1} \cdot \xi = \partial_x \xi \, .$$

By starting from this preliminary remark, one need not introduce further assumptions, but has only to apply the general methods of the PN theory. The analysis of the Toda lattice, although related in a quite natural way with the previous example, has been postponed to a forthcoming paper, since it actually requires a slight but rather subtle extension of the theory of the Poisson cocycle (as a matter of fact, this extension has given not little trouble to the authors).

At the end of the paper we have collected four Appendices: the first three contain some remarks and proofs which are essential for the completeness of the theory but which would have interrupted its development if given in the text. Instead, Appendix D contains a few technical remarks concerning the applications. In particular, App. A contains all the notations used in the paper and a few practical rules (suited for the applications) allowing to set into explicit form the abstract conditions given in the text. To simplify the citations, the Appendices are divided in short sections which are referred to in the text as bibliographical citations, such as [A.2] to mean the second section of the Appendix A.

2. Four remarkable manifolds

On any differentiable manifold M (modelled on a Banach space) three classes of second-order tensors are noteworthy: the presymplectic tensors, the Poisson tensors and the Nijenhuis tensors. We recall that a presymplectic tensor is a skew-symmetric tensor $\Omega : \mathcal{X}(M) \longrightarrow \mathcal{X}^*(M)$, of type (0,2) with constant rank and with vanishing exterior derivative

(2.1) $$\Omega + \Omega^* = 0 \qquad d\Omega = 0$$

(2.2) $$d\Omega(\varphi,\psi) := L_\varphi(\Omega)\psi - L_\psi(\Omega)\varphi - \Omega[\varphi,\psi] + d\langle\Omega\varphi,\psi\rangle$$

and that a Poisson tensor is a skewsymmetric tensor $P : \mathcal{X}^*(M) \longrightarrow \mathcal{X}(M)$, of type (2,0), with constant rank and with vanishing Schouten bracket [B.1]:

(2.3) $$P + P^* = 0 \qquad [P,P] = 0$$

(2.4) $$[P,P](\alpha,\beta) := P\big(L_{P\alpha}(\beta) + d\langle\alpha, P\beta\rangle\big) + L_{P\beta}(P)\alpha$$

Likewise, a Nijenhuis tensor is a tensor $N : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$, of type (1,1), with constant rank and with vanishing torsion tensor [B.2] :

(2.5) $$\mathbb{T}(N) = 0$$

(2.6) $$\mathbb{T}(N)(\varphi,\psi) = [N\varphi, N\psi] - N[N\varphi,\psi] - N[\varphi,N\psi] + N^2[\varphi,\psi]$$

They separately give M the structure of presymplectic manifolds, of Poisson manifold and of Nijenhuis manifold.

The geometrical structures we are looking for on the manifold M arise from a suitable coupling of such tensors. It is suggested by the remark that there exists a unique way of composing Ω, P, N in pairs so to obtain new second-order tensors

$$(2.7) \qquad \bar{N} := P\Omega \qquad \bar{P} := NP \qquad \bar{\Omega} := \Omega N$$

Let us demand the new tensors to be a Nijenhuis, a Poisson and a presymplectic tensor respectively. Then, the readily proved identities of Appendix B3 show that the pair (P,Ω) must obey the condition

$$(2.8) \qquad d(\Omega P \Omega) = 0$$

while the pairs (P,N) and (Ω,N) must obey the conditions

$$(2.9) \qquad NP - PN^* = 0 \qquad R(P,N) = 0$$ sufficient condition for $[NP, NP] = 0$

$$(2.10) \qquad \Omega N - N^*\Omega = 0 \qquad S(\Omega,N) = 0$$

respectively, where the tensors $R(P,N)$ and $S(\Omega,N)$ are defined by :

$$(2.11) \qquad R(P,N)(\alpha,\varphi) := L_{P\alpha}(N)\varphi - P\,L_{\varphi}(N^*\alpha) + P\,L_{N\varphi}(\alpha)$$

$$(2.12) \qquad S(\Omega,N)(\varphi,\psi) := L_{N\psi}(\Omega)\varphi - L_{N\varphi}(\Omega)\psi + \Omega N[\varphi,\psi] + d\langle\Omega\psi,\varphi\rangle$$

N

This leads to set the following definitions :

<u>Definition 2.1 ($P\Omega$ manifold)</u>. A $P\Omega$ manifold is a differentiable manifold M endowed with a Poisson tensor P and with a presymplectic tensor Ω fulfilling the coupling condition $d(\Omega P \Omega) = 0$.

<u>Definition 2.2 (PN manifold)</u>. A PN manifold is a differentiable manifold M endowed with a Poisson tensor P and with a Nijenhuis tensor N fulfilling the coupling conditions $NP-PN^* = 0$ and $R(P,N) = 0$.

<u>Definition 2.3 (ΩN manifold)</u>. A ΩN manifold is a differentiable manifold M endowed with a presymplectic tensor Ω and with a Nijenhuis tensor N fulfilling the coupling conditions $\Omega N-N^*\Omega = 0$ and $S(\Omega,N) = 0$.

The reason for dealing with these manifolds simultaneously is that they are intimately related. Indeed, let M be any $P\Omega$ manifold, and let us form the new pairs $(P,\bar{N})$ and $(\Omega,\bar{N})$. Then it can be readily proved (see Appendix B3) that these pairs endow M with the structure of both PN and Ω N manifold, without any further condition on P and Ω. Thus

<u>Proposition 2.1 (relation among $P\Omega$, PN and ΩN manifolds)</u>

Any $P\Omega$ manifold is canonically endowed both with a PN and with a ΩN structure, defined by the pairs $(P,\bar{N}:= P\Omega)$ and $(\Omega,\bar{N}:= P\Omega)$ respectively. Conversely, any PN or ΩN manifold is canonically endowed with a $P\Omega$ structure, defined by the pairs $(P,\Omega: = P^{-1}N)$ and $(\Omega,P: N\Omega^{-1})$ respectively, provided P and Ω are kernel-free.

Such manifolds are the main object of interest of the present paper, where many of their properties are studied. The following one gives an idea of the wealth of structures which are implicitly defined on them. Let M be a PN or a ΩN manifold, and let us form the new pairs $(\bar{P},N)$ and $(\bar{\Omega},N)$. In Appendix B3 it is shown that these pairs define a second PN or ΩN structure on M, without any further condition on the original pairs (P,N) or (Ω,N). By iteration, we have

<u>Proposition 2.2</u> <u>(hierarchy of PN and ΩN structures)</u>.
Any PN manifold is canonically equipped with a hierarchy of Poisson structures defined by the recursion relation

(2.13) $$P_{i+1} := N.P_i \qquad P_1 := P$$

Moreover, these structures obey the "involution relations"

(2.14) $$[P_i , P_j] = 0$$

that is the Schouten bracket of any pair of Poisson tensors of the hierarchy vanishes. Likewise, any $P\Omega$ manifold is canonically equipped with a hierarchy of presymplectic structures defined by the recursion relation

(2.15) $$\Omega_{i+1} := \Omega_i\, N \qquad \Omega_1 := \Omega$$

Finally, both manifolds are endowed with a hierarchy of Nijenhuis tensors defined by

(2.16) $$N^{i+1} := N\, N^i \qquad N^1 := N$$

Any pair (P_j, N^k) or (Ω_j, N^k) defines a different PN or ΩN structure on M .

The missing details in the proof of this proposition are given in Appendix B3 . The properties (2.14), in particular, deserves some further attention. Since the Poisson condition $[P,P] = 0$ is quadratic in P, it is clear that we cannot add, in general, Poisson tensors to obtain new Poisson tensors. Indeed, we find

(2.17) $$[P+Q, P+Q] = [P,P] + 2[P,Q] + [Q,Q]$$

and so only if (2.14) is fulfilled we obtain $[P + Q, P + Q] = 0$. More in general, by the homogeneity of the Poisson condition, we conclude that if (2.14) is fulfilled we have a whole family

(2.18) $$P_\lambda := P + \lambda Q \qquad \lambda \in R$$

of Poisson structures defined on M. This remarkable fact suggests to set the following (final) definition:

Definition 2.4 (PQ or "twofold Hamiltonian" manifolds [7-10])
A PQ manifold is a differentiable manifold M endowed with a pair of Poisson tensors P and Q fulfilling the coupling condition $[P,Q] = 0$.

3. A noteworthy class of vector fields on PN manifolds

As is known, on a Poisson manifold the (locally) Hamiltonian vector fields are defined as the fields associated with the closed one-forms :

$$(3.1) \qquad \varphi_\alpha := P\alpha \qquad\qquad d\alpha = 0$$

Such fields make a Lie algebra (with respect to the commutator of fields) and leave the Poisson tensor P invariant

$$(3.2) \qquad L_{\varphi_\alpha}(P) = 0$$

The closed one-forms, in turn, make a Lie algebra with respect to the Poisson brackets defined by P

$$(3.3) \quad \{\alpha,\beta\}_P := L_{P\beta}(\alpha) - L_{P\alpha}(\beta) + d\langle\beta, P\alpha\rangle$$

and the two algebras are homomorphic on account of the identity [B.4]

$$(3.4) \qquad [\varphi_\alpha,\varphi_\beta] = P\,.\{\alpha,\beta\}_P$$

On a PN manifold, a narrower class of vector fields and one-forms can be defined, playing a central role in the theory of the integrable systems. It enjoies all the properties of the Hamiltonian vector fields, besides that of being invariant with respect to N. Its study makes the object of the present section.

Let M be a PN manifold, N^* the dual tensor of N, and α a closed one-form . By using the identity (A.1.12)

$$(3.5)\quad d(N^*\alpha).\varphi := L_\varphi(N^*\alpha) - d\langle N^*\alpha, \varphi\rangle$$
$$= (L_{N\varphi}(\alpha) - d\langle\alpha, N\varphi\rangle) + (L_\varphi(N^*\alpha) - L_{N\varphi}(\alpha))$$
$$= d\alpha \,.\, N\varphi + L_\varphi(N^*\alpha) - L_{N\varphi}(\alpha)$$

one readily verifies that $N^*\alpha$ is no longer closed, unless the further condition

$$(3.6)\qquad L_\varphi(N^*\alpha) - L_{N\varphi}(\alpha) = 0$$

is fulfilled by α . This leads to consider the subspace $\mathcal{X}^*_N(M)$ of the closed one-forms obeying (3.6). It can be shown that $\mathcal{X}^*_N$ is a Lie algebra with respect to the Poisson brackets (3.3)

$$(3.7)\qquad \{\mathcal{X}^*_N, \mathcal{X}^*_N\}_P \subset \mathcal{X}^*_N$$

that it is invariant with respect to N^*

$$(3.8)\qquad N^*(\mathcal{X}^*_N) \subset \mathcal{X}^*_N$$

and that the "exchange rules"

$$(3.9)\qquad N^*.\{\alpha,\beta\}_P = \{N^*\alpha,\beta\}_P = \{\alpha, N^*\beta\}_P$$

hold in $\mathcal{X}^*_N$. Indeed, if $\alpha \in \mathcal{X}^*_N$, the coupling condition $R(P,N)=0$ entails

$$(3.10) \qquad L_{P\alpha}(N) = 0$$

whence

$$(3.11) \qquad L_{P\alpha}(N^*) = 0$$

So

$$(3.12) \quad N^*\{\alpha,\beta\}_P - \{N^*\alpha,\beta\}_P = N^*.\ (L_{P\beta}(\alpha)-d\alpha.\ P\beta) - (L_{P\beta}(N^*\alpha)-dN^*\alpha.\ P\beta)$$

$$\overset{(3.6)}{=} N^* L_{P\beta}(\alpha) - L_{P\beta}(N^*\alpha)$$

$$= - L_{P\beta}(N^*)\ .\ \alpha$$

$$\overset{(3.11)}{=} 0$$

and

$$(3.13) \quad \{N^*\alpha,\beta\}_P - \{\alpha, N^*\beta\}_P = d\langle N^*\alpha, P\beta\rangle - d\langle\alpha, PN^*\beta\rangle$$

$$= d\langle\alpha, (NP - PN^*)\ .\beta\rangle$$

$$= 0$$

Consequently, one finds

$$(3.14) \quad dN^*\{\alpha,\beta\}_P = d\{N^*\alpha,\beta\}_P = d\left(d\langle N^*\alpha,\ P\beta\rangle\right) = 0$$

showing that $\{\alpha,\beta\}_P \in \mathfrak{X}^*_N$. Finally, the invariance property follows from

$$(3.15)\quad L_{\varphi}(N^{*}.\, N^{*}\alpha) - L_{N\varphi}(N^{*}\alpha) =$$

$$= N^{*}.\,(L_{\varphi}(N^{*}\alpha) - L_{N\varphi}(\alpha)) + (L_{\varphi}(N^{*}).N^{*} - L_{N\varphi}(N^{*})).\alpha$$

$$= N^{*}.\,(L_{\varphi}(N^{*}\alpha) - L_{N\varphi}(\alpha))$$

on account of the identity

$$(3.16)\quad L_{\varphi}(N^{*})\,.\,N^{*} - L_{N\varphi}(N^{*}) = 0$$

fulfilled by N^{*} [B2].

The special class of Hamiltonian vector fields, previously referred to, is then given by

$$(3.17)\quad \mathcal{X}_{PN}(M) := P.(\mathcal{X}^{*}_{N}(M))$$

As a consequence of the properties of $\mathcal{X}^{*}_{N}$, it is readily seen that $\mathcal{X}_{PN}$ is a subalgebra of $\mathcal{X}(M)$, that it is invariant with respect to N, and that the exchange rules

$$(3.18)\quad N\,.\,[\varphi_{\alpha},\varphi_{\beta}] = [\bar{N}\varphi_{\alpha},\varphi_{\beta}] = [\varphi_{\alpha}, N\varphi_{\beta}]$$

hold in $\mathcal{X}_{PN}$. Indeed :

$$(3.19)\quad [\varphi_{\alpha},\varphi_{\beta}] = [\bar{P}\alpha, P\beta] \overset{[B1]}{=} P\{\alpha,\beta\}_{P} \in \mathcal{X}_{PN}$$

$$(3.20)\quad N(\mathcal{X}_{PN}) = NP(\mathcal{X}^{*}_{N}) = PN^{*}(\mathcal{X}^{*}_{N}) \subset P(\mathcal{X}^{*}_{N}) = \mathcal{X}_{PN}$$

$$(3.21)\quad [\bar{N}\varphi_{\alpha},\varphi_{\beta}] = [\bar{N}P\alpha, P\beta] = [\bar{P}N^{*}\alpha, P\beta] = P\{N^{*}\alpha,\beta\} = NP\{\alpha,\beta\} = N[\varphi_{\alpha},\varphi_{\beta}]$$

Moreover, ——— the identities

$$(3.22) \qquad L_{P\alpha}(N).\varphi \overset{(R(P,N)=0)}{=} P.\,(L_\varphi(N^*\alpha) - L_{N\varphi}(\alpha)) \overset{(3.6)}{=} 0$$

$$(3.23) \qquad L_{P\alpha}(P).\beta \overset{[P,P]=0}{=} -P.\,d\alpha.\,P\beta = 0$$

show that $\mathcal{X}_{PN}$ may be identified with the algebra of the vector fields leaving both N and P invariant

$$(3.24) \qquad L_{\varphi_\alpha}(N) = 0 \qquad L_{\varphi_\alpha}(P) = 0$$

(at least, in the particular case of P being kernel-free)

Due to the importance of $\mathcal{X}^*_N$ and of $\mathcal{X}_{PN}$, let us summarize their properties into the following:

Proposition 3.1 (algebra of fundamental fields and forms)

On a PN manifold M, the one-forms obeying the conditions

$$(3.25) \qquad d\alpha = 0 \qquad dN^*\alpha = 0$$

make a Lie algebra $\mathcal{X}^*_N(M)$ (with respect to the Poisson bracket defined by P), called the algebra of the fundamental forms. The corresponding vector fields

$$(3.26) \qquad \varphi_\alpha := P\alpha \qquad d\alpha = 0 \qquad dN^*\alpha = 0$$

are the fundamental fields of the manifold. They leave both P and N invariant

$$(3.27) \qquad L_{\varphi_\alpha}(P) = 0 \qquad L_{\varphi_\alpha}(N) = 0$$

and they make a Lie algebra $\mathcal{X}_{PN}(M)$ homomorphic to $\mathcal{X}^*_N$:

$$(3.28)\qquad [\varphi_\alpha, \varphi_\beta] = P\{\alpha,\beta\}_P$$

Furthermore, both algebras are invariant with respect to N, which commutes with the Lie algebra structure :

$$(3.29)\qquad N^*.\{\alpha,\beta\}_P = \{N^*\alpha,\beta\}_P = \{\alpha, N^*\beta\}_P \qquad \alpha,\beta\in\mathcal{X}^*_N$$

$$(3.30)\qquad N\,.\,[\varphi_\alpha,\varphi_\beta] = [N\varphi_\alpha,\varphi_\beta] = [\varphi_\alpha, N\varphi_\beta] \qquad \varphi_\alpha,\varphi_\beta\in\mathcal{X}_{PN}$$

Three properties of the fundamental fields and one-forms, readily deduced from the previous ones, are worth recording. They concern the chains

$$(3.31)\qquad \alpha_{j+1} = N^*\alpha_j \qquad \alpha_1\in\mathcal{X}^*_N$$

$$(3.32)\qquad \varphi_{j+1} = N\,\varphi_j \qquad \varphi_1 = P\alpha_1\in\mathcal{X}_{PN}$$

and they state that :

$$(3.33)\qquad \varphi_j = P_{j-\kappa+1}\;\alpha_\kappa \qquad (1\leq\kappa\leq j)$$

$$(3.34)\qquad \{\alpha_j,\alpha_\kappa\}_P = 0 \qquad [\varphi_j,\varphi_\kappa] = 0$$

$$(3.35)\qquad \langle \alpha_\kappa, \varphi_j\rangle = 0$$

The first one, proved by

$$(3.36)\qquad \varphi_j = N^{j-1}P\alpha = N^{j-\kappa}\circ P\,.\,N^{*\kappa-1}\alpha = P_{j-\kappa+1}\,.\,\alpha_\kappa \quad (1\leq\kappa\leq j)$$

means that the fields φ_j are Hamiltonian with respect to all the Poisson tensors $P_{j-\kappa+1}$ (of degree not greater than j) of the hierarchy

$$(3.37)\qquad P_{j+1} = N \cdot P_j \qquad\qquad P_1 = P$$

canonically associated with the PN manifold. Thus the fundamental fields admit several Hamiltonian formulations. The second and the third ones, which readily follow from the exchange rules (3.29) and (3.30), and from

$$(3.38)\qquad \langle \alpha_\kappa, \varphi_j \rangle \overset{(3.33)}{=} \langle \alpha_\kappa , P_{j-\kappa+1}\, \alpha_\kappa \rangle = 0$$

mean that the chain (3.31) and (3.32) are involutive. In particular, if the one-forms α_κ are exact

$$(3.39)\qquad \alpha_\kappa := d I_\kappa$$

the "orthogonality conditions" (3.35) entail that the functions I_κ are kept invariant by the fields φ_j

$$(3.40)\qquad L_{\varphi_j}(I_\kappa) := \langle \alpha_\kappa, \varphi_j \rangle = 0$$

and that they are in involution with respect to the Poisson brackets

$$(3.41)\qquad \{ I_\kappa , I_j \} := \langle \alpha_\kappa, P\alpha_j \rangle = 0 \,.$$

This simple result gives the reason for the present interest in PN manifolds. It suggests that, under suitable "completeness conditions" (to be precised below), every chain (3.32) may define an integrable Hamiltonian system, in the sense of Arnold-Liouville. The missing conditions will be supplied in the next section, where the converse statement will also proved to be correct : namely, any integrable Hamiltonian system can be considered as a fundamental field of a suitable PN manifold, and encompassed into a chain of the kind (3.32). Thus, we can state the following proposition:

Proposition 3.2 ("Lenard recursion relations")

Let M be a PN manifold, α any fundamental form, and φ_j and α_k the vector fields and the one-forms engendered by α according to the iterative scheme

$$\varphi_{j+1} = N\varphi_j \qquad \varphi_1 = P\alpha \tag{3.42}$$

$$\alpha_{j+1} = N^*\alpha_j \qquad \alpha_1 = \alpha \tag{3.43}$$

Then, the fields φ_j are Hamiltonian with respect to all the Poisson tensors of the hierarchy canonically associated with M, of degree not greater than j. The associated one-forms α_k belong to the chain (3.43) engendered by α. Moreover, both the chains (3.42) and (3.43) obey the commutation relations

$$[\varphi_j, \varphi_k] = 0 \qquad \{\alpha_j, \alpha_k\}_P = 0 \tag{3.44}$$

and the "orthogonality conditions"

$$\langle \alpha_k, \varphi_j \rangle = 0 \tag{3.45}$$

If the one-forms α_k are exact

$$\alpha_k := dI_k \tag{3.46}$$

these conditions mean that the functions I_k make a system of involutive integrals for all the fields of the chain (3.42):

$$L_{\varphi_j}(I_k) = 0 \qquad \{I_j, I_k\}_P = 0 \tag{3.47}$$

4. Integrable Hamiltonian systems as fundamental fields of PN manifolds

In this section, M is taken to be of finite-dimension and P to be kernel-free. In particular, the dimension of M is even, dim M = $2n$, and the manifold is symplectic. Under these assumptions (not really restrictive, in view of Prop. 9.1 of Sec. 9) we study the eigenvalue problem for N, and we show that :

(i) N has at most n distinct eigenvalues

(ii) these eigenvalues are in involution with respect to the Poisson brackets defined by P

Hence, we introduce the concept of PN manifolds of maximal rank (as those for which N has exactly n distinct eigenvalues) and we show that the study of the integrable Hamiltonian fields coincides with the study of the fundamental fields of such manifolds. In our opinion, this result gives the main support for the study of PN manifolds.

Let M be a PN manifold fulfilling the stated conditions. Since P is regular, N can be written as the product $N = P\Omega$ of P with a presymplectic tensor Ω (Prop. 2.1 of Sec. 2). Therefore, the eigenvalue problem for N takes the form

$$(4.1) \qquad (\Omega - \lambda P^{-1})\varphi = 0$$

showing that the eigenspaces of N are the kernels of the skewsymmetric tensors $(\Omega - \lambda P^{-1})$ associated with the different eigenvalues λ. The dimension of these kernels is always even. So, at least two eigenvectors are associated with any eigenvalue λ and the number of distinct eigenvalues cannot exceed n . To show that these eigenvalues are in involution, we use the following noteworthy relation

$$(4.2) \qquad N^* . \, dI_\kappa = dI_{\kappa+1}$$

connecting the gradients of the traces

$$(4.3) \qquad I_\kappa(m) := \frac{1}{\kappa} \operatorname{Tr}(N_m)^\kappa$$

If follows from :

$$(4.4) \qquad \langle dI_{\kappa+1}, \varphi \rangle = L_\varphi \left(\operatorname{Tr} \frac{1}{\kappa+1} N^{\kappa+1}\right)$$

$$= \operatorname{Tr}\left(N^{*\kappa-1} L_\varphi(N^*) . N^*\right)$$

$$\overset{(3.16)}{=} \operatorname{Tr}\left(N^{*\kappa-1} L_{N\varphi}(N^*)\right)$$

$$= \frac{1}{\kappa} \operatorname{Tr} L_{N\varphi}(N^{*\kappa})$$

$$= L_{N\varphi}(I_\kappa)$$

$$= \langle dI_\kappa, N\varphi \rangle$$

$$= \langle N^* dI_\kappa, \varphi \rangle$$

Consequently, the one-forms $\alpha_\kappa := dI_\kappa$ belong to the chain engendered by $\alpha_1 = dI_1$, and so

$$(4.5) \qquad \{I_\kappa, I_j\}_P = 0$$

by Prop. (3.2) (observe that α_1 is fundamental, since $d\alpha_1 = d N^* \alpha_1 = 0$).

In view of the previous result, the case of N having exactly n distinct eigenvalues is in some sense exceptional. This justifies to deal with it apart. Firstly, let us set the following definition :

Definition 4.1 (PN manifold of maximal rank)

A PN manifold of maximal rank is a (finite-dimensional) PN manifold M endowed with a kernel-free Poisson tensor P, and with a Nijenhuis tensor N having only double eigenvalues. The map $J : M \longrightarrow R^n$, associating the eigenvalues $(\lambda_1(m), \ldots, \lambda_n(m))$ with any point m, is the momentum mapping of the manifold. The open subset M' where J has its maximal rank

$$(4.6) \qquad M' := \left\{ m \in M \; : \; rk(dJ(m)) = n \right\}$$

is the set of the regular points of M.

Let then φ be any fundamental field of M. Observe that φ, leaving N invariant (Prop. 3.1 of Sec. 3), leaves invariant also the traces I_k and the open subset M', since

$$(4.7) \qquad L_\varphi(I_k) = \frac{1}{k} L_\varphi(\mathrm{Tr}\, N^k) = \mathrm{Tr}\left(N^{k-1} L_\varphi(N)\right) = 0$$

so that the rank of J is constant along the orbits of φ. Assume $M' \neq \emptyset$, and take the restriction of φ to M'. Such field is Hamiltonian and it admits the n independent involutive integrals $(I_1, \ldots, I_n)\big|_{M'}$. Consequently, it is integrable in the sense of Arnold-Liouville [4]. Conversely, let such a system be given on a symplectic manifold (M, P^{-1}), and let $(I_1, \ldots, I_n)$ be its n independent integrals in involution.

[Without loss of generality,] let us assume that one of such integrals, say I_1, is the Hamiltonian of the given field. Moreover, let $S_c := \{ m \in M : I_\kappa = c_\kappa \}$ be the compact connected submanifold corresponding to the regular value $c = (c_1, \ldots, c_n)$ of the integrals. By the Arnold theorem, there exists an open tubular neighbourhood M' of S_c which is covered by a system of canonical coordinates $(I_\kappa, \theta_\kappa)$, whose first n coordinates are the integrals I_κ [11]. Set

(4.8) $$\omega_2 = \frac{1}{2} \sum_{1}^{n}{}_{\kappa} I_\kappa^2 \, d\theta_\kappa$$

and consider the presymplectic tensor Ω defined by

(4.9) $$\Omega = d\,\omega_2$$

Then, it is straightforward to show that Ω and P (the inverse of the symplectic tensor of M) give M' the structure of a $P\Omega$ manifold, since

(4.10) $$\Omega P \Omega = d\,\omega_3$$

where

(4.11) $$\omega_3 = \frac{1}{3} \sum_{1}^{n}{}_{\kappa} I_\kappa^3 \, d\theta_\kappa \; .$$

(Hence, $\Omega P \Omega$ is closed as is required by Def.2.1 of Sec.2). Moreover, the Nijenhuis tensor $N := P\Omega$ canonically associated with the $P\Omega$ structure admits the integrals I_κ as its eigenvalues and it is invariant with respect to φ, since

(4.12) $$L_\varphi(N) = L_\varphi(P) \, . \, \Omega + P L_\varphi(\Omega) = 0$$

being separately $L_\varphi(P) = 0$ (since φ is Hamiltonian) and $L_\varphi(\Omega) = 0$ (since Ω has been constructed by means of the integrals I_κ and of their conjugate variables θ_κ). Hence, φ is a fundamental field of the PN structure defined by P and N, on account of Prop. 3.1 and of the lack of kernel of P. Given any integrable Hamiltonian vector field (in the sense of Arnold-Liouville), we have thus succeeded in constructing a PN structure of maximal rank of which the given field turns out to be a fundamental field. Therefore, we can state :

Proposition 4.1 (Integrable Hamiltonian systems geometrically characterized)

Any fundamental field of a PN manifold of maximal rank is an integrable Hamiltonian field in the open subset M' of the regular points. Its integrals in involution are the traces

$$I_\kappa(m) := \frac{1}{\kappa} \operatorname{Tr}(N_m)^\kappa \tag{4.13}$$

of the powers of the Nijenhuis tensor. Conversely, given ~~any~~ integrable Hamiltonian vector field (in the sense of Arnold-Liouville), it is possible to find, in the open tubular neighbourhood M' of the regular level surface $S_c = \{ m \in M : I_\kappa = c_\kappa \}$ whose existence is assured by the Arnold's theorem, a PN structure of maximal rank of which the given field turns out to be a fundamental field. This structure is defined by the Poisson tensor P (inverse of the symplectic tensor of the manifold) and by the presymplectic tensor Ω given by

(4.14) $$\Omega = d\omega$$

where

(4.15) $$\omega = \frac{1}{2} \sum_{1}^{n}{}_{\kappa} I_{\kappa}^{2} d\theta_{\kappa}$$

In (4.15), the θ_κ are the functions canonically conjugated to the integrals I_κ in M'. Once more, the traces of the Nijenhuis tensor and of its powers turn out to be integrals in involutions of the given system. In short: the integrable Hamiltonian vector fields coincide with the fundamental fields of PN manifolds of maximal rank.

Although this result may be of not great relevance from a practical point of view, since the construction of the PN structure is a problem of the same difficulty as the explicit integration of the given field, what seems promising is that the integrability condition has been reduced to a condition on the degeneracy of the spectrum of the Nijenhuis tensor. In this form, it keeps its meaning also for infinite-dimensional manifolds. Indeed, specific examples seem to support the validity of this criterion also in the context of infinite dimensional-manifolds [11,13]. If this conjecture could be proved to be correct, there would be a workable criterion replacing the classical Liouville condition.

5. The spectral problem

By pushing the analysis of the spectral problem for N a little more forward, it is possible to point out an integration method of the nonlinear equations

$$(5.1) \qquad \dot{m}(t) = \varphi(m(t))$$

associated with the fundamental fields, bearing strong analogies with the well-known inverse scattering theory. Indeed, one can show that the solution of (5.1) can be reduced to the solution of the linear eigenvalue problem

$$(5.2) \qquad N^{*}\alpha = \lambda\alpha$$

for the dual Nijenhuis tensor N^{*}.

Assume that N has everywhere n distinct eigenvalues, and set, for simplicity, $M' = M$. The eigenspaces of N at different points of M define n distinct two-dimensional distributions. Let φ and ψ be two vector fields spanning the eigenspace associated with the same eigenvalue λ. Then from the identity

$$\begin{aligned}(5.3) \quad T(N)(\varphi,\psi) :&= N^2[\varphi,\psi] - N[N\varphi,\psi] - N[\varphi,N\psi] + [N\varphi,N\psi] \\ &= N^2[\varphi,\psi] - N[\lambda\varphi,\psi] - N[\varphi,\lambda\psi] + [\lambda\varphi,\lambda\psi] \\ &= (N-\lambda I)^2[\varphi,\psi]\end{aligned}$$

and from the Nijenhuis condition, we get

$$(5.4) \qquad (N-\lambda I)^2 \cdot [\varphi,\psi] = 0$$

or

$$(5.5) \qquad (N-\lambda I)[\varphi,\psi] = 0$$

since N is a semisimple tensor (on account of the maximality assumption, N has two distinct eigenvectors for any eigenvalue λ). Eq.(5.5) shows that the commutator $[\varphi,\psi]$ belongs itself to the distribution spanned by φ and ψ. If φ and ψ belong to two eigenspaces corresponding to different eigenvalues λ and μ, from the Nijenhuis condition we obtain

$$(5.6) \qquad (N-\lambda I)^2 (N-\mu I)^2 \cdot [\varphi,\psi] = 0$$

showing that $[\varphi,\psi]$ belongs to the four-dimensional distribution spanned by the eigenvectors associated either with λ or μ. So, by the Frobenius theorem, around any point $m \in M$ there exists a local chart of coordinates $(x^1 y^1, x^2 y^2, \ldots, x^n y^n)$ such that the vector fields $\xi_i = \frac{\partial}{\partial x^i}$ and $\eta_i = \frac{\partial}{\partial y^i}$ span the eigenspace associated with the eigenvalue λ_i, for any $i = 1,2,\ldots,n$. In this system of coordinates, the eigenvalues λ_i depend only on the coordinates spanning the corresponding eigenspace :

$$(5.7) \qquad \lambda_i = \lambda_i(x_i y_i)$$

and the components (X^ℓ, Y^ℓ) of any fundamental field fulfil the constraints

$$(5.8) \qquad \frac{\partial X^\ell}{\partial x^j} = \frac{\partial X^\ell}{\partial y^j} = \frac{\partial Y^\ell}{\partial x^j} = \frac{\partial Y^\ell}{\partial y^j} = 0 \qquad (j \neq \ell)$$

Eq.(5.7) can be proved by using the identity

$$(5.9)\quad N^2[\xi_j,\xi_\ell] - N[N\xi_j,\xi_\ell] - N[\xi_j,N\xi_\ell] + [N\xi_j,N\xi_\ell]$$

$$= -N[\lambda_j\xi_j,\xi_\ell] - N[\xi_j,\lambda_\ell\xi_\ell] + [\lambda_j\xi_j,\lambda_\ell\xi_\ell]$$

$$= (\lambda_\ell-\lambda_j)\left(\frac{\partial\lambda_j}{\partial x^\ell}\xi_j + \frac{\partial\lambda_\ell}{\partial x^j}\xi_\ell\right)$$

and the companions with different basis vectors, and (5.8) readily follows from the invariance condition $L_\varphi(N) = 0$. Therefore, the system (5.1) immediately splits into the n decoupled second-order systems

$$(5.10)\qquad \dot{x}^\ell = X^\ell(x^\ell,y^\ell) \qquad \dot{y}^\ell = Y^\ell(x^\ell,y^\ell)$$

which are solved by using the integrals (5.7).

The problem is thus to find the coordinates (x^i,y^i). To this end, let us consider the dual eigenvalue problem

$$(5.11)\qquad N^*\alpha = \lambda\alpha$$

A careful analysis shows that :

(i) the gradients of the eigenvalues λ_k are eigenvectors of N^*

$$(5.12)\qquad N^*\cdot d\lambda_k = \lambda\, d\lambda_k$$

(this is merely a different form of the recursion relation (4.2)).

(ii) in the neighbourhood of any point $m \in M$, there exist other n functions $(\mu_1,\dots,\mu_n)$ whose gradients give the remaining eigenvectors

$$N^{*} \circ d\mu_{\kappa} = \lambda_{\kappa}\, d\mu_{\kappa} \tag{5.13}$$

(iii) the functions (λ,μ) obey the canonical commutation relations

$$\{\lambda_j,\lambda_\ell\}=0 \qquad \{\lambda_j,\mu_\ell\}=\delta_{j\ell} \qquad \{\mu_j,\mu_\ell\}=0 \tag{5.14}$$

This means that, in the neighbourhood of every point $m \in M$, there exists a system of coordinates (λ,μ) which not only span the eigenspaces of N but also are canonical. They will give the nonlinear equations (5.1) the simple form

$$\dot{\lambda}_\ell = 0 \qquad\qquad \dot{\mu}_\ell = \Phi_\ell(\lambda_\ell,\mu_\ell) \tag{5.15}$$

So, we perceive the following integration method. Suppose that the linear problem (5.11) has been completely solved, so that the spectral functions λ^ℓ and μ^ℓ are known as functions of any local set of coordinates ξ^α on M :

$$\lambda^\ell = \lambda^\ell(\xi^\alpha) \qquad\qquad \mu^\ell = \mu^\ell(\xi^\alpha) \;. \tag{5.16}$$

Once the spectral functions have been evaluated at the initial time

$$\lambda_o = \lambda(\xi(t_o)) \qquad \mu_o = \mu(\xi(t_o)) \tag{5.17}$$

we let them evolve according to (5.15), up to the time t. Then the "inverse spectral transform"

$$\xi^\alpha(t) = \xi^\alpha(\lambda(t),\mu(t)) \tag{5.18}$$

gives the solution of the Eq.(5.1) we were looking for.

6. Reduction theory: the restriction problem.

Having explained the reasons compelling us to study PN manifolds, we turn toward the constructive aspects of the theory. Schematically, the open questions which confront us are:

i) to construct explicitly PN manifolds
ii) to select PN manifolds of maximal rank
iii) to encompass known examples into the theory.

In this second part of the paper, we shall try to answer the second question,at least so to develop the basic technique, the reduction technique, which seems essential to deal with this problem. The study of the local structure of the PN manifolds leads to identify a certain number of integrable distributions, defined directly in terms of the tensors P and N. Among the submanifolds defined by these distributions (either immersed submanifolds or quotient submanifolds), someones are recognized as irreducible, in the sense that no further integrable distribution splits them in lower dimensional components. On the ground of the experience, it seems that the PN manifold of maximal rank, leading to integrable Hamiltonian systems, are to be looked for among these irreducible submanifolds. However, we do not know any sound theoretical explanation of this fact. We only point out the existence of these submanifolds, by giving a systematic procedure for finding them and for recovering their PN structure.

Let M be a $P\Omega$ or a PN manifold. The first problem of the reduction theory (the restriction problem) is to find the immersed submanifolds S inheriting a $P\Omega$ or a PN structure from M, as well as any submanifold of a Euclidean space inherits a Riemannian structure from the ambient space. Its complete solution will be given in Secs 8 and 9, for PΩ and PN manifolds respectively. In this section we deal with the preliminary problem of defining the restriction process and of finding the conditions under which it takes place. To this end, we introduce a parametrization of the submanifold S. It consists in a pair $(M', f : M' \longrightarrow M)$, formed by a second manifold M' (the parameter space) and by an injective immersion $f : M' \longrightarrow M$ such that $f(M') = S$. This map connects the vector fields and the one-forms defined on M' with those defined on S, according to the well-known relations

$$(6.1) \qquad \varphi(f(m')) = df(m') \circ \varphi(m')$$

$$(6.2) \qquad \alpha'(m') = \delta f(m') \circ \alpha(f(m'))$$

Thus it defines two maps, hereafter denoted by $df : \mathcal{X}(M') \rightarrow \mathcal{X}(S,M)$ and by $\delta f : \mathcal{X}^*(S,M) \rightarrow \mathcal{X}^*(M')$, between $\mathcal{X}(M')$ and $\mathcal{X}^*(M')$ and the linear spaces $\mathcal{X}(S,M)$ and $\mathcal{X}^*(S,M)$ of the restrictions to S of the vector fields and one-forms defined on M. We remark that df is an injective map, whose image is the algebra $\mathcal{X}(S)$ of the vector fields tangent to S, and that δf is a surjective map, whose kernel is the annihilator $\mathcal{X}(S)^o$ of $\mathcal{X}(S)$.

A process of restriction to S may then be defined as any prescription allowing to associate, by means of df and δf, tensors defined on $\mathcal{X}(M')$ or $\mathcal{X}^*(M')$ with tensors defined on $\mathcal{X}(S,M)$ or $\mathcal{X}^*(S,M)$, in such a way to maintain their properties. Different prescriptions, defining different processes of reduction, are possible (see Sec.10). According to the simplest one, considered in this section, we associate the tensor

$$(6.3) \qquad \Omega' := \delta f \circ \Omega \circ df$$

with any tensor Ω of type (0,2), and the tensor

(6.4) $\quad N' := df^{-1} \,.\, N \,.\, df$

with any tensor N of type (1,1) fulfilling the conditions

(6.5) $\quad N(\mathfrak{X}(S)) \subset \mathfrak{X}(S)$

(see Fig.s 1a-1b). As for the reduction of the tensor P of type (2,0), one has to consider the linear subspace

(6.6) $\quad \mathfrak{X}^*_P(S) := \left\{ \alpha \in \mathfrak{X}^*(S,M) : P\alpha \in \mathfrak{X}(S) \right\}$

of the one-forms defined on S and mapped by P into vector fields tangent to S. If it fulfils the conditions

(6.7) $\quad \mathfrak{X}^*_P(S) + \mathfrak{X}(S)^o = \mathfrak{X}^*(S,M)$

(6.8) $\quad \mathfrak{X}^*_P(S) \cap \mathfrak{X}(S)^o \subset \operatorname{Ker} P$

the restriction of δf to $\mathfrak{X}^*_P(S)$ is a surjective map onto $\mathfrak{X}^*(M')$, and its kernel is contained into the kernel of P, as is proved by

(6.9)
$$\begin{aligned} \delta f(\mathfrak{X}^*_P(S)) &= \delta f(\mathfrak{X}^*_P(S) + \mathfrak{X}(S)^o) \\ &\overset{(6.7)}{=} \delta f(\mathfrak{X}^*(S,M)) \\ &= \mathfrak{X}^*(M') \end{aligned}$$

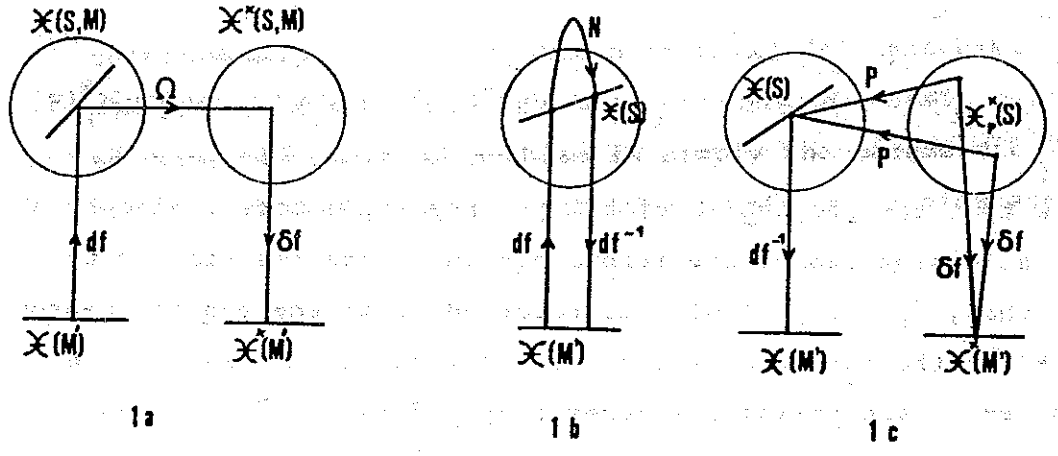

Fig.1 The restriction of Ω,P,N

and by

$$(6.10)\qquad \mathrm{Ker}\,\delta f \cap \mathfrak{X}^*_P(S) = \mathfrak{X}(S)^o \cap \mathfrak{X}^*_P(S) \overset{(6.8)}{\subset} \mathrm{Ker}\,P\,.$$

Therefore, all the one-forms $\alpha \in \mathfrak{X}^*_P(S)$ corresponding to the same one-form $\alpha' \in \mathfrak{X}^*(M')$, $\alpha' = \delta f \circ \alpha$, are mapped by P into the same vector field tangent to S (see Fig.1c). Consequently, the tensor

$$(6.11)\qquad P' := df^{-1} \circ P \circ \delta f \Big|^{-1}_{\mathfrak{X}^*_P(S)}$$

is well-defined, whatever right-inverse $\delta f \Big|^{-1}_{\mathfrak{X}^*_P(S)}$ of $\delta f \Big|_{\mathfrak{X}^*_P(S)}$ may be used.

Eq.s (6.3), (6.4) and (6.11) define the process of restriction used in this paper, and (6.5), (6.7) and (6,8) give the conditions under which it can take place. Furthermore, the readily proved identities [C.2]

$$(6.12)\qquad d\Omega'(\varphi',\psi') = \delta f \circ d\Omega\,(df\,.\varphi',\ df\,.\psi')$$

$$(6.13)\qquad T(N')(\varphi',\psi') = df^{-1} \circ T(N)\,(df\,.\varphi',\ df\,.\psi')$$

$$(6.14)\qquad [P',\, P'](\delta f\,.\alpha,\, \delta f\,.\beta) = df^{-1}.[P, P](\alpha,\beta) \qquad \alpha,\beta \in \mathfrak{X}^*_P(S)$$

show that the properties of being a presymplectic, a Nijenhuis and a Poisson tensor are maintained under the restriction to S. For that, Ω', N', P' will be referred to as the reduced tensors on M' of the corresponding tensors defined on M. As for the problem of ascertaining whether they define a PΩ or a PN structure on M', we observe that it is [C.2]

$$(6.15)\qquad (N'P'-P'N'^{*}) \,.\, \delta f \,.\, \alpha = df^{-1} \,.\, (NP - PN^{*}) \,.\, \alpha$$

$$(6.16)\qquad R(P', N')(\delta f.\alpha, \varphi') = df^{-1}.\, R(P,N)(\alpha, df\, .\varphi') \qquad \alpha \in \mathcal{X}^{*}_{P}(S)$$

and that

$$(6.17)\qquad d(\Omega' P'\Omega')(\varphi', \psi') = \delta f \,.\, d(\Omega P\Omega)\ (df\, .\varphi', df\, .\psi')$$

if the further condition

$$(6.18)\qquad \Omega\,(\mathcal{X}(S)) \subset \mathcal{X}^{*}_{P}(S)$$

is assumed. Thus we can conclude that also the property of being a PN or a $P\Omega$ manifold is maintained under restriction. So, we have the following Lemmas (the $P\Omega$ case being kept distinct from the PN case for further reference) :

Lemma 6.1 (Restriction lemma for PN manifolds). Let S be any immersed submanifold of a PN manifold M. Assume that

$$(R1)\qquad N(\mathcal{X}(S)) \subset \mathcal{X}(S)$$

$$(R2)\qquad \mathcal{X}^{*}_{P}(S) + \mathcal{X}(S)^{o} = \mathcal{X}^{*}(S,M)$$

$$(R3)\qquad \mathcal{X}^{*}_{P}(S) \cap \mathcal{X}(S)^{o} \subset \mathrm{Ker}\ P$$

Then S inherits from M a PN structure defined by the tensors

$$(R4)\qquad N' := df^{-1} \,.\, N \,.\, df$$

$$(R5)\qquad P' := df^{-1} \,.\, P \,.\, \delta f \Big|^{-1}_{\mathcal{X}^{*}_{P}(S)}$$

where $(M', f : M' \longrightarrow M)$ is any parametrization of S.

<u>Lemma 6.2 (Restriction lemma for $P\Omega$ manifolds).</u> Let S be any immersed submanifold of a $P\Omega$ manifold M. Assume that

(R6) $\qquad \Omega(\mathcal{X}(S)) \subset \mathcal{X}^*_P(S)$

(R7) $\qquad \mathcal{X}^*_P(S) + \mathcal{X}(S)^o = \mathcal{X}^*(S,M)$

(R8) $\qquad \mathcal{X}^*_p(S) \cap \mathcal{X}(S)^o \subset \operatorname{Ker} P$

Then, S inherits from M a $P\Omega$ structure defined by the tensors

(R9) $\qquad \Omega' := \delta f . \Omega . df$

(R10) $\qquad P' := df^{-1} . P . \delta f \Big|^{-1}_{\mathcal{X}^*_P(S)}$

<u>Remark.</u> The conditions of the lemma (6.1) may be slightly narrowed, if one observes that it is really unnecessary to consider the <u>whole</u> subspace $\mathcal{X}^*_P(S)$ of the one-forms mapped by P into $\mathcal{X}(S)$. It would suffice to consider a subspace $V \subset \mathcal{X}^*_P(S)$ fulfilling ——— the same conditions (R2)(R,3) of $\mathcal{X}^*_P(S)$. So, the lemma could be stated as follows : Let S be a submanifold of a Poisson manifold M and let V be a subspace of $\mathcal{X}^*_P(S)$ such that :

(6.19) $\qquad V + \mathcal{X}(S)^o = \mathcal{X}^*(S,M)$

(6.20) $\qquad V \cap \mathcal{X}(S)^o \subset \operatorname{Ker} P$

Then S inherits a Poisson structure from M defined by the tensor

(6.21) $$P' := df^{-1} \cdot P \quad \delta f\big|_{V}^{-1}$$

The same remark holds for lemma 6.2. However, although these generalizations are of interest in some cases, we have not to use them in this paper and so we skip over their discussion.

7. Reduction theory : the projection problem

The second problem of the reduction theory (the projection problem) is to find the integrable distributions D of M admitting a quotient manifold M/D which inherits a PΩ or a PN structure from M. To specify this problem, we need the concepts of projectable vector fields and of projectable one-forms. Let $(M', g : M \longrightarrow M')$ be a parametrization of M/D , formed by a second manifold M' and by a surjective sybmersion $g : M \longrightarrow M'$ such that the leaves of the distribution are the fibers of the submersion. We recall that a vector field $\varphi \in \mathfrak{X}(M)$ is said to be projectable if

$$(7.1) \qquad dg(m_1) \cdot \varphi(m_1) = dg(m_2) \cdot \varphi(m_2)$$

for any pair of points m_1 and m_2 belonging to the same leave [14]. If such leaves are connected, this condition may be replaced by [C.1]

$$(7.2) \qquad [\varphi, \xi] = \xi'$$

where ξ and ξ' are vector fields spanning D. Thus, a vector field φ is projectable if its Lie derivative along any vector field ξ tangent to the leaves is itself a vector tangent to the leaves. Such a field defines a unique vector field $\varphi' \in \mathfrak{X}(M')$, given by

$$(7.3) \qquad \varphi'(g(m)) \;:=\; dg(m) \cdot \varphi(m) \quad ,$$

which is called the projection of φ on M'. Likewise, a one-form $\alpha \in \mathfrak{X}^*(M)$ may be said to be projectable if there exists a one-form $\alpha' \in \mathfrak{X}^*(M')$ such that at any point $m \in M$

$$(7.4) \qquad \alpha(m) = \delta g(m) \,.\, \alpha'(g(m))$$

If the leaves of the distribution are connected, this happens only if [C.1]

$$(7.5) \qquad \langle \alpha , \xi \rangle = 0$$

$$(7.6) \qquad L_\xi(\alpha) = 0$$

Thus α is projectable if it is constant along the leaves and if it annihilates their tangent spaces.

Let $\mathfrak{X}_D(M)$ and $\mathfrak{X}^*_D(M)$ be the linear subspaces of the projectable vector fields and one-forms, defined by (7.2) and (7.5)-(7.6) respectively, and let $dg : \mathfrak{X}_D(M) \longrightarrow \mathfrak{X}(M')$ and $\delta g : \mathfrak{X}^*(M') \longrightarrow \mathfrak{X}^*(M)$ be the linear mappings defined by (7.3) and (7.4). We remark that dg is a surjective map whose kernel is the subalgebra Z_D of the vector fields spanning D, and that δg is an injective map whose image is $\mathfrak{X}^*_D(M)$. Then, the tensors P and N, of type (2,0) and (1,1) respectively, are said to be projectable on M' if P maps projectable one-forms into projectable vector fields

$$(7.7) \qquad P(\mathfrak{X}^*_D(M)) \subset \mathfrak{X}_D(M)$$

and if N^* maps the projectable one-forms into themselves

$$(7.8) \qquad N^*(\mathfrak{X}^*_D(M)) \subset \mathfrak{X}^*_D(M)$$

In this case, we say that the tensors

$$(7.9) \qquad P' := dg \cdot P \cdot \delta g$$

$$(7.10) \qquad N'^{*} := \delta g^{-1} \cdot N^{*} \cdot \delta g$$

are the reduced tensor of P and N^* on M' (see Fig.s 2a-2b). As for a tensor Ω of type (0,2), one readily proves (exactly as in Sec.6) that if the linear subspace

$$(7.11) \qquad \mathfrak{X}_{\Omega}(M) := \left\{ \varphi \in \mathfrak{X}_D(M) : \quad \Omega \cdot \varphi \in \mathfrak{X}^*_D(M) \right\}$$

of the projectable vector fields mapped by Ω into projectable one-forms fulfils the conditions

$$(7.12) \qquad \mathfrak{X}_{\Omega}(M) + Z_D = \mathfrak{X}_D(M)$$

$$(7.13) \qquad \mathfrak{X}_{\Omega}(M) \cap Z_D \subset \operatorname{Ker}\Omega$$

then the tensor (see Fig.2c)

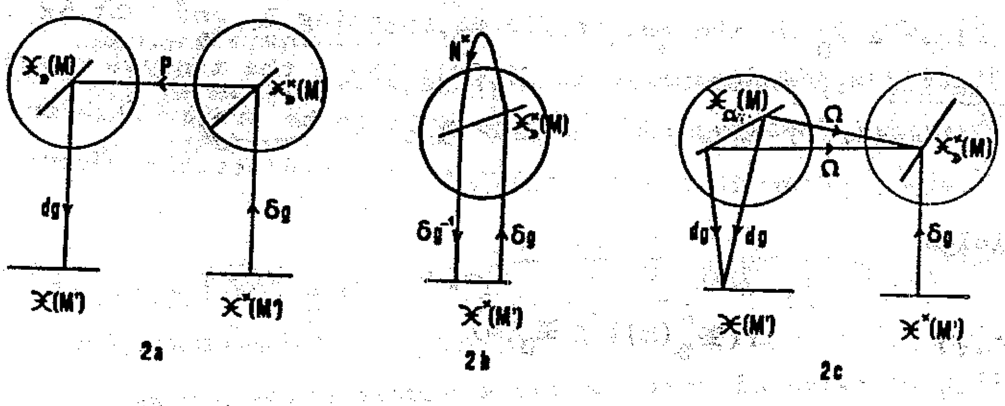

Fig.2 The projection of P, N^*, Ω

$$(7.14) \qquad \Omega' := \delta g^{-1} \cdot \Omega \cdot dg \Big|^{-1}_{\mathfrak{X}_{\Omega}(M)}$$

is well-defined, whatever right-inverse $dg \Big|^{-1}_{\mathfrak{X}_{\Omega}(M)}$ of $dg \Big|_{\mathfrak{X}_{\Omega}(M)}$ may be used. It will be referred to as the reduced tensor of Ω on M'.

Eq.s (7.9), (7.10) and (7.14) define the process of projection used in this paper, and (7.7), (7.8), (7.12) and (7.13) give the conditions under which it can take place. Furthermore, the readily proved identities [C.2]

$$(7.15) \qquad [\bar{P}', P'](\alpha', \beta') = dg \cdot [\bar{P}, P](\delta g \cdot \alpha', \delta g \cdot \beta')$$

$$(7.16) \qquad T(N')(dg \cdot \varphi, dg \cdot \psi) = dg \cdot T(N)(\varphi, \psi) \qquad \varphi, \psi \in \mathfrak{X}_D(M)$$

$$(7.17) \qquad d\Omega'(dg \cdot \varphi, dg \cdot \psi) = \delta g^{-1} \cdot d\Omega(\varphi, \psi) \qquad \varphi, \psi \in \mathfrak{X}_D(M)$$

show that the properties of being a Poisson, a Nijenhuis and a presymplectic tensor are maintained under projection. As for the coupling conditions between the reduced tensors, the identities [C.2]

$$(7.18) \qquad (N'P' - P'N'^*) \cdot \alpha' = dg \cdot (NP - PN^*)(\delta g \cdot \alpha')$$

$$(7.19) \qquad R(P', N')(\alpha', dg \cdot \varphi) = dg \cdot R(P, N)(\delta g \cdot \alpha', \varphi) \quad \varphi \in \mathfrak{X}_D(M)$$

and

$$(7.20) \qquad d(\Omega' P' \Omega')(dg \cdot \varphi, dg \cdot \psi) = \delta g^{-1} \cdot d(\Omega P \Omega)(\varphi, \psi) \quad \varphi, \psi \in \mathfrak{X}_D(M)$$

holding under the further assumption

$$(7.21) \qquad P(\mathfrak{X}^*_D(M)) \subset \mathfrak{X}_\Omega(M)$$

show that also the property of being a PN or a PΩ manifold is maintained under projection. So, we have the following lemmas :

<u>Lemma 7.1 (Projection lemma for PN manifolds).</u> Let D be any integrable distribution of a PN manifold M. Assume its leaves to be connected, the quotient space M/D to be a quotient manifold M', and the canonical projection $g : M \longrightarrow M'$ to be a surjective submersion. Then, if

(P1) $P(\mathcal{X}^*_D(M)) \subset \mathcal{X}_D(M)$

(P2) $N^*(\mathcal{X}^*_D(M)) \subset \mathcal{X}^*_D(M)$

M' inherits from M a PN structure defined by the tensors

(P3) $P' := dg \,.\, P \,.\, \delta g$

(P4) $N'^* = \delta g^{-1} \,.\, N^* \,.\, \delta g$

<u>Lemma 7.2 (Projection lemma for PΩ manifolds).</u> Let D be any integrable distribution of a PΩ manifold, obeying the same standard conditions as in the previous lemma. Then, if

(P5) $P(\mathcal{X}^*_D(M)) \subset \mathcal{X}_\Omega(M)$

(P6) $\mathcal{X}_\Omega(M) + Z_D = \mathcal{X}_D(M)$

(P7) $\mathcal{X}_\Omega(M) \cap Z_D \subset \operatorname{Ker}\Omega$

M' inherits from M a PΩ structure defined by the tensors

(P8) $P' := dg \,.\, P \,.\, \delta g$

(P9) $\Omega' := \delta g^{-1} \,.\Omega \,.\, dg\Big|^{-1}_{\mathcal{X}_\Omega(M)}$

8. The reduction of $P\Omega$ manifolds

The technique for reducing the $P\Omega$ and the PN manifolds consists in a suitable sequence of restrictions and projections, aiming to eliminate the kernels of P, Ω and N. For the success of the technique, it is essential that the images and the kernels of P and Ω fulfil the conditions

$$\text{(8.1)} \qquad \text{Ker } P = (\text{Im } P)^{o}$$

$$\text{(8.2)} \qquad \text{Im } \Omega = (\text{Ker } \Omega)^{o}$$

and that there exists a minimal finite integer r, called the Riesz index of the tensor N, such that simultaneously

$$\text{(8.3)} \qquad \text{Im } N^{r+1} = \text{Im } N^{r} \qquad \text{Ker } N^{r+1} = \text{Ker } N^{r}$$

As is known, (8.1) follows from the skewsymmetry of P, while Eq.s (8.2) and (8.3) may fail for infinite-dimensional manifolds, where, a priori, one simply has

$$\text{(8.4)} \qquad \text{Im } \Omega \subset (\text{Ker } \Omega)^{o}$$

and where the index r may become infinite or to be different for the sequence of the kernels with respect to the sequence of the images [15]. Without further probing these questions, we simply assume these conditions to be fulfilled by the tensors Ω and N to be considered below.

So, let us consider firstly a $P\Omega$ manifold, and let S be any characteristic leaf of P, that is any connected maximal integral submanifold of the distribution spanned by the vector fields $\varphi_\alpha = P\alpha$, $\alpha \in \mathfrak{X}^*(M)$ (the integrability of this distribution is a well-known consequence of the condition $[\bar{P}, \underline{P}] = 0$ and of the constancy of the rank of P). In this case

(8.5) $\mathfrak{X}(S) = \mathrm{Im}\ P$

(8.6) $\mathfrak{X}(S)^o \overset{(\underline{8.1})}{=} \mathrm{Ker}\ P$

(8.7) $\mathfrak{X}^*_P(S) = \mathfrak{X}^*(S,M)$

so that the conditions (R.6), (R.7), (R.8) of the Restriction Lemma 6.2 are trivially verified. Consequently, S inherits a $P\Omega$ structure from M. If $(M', f : M' \longrightarrow M)$ is any parametrization of S, the reduced structure is defined by the tensors Ω' and P' given by (R.9) and (R.10). In particular, P' is kernel-free, since $\mathrm{Ker}\ P = \mathrm{Ker}\ \delta f$, so that by the restriction on the characteristic leaf we have get rid of the kernel of P.

To further reduce the $P\Omega$ structure defined on M', let us consider the <u>characteristic distribution of</u> Ω', spanned by the vector fields $\xi' \in \mathrm{Ker}\ \Omega'$ (its integrability is a well-known consequence of the condition $d\Omega' = 0$ and of the constancy of the rank of Ω'). Assume its leaves to be connected, the quotient space $M'/\mathrm{Ker}\ \Omega'$ to be a quotient manifold M'', and the canonical projection $g : M' \longrightarrow M''$ to be a surjective submersion, according to the standard assumptions of the Projection Lemma 7.2. To verify that the conditions (P.5), (P.6), (P.7), allowing the reduction process on the quotient manifold M'' to take place, we remark that

(8.8) $\qquad Z_D = \mathrm{Ker}\,\Omega'$

and that the projectable vector fields are characterized by the condition

(8.9) $\qquad \Omega'[\varphi', \xi'] = 0$

Hence, we deduce that

(8.10) $\qquad \Omega'(\mathfrak{X}_D(M')) \subset \mathfrak{X}^*_D(M')$

by setting $\alpha' = \Omega'\varphi'$ and by using the identities

(8.11) $\qquad \langle \alpha', \xi' \rangle = \langle \Omega'\varphi', \xi' \rangle = -\langle \Omega'\cdot\xi', \varphi' \rangle \overset{(8.8)}{=} 0$

(8.12) $$L_{\xi'}(\alpha') = L_{\xi'}(\Omega'\varphi')$$

$$\overset{[A.1]}{=} d\Omega'(\xi', \varphi') + L_{\varphi'}(\Omega'\xi') - \Omega' L_{\varphi'}(\xi') + d\langle \Omega'\varphi', \xi' \rangle$$

$$\overset{(8.8)}{=} + \Omega'[\varphi', \xi']$$

$$\overset{(8.9)}{=} 0$$

This proves that $\mathfrak{X}_{\Omega'}(M') = \mathfrak{X}_D(M')$, so that Ω' verifies the conditions (P6) and (P7) of the Lemma 7.2. To further prove that P obeys the condition (P5), remark that the assumption (8.2) allows to replace (8.10) by the stronger relation

(8.13) $\qquad \Omega'(\mathfrak{X}_D(M')) = \mathfrak{X}^*_D(M')$

(indeed, (8.2) allows to represent any form α' such that $\langle\alpha', \xi'\rangle = 0$ as $\alpha' = \Omega'\varphi'$, for some $\varphi' \in \mathfrak{X}(M')$; (8.13) follows then from (8.12)). Hence, we deduce

$$
\begin{aligned}
(8.14)\quad \Omega' \cdot L_{\xi'}(P'\alpha') &= \Omega' \cdot L_{\xi'}(P'\Omega'\varphi') \\
&= L_{\xi'}(\Omega' P'\Omega'\varphi') - L_{\xi'}(\Omega') \cdot P'\Omega'\varphi' \\
&\overset{[8.1]}{=} d(\Omega' P'\Omega')(\varphi', \xi') - \Omega' P'\Omega' \cdot L_{\xi'}(\varphi') + d\Omega'(P'\Omega'\varphi', \xi') \\
&\overset{(8.9)}{=} d(\Omega' P'\Omega')(\varphi', \xi') \\
&= 0
\end{aligned}
$$

showing that $P'\alpha'$ is a projectable vector field whenever α' is a projectable one-form, as it was required. Therefore, we conclude that M'' inherits a $P\Omega$ structure from M', defined by tensors Ω'' and P'' given by (P.8) and (P.9), and that Ω'' is kernel-free, since Ker dg = Ker Ω'. This does not imply, however, that P'' be kernel-free. In fact, P'' might have acquired a kernel in the passage from M' to M''. If this is the case, we iterate the process: first, we perform a restriction on a characteristic leaf of P'', and then we project the characterisric distribution of Ω''', and so on, up to arrive to a kernel-free $P\Omega$ structure. In the next section, it will be shown that this iterative process actually stops after r steps, on account of condition (8.3). Summarizing, we can state the following Proposition :

<u>Proposition 8.1 (Reduction of $P\Omega$ manifolds)</u>. Let M be a $P\Omega$ manifold and assume P to have a kernel. Then any characteristic leaf of P, parametrized by $(M', f : M' \longrightarrow M)$, inherits a reduced $P\Omega$ structure from M, defined by tensors P' and Ω' given by (R.9) and (R.10). In particular, P' is kernel-free. Furthermore, suppose Ω' to have a kernel, and assume that it fulfils the condition (8.2). Assume the leaves of the characteristic distribution of Ω' to be connected, the quotient space $M'/\mathrm{Ker}\,\Omega'$ to be a quotient manifold M'', and the canonical projection $g : M' \longrightarrow M''$ to be a surjective submersion. Then, M'' inherits a $P\Omega$ structure from M', defined by tensors P'' and Ω'' given by (P.8) and (P.9). In particular, Ω'' is kernel-free. If also P'' is kernel-free, the reduction process stops. Otherwise, the process may be iterated up to arrive to a kernel-free $P\Omega$ structure.

9. The reduction of PN manifolds

Let M be a PN manifold (not necessarily a $P\Omega$ manifold). As well as for the $P\Omega$ manifolds, the first reduction is over the characteristic leaves of P. Indeed, by the coupling condition $PN^* = NP$, the characteristic distribution of P is invariant with respect to N

$$N(\varkappa(S)) \subset \varkappa(S) \tag{9.1}$$

so that N can be reduced according to the prescriptions of lemma (6.1). Each characteristic leaf of P is thus a regular PN manifold, whose structure is defined by tensors P' and N' (with P' kernel-free), given by Eq.s (R.4) and (R.5). Further on, we shall get rid of the apices to simplify the notation, and we shall denote the reduced tensors simply by P and N. Accordingly, the reduced manifold will be once more denoted by M, as though the first reduction had never been performed (nevertheless, one has to recall that P is now kernel-free).

To pursue the reduction, assume N to obey the condition (8.3), and consider the distributions spanned by the vector fields belonging to Im N^r and to Ker N^r. They will be referred to as the characteristic distributions of N (the active and the null distributions respectively, if it is required to distinguish them). In view of the regularity of P, these distributions coincide with the characteristic distributions of the tensors

$$P_r := N^r \cdot P \tag{9.2}$$

$$\Omega_r := P^{-1} \cdot N^r \tag{9.3}$$

that is

$$(9.4) \qquad \mathrm{Im}\, N^{r} = \mathrm{Im}\, P_{r} \qquad \mathrm{Ker}\, N^{r} = \mathrm{Ker}\, \Omega_{r}$$

So, they are integrable, since P_r is a Poisson tensor and Ω_r is a presymplectic tensor, by Prop.2.2 of Sec.2. Moreover, they are also transversal, on account of the following

Lemma 9.1 (Splitting lemma [15]). Let N be any tensor of type (1,1) on M, admitting a finite Riesz index r, in the sense of condition (8.3). Then, the subspace $\mathrm{Im}\, N^r$ and $\mathrm{Ker}\, N^r$ split the algebra $\mathcal{X}(M)$ into the direct sum

$$(9.5) \qquad \mathcal{X}(M) = \mathrm{Im}\, N^{r} \oplus \mathrm{Ker}\, N^{r}$$

Owing to this remarkable property, it is now easy to show that the regular PN structure of M may be reduced either by restriction on the leaves of the distribution $\mathrm{Im}\, N^r$, or by projection on the quotient manifold of the distribution $\mathrm{Ker}\, N^r$. In both cases, moreover, the reduced tensors turn out to be kernel-free. Indeed, in the case of the restriction, one has

$$(9.6) \qquad \mathcal{X}(S) = \mathrm{Im}\, N^{r} \qquad \mathcal{X}(S)^{o} = \mathrm{Ker}\, N^{*r} \qquad \mathcal{X}^{*}_{P}(S) = \mathrm{Im}\, N^{*r}$$

since the coupling condition

$$(9.7) \qquad N^{r} \cdot P = P \cdot N^{*r}$$

and the regularity of P entail

$$(9.8) \qquad P(\mathrm{Ker}\, N^{*r}) = \mathrm{Ker}\, N^{r} \qquad P(\mathrm{Im}\, N^{*r}) = \mathrm{Im}\, N^{r}$$

Consequently, conditions (R1), (R2) and (R3) are readily verified on account of the splitting lemma. Moreover, since

$$(9.9) \qquad \mathrm{Ker}\ N' \overset{(R.4)}{=} df^{-1}(\mathrm{Im}\ df \cap \mathrm{Ker}\ N)$$

and

$$(9.10) \qquad \mathrm{Ker}\ P' \overset{(R.5)}{=} \delta f(\mathrm{Ker}\,\delta f \cap \mathfrak{X}^*_P(S))$$

one can readily check that N' and P' are kernel-free, being

$$(9.11) \quad \mathrm{Im}\ df \cap \mathrm{Ker}\ N = \mathrm{Im}\ N^r \cap \mathrm{Ker} N \subset \mathrm{Im}\ N^r \cap \mathrm{Ker}\ N^r \overset{(9.5)}{=} \emptyset$$

$$(9.12) \quad \mathrm{Ker}\,\delta f \cap \mathfrak{X}^*_P(S) = \mathrm{Ker}\ N^{*r} \cap \mathfrak{X}^*_P(S) = \mathrm{Ker} N^{*r} \cap \mathrm{Im} N^{*r} \overset{(9.5)}{=} \emptyset$$

In the case of the projection, on the contrary, one has first to remark that any projectable one-form α, fulfilling the conditions

$$(9.13) \qquad \langle \alpha, \xi \rangle = 0 \qquad L_\xi(\alpha) = 0$$

for $\xi \in \mathrm{Ker}\ N^r$, belongs necessarily to $\mathrm{Im}\ N^{*r}$, so that

$$(9.14) \qquad \alpha = N^{*r} \cdot \beta$$

for some $\beta \in \mathfrak{X}^*(M)$. Thus one readily obtains

$$(9.15) \qquad \langle N^*\alpha, \xi \rangle = 0$$

and

$$(9.16) \qquad L_\xi(N^*\alpha) = N^*.L_\xi(\alpha) + L_\xi(N^*) \,.\, N^{*r}\beta$$

$$\overset{[B.2]}{=} L_{N^r\varphi}(N^*) \,.\, \beta$$

$$= 0$$

showing that

$$(9.17) \qquad N^*(\mathfrak{X}^*_D(M)) \subset \mathfrak{X}^*_D(M)$$

Likewise, one has

$$(9.18) \qquad N^r \,.\, L_\xi(P\alpha) = -L_{P\alpha}(N^r\xi) + L_{P\alpha}(N^r)\,.\,\xi$$

$$\overset{R(P,N)=0}{=} P \,.\, L_\xi(N^{*r}\alpha) - P.L_{N^r\xi}(\alpha)$$

$$= P \,.\, L_\xi(N^{*r}\alpha)$$

$$= 0$$

since, by (9.17), $N^{*r}\alpha$ is still a projectable one-form : thus, one proves that

$$(9.19) \qquad P(\mathfrak{X}^*_D(M)) \subset \mathfrak{X}_D(M)$$

Since the conditions (P.1) and (P.2) of the Projection Lemma (7.1) are fulfilled, the PN structure can be projected. Moreover, the reduced tensors P' and N' are both kernel-free. Indeed :

$$(9.20)\quad \mathrm{Ker}\ P' \overset{(P3)}{=} \delta g^{-1}(P^{-1}(\mathrm{Ker}\ N^r) \cap \mathrm{Im}\ N^{*r}) \overset{(9.8)}{=} \delta g^{-1}(\mathrm{Ker}\ N^{*r} \cap \mathrm{Im}\ N^{*r}) \overset{(9.5)}{=} \emptyset$$

and

$$(9.21)\quad \mathrm{Ker}\ N'^{*} \overset{(P4)}{=} \delta g^{-1}(\mathrm{Ker}\ N^{*} \cap \mathrm{Im}\ N^{*r}) \subset \delta g^{-1}(\mathrm{Ker}\ N^{*r} \cap \mathrm{Im}\ N^{*r}) \overset{(9.5)}{=} \emptyset$$

Summarizing, we can state the following

<u>Proposition 9.1 (Reduction of PN manifolds)</u>. Let M be a PN manifold and assume P to have a Kernel. Then any characteristic leaf of P, parametrized by $(M', f : M' \longrightarrow M)$, inherits a reduced PN structure from M, defined by tensors N' and P' given by (R4) and (R5). In particular P' is kernel-free. Suppose N' to have a finite Riesz index r', and let S' be any integral leaf of the active characteristic distribution $\mathrm{Im}\ N'^{r'}$, parametrized by $(M'', f': M'' \longrightarrow M')$. Then M'' is an irreducible PN manifold, whose structure is defined by kernel-free tensors N'' and P'', given once more by (R4) and (R5). Furthermore, assume the leaves of the null characteristic distribution $\mathrm{Ker}\ N'^{r'}$ to be connected, the quotient space $M'/\mathrm{Ker}\ N'^{r'}$ to be a quotient manifold M''', and the canonical projection $g' : M' \longrightarrow M'''$ to be a surjective submersion . Then, M''' inherits an irreducible PN structure from

M', defined by kernel-free tensors N'' and $P''{}'$ given by (P3) and (P4). In both cases, after two steps, the PN structure is completely reduced.

Usually, the last projection can be performed when the leaves of the characteristic distributions $\mathrm{Im}\, N'^{r'}$ and $\mathrm{Ker}\, N'^{r'}$ are not only locally but also globally transversal, so that M'' and $M''{}'$ are diffeomorphic manifolds. Consequently, also their PN structures are diffeomorphic. Nevertheless, this does not entail the equivalence of the two methods from a computational point of view. Usually, the projection turns out to be more profitable, for reasons which will become clear during the study of the applications. This was the motivation for developing side by side the restriction technique and the projection technique.

10. A digression

In this section, we discuss a different method of reduction for Nijenhuis tensors, whose knowledge is not required for the further developments of the theory, but which reveal itself profitable for the applications. It is included here in order to show a variant of the previous reduction techniques.

To enhance the algebraic essence of the method, it is suitable to regard the Nijenhuis tensor as a linear map $N : \mathcal{G} \longrightarrow \mathcal{G}$ defined over an unspecified Lie algebra $\mathcal{G}$ and obeying the condition

$$T(N)(\varphi,\psi) := [N\varphi, N\psi] - N[\varphi, N\psi] - N[N\varphi,\psi] + N^2[\varphi,\psi] = 0 \tag{10.1}$$

with respect to the commutator of the algebra. As usual, the index of N will be assumed to be finite, ind $N = r$, so that the algebra $\mathcal{G}$ splits into the direct sum

$$\mathcal{G} = \operatorname{Im} N^r \oplus \operatorname{Ker} N^r \tag{10.2}$$

where both Im N^r and Ker N^r are Lie subalgebras on account of the Nijenhuis condition $T(N^r) = 0$. Our aim is to show that <u>any</u> <u>subalgebra</u> $\mathcal{H}$ of $\mathcal{G}$ which is transversal to Ker N^r in the sense that

$$\mathcal{G} = \mathcal{H} \oplus \operatorname{Ker} N^r \qquad [\mathcal{H},\mathcal{H}] \subset \mathcal{H} \tag{10.3}$$

inherits a reduced Nijenhuis structure from $\mathcal{G}$, even if it is <u>not</u> invariant with respect to N (remark that the reduction lemma of Sec.6 fails in this case).

To specify the statement, let us call φ_I and φ_K the components of any element φ of $\mathfrak{G}$ relative to the splitting (10.2)

(10.4) $\varphi = \varphi_I + \varphi_K \qquad (\varphi_I \in \mathrm{Im}\, N^r ; \varphi_K \in \mathrm{Ker}\, N^r)$

and let us denote by π the canonical projection on Im N^r, by π_H its restriction to $\mathcal{H}$ and by π_H^{-1} the inverse of π_H, which exists by (10.3) :

(10.5) $\pi : \varphi \mapsto \varphi_I \qquad \pi_H := \pi|_{\mathcal{H}} \qquad \pi_H \, \pi_H^{-1} = id_{\mathrm{Im}\, N^r}$

The idea of the method is to modify the Nijenhuis tensor N so — to construct a new tensor $\bar{N} : \mathfrak{G} \longrightarrow \mathfrak{G}$ which is still a Nijenhuis tensor and which keeps $\mathcal{H}$ invariant. Such a tensor is

(10.6) $\bar{N} := \pi_H^{-1} \cdot N \cdot \pi$

The proof runs as follows. From the Nijenhuis condition

(10.7) $T(N^{r+i})(\varphi_K , \psi_I) = N^{r+i} \left(N^{r+i} [\varphi_K , \psi_I]_I - [\varphi_K , N^{r+i} \psi_I]_I \right) = 0$

and from the splitting lemma, one first proves that

(10.8) $N^{r+i} [\varphi_K , \psi_I]_I - [\varphi_K , N^{r+i} \psi_I]_I = 0$

for any $i > 0$. Then, from

(10.9) $$N^{l}\left(N\,[\varphi_K , \psi_J]_I - [\varphi_K , N\psi_J]_I \right) =$$

$$= N^{l+1}\,[\varphi_K , \psi_J]_I - N^{l}\,[\varphi_K , N\psi_J]_I$$

$$\overset{(10.8)}{=} [\varphi_K , N^{l+1}\psi_J]_I - [\varphi_K , N^{l+1}\psi_J]_I$$

$$= 0$$

one obtains the stronger relation

(10.10) $$N\,[\varphi_K , \psi_J]_I - [\varphi_K , N\psi_J]_I = 0$$

Together with the obvious relation

(10.11) $$\pi\,\bar{N} = N\,\pi$$

it allows to construct the following chain of identities:

(10.12) $$\pi\,T(\bar{N})(\varphi, \psi) =$$

$$= \pi\,[\bar{N}\varphi, \bar{N}\psi] - \pi\,\bar{N}\,[\bar{N}\varphi, \psi] - \pi\,\bar{N}\,[\varphi, \bar{N}\psi] + \pi\,\bar{N}^2\,[\varphi, \psi]$$

$$\overset{(10.11)}{=} [(\bar{N}\varphi)_J , (\bar{N}\psi)_J] + [(\bar{N}\varphi)_K , (\bar{N}\psi)_I]_I + [(\bar{N}\varphi)_I , (\bar{N}\psi)_K]_I$$

$$- N\,[(\bar{N}\varphi)_J , \psi_J] - N\,[(\bar{N}\varphi)_K , \psi_I]_I - N\,[(\bar{N}\varphi_I)_I , \psi_K]_I$$

$$- N\,[\varphi_J , (\bar{N}\psi)_J] - N\,[\varphi_I , (\bar{N}\psi)_K]_I - N\,[\varphi_K , (\bar{N}\psi)_I]_I$$

$$+ N^2\,[\varphi_J , \psi_J] + N^2\,[\varphi_I , \psi_K]_I + N^2\,[\varphi_K , \psi_I]_I$$

$$\overset{(10.11)}{=} [N\varphi_I, N\psi_I] + [(\bar{N}\varphi)_K, N\psi_I]_I + [N\varphi_I, (\bar{N}\psi)_K]_I$$

$$- N[N\varphi_I, \psi_I] - N[(\bar{N}\varphi)_K, \psi_I]_I - N[N\varphi_I, \psi_K]_I$$

$$- N[\varphi_I, N\psi_I] - N[\varphi_I, (\bar{N}\psi)_K]_I - N[\varphi_K, N\psi_I]_I$$

$$+ N^2[\varphi_I, \psi_I] + N^2[\varphi_I, \psi_K]_I + N^2[\varphi_K, \psi_I]_I$$

$$\overset{(10.10)}{=} \mathbb{T}(N)(\varphi_I, \psi_I) - N[\varphi_K, N\psi_I]_I - N[N\varphi_I, \psi_K]_I + N^2[\varphi_I, \psi_K]_I +$$

$$+ N^2[\varphi_K, \psi_I]_I \overset{(10.10)}{=} \mathbb{T}(N)(\varphi_I, \psi_I) = 0$$

showing that $\pi \cdot T(\bar{N})(\varphi,\psi)$ must vanish. Since $T(\bar{N})(\varphi,\psi)$ belongs to $\mathcal{H}$ ($\mathcal{H}$ being a subalgebra) and since π is kernel-free when restricted to $\mathcal{H}$, this is possible only if

$$(10.13) \qquad \mathbb{T}(\bar{N})(\varphi,\psi) = 0$$

showing that $\bar{N}$ is a new Nijenhuis tensor on $\mathcal{G}$, <u>leaving $\mathcal{H}$ invariant</u>. Its restriction to $\mathcal{H}$

$$(10.14) \qquad N_H := \pi_H^{-1} \cdot N \cdot \pi_H$$

defines (by lemma 6.1) the reduced structure we were looking for. The new technique is thus a composite technique, where two processes, a projection and a restriction, are performed simultaneously. Let us summarize it into the following proposition :

Proposition 10.1 (algebraic technique for reducing Nijenhuis tensors).

Let N be a Nijenhuis tensor, with a finite index r, defined on a Lie algebra $\mathcal{G}$, and let $\mathcal{H} \subset \mathcal{G}$ be a subalgebra transversal to Ker N^r :

$$(10.15) \qquad \mathcal{G} = \mathcal{H} \oplus \mathrm{Ker}\, N^r \qquad\qquad [\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$$

Let π, $\hat{\pi}_H$, π_H^{-1} be respectively the canonical projection on $\mathrm{Im} N^r$, its restriction to $\mathcal{H}$ and the inverse of $\hat{\pi}_H$, relative to the splitting

$$(10.16) \qquad \mathcal{G} = \mathrm{Im}\, N^r \oplus \mathrm{Ker}\, N^r$$

Then, the (kernel-free) tensor

$$(10.17) \qquad N_H := \pi_H^{-1}\, N\, \pi_H$$

endows H with a Nijenhuis structure.

To show how this result fits the theory of the Nijenhuis manifolds, let S be any submanifold of a Nijenhuis manifold M and assume that :

i) N has a finite constant index r on S

ii) at each point $m \in S$, the tangent space $T_m S$ and the subspace $\mathrm{Ker}(N_m)^r$ split $T_m M$:

$$(10.18) \qquad T_m M = T_m S \oplus \mathrm{Ker}\,(N_m)^r$$

In this case, take $\mathcal{G}$ and $\mathcal{H}$ to be $\mathcal{X}(S,M)$ and $\mathcal{X}(S)$ respectively, and let $\pi : \mathcal{X}(S,M) \to \mathrm{Im}\, N^r$ be the projection over Im N^r

associated with the splitting

(10.19) $$\mathcal{X}(S, M) = \mathrm{Im}\, N^2 \oplus \mathrm{Ker}\, N^2$$

Then, construct the tensor

(10.20) $$N_S := \pi_S^{-1}\, N\, \pi_S$$

where π_S is the restriction to $\mathcal{X}(S)$ of π. It is a new Nijenhuis tensor endowing S with a Nijenhuis structure (even if S is not invariant with respect to the older tensor N). An explicit example of this procedure will be worked out in the next section.

11. A first example of reduction of a PN manifold

Before considering in a general context the construction of $P\Omega$ and PN manifolds and their reduction, it may be suitable to discuss a simple case. It is intended both as a first study of the distributions and of the submanifolds entering into the reduction process and as a guide to the more general example of Sec.16.

Let V be any finite-dimensional vector space over R (or C), and M the affine manifold of 2x2 matrices whose entries are C^{∞} functions u^K_j : $R \to \mathrm{End}\, V$ obeying given asymptotic conditions for $|x| \to \infty$. The "points" of the manifold M are thus the matrices

$$(11.1) \qquad u := \begin{bmatrix} u_1(x) & u_2(x) \\ u_3(x) & u_4(x) \end{bmatrix}$$

whose entries (henceforth denoted by a single index to simplify the notations) are matrix-valued functions. Constant vector fields and one-forms on H are then given by matrices

$$(11.2) \qquad \varphi := \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{bmatrix} \qquad\qquad \alpha := \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

whose entries are rapidly decreasing matrix-valued functions for $|x| \to \infty$. The natural pairing $\langle\, ,\rangle$ between vector fields and one-forms is given by

$$(11.3) \qquad \langle \alpha, \varphi \rangle = \mathrm{Tr} \int_{-\infty}^{+\infty} (\alpha_1 \varphi_1 + \alpha_2 \varphi_3 + \alpha_3 \varphi_2 + \alpha_4 \varphi_4)\, dx$$

the symbol Tr meaning the trace of the products $\alpha_i \varphi_j$.

As it will be shown from a general point of view in Sec.s 14 and 15, the manifold M is naturally endowed with a twofold Hamiltonian structure defined by the Poisson tensors

$$(11.4) \qquad P_u \alpha := \alpha_x + [u, \alpha]$$

$$(11.5) \qquad Q_u \alpha := [a, \alpha]$$

where a is any fixed matrix with constant entries, and the symbol $[\,,\,]$ denotes the commutator of matrices. In particular, we choose

$$(11.6) \qquad a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and we aim to show that a straightforward application of the reduction techniques allows to recover the integrability structure of the (non-abelian) KdV equation.

<u>11.1</u> Since the tensor P is kernel-free (on account of the asymptotic condition for α), the manifold M can be dealt with either as a PN or as a PΩ manifold, by introducing the presymplectic tensor Ω and the Nijenhuis tensor N given by

$$(11.7) \qquad \Omega := P^{-1} \qquad\qquad N := Q\, P^{-1} = Q\, \Omega$$

In both cases, however, the first step is the restriction to a characteristic leaf of the Poisson tensor Q. Since by (11.5), (11.6) Q has the explicit form

(11.8) $\varphi = Q\alpha \quad : \quad \varphi_1 = \alpha_3 \quad \varphi_2 = \alpha_4 - \alpha_1 \quad \varphi_3 = 0 \quad \varphi_4 = -\alpha_3$

the distribution Im Q is defined by

(11.9) $\operatorname{Im} Q = \{ \varphi : \varphi_1 + \varphi_4 = 0 \ , \ \varphi_3 = 0 \}$

Its integral leaves are the affine hyperplanes

(11.10) $u_1 + u_4 = c_1 \qquad u_3 = c_2$

c_1 and c_2 being arbitrary matrices. For further developments, the choice of c_1 and c_2 is largely immaterial, at least if $c_2 \neq 0$ (the leaf $c_2 = 0$ is in some sense singular, the index of N being greater on it): thus we choose

(11.11) $S = \{ u \in M : u_1 + u_4 = 0 \ , \ u_3 = 1 \}$

Clearly, S is a "two-dimensional" affine hyperplane (over the ring of the functions $f : R \to \operatorname{End} V$) admitting the parametric representation

(11.12) $u \in S \qquad u = u'_1 \sigma_3 + u'_2 \sigma^+ + \sigma^-$

where the matrices σ_3, σ^+ and σ^- are given by :

(11.13) $\sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma^+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \sigma^- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

The pair of functions (u'_1, u'_2) may thus be called the "coordinates" of the point $u \in S$, with respect to the affine frame (11.13). Accordingly, a tangent vector φ to S at the point u may be written in the form

(11.14) $$\varphi = \varphi_1' \sigma_3 + \varphi_2' \sigma^+$$

where (φ_1', φ_2') are the components of φ in the frame (11.13), so that for the one-forms we get

(11.15) $$\langle \alpha, \varphi \rangle = \int_{-\infty}^{+\infty} \left(\varphi_1' \operatorname{Tr}(\alpha \sigma_3) + \varphi_2' \operatorname{Tr}(\alpha \sigma^+) \right) dx$$

Thus, the components of the one-form α in this coordinate system are

(11.16) $$\alpha_1' = \operatorname{Tr}(\alpha \sigma_3) = \alpha_1 - \alpha_4 \qquad \alpha_2' = \operatorname{Tr}(\alpha \sigma^+) = \alpha_3$$

Let us now restrict to S the PΩ structure of M, defined by the tensors Q and $\Omega = P^{-1}$. As for the Poisson tensor Q, we have to use the formula

(11.17) $$\varphi' = df^{-1} \, Q \, \delta f^{-1} \alpha'$$

(where f : $(u_1', u_2') \longmapsto u$ symbolically denotes the parametrization (11.12)), namely we have to express the components (φ_1', φ_2') of the tangent vector as functions of the components (α_1', α_2') of α . Explicitly, we get

(11.18) $$\begin{cases} \varphi_1' \overset{(11.14)}{=} \varphi_1 \overset{(11.8)}{=} \alpha_3 \overset{(11.16)}{=} \alpha_2' \\ \varphi_2' = \varphi_2 = \alpha_4 - \alpha_1 = -\alpha_1' \end{cases}$$

showing that the restricted tensor Q' has, in the affine coordinates (u_1', u_2'), the canonical form

$$(11.19)\qquad \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \left[\begin{array}{c|c} \cdot & 1 \\ \hline -1 & \cdot \end{array}\right] \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix}$$

As for the presymplectic tensor Ω, we have to use the formula

$$(11.20)\qquad \alpha' = \delta f \cdot \Omega \cdot df \;\; \varphi'$$

namely we have to express the components (α_1', α_2') of α as functions of the components (φ_1', φ_2') of φ. This can be done by using the following three sets of equations : first, the equations

$$(11.21)\qquad \alpha' = \delta f \cdot \alpha \qquad \alpha_1' = \alpha_1 - \alpha_4 \qquad \alpha_2' = \alpha_3$$

giving the components (α_1', α_2') in terms of the entries of the matrix α; next, the equations

$$(11.22)\qquad \varphi = \Omega^{-1}\alpha : \qquad \begin{aligned} \varphi_1 &= \alpha_{1x} + u_2\alpha_3 - \alpha_2 + [u_1', \alpha_1] \\ \varphi_2 &= \alpha_{2x} + \{u_1', \alpha_2\} - \alpha_1 u_2' + u_2'\alpha_4 \\ \varphi_3 &= \alpha_{3x} + \alpha_1 - \alpha_4 - \{u_1, \alpha_3\} \\ \varphi_4 &= \alpha_{4x} + \alpha_2 - \alpha_3 u_2 - [u_1', \alpha_4] \end{aligned}$$

(where the bracket $\{\ ,\ \}$ denotes the anticommutator of matrices) giving the entries of α in terms of the entries of the tangent vector φ; and, finally, the equations

$$(11.23)\qquad \varphi = df \cdot \varphi' \qquad \varphi_1 = \varphi'_1 \qquad \varphi_2 = \varphi'_2 \qquad \varphi_3 = 0 \qquad \varphi_4 = -\varphi'_1$$

giving the entries of the tangent vectors as functions of its "affine components" (φ'_1, φ'_2). Simple computations allow to split the previous system as follows

$$(11.24)\qquad \begin{cases} \alpha_3 = \alpha'_2 \\ \alpha_1 - \alpha_4 = \alpha'_1 \\ (\alpha_1 + \alpha_4)_x + [u'_2, \alpha_3] + [u'_1, \alpha_1 - \alpha_4] = 0 \\ \alpha_2 - \alpha_{1x} - u'_2\, \alpha_2 + [\alpha_2, u'_1] = -\varphi'_1 \end{cases}$$

and

$$(11.25)\qquad \begin{cases} \alpha'_1 = -\alpha'_{2x} + \{u'_1, \alpha'_2\} \\ \mathcal{L}(\alpha'_2) = \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \end{cases}$$

where the linear integro-differential operator $\mathcal{L}$ is given by

$$(11.26)\qquad \begin{cases} 2\mathcal{L}(\alpha'_2) := -\alpha'_{2xxx} + 2(u''\alpha'_2)_x + \{\alpha'_{2x}, u''\} + [\alpha'_2, u'']_x + \left[u'', \int_{-\infty}^{x} [\alpha'_2, u'']\, dx\right] \\ u'' := u'_{1x} + u'_2 + u'^2_1 \end{cases}$$

The system (11.24) allows to express $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ as functions of $(\alpha'_1, \alpha'_2, \varphi'_1)$ and its solution does not enter the process of reduction; the second system (11.25), on the contrary, gives directly the link between (α'_1, α'_2) and (φ'_1, φ'_2) we were looking for and, consequently, it defines the reduced presymplectic tensor Ω' :

$$(11.27)\qquad \begin{cases} \alpha'_2 = \mathcal{L}^{-1}\left(\varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\}\right) \\ \alpha'_1 = -\alpha'_{2x} + \{u'_1, \alpha'_2\} \end{cases}$$

In particular, we see that the kernel of Ω' is given by

$$(11.28)\qquad \mathrm{Ker}\,\Omega' = \left\{ (\varphi'_1, \varphi'_2) : \ \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} = 0 \right\}$$

<u>Remark.</u> The existence of Ker Ω' could have been determined without constructing Ω' and without an explicit introduction of the parametrization of S; indeed, eq.(11.20) entails that

$$(11.29)\qquad \begin{aligned} df\,(\mathrm{Ker}\,\Omega') &= \Omega^{-1}(\mathrm{Ker}\,\delta f) \cap \mathrm{Im}\,Q \\ &= \Omega^{-1}(\mathrm{Im}\,df)^{\circ} \cap \mathrm{Im}\,Q \\ &= \Omega^{-1}(\mathrm{Im}\,Q)^{\circ} \cap \mathrm{Im}\,Q \end{aligned}$$

Then, on account of (11.9) and of the explicit form of Ω^{-1}, it follows that

$$(11.30)\quad \Omega^{-1}(\mathrm{Im}\,Q)^{\circ} \cap \mathrm{Im}\,Q = \left\{ \varphi : \ \varphi_1 + \varphi_4 = 0,\ \varphi_3 = 0,\ \varphi_{1x} + \varphi_2 + \{u_1, \varphi_1\} = 0 \right\}$$

giving directly (11.28).

11.2 One of the main outcomes of the previous computation is to show that the leaves of the characteristic distribution $\mathrm{Ker}\,\Omega'$ of Ω' are defined by the equation

$$u'_{1x} + u'_2 + u'^2_1 = u'' \tag{11.31}$$

where u'' is any given matrix-valued function $u'' : R \longrightarrow \mathrm{End}\,V$. Different leaves of $\mathrm{Ker}\,\Omega'$ correspond to different choices of u''. Consequently, we can consider the Eq.(11.31) as defining the surjective submersion $g : M' \longrightarrow M'/\ker\Omega'$ required by the second step of the reduction process, and u'' as the coordinate on the quotient manifold (the topology of this manifold will be more carefully investigated in Sec.16). If we call φ'' the component of a tangent vector to $M'' = M'/\ker\Omega'$ at the point u'', from (11.31) we get

$$\varphi'' = \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \tag{11.32}$$

and consequently

$$\begin{aligned} \mathrm{Tr}\int_{-\infty}^{+\infty} \alpha''\varphi''\,dx &= \mathrm{Tr}\int_{-\infty}^{+\infty} \alpha''(\varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\})\,dx \\ &= \mathrm{Tr}\int_{-\infty}^{+\infty} \varphi'_1\left(-\alpha''_x + \{\alpha'', u'_1\}\right)dx + \mathrm{Tr}\int_{-\infty}^{+\infty} \varphi'_2\,\alpha''\,dx \end{aligned} \tag{11.33}$$

so that

$$\alpha'_1 = -\alpha''_x + \{\alpha'', u'_1\} \qquad \alpha'_2 = \alpha'' \tag{11.34}$$

Eq.s (11.32) and (11.34) define the mappings $dg : (\varphi'_1, \varphi'_2) \longmapsto \varphi''$ and $\delta g : \alpha'' \longmapsto (\alpha'_1, \alpha'_2)$ associated with the projection $g : (u'_1, u'_2) \longmapsto u''$. Then, by using the equation

(11.35) $$Q'' := dg \cdot Q' \cdot \delta g$$

we readily obtain

(11.36) $$\begin{aligned} \varphi'' &\overset{(11.32)}{=} \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \\ &\overset{(11.19)}{=} \alpha'_{2x} - \alpha'_1 + \{u'_1, \alpha'_2\} \\ &\overset{(11.34)}{=} \alpha''_x + \alpha''_x - \{\alpha'', u'_1\} + \{u'_1, \alpha''\} \\ &= 2\,\alpha''_x \end{aligned}$$

giving the <u>final form</u> of the reduced Poisson tensor Q'' over the quotient manifold $M'' = M'/\ker\Omega'$. Indeed, on account of the boundary conditions, this tensor is clearly kernel-free, so that the process of reduction ends at this stage, according to the remark of Sec. 8.

As for Ω'', we have to use the formula

(11.37) $$\alpha'' = \delta g^{-1} \cdot \Omega'' \cdot dg^{-1} \cdot \varphi''$$

namely we must eliminate φ' and α' from the equations

(11.38) $$\alpha' = \delta g \cdot \alpha'' \qquad \alpha' = \Omega' \varphi' \qquad \varphi'' = dg\ \varphi'$$

Explicitly, this means that we must solve the following system of three sets of equations

(11.39) $$\alpha'_1 = -\alpha''_x + \{\alpha'', u'_1\} \qquad \alpha'_2 = \alpha''$$

$$(11.40) \qquad \begin{cases} \alpha'_2 = \mathcal{L}^{-1}\left(\varphi'_{1x} - \varphi'_2 + \{u'_1, \varphi'_1\}\right) \\ \alpha'_1 = -\alpha'_{2x} + \{u'_1, \alpha'_2\} \end{cases}$$

$$(11.41) \qquad \varphi'' = \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\}$$

either with respect to α'' or with respect to φ''. In this way we obtain either the reduced symplectic tensor or its inverse. So, for example, the equation

$$(11.42) \qquad \begin{aligned} \varphi'' &= \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \\ &= \mathcal{L}(\alpha'_2) \\ &= \mathcal{L}(\alpha'') \\ &\overset{(11.56)}{=} -\frac{1}{2}\alpha''_{xxx} + (u''\alpha'')_x + \frac{1}{2}\{\alpha''_x, u''\} + \frac{1}{2}[\alpha'', u'']_x - \frac{1}{2}\left[u'', \int_{-\infty}^{x}[u'', \alpha'']\,dx\right] \end{aligned}$$

gives the second Hamiltonian structure on $M'' \simeq M'/\ker\Omega'$. The tensors Q'' and Ω''^{-1} define the integrability structure of the hierarchy of evolution equations whose first member is the well-known non-abelian KdV equation; this hierarchy is characterized by the Nijenhuis tensor $N'' := \Omega''^{-1} \cdot Q''^{-1}$ given by

$$(11.43) \qquad N''\varphi'' = -\varphi''_{xx} + 2\left(u''\int_{-\infty}^{x}\varphi''\,dx\right)_x + \{u'', \varphi''\} - \left[\int_{-\infty}^{x}\varphi''\,dx, u''\right]_x + \left[u'', \int_{-\infty}^{x}\left[\int_{-\infty}^{x'}\varphi''\,dx', u''\right]dx\right]$$

and the non-abelian KdV equation

$$(11.44) \qquad u''_t = -u''_{xxx} + 3\{u'', u''_x\}$$

is obtained by choosing $\varphi'' = u''_x$.

11.3 Now, we discuss the reduction of the manifold M from the point of view of the PN theory. Let $(Q', N' := Q'\Omega')$ be the Poisson and Nijenhuis tensors restricted to the submanifold S (11.11). The first step is to compute the index of N' at the points of S; since Q' is kernel-free, it is

(11.45) $$\mathrm{Ker}\, N' = \mathrm{Ker}\, \Omega' \overset{(11.28)}{=} \{ (\varphi_1', \varphi_2') : \varphi_{1x}' + \varphi_2' + \{u_1', \varphi_1'\} = 0 \}$$

and, by (11.8) and (11.27)

(11.46) $$\mathrm{Im}\, N' = Q'(\mathrm{Im}\, \Omega') = \{ (\varphi_1', \varphi_2') : \varphi_{1x}' - \varphi_2' - \{u_1', \varphi_1'\} = 0 \}$$

Clearly these two distributions are transversal on account of the homogeneous boundary conditions on (φ_1', φ_2'); therefore, the index of N' is $r = 1$ at any point $u \in S$, and the characteristic leaves of N' are given by

(11.47) $$S_K := \{ (u_1', u_2') : u_{1x}' + u_2' + u_1'^2 = u'' \}$$

(11.48) $$S_I := \{ (u_1', u_2') : u_{1x}' - u_2' - u_1'^2 = v'' \}$$

on account of (11.45) and (11.46). In the last equations u'' and v'' are arbitrary matrix-valued functions, playing the role of the parameters specifying the leaves. Taken together, Eq.(11.47) and (11.48) can be looked at as the formulas which define a change of coordinates: from the old "affine coordinates" (u_1', u_2') to the new "fibered coordinates" (u'', v'') (the reason for this terminology will be made clear in Sec.16). By calling (φ'', ψ'') and (α'', β'') the components of tangent and cotangent vectors in the new system of coordinates, from Eqs (11.47-48) we get

(11.49) $$\begin{cases} \varphi'' = \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \\ \psi'' = \varphi'_{1x} - \varphi'_2 - \{u'_1, \varphi'_1\} \end{cases}$$

and hence, by duality,

(11.50) $$\alpha'_1 = -(\alpha'' + \beta'')_x + \{u'_1, \alpha'' - \beta''\} \qquad \alpha'_2 = \alpha'' - \beta''$$

The "components" of the reduced tensor Q' in this new coordinate system are thus given by

(11.51) $$\begin{aligned} \varphi'' &\overset{(11.49)}{=} \varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\} \\ &\overset{(11.18)}{=} \alpha'_{2x} - \alpha'_1 + \{u'_1, \alpha'_2\} \\ &\overset{(11.50)}{=} 2\alpha''_x \end{aligned}$$

and by

(11.52) $$\begin{aligned} \psi'' &= \varphi'_{1x} - \varphi'_2 - \{u'_1, \varphi'_1\} \\ &= \alpha'_{2x} + \alpha'_1 - \{u'_1, \alpha'_2\} \\ &= -2\beta''_x \end{aligned}$$

so that Q' takes the explicit form

(11.53) $$Q' = 2\left[\begin{array}{c|c} \partial_x & 0 \\ \hline 0 & -\partial_x \end{array}\right]$$

To compute the components of Ω' (and hence of N') it is suitable to use the inverse transformation :

(11.54) $\quad u'_1 = \frac{1}{2}\int_{-\infty}^{x}(u''+v'')\,dx \qquad u'_2 = \frac{1}{2}(u''-v'') - \frac{1}{4}\left(\int_{-\infty}^{x}(u''+v'')\,dx\right)^2$

entailing

(11.55) $$\begin{cases} \alpha'' = \frac{1}{2}\alpha'_2 + \frac{1}{2}\int_{-\infty}^{x}(\{u'_1,\alpha'_2\} - \alpha'_1)\,dx \\ \beta'' = -\frac{1}{2}\alpha'_2 + \frac{1}{2}\int_{-\infty}^{x}(\{u'_1,\alpha'_2\} - \alpha'_1)\,dx \end{cases}$$

Therefore we obtain

(11.56) $$\begin{aligned} \alpha'' &\overset{(11.55)}{=} \frac{1}{2}\alpha'_2 + \frac{1}{2}\int_{-\infty}^{x}(\{u'_1,\alpha'_2\} - \alpha'_1)\,dx \\ &\overset{(11.27)}{=} \mathcal{L}^{-1}(\varphi'_{1x} + \varphi'_2 + \{u'_1, \varphi'_1\}) \\ &= \mathcal{L}^{-1}(\varphi'') \end{aligned}$$

and

(11.57) $$\beta'' \overset{(11.55)}{=} -\frac{1}{2}\alpha'_2 + \frac{1}{2}\int_{-\infty}^{x}(\{u'_1,\alpha'_2\} - \alpha'_1)\,dx \overset{(11.27)}{=} 0$$

showing that Ω' takes the explicit form

(11.58) $$\Omega' = \left[\begin{array}{c|c} \mathcal{L}^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $\mathcal{L}^{-1}$ does not depend on the second fibered coordinate v'' (by (11.26)). The study of the further restriction on the leaves of Im N' or projection on S/Ker N' becomes now trivial, the reduced tensors being simply given by the first diagonal block in (11.53) and (11.58) respectively. The interesting feature of this example is that it is completely general : in Sec.16 we shall show that the general case for nxn matrices can be dealt with exactly in the same way, leading to the so-called general Gel'fand-Dikii equations.

11.4. To end this section, we aim to give an example also of the reduction technique of Sec.10. For this reason, we consider the submanifold $F \subset M$ of 2x2 traceless Frobenius matrices defined by

$$(11.59) \qquad F := \{ u \in M : \; u_1 = u_4 = 0 \;, u_3 = 1 \} \iff u \in F : \; u = \begin{bmatrix} 0 & v \\ 1 & 0 \end{bmatrix}$$

The deep reason to do that will be made clear in Sec.16; here, we limit ourselves to show that F obeys the conditions stated in Sec.10, allowing to perform the reduction. To this end, we first remark that, at the points of F, the tensors Q and P take the simplified form (see Eq (11.22))

$$(11.60) \qquad \varphi = Q\alpha \; : \qquad \varphi_1 = \alpha_3 \qquad \varphi_2 = \alpha_4 - \alpha_1 \qquad \varphi_3 = 0 \qquad \varphi_4 = -\alpha_3$$

$$(11.61) \qquad \varphi = P\alpha \; : \qquad \begin{aligned} \varphi_1 &= \alpha_{1x} + v\alpha_3 - \alpha_2 \\ \varphi_2 &= \alpha_{2x} - \alpha_1 v + v\alpha_4 \\ \varphi_3 &= \alpha_{3x} + \alpha_1 - \alpha_4 \\ \varphi_4 &= \alpha_{4x} + \alpha_2 - \alpha_3 v \end{aligned}$$

Simple computations show that

$$(11.62)\quad \mathrm{Im}\, N = \mathrm{Im}\, Q = \{ \varphi : \varphi_1 + \varphi_4 = 0 \ , \ \varphi_3 = 0 \}$$

$$(11.63)\quad \mathrm{Im}\, N^2 = Q\, P^{-1}(\mathrm{Im}\, N) = \{ \varphi : \ \varphi_1 + \varphi_4 = 0, \ \varphi_3 = 0 \ , \ \varphi_{1x} - \varphi_2 = 0 \}$$

$$(11.64)\quad \mathrm{Ker}\, N = P(\mathrm{Ker}\, Q) = \{ \varphi : \varphi_3 = 0 \ , 2\varphi_2 + \varphi_{1x} - \varphi_{4x} - [v, \int_{-\infty}^{x} (\varphi_1 + \varphi_4)\, dx] = 0 \}$$

$$(11.65)\quad \mathrm{Ker}\, N^2 = P\, Q^{-1}(\mathrm{Ker}\, N) = \{ \varphi \quad \varphi_2 = \Phi(\varphi_1, \varphi_3, \varphi_4) \}$$

where the last constraint is given by :

$$(11.66)\quad 4\,\Phi(\varphi_1, \varphi_3, \varphi_4) := -2(\varphi_1 - \varphi_4)_x + \varphi_{3xx} + \{ v, \int_{-\infty}^{x} \varphi_3\, dx \} +$$
$$+ [\varphi_3, v] - [v, \int_{-\infty}^{x} [v, \int_{-\infty}^{x'} \varphi_3\, dx']\, dx] + [v, \int_{-\infty}^{x} (\varphi_1 + \varphi_4)\, dx]$$

Thus one verifies that Ker N_u^2 is transversal at any point $u \in F$ both to Im N_u^2 and $T_u F$.

Having verified that the reduction method of Sec.10 can be applied to the submanifold F, we proceed by constructing the projection π. To this end, let us remark that the matrices φ_I, φ_K, φ_F belonging to $\mathrm{Im} N_u^2$, $\mathrm{Ker} N_u^2$ and $T_u F$ can be given the <u>parametric form</u> :

$$(11.67)\quad \varphi_I \in \mathrm{Im}\, N_u^2 \qquad \varphi_I = \left[\begin{array}{c|c} \lambda & \lambda_x \\ \hline 0 & -\lambda \end{array} \right]$$

$$(11.68)\quad \varphi_K \in \mathrm{Ker}\, N_u^2 \qquad \varphi_K = \left[\begin{array}{c|c} \mu & \Phi(\mu, \nu, \varrho) \\ \hline \nu & \varrho \end{array} \right]$$

$$(11.69)\quad \varphi_F \in T_u F \qquad \varphi_F = \left[\begin{array}{c|c} 0 & \psi \\ \hline 0 & 0 \end{array} \right]$$

so that the equation $\varphi = \varphi_I + \varphi_K$ means

(11.70) $\varphi_1 = \lambda + \mu \quad , \quad \varphi_2 = \lambda_x + \Phi(\mu, \nu, \rho) \quad \varphi_3 = \nu \quad \varphi_4 = \rho - \lambda$

Its solution is

(11.71) $2\lambda = \int_{-\infty}^{x} (\varphi_2 - \Phi(\varphi_1, \varphi_3, \varphi_4))\, dx \quad , \quad \mu = \varphi_1 - \lambda \; , \; \nu = \varphi_3 \; , \; \rho = \varphi_4 + \lambda$

The projection π is then given by

(11.72) $\pi : \varphi \mapsto \varphi_I \; : \qquad \varphi_I = \left[\begin{array}{c|c} \lambda(\varphi_1, \varphi_2, \varphi_3, \varphi_4) & \lambda_x(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \\ \hline 0 & -\lambda(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \end{array} \right]$

and its restriction π_F to S by

(11.73) $\pi_F : \varphi_F \mapsto \varphi_I \qquad \varphi_I = \frac{1}{2} \left[\begin{array}{c|c} \int_{-\infty}^{x} \psi\, dx & \psi \\ \hline 0 & -\int_{-\infty}^{x} \psi\, dx \end{array} \right]$

so that

(11.74) $\pi_F^{-1} : \varphi_I \longmapsto \varphi_F \qquad \varphi_F : \left[\begin{array}{c|c} 0 & 2\lambda_x \\ \hline 0 & 0 \end{array} \right]$

The reduction of N is now straightforward, the reduced tensor N_F being given by

(11.75) $N_F := \pi_F^{-1} \, Q \, \Omega \, \pi_F$

Since the equations $\alpha = \Omega \, \pi_F \, \bar{\varphi}_F$ and $\varphi_F = \pi_F^{-1} Q \alpha$ entail

(11.76) $\begin{cases} 2\bar{\psi} = -\alpha_{3xxx} + \{\alpha_3, v\}_x + \{v, \alpha_{3x}\} + \left[v, \int_{-\infty}^{x} [\alpha_3, v]\, dx \right] \\ \alpha_3 = \int_{-\infty}^{x} (\alpha_4 - \alpha_1)\, dx \end{cases}$

and

$$(11.77) \qquad \alpha_4 - \alpha_1 = \psi$$

respectively, one directly obtains :

$$(11.78) \quad \bar{\psi} = N_F^{-1}(\psi) = -\frac{1}{4}\psi_{xx} + \frac{1}{2}\left\{\int_{-\infty}^{x}\psi\, dx, v\right\}_x + \frac{1}{2}\{v, \psi\} + \frac{1}{2}\left[v, \int_{-\infty}^{x}\left[\int_{-\infty}^{x'}\psi\, dx', v\right]dx\right]$$

It is now possible to compare advantages and drawbacks of the two methods of reduction, the <u>geometric</u> one of Sec.8-9 and the <u>algebraic</u> one of Sec.10. The first one has the advantage of giving a clear picture of the reduction process at any stage, and of suggesting <u>systematically</u> what are the submanifolds carrying a reduction; but it forces us to make repeated changes of coordinates which become more and more cumbersome when the dimension of the manifold increases. The second one, on the contrary, does not require any change of coordinates and so it is more direct, but it does not provide any specific indications on how to find the submanifolds carrying the reduction. Roughly speaking, we can say that the geometric method is more systematic, while the algebraic one is more efficient.

12. Lie groups as PΩ manifolds

In this section we consider the particular problem of the construction of invariant Poisson tensors on a Lie group, and we show that any Lie group admitting an invariant Poisson tensor is a PΩ manifold. From this result we shall derive a simple algebraic method to construct PΩ structures on a Lie group : it will be used in Sec.15 to obtain the PΩ structure associated with the equations solvable by the inverse scattering technique.

For further convenience, the main notations and some noteworthy identities to be used in this section and in the following ones are now summarized. If H is a (Banach-)Lie group, $\mathcal{H}$ its Lie algebra and $\mathcal{H}^*$ the dual of $\mathcal{H}$, we shall use the symbols $(h, k, \ldots)$, $(\xi, \eta, \theta, \ldots)$ and $(\lambda, \mu, \nu, \ldots)$ to denote elements of H, $\mathcal{H}$ and $\mathcal{H}^*$ respectively; e stands for the identity of the group and L_h, R_h stand for the left and right-translations, as usual. The adjoint and the coadjoint representations are denoted by $Ad_h := dR_h(e)^{-1}.dL_h(e)$ and $Ad_h^* := \delta R_h(e)^{-1}.\ \delta L_h(e)$, and their generators by

$$(12.1)\quad ad_\xi \eta = [\xi, \eta] := \frac{d}{dt} Ad_{\exp t\xi}\, \eta \Big|_{t=0} \qquad ad_\xi^* \mu := \frac{d}{dt} Ad^*_{\exp t\xi}\, \mu \Big|_{t=0}$$

Finally, φ_ξ and $\hat{\varphi}_\xi$ are the left-invariant and right-invariant vector fields, defined by

X^λ $$(12.2)\qquad \varphi_\xi(h) := dL_h(e).\,\xi \qquad \hat{\varphi}_\xi(h) := dR_h(e).\,\xi$$ X^ξ

and α_μ, $\hat{\alpha}_\mu$ are the corresponding invariant one-forms

α^λ $$(12.3)\qquad \alpha_\mu(h) := \delta L_h(e)^{-1}\mu \qquad \hat{\alpha}_\mu(h) := \delta R_h(e)^{-1}\mu$$ α^ξ

They are related by the well-known relations

$$(12.4) \quad \langle \alpha_\mu , \varphi_\xi \rangle = \langle \mu , \xi \rangle = \langle \hat{\alpha}_\mu , \hat{\varphi}_\xi \rangle$$

$[X^\rho Y^\rho] = [X,Y]^\rho$

$$(12.5) \quad [\hat{\varphi}_\xi , \hat{\varphi}_\eta] = \hat{\varphi}_{ad_\xi \eta} \; ; [\varphi_\xi , \hat{\varphi}_\eta] = 0 \; ; [\varphi_\xi , \varphi_\eta] = -\varphi_{ad_\xi \eta}$$

$$(12.6) \quad L_{\varphi_\xi}(\alpha_\mu) = -\alpha_{ad^*_\xi \mu} , \; L_{\varphi_\xi}(\hat{\alpha}_\mu) = 0 \; , \; L_{\hat{\varphi}_\xi}(\hat{\alpha}_\mu) = \hat{\alpha}_{ad^*_\xi \mu}$$

$$(12.7) \quad d\alpha_\mu \, \varphi_\xi = -\alpha_{ad^*_\xi \mu} \qquad d\hat{\alpha}_\mu \, \hat{\varphi}_\xi = \hat{\alpha}_{ad^*_\xi \mu}$$

of common use henceforth.

Now, let $P : \mathcal{X}^*(H) \longrightarrow \mathcal{X}(H)$ be a right-invariant second-order tensor of type (2,0) on H, mapping right-invariant one-forms into right-invariant vector fields according to

$\boxed{\Lambda^\rho_r \xi^\rho = (r\xi)^\rho}$

$$(12.8) \quad P \hat{\alpha}_\mu = \hat{\varphi}_{P_e \mu}$$

X_ξ

where $\boxed{P_e : \mathcal{H}^* \longrightarrow \mathcal{H}}$ is the evaluation of P at the identity of the group. By using the identities

$$(12.9) \quad \langle \hat{\alpha}_\mu , P \hat{\alpha}_\nu \rangle \overset{(12.4)}{=} \langle \mu , P_e \nu \rangle$$

$$(12.10) \quad \langle L_{P\hat{\alpha}_\lambda}(\hat{\alpha}_\mu) , P\hat{\alpha}_\nu \rangle \overset{(12.6)}{=} \langle \hat{\alpha}_{ad^*_{P_e\lambda}\mu} , P\hat{\alpha}_\nu \rangle \overset{(12.4)}{=} \langle \mu , [P_e\lambda , P_e \nu] \rangle$$

$$\langle \mathcal{L}_{X_{\xi_1}} \xi_2^\rho , X_{\xi_3} \rangle = \langle \xi_2 , [r\xi_1 , r\xi_3] \rangle$$

one easily shows that P is a Poisson tensor if P_e fulfils the following two conditions

$$(12.11) \quad \langle \mu , P_e \nu \rangle = -\langle \nu , P_e \mu \rangle$$

$$(12.12) \quad \langle \lambda , [P_e \mu , P_e \nu] \rangle + \text{cyclic permutation} = 0$$

Yang-Baxter equation

The tensors $P_e : \mathcal{H}^* \longrightarrow \mathcal{H}$ fulfilling them are said Poisson cocycles (or P-cocycles for short). Similarly, one can show that any $\mathcal{H}^*$-cocycle of H [16], i.e. any tensor $\boxed{\Omega_e : \mathcal{H} \to \mathcal{H}^*}$ linear map ~~of type~~ (0,2), such that

$$(12.13) \qquad \langle \Omega_e \xi, \eta \rangle = - \langle \Omega_e \eta, \xi \rangle$$

symplectic cocycle cf Souriau p 116

$$(12.14) \qquad \langle \Omega_e [\xi, \eta], \theta \rangle + \text{cyclic permutation} = 0$$

for any $\xi, \eta, \theta \in \mathcal{H}$, defines on the group H a tensor $\Omega : \mathcal{X}(H) \longrightarrow \mathcal{X}^*(H)$ of type (0,2)

$$(12.15) \qquad \Omega\, \varphi_{\xi} := \alpha_{\Omega_e \xi}$$

which is left-invariant and presymplectic. For further reference, we note that conditions (12.12) and (12.14) can be given the form

$$(12.16) \qquad [P_e \mu, P_e \nu] = P_e \left(ad^*_{P_e \nu} \mu - ad^*_{P_e \mu} \nu \right) \qquad \boxed{[r\xi_1, r\xi_2] - r[\xi_1, \xi_2]_r = 0}$$

cf Souriau p 114

$$(12.17) \qquad \Omega_e [\xi, \eta] = - ad^*_{\xi} \Omega_e \eta + ad^*_{\eta} \Omega_e \xi$$

showing, in particular, that the subspaces

$$(12.18) \qquad \mathcal{H}' = P_e(\mathcal{H}^*) \qquad \mathcal{H}'_{\Omega} := \mathrm{Ker}\, \Omega_e$$

are Lie subalgebras of $\mathcal{H}$.

The main result we want to point out in this section is that any pair (P, Ω) formed by a right-invariant Poisson tensor P and by a left-invariant presymplectic tensor Ω (or viceversa by a left-invariant Poisson tensor P and by a right-invariant presymplectic tensor Ω) defines a $P\Omega$ structure on H, without any further condition on P_e and Ω_e . Since any Lie group has an infinite set of $\mathcal{H}^*$-cocycles, e.g. those defined by

(12.19) $$\Omega_e^{(\mu)}\, \xi := ad^*_\xi\, \mu$$

for any given element $\mu \in \mathcal{H}^*$, this result reduces the problem of the construction of a $P\Omega$ structure on H to the solution of the cocycle conditions (12.11) (12.12), that is to a purely algebraic problem.

To prove this statement we have to make two remarks, concerning respectively the left-invariant one-forms α_μ and the vector fields

(12.20) $$\psi_\mu := P\, \alpha_\mu = U_\mu = \#_{\Lambda^{\varrho}_{P_e}} \mu^\lambda$$

which turn out to be associated with them. According to the first remark, the forms α_μ are in involution with respect to the Poisson tensor P, since

(12.21) $$\langle \{\alpha_\mu, \alpha_\nu\}_P, \varphi_\xi \rangle := \langle d\alpha_\mu\; P\alpha_\nu - d\alpha_\nu\; P\alpha_\mu +$$

$$+ d\langle \alpha_\mu . P\alpha_\nu \rangle , \varphi_\xi \rangle$$

$$= -\langle d\alpha_\mu\; \varphi_\xi, P\alpha_\nu \rangle + \langle d\alpha_\nu\; \varphi_\xi, P\alpha_\mu \rangle + L_{\varphi_\xi} \langle \alpha_\mu, P\alpha_\nu \rangle$$

$$= \langle L_{\varphi_\xi}(\alpha_\mu) - d\alpha_\mu\; \varphi_\xi, P\alpha_\nu \rangle + \langle d\alpha_\nu\; \varphi_\xi - L_{\varphi_\xi}(\alpha_\nu), P\alpha_\mu \rangle = 0$$

having used the obvious identity

$0 = \{\alpha_\xi, \alpha_\eta\}_\varrho = i_{\#_\varrho \eta^\lambda} d\xi^\lambda - i_{\#_\varrho \xi^\lambda} d\eta^\lambda + d\langle \xi^\lambda, \#_\varrho \eta^\lambda \rangle = \mathcal{L}_{\#_\varrho \eta^\lambda} \xi^\lambda - \mathcal{L}_{\#_\varrho \xi^\lambda} \eta^\lambda - d\langle \xi^\lambda, \#_\varrho \eta^\lambda \rangle$

$0 = \langle \mathcal{L}_{U_\eta} \xi^\lambda, X^\lambda \rangle - \langle \mathcal{L}_{U_\xi} \eta^\lambda, X^\lambda \rangle - X^\lambda . \langle \xi^\lambda, U_\eta \rangle$

$(= -\langle \xi^\lambda, [U_\eta, X^\lambda] \rangle + \langle \eta^\lambda, [U_\xi, X^\lambda] \rangle - X^\lambda . \langle \xi^\lambda, U_\eta \rangle$ (12.21)

(4) $0 = -\langle \xi^\varrho, [Z, X^\varrho] \rangle + \langle \eta^\varrho, [Z, X^\varrho] \rangle - X^\varrho . \langle \xi^\varrho, Z_\eta \rangle$ (p 20-21)

(12.22) $$L_{\varphi_\xi}(P) = 0$$

which expresses the right-invariance of P. According to the second remark, the vector fields ψ_μ are in involution and Ad*-equivariant with respect to right-translations, i.e. they fulfil the conditions

(12.23) $$[\psi_\mu, \psi_\nu] = 0$$

(12.24) $$dR_k(h)\, \psi_\mu(h) = \psi_{Ad^*_k \mu}(h)$$

for any $k \in H$. This is proved by using the identity

(12.25) $$[\psi_\mu, \psi_\nu] = P\, \{\alpha_\mu, \alpha_\nu\}_P$$

characteristic of Poisson tensors ([B.1]), the invariance of P and the Ad*-equivariance of α_μ with respect to right-translations :

(12.26) $$P_{R_k(h)} = dR_k(h)\; P_h\; \delta R_k(h)$$

(12.27) $$\alpha_\mu(h) = \delta R_k(h)\; \alpha_{Ad^*_k \mu}(R_k(h))$$

In particular, the equivariance property (12.24) entails the following commutation relation between the vector fields φ_ξ and ψ_μ

(12.28) $$[\varphi_\xi, \psi_\mu] = \psi_{ad^*_\xi \mu}$$

so that from (12.5-3) and (12.23) we have

(12.29) $[\varphi_\xi, \varphi_\eta] = -\varphi_{ad_\xi \eta}$; $[\varphi_\xi, \psi_\mu] = \psi_{ad^*_\xi \mu}$; $[\psi_\mu, \psi_\nu] = 0$

They are the basic relations on which rests the proof of our statement; they show that the fields φ_ξ and ψ_μ make a Lie algebra isomorphic to the algebra of the semi-direct product $H \ni_{Ad^*} \mathcal{H}^*$ of the group H by the dual $\mathcal{H}^*$ of its Lie algebra.

To prove the statement, let us introduce the tensor $N := P\Omega$ and let us observe that

(12.30) $$N\varphi_\xi = P\Omega\,\varphi_\xi = P\,\alpha_{\Omega_e\xi} = \psi_{\Omega_e\xi}$$

We have to compute its torsion and to see when it vanishes. We obtain :

(12.31) $$T(N)(\varphi_\xi\ \varphi_\eta) :=$$

$$= [N\varphi_\xi, N\varphi_\eta] - N[\varphi_\xi, N\varphi_\eta] - N[N\varphi_\xi, \varphi_\eta] + N^2[\varphi_\xi, \varphi_\eta]$$

$$= [\psi_{\Omega_e\xi}, \psi_{\Omega_e\eta}] - N\left([\varphi_\xi, \psi_{\Omega_e\eta}] + [\psi_{\Omega_e\xi}, \varphi_\eta] + N\varphi_{ad_\xi\eta}\right)$$

$$= -N\cdot\psi_{(ad^*_\xi\Omega_e\eta - ad^*_\eta\Omega_e\xi + \Omega_e[\xi,\eta])} \overset{(12.17)}{=} 0$$

without any further condition on the cocycles Ω_e and P_e. Thus we have proved that H is endowed with a $P\Omega$ structure by any pair of cocycles Ω_e and P_e. Summarizing we have the following

Proposition 12.1 (Group-theoretical $P\Omega$ manifolds). Let H be a (finite or infinite-dimensional) Lie group admitting a Poisson cocycle P_e, i.e. a tensor $P_e : \mathcal{H}^* \to \mathcal{H}$ fulfilling the conditions

(12.32) $\langle \mu, P_e \nu \rangle = -\langle \nu, P_e \mu \rangle$

(12.33) $\langle \lambda, [P_e \mu, P_e \nu] \rangle$ + cyclic permutation $= 0$

Then the tensors P and Ω defined by

(12.34) $$P \hat{\alpha}_\mu := \hat{\varphi}_{P_e \mu}$$

(12.35) $$\Omega \varphi_\xi := \alpha_{\Omega_e \xi}$$

for any $\mathcal{H}^*$-cocycle Ω_e endow H with the structure of PΩ-manifold. Therefore any Lie group admitting one P-cocycle has as many PΩ structures as $\mathcal{H}^*$-cocycles.

Remark. Due to the right-invariance, the characteristic leaves of P are the right-cosets of the connected subgroup $H' \subset H$ whose algebra is

(12.36) $$\mathcal{H}' := P_e(\mathcal{H}^*)$$

13. The reduction of group-theoretical $P\Omega$ manifolds

Let us now perform the reduction of the group-theoretical $P\Omega$ structure previously defined, according to the general scheme outlined in Sec.8 for arbitrary $P\Omega$ manifolds. A clear understanding of this process will be essential to deal correctly with the applications in Sec.16.

We recall that the first step is the reduction over the characteristic leaves of P, which are, in the present case, the right-cosets of the connected subgroup H' whose algebra is $\mathcal{H}' = P_e(\mathcal{H}^*)$. Since these leaves are all diffeomorphic to H', we can limit ourselves to study the reduced structure on this subgroup. Let $f : H' \longrightarrow H$ be the canonical immersion of H' into H. The reduced tensors P' and Ω' can be computed either by using the general formulas (6.R9) and (6.R10) or, more easily in the present context, by first reducing the cocycles P_e and Ω_e to the subalgebra $\mathcal{H}'$, according to

$$(13.1) \qquad \Omega'_e := \delta f(e)\, \Omega_e\, df(e) \qquad\qquad P'_e := df(e)^{-1}\, P_e\, \delta f(e)^{-1}$$

(where $\delta f(e)^{-1}$ is any right-inverse of $\delta f(e) : \mathcal{H}^* \longrightarrow \mathcal{H}'^*$), and then by using the group-theoretical structure of H' to recover the whole $P\Omega$ structure. Thus, the first reduction simply amounts to replace the group H by the subgroup H', and the cocycles Ω_e and P_e by the cocycles Ω'_e and P'_e. Further on, in order to take the notation as simple as possible, we shall continue to use H, Ω_e, P_e without apices, to denote H', Ω'_e, P'_e, recalling that P_e is now a kernel-free Poisson cocycle.

To continue the reduction process, two different ways can be followed, according to H is regarded as a $P\Omega$ or a

PN manifold. They both provide valuable insights into the reduction process. According to the first point of view, we have to determine the characteristic distribution of Ω and to pass to the quotient manifold $H/\mathrm{Ker}\Omega$. This problem is simply solved by considering the left-cosets of the connected subgroup H_Ω whose algebra is

$$\mathcal{H}_\Omega := \mathrm{Ker}\ \Omega_e \tag{13.2}$$

Due to the left-invariance of Ω, they coincide with the leaves we are looking for and, accordingly, the reduced manifold is the homogeneous space H/H_Ω. According to the second point of view, we have to determine the characteristic leaves of $N: = P\Omega$. The first step is to characterize the leaves of the distributions $\mathrm{Ker}N$ and $\mathrm{Im}N$ and their intersections. This can be done as follows.

As for the leaves of the distribution $\mathrm{Ker}N$, it suffices to remark that $\mathrm{Ker}N = \mathrm{Ker}\,\Omega$, since P is kernel-free. So, such leaves are the left-cosets of the connected subgroup H_Ω. As for the leaves of the distribution $\mathrm{Im}N$, we need a deeper analysis of the properties of the groups admitting kernel-free Poisson cocycles. In particular, we have to introduce two new geometrical objects strictly related with P_e, that is a map $J : H \rightarrow \mathcal{H}^*$ and an action $\phi: \mathcal{H}^* \times H \longrightarrow \mathcal{H}^*$ of H on $\mathcal{H}^*$, which are called respectively the <u>momentum mapping</u> and the <u>canonical action</u> associated with the kernel-free Poisson cocycle. To introduce J, let us consider the vector-valued one-form $\theta: \mathfrak{X}(H) \longrightarrow \mathcal{H}^*$, with values in $\mathcal{H}^*$, uniquely defined by

$$\langle \theta(h), \psi_\mu(h) \rangle = \mu \tag{13.3}$$

(recall that now the vector fields ψ_μ span the whole tangent spaces $T_h H$ since P_e is kernel-free). This form is closed and Ad^*-equivariant (with respect to right-translations on the group), owing to the commutativity of the fields ψ_μ and of their Ad^*-equivariance. This means that θ obeys the conditions

(13.4) $$d\theta = 0$$

(13.5) $$< \delta R_\kappa(h)\ \theta(R_\kappa(h)),\ \psi_\mu(h) > = Ad^*_\kappa < \theta(h),\ \psi_\mu(h) >$$

as is proved by :

(13.6) $$< d\theta\ \psi_\mu\ ,\ \psi_\nu > := L_{\psi_\mu} < \theta, \psi_\nu > - L_{\psi_\nu} < \theta, \psi_\mu > + < \theta, [\psi_\mu, \psi_\nu] > = 0$$

(13.7) $$< \delta R_\kappa(h)\cdot \theta(R_\kappa(h)),\ \psi_\mu(h) > \overset{(12.22)}{=} < \theta(R_\kappa(h)),\ \psi_{Ad^*_\kappa \mu}(R_\kappa(h)) > = Ad^*_\kappa \mu$$

Hence, there exists a local map $J : U \to \mathcal{H}^*$, defined on a neighbourhood U of the identity e, such that

(13.8) $$dJ(h)\cdot \varphi(h) = < \theta(h),\ \varphi(h) >$$

for any vector field $\varphi \in \mathfrak{X}(H)$. It will be referred to as the (local) momentum mapping canonically associated with the Poisson cocycle P_e. By using Eq.s (13.8), (13.4) and the explicit form (12.20) of the fields ψ_μ, one readily obtains the equation

(13.9) $$dJ(h)\cdot \hat{\varphi}_\xi(h) = Ad^*_h \cdot P_e^{-1}\cdot \xi$$

allowing to locally compute the momentum mapping, for any given cocycle P_e .

On account of Eq.(13.5), the mapping J obeys the condition

$$(13.10) \qquad dJ\left(R_k(h)\right)\, dR_k(h) = Ad^*_k\ dJ(h)$$

entailing

$$(13.11) \qquad J\left(R_k(h)\right) = Ad^*_k\ J(h) + J(k)$$

((13.11) is obtained by integrating (13.10) with the condition J(e) = 0). Consequently, the momentum mapping verifies the coadjoint cocycle identity [17] and, therefore, the equation

$$(13.12) \qquad \Phi(\mu;k) \equiv \Phi_k(\mu) := J(k) + Ad^*_k\, \mu$$

defines a (local) right-action $\Phi : \mathcal{H}^* x U \to \mathcal{H}^*$ making J equivariant:

$$(13.13) \qquad \Phi_k\left(J(h)\right) = J\left(R_k(h)\right)$$

for any $h,k \in U$ such that $R_k(h) \in U$. It will be referred to as the (local) right-action of H on $\mathcal{H}^*$ canonically associated with the Poisson cocycle.

Although J and Φ have been so far introduced locally on U, they can be globally extended on the whole group H. Indeed, let k be any fixed element of U and $U' := R_k(U)$. We define a function $J' : U' \longrightarrow \mathcal{H}^*$ by

$$(13.14) \qquad J'\left(R_k(h)\right) := Ad^*_k\ J(h) + J(k)$$

Clearly, J and J' coincide on the intersection $U \cap U'$

$$(13.15) \qquad J\big|_{U\cap U'} = J'\big|_{U\cap U'}$$

so that J' defines a continuation of J from U to $U \cup U'$ (relative to the choice of k). To claim that, by iteration, J can be extended on the whole group H, we have to show that the result of the continuation does not depend on the way the continuation is performed. To this end, let k_1 and k_2 be two elements of U so near to the identity that k_1^{-1} and $k_1^{-1} \cdot k_2$ also belong to U. Hence, for any $h \in U$ we find :

$$\begin{aligned}(13.16) \quad J''(R_{k_1^{-1}k_2}(R_{k_1}(h))) &:= Ad^*_{k_1^{-1}k_2} J'(R_{k_1}(h)) + J'(k_1^{-1}k_2) \\ &= Ad^*_{k_2} Ad^*_{k_1^{-1}}\left(Ad^*_{k_1} J(h) + J(k_1)\right) + Ad^*_{k_2} J(k_1^{-1}) + J(k_2) \\ &= \left(Ad^*_{k_2} J(h) + J(k_2)\right) + Ad^*_{k_2}\left(Ad^*_{k_1^{-1}} J(k_1) + J(k_1^{-1})\right) \\ &= \left(Ad^*_{k_2} J(h) + J(k_2)\right) + Ad^*_{k_2} J(e) \\ &= Ad^*_{k_2} J(h) + J(k_2)\end{aligned}$$

showing that the continuation of J from U to $U_2 = R_{k_2}(U) = R_{k_1^{-1}k_2}(U_1)$ obtained either by passing through $U_1 = R_{k_1}(U)$ or directly coincide. Hence the function J is globally defined on the whole group H, where it globally verifies the conditions (13.8) and (13.11).

To point out how Φ and J allow to identify the characteristic leaves of N, and to perform the reduction of the PN structure, let us consider the connected subgroup H_Ω associated with the $\mathcal{H}^*$-cocycle Ω_e, and let us take the restrictions $\hat{J}$ and $\hat{\Phi}$ of J and Φ to $\mathcal{H}^*_\Omega$. They are defined by

$$(13.17) \qquad \hat{J} := \delta i(e)\; J$$

$$(13.18) \qquad \hat{\Phi}(\delta i(e)\mu\,; K) := \delta i(e)\;\Phi(\mu\,; i(K))$$

where $i : H_\Omega \longrightarrow H$ is the canonical immersion of H_Ω into H. From (13.13), it follows that

$$(13.19) \qquad \hat{\Phi}_K(\hat{J}(h)) = \hat{J}(R_K(h))$$

showing that $\hat{J}$ is equivariant with respect to $\hat{\Phi}$, as well as J is equivariant with respect to Φ. By differentiating (13.17) and (13.19) one obtains

$$(13.20) \qquad d\hat{J}(h)\,\psi_\mu(h) = \delta i(e)\, dJ(h)\cdot\psi_\mu(h) \overset{(13.9)}{=} \delta i(e)\,\mu$$

$$(13.21) \qquad d\hat{J}(h)\;\varphi_\xi(h) = d\hat{\Phi}_{\hat{J}(h)}(e)\cdot\xi$$

and, therefore, one proves that

$$(13.22) \qquad d\hat{J}\;\psi_\mu = 0$$

if $\mu\in(\mathcal{H}_\Omega)^\circ$, and that

$$(13.23) \qquad d\hat{J}(h)\;\varphi_\xi(h) = 0$$

if ξ belongs to the algebra $\mathcal{H}_\mu$ of the <u>isotropy group</u> of $\mu = \hat{J}(h)$, defined by

$$(13.24) \qquad H_\mu = \{\, K\in H_\Omega : \hat{\Phi}(\mu; K) = \mu \;,\; \mu\in\mathcal{H}^*_\Omega \,\}$$

Indeed, the annihilator $(\mathcal{H}_\Omega)^o$ of $\mathcal{H}_\Omega$ is the kernel of $\delta(\cdot)$ and $\mathcal{H}_\mu$ is the kernel of $d\hat{\Phi}_\mu(e)$.

To interpret the basic Eq.s (13.22) and (13.23) from the point of view of the reduction theory, let us observe that the vector fields ψ_μ, with $\mu \in (\mathcal{H}_\Omega)^o$, and φ_ξ, with $\xi \in \mathcal{H}_\Omega$, span respectively the distributions $\mathrm{Im}\,\bar{N}$ and Ker N, since $N\varphi_\xi = \psi_{\Omega_e \xi}$ and $\mathrm{Ker}\,N = \mathrm{Ker}\,\Omega$ (recall that $\mathrm{Im}\,\Omega_e = (\mathrm{Ker}\,\Omega_e)^o = \mathcal{H}_\Omega^o$, that P is kernel-free, and that Ω is left-invariant). Consequently, Eq.(13.22) means that the map $\hat{J}: H \longrightarrow \mathcal{H}_\Omega^*$ is constant along the leaves of the distribution Im N (assumed to be connected), and Eq.(13.23) means that the subspace $\mathrm{Im}N_h \cap \mathrm{Ker}N_h$, at the point h, is spanned by the left-invariant vector fields φ_ξ associated with the elements ξ of the algebra $\mathcal{H}_\mu$ of the isotropy group of $\mu = \hat{J}(h)$.

These two results allow to simply describe the reduction of the PN structure of H as follows. Let us consider any integral leaf of the distribution Im N. By the first result, this leaf coincides with a level surface, say $\hat{J}^{-1}(\mu)$, of the momentum mapping $\hat{J}: H \longrightarrow \mathcal{H}_\Omega^*$. This leaf inherits a PN structure from H, according to the reduction theorem of Sec.9. The reduced tensors N' and P', however, may have a kernel, since we have performed the reduction over Im N instead than over $\mathrm{Im}\,N^r$ (r being the Riezs index of the Nijenhuis tensor). The kernel of N', in particular, is spanned by the vector fields belonging to $\mathrm{Im}\,N \cap \mathrm{Ker}\,N$, evaluated at the points of $\hat{J}^{-1}(\mu)$. By the second result, this kernel is spanned by the vector fields φ_ξ associated with the algebra of the isotropy group of μ. Consequently, their integral leaves are the orbits of the points of $\hat{J}^{-1}(\mu)$ under the right-action of the isotropy group H_μ on H. The quotient space $\hat{J}^{-1}(\mu)/\mathrm{Ker}N'$, entering into the reduction process, coincides then with the quotient space $\hat{J}^{-1}(\mu)/H_\mu$. Assume this space to fulfil the standard assumptions of the projection lemma. Then it inherits a PN structure from H, according to the corollary of Prop.(9.1). If the Riezs index of N is r = 2, this structure cannot be further reduced. Otherwise, we have to iterate the process according to the general scheme of Sec.9. In this case, however, it is no longer possible to give a group theoretical interpretation of the reduction process, the reduced manifold being no longer a group.

For further reference, we collect the main results of the previous discussion into the following :

Proposition 13.1 (Reduction of group-theoretical PΩ manifolds)

Let H be a Lie group admitting a Poisson cocycle $P_e : \mathcal{H}^* \to \mathcal{H}$, and let Ω_e be any $\mathcal{H}^*$-cocycle of H. To reduce the corresponding PΩ structure of H, let us first consider the connected subgroup H' whose algebra is

$$(13.25) \qquad \mathcal{H}' := P_e(\mathcal{H}^*)$$

and let us reduce the cocycles P_e and Ω_e to H', by means of the formulas

$$(13.26) \qquad \Omega'_e := \delta f(e)\, \Omega_e\, df(e) \qquad P'_e := df^{-1}(e)\, P_e\, \delta f(e)^{-1}$$

where f : H' $\longrightarrow$ H is the canonical immersion of H' into H. They endow H' with a reduced PΩ structure defined by tensors P' and Ω' (with P' kernel-free) obtained by the general prescription of Prop.12.1. To further reduce the PΩ structure of H', in the case of Ω' having a kernel, let us consider the new connected subgroup H'_Ω of H' whose algebra is

$$(13.27) \qquad \mathcal{H}'_\Omega := \operatorname{Ker} \Omega'_e$$

and let us construct the local momentum mapping associated with the Poisson cocycle P'_e by integrating the equation

$$(13.28) \qquad dJ'(h') = Ad^*_{h'} \, P'^{-1}_e \, dR_{h'}(e)^{-1}$$

with the conditions $J'(e) = 0$. J' is globally defined. Associate with it the action Φ': $H' \times H'^* \longrightarrow H'^*$ defined by

$$(13.29) \qquad \Phi'(\mu'; h') := J'(h') + Ad^*_{h'} \mu'$$

and reduce J' and Φ' on $\mathcal{H}'^*_\Omega$ by means of the formulas

$$(13.30) \qquad \hat{J}'(h') := \delta i(e) \; J'(h')$$

$$(13.31) \qquad \hat{\Phi}'(\delta i(e)\mu'; k') := \delta i(e) \; \Phi'(\mu'; i(k'))$$

where $i : H'_\Omega \longrightarrow H'$ is the canonical immersion of H'_Ω into H'. Then the level surfaces of $\hat{J}'$ (assumed to be connected) are the integral leaves of the characteristic distribution $\operatorname{Im}\overleftarrow{N}'$, which are consequently parametrized by the covectors $\mu' \in \mathcal{H}'^*_\Omega$. To perform the reduction, let us fix any covector $\mu \in \mathcal{H}'^*_\Omega$, and let H'_μ be its isotropy group with respect to the action $\hat{\Phi}'$ of H'_Ω on $\mathcal{H}'^*_\Omega$. Assume the orbits of H'_μ on $\hat{J}'^{-1}(\mu)$ to be connected, the quotient space $\hat{J}'^{-1}(\mu)/H'_\mu$ to be a quotient manifold and the canonical projection $g : \hat{J}'^{-1}(\mu) \longrightarrow \hat{J}'^{-1}(\mu)/H'_\mu$

to be a surjective submersion according to the standard assumptions of the projection lemma. Then $\hat{J}'^{-1}(\mu)/H'_\mu$ inherits a reduced PN structure from H'. It can be computed by first restricting the PN structure of H' to $\hat{J}'^{-1}(\mu)$, according to the restriction lemma, and then by projecting the restricted structure along the orbits of H', according to the projection lemma. If the reduced structure turns out to be kernel-free the process ends; otherwise it must be iterated up to arrive to a kernel-free PN structure.

Remark. Since $\mathcal{H}'^*_\Omega \simeq \mathcal{H}'^*/(\mathcal{H}'_\Omega)^0$, in order to find the level surface $\hat{J}'^{-1}(\mu)$ and the isotropy group of μ with respect to $\hat{\Phi}'$, once J', Φ' and $\mathcal{H}'_\Omega$ are given, we can avoid Eq.s (13.30), (13.31), replacing them by the equations

(13.32) $$< J'(k') - \mu , \mathcal{H}'_\Omega > = 0$$

(13.33) $$< \Phi'_{k'}(\mu) - \mu , \mathcal{H}'_\Omega > = 0$$

14. PN structures on the dual of the Lie algebra

In this section we aim to study the Poisson cocycles from the point of view of the dual space $\mathcal{H}^*$, rather than from the point of view of the group H. The first noteworthy result is that P_e allows to define a $P\Omega$ structure on $\mathcal{H}^*$ as well as on H. To explain this result, let us recall that $\mathcal{H}^*$ carries a universal Poisson structure (not of constant rank), called the Lie-Kirillov Poisson structure of $\mathcal{H}^*$ [18,19] and defined by

$$K_\mu \, \xi := ad^*_\xi \, \mu \tag{14.1}$$

Furthermore, let us remark that, by means of the natural identifications $\mathcal{H} = T^*_\mu \mathcal{H}^*$ and $\mathcal{H}^* = T_\mu \mathcal{H}^*$, the cocycle P_e can now be considered as defining a (constant) presymplectic tensor field Ω_μ on $\mathcal{H}^*$ according to

$$\Omega_\mu \, \nu := P_e \, \nu \tag{14.2}$$

Let us form the product

$$N_\mu \, \nu := P_\mu \, \Omega_\mu \, \nu = ad^*_{P_e \nu} \, \mu \tag{14.3}$$

Then, it is a simple matter to see that the cocycle condition (12.12) implies that the torsion of N_μ vanishes so that the pair (P_μ, Ω_μ) endows $\mathcal{H}^*$ with a $P\Omega$ structure. Indeed, by using the local form (A.2.7) of the Nijenhuis condition we find

$$(14.4)\quad N'_\mu(\nu_1;N_\mu\nu_2)-N'_\mu(\nu_2;N_\mu\nu_1)-N_\mu\left(N'_\mu(\nu_1;\nu_2)-N'_\mu(\nu_2;\nu_1)\right)$$

$$= ad^*_{P_e\nu_1}\,ad^*_{P_e\nu_2}\,\mu - ad^*_{P_e\nu_2}\,ad^*_{P_e\nu_1}\,\mu - ad^*_{P_e(ad^*_{P_e\nu_1}\nu_2-ad^*_{P_e\nu_2}\nu_1)}\,\mu$$

$$\overset{(11.46)}{=} ad^*_{P_e\nu_1}\,ad^*_{P_e\nu_2}\,\mu - ad^*_{P_e\nu_2}\,ad^*_{P_e\nu_1}\,\mu + ad^*_{ad_{P_e\nu_1}P_e\nu_2}\,\mu$$

$$= 0$$

where the Jacobi's identity has been taken into account. To look for Poisson cocycles is then equivalent to look for presymplectic tensors on $\mathcal{H}^*$ converting the Lie-Kirillov structure of $\mathcal{H}^*$ into a $P\Omega$ structure.

Of course, there exists a close connection between the $P\Omega$ structure on the group H and that just defined on $\mathcal{H}^*$. It may be outlined as follows. Let us fix any point $\mu_0 \in \mathcal{H}^*$, and let us consider the orbit $\mathcal{O}_H(\mu_0)$ of the coadjoint representation passing through it. This orbit is a characteristic leaf of the Poisson-Kirillov tensor (14.1) and, consequently, it carries a reduced $P\Omega$ structure, according to the reduction lemma (8.1). Furthermore, let us consider the $P\Omega$ structure defined on H by P_e and by the $\mathcal{H}^*$-cocycle $\Omega_e^{(\mu_0)}$ defined by

$$(14.5)\qquad \Omega_e^{(\mu_0)}\,\xi := ad^*_\xi\,\mu_0$$

Then, the reduced $P\Omega$ structure on the orbit of μ_0 is the projection of the last $P\Omega$ structure on H, through the natural submersion

$$(14.6)\qquad g : h \mapsto \mu = Ad^*_h\,\mu_0 \in \mathcal{O}_H(\mu_0)$$

Let us now consider the connected subgroup H' defining the characteristic leaves of the Poisson tensor P on H, and the dual of its algebra $\mathcal{H}' = P_e(\mathcal{H}^*)$. As previously explained, H' carries a $P\Omega$ structure defined by the cocycles P'_e and Ω'_e (with P'_e kernel-free) given by Eq. (13.26). Consequently, the dual $\mathcal{H}'^* \simeq \mathcal{H}^*/\mathrm{Ker}P_e$ of its algebra carries a first $P\Omega$ structure (hereafter, to be referred to as the Kirillov $P\Omega$ structure of $\mathcal{H}'^*$), constructed according to the previous remarks.

It is now important to observe that $\mathcal{H}'^*$ carries a second distinguished structure, namely a PQ or "twofold Hamiltonian structure", defined by the tensors

$$(14.7) \qquad P_\mu \xi := P_e'^{-1} \xi + ad_\xi^* \mu$$

$$(14.8) \qquad Q_\mu \xi := \Omega'_e \xi$$

Indeed, the cocycle conditions (12.11) and (12.12) entail that P_μ is a Poisson tensor and that its Schouten bracket with Q_μ vanishes. This can be proved by computing the Fréchet derivative of P

$$(14.9) \qquad P'_\mu(\xi;\nu) = ad_\xi^* \nu$$

and by using the local forms (A.2 6) and (A.2 10) of the Schouten brackets $[\bar{P}, P]$ and $[\bar{P}, Q]$. One finds

$$(14.10) \quad \langle \xi, P'_\mu(\eta; P_\mu \theta)\rangle + \text{cyclic permutation} =$$

$$= \langle \xi, ad_\eta^* P_e'^{-1}\theta - P_e'^{-1} ad_\theta \eta - ad_\theta^* P_e'^{-1} \eta \rangle$$

$$+ \langle \xi, ad_\eta^* ad_e^* \mu - ad_{ad_\theta \eta}^* \mu - ad_\theta^* ad_\eta^* \mu \rangle$$

$$\overset{(1?.17)}{=} \langle \xi, ad_\eta^* ad_\theta^* \mu - ad_{ad_\theta \eta}^* \mu - ad_\theta^* ad_\eta^* \mu \rangle$$

so that the condition is verified on account of Jacobi's identity, and

$$(14.11) \quad \langle \xi, P'_\mu(\eta; Q_\mu \theta)\rangle + \text{cyclic permutation} =$$

$$= \langle \xi, \mathrm{ad}^*_\eta \Omega'_e \theta - \Omega'_e \mathrm{ad}_\theta \eta + \mathrm{ad}^*_\theta \Omega'_e \eta \rangle$$

$$\overset{(14.7)}{=} 0$$

as it was required.

Of course, also this second structure is deeply related with the geometry of the Poisson cocycles, as is shown by the remark that the tensor (14.7) is simply the infinitesimal generator of the (right-)action

$$(14.12) \qquad \Phi'(\mu; h) := J'(h) + \mathrm{Ad}^*_h \mu$$

making the momentum mapping J' equivariant. This is proved by

$$(14.13) \qquad d\,\Phi'_\mu(e)\cdot \xi = dJ'(e)\,\xi + \mathrm{ad}^*_\xi \mu = P_\mu \xi$$

on account of the relation

$$(14.14) \qquad dJ'(e) = P'^{-1}_e$$

connecting the momentum mapping and the Poisson cocycle according to the definition (13.28). So we must expect to be able to derive this second structure directly from the group-theoretical $P\Omega$ structure. This can be done as follows.

Let P_h and Ω_h be the group-theoretical $P\Omega$ tensors, evaluated at any point $h \in H'$, and let

(14.15) $$Q_h := P_h \, \Omega_h \, P_h$$

be the evaluation at the same point of the second Poisson tensor Q of the hierarchy canonically associated with P and Ω (see Prop.2.2 of Sec.2). Then we state that it is :

(14.16) $$P_\mu = - dJ'(h) \, P_h \, \delta J'(h)$$

(14.17) $$Q_\mu = - dJ'(h) \cdot Q_h \, \delta J'(h)$$

where $\mu = J'(h)$. This means that the twofold Hamiltonian structures previously pointed out on $\mathcal{H}'^*$ and H' are J'-related. To prove (14.16) and (14.17) it suffices to observe that, according to the definition (13.28) of J',

(14.18) $$P_h \, \delta J'(h) \cdot \xi = - \varphi_\xi (h)$$

and that from the relation (13.11) one gets

(14.19) $$dJ'(h) \, \varphi_\xi (h) = dJ'(e) \, \xi + ad^*_\xi \, J(h)$$

Then at any point $\mu = J'(h)$ it is :

(14.20) $$- dJ'(h) \cdot P_h \, \delta J'(h) \, \xi \overset{(14.18)}{=} dJ'(h) \, \varphi_\xi (h)$$
$$\overset{(14.19)}{=} P_e^{-1} \xi + ad^*_\xi \, J'(h)$$
$$= P_\mu \, \xi$$

$$(14.21)\quad dJ'(h)\, Q_h\, \delta J'(h)\, \xi = dJ'(h)\, P_h \Omega_h P_h\, \delta J'(h)\, \xi$$

$$= -dJ'(h)\, P_h \Omega_h\, \varphi_\xi(h)$$

$$\overset{(12.30)}{=} -dJ'(h)\, \psi_{\Omega,\xi}(h)$$

$$\overset{(13.9)}{=} -\Omega_e\, \xi$$

$$= -Q_\mu\, \xi$$

Summarizing the previous results we get the following

Proposition 14.1 (Hamiltonian structures on the dual of a Lie algebra)

Let $\mathcal{H}^*$ be the dual of the algebra of a Lie group H admitting a Poisson cocycle P_e. Then $\mathcal{H}^*$ carries a natural $P\Omega$ structure (the "Kirillov $P\Omega$ structure") defined by the tensors

$$(14.22)\qquad K_\mu\, \xi := ad^*_\xi\, \mu$$

$$(14.23)\qquad \Omega_\mu\, \nu := P_e\, \nu$$

Moreover, the dual $\mathcal{H}'^* \simeq \mathcal{H}^*/\mathrm{Ker}\, P_e$ of the connected subgroup H' whose algebra is

$$(14.24)\qquad \mathcal{H}' := P_e(\mathcal{H}^*)$$

carries also a PQ or twofold Hamiltonian structure, defined by the tensors

$$(14.25) \qquad P_\mu \, \xi := P_e^{\prime -1} \xi + ad^*_\xi \, \mu$$

$$(14.26) \qquad Q_\mu \, \xi := \Omega'_e \, \xi$$

where P'_e and Ω'_e are the cocycles reduced over H'. This structure is J'-related with the twofold Hamiltonian structure canonically associated with H'.

15. The PN structure of the equations solvable by the inverse scattering method (in one space-dimension)

In this section we show the explicit construction of a family of $P\Omega$ structures generalizing that of Sec.11, on a particular Lie group H. Its reduction will be discussed in Sec.16.

Let V be a finite-dimensional vector space over R (or C), $W := V^{n+1}$ the Cartesian product of (n+1) replicas of V, G_1 = Aut W the group of the automorphisms of W and H the space of C^∞ functions $h : R \longrightarrow \mathrm{Aut}W$ fulfilling the asymptotic conditions $h \longrightarrow id_W$ for $|x| \longrightarrow \infty$. Any such a function will be written in the matrix form

$$(15.1) \qquad h = e_j h^j_k(x) e^k = h_k e^k = e_j h^j \qquad (h^j_k(x) \in \mathrm{Aut}\, V)$$

where $e_j = (0,\ldots 1,\ldots,0)^T$ and $e^K = (0,\ldots 1,\ldots,0)$ are the elements of the standard basis for column and row vectors respectively, $1 := id_V$ is the identity function on V, and $h_k := e_j h^j_k$ and $h^j := h^j_k e^k$ denote the k-th column and the j-th row of the matrix h. The space H can be considered as a manifold modelled on the Fréchet space E of the C^∞ functions $\varphi : R \longrightarrow \mathrm{End}\, W$ rapidly decreasing for $|x| \to \infty$, endowed with the Schwartz topology [20]. It becomes a Lie group by defining the product h.k pointwise by

$$(15.2) \qquad (h.k)(x) := h(x)\; k(x)$$

that is by considering H as the direct product of an infinite number of replicas of G = Aut W, any replica $H_x = G$ being labelled by a point $x \in R$. The identity of H is the mapping

$e : R \to \operatorname{Aut} W$ which associates the identity matrix at any point x. The algebra $\mathcal{H} = T_e H$ of H is identified with the space E of the rapidly decreasing functions $\xi : R \longrightarrow \operatorname{End} W$ and the structure of Lie algebra is defined by the usual commutator of matrices

$$(15.3) \qquad \operatorname{ad}_\xi \eta = [\xi, \eta] := \xi\eta - \eta\xi$$

As a dual space of $\mathcal{H}$, $\mathcal{H}^* = T_e^* H$, we consider the space $\hat{E}$ of the matrix-valued functions $\mu : R \to \operatorname{End} W$ fulfilling the condition

$$(15.4) \qquad \int_{-\infty}^{+\infty} \mu(x)\, dx = 0$$

The two spaces $\hat{E}$ and E are put in duality by the pairing $\langle\,,\,\rangle : \hat{E} \times E \to R$

$$(15.5) \qquad \langle \mu, \xi \rangle := \operatorname{Tr} \int_{-\infty}^{+\infty} \mu(x) \cdot \xi(x)\, dx$$

where $\operatorname{Tr} \mu.\xi := \operatorname{Tr} (\mu^i_j\, \xi^j_i)$. This pairing is separating in both arguments [21]. Finally, the right-invariant fields $\hat{\varphi}_\xi$ and the left-invariant fields φ_ξ are simply defined by

$$(15.6) \qquad \hat{\varphi}_\xi(h) := \xi h \qquad\qquad \varphi_\xi(h) := h.\xi$$

Having defined the ambient space where to set up the construction of the $P\Omega$ structure, the first step is the definition of the Poisson cocycle $P_e : \mathcal{H}^* \to \mathcal{H}$. Let $D : \mathcal{H} \to \mathcal{H}^*$ be the linear and continuous operator (in the topology we have considered) defined by

$$(15.7) \qquad D\,\xi := \xi_x$$

Clearly, D is kernel-free (on account of the asymptotic conditions for ξ) and its range is the whole dual space $\mathcal{H}^*$ (by (15.4)); moreover, D is an $\mathcal{H}^*$-cocycle on the group H since

$$(15.8) \quad \langle D\xi_1, \xi_2 \rangle + \langle D\xi_2, \xi_1 \rangle = \mathrm{Tr} \int_{-\infty}^{+\infty} (\xi_{1x}\xi_2 + \xi_{2x}\xi_1)\, dx = 0$$

and

$$(15.9) \quad \langle \xi_1, D[\xi_2, \xi_3] \rangle + \cdots + \cdots = 2\,\mathrm{Tr} \int_{-\infty}^{+\infty} (\xi_1\xi_2\xi_3 - \xi_1\xi_3\xi_2)\, dx = 0$$

Thus $P_e := D^{-1} : \mathcal{H}^* \to \mathcal{H}$ is a (kernel-free) Poisson cocycle

$$(15.10) \qquad P_e\,\mu := \int_{-\infty}^{x} \mu(x)\, dx$$

On account of the form (15.6) of the right-invariant fields and of the corresponding right-invariant one-forms

$$(15.11) \qquad \hat{\alpha}_\mu(h) := h^{-1}\mu$$

it follows that the kernel-free Poisson tensor $P : \mathfrak{X}^*(H) \to \mathfrak{X}(H)$ defined by (12.8) takes the explicit form

$$(15.12) \qquad P_h\,\alpha := \left(\int_{-\infty}^{x} h\,\alpha\, dx\right) h \qquad \alpha \in \mathfrak{X}^*(H)$$

To complete the $P\Omega$ structure, we choose the following $\mathcal{H}^*$-cocycle :

$$(15.13) \qquad \Omega_e\,\xi := [a, \xi]$$

where a is any arbitrary matrix with constant entries. Thus the left-invariant presymplectic tensor $\Omega : \mathcal{X}(H) \longrightarrow \mathcal{X}^*(H)$, defined by (12.15), takes the explicit form

$$(15.14) \qquad \Omega_h \varphi := [a, h^{-1}\varphi]\, h^{-1} \qquad \varphi \in \mathcal{X}(H)$$

Since Ω is left-invariant and P right-invariant, the tensor $N := P \cdot \Omega : \mathcal{X}(H) \to \mathcal{X}(H)$ is a Nijenhuis tensor for any choice of the matrix a

$$(15.15) \qquad N_h \varphi := P_h \Omega_h \varphi = \left(\int_{-\infty}^{x} h\, [a, h^{-1}\varphi]\, h^{-1} dx \right) h$$

Thus the group H is endowed with a whole family of $P\Omega$ and PN structures (defined by (15.13), (15.14), (15.15)), each element of this family being parametrized by a matrix a.

Since P is kernel-free, the subgroup H' defining its characteristic leaves coincide with H. Hence, we are not required to perform the first reduction over H', and the momentum mapping J and the action Φ are defined on the whole H. By integrating the equation (13.9) with $Ad_n^* \mu = h^{-1}\mu h$ and $P_e^{-1} = \partial_x$, that is the equation

$$(15.16) \qquad dJ(h)\varphi = h^{-1} \partial_x (\varphi h^{-1})\, h$$

and by requiring the condition $J(e) = 0$ we get

$$(15.17) \qquad J(h) = h^{-1} h_x$$

The action Φ making J equivariant is then given, on account of (13.12) by

(15.18) $$\Phi_h(\mu) := h^{-1}h_x + h^{-1}\mu h$$

Finally, let us compute the $P\Omega$ and the PQ structure of $\mathcal{H}^*$. According to the discussion of the previous section they are given by

(15.19) $$K_\mu \xi := ad^*_\xi \mu = [\mu, \xi]$$

(15.20) $$\Omega_\mu \nu := P_e \nu = \int_{-\infty}^{x} \nu(x)\,dx$$

and by

(15.21) $$P_\mu \xi := P_e^{-1}\xi + ad^*_\xi \mu = \xi_x + [\mu, \xi]$$

(15.22) $$Q_\mu \xi := \Omega_e \xi = ad^*_\xi a = [a, \xi]$$

respectively. Clearly, (15.21) and (15.22) reduce themselves to the structure discussed in Sec.11 for the particular case of (2x2) matrices. One can recognize in (15.19), (15.20) the integrability structure of the so-called "chiral field equations", and in (15.21),(15.22) that of the so-called "Zacharov-Shabat matrix spectral problem" [22-24]. Indeed, from the study of these tensors, pursued along the lines pointed out in Sec.s 13 and 14, one can recover the main classes of the equations solvable by means of the inverse spectral transform technique. An explicit example will be worked out, in detail, in the next section.

16. The PN structure of the Gel'fand-Dikii equations

In this section, the PΩ structure previously constructed is reduced in correspondence with the particular choice

$$(16.1) \qquad a := e_0 e^* = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix}$$

of the arbitrary matrix a. According to the general scheme of Sec.13, we have to perform the following steps :

i) to specify the algebra $\mathcal{H}_\Omega$ of the isotropy subgroup of a by solving the equation

$$(16.2) \qquad [a, \eta] = 0$$

ii) to fix arbitrarily an element $b \in \mathcal{H}_\Omega^*$ (i.e. a matrix b which is not orthogonal to $\mathcal{H}_\Omega$) and to construct the level surface $S_b = \hat{J}^{-1}(b)$ of the momentum mapping restricted to the subgroup H_Ω, by solving the equation (13.32):

$$(16.3) \qquad \langle J(h) - b, \eta \rangle = 0 \qquad \forall \eta \in \mathcal{H}_\Omega$$

iii) to find the isotropy subgroup of b in H_Ω, with respect to the reduced equivariant action $\hat{\Phi}: H_\Omega \times \mathcal{H}_\Omega^* \longrightarrow \mathcal{H}_\Omega^*$ introduced in Sec.13, by solving the equations

$$(16.4) \qquad \begin{cases} Ad_h^* a = a & : \quad h^{-1} a h = a \\ \langle \Phi_h(b) - b, \eta \rangle = 0 & : \quad \mathrm{Tr}\,(h^{-1} b h + h^{-1} h_x - b)\eta = 0 \end{cases}$$

The former means the $h \in H_\Omega$; the latter defines the isotropy subgroup of b in H_Ω . As shown in Sec.13, this subgroup keeps the level surface S_b invariant

$$R_h(S_b) \subset S_b \qquad \text{if} \quad h \in H_b \tag{16.5}$$

and allows to define the reduced phase space as the quotient space S_b/H_b .

iv) Once this space has been identified, one has finally to reduce the PΩ structure on S_b/H_b, first by performing the restriction to S_b and then by passing to the quotient on S_b/H_b. The details of this procedure will be specified below.

16.1. The first part of this program is readily performed if we remark that the algebra $\mathcal{H}_\Omega$ of the isotropy group of a is spanned by the matrices $\{I, e_a e^{b+1} \ (a,b = 0,1,\dots n-1)\}$. The abstract Eq.(16.3), which must be solved in order to determine the level surfaces S_b of the momentum mapping, then reads

$$\int_{-\infty}^{+\infty} \mathrm{Tr}\,(h^{-1}h_x - b)\,dx = 0 \qquad \int_{-\infty}^{+\infty} \mathrm{Tr}\,(h^{-1}h_x - b)\,e_a e^{b+1}\,dx = 0 \tag{16.6}$$

Let us choose, at this point, the arbitrary matrix b . As already remarked, this choice is in principle a matter of taste (a reduction being associated with any choice of b); however, it largely influences the possibility of carrying out in practice the reduction process. A particularly convenient choice is the following one

(16.7) $$b := e_{a+1} e^{a} = \begin{bmatrix} 0 & & & 0 \\ 1 & & & \vdots \\ & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

It leads to the integrability structure of the equations associated with the matrix Gel'fand-Dikii-type spectral problem. The reason for the choice (16.7) is that it leads to a simple characterization of the level surface S_b. Indeed, (16. 6) becomes

(16.8) $$\mathrm{Tr}\ h^{-1} h_x = 0$$

(16.9) $$\mathrm{Tr}\ e^{b+1} h^{-1} (h_{ax} - h_{a+1}) = 0 \qquad (a, b = 0, 1, \ldots, n-1)$$

where

(16.10) $$h_a := h\, e_a$$

is the a-th column of the matrix h.

By the first condition, the matrix h is a unimodular matrix (up to a scale factor). By the second condition, the vectors $h^{-1}(h_{ax}-h_{a+1})$ are orthogonal to the base vectors e^{b+1} for $b = 0,1,\ldots,n-1$. Consequently, they are all proportional to e_o, and we can write

(16.11) $$h^{-1} (h_{ax} - h_{a+1}) = e_o\, v_a$$

where the functions v_a are arbitrary Lagrangean multipliers. By (16.11), the columns of the matrix h obey the iterative condition

(16.12) $$h_{a+1} = h_{ax} - h_o\, v_a$$

showing that the matrix h can be recursively obtained from the first column and from the Lagrangean multipliers v_a. The general solution is

(16.13) $$h = w . t$$

where w is the unimodular (i.e., such that $\mathrm{Tr}\, w^{-1} w_x = 0$) Wronskian matrix associated with the first column h_o of h

(16.14) $$w := Wr\,(h_o) = (\partial_i h_o)\, e^i$$

and t is the upper triangular matrix (with unit diagonal entries) defined by ($j \geq k$)

(16.15) $$t = e_k\, t^k_j\, e^j \qquad t^k_j := -\sum_{k}^{j}{}_{\ell} \binom{\ell}{k} \partial_{\ell-k}\, v_{j-\ell-1} \quad (v_{-1} \equiv -1)$$

This can be proved as follows. From (16.12) we get

(16.16) $$\begin{aligned} h_j &= -\sum_{o}^{j}{}_{\ell}\; \partial_\ell\, (h_o\, v_{j-\ell-1}) \\ &= -\sum_{o}^{j}{}_{\ell} \sum_{o}^{\ell}{}_{m} \binom{\ell}{m} (\partial_m h_o)\, (\partial_{\ell-m}\, v_{j-\ell-1}) \\ &= -\sum_{o}^{j}{}_{m} \sum_{m}^{j}{}_{\ell} \binom{\ell}{m} (\partial_m h_o)\, (\partial_{\ell-m}\, v_{j-\ell-1}) \end{aligned}$$

so that the matrix h is given by

$$(16.17)\quad h = \sum_{0}^{n}{}_j\, h_j\, e^j = -\sum_{0}^{n}{}_j \sum_{0}^{j}{}_m \sum_{m}^{j}{}_\ell \binom{\ell}{m} (\partial_m h_o)(\partial_{\ell-m}\, v_{j-\ell-1})$$
$$= -\sum_{0}^{n}{}_m (\partial_m h_o) \sum_{m}^{n}{}_j \sum_{m}^{j}{}_\ell \binom{\ell}{m} (\partial_{\ell-m}\, v_{j-\ell-1})$$
$$= \sum_{0}^{n}{}_m w_m\, t^m = w\, t$$

As it will be shown in the following, the matrices t appearing in (16.13) make a Lie group T. We can then summarize the previous result by saying that :

<u>Proposition 16.1</u>. The level surface S_b of the momentum mapping corresponding to the choice (16.7) is diffeomorphic to the product of the manifold W of the unimodular Wronskian matrices and of the subgroup T of the group of upper triangular matrices specified by (16.15). The diffeomorphism is given by

$$(16.18)\qquad h = w \cdot t$$

<u>16.2</u>. Having characterized in this way the surface S_b, let us proceed in our program by finding the isotropy subgroup of b. To this end, we must solve Eq.s (16.4), or explicitly $(a, b = 0, 1, \dots, n-1)$:

$$(16.19)\qquad h^{-1} a\, h = a$$

$$(16.20)\qquad \mathrm{Tr}\,(h^{-1} b\, h + h^{-1} h_x - b) = 0$$

$$(16.21)\qquad \mathrm{Tr}\; e^{b+1} (h^{-1} b h + h^{-1} h_x - b)\; e_a = 0$$

This system is discussed in Appendix D.1, where it is shown that the matrices h are upper triangular matrices (with unit diagonal entries) given by

$$(16.22) \qquad h_j^k = 0 \ (k<j) \ ; \ h_j^j = 1 \ ; \ h_j^k = -\sum_{\ell=k}^{j} \binom{\ell}{k} \left(\partial_{\ell-k} \, v_{j-\ell-1}\right) \qquad (j>k)$$

where v_a are arbitrary Lagrangean multipliers. One verifies in this way that the isotropy subgroup of b is <u>exactly</u> the group T previously determined. This important result allows to immediately solve the problem of finding the quotient space S_b/H_b. Indeed, Eq. (16.18) says that S_b is obtained by letting the group $H_b = T$ act (by right-translation) on the manifold W of the Wronskian unimodular matrices. Moreover, Eq.(16.14) says that just a unique Wronskian matrix corresponds to any point $h \in S_b$. The action being free, we have

<u>Proposition 16.2</u>. The quotient space S_b/H_b, associated with the choice (16.7) of the matrix b, is diffeomorphic to the manifold of the unimodular Wronskian matrices. The canonical projection $g : S_b \longrightarrow S_b/H_b$ is given by

$$(16.23) \qquad g : h \longmapsto w = Wr(h_o) = (\partial_j h_o) e^j$$

<u>16.3</u>. At this point, we have at our disposal all the elements which are needed to perform the reduction, namely :

i) the manifold S_b and its parametrization $f : W \times T \longrightarrow S_b$ given by $h = wt$.

ii) the quotient manifold $S_b/H_b \simeq W$ and the canonical projection $g : W \times T \longrightarrow W$ simply given by $g : (w,t) \longmapsto w$.

So, for example, it is now quite simple to obtain the Nijenhuis tensor (or "recursion operator") associated with the Gel'fand-Dikii equations [25]. As previously explained, we proceed in two steps: first, we restrict the Nijenhuis tensor

(16.24) $$\bar{\varphi} = N\varphi = Q\Omega\varphi \overset{(15.15)}{=} \left(\int_{-\infty}^{x} h\left[a, h^{-1}\varphi\right] h^{-1} dx \right) h$$

to the leaf S_b, then we pass to the quotient over S_b/H_b. The first step is accomplished by

(16.25) $$\bar{\varphi}_w\, t + w\, \bar{\varphi}_t =$$

$$= \left(\int_{-\infty}^{x} w\left[t a t^{-1}, w^{-1}\varphi_w + \varphi_t t^{-1} \right] w^{-1} dx \right) w t$$

$$= \left(\int_{-\infty}^{x} w\left[a, w^{-1}\varphi_w \right] w^{-1} dx \right) w t$$

Since

(16.26) $$\left(\bar{\varphi}_w\, t + w\, \bar{\varphi}_t \right) e_0 = \bar{\varphi}_w \left(t e_0 \right) + w\left(\bar{\varphi}_t e_0 \right) = \bar{\varphi}_w\, e_0$$

Eq.(16.25) splits into

(16.27) $$\bar{\varphi}_w = \mathcal{W}_l\left(\left(\int_{-\infty}^{x} w\left[a, w^{-1}\varphi_w \right] w^{-1} dx \right) w\, e_0 \right)$$

and

$$(16.28) \quad \bar{\varphi}_t = w^{-1}\left(\int_{-\infty}^{x} w\,[a, w^{-1}\varphi_w]\,w^{-1}dx \right) w t \;+$$

$$- \; w^{-1}\; Wr\left(\left(\int_{-\infty}^{x} w\,[a, w^{-1}\varphi_w]\,w^{-1}dx\right) w e_0 \right)$$

These equations define the reduced Nijenhuis tensor on the leaf S_b in a component-wise form adapted to the splitting of the leaf into Wronskian and triangular matrices. The last step, namely the projection over the space of the unimodular Wronskian matrices, is now trivial, the reduced tensor $N_W : \varphi_w \mapsto \bar{\varphi}_w$ being simply given by

$$(16.29) \quad \bar{\varphi}_w = Wr\left(\left(\int_{-\infty}^{x} w\,[a, w^{-1}\varphi_w]\,w^{-1}dx\right) w e_0 \right)$$

Since it can be shown [25 , p.619-621] that N_W is kernel-free, the reduction process ends, no further reduction being possible.

16.4. Let us now analyze the problem of the reduction of the Poisson structure. In order to avoid cumbersome calculations in dealing with this problem, it is useful to make the following remarks.

Let us use the momentum mapping $J : H \longrightarrow \mathcal{H}^*$ given by

$$(16.30) \qquad u = h^{-1} h_x$$

to set our study in the framework of the dual $\mathcal{H}^*$ of the algebra of H (remark : we have slightly changed the notations, by using u in place of μ , in order to emphasize that from now on $\mathcal{H}^*$ will be regarded as a manifold rather than as the dual of a Lie algebra). As it has been explained in Sec.14, the diffeomorphism (16.30) maps $S_b \subset H$ into the characteristic leaf S_b^* (passing at $b \in \mathcal{H}^*$) of the Poisson tensor on $\mathcal{H}^*$ defined by :

$$(16.31) \qquad \varphi = Q\alpha \ : \qquad \varphi = [\, a, \alpha \,]$$

A straightforward computation shows that this leaf is the <u>affine hyperplane</u> defined by

$$(16.32) \qquad u \in S_b^* \ : \qquad u = \begin{bmatrix} u_0 & u_1 & & u_{n-1} & u^0 \\ 1 & & & & u^1 \\ & 1 & & & \vdots \\ & & \ddots & & u^{n-1} \\ & & & 1 & -u_0 \end{bmatrix}$$

where the 2n functions (u_a, u^a) , appearing in the first row and in the last column of u will be referred to as the <u>affine coordinates</u> of the point $u \in S_b^*$. Hence, we see that the manifold S_b is diffeomorphic not only to the Cartesian product WxT but also to the affine hyperplane S_b^*. Conversely, by using the properties of S_b, we can easily conclude that S_b^* is diffeomorphic to the Cartesian product of the group T by the vector space F of the <u>traceless Frobenius matrices</u> [26]

$$(16.33) \qquad v = \begin{bmatrix} 0 & & & & v^0 \\ 1 & & & & v^1 \\ & 1 & & & \vdots \\ & & \ddots & & v^{n-1} \\ & & & 1 & 0 \end{bmatrix}$$

defining a $\mathcal{G}(\mathcal{H}^*)$- module of dimension n, naturally parametrized by the functions v^a. Indeed, by using the diffeomorsphim (16.18) : $h = wt$, from (16.30)we readily obtain

(16.34) $$u = h^{-1}h_x = t^{-1}w^{-1}(w_x t + w t_x)$$
$$= t^{-1}(w^{-1}w_x)t + t^{-1}t_x$$
$$= t^{-1} v\, t + t^{-1}t_x$$

where we have taken into account that the momentum-mapping $J : H \longrightarrow \mathcal{H}^*$ maps the manifold W of the unimodular Wronskian matrices onto the linear space F of the traceless Frobenius matrices. Eq. (16.34) says that the characteristic leaf S^*_b in $\mathcal{H}^*$ is simply the disjoint union of the orbits of the points of F under the action (15.18) of the group T. Symbolically, we can write

(16.35) $$S^*_b = \Phi_T(F)$$

The reduction of the Poisson structure can then be performed as follows : starting from the PΩ structure on $\mathcal{H}^*$ defined by (see Sec. 15)

(16.36) $$\varphi = Q\alpha \quad : \qquad \varphi = [a, \alpha]$$

(16.37) $$\varphi = \Omega^{-1}\alpha \quad : \qquad \varphi = \alpha_x + [u, \alpha]$$

we first restrict to S^*_b and then pass to the quotient on $S^*_b/T \simeq F$. The advantage of passing on the dual $\mathcal{H}^*$ is now clear: by so doing, in fact, we have "linearized" the quotient manifold, replacing the unimodular Wronskian matrices by the traceless Frobenius matrices.

16.5. To perform the reduction, it may be suitable, from a purely technical point of view, to replace the matrix formalism which has been used so far by a coordinate approach. Let us remark that all the manifolds we shall deal with, that is : the characteristic leaf S^*_b , the isotropy subgroup T and the reduced space F, are naturally endowed with a global coordinate system. Indeed:

i) the manifold S^*_b, defined by (16.32), is naturally parametrized by the 2n functions $(u_a := u^o_a,\ u^\alpha := u^\alpha_n)$, namely by the entries of the first row and of the last column of the matrix u. Further on, such coordinates will be referred to as the "affine coordinates" on S^*_b.

ii) the group T is naturally parametrized by the n coordinates v_a (the "Lagrangean multipliers") by means of which the entries t^j_k are given by (16.15). Further on, the coordinates v_a will be referred to as the "group-theoretical parameters".

iii) the quotient space F, finally, is naturally parametrized by the n coordinates $(v^a := v^a_n)$, that is by the entries of the last column of the Frobenius matrix v.

Taken together, the 2n coordinates (v_a, v^a) define a second system of coordinates on S^*_b which can be referred to as the coordinates adapted to the fibration of S^*_b. By using the relation (16.34), it is not difficult to show that the fibered coordinates (v_a, v^a) are linked to the affine coordinates (u_a, u^a) through the relations

$$(16.39) \qquad \begin{cases} v_a = u_a \\ v^a = -\sum_{a}^{n+1}{}_{\ell} \sum_{\ell}^{n+1}{}_{k} \binom{\ell}{a} \left(\partial_{\ell-a} u_{k-\ell-1}\right) u^k \end{cases}$$

where, to simplify the notations, we have set

$$(16.40) \qquad u^n := 0 \qquad u^{n+1} := -1 \qquad u_{-1} := -1$$

(Eq. (16.39) is proved in Appendix D.2).

Both coordinate systems on S_b^* are well-suited for the reduction: the affine coordinates (u_a, u^a) are especially suitable in order to perform the restriction, whereas the fibered coordinates are more useful for the projection process. Accordingly, the reduction process will be divided into three phases :

i) first, we shall perform the restriction by using the affine coordinates

ii) next, we shall perform a change of coordinates, passing to the fibered coordinates by means of (16.39)

iii) finally, we shall perform the projection process, which will turn out to be particularly simple in the fibered coordinates.

16.6. We begin by restricting the $P\Omega$ structure of $\mathcal{H}^*$ to the affine hyperplane S_b^*, by using the affine coordinates (u_a, u^a). Since the parametric equation of S_b^* is

$$(16.41) \qquad u = \sum_{0}^{n-1}{}_{a} \left(u_a \sigma^a + u^a \sigma_a\right) + b$$

where the matrices

(16.42) $\sigma^a := e_o e^a - \delta^a_o e_n e^n \qquad \sigma_a := e_a e^n$

define the natural basis associated with the affine coordinates, any tangent vector φ at u can be written as

(16.43) $$\varphi = \sum_o^{n-1}{}_a \left(\varphi_a \sigma^a + \varphi^a \sigma_a \right)$$

where the arbitrary functions (φ_a, φ^a) play the role of "affine components" of the tangent vector. Then, from

(16.44) $$\mathrm{Tr}\, \alpha\varphi = \varphi_a \cdot \mathrm{Tr}\left(\alpha\sigma^a\right) + \varphi^a \cdot \mathrm{Tr}\left(\alpha\sigma_a\right) = \varphi_a\left(\alpha^a_o - \delta^a_o \alpha^n_n\right) + \varphi^a \alpha^n_a$$

we see that any covector α at $u \in S^*_b$ is characterized by the "affine (covariant) components"

(16.45) $\alpha^a := \alpha^a_o - \delta^a_o \alpha^n_n \qquad \alpha_a := \alpha^n_a$

To have the affine components also of the tensor Q, let us write explicitly Eq.s (16.41) and (16.42) in the form

(16.46a) $u^o_a = u_a \qquad u^a_n = u^a \qquad u^n_n = -u_o \qquad u^{a+1}_a = 1$

(16.46b) $\varphi^o_a = \varphi_a \qquad \varphi^a_n = \varphi^a \qquad \varphi^n_n = -\varphi_o$

(the remaining components being zero), and let us remark that in the natural coordinates u^j_k the Poisson tensor (16.36) takes the form

(16.47) $\varphi^o_a = \alpha^n_a \qquad \varphi^a_n = \delta^a_o \alpha^n_n - \alpha^a_o \qquad \varphi^n_n = -\alpha^n_o \qquad \varphi^{a+1}_a = 0$

So, the usual transformation scheme for Poisson tensor entails that

(16.48) $$\varphi_a \overset{(16.46b)}{=} \overset{o}{\varphi}_a \overset{(16.47)}{=} \alpha^m_a \overset{(16.45)}{=} \alpha_a$$

(16.49) $$\varphi^a = \varphi^a_m = \delta^a_m \alpha^m_m - \alpha^a_o = -\alpha^a$$

Eq.s(16.48) and (16.49) define the Poisson tensor restricted to S^*_b : they clearly show that the restriction of $\varphi = [\alpha, \alpha]$ to S^*_b is simply given by the canonical Poisson tensor

(16.50) $$\left[\begin{array}{c}\varphi_a \\ \hline \varphi^a\end{array}\right] \quad \left[\begin{array}{c|c} 0 & \mathbb{I} \\ \hline -\mathbb{I} & 0 \end{array}\right] \quad \left[\begin{array}{c}\alpha^a \\ \hline \alpha_a\end{array}\right]$$

i.e. that the affine coordinates on S^*_b are canonical coordinates.

The restriction of the symplectic tensor Ω is carried out in the same way. However, one has to recall that the explicit form of Ω^{-1} is known

(16.51) $$\varphi = \alpha_x + [u, \alpha]$$

so that, in practice, for any given pair u, φ given by (16.41) and (16.43), one has to compute the corresponding one-form α by solving Eq.(16.51), and then to determine its affine components by means of (16.45). Explicitly, this would amount to solve (at least in principle) the following set of equations

$$
(16.52)\quad
\begin{cases}
\alpha^{o}_{ax} + u_b\,\alpha^{b}_{a} + u^{o}\,\alpha^{n}_{a} - \alpha^{o}_{o}\,u_a - \alpha^{o}_{a+1} = \varphi_a \\[2ex]
\alpha^{a}_{nx} + \alpha^{a-1}_{n} + \delta^{a}_{o}\,u_b\,\alpha^{b}_{n} + u^{a}\,\alpha^{n}_{n} - \alpha^{a}_{b}\,u^{b} - \alpha^{a}_{n}\,u_o = \varphi^{a} \\[2ex]
\alpha^{a+1}_{bx} + \alpha^{a}_{b} + u^{a-1}\,\alpha^{n}_{b} - \delta^{a}_{n-1}\,u_o\,\alpha^{n}_{b} - \alpha^{a+1}_{b+1} - \alpha^{a+1}_{o}\,u_b = 0 \\[2ex]
\alpha^{o}_{ox} + \alpha^{n}_{nx} + u_a\,\alpha^{a}_{o} + u^{o}\,\alpha^{n}_{o} - \alpha^{o}_{1} - \alpha^{o}_{o}\,u_o + \alpha^{n-1}_{n} - u_o\,\alpha^{n}_{n} - \alpha^{n}_{a}\,u^{a} + \alpha^{n}_{n}\,u_o = 0
\end{cases}
$$

which are equivalent to (16.51), and then to put its solutions α^{i}_{k} into

$$
(16.53)\qquad \alpha_a = \alpha^{n}_{a} \qquad\qquad \alpha^{a} = \alpha^{a}_{o} - \alpha^{n}_{n}\,\delta^{a}_{o}
$$

Since the explicit solution of this system is not required to pursue the reduction process, we shall skip over its discussion, and we are content with regarding Eq.s (16.52)(16.53) as the "parametric definition" of the reduced presymplectic tensor Ω' (compare Sec.11, where this system is discussed for n = 1).

<u>16.7</u>. To write the components of the reduced structures in the fibered coordinates is now straightforward. In fact, if (ψ_a, ψ^a) and (β_a, β^a) are the components of φ and α in this system of coordinates, from

$$
(16.54)\qquad v_a = u_a \qquad\qquad v^{a} = -\sum_{a}^{n+1}{}_{\ell}\ \sum_{\ell}^{n+1}{}_{k} \binom{\ell}{a} \left(\partial_{\ell-a}\, u_{k-1-\ell}\right) u^{k}
$$

it follows (Appendix D.2) that

$$
(16.55)\qquad
\begin{cases}
\psi_a = \varphi_a \\[2ex]
\psi^{a} = -\sum_{a}^{n+1}{}_{\ell}\ \sum_{\ell}^{n+1}{}_{k} \binom{\ell}{a} \left(\left(\partial_{\ell-a}\,\varphi_{k-1-\ell}\right) u^{k} + \left(\partial_{\ell-a}\, u_{k-1-\ell}\right) \varphi^{k} \right)
\end{cases}
$$

and

$$(16.56)\quad \begin{cases} \alpha_a = -\sum_{0}^{a}{}_{\ell} \sum_{0}^{\ell}{}_{j} \binom{\ell}{j} \beta_j \, \partial_{\ell-j} u_{a-\ell-1} \\ \alpha^a = -\sum_{0}^{m}{}_{\ell} \sum_{0}^{\ell}{}_{j} (-1)^{\ell-j} \binom{\ell}{j} \partial_{\ell-j} (u^{j+\ell+1} \beta_j) + \beta^a \qquad (\beta_m \equiv 0) \end{cases}$$

Then the Poisson tensor Ω, reduced to S^*_b, takes the form

$$(16.57)\quad \psi_a = \varphi_a = \alpha_a = -\sum_{0}^{a}{}_{b} \sum_{b}^{a}{}_{\ell} \binom{\ell}{b} \beta_b \, \partial_{\ell-b} u_{a-\ell-1}$$

$$\begin{aligned} \psi^a &= -\sum_{a}^{m+1}{}_{\ell} \sum_{\ell}^{m+1}{}_{k} \binom{\ell}{a} \Big((\partial_{\ell-a} \varphi_{k-1-\ell}) u^k + (\partial_{\ell-a} u_{k-1-\ell}) \varphi^k \Big) \\ &= -\sum_{a}^{m+1}{}_{\ell} \sum_{\ell}^{m+1}{}_{k} \binom{\ell}{a} \Big((\partial_{\ell-a} \alpha_{k-1-\ell}) u^k - (\partial_{\ell-a} u_{k-1-\ell}) \alpha^k \Big) \\ &= \sum_{a}^{m+1}{}_{\ell} \sum_{\ell}^{m+1}{}_{k} \sum_{0}^{k-\ell-1}{}_{p} \sum_{0}^{p}{}_{q} \binom{\ell}{a} \binom{p}{q} \partial_{\ell-a} \big(\beta_q \partial_{p-q} u_{k-\ell-p-2} \big) u^k + \\ &\quad - \sum_{a}^{m+1}{}_{\ell} \sum_{\ell}^{m+1}{}_{k} \sum_{0}^{m}{}_{p} \sum_{0}^{p}{}_{q} (-1)^{p-q} \binom{\ell}{a} \binom{p}{q} (\partial_{\ell-a} u_{k-1-\ell}) \partial_{p-q} (u^{p+q+1} \beta_q) + \\ &\quad + \sum_{a}^{m+1}{}_{\ell} \sum_{\ell}^{m+1}{}_{k} \binom{\ell}{a} (\partial_{\ell-a} u_{k-\ell-1}) \beta^a \end{aligned}$$

where the affine coordinates (u^a, u_a) are to be replaced by the fibered coordinates (v^a, v_a) by means of the inverse transformation of (16.54); by so doing, we would obtain the explicit expression, written in fibered coordinates, of the canonical Poisson tensor (16.50).

As for the presymplectic tensor, it can be obtained (at least in principle) by solving the system

$$\psi_a = \alpha^{0}_{ax} + u_b\,\alpha^{b}_{a} + u_0\,\alpha^{n}_{a} - \alpha^{0}_{a+1} - \alpha^{0}_{0}\,u_a \tag{16.58}$$

$$\psi^a = -\sum_{a}^{n+1}{}_{\ell} \sum_{\ell}^{n+1}{}_{k} \binom{\ell}{a} \partial_{\ell-a}\Big(\alpha^{0}_{(k-1-\ell)x} + u_b\,\alpha^{b}_{k-\ell-1} + u^0\alpha^{n}_{k-\ell-1} + {} - \alpha^{0}_{k-\ell} - \alpha^{0}_{b}\,u^b\Big)u^k - \sum_{a}^{n+1}{}_{\ell}\sum_{\ell}^{n-1}{}_{k}\binom{\ell}{a}\Big(\partial_{\ell-a}\,u_{k-\ell-1}\Big) \cdot \Big(\alpha^{k}_{nx} + \alpha^{a-1}_{n} + \delta^{k}_{0}\,u_b\,\alpha^{b}_{n} + u^k\alpha^{n}_{n} - \alpha^{k}_{b}\,u^b - \alpha^{k}_{n}\,u_0\Big)$$

with respect to the α^{j}_{k} which, moreover, are related by the constraints (16.52.3-4)

$$\alpha^{a+1}_{bx} + \alpha^{a}_{b} + u^{a-1}\alpha^{n}_{b} - \delta^{a}_{n-1}u_0\alpha^{n}_{b} - \alpha^{a+1}_{b+1} - \alpha^{a+1}_{0}\,u_b = 0 \tag{16.59}$$

$$(\alpha^{0}_{0} + \alpha^{n}_{n})_x + u_a\,\alpha^{a}_{0} + u^0\alpha^{n}_{0} - \alpha^{0}_{1} - \alpha^{0}_{0}\,u_0 + \alpha^{n-1}_{n} - u_0\alpha^{n}_{n} - \alpha^{n}_{a}\,u^a + \alpha^{n}_{n}\,u_0 = 0$$

and then by solving, with respect to (β_a, β^a), the system

$$\alpha^{n}_{a} = -\sum_{0}^{a}{}_{\ell}\sum_{0}^{\ell}{}_{j}\binom{\ell}{j}\beta_j\,\partial_{\ell-j}\,u_{a-\ell-1} \tag{16.60}$$

$$\alpha^{a}_{0} - \delta^{a}_{0}\,\alpha^{n}_{n} = -\sum_{0}^{n}{}_{\ell}\sum_{0}^{\ell}{}_{j}(-1)^{\ell-j}\binom{\ell}{j}\Big(\partial_{\ell-j}\,u^{j+\ell+1}\beta_j\Big) + \beta^a$$

In this way, we would obtain (β_a, β^a) in terms of (ψ_a, ψ^a). However, it is important to remark that the solution of (16.58)(16.59) and (16.60) is not required to obtain the projected structure on S^*_b/H_b, since, as it has been previously remarked, the explicit form of the presymplectic tensor on S^*_b is not needed to carry out the projection process.

16.8. The last step is the projection onto the quotient manifold $S^*_b/H_b \simeq F$. For the sake of clarity, we distinguish between the coordinates of a point of F when F is thought of as a manifold, and the coordinates of the same point when F is thought of as a submanifold of $F\times T \simeq S_b^*$. In the first case, we use (z^a, ψ^a, γ_a) to denote the coordinates of the point and the components of tangent vectors and one-forms respectively; in the second case, we continue to use $(v^a, \varphi^a, \beta_a)$. By this convention, the canonical projection $g : S_b^* \longrightarrow S_b^*/H_b \simeq F$ and the tangent and cotangent mappings dg and δg take the form :

$$g : (v^a, v_a) \mapsto z^a = v^a \tag{16.61}$$

$$dg : (\psi^a, \psi_a) \mapsto \chi^a = \psi^a \tag{16.62}$$

$$\delta g : \gamma_a \mapsto (\beta_a = \gamma_a \ , \beta^a = 0) \tag{16.63}$$

The projection of the Poisson tensor Q is thus obtained from

$$\begin{aligned}
\chi^a &\overset{(16.62)}{=} \psi^a \\
&\overset{(16.57)}{=} \sum_a^{m+1}{}_\ell \sum_\ell^{m+1}{}_k \sum_0^{k-\ell-1}{}_p \sum_0^{p}{}_q \binom{\ell}{a}\binom{p}{q} \partial_{\ell-a}\left(\beta_q \cdot \partial_{p-q} u_{k-\ell-p-2}\right) u^k \\
&\quad - \sum_a^{m+1}{}_\ell \sum_\ell^{m+1}{}_k \sum_0^{m}{}_p \sum_0^{p}{}_q (-1)^{p-q}\binom{\ell}{a}\binom{p}{q}\left(\partial_{\ell-a} u_{k-1-\ell}\right) \partial_{p-q}\left(u^{p+q+1}\beta_q\right) \\
&\quad + \sum_a^{m+1}{}_\ell \sum_\ell^{m+1}{}_k \binom{\ell}{a}\left(\partial_{\ell-a} u_{k-1-\ell}\right)\beta^q \\
&\overset{(16.63)}{=} \sum_a^{m+1}{}_\ell \sum_\ell^{m+1}{}_k \sum_0^{k-\ell-1}{}_p \sum_0^{p}{}_q \binom{\ell}{a}\binom{p}{q} \partial_{\ell-a}\left(\gamma_q \, \partial_{p-q} u_{k-\ell-p-2}\right) u^k \\
&\quad - \sum_a^{m+1}{}_\ell \sum_\ell^{m+1}{}_k \sum_0^{m}{}_p \sum_0^{p}{}_q (-1)^{p-q}\binom{\ell}{a}\binom{p}{q}\left(\partial_{\ell-a} u_{k-1-\ell}\right) \partial_{p-q}\left(u^{p+q+1}\gamma_q\right)
\end{aligned} \tag{16.64}$$

where the affine coordinates (u^a, u_a) must be replaced by z^a by means of

$$(16.65) \qquad z^a = v^a = -\sum_{a}^{n+1}{}_k \sum_{a}^{k}{}_l \binom{l}{a} \left(\partial_{l-a}\, u_{k-l-1} \right) u^k$$

That this is possible is a consequence of the projection lemma (7.2). This lemma, indeed, guarantees that the r.h.s. of (16.64) depends on (u^a, u_a) only through formations of the type (16.65), so that only the coordinates z^a are contained in the final form of Q.

In other words, this means that we can put in Eq.s (16.64) <u>any</u> solution of Eq.(16.65) without affecting the final form of the reduced tensor. For this reason, it is obvious to look for the simplest solution of (16.65) corresponding to a given z^a: in our case, it is clearly given by

$$(16.66) \qquad u^a = z^a \qquad u_a = 0 \qquad (u_{-1} := -1)$$

By such a choice, (16.64) becomes $(z^n := 0,\ z^{n+1} := -1)$:

$$(16.67) \qquad \chi^a = -\sum_{0}^{n-a}{}_c \sum_{a}^{n-c}{}_b \binom{b}{a} \left(\partial_{b-a}\, \gamma_c \right) z^{b+c+1} + \\ + \sum_{0}^{n-a}{}_c \sum_{c}^{n-a}{}_b \binom{b}{c} (-1)^{b-c}\, \partial_{b-c} \left(z^{a+b+1} \gamma_c \right)$$

This is the final form of the reduced Poisson tensor on the quotient space $S_b^*/H_b \simeq F$, and it corresponds to the <u>first Hamiltonian structure</u> of the non abelian Gel'fand-Dikii equations [17].

<u>Remark</u> – From a geometrical point of view, the meaning of the choice (16.66) is clear. Indeed, the leaf S_b^* is a trivial principal fiber bundle, having the base F and the structural group T; then, for any fixed point on the base F with coor-

dinates z^a, to take the solution (16.66) means to choose the point of the fiber over (z^a) belonging to the null section $F\times\{e\}$ of the fiber bundle.

The particular choice of the point on the null section turns out to be particularly useful also for the projection of the presymplectic tensor. By putting $u_a = 0$, $u^a = z^a$ in (16.58) (16.59) (16.60), the projection process amounts to solve the system

$$(16.68)\quad \begin{cases} \chi^a = \alpha^a_{nx} + \alpha^{a-1}_n + z^a \alpha^n_n - \alpha^a_b z^b \\ \psi_a = \alpha^0_{ax} + z^0 \alpha^n_a - \alpha^0_{a+1} \end{cases}$$

where the α^i_k are related by the constraints (16.59):

$$(16.69)\quad \begin{cases} \alpha^{a+1}_{bx} + \alpha^a_b + z^{a-1} \alpha^n_b - \alpha^{a+1}_{b+1} = 0 \\ (\alpha^0_0 + \alpha^n_n)_x + z^0 \alpha^n_0 - \alpha^0_1 + \alpha^{n-1}_n - \alpha^n_a z^a = 0 \end{cases}$$

and to put its solution into the equations

$$(16.70)\quad \begin{cases} \alpha^n_a = -\gamma_a \\ \alpha^a_0 - \delta^a_0\, \alpha^n_n = -\sum_{j=0}^{n-a} \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h}\, \partial_{j-h}\left(z^{a+j+1} \gamma_h\right) \end{cases}$$

which are obtained from (16.60) (16.63) (16.66). Furthermore, it may be suitable to recall that ψ_a can be <u>arbitrarily</u> chosen, since any choice of ψ_a corresponds to take a different right-inverse dg^{-1} in the projection process defined by (16.61)-(16.63). Clearly, the simplest choice is $\psi_a = 0$. By such a choice, the second equation (16.68) and the first equation of the system (16.69) can be written as a unique matrix equation for the column vectors α_a of the matrix α :

$$(16.71) \qquad (\partial_x + u)\,\alpha_a = \alpha_{a+1}$$

Thus one must solve the following set of equations

$$(16.71.1) \qquad \chi^a = \alpha^a_{nx} + \alpha^{a-1}_n + z^a \alpha^n_n - \alpha^a_b z^b$$

$$(16.71.2) \qquad \alpha_{a+1} = \alpha_{ax} + u\,\alpha_a$$

$$(16.71.3) \qquad \alpha^0_{0x} + \alpha^n_{nx} + z^0 \alpha^n_0 - \alpha^0_1 + \alpha^{n-1}_n - \alpha^n_a z^a = 0$$

$$(16.71.4) \qquad \alpha^n_a = -\gamma_a \;;\; \alpha^a_0 - \delta^a_0 \alpha^n_n = -\sum_0^{n-a}{}_j \sum_0^j{}_h (-1)^{j-h} \binom{j}{h} \partial_{j-h}\left(z^{a+j+1} \gamma_h\right)$$

To this end, two different methods of solution can be used. One can think of giving χ^a and then of solving the equations starting from (16.71.1) up to (16.71.4), so to obtain γ_a. Otherwise, one can proceed in the opposite direction, obtaining χ^a in terms of γ_a. In the first case, one obtains the reduced symplectic tensor on $S^*_b/H_b \simeq F$, in the second case its inverse. We will discuss in detail the second method of reduction. To this end, we observe that (16.71.4) allows to express the first column α_0 of α by means of γ_a and α^0_0:

$$(16.72) \quad \alpha_0 = \gamma_0 e_n + \alpha^0_0 e_0 - \sum_1^{n-1}{}_a \sum_0^{n-a}{}_j \sum_0^j{}_h (-1)^{j-h} \binom{j}{h} \partial_{j-h}\left(z^{a+j+1} \gamma_h\right) e_a$$

Then the whole matrix α can be given in terms of γ_a and α^0_0 by means of the iterative relation (16.71.2), whose solution is (Appendix D.3)

$$(16.73) \qquad \alpha = \sum_0^n{}_i \sum_0^i{}_j \binom{i}{j} u_{(j)} \left(\partial_{i-j}\, \alpha_0\right) e^i$$

where the matrices $u_{(j)}$ are recursively obtained from

$$(16.74) \qquad u_{(j+1)} = \left(\partial_x u_{(j)}\right) + u \cdot u_{(j)} \qquad\qquad u_{(0)} = \mathbb{I}$$

To obtain α_0^0, we make use of the last constraint (16.71.3), whose solution is (Appendix D.4)

$$(16.75) \qquad \alpha_0^0 = -\frac{1}{n+1} \sum_1^n{}_k \Bigg[\sum_0^{n-1}{}_j \Bigg(\binom{n}{j} u^n_{(j)k}\, \partial_{n-j}\, \alpha_0^k + \binom{n}{j} \int_{-\infty}^x u^{n-1}_{(j)k}\, \partial_{n-j}\, \alpha_0^k \, dx + {}$$
$$- \sum_j^{n-1}{}_a \binom{a}{j} \int_{-\infty}^x u^n_{(j)k}\, \partial_{a-j}\, \alpha_0^k \, dx \Bigg) - u^n_{(n)k}\, \alpha_0^k - \int_{-\infty}^x u^{n-1}_{(n)k}\, \alpha_0^k \, dx \Bigg]$$

where the α_0^k are given by (16.72)

$$(16.76) \qquad \alpha_0^k = \gamma_0\, \delta_n^k - \sum_1^{n-1}{}_a \sum_0^{n-a}{}_j \sum_0^j{}_h (-1)^{j-h} \binom{j}{h}\, \partial_{j-h}\left(z^{a+j+1}\, \gamma_a\right) \delta_a^k \quad (k>0)$$

At this point, since the matrix α is completely expressed in terms of the components γ_a, Eq.(16.71.1) gives χ^a in terms of γ_a: thus, we have explicitly obtained the <u>second Poisson tensor</u> Ω^{-1} on S^*_b/H_b. Actually, it is the second Hamiltonian structure of the non-abelian Gel'fand-Dikii equations[8,28,29]. Since the final form of Ω^{-1} is clearly very complicated, we do not write it explicitly in a compact form, but we give an explicit example of construction of this tensor for the simple case n = 2.

<u>Example</u>. For n = 2, the quotient space S^*_b/H_b is the space of traceless Frobenius matrices

$$(16.77) \qquad u = \begin{bmatrix} 0 & 0 & z^0 \\ 1 & 0 & z^1 \\ 0 & 1 & 0 \end{bmatrix}$$

One account of (16.71.4), the first column α_0 of α is given by

$$(16.78) \qquad \alpha_0 = \begin{bmatrix} \alpha_0^0 \\ \gamma_1 - \gamma_{0x} \\ \gamma_0 \end{bmatrix}$$

so that α is obtained by iteration

$$(16.79)\qquad \alpha_1 := \alpha_{0x} + u\,\alpha_0 = \begin{bmatrix} \alpha^0_{0x} + z^0\gamma_0 \\ \gamma_{1x} - \gamma_{0xx} + \alpha^0_0 + z^1\gamma_0 \\ \gamma_1 \end{bmatrix}$$

$$(16.80)\qquad \alpha_2 := \alpha_{1x} + u\,\alpha_1 = \begin{bmatrix} \alpha^0_{0xx} + (z^0\gamma_0)_x + z^0\gamma_1 \\ \gamma_{1xx} - \gamma_{0xxx} + 2\alpha^0_{0x} + (z^1\gamma_0)_x + z^0\gamma_0 + z^1\gamma_1 \\ 2\gamma_{1x} - \gamma_{0xx} + \alpha^0_0 + z^1\gamma_0 \end{bmatrix}$$

By requiring (Eq.(16.71.3)) that the matrix $\varphi = \alpha_x + [u, \alpha]$ be traceless, one obtains

$$(16.81)\qquad (\alpha^0_0 + \alpha^2_2)_x + z^0\alpha^2_0 - \alpha^0_1 + \alpha^1_2 - \alpha^2_0 z^0 - \alpha^2_1 z^1 = 0$$

so that

$$(16.82)\qquad 3\alpha^0_0 = -2\gamma_{0xx} - 3\gamma_{1x} - 2z^1\gamma_0 + \int_{-\infty}^{x} \left([\gamma_0, z^0] + [\gamma_1, z^1] \right) dx$$

The matrix α is thus completely expressed in terms of γ_0 and γ_1. Finally, since it is (Eq.(16.71.1))

$$(16.83)\qquad \chi^0 = \alpha^0_{2x} + z^0\alpha^2_2 - \alpha^0_0 z^0 - \alpha^0_1 z^1$$

$$\chi^1 = \alpha^1_{2x} + \alpha^0_2 + z^1\alpha^2_2 - \alpha^1_0 z^0 - \alpha^1_1 z^1$$

the reduced Poisson tensor Ω^{-1} takes the form

$$(16.84)\qquad \begin{bmatrix}\chi^0\\ \chi^1\end{bmatrix} = \frac{1}{3}\left[\begin{array}{c|c} A & B \\ \hline -B^* & C\end{array}\right]\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}$$

where the operators A,B,C are given by

$$(16.85)\qquad A: = 2\partial_5\cdot -2\left(\partial_3(z^1\cdot)+\partial_3\cdot z^1\right) - 3\partial_2\cdot z^0 + 3\partial_2(z^0\cdot) + 2\partial_1(z^1_x\cdot) + 3(z^1\cdot z^0 - z^0\cdot z^1) + \partial_2[\cdot,z^0] + [z^0,z^1\cdot] + [\partial_2\cdot, z^0] + [z^0,\int_{-\infty}^{x}[\cdot,z^0]\,dx] + [z^0,\cdot]z^1$$

$$(16.86)\qquad B: = -3\partial_4\cdot + 3\partial_2\cdot z^1 + 6\partial_1\cdot z^0 + 3\partial_1(z^0\cdot) + \partial_2[\cdot,z^1] + 3[z^0,\partial_1\cdot] + [z^0,\int_{-\infty}^{x}[\cdot,z^1]\,dx] + [z^1,\cdot]z^1$$

$$(16.87)\qquad C: = -6\partial_3 + 3\{z^1,\partial_1\cdot\} + 3\cdot z^1_x + 3[z^0,\cdot] + [z^1,\int_{-\infty}^{x}[\cdot,z^1]\,dx]$$

As for the first Poisson tensor, it is easily obtained from (16.67)

$$(16.88)\qquad \begin{bmatrix}\chi^0\\ \chi^1\end{bmatrix} = \left[\begin{array}{c|c} [z^1,\cdot] & 3\partial_x \\ \hline 3\partial_x & 0\end{array}\right]\cdot\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}$$

so thatits inverse is

$$(16.89)\qquad \begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix} = \frac{1}{9}\left[\begin{array}{c|c} 0 & 3\int_{-\infty}^{x}\cdot\,dx \\ \hline 3\int_{-\infty}^{x}\cdot\,dx & -\int_{-\infty}^{x}[z^1,\int_{-\infty}^{x'}\cdot\,dx']\,dx\end{array}\right]\begin{bmatrix}\chi^0\\ \chi^1\end{bmatrix}$$

Then the Nijenhuis tensor for n = 2 has the following form :

$$(16.90)\qquad \bar\chi = N\chi :\quad \begin{bmatrix}\bar\chi^0\\ \bar\chi^1\end{bmatrix} = \frac{1}{9}\left[\begin{array}{c|c} A & B \\ \hline -B^* & C\end{array}\right]\left[\begin{array}{c|c} 0 & 3\int_{-\infty}^{x}\cdot\,dx \\ \hline 3\int_{-\infty}^{x}\cdot\,dx & -\int_{-\infty}^{x}[z^1,\int_{-\infty}^{x'}\cdot\,dx']\,dx\end{array}\right]\begin{bmatrix}\chi^0\\ \chi^1\end{bmatrix}$$

Remark 16.1 - A brief inspection of the form (16.67) of the reduced tensor Q allows to conclude that Q is kernel-free. Since we are guaranteed a priori that the reduced tensor Ω^{-1} is kernel-free, we can conclude that Q and Ω^{-1} define an irreducible structure. To explicitly show that Q is kernel-free, let us write (16.67) in the form

$$(16.91) \quad \chi^a = Q^{ac}\gamma_c = -\sum_{c=0}^{n-a}\sum_{b=a}^{m-c}\binom{b}{a}(\partial_{b-a}\gamma_c)\,z^{b+c+1} + \sum_{c=0}^{m-a}\sum_{b=c}^{n-a}(-1)^{b-c}\binom{b}{c}\partial_{b-c}\left(z^{a+b+1}\gamma_c\right)$$

Since $c \leq n-a$, the operator Q is of triangular type; moreover, by (16.67) one easily verifies that

$$(16.92) \quad Q^{a,n-a} = -z^{n+1} + z^{n+1} = 0$$

and that

$$(16.93) \quad Q^{a,n+1-a} = -\sum_{b=a}^{n+1}\binom{b}{a}(\partial_{b-a}\,\cdot\,)\,z^{n+b-a} + \sum_{b=n-a-1}^{m-a}(-1)^{b+a+1-n}\binom{b}{n-a-1}\partial_{b+a+1-n}\left(z^{a+b+1}\,\cdot\,\right) = (n+1)\partial_x$$

so that Q takes the form

$$(16.94) \quad Q = \begin{bmatrix} & & & (n+1)\partial_x \\ & * & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & 0 \\ (n+1)\partial_x & & & \end{bmatrix}$$

Remark 16.2 (The connection with the Gel'fand-Dikii spectral problem)

As it has been previously remarked, the momentum mapping $J : H \longrightarrow \mathcal{H}^*$ is defined by

$$(16.95) \qquad J : h \longmapsto u = h^{-1} h_x$$

When restricted to the quotient manifold of the unimodular Wronskian matrices, this map associates a traceless Frobenius matrix v with any Wronskian matrix w according to

$$(16.96) \qquad w_x = w . v$$

Let us compute, in a coordinate system, the restricted map, by using the first n elements $w_o^a := w^a$ $(a = 0, a, \ldots n-1)$ of the first column as coordinates for w. Then by writing the matrix equation (16.96) by columns, one easily proves that the parameters w^a of the Wronskian matrices and v^a of the Frobenius matrices are related by the equation

$$(16.97) \qquad \partial_{n+1} w^b = \sum_{0}^{n-1}{}_a \left(\partial_a w^b\right) v^a$$

corresponding to the well-known (matrix) Gel'fand-Dikii spectral problem. Thus, from the present geometric point of view, the spectral problem is simply the parametric form of the momentum mapping restricted to the quotient space.

17. Conclusive remarks.

In this final section we make a few comments on some aspects of the PN manifold theory which seem to us particularly interesting and on some questions which have not been considered in this paper for the sake of brevity.

(i) The reduction method is a "tensor process", in this sense that the properties of the PΩ or PN structures are maintained under reduction, either by restriction or by projection. From a practical point of view, this means that one has not to verify some properties of the "final" reduced structures, whose checking would be very cumbersome in most cases.

(ii) The reduction method allows to acknowledge that different integrability structures corresponding to quite different non linear evolution equations are actually obtained by reducing over different submanifolds a few structures, which are moreover in general quite simple [35].

(iii) Once the PΩ or PN structure has been chosen, the reduction process does not require further ad hoc assumptions, since the structure itself yelds the characteristic distributions where to perform the reduction (the particular choice of their integral submanifolds being in general immaterial). Moreover, one can remark that the PN structure is "irreducible" when it is defined over the characteristic submanifold of N^r (under the assumption that a constant and finite Riesz index r exists), since the integral manifolds of the distribution Im N^r are not foliated by invariant submanifolds of N. However, it may happen that there are submanifolds which are invariant for a power N^i $(i > 1)$ of N (e.g., see [31] for the case of the KdV equation): in this case, it is possible to obtain further integrability structures by reducing N^i instead of N.

(iv) By considering in particular the family of PΩ structures defined in Sec. 15, parametrized by the arbitrary matrix a, different integrability structures are obtained in correspondence with different choices of a. In particular, one can show that, by suitable choices of a, hyerarchies of integrable Hamiltonian equations are obtained which are formally different but in some sense equivalent, since they are related by "generalized Miura transformations" exactly as the KdV and the modified KdV equation are related by the well-known Miura transformation. This problem, and the explicit construction of the generalized Miura transformations, will be considered in a forthcoming paper.

(v) At last, we remark that the Poisson cocycle and the $\mathcal{H}^*$-cocycle conditions on which rests the purely algebraic construction of a group-theoretical $P\Omega$ manifold are only sufficient conditions, so that they can be generalized. As a matter of fact, it will be shown in detail in a forthcoming paper that a non-trivial generalization of the construction given in Secs.12-13 is actually possible. It allows to obtain the group-theoretical structure giving rise, by reduction, to the integrability structure of the equations for the non-abelian Toda lattice.

pseudo cocycle = r-matrices

Appendix A : Conventions and Notations

A.1 From a theoretical point of view, tensor fields on a manifold M are usually regarded as real-valued multilinear functions $T : \mathfrak{X}(M)^{t} \times \mathfrak{X}(M)^{s} \longrightarrow R$. This point of view, however, is inappropriate for the kind of applications we are considering in this paper. For example, we are often required to consider as tensor objects on suitable infinite-dimensional manifolds some integro-differential operators such as

$$\text{(A.1.1)} \qquad N_u \varphi = \varphi_{xx} + 2u\varphi + u_x \int_{-\infty}^{x} \varphi \, dx$$

Clearly, it would be quite artful to replace this operator by an object such as

$$\text{(A.1.2)} \qquad < N_u \varphi, \alpha > = \int_{-\infty}^{+\infty} \alpha \left(\varphi_{xx} + 2u\varphi + u_x \int_{-\infty}^{x'} \varphi \, dx' \right) dx$$

with the only aim to regard it as a real-valued multilinear function. Consequently, in this paper we prefer to regard tensor fields as $\mathcal{F}$-multilinear maps defined on $\mathfrak{X}^*(M)^{t} \times \mathfrak{X}(M)^{s}$ and taking their values either in $\mathfrak{X}(M)$ or in $\mathfrak{X}^*(M)$ [30].This change of point of view only requires some minor changes of the usual conventions, as explained below.

Let M be a differentiable manifold, and $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$ be the spaces of the vector fields $\varphi: M \longrightarrow TM$ and of the one-forms $\alpha: M \longrightarrow T^*M$. In the whole paper, the symbols

Ω (for presymplectic) : $\mathfrak{X}(M) \longrightarrow \mathfrak{X}^*(M)$
P (for Poisson) : $\mathfrak{X}^*(M) \longrightarrow \mathfrak{X}(M)$
N (for Nijenhuis) : $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

stand systematically for tensor fields of type (0,2), (2,0) and (1,1) on M (sometimes, when we have to consider simultaneously a pair of tensor fields of type (2,0), we use the symbols P and Q for them). The product of tensors is defined as the product of linear maps and denoted, for example, by $P \circ \Omega$ or simply $P\Omega$. The inverse P^{-1} of the tensor P, if it exists, is defined in the usual way. The dual of the tensor $N : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ is the tensor $N^* : \mathcal{X}^*(M) \longrightarrow \mathcal{X}^*(M)$ defined by

(A.1.3) $$\langle N^*\alpha, \varphi \rangle = \langle \alpha, N\varphi \rangle$$

for any α and φ. In the same way the duals $P^* : \mathcal{X}^*(M) \longrightarrow \mathcal{X}(M)$ and $\Omega^* : \mathcal{X}(M) \longrightarrow \mathcal{X}^*(M)$ of the tensor P and Ω are defined by

(A.1.4) $$\langle \alpha, P\beta \rangle = \langle \beta, P^*\alpha \rangle \qquad \langle \Omega\varphi, \psi \rangle = \langle \Omega^*\psi, \varphi \rangle$$

The tensors P and Ω are said skewsymmetric if $P = -P^*$ and $\Omega = -\Omega^*$. The evaluation at the point $m \in M$ of Ω, P, N is denoted by a subscript, so that we write

(A.1.5) $$\alpha(m) = \Omega_m \cdot \varphi(m) \qquad \psi(m) = N_m \cdot \varphi(m) \qquad \varphi(m) = P_m \cdot \alpha(m)$$

to denote the linear maps induced by the tensor fields Ω, N, P at the point m.

The Lie derivative and the exterior derivative of Ω, N, P are then introduced by a quite natural use of the correspondence among maps and multilinear forms. So, for example, if $\omega : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow R$ is the two-form associated with the tensor Ω by

(A.1.6) $$\omega(\varphi, \psi) = \langle \Omega\varphi, \psi \rangle$$

we require that

(A.1.7) $$L_\varphi(\omega)(\psi, \chi) =: \langle L_\varphi(\Omega) \cdot \psi, \chi \rangle$$

(A.1.8) $$d\omega(\varphi, \psi, \chi) =: \langle d\Omega(\varphi, \psi), \chi \rangle$$

Of course the Lie derivative defined by (A.1.7) verifies the Leibnitz rule

(A.1.9) $$L_\varphi(\Omega\psi) = L_\varphi(\Omega)\,.\,\psi + \Omega\,.\,L_\varphi(\psi)$$

where

(A.1.10) $$L_\varphi(\psi) := [\psi, \varphi]$$

is the commutator of the vector fields ψ and φ. The well-known Cartan identity relating the Lie derivative and the exterior derivative of a two-form is now expressed, in terms of the tensor Ω, by the relation

(A.1.11) $$d\Omega(\varphi,\psi) = L_\varphi(\Omega)\cdot\psi - L_\psi(\Omega)\cdot\varphi - \Omega\,.[\varphi,\psi] + d<\Omega\varphi,\psi>$$

In the same way one proves the identity

(A.1.12) $$d\alpha\cdot\varphi = L_\varphi(\alpha) - d<\alpha,\varphi>$$

Finally, if $f : U\subset M \longrightarrow U'\subset M'$ is a differentiable mapping, we denote by $df(m): T_m M \longrightarrow T_{f(m)}M'$ its differential at the point $m\in U$, by $\delta f(m): T^*_{f(m)}M' \longrightarrow T^*_m M$ the dual map defined by

(A.1.13) $$<\delta f(m)\,\beta(f(m)),\, \varphi(m)>_M = <\beta(f(m)),\, df(m)\,\varphi(m)>_{M'}$$

and by $df : \mathfrak{X}(U,M) \longrightarrow \mathfrak{X}(U',M')$ and $\delta f: \mathfrak{X}^*(U',M') \longrightarrow \mathfrak{X}^*(U,M)$ the maps it induces between vector fields and one-forms defined on M and M' (or, more exactly, between their restrictions to U and U'). Consequently, if $f : U\subset M \longrightarrow U'\subset M'$ is a local diffeomorphism, the transformation formulas for Ω, P, N take the form

(A.1.14) $$\Omega' := \delta f^{-1}.\,\Omega\,.\,df^{-1}$$

(A.1.15) $$N' := df\,.\,N\,.\,df^{-1}$$

(A.1.16) $$N'^* := \delta f^{-1}.\,N^*\,.\,\delta f$$

(A.1.17) $$P' := df\,.\,P\,.\,\delta f$$

with the obvious meaning of the symbols.

A.2 In many applications, the manifold M is simply a Banach space or an affine hyperplane of a Banach space (typically, in problem where the boundary conditions are non-homogeneous). For this reason, it is useful to have a formulation of the theory of the PN manifolds also in the language of the differential calculus on Banach spaces. Therefore, we shall now give the local version of the abstract definitions introduced in Sec. 2.

Let us identify M with an open set U of a Banach space E (the "local chart"); the tensor fields Ω, N and P are then represented by mappings $\Omega: U\times E \to E^*$, $N: U\times E \to E$ and $P: U\times E^* \to E$, which are linear in the second argument. If φ and α stand for generic elements of E and E^* (identified with constant vector fields and one-forms on U) and if u is any point of U, we can denote the previous mappings as follows

$$\alpha = \Omega_u \cdot \varphi \qquad \psi = N_u \cdot \varphi \qquad \varphi = P_u \cdot \alpha \tag{A.2.1}$$

by putting into evidence the linear dependence on the vector φ or on the covector α. Accordingly, we denote by $\Omega'_u(\varphi;\psi)$, $N'_u(\varphi;\psi)$ and $P'_u(\alpha;\psi)$ the (partial) Fréchet derivative with respect to u of the mappings Ω_u, N_u, P_u, evaluated at the point u and in the direction ψ (keeping φ and α constant). So, for example, by assuming the continuity with respect to ψ, the derivative Ω'_u can be computed according to

$$\Omega'_u(\varphi;\psi) := \frac{d}{d\varepsilon}\, \Omega_{u+\varepsilon\psi}\, \varphi \Big|_{\varepsilon=0} \tag{A.2.2}$$

For fixed u, Ω'_u is a bilinear operator in φ and ψ. In particular, we can take the adjoint with respect to ψ, which we shall denote by $\Omega'^*_u : E\times E \to E^*$. So, by definition, it is

$$\langle \Omega'_u(\varphi;\psi), \chi \rangle = \langle \Omega'^*_u(\varphi;\chi), \psi \rangle \tag{A.2.3}$$

By using such notations (and the corresponding ones for P and N) and by letting $\phi : U \rightarrow E$ and $A : U \rightarrow E^*$ to denote the local representatives of arbitrary vector fields and one-forms on M, the Lie and the exterior derivatives previously introduced can be given the following local forms :

$$
\begin{aligned}
&[\phi,\psi](u) := \phi'_u \cdot \Psi(u) - \Psi'_u \cdot \Phi(u) \\
&L_\phi(A)(u) := A'_u \cdot \phi(u) + \phi'^*_u \cdot A(u) \\
(A.2.4)\quad &L_\phi(\Omega)_u \varphi := \Omega'_u(\varphi; \Phi(u)) + \Phi'^*_u \Omega_u \varphi + \Omega_u \Phi'_u \varphi \\
&L_\phi(P)_u \alpha := P'_u(\alpha; \Phi(u)) - \Phi'_u P_u \alpha - P_u \Phi'^*_u \alpha \\
&L_\phi(N)_u \varphi := N'_u(\varphi; \Phi(u)) + N_u \Phi'_u \varphi - \Phi'_u N_u \varphi \\
&(dA)_u \cdot \varphi := (A'_u - A'^*_u) \cdot \varphi \\
&(d\Omega)_u(\varphi, \psi) := \Omega'_u(\varphi;\psi) - \Omega'_u(\psi;\varphi) + \Omega'^*_u(\psi;\varphi)
\end{aligned}
$$

They allow to reset the "closure conditions" for Ω, P and N respectively (namely: $d\Omega = 0$, $[P,P] = 0$ and $T(N) = 0$) into the local form [31]

$d\Omega = 0$ (A.2.5) $< \Omega'_u(\varphi;\psi), \chi > + \cdots\cdots + \cdots\cdots = 0$

$[P,P] = 0$ (A.2.6) $< \alpha, P'_u(\beta; P_u \gamma) > + \cdots\cdots + \cdots\cdots = 0$

$T(N) = 0$ (A.2.7) $N'_u(\varphi; N_u \psi) - N'_u(\psi; N_u \varphi) - N_u N'_u(\varphi;\psi) + N_u N'_u(\psi;\varphi) = 0$

(where the dots mean cyclic permutation over the arguments) while the "coupling conditions" defining $P\Omega$, PN and PQ manifolds become respectively :

$P\Omega$ (A.2.8) $< P'_u(\Omega_u \varphi;\psi), \Omega_u \chi > + < \Omega'_u(\varphi; P_u \Omega_u \psi), \chi > + \cdots + \cdots = 0$

PN (A.2.9)

$$\begin{cases} N_u \cdot P_u = P_u N_u^* \\ <\alpha, N_u'(P_u\beta,\varphi) - N_u'(\varphi, P_u\beta)> + \\ + <\beta, N_u'(\varphi, P_u\alpha) + N_u \cdot P_u'(\alpha,\varphi) - P_u'(\alpha, N_u\varphi)> = 0 \end{cases}$$

and $<\beta, R(P,N)(\alpha,\varphi)_u> = 0$

PQ (A.2.10) $<\alpha, Q_u'(\beta, P_u\gamma) + P_u'(\beta, Q_u\gamma)> + \cdots + \cdots = 0$

Let now $f : U \to \bar{U}$ be the local representative of a diffeomorphism on M. Then, the transformation formulas (A.1.14-17) take the local form

(A.2.11) $\bar{\Omega}_{\bar{u}} := f_u'^{*-1} \cdot \Omega_u \cdot f_u'^{-1}$

(A.2.12) $\bar{N}_{\bar{u}} := f_u' \cdot N_u \cdot f_u'^{-1}$

(A.2.13) $\bar{P}_{\bar{u}} := f_u' \cdot P_u \cdot f_u'^{*}$

where $f_u'^{*}$ is the dual map of the Fréchet derivative of f, defined by

(A.2.14) $<\alpha, f_u' \varphi> = <f_u'^{*}\alpha, \varphi>$

Accordingly, we are led to define a local PN manifold as an open set U of a Banach space E endowed with two mappings $N_u : E \to E$ and $P_u : E^* \to E$ fulfilling the "clusure conditions" (A.2.7) and (A.2.6) and the "coupling conditions" (A.2.9) and transforming themselves with respect to local diffeomorphism according to (A.2.12) and (A.2.13). Local PΩ and PQ manifolds are defined in the same way.

Appendix B: $P\Omega$ and PN manifolds.

B.1 Poisson tensors

The Schouten bracket is a composition law defined over the graded algebra of the skewsymmetric controvariant tensors of any order [31,33]. For second-order tensors P and Q, it reduces to :

$$(B.1.1)\quad \begin{aligned} 2![P,Q](\alpha,\beta) := {} & L_{P\beta}(Q)\cdot\alpha + Q\cdot L_{P\alpha}(\beta) + Q\, d\langle\alpha, P\beta\rangle \\ & + L_{Q\beta}(P)\cdot\alpha + P\cdot L_{Q\alpha}(\beta) + P\, d\langle\alpha, Q\beta\rangle \\ = {} & L_{P\beta}(Q)\alpha + Q\cdot d\beta\cdot P\alpha + L_{Q\beta}(P)\cdot\alpha + P\cdot d\alpha\cdot Q\beta \end{aligned}$$

whence, for $Q = P$,

$$(B.1.2)\quad [P,P](\alpha,\beta) := L_{P\beta}(P)\cdot\alpha + P\cdot L_{P\alpha}(\beta) + P\, d\langle\alpha, P\beta\rangle$$

Equivalent forms are

$$(B.1.3)\quad \langle\alpha, [P,P](\beta,\gamma)\rangle = \langle L_{P\alpha}(\gamma), P\beta\rangle + \cdots + \cdots$$

$$(B.1.4)\quad [P,P](\alpha,\beta) = P\,.\,d\beta\,.\,P\alpha + L_{P\beta}(P)\,.\alpha$$

$$(B.1.5)\quad [P,P](\alpha,\beta) = -[P\alpha, P\beta] - P\,.\{\alpha,\beta\}_P$$

where

$$(B.1.6)\quad \{\alpha,\beta\}_P := L_{P\beta}(\alpha) - L_{P\alpha}(\beta) + d\langle\beta, P\alpha\rangle = d\alpha\,.\,P\beta - d\beta\,.\,P\alpha + d\langle\alpha, P\beta\rangle\,.$$

If $[P,P] = 0$, the tensor P is said to be a Poisson tensor, the bracket (B.1.6) is called the Poisson bracket of the one-forms α and β (with respect to P) and the vector fields $\varphi_\alpha = P\alpha$ (associated with closed one-forms α) are said to be (locally) Hamiltonian (with respect to P). By (B.1.5), the

$$P_\alpha(Q)\cdot\beta = L_{P\alpha}(Q)\,\beta + P\cdot d\alpha\cdot Q\beta + Q\cdot d\alpha\cdot P\beta$$

Lie algebra of the vector fields $\varphi_\alpha = P\alpha$ (with respect to the comutator of fields) is homomorphic to the Lie algebra of the one-forms α (with respect to the Poisson bracket (B.1.6.)):

$$(B.1.7) \qquad [P\alpha,\ P\beta] = -P\cdot\{\alpha,\beta\}_P$$

The closed one-forms make a special subalgebra, since the Poisson bracket of two closed one-forms is an exact one-form:

$$(B.1.8) \quad d\alpha = d\beta = 0 \ \Rightarrow\ \{\alpha,\beta\}_P = d\langle\alpha,\ P\beta\rangle$$

The corresponding algebra of vector fields is the Lie algebra of the Hamiltonian vector fields. By (B.1.4), it coincides with the algebra of the vector fields keeping P invariant ($[P,P] = 0$ and $d\beta = 0$ entail $L_{P\beta}(P) = 0$).

As for the proofs of (B.1.3), (B.1.4) and (B.1.5), we limit ourselves to show the equivalence of (B.1.3) with (B.1.2) (the other proofs being straightforward). It follows from :

$$\begin{aligned}
(B.1.9) \quad & \langle L_{P\alpha}(\beta),\ P\gamma\rangle + \langle L_{P\beta}(\gamma),\ P\alpha\rangle + \langle L_{P\gamma}(\alpha),\ P\beta\rangle = \\
&= -\langle\gamma,\ P\cdot L_{P\alpha}(\beta)\rangle + L_{P\beta}(\langle\gamma,\ P\alpha\rangle) - \langle\gamma,\ L_{P\beta}(P\alpha)\rangle + \\
&\quad + L_{P\gamma}(\langle\alpha,\ P\beta\rangle) - \langle\alpha,\ L_{P\gamma}(P\beta)\rangle \\
&= -\langle\gamma,\ P\cdot L_{P\alpha}(\beta)\rangle - \langle L_{P\beta}(\alpha), P\gamma\rangle - \langle\gamma, L_{P\beta}(P\alpha)\rangle + \langle d\langle\alpha, P\beta\rangle, P\gamma\rangle \\
&= -\langle\gamma,\ P\cdot L_{P\alpha}(\beta) - P\cdot L_{P\beta}(\alpha) + L_{P\beta}(P\alpha) + P\cdot d\langle\alpha, P\beta\rangle\rangle \\
&= -\langle\gamma,\ [P,P](\alpha,\beta)\rangle \quad .
\end{aligned}$$

This equivalence clearly shows that the Schouten bracket $[P,P]$ is a skewsymmetric third-order tensor of type (3,0).

B.2 Nijenhuis tensors

Let $N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a tensor field of type (1,1). Its torsion tensor $T(N)$ is defined by [34]

(B.2.1) $$T(N)(\varphi,\psi) := [N\varphi, N\psi] - N[\varphi, N\psi] - N[N\varphi,\psi] + N^2[\varphi,\psi]$$

It is a third-order tensor of type (1,2), sometimes called the Nijenhuis tensor of N. For the sake of brevity, in this paper we call Nijenhuis tensor a second-order tensor N of type (1,1) whose torsion vanishes, and we call Nijenhuis condition the condition $T(N) = 0$.

By using Lie derivative instead of commutators, we get

(B.2.2) $$\begin{aligned} T(N)(\varphi,\psi) &= L_{N\psi}(N\varphi) - N.L_{N\psi}(\varphi) - N L_{\psi}(N\varphi) + \\ &\quad + N^2 L_{\psi}(\varphi) \\ &= \left(L_{N\psi}(N) - N.L_{\psi}(N)\right)\varphi \end{aligned}$$

so that the Nijenhuis condition can be written as

(B.2.3) $$L_{N\psi}(N) - N\,L_{\psi}(N) = 0$$

By introducing the dual tensor N^* of N, the Nijenhuis condition can be written also in the form

(B.2.4) $$L_{N\psi}(N^*) - L_{\psi}(N^*).N^* = 0$$

as is proved by

(B.2.5) $$\begin{aligned} \langle\alpha, T(N)(\varphi,\psi)\rangle &\overset{(B.2.2)}{=} \langle\alpha, L_{N\psi}(N)\varphi - N.L_{\psi}(N)\varphi\rangle \\ &= \langle L_{N\psi}(N^*)\alpha, \varphi\rangle - \langle N^*\alpha, L_{\psi}(N)\varphi\rangle \\ &= \langle (L_{N\psi}(N^*) - L_{\psi}(N^*).N^*)\alpha, \varphi\rangle \end{aligned}$$

B.3 Some note-worthy identities

Let N, P, Ω be three second-order tensorfields on a manifold M, respectively of type (1,1), (2,0), (0,2), P and Ω being assumed to be skew-symmetric. We recall that to N, P, Ω three remarkable third-order tensors can be separately associated

(B.3.1) $\quad \mathbb{T}(N)(\varphi,\psi) := \left(L_{N\psi}(N) - N\, L_{\psi}(N)\right)\cdot\varphi$

(B.3.2) $\quad [P,P](\alpha,\beta) := P\left(L_{P\alpha}(\beta) - L_{P\beta}(\alpha) + d\langle\alpha, P\beta\rangle\right) + L_{P\beta}(P\alpha)$

(B.3.3) $\quad d\Omega(\varphi,\psi) := L_{\varphi}(\Omega)\psi - L_{\psi}(\Omega)\varphi + d\langle\Omega\varphi,\psi\rangle$

whose vanishing defines respectively the Nijenhuis tensors, the Poisson tensors and the presymplectic tensors. Moreover, by taking the tensors N, P, Ω in pairs, three other third-order tensors can be defined, whose vanishing entail that

(B.3.4) $\quad \bar{N} := P\Omega \qquad \bar{P} := NP \qquad \bar{\Omega} := \Omega N$

are respectively a Nijenhuis tensor, a Poisson tensor and a presymplectic tensor (if the skew-symmetry of NP and of ΩN is understood). These noteworthy tensors are:

(B.3.5) $\quad R(P,N)(\alpha,\varphi) := L_{P\alpha}(N)\varphi - P\, L_{\varphi}(N^{*}\alpha) + P\, L_{N\varphi}(\alpha)$

(B.3.6) $\quad S(\Omega,N)(\varphi,\psi) := d\Omega(N\psi,\varphi) - d\Omega(N\varphi,\psi) + d(\Omega N)(\varphi,\psi)$

(B.3.7) $\quad d(\Omega P\Omega)(\varphi,\psi) := L_{\varphi}(\Omega P\Omega)\psi - L_{\psi}(\Omega P\Omega)\varphi + d\langle\Omega P\Omega\varphi,\psi\rangle$

As it has been discussed in Sec.2, the mutual relations among the three $P\Omega$, PN, ΩN structures which are defined on M and the existence of hierarchies of PN and ΩN structures are due to five remarkable identities among the previous six third-order tensors. These identities are :

(B.3.8) $$T(P\Omega)(\varphi,\psi) = [P,P](\Omega\psi,\Omega\varphi) - P\, S(\Omega,P\Omega)(\varphi,\psi)$$

(B.3.9) $$R(P,P\Omega)(\alpha,\varphi) = [P,P](\Omega\varphi,\alpha) + P\, d\Omega(P\alpha,\varphi)$$

concerning the pair (Ω, P) ;

(B.3.10) $$[NP,NP](\alpha,\beta) = N\Big([P,P](N^{*}\alpha,\beta) - R(P,N)(\alpha,P\beta)\Big) + T(N)(P\alpha,P\beta)$$

(B.3.11) $$R(NP,N)(\alpha,\varphi) = N.\, R(P,N)(\alpha,\varphi) + T(N)(\varphi,P\alpha)$$

concerning the pair (P,N) and

(B.3.12) $$S(\Omega N,N)(\varphi,\psi) = d\Omega(N\psi,N\varphi) + \Omega\, T(N)(\varphi,\psi)$$

on the pair (Ω, N). Before proving them, it is suitable to show how they work. Let us distinguish three cases :

i) <u>PΩ manifolds</u>

given P and Ω fulfilling the conditions $[P,P] = 0$, $d\Omega = 0$ and $d(\Omega P\Omega) = 0$, by (B.3.6)(B.3.8)(B.3.9) we get $S(\Omega, P\Omega) = 0$, $T(P\Omega) = 0$ and $R(P, P\Omega) = 0$ showing that $N := P\Omega$ is a Nijenhuis tensor, well-coupled both with Ω and P. Therefore, PΩ structures are the basic ones, inducing the existence of PN and ΩN structures as well.

ii) <u>ΩN manifolds</u>

given Ω and N fulfilling the conditions $d\Omega = 0$, $T(N) = 0$ and $S(\Omega, N) = 0$, by (B.3.6)(B.3.12) we get $d(\Omega N) = 0$ and $S(\Omega N, N) = 0$ showing that $\bar{\Omega} := \Omega N$ is a new presymplectic tensor well-coupled with N. By iteration, we realize the existence of a hierarchy of ΩN structures defined by $\Omega_j := \Omega N^j$.

iii) <u>PN manifolds</u>

$NP = P^t N$

given P and N fulfilling the conditions $[P, P] = 0$, $T(N) = 0$ $R(P, N) = 0$, by (B.3.10)(B.3.11) we get $[NP, NP] = 0$ and $R(NP, N) = 0$, showing that $\bar{P} := NP$ is a new Poisson tensor well-coupled with N. By iteration, we realize the existence of a hierarchy of PN structures, defined by $P_j := N^j P$. To prove the further property of this hierarchy of being in involution, $[P_j, P_k] = 0$, it suffices to use the following generalized form of (B.3.10)

$NQ = Q^t N$

(B.3.13) $$2[NP, Q](\alpha,\beta) = 2N\,[P,Q](\alpha,\beta) + Q\,R^*(P,N)(\alpha,\beta) + R(Q,N)(\beta, P\alpha) - R(Q,N)(\alpha, P\beta)$$

where $R^*(P,N) : \mathfrak{X}^*(M) \times \mathfrak{X}^*(M) \rightarrow \mathfrak{X}^*(M)$ is defined by

(B.3.14) $$< R^*(P,N)(\alpha,\beta) ; \varphi > = < \beta, R(P,N)(\alpha,\varphi) >$$

Without making any further comments for the sake of brevity, we limit ourselves to sketch the proof of the previous identities

$$(B.3.15)\quad \mathbb{T}(P\Omega)(\varphi,\psi) := L_{P\Omega\varphi}(P\Omega)\psi - P\Omega\, L_{\varphi}(P\Omega)\psi$$

$$= L_{P\Omega\varphi}(P\Omega\psi) - P\Omega\, L_{P\Omega\varphi}(\psi) - P\Omega\, L_{\varphi}(P\Omega)\psi$$

$$\overset{(B.3.2)}{=} [P,P](\Omega\psi,\Omega\varphi) - P\cdot\Big(L_{P\Omega\psi}(\Omega\varphi) +$$

$$+ d\langle\Omega\psi, P\Omega\varphi\rangle - L_{P\Omega\varphi}(\Omega)\psi + \Omega\, L_{\varphi}(P\Omega)\psi\Big)$$

$$\overset{(B.3.3)}{=} [P,P](\Omega\psi,\Omega\varphi) - P\cdot\big(d\Omega(P\Omega\psi,\varphi) +$$

$$- d\Omega(P\Omega\varphi,\psi)\big) - P\,\Big(L_{\varphi}(\Omega P\Omega\psi) +$$

$$- L_{\psi}(\Omega P\Omega\varphi) - \Omega P\Omega\cdot L_{\varphi}(\psi) - d\langle\Omega P\Omega\psi,\varphi\rangle\Big)$$

$$\overset{(B.3.6-7)}{=} [P,P](\Omega\psi,\Omega\varphi) - P\cdot S(\Omega, P\Omega)(\varphi,\psi)$$

$$(B.3.16)\quad R(P,P\Omega)(\alpha,\varphi) := L_{P\alpha}(P\Omega)\varphi - P\, L_{\varphi}(\Omega P\alpha) + P\, L_{P\Omega\varphi}(\alpha)$$

$$= L_{P\alpha}(P\Omega\varphi) - P\Big(\Omega\, L_{P\alpha}(\varphi) + L_{\varphi}(\Omega P\alpha) - L_{P\Omega\varphi}(\alpha)\Big)$$

$$\overset{(B.3.9)}{=} [P,P](\Omega\varphi,\alpha) + P\,\Big(L_{P\alpha}(\Omega\varphi) - d\langle\Omega\varphi, P\alpha\rangle$$

$$- \Omega\, L_{P\alpha}(\varphi) - \Omega\, L_{\varphi}(P\alpha) - L_{\varphi}(\Omega)(P\alpha)\Big)$$

$$\overset{(B.3.3)}{=} [P,P](\Omega\varphi,\alpha) + P\, d\Omega(P\alpha,\varphi)$$

$$(B.3.17)\quad [NP,NP](\alpha,\beta) := NP\,\Big(L_{NP\alpha}(\beta) - L_{NP\beta}(\alpha) + d\langle\alpha, NP\beta\rangle\Big) +$$

$$+ L_{NP\beta}(NP\alpha)$$

$$= NP \cdot d\langle N^*\alpha, P\beta\rangle + NP\,\left(L_{NP\alpha}(\beta) - L_{NP\beta}(\alpha)\right) +$$

$$+ L_{NP\beta}(NP\alpha)$$

$$\overset{(B.3.2)}{=} N\,[P,P](N^*\alpha,\beta) - NP\,L_{NP\beta}(\alpha) + L_{NP\beta}(NP\alpha) +$$

$$+ NP\,L_{P\beta}(N^*\alpha) - N\,L_{P\beta}(PN^*\alpha)$$

$$= N\,[P,P](N^*\alpha,\beta) + L_{NP\beta}(N)\cdot P\alpha +$$

$$+ N\cdot\left(-P\,L_{NP\beta}(\alpha) + P\,L_{P\beta}(N^*\alpha) + L_{NP\beta}(P\alpha) - L_{P\beta}(PN^*\alpha)\right)$$

$$\overset{(B.3.1)}{=} N\,[P,P](N^*\alpha,\beta) + \mathcal{T}(N)(P\alpha,P\beta) +$$

$$+ N\cdot\left(L_{NP\beta}(P)\alpha - L_{P\beta}(P)\,N^*\alpha + L_{P\beta}(N)\,P\alpha\right)$$

$$\overset{(B.3.5)}{=} N\,[P,P](N^*\alpha,\beta) + \mathcal{T}(N)(P\alpha,P\beta) - N\cdot R(P,N)(\alpha,P\beta)$$

(B.3.18) $$R(NP,N)(\alpha,\varphi) := L_{NP\alpha}(N)\,\varphi - NP\,L_\varphi(N^*\alpha) + NP\,L_{N\varphi}(\alpha)$$

$$\overset{(B.3.5)}{=} L_{NP\alpha}(N)\,\varphi + N\left(R(P,N)(\alpha,\varphi) - L_{P\alpha}(N)\,\varphi\right)$$

$$\cancel{-P\,L_{N\varphi}(\alpha)) + NP\,L_{N\varphi}(\alpha)}$$

$$\overset{(B.3.1)}{=} N\,R(P,N)(\alpha,\varphi) + \mathcal{T}(N)(\varphi,P\alpha)$$

(B.3.19) $$S(\Omega N,N)(\varphi,\psi) := d(\Omega N)(N\psi,\varphi) - d(\Omega N)(N\varphi,\psi) + d(\Omega N^2)(\varphi,\psi)$$

$$\overset{(B.3.3)}{=} L_{N\psi}(\Omega N)\,\varphi - L_\varphi(\Omega N)\cdot N\psi + d\langle\Omega N^2\psi,\varphi\rangle$$

$$- L_{N\varphi}(\Omega N)\,\psi + L_\psi(\Omega N)\cdot N\varphi - d\langle\Omega N^2\varphi,\psi\rangle$$

$$+ L_\varphi(\Omega N^2)\,\psi - L_\psi(\Omega N^2)\,\varphi + d\langle\Omega N^2\varphi,\psi\rangle$$

$$\overset{(B.3.3)}{=} d\Omega(N\psi, N\varphi) + \Omega\cdot\left(N\cdot L_{N\varphi}(\psi) + L_{N\psi}(N)\,\varphi + N^2\cdot L_{\psi}(\varphi)\right)$$

$$\overset{(B.3.1)}{=} d\Omega(N\psi, N\varphi) + \Omega.\, \mathcal{T}(N)(\varphi, \psi)$$

At last, we prove the generalized form (B.3.13) of the identity (B.3.10):

(B.3.20)

use generalized (B.3.2)

$$2[NP,Q](\alpha,\beta) := L_{Q\beta}(NP)\alpha + NP\, L_{Q\alpha}(\beta) - NP\, d\langle\beta, Q\alpha\rangle$$

$$+ L_{NP\beta}(Q)\alpha + Q\, L_{NP\alpha}(\beta) - Q\, d\langle\beta, NP\alpha\rangle$$

$$= N.\Big(L_{Q\beta}(P)\alpha + P\, L_{Q\alpha}(\beta) - P\, d\langle\beta, Q\alpha\rangle$$

$$+ L_{P\beta}(Q)\alpha + Q\cdot L_{P\alpha}(\beta) - Q\, d\langle\beta, P\alpha\rangle\Big) +$$

$$+ L_{Q\beta}(N)\, P\alpha + L_{NP\beta}(Q)\alpha - N\cdot L_{P\beta}(Q)\alpha$$

$$+ Q\, L_{NP\alpha}(\beta) - Q\, N^*.\, L_{P\alpha}(\beta) +$$

$$+ Q\, N^*\, d\langle\beta, P\alpha\rangle - Q\,.\, d\langle\beta, NP\alpha\rangle$$

$$\overset{(B.1.1)}{=} 2N\, \overset{P\ \alpha}{[Q,P]}(\alpha,\beta) +$$

$$+ L_{Q\beta}(N)\, P\alpha + L_{NP\beta}(Q)\alpha - N\, L_{P\beta}(Q)\alpha$$

$$- Q\, L_{P\alpha}(N^*\beta) + Q\cdot L_{P\beta}(N^*)\alpha + Q.L_{NP\alpha}(\beta)$$

$$+ Q\cdot\Big(L_{P\alpha}(N^*)\beta - L_{P\beta}(N^*)\alpha + d\langle\alpha, NP\beta\rangle +$$

$$- N^*.\, d\langle\alpha, P\beta\rangle\Big)$$

$$\overset{(B.3.14)}{=} 2\, N[\overset{P,Q}{Q,P}](\alpha,\beta) + Q \cdot R^*(P,N)(\alpha,\beta)$$

$$+ \left(L_{Q\beta}(N)\, P\alpha - Q\, L_{P\alpha}(N^*\beta) + Q\, L_{NP\alpha}(\beta) \right)$$

$$- \left(- L_{NP\beta}(Q)\,\alpha + N\, L_{P\beta}(Q)\,\alpha - Q \cdot L_{P\beta}(N^*)\,\alpha \right) \qquad \boxed{NQ = Q^t N}$$

$$\overset{(B.3.5)}{=} 2\, N[\overset{P,Q}{Q,P}](\alpha,\beta) + Q \cdot R^*(P,N)(\alpha,\beta) +$$

$$+ R(Q,N)(\beta, P\alpha) - R(Q,N)(\alpha, P\beta) .$$

where we have taken into account the explicit form of $R^*(P,N)$ following from:

(B.3.21) $\langle \beta, R(P,N)(\alpha,\varphi) \rangle := \langle \beta, L_{P\alpha}(N)\varphi \rangle - \langle \beta, P.L_\varphi(N^*\alpha) \rangle$

$$+ \langle \beta, P.L_{N\varphi}(\alpha) \rangle$$

$$= \langle L_{P\alpha}(N^*)\beta, \varphi \rangle + L_\varphi \langle N^*\alpha, P\beta \rangle - \langle N^*\alpha, L_\varphi(P\beta) \rangle$$

$$- L_{N\varphi} \langle \alpha, P\beta \rangle + \langle \alpha, L_{N\varphi}(P\beta) \rangle$$

$$= \langle L_{P\alpha}(N^*)\beta + d\langle \alpha, NP\beta \rangle - N^* d \langle \alpha, P\beta \rangle, \varphi \rangle$$

$$+ \langle N^*\alpha, L_{P\beta}(\varphi) \rangle - \langle \alpha, L_{P\beta}(N\varphi) \rangle$$

$$= \langle L_{P\alpha}(N^*)\beta - L_{P\beta}(N^*)\alpha + d\langle \alpha, NP\beta \rangle +$$

$$- N^* d \langle \alpha, P\beta \rangle, \varphi \rangle = \langle {}^t R(P,N)(\alpha,\beta), \varphi \rangle$$

$$\langle \alpha, \mathcal{L}_x(N) y \rangle = \langle \mathcal{L}_x^t(N)\alpha, y \rangle$$

Appendix C

C.1. Projectable fields and projectable one-forms.

Let D be an integrable distribution of M admitting a quotient manifold M/D. Let $(M', g : M \longrightarrow M')$ be a parametrization of M/D. We recall that a field $\varphi \in \mathcal{X}(M)$ is projectable if

$$dg(m_1)\ \varphi(m_1) = dg(m_2)\ \varphi(m_2) \tag{C.1.1}$$

for any pair (m_1, m_2) of the same leaf [14].

A one-form $\alpha \in \mathcal{X}^*(M)$ is said to be projectable if there exists a one-form $\alpha' \in \mathcal{X}^*(M')$ such that

$$\alpha = \delta g\ \alpha' \tag{C.1.2}$$

We denote by $\mathcal{X}_D(M)$ and $\mathcal{X}^*_D(M)$ the subspaces of the projectable vector fields and projectable one-forms, and by $Z_D \subset \mathcal{X}_D(M)$ the algebra of vector fields tangent to the fiber

$$Z_D := \{ \xi \in \mathcal{X}(M) : dg \cdot \xi = 0 \} \tag{C.1.3}$$

Then one can show that a projectable vector field is completely characterized by the condition

$$dg\ [\varphi, \xi] = 0 \qquad \xi \in Z_D \tag{C.1.4}$$

and that $\mathcal{X}_D(M)$ is a subalgebra of $\mathcal{X}(M)$, whereas projectable one-forms are completely characterized by the conditions

$$\langle \alpha, \xi \rangle = 0 \qquad \xi \in Z_D \tag{C.1.5}$$

(C.1.6) $$L_\xi(\alpha) = 0 \qquad \xi \in Z_D$$

Indeed, if $\varphi \in \mathcal{X}_D(M)$ and $\xi \in Z_D$, it is

(C.1.7) $$dg\,[\varphi, \xi] = dg\; L_\xi(\varphi) = L_{dg\,\xi}(\varphi') = 0.$$

On the other hand, if φ fulfils the condition (C.1.4), the map $\Phi_\varphi : F_{m'} \to T_{m'}M'$ ($F_{m'}$ being the fiber over $m' \in M'$) defined by:

(C.1.8) $$\Phi_\varphi(m) := dg(m) \cdot \varphi(m)$$

has a vanishing differential at any point $m \in F_{m'}$: thus Φ_φ is constant along the fiber (which is a connected submanifold) and φ is a projectable field. At last, projectable fields make a Lie algebra, since if $\varphi, \psi \in \mathcal{X}_D(M)$, then

(C.1.9) $$\begin{aligned} dg \cdot \big[[\varphi, \psi], \xi\big] &= -\,dg\; L_\varphi\big(L_\xi(\psi)\big) + dg\; L_\psi\big(L_\xi(\varphi)\big) \\ &= -\,L_{\varphi'}\big(dg\; L_\xi(\psi)\big) + L_{\psi'}\big(dg\; L_\xi(\varphi)\big) \\ &\overset{(C.1.4)}{=} 0 \end{aligned}$$

If α is a projectable one-form, the first condition (C.1.5) clearly follows from (C.1.3), the second condition (C.1.6) follows from the fact that projectable fields form a basis of T_mM at any point $m \in M$ and that

(C.1.10) $$\begin{aligned} \langle L_\xi(\alpha), \varphi \rangle &= L_\xi \langle \alpha, \varphi \rangle - \langle \alpha, L_\xi(\varphi) \rangle \\ &\overset{(C.1.2)}{=} L_\xi \langle \alpha', \varphi' \rangle - \langle \alpha', dg\; L_\xi(\varphi) \rangle \\ &\overset{(C.1.4)}{=} 0. \end{aligned}$$

Conversely, the condition (C.1.5) and the injectivity of the mapping $\delta g(m) : T^*_{m'} M' \longrightarrow (T_m F_{m'})^\circ \subset T^*_m M$ imply the existence of a unique map $\Psi : F_m \longrightarrow T^*_{m'} M'$ such that

$$(C.1.11) \qquad \alpha(m) = \delta g(m)\ \Psi(m)\,.$$

Since this map is constant along the fiber $F_{m'}$ on account of the condition (C.1.5), Ψ factorizes as follows

$$(C.1.12) \qquad \Psi(m) = \alpha'(g(m))$$

showing that α is a projectable one-form.

C.2 The PN and PΩ structures on the reduced manifolds

As it has been showed in Sec 6, if $S \subset M$ is an immersed submanifold of M parametrized by $(M', f : M' \longrightarrow M)$ and if N, P, Ω are tensor fields on M, respectively of type (1,1), (2,0), (0,2), the reduced tensors N', P', Ω' on M'are given by

(C.2.1) $$N' \varphi' := df^{-1}\, N\, df\, \varphi'$$

(C.2.2) $$P' \alpha' := df^{-1}\, P\, \delta f\big|^{-1}_{\mathcal{X}^*_p(S)}\, \alpha'$$

(C.2.3) $$\Omega' \varphi' := \delta f\, \Omega\, df\, \varphi'$$

for any $\varphi' \in \mathcal{X}(M')$ and $\alpha' \in \mathcal{X}^*(M')$, $\delta f\big|^{-1}_{\mathcal{X}^*_p(S)}$ being any right-inverse of $\delta f\big|_{\mathcal{X}^*_p(S)}$. By (6.8), it follows that for any one-form $\alpha \in \mathcal{X}^*_p(S)$ it is

(C.2.4) $$P\, \delta f\big|^{-1}_{\mathcal{X}^*_p(S)}\, \delta f\big|_{\mathcal{X}^*_p(S)}\, \alpha = P\, \alpha$$

so that (C.2.2) can be given the form

(C.2.2') $$df\, P'\, \delta f\, \alpha = P \alpha \qquad \alpha \in \mathcal{X}^*_p(S)$$

(i) The property of being a Nijenhuis tensor, a Poisson tensor and a presymplectic tensor are maintained under restriction, on account of the following identities

(C.2.5) $$T(N')(\varphi', \psi') = df^{-1}\, T(N)(df\, \varphi', df\, \psi')$$

(C.2.6) $$[P', P'](\delta f \alpha, \delta f\, \beta) = df\, [P, P](\alpha, \beta) \qquad \alpha, \beta \in \mathcal{X}^*_p(S)$$

$$\text{(C.2.7)} \qquad d\Omega'(\varphi',\psi') = \delta\rho \cdot d\Omega(\varphi,\psi)$$

The identity (C.2.5) follows from

$$\text{(C.2.8)} \quad \mathbb{T}(N)(d\rho\,\varphi', d\rho\,\psi') := L_{N d\rho\cdot\varphi'}(N)(d\rho\cdot\varphi') - N\, L_{d\rho\cdot\varphi'}(N)(d\rho\cdot\varphi')$$

$$\overset{(C.2.1)}{=} L_{d\rho\cdot N'\varphi'}(d\rho\, N'\varphi') - N \cdot L_{d\rho\, N'\varphi'}(d\rho\,\varphi') +$$

$$- N\, L_{d\rho\,\varphi'}(d\rho\, N'\varphi') + N^2\, L_{d\rho\,\varphi'}(d\rho\,\varphi')$$

$$= d\rho \cdot (L_{N'\varphi'}(N')\varphi' - N' \cdot L_{\varphi'}(N')\varphi')$$

$$= d\rho \cdot \mathbb{T}(N')(\varphi',\psi')$$

where the well-known property of the Lie- derivative

$$\text{(C.2.9)} \qquad d\rho \cdot L_{\varphi'}(\psi') = L_{d\rho\cdot\varphi'}(d\rho\,\psi')$$

has been used.

On account of the identity (B.1.3), to prove (C.2.6), is equivalent to prove that

$$\text{(C.2.10)} \quad \langle L_{P'\alpha'}(\beta'), P'\gamma' \rangle + \cdots + \cdots = 0$$

Indeed, for any triple $\alpha,\beta,\gamma \in \mathfrak{X}^*_P(S)$, it is

$$\text{(C.2.11)} \quad \langle L_{P\alpha}(\beta), P\gamma \rangle + \cdots + \cdots =$$

$$\overset{(C.2.2')}{=} \langle \delta\rho \cdot L_{d\rho\, P'\delta\rho\,\alpha}(\beta), P'\,\delta\rho\,\gamma \rangle + \cdots + \cdots =$$

$$= \langle L_{P'\alpha'}(\beta'), P'\gamma' \rangle + \cdots + \cdots$$

where $\alpha' = \delta f\, \alpha$ and so on, and the well-known property of the Lie derivative

$$(C.2.12) \qquad \delta f \cdot L_{df\,\varphi'}(\alpha) = L_{\varphi'}(\delta f\, \alpha)$$

has been used. At last, (C.2.7) follows from

$$(C.2.13) \quad d\Omega'(\varphi', \psi') := L_{\varphi'}(\Omega'\psi') - \Omega' \cdot L_{\varphi'}(\psi') - L_{\psi'}(\Omega'\varphi') + d\langle \Omega'\varphi', \psi' \rangle$$

$$\overset{(C.2.3)}{=} L_{\varphi'}(\delta f\, \Omega\, \psi) - \delta f\, \Omega\, df\, L_{\varphi'}(\psi') +$$

$$- L_{\psi'}(\delta f\, \Omega\, df\, \varphi') + d\langle \delta f\, \Omega\, df\, \varphi', \psi' \rangle$$

$$\overset{(C.2.9\text{-}12)}{=} \delta f \cdot \left(L_{\varphi}(\Omega\psi) - \Omega\, L_{\varphi}(\psi) - L_{\psi}(\Omega\varphi) + d\langle \Omega\varphi, \psi \rangle \right)$$

$$= \delta f \cdot d\Omega(\varphi, \psi)$$

(ii) If P and N define a PN structure on M, this structure is maintained under restriction, since for any $\alpha, \beta \in \mathcal{X}^*_P(S)$

$$(C.2.14) \quad (NP - PN^*)\,\alpha = df\,(N'P' - P'N'^*)(\delta f\, \alpha)$$

$$(C.2.15) \quad R^*(P', N')(\delta f\, \alpha, \delta f\, \beta) = \delta f\; R^*(P, N)(\alpha, \beta)$$

Indeed, it is

$$(C.2.16) \quad (NP - PN^*)\,\alpha = N\, df\, P'\, \delta f\, \alpha - df\, P'\, \delta f\, N^*\alpha$$

$$= df\, N'P'\, \delta f\, \alpha - df\, P'N'^*\, \delta f\, \alpha$$

$$= df\, (N'P' - P'N'^*)(\delta f\, \alpha)$$

where we have taken into account that $N^*.(\mathcal{X}_P^*(S)) \subset \mathcal{X}_P^*(S)$ and that N'^* is defined by

(C.2.17) $$N'^* \delta\rho\, \alpha = \delta\rho\, N^* \alpha$$

To prove the second identity, let us observe that for any $\alpha, \beta \in \mathcal{X}_P^*(S)$ it is

(C.2.18) $$\begin{aligned} \delta\rho\, R^*(P,N)(\alpha,\beta) &= \delta\rho\cdot\big(L_{P\alpha}(N^*\beta) - N^* L_{P\alpha}(\beta) - L_{P\beta}(N^*\alpha) + \\ &\quad + N^* L_{P\beta}(\alpha) + d\langle\alpha, NP\beta\rangle - N^* d\langle\alpha, P\beta\rangle\big) \\ &= L_{d\rho^{-1}P\alpha}(N'^*\delta\rho\,\beta) - N'^* L_{d\rho^{-1}P\alpha}(\delta\rho\,\beta) + \\ &\quad - L_{d\rho^{-1}P\beta}(N'^*\delta\rho\,\alpha) + N'^* L_{d\rho^{-1}P\beta}(\delta\rho\,\alpha) + \\ &\quad + \delta\rho\, d\langle\alpha, d\rho\, N'P'\,\delta\rho\,\beta\rangle - N'^*\delta\rho\, d\langle\alpha, P\beta\rangle \\ &= L_{P'\alpha'}(N'^*\beta') - N'^* L_{P'\alpha'}(\beta') - L_{P'\beta'}(N'^*\alpha') \\ &\quad + N'^* L_{P'\beta'}(\alpha') + d\langle\alpha', N'P'\beta'\rangle - N'^* d\langle\alpha, P'\beta'\rangle \\ &= R^*(P',N')(\alpha',\beta') \end{aligned}$$

where $\alpha' = \delta\rho\,\alpha$, $\beta' = \delta\rho\,\beta$ and we have taken into account that

(C.2.19) $$N'P'\,\delta\rho\,\alpha = (d\rho^{-1} N\, d\rho)(d\rho^{-1} P\alpha) = d\rho^{-1}(NP)\alpha \qquad \alpha \in \mathcal{X}_P^*(S)$$

and that

(C.2.20) $$\delta\rho\, d\langle\alpha, \varphi\rangle = d\langle\alpha', \varphi'\rangle$$

(iii) If P and Ω define a PΩ structure on M, and if

(c.2.21) $$\Omega(\mathfrak{X}(S)) \subset \mathfrak{X}^*_P(S)$$

then the PΩ structure is maintained under restriction, since the following identity holds

(c.2.22) $$d(\Omega' P' \Omega')(\varphi', \psi') = \delta f \cdot d(\Omega P \Omega)(df\, \varphi', df\, \psi')$$

Indeed, (C.2.22) follows from (C.2.7) and from

(c.2.23) $$\begin{aligned} \Omega' P' \Omega' &= (\delta f\, \Omega\, df) \cdot (df^{-1} P\ \delta f\big|^{-1}_{\mathfrak{X}^*_P(S)}) (\delta f\, \Omega\, df) \\ &= \delta f\ \Omega\ \left(P\ \delta f\big|^{-1}_{\mathfrak{X}^*_P(S)}\ \delta f\right) \cdot \Omega\, df \\ &\overset{(c.2.4)(c.2.11)}{=} \delta f \cdot (\Omega P \Omega)\, df \end{aligned}$$

(iv) In a quite similar way as for the restriction, we could prove that the properties of being a Nijenhuis, a Poisson or a presymplectic tensor are maintained under projection, as well as the PN or PΩ structure. One has to proceed exactly as in the case of the restricted structures, taking into account that the properties (C.2.9) (C.2.12) of the Lie derivatives must be replaced by the property that the Lie derivative of a projectable one-form along a projectable field is itself a projectable one form:

(C.2.24) $$L_\varphi(\delta g \cdot \alpha') = \delta g \cdot L_{dg\,\varphi}(\alpha') \qquad \varphi \in \mathfrak{X}_D(M)$$

that projectable fields fulfil the well-known condition

(C.2.25) $$d g \; L_{\varphi}(\psi) = L_{dg\,\varphi}(dg \cdot \psi) \qquad \varphi, \psi \in \mathfrak{X}_D(M)$$

and that for any projectable field φ it is

(C.2.26) $$\Omega \; dg\Big|_{\mathfrak{X}_\Omega}^{-1} dg \cdot \varphi = \Omega \; \varphi \qquad \varphi \in \mathfrak{X}_D(M)$$

For that, the details of the proof are omitted for the sake of brevity.

Appendix D

D.1 The isotropy group of b is a subgroup of the group of upper triangular matrices

The isotropy group of the matrix $b := e_{a+1}e^{a}$ is given by the matrices h such that

(D.1.1) $$h^{-1} a h = a \qquad a := e_0 e^n$$

(D.1.2) $$\mathrm{Tr}\,(h^{-1} h_x) = 0$$

(D.1.3) $$\mathrm{Tr}\, e^{b+1} (h^{-1} b h + h^{-1} h_x - b)\, e_a = 0 \quad (a, b = 0, 1, \ldots, n-1)$$

Eq. (D.1.3) can be given the form

(D.1.4) $$\mathrm{Tr}\ e^{b+1}\, h^{-1} (b\, h_a + h_{ax} - h_{a+1}) = 0$$

entailing that

(D.1.5) $$h^{-1} (b\, h_a + h_{ax} - h_{a+1}) = e_0 v_a$$

where v_a (a = 0,...,n-1) are arbitrary Lagrangean multipliers. Thus we can write

(D.1.6) $$h_{a+1} = h_{ax} + b h_a - h_0 v_a$$

On the other hand, (D.1.1) entails that $h_0 = h^n_n e_0$, so that we can conclude that h is an upper triangular matrix whose diagonal entries are $h^i_i = h^n_n$.

Indeed, the first property is readily seen by recursion, since from (D.1.6) it follows that if $h^c_a = 0$ $(c > a)$ then

(D.1.7) $$h^{c+1}_{a+1} = h^{c+1}_{a\,x} + h^{c}_{a} = 0$$

Furthermore, (D.16) entails that

(D.1.8) $$h^{a+1}_{a+1} = h^{a+1}_{a\,x} + h^{a}_{a} - h^{a+1}_{o}\, v_a$$

$$= h^{a}_{a}$$

At last, it follows from (D.1.2) that

(D.1.9) $$\mathrm{Tr}\; h^{-1} h_x = m.(h^{m}_{m})^{-1}.\, h^{m}_{m\,x} = 0$$

so that we can choose $h^{i}_{i} := 1$: the matrices h are thus given by upper triangular matrices with unit entries on the diagonal. The iterated solution of (D.1.6) is given by

(D.1.10) $$h_a = -\sum_{o}^{a}{}_{\kappa} \sum_{\kappa}^{a}{}_{\ell} \binom{\ell}{\kappa} e_\kappa\, \partial_{\ell-\kappa}\, v_{a-\ell-1} \qquad (v_{-1} := -1)$$

as it can be verified straightforwardly.

D.2 The relation between the affine coordinates (u_a, u^a) and the fibered coordinates (v_a, v^a)

Let us recall that $u \in S^{*}_{b}$, $v \in F$ and $t \in T$ are given respectively by

(D.2.1) $$u := e_{b+1}\, e^{b} + u^{a}\, e_a\, e^{m} + u_a\, e_o\, e^{a}$$

(D.2.2) $$v := e_{b+1}\, e^{b} + v^{a}\, e_a\, e^{m}$$

(D.2.3) $$t^{\kappa}_{j} := -\sum_{\kappa}^{j}{}_{\ell} \binom{\ell}{\kappa} \left(\partial_{\ell-\kappa}\, v_{j-\ell-1} \right) \qquad (v_{-1} := -1\,;\ j \geq \kappa)$$

Then from

(D.2.4) $$u = h^{-1} h_x \qquad h = w t \qquad w^{-1} w_x = v$$

it follows that

(D.2.5) $$t u = v t + t_x$$

Since

(D.1.6) $$e^{o}\, t\, u\, e_a = t^{o} u\, e_a \overset{(D.2.1)}{=} u_a + t^{o}_{a+1}$$

(D.2.7) $$e^{o}(v t + t_x) \overset{(D.2.2)}{=} v^{o} t^{n}_{a} + t^{o}_{ax} = t^{o}_{ax}$$

the affine coordinate u_a and the fibered coordinate v_a can be identified, as follows from

(D.2.8) $$u_a = t^{o}_{ax} - t^{o}_{a+1} \overset{(D.2.3)}{=} v_a$$

On the other hand, since

(D.2.9) $$e^{a}(t u)\, e_n \overset{(D.2.1)}{=} t^{a}_{b} u^{b}$$

(D.2.10) $$e^{a}(v t + t_x)\, e_n \overset{(D.2.2)}{=} t^{a-1}_{n} + t^{a}_{nx} + v^{a}$$

it follows that

$$v^a = t^a_b u^b - t^{a-1}_n - t^a_{nx}$$

(D.2.11)
$$= -\sum_a^{n-1}{}_b \sum_a^b{}_\ell \binom{\ell}{a}(\partial_{\ell-a} u_{b-\ell-1}) u^b +$$
$$+ \sum_{a-1}^n{}_\ell \binom{\ell}{a-1}(\partial_{\ell-a+1} u_{n-\ell-1}) + \sum_a{}_\ell \binom{\ell}{a}(\partial_{\ell-a} u_{n-\ell-1})$$
$$= -\sum_a^{n+1}{}_\ell \sum_\ell^{n+1}{}_\kappa \binom{\ell}{a}(\partial_{\ell-a} u_{\kappa-1-\ell}) u^\kappa$$

where $u^n := 0$, $u^{n+1} := -1$, $u_{-1} = v_{-1} := -1$.

In conclusion, the relation between affine and fibered coordinates is given by

(D.2.12)
$$\begin{cases} v_a = u_a \\ v^a = -\sum_a^{n+1}{}_\ell \sum_\ell^{n+1}{}_\kappa \binom{\ell}{a}(\partial_{\ell-a} u_{\kappa-1-\ell}) u^\kappa \end{cases}$$

The differential of this map is obviously

(D.2.13)
$$\begin{cases} \psi_a = \varphi_a \\ \psi^a = -\sum_a^{n+1}{}_\ell \sum_\ell^{n+1}{}_\kappa \binom{\ell}{a}\left((\partial_{\ell-a} \varphi_{\kappa-\ell-1}) u^\kappa + (\partial_{\ell-a} u_{\kappa-1-\ell}) \varphi^\kappa\right) \end{cases}$$

Then by substituting (D.2.13) into

(D.2.14)
$$\mathrm{Tr} \sum_0^{n-1}{}_a \int_{-\infty}^{+\infty} (\psi_a \beta^a + \psi^a \beta_a)\, dx = \mathrm{Tr} \sum_0^{n-1}{}_a \int_{-\infty}^{+\infty} (\varphi_a \alpha^a + \varphi^a \alpha_a)\, dx$$

by integrating by parts and taking into account algebraic identities such as

(D.2.15)
$$\sum_a^n{}_\ell \sum_0^{n-\ell}{}_\kappa \equiv \sum_0^{n-a}{}_\kappa \sum_a^{n-\kappa}{}_\ell$$

one obtains the explicit form

$$\text{(D.2.15)} \quad \begin{cases} \alpha^a = -\sum_{0}^{n}{}_{\ell} \sum_{0}^{\ell}{}_{j} (-1)^\ell \binom{\ell}{j} \partial_{\ell-j} \left(u^{j+\ell+1} \beta_j \right) + \beta^a & (\beta_n := 0) \\ \alpha_a = -\sum_{0}^{n}{}_{\ell} \sum_{0}^{\ell}{}_{j} \binom{\ell}{j} \beta_j \, \partial_{\ell-j} \left(u_{a-\ell-1} \right) & \end{cases}$$

of the dual map (the details are omitted for brevity, since they are straightforward but lengthy).

D.3 ——— The solution of the equation

(D.3.1) $$\alpha_{a+1} = (\partial_x + u)\,\alpha_a$$

is given, for any arbitrary matrix u, by the matrix

(D.3.2) $$\alpha = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{i}{j} u_{(j)} \partial_{i-j}\, \alpha_o \cdot e^i$$

where α_o is the first column of α and the matrices $u_{(j)}$ are recursively determined by $(j = 0,1,\ldots,n-1)$:

(D.3.3) $$u_{(j+1)} = \partial_x u_{(j)} + u\; u_{(j)} \qquad u_{(o)} = \mathbb{1}$$

Proof. Since (D.3.2) implies that the a-ieme column of α is given by

(D.3.4) $$\alpha_a = \sum_{j=0}^{a} \binom{a}{j} u_{(j)} \partial_{a-j}\, \alpha_o$$

the property holds for a = 0. For a > 0, we obtain

(D.3.5) $$\alpha_{ax} + u\,\alpha_a =$$

$$\overset{(D.3.4)}{=} \sum_{j=0}^{a} \binom{a}{j} u_{(j)x}\, \partial_{a-j}\, \alpha_o + \sum_{j=0}^{a} \binom{a}{j} u_{(j)}\, \partial_{a+1-j}\, \alpha_o + \sum_{j=0}^{a} \binom{a}{j} u\, u_{(j)}\, \partial_{a-j}\, \alpha_o$$

$$= \sum_{j=0}^{a} \binom{a}{j} \left(u_{(j)x} + u\, u_{(j)}\right) \partial_{a-j}\, \alpha_o + \sum_{j=0}^{a} \binom{a}{j} u_{(j)}\, \partial_{a+1-j}\, \alpha_o$$

$$\overset{(D.3.3)}{=} \sum_{j=0}^{a} \binom{a}{j} u_{(j+1)}\, \partial_{a-j}\, \alpha_o + \sum_{j=0}^{a} \binom{a}{j} u_{(j)}\, \partial_{a+1-j}\, \alpha_o$$

$$= \sum_{k=0}^{a+1} \binom{a+1}{k} u_{(k)}\, \partial_{a+1-k}\, \alpha_o$$

$$\overset{(D.3.4)}{=} \alpha_{a+1} \,.$$

where the identity

$$(D.3.6) \qquad \binom{a}{k-1} + \binom{a}{k} = \binom{a+1}{k}$$

has been used. Therefore the property (D.3.2) holds for any $a < n$. The previous result holds for any matrix u. As a particular case, if u is a "traceless Frobenius matrix", i.e. if

$$(D.3.7) \qquad u = v := e_{a+1}\, e^{a} + v^{a} e_{a}\, e^{m}$$

then the matrices $u_{(j)}$ are such that ($j = 0,1,\ldots,n-1$)

$$(D.3.8) \qquad u^{k}_{(j)o} = \begin{cases} 0 & k > j \\ 1 & k = j \end{cases}$$

The property (D.3.8) holds trivially for $j = 0$. If it holds for $j > 0$, then

$$(D.3.9) \quad u^{k}_{(j+1)o} \overset{(D.3.3)}{=} \partial_x u^{k}_{(j)o} + v \cdot u_{(j)o} \overset{(D.3.7)}{=} \partial_x u^{k}_{(j)o} + u^{k-1}_{(j)o} + v^{k} u^{m}_{(j)o}$$

so that if $k > j+1$

$$(D.3.10) \qquad u^{k}_{(j+1)o} \overset{(D.3.8)}{=} 0$$

and

$$(D.3.11) \qquad u^{j+1}_{(j+1)o} = \partial_x u^{j+1}_{(j)o} + u^{j}_{(j)o} + v^{j+1} u^{m}_{(j+1)o} \overset{(D.3.8)}{=} 1$$

Then the property (D.3.8) holds for any integer $j \leq n-1$. In particular, from (D.3.8) it follows that

(D.3.12) $\quad u^{m-1}_{(j)0} = \delta^{m-1}_j$

(D.3.13) $\quad u^{m}_{(j)0} = \delta^{m}_j$.

D.4 The equation

(D.4.1) $$\alpha^m_{mx} + \alpha^{m-1}_m - \alpha^m_a\, v^a_m = 0$$

where v is a traceless Frobenius matrix and α is solution of the equation (D.3.1) has the following solution :

(D.4.2) $$\alpha^0_0 = -\frac{1}{n+1}\sum_{k=1}^{m}\Big(\sum_{j=0}^{n-1}\Big(\binom{n}{j}u^m_{(j)k}\partial_{m-j}\alpha^k_0 + \binom{n}{j}\int_{-\infty}^{x}u^{m-1}_{(j)k}\partial_{m-j}\alpha^k_0\,dx + $$
$$-\sum_{a=j}^{m-1}\binom{a}{j}\int_{-\infty}^{x}u^m_{(j)k}\partial_{a-j}\alpha^k_0\,dx\Big) - u^m_{(n)k}\alpha^k_0 - \int_{-\infty}^{x}u^{m-1}_{(n)k}\alpha^k_0\,dx\Big) .$$

Proof: On account of Eq.(D.3.2) , Eq.(D.4.1) takes the form

(D.4.3) $$\sum_{j=0}^{m}\binom{m}{j}\partial_x\left(u^m_{(j)}\partial_{m-j}\alpha_0\right) + \sum_{j=0}^{m} u^{m-1}_{(j)}\partial_{m-j}\alpha_0 - \sum_{a=0}^{m-1}\sum_{j=0}^{a}\binom{a}{j}u^m_{(j)}(\partial_{a-j}\alpha_0)v^a = 0$$

By writing on the left-hand side only the terms with α^0_0 , we obtain

(D.4.4) $$\sum_{j=0}^{m}\binom{n}{j}\left(\partial_x\left(u^m_{(j)0}\partial_{m-j}\alpha^0_0\right) + u^{m-1}_{(j)0}\partial_{m-j}\alpha^0_0\right) +$$
$$-\sum_{a=0}^{n-1}\sum_{j=0}^{a}\binom{a}{j}u^m_{(j)0}\left(\partial_{a-j}\alpha^0_0\right)v^a =$$
$$= -\sum_{j=0}^{m}\sum_{k=1}^{m}\binom{n}{j}\left(\partial_x\left(u^m_{(j)k}\partial_{m-j}\alpha^k_0\right) + u^{m-1}_{(j)k}\partial_{m-j}\alpha^k_0\right) +$$
$$+\sum_{a=0}^{m-1}\sum_{j=0}^{a}\sum_{k=1}^{m}\binom{a}{j}u^m_{(j)k}\left(\partial_{a-j}\alpha^k_0\right)v^a$$

On account of (D.3.12) and (D.3.13), the left-hand side of (D.4.4) becomes

$$\text{(D.4.5)} \quad \partial_x \alpha_0^0 + \binom{n}{n-1} \partial_x \alpha_0^0 = (n+1)\, \partial_x \alpha_0^0$$

so that one easily obtains Eq. (D.4.2), by using the identity

$$\text{(D.4.6)} \quad \sum_{a=0}^{n-1} \sum_{j=0}^{a} \sum_{k=1}^{n} = \sum_{j=0}^{n-1} \sum_{k=1}^{n} \sum_{a=j}^{n-1}$$

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