Università di Milano–Bicocca
Quaderni di Matematica

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Quaderno n. 10/2008 (arxiv:math/0806.1831v1)

Stampato nel mese di giugno 2008
presso il Dipartimento di Matematica e Applicazioni,
Università degli Studi di Milano-Bicocca, via R. Cozzi 53, 20125 Milano, ITALIA.

Disponibile in formato elettronico sul sito www.matapp.unimib.it.
Segreteria di redazione: Ada Osmetti - Giuseppina Cogliandro
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Esemplare fuori commercio per il deposito legale agli effetti della Legge 15 aprile 2004 n.106.

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June 11, 2008

Abstract

We study the intrinsic geometry of a one-dimensional complex space provided with a Kähler metric in the sense of Grauert. We show that if κ is an upper bound for the Gaussian curvature on the regular locus, then the intrinsic metric has curvature $\leq \kappa$ in the sense of Alexandrov.

Contents

1 Introduction

Let Ω be a domain in affine space $\mathbb{C}^n$ and let $X \subset \Omega$ be a one-dimensional analytic subset. Denote by $\langle\ ,\ \rangle$ the flat metric on $\mathbb{C}^n$ and by g the induced Riemannian metric on the regular part X_{reg} of X. Define a distance on X by setting $d(x, y)$ equal to the infimum of the lengths of curves lying in X and joining x to y. If X is smooth d is simply the Riemannian distance associated to g. Gauss equation together with the Kähler property of g ensures that the Gaussian curvature of g is nonpositive. What can be said if X contains singularities? The purpose of the present paper is to show that the same statement

*Partially supported by MIUR PRIN 2005 *"Spazi di moduli e Teorie di Lie"*.

still holds provided Gaussian curvature is replaced by Alexandrov curvature. Namely (X, d) is an inner metric space of nonpositive curvature in the sense of Alexandrov.

More generally the following situation is considered. Let X be a one-dimensional connected reduced complex space and let ω be a Kähler form on X in the sense of Grauert [12]. The Kähler form ω and the complex structure define a Riemannian metric g on X_{reg}. Use this g to compute the length of paths in X and for $x, y \in X$ define $d(x, y)$ as the infimum of the lengths of paths in X from x to y (see §2 for precise definitions). We refer to d as the intrinsic distance of (X, ω). It turns out that d is an intrinsic distance on X inducing the original topology. Our results can then be summarised in the following statement.

Theorem 8. *Let X be a one-dimensional connected reduced complex space. Let ω be a Kähler metric on X in the sense of Grauert and let d be the intrinsic distance of (X, ω). If κ is an upper bound for the Gaussian curvature of g on X_{reg}, then (X, d) is a metric space of curvature $\leq \kappa$ in the sense of Alexandrov.*

The proof is delicate but rather elementary. The plot of the paper is the following.

In §2 we recall the definition of Kähler forms on a singular space, define the intrinsic distance in the one-dimensional case and prove some basic properties. Many statements hold in more general situations, but we restrict from the beginning to the one-dimensional case in order not to burden the presentation. At the end we show that to investigate local problems one might restrict consideration to the case in which X is a one-dimensional analytic subset in $\mathbb{C}^n$ provided with a general Kähler metric. Appropriate conventions and notations are fixed to be used in the study of this particular case under the additional hypothesis that there is only one singularity which is (analytically) irreducible. This study occupies §§3–6.

In §3 we consider differentiability properties of segments $\alpha : [0, L] \to X$. Since $X \subset \mathbb{C}^n$ we can consider the tangent vector $\alpha(t)$ at least when $\alpha(t) \in X_{\text{reg}}$. The main point is a Hölder estimate for $\dot{\alpha}$ (Theorem 2). This is proved by expressing the second fundamental form of $X \subset \mathbb{C}^n$ in terms of the normalisation map. Here is where the Kähler property is used. Next we make various observations regarding the asymptotic behaviour of the distance d and of the tangents to segments close to a singular point.

In §4 we study regularity properties of segments through the normalisation. This is useful to compute angles between the tangent vectors at a singular point.

§5 is the most technical section. We study uniqueness properties of geodesics near the singular point. We construct a decreasing sequence of radii $r_1 > r_2 > r_3 > r_4 > r_5 > r_6$ such that the geodesic balls centred at the singular point have better and better uniqueness properties. As a first step (Prop. 9 and Theorem 3) we show that if two segments have the same endpoints then the singular point lies in the interior of the closed curve formed by the segments. To prove this we combine extrinsic and intrinsic information. The former amounts to the Hölder estimate alluded to above and the finiteness of the area of X (Lelong theorem). The latter is provided by Gauss–Bonnet and Rauch theorems. The

Jordan separation theorem is used on several occasions. Th next step (Theorem 4) is to show that if two points sufficiently close to the singularity are joined by two distinct segments one of them has to pass through the singular point. Here the argument is based on the winding number and the fact that X is a ramified covering of the disc.

In §6 we prove that sufficiently small balls centred at the singular point are geodesically convex (Cor. 5). On the way we prove (using an idea from [17]) that the distance from a singular point is C^1 in a (deleted) neighbourhood of it. We establish various technical properties of segments emanating from the singular point and the angle their tangent vectors form at the singular point. In particular we study "sectors" with vertex at the singular point (Lemma 25) and establish their convexity (Theorem 5).

In §7 we recall the main concepts of the intrinsic geometry of metric spaces in the sense of A.D. Alexandrov. Next, by combining the information on sectors and angles collected before, we show that a sufficiently small ball centred at a singular point is a CAT(κ)–space. This completes the proof of Theorem 8 in the case of an irreducible singularity. The case of reducible singularities is dealt with by reasoning as in Reshetnyak gluing theorem and invoking the result in the irreducible case.

At the end of the paper we observe that the statement corresponding to Theorem 8 with lower bounds on curvature instead of upper bounds is false. In particular a Kähler curve (X, d) can have curvature bounded below in the sense of Alexandrov only if X is smooth (Theorem 9).

Acknowledgements The author wishes to thank Prof. Giuseppe Savaré for turning his attention to the Alexandrov notions of curvature and both him and Prof. Gian Pietro Pirola for various interesting discussions on subjects connected with this work. He also acknowledges generous support from MIUR PRIN 2005 "Spazi di moduli e Teorie di Lie".

2 Intrinsic distance

A Let X be *complex curve*, that is a one-dimensional reduced complex space. By definition for any point $x \in X$ there is an open neighbourhood U of x in X, a domain Ω in some affine space $\mathbb{C}^n$ and a map $\tau : U \to \Omega$ that maps U biholomorphically onto some one-dimensional analytic subset $A \subset \Omega$. We call the quadruple (U, τ, A, Ω) a *chart* around x.

Definition 1. *A* Kähler form *on X is a Kähler form ω on X_{reg} with the following property: for any $x \in X_{\text{sing}}$ there is are a chart (U, τ, A, Ω) around x and a Kähler form ω' on Ω such that $\tau^*\omega' = \omega$ on $U \cap X_{\text{reg}}$. We call ω' a* local extension *of ω.*

This definition is due to Grauert [12, §3.3]. A *Kähler curve* is a complex curve with a fixed Kähler form. Let (X, ω) be a Kähler curve. Denote by J the complex structure on X_{reg}. Then $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric on X_{reg}. Denote by $|v|_g$ the norm of $v \in T_x X_{\mathrm{reg}}$ with respect to g. A path $\alpha : [a,b] \to X$ is of class C^1 if $\tau \circ \alpha$ is C^1 for any chart. For a piecewise C^1 path α the length is defined by

$$L(\alpha) = \int_{\alpha^{-1}(X_{\mathrm{reg}})} |\dot{\alpha}(t)|_g dt. \tag{1}$$

Lemma 1. *Let (U, τ, A, Ω) be a chart and ω' a Kähler form on Ω extending ω, with g' the corresponding metric. If $\alpha : [a,b] \to U$ is a piecewise C^1 path and $\beta = \tau \circ \gamma$, then*

$$L(\alpha) = \int_a^b |\dot{\beta}(t)|_{g'} dt \tag{2}$$

Proof. Let $E = \alpha^{-1}(X_{\mathrm{reg}})$, $F = I \setminus E$, $B = F^0$, $D = \partial F$. Then $I = E \sqcup B \sqcup D$. Since X has isolated singularities α and β are constant on the connected components of F, so $\dot{\beta} \equiv 0$ on B. The set D is countable, so has zero measure. Therefore

$$\int_a^b |\dot{\beta}|_{g'} dt = \int_E |\dot{\beta}|_{g'} = \int_E |\dot{\alpha}|_g = L(\alpha).$$

□

For $x, y \in X$ set

$$d(x,y) = \inf\{L(\alpha) : \alpha \text{ piecewise } C^1 \text{ path in } X \text{ with } \alpha(0) = x,\ \alpha(1) = y\}. \tag{3}$$

For $r > 0$ set also $\mathfrak{B}(x,r) = \{y \in X : d(x,y) < r\}$. Recall the following fundamental result of Łojasiewicz.

Theorem 1 (Łojasiewicz, [16, §18, Prop. 3, p.97])**.** *Let A be an analytic subset in a domain $\Omega \subset \mathbb{C}^n$ and $z_0 \in A$. Then there are $C > 0$, $\mu \in (0,1]$ and a neighbourhood V of z_0 in A such that for any $z, z' \in V$ there is a real analytic path $\beta : [0,1] \to A$ joining z to z' with $\int_0^1 |\dot{\beta}| dt \leq C|z - z'|^\mu$. (Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^n$.)*

Proposition 1. *If (X, ω) is a connected Kähler curve, then d is a distance on X inducing the original topology.*

Proof. We start by showing that $d(x,y)$ is finite for any $(x,y) \in X \times X$. If x and y belong to the same connected component of X_{reg} this is obvious. Assume that $x \in X_{\mathrm{sing}}$. Let (U, τ, A, Ω) be a chart around x and ω' a local extension of ω. By restricting U we may assume that there is a constant $C > 0$ such that $C^{-1}|d\tau(v)| \leq |v|_g \leq C|d\tau(v)|$ for any $v \in TU_{\mathrm{reg}}$. If $\alpha : [a,b] \to U$ is a piecewise C^1 curve and $\beta = \tau \circ \alpha$, then $C^{-1}L(\beta) \leq L(\alpha) \leq CL(\beta)$, where the length of β is computed with respect to the Euclidean norm. By Łojasiewicz Theorem for any point $y \in U$ there is a piecewise C^1 path α in A joining $\tau(x)$ to $\tau(y)$. Then $\beta = \tau^{-1} \circ \alpha$ is a path in X joining x to y with $L(\beta) \leq C \cdot L(\alpha) < +\infty$ hence $d(x,y) < +\infty$ for all $y \in U$. Because the length functional L is additive with respect to the concatenation of paths, it follows that $d(x,y) < +\infty$ for all y in some irreducible component of X that passes through x. Since X is connected this yields finiteness of d.

At this point one might apply the general machinery of [19, p.123ff] or [7, p.26ff]. The class of piecewise C^1 paths is closed under restriction, concatenation and C^1 reparametrisations. Moreover L is invariant under C^1 reparametrisation, it is an additive function on the intervals and $L(\alpha\big|_{[a,t]})$ is a continuous function of $t \in [a,b]$. It follows that d is a distance on X.

Let $V \subset X$ be an open set (for the original topology) and let $x \in V$. Fix a chart (U, τ, A, Ω) around x and a local extension ω'. Denote by d_Ω the Riemannian distance of (Ω, ω'). Let U' be a neighbourhood of x with compact closure in $U \cap V$. Since $\tau(x) \notin \tau(\partial U')$, $\varepsilon = d_\Omega\big(\tau(x), \tau(\partial U')\big) > 0$. If $\alpha : [a,b] \to X$ is a continuous path with $\alpha(a) = x$ and $\alpha(b) \notin U'$ set $c = \sup\{t \in [a,b] : \alpha(t) \in U'\}$. Then

$$L(\alpha) \geq L(\alpha\big|_{[a,c]}) = L(\tau \circ \alpha\big|_{[a,c]}) \geq d_\Omega\big(\tau(x), \tau(\partial U')\big) = \varepsilon.$$

Hence $\mathfrak{B}(x, \varepsilon) \subset U' \subset V$. This shows that the metric topology is finer than the original one.

Conversely we show that for any $x \in X$ and $\delta > 0$ the metric ball $\mathfrak{B}(x,\delta)$ is open in the original topology. Let again (U, τ, A, Ω) be a chart around x and let ω' be a local extension and assume that there is a constant $C > 0$ such that $C^{-1}|d\tau(v)| \leq |v|_g \leq C|d\tau(v)|$ for any $v \in TU_{\text{reg}}$. Thanks to Łojasiewicz Theorem by restricting U and Ω we can assume that for any $z, z' \in A$ there is a C^1 path joining z and z' and having Euclidean length $\leq C'|z - z'|^\mu$. For $x' \in \mathfrak{B}(x, \delta)$ put $\delta' = \sqrt[\mu]{(\delta - d(x,x'))/CC'} > 0$. Then the set $\tau^{-1}(\{z \in \Omega : |z - \tau(x')| < \delta'\})$ is contained in $\mathfrak{B}(x', \delta - d(x,x')) \subset \mathfrak{B}(x, \delta)$. Therefore $\mathfrak{B}(x,\delta)$ is open in the original topology and the two topologies coincide. □

Starting from the metric space (X, d) one can define a new length functional L_d by the formula

$$L_d(\gamma) = \sup \sum_{i=1}^{N} d(\gamma(t_{i-1})\gamma(t_i)) \tag{4}$$

the supremum being over all partitions $t_0 < \ldots < t_N$ of the domain of γ. By definition $d(x,y) \leq L_d(\gamma)$ for any continuous path joining x to y, while the inequality $L_d(\gamma) \leq L(\gamma)$ holds for any piecewise C^1 path. The distance d is *intrinsic* if $d(x,y) = \inf\{L_d(\gamma) : \gamma \in C([0,1], X), \gamma(0) = x, \gamma(1) = y\}$.

Proposition 2. *The distance d is intrinsic.*

Proof. This is proved for general length structures in [7, Prop. 2.4.1 p.38]. □

Definition 2. *We call d the* intrinsic distance *of the Kähler curve (X, ω).*

For geodesics in the metric space (X,d) we adopt the following terminology. A *shortest path* is a map $\gamma : [a,b] \to X$ such that $L_d(\gamma) = d(\gamma(a), \gamma(b))$. *Minimising geodesic* is synonymous of shortest path. One can reparametrise a shortest path in such a way that $d(\gamma(t), \gamma(t')) = |t - t'|$ for any t, t'. In this case we say that γ has *unit speed*. A *segment* is by definition a unit speed shortest path. More generally, we say that γ is parametrised with constant speed c if $d(\gamma(t), \gamma(t')) = |t - t'|$ for any t, t'. If I is any interval a path $\gamma : I \to X$ is a *geodesic* if for any $t \in I$ there is a compact neighbourhood $[t_0, t_1]$ of t in I such that $\gamma\big|_{[t_0,t_1]}$ is a shortest path with constant speed.

Lemma 2 ([7, Prop. 2.5.19 p.49]). *If the ball $\mathfrak{B}(x, r)$ is relatively compact in X, for any $y \in \mathfrak{B}(x, r)$ there is a segment from x to y.*

(X_{reg}, g) is a (smooth) Riemann surface with a *noncomplete* smooth Kähler metric. For $x \in X_{\text{reg}}$ denote by $U_x X$ the unit sphere in $T_x X$. Let $UX_{\text{reg}} = \bigcup_{x \in X_{\text{reg}}} U_x X$ be the unit tangent bundle. We denote by $(t, v) \mapsto \gamma^v(t)$ the geodesic flow: that is $\gamma^v(t) = \exp_x(tv)$ where $x = \pi(v)$. Let $\mathscr{U} \subset \mathbb{R} \times TX_{\text{reg}}$ be the maximal domain of definition of the geodesic flow of (X_{reg}, g). It is an open neighbourhood of $\{0\} \times TX_{\text{reg}}$ in $\mathbb{R} \times TX_{\text{reg}}$. Let $\mathscr{D} \subset TX_{\text{reg}}$ denote the maximal domain of definition of the exponential: $\mathscr{D} = \{v \in TX_{\text{reg}} : (1, v) \in \mathscr{U}\}$. For $x \in X_{\text{reg}}$ set $\mathscr{D}_x = \mathscr{D} \cap T_x X$. Then $\mathscr{D}_x$ is the maximal domain of definition of $\exp_x$. Both $\mathscr{D}$ and $\mathscr{D}_x$ are open in TX_{reg} and $T_x X$ respectively and the maps $\exp : \mathscr{D} \to X_{\text{reg}}$ and $\exp_x : \mathscr{D}_x \to X_{\text{reg}}$ are defined and smooth. For $v \in U_x X$ set

$$T_v = \sup\{t > 0 : tv \in \mathscr{D}_x\}. \tag{5}$$

Denote by $B_x(0, r)$ the ball in $T_x X$ with respect to g_x.

Definition 3. *For $x \in X_{\text{reg}}$ the* injectivity radius *at x, denoted* inj_x*, is the least upper bound of all $\delta > 0$ such that $B_x(0, \delta) \subset \mathscr{D}_x$ and $\exp_x\big|_{B_x(0,\delta)}$ is a diffeomorphism onto its image.*

Lemma 3. *For any $x \in X_{\text{reg}}$, $\text{inj}_x \leq d(x, X_{\text{sing}})$.*

Proof. Let $\gamma : I \to X$ be a piecewise C^1 path in $\mathcal{P}$ joining x to some singular point x_0. For $\delta \in (0, \text{inj}_x)$ put $U_\delta = \exp_x(B_x(0, \delta))$. Since $\overline{U_\delta} \subset X_{\text{reg}}$ and x_0 is singular, there is some $t \in I$ such that $\gamma(t) \in \partial U_\delta$. Let t_0 be the smallest such number. Then $\gamma\big|_{[0,t_0]}$ is a path entirely contained in X_{reg}. It follows from Gauss Lemma [11, Prop. 3.6, p.70] that $L(\gamma\big|_{[0,t_0]}) \geq \delta$. Therefore also $L(\gamma) \geq \delta$. Since γ, x_0 and $\delta < \text{inj}_x$ are arbitrary we get $d(x, X_{\text{sing}}) \geq \text{inj}_x$. ☐

Lemma 4. *Let $x \in X_{\text{reg}}$ and $y \in \mathfrak{B}(x, \text{inj}_x)$.*

1. *The intrinsic distance equals the Riemannian distance in (X_{reg}, g):*

$$\begin{aligned} d(x, y) = \inf\{L(\gamma) : \gamma \textit{ piecewise } C^1 \textit{ path in } X_{\text{reg}} \\ \textit{with } \gamma(0) = x, \gamma(1) = y\}. \end{aligned} \tag{6}$$

2. $\mathfrak{B}(x, \text{inj}_x) = \exp_x(B_x(0, \text{inj}_x))$.
3. *There is a unique segment joining x to y and it coincides with the minimising Riemannian geodesic in (X_{reg}, g) from x to y.*

4. A geodesic γ in (X,d) is smooth on $\gamma^{-1}(X_{\mathrm{reg}})$ and there $\nabla_{\dot\gamma}\dot\gamma = 0$.

Proof. It follows from the previous lemma and the hypothesis $d(x,y) < \mathrm{inj}_x$ that paths passing through singular points do not contribute to the infimum in (3). This proves (6). From this follows that y lies in $\exp_x(B_x(0,\mathrm{inj}_x))$. So $\mathfrak{B}(x,\mathrm{inj}_x) \subset \exp_x(B_x(0,\mathrm{inj}_x))$. The reverse implication is obvious. This proves 2. In particular $y \in \exp_x(B_x(0,\mathrm{inj}_x))$, so there are $v \in U_xX$ and $r \in (0,\mathrm{inj}_x)$ such that $y = \exp_x rv$. It follows from Gauss Lemma that the inf in (6) is attained only on the path $\gamma(t) = \exp_x tv$, $t \in [0,r]$. So $L(\gamma) = d(x,y)$. But $d(x,y) \le L_d(\gamma) \le L(\gamma)$ so $L_d(\gamma) = L(\gamma)$ and γ is a segment also in (X,d). We have to prove that it is the unique one. Since $d(x,y) < \mathrm{inj}_x$ it follows from 2 that any other segment α must lie in $\exp_x(B_x(0,\mathrm{inj}_x)) \subset X_{\mathrm{reg}}$. If α is smooth we can again apply Gauss Lemma. So it is enough to show that α is differentiable, which will yield 4 at once. This is a local problem, so we just prove that $\alpha\big|_{[0,t_0]}$ is smooth for some $t_0 > 0$. By Whitehead theorem [11, Prop. 4.2 p.76] there is a neighbourhood W of x such that for any $z \in W$, $W \subset \mathfrak{B}(z,\mathrm{inj}_z)$. Let t_0 be small enough so that $\alpha([0,t_0]) \subset W$. Put $x_0 = \alpha(t_0)$ and let β be the unique minimising Riemannian geodesic from x to x_0. We already know that $L(\beta) = d(x,x_0) = t_0$. For $t \in (0,t_0)$ let β_1 and β_2 be the unique Riemannian geodesics from x to $\alpha(t)$ and from $\alpha(t)$ to x_0 respectively. Both of them are also shortest paths, by the above. Moreover

$$t_0 = L_d(\alpha) \ge d(x,\alpha(t)) + d(\alpha(t),x_0) = L(\beta_1) + L(\beta_2) \ge L(\beta) = t_0.$$

So $L(\beta_1 * \beta_2) = L(\beta_1) + L(\beta_2) = L(\beta)$. Since the concatenation $\beta_1 * \beta_2$ is piecewise smooth $\beta_1 * \beta_2 = \beta$. This means that $\alpha(t)$ lies on $\beta([0,t_0])$. Since t is arbitrary we get $\alpha\big|_{[0,t_0]} = \beta$. In particular α is smooth. ☐

Proposition 3. *On piecewise C^1 paths the functional L_d agrees with L.*

Proof. The inequality $L_d \le L$ is obvious from the definition of d. For the reverse inequality consider a piecewise C^1 path $\gamma : [0,1] \to X$ and assume at first that $\gamma([0,1]) \subset X_{\mathrm{reg}}$. Since $L_d(\gamma) \le L(\gamma) < \infty$ the limit

$$\lim_{h\to 0} \frac{d(\gamma(t+h),\gamma(t))}{|h|}.$$

exists for a.e. $t \in [0,1]$. It is called *metric derivative* and denoted by $|\dot\gamma(t)|_d$. It is an integrable function of t and

$$L_d(\gamma) = \int_0^1 |\dot\gamma(t)|_d \; dt.$$

(See [19, p.106-109] or [3, p.59ff].) So it is enough to check that $|\dot\gamma|_d = |\dot\gamma|_g$. This is accomplished as follows. Put $x = \gamma(t)$. For small h we can write $\gamma(t+h) = \exp_x(z(h))$ where $z = z(h)$ is some C^1 path in T_xX with $z(0) = 0$ and and $\dot z(0) = d(\exp_x)_0(\dot z(0)) = \dot\gamma(t)$. Since $d(\gamma(t+h),\gamma(t)) = |z(h)|_g$

$$|\dot\gamma(t)|_d = \lim_{h\to 0}\frac{|z(h)|_g}{|h|} = |\dot z(0)|_g = |\dot\gamma(t)|_g.$$

This proves that $L_d = L$ for paths that do not meet $X_{\rm sing}$. (There is a proof for C^1 Finsler manifolds due to Busemann and Mayer. It can be found in [8] or at pp. 134-140 of Rinow's book [19].) For a general path one can reason as in Lemma 1: let $E = \gamma^{-1}(X_{\rm reg})$, $F = I \setminus E$, $B = F^0$, $D = \partial F$. Then $I = E \sqcup B \sqcup D$. Since γ is constant on the connected components of F, if $[a, b]$ is one such component then $L_d(\gamma_{[a,b]} = L(\gamma_{[a,b]}) = 0$. The result follows from additivity of both functionals. □

Corollary 1. *The functional L is lower semicontinuous on the set of piecewise C^1 paths with respect to the topology of pointwise convergence.*

Proof. It easily follows from the definition that L_d is lower semicontinuous on $C^0([0, 1], X)$ with respect to pointwise convergence [7, Prop. 2.3.4(iv)]. □

The construction of the intrinsic distance is *local* in the following sense.

Lemma 5. *For any point $x_0 \in X$ and any neighbourhood U of x_0 in X there is a smaller neighbourhood $U' \subset U$ such that for any $x, y \in U'$ there is a segment γ from x to y and any such segment is contained in U'. In particular the intrinsic distance of (X, ω) and that of $(U, \omega\big|_U)$ coincide on U'.*

Proof. Let $\varepsilon > 0$ be such that $\overline{\mathfrak{B}(x_0, 4\varepsilon)}$ is a compact subset of U. Put $U' = \underline{\mathfrak{B}(x, \varepsilon)}$. If $x, y \in U'$ then $d(x, y) \le d(x, x_0) + d(x_0, y) < 2\varepsilon$. By Lemma 2 since $\overline{\mathfrak{B}(x, 2\varepsilon)}$ is compact there is a segment from x to y. Now if $\gamma = \gamma(t)$ is *any* such segment $d(\gamma(t), x_0) \le d(\gamma(t), x) + d(x, x_0) \le L(\gamma) + d(x, x_0) \le 3\varepsilon$. So $\gamma(t)$ lies in U. □

Corollary 2. *Let (X, ω) be a Kähler curve and let d be the intrinsic distance. If (U, τ, A, Ω) is a chart around $x \in X$ and ω' is a local extension of ω there is a neighbourhood $U' \subset U$ of x such that $\tau\big|_{U'}$ is a biholomorphic isometry between (U', d) and $\tau(U') \subset A$ provided with the intrinsic distance obtained from ω'.*

It follows that to study *local* properties of the metric spaces (X, d) it is enough to consider the special case in which X is an analytic set in a domain of $\mathbb{C}^n$ with the metric induced from some Kähler metric of the domain. This situation, under the additional hypothesis that the singularity be analytically irreducible, is the object of §§3–6, throughout which we will make the following assumptions and use the following notation.

$\langle\ ,\ \rangle$ is the standard Hermitian product on $\mathbb{C}^n$,
$v \cdot w = \mathrm{Re}\langle v, w\rangle$ is the corresponding scalar product,
$|\cdot|$ is the corresponding norm.
Given two nonzero vectors v, w in a Euclidean space

$$\sphericalangle(v, w) = \arccos \frac{v \cdot w}{|v| \cdot |w|}$$

is the unoriented angle between them.
Ω' is an open polydisc centred at $0 \in \mathbb{C}^n$,

$A \subset \Omega'$ is an analytic curve,
ω is a smooth Kähler form on Ω',
g is the corresponding Kähler metric,
g_x is the value of g at $x \in \Omega'$,
$|\cdot|_g$ or $|\cdot|_x$ denotes the corresponding norm,
$g_0 = \langle\, , \,\rangle$,
Ω is an open subset of Ω' with $\overline{\Omega} \Subset \Omega'$,
$X := A \cap \Omega$,
d is the intrinsic distance of $(X, \omega|_X)$,
$\mathfrak{B}(x,r)$ is the ball in (X,d),
$\mathfrak{B}^*(x,r) = \mathfrak{B}(x,r) \setminus \{0\}$,
$B_x(0,r) = \{w \in T_xX : |w|_x < r\}$.
$X_{\text{sing}} = \{0\}$,
X is analytically irreducible at 0,
$m = \operatorname{mult}_0 X$ is the multiplicity of X at 0,
$K(x)$ is the Gaussian curvature of (X_{reg}, g) at $x \in X_{\text{reg}}$ and

$$\kappa = \sup_{x \in X_{\text{reg}}} K(x). \tag{7}$$

$\Delta = \{z \in \mathbb{C} : |z| < 1\}$,
$\Delta^* = \Delta \setminus \{0\}$,
$\Delta' \subset \mathbb{C}$ is an open subset containing $\overline{\Delta}$,
$\varphi : \Delta' \to X'$ is the normalisation map,
$\varphi(\Delta) = X$.
There is a holomorphic map $\psi = (\psi_1, \ldots, \psi_n) : \Delta' \to \mathbb{C}^n$ such that

$$\varphi(z) = z^m \psi(z) \qquad \psi_1(z) \equiv 1 \qquad \psi_j(0) = 0 \ j > 1. \tag{8}$$

$R : \Delta' \to \mathbb{C}^n$ is the holomorphic map defined by

$$R(z) := \frac{\psi(z) - \psi(0)}{mz} + \frac{\psi'(z)}{m} \tag{9}$$

$$\varphi'(z) = mz^{m-1}(e_1 + zR(z)) \tag{10}$$

$e_1 = (1, 0, \ldots, 0)$.
$c_0 > 0$ is a constant such that

$$\sup_\Delta |\varphi'| \leq c_0 \qquad \sup_\Delta |R| \leq c_0 \tag{11}$$

$$\forall x \in \Omega, \forall v \in \mathbb{C}^n, \quad \begin{cases} \frac{1}{c_0}|v| \leq |v|_x \leq c_0|v| \\ |v|_x \leq |v|(1 + c_0|x|). \end{cases} \tag{12}$$

From (11) it follows that for any $z \in \Delta$

$$|\varphi(z)| \leq c_0|z|. \tag{13}$$

$\pi : \mathbb{C}^n \to \mathbb{C} \times \{0\}$ is the projection on the first coordinate,
$u := \pi \circ \varphi : \Delta \to \Delta$ is the standard $m : 1$ ramified covering: $u(z) = z^m$.
For $\theta_0 \in \mathbb{R}$ and $\alpha \in (0, \pi]$ put

$$S(\theta_0, \alpha) = \{\rho e^{i\theta} : \rho \in (0,1), |\theta - \theta_0| < \alpha\} \subset \Delta. \tag{14}$$

Then

$$u^{-1}\big(S(\theta_0, \alpha)\big) = \bigsqcup_{j=0}^{m-1} S\left(\frac{\theta_0 + 2\pi j}{m}, \frac{\alpha}{m}\right) \tag{15}$$

and

$$u_j := u\Big|_{S\left(\frac{\theta_0+2\pi j}{m}, \frac{\alpha}{m}\right)} \tag{16}$$

is a biholomorphism onto $S(\theta_0, \alpha)$. The Whitney tangent cone of X at 0 is

$$C_0 X = \mathbb{C} \times \{0\} \subset \mathbb{C}^n \tag{17}$$

(see e.g. [9, p.122, p.80]).
If $\gamma : [0, L] \to X$ is a path, $\gamma^0(t) = \gamma(L - t)$.

3 Regularity of geodesics

Lemma 6 ([16, Lemma 1, p.86])**.** *Let m be a positive integer and $K > 0$. Put $Z = \{(a_1, \ldots, a_m, x) \in \mathbb{C}^{m+1} : \ x^m + \sum_{j=1}^m a_j x^{m-j} = 0, \ |a_j| \le K\}$. Then there is an $M = M(m, K) > 0$ with the following property. Let $\alpha(t) = (a(t), x(t))$ be a continuous path $\alpha : [0,1] \to Z$ and $L > 0$ such that $|a(t) - a(t')| \le L|t - t'|$ for $t, t' \in [0, 1]$. Then*

$$|x(t) - x(t')| \le M L^{1/m} |t - t'|^{1/m} \qquad \forall t, t' \in [0,1]. \tag{18}$$

Proposition 4. *There is a constant $c_1 > 1$ such that for any $z, z' \in \Delta$*

$$\frac{1}{c_1} d(\varphi(z), \varphi(z')) \le |z - z'| \le c_1 d(\varphi(z), \varphi(z'))^{1/m}. \tag{19}$$

Proof. Recall that Ω' is a polydisc, say $\Omega' = P(0)_{K,\ldots,K}$ and X is compactly contained in Ω'. Let $z, z' \in \Delta$ and $x = \varphi(z), x' = \varphi(z')$. For $\varepsilon > 0$ let $\gamma : [0,1] \to X$ be a piecewise C^1 path with $L := L(\gamma) < d(x, x') + \varepsilon$. We can assume that γ has constant speed equal to L, so $d(\gamma(t), \gamma(t')) \le L|t - t'|$. On the other hand we trivially have $|\gamma(t) - \gamma(t')| \le d(\gamma(t), \gamma(t'))$. Put $a_m(t) = -\pi\big(\gamma(t)\big)$, $x(t) = \varphi^{-1}(\gamma(t))$ and $\alpha(t) = (0, \ldots, 0, a_m(t), x(t))$. Then

$$a_m(t) = -\pi \circ \varphi(x(t)) = -u(x(t)) = -x^m(t) \qquad x^m(t) + a_m(t) = 0$$
$$|a_m(t) - a_m(t')| = |\pi(\gamma(t)) - \pi(\gamma(t'))| \le$$
$$\le |\gamma(t) - \gamma(t')| \le d(\gamma(t), \gamma(t')) \le L|t - t'|.$$

Therefore by Lemma 6 applied to α

$$|x(t) - x(t')| \leq ML^{1/m}|t - t'|^{1/m}.$$

For $t = 0$ and $t = 1$ we get

$$|z - z'| \leq ML^{1/m} \leq M\big(d(x, x') + \varepsilon\big)^{1/m}.$$

Letting $\varepsilon \to 0$ we get $|z-z'| \leq Md(\varphi(z), \varphi(z'))^{1/m}$. On the other hand it follows from (11) that $d(\varphi(z), \varphi(z')) \leq c_0|z - z'|$, so $c_1 = \max\{c_0, M\}$ works. □

Corollary 3. *For any r with $0 < r < c_1^{-m}$*

$$\mathfrak{B}(0, r) \subset \varphi(B(0, c_1 r^{1/m})) \subset \mathfrak{B}(0, c_1^2 r^{1/m}). \tag{20}$$

For $x \in X_{\mathrm{reg}}$ let $(T_x X)^\perp$ denote the orthogonal complement of $T_x X \subset \mathbb{C}^n$ with respect to the scalar product g_x. If $w \in \mathbb{C}^n$, $w^\perp$ denotes the g_x–orthogonal projection of w on $(T_x X)^\perp$. Let $B_x : T_x X \times T_x X \to (T_x X)^\perp$ be the second fundamental form of X_{reg}. Since g is Kähler and X_{reg} is a complex submanifold B_x is complex linear. If v is a nonzero vector in $T_x X$ put

$$|B_x| = \frac{|B_x(v, v)|_x}{|v|_x^2}. \tag{21}$$

Since $T_x X$ is complex one-dimensional the choice of v is immaterial. Denoting by $K_\Omega(T_x X)$ the sectional curvature of (Ω, g) on the 2-plane $T_x X$, Gauss equation yields

$$K(x) = K_\Omega(T_x X) - 2|B_x|^2. \tag{22}$$

(See e.g. [15] p. 175-176.)

Proposition 5. *There is a constant c_2 such that*

$$|B_{\varphi(z)}| \leq \frac{c_2}{|z|^{m-1}} \qquad \forall z \in \Delta \tag{23}$$

$$|B_x| \leq \frac{c_2}{d(x, 0)^{1-1/m}} \qquad \forall x \in X_{\mathrm{reg}}. \tag{24}$$

Proof. By (8) we have $\varphi(z) = z^m\psi(z)$, so $\varphi'(z) = z^{m-1}v(z)$ where $v(z) = m\psi(z) + z\psi'(z)$. Since φ is a holomorphic immersion on $\Delta' \setminus \{0\}$, $v(z) \neq 0$ and $T_{\varphi(z)}X = \mathbb{C}\varphi'(z) = \mathbb{C}v(z)$ for any $z \neq 0$. But also $v(0) = me_1 \neq 0$. So $v : \Delta' \to \mathbb{C}^m$ is continuous and nonvanishing, hence $\inf_\Delta |v|_g > 0$ and $\sup_\Delta |v|_g < +\infty$. Similarly $\sup_\Delta |v'|_g < +\infty$. Let C be such that

$$\inf_\Delta |v|_g \geq \frac{1}{C} \qquad \sup_\Delta |v|_g \leq C \qquad \sup_\Delta |v'|_g \leq C.$$

Then we have

$$B_{\varphi(z)}(v(z), v(z)) = \frac{1}{z^{m-1}} B_{\varphi(z)}(\varphi'(z), v(z)) = \frac{1}{z^{m-1}}\big(v'(z)\big)^\perp$$

$$\big|B_{\varphi(z)}\big| = \frac{|(v'(z))^\perp|_{\varphi(z)}}{|z|^{m-1}|v|^2_{\varphi(z)}} \leq \frac{|v'(z))|_{\varphi(z)}}{|z|^{m-1}|v|^2_{\varphi(z)}} \leq \frac{C^3}{|z|^{m-1}}. \tag{25}$$

This proves (23). For $x \in X_{\mathrm{reg}}$ let $z = \varphi^{-1}(x)$ and consider the path $\gamma(t) = \varphi(tz)$, $t \in [0,1]$. Then

$$\begin{gathered}
\dot\gamma(t) = \varphi'(tz)z = (tz)^{m-1}v(tz)z = z^m t^{m-1} v(tz)\\
|\dot\gamma(t)|_x = |z|^m t^{m-1}|v(tz)|_x \le C|z|^m\\
d(x,0) \le L(\gamma) = \int_0^1 |\dot\gamma(t)|\,dt \le C|z|^m\\
|z| \ge \sqrt[m]{\frac{d(x,0)}{C}}.
\end{gathered}$$

So (24) follows from (23). $\square$

Remark 1. *The map v above is a holomorphic vector field along the map $\varphi : \Delta \to X$. On the other hand the push forward of v to X, that is $v \circ \varphi^{-1}$, is only weakly holomorphic. In fact any holomorphic vector field on X has to vanish at 0 if X is singular [20, Thm. 3.2].*

Lemma 7. *If $a, b \ge 0$ and $s \in (0,1)$ then $|a^s - b^s| \le |a-b|^s$.*

Proof. Assume $a \ge b$. The function $\eta(x) = (b+x)^s - x^s$ belongs to $C^0([0,+\infty)) \cap C^1((0,+\infty))$. Since $s < 1$, $\eta'(x) = s[(b+x)^{s-1} - x^{s-1}] \le 0$. So $\eta(a-b) = a^s - (a-b)^s \le b^s$. $\square$

Theorem 2. *There is a constant c_3 such that for any unit speed geodesic $\gamma : [0,L] \to X$ with $\gamma((0,L]) \subset X_{\mathrm{reg}}$ we have*

$$||\dot\gamma||_{C^{0,1/m}} \le c_3 \tag{26}$$

Here the Hölder norm is computed using the Euclidean distance on $\mathbb{C}^n$.

Proof. By 4 of Lemma 4 $\gamma\big|_{(0,L]}$ is a Riemannian geodesic of X_{reg}. Hence the acceleration $\ddot\gamma(t)$ is orthogonal to $T_{\gamma(t)}X$ with respect to the scalar product g. So for $t > 0$, $\ddot\gamma(t) = \big(\ddot\gamma(t)\big)^\perp = B_{\gamma(t)}\big(\dot\gamma(t), \dot\gamma(t)\big)$. Using (24), $|\dot\gamma| \equiv 1$ and (12) we get

$$|\ddot\gamma(t)| \le c_0|\ddot\gamma(t)|_x = c_0\big|B_{\gamma(t)}\big(\dot\gamma(t),\dot\gamma(t)\big)\big|_x = c_0\big|B_{\gamma(t)}\big| \le \frac{C}{d(\gamma(t),0)^{1-1/m}}$$

where $C = c_0 c_2$. Set $a := d(\gamma(0),0)$ and $\beta = 1 - 1/m$. For $t > 0$ we have

$$d(\gamma(t),0) \ge \big|d(\gamma(t),\gamma(0)) - d(\gamma(0),0)\big| = |t-a|$$

$$|\ddot\gamma(t)| \le \frac{C}{|t-a|^\beta}. \tag{27}$$

We claim that

$$|\dot\gamma(t) - \dot\gamma(s)| \le 2mC|t-s|^{1/m} \tag{28}$$

for any pair of numbers s, t such that $0 < s \leq t \leq 1$. Indeed

$$|\dot\gamma(t) - \dot\gamma(s)| \leq \int_s^t |\ddot\gamma(\tau)| d\tau \leq C \int_s^t \frac{d\tau}{|\tau - a|^\beta} = C\big(I_a(t) - I_a(s)\big)$$

where we put $I_a(t) = \int_0^t |\tau - a|^{-\beta} d\tau$. A simple computation shows that

$$I_a(t) - I_a(s) = \begin{cases} m\big(|s-a|^{1/m} - |t-a|^{1/m}\big) & \text{for } 0 < s \leq t \leq a \\ m\big(|s-a|^{1/m} + |t-a|^{1/m}\big) & \text{for } 0 < s \leq a \leq t \\ m\big(|t-a|^{1/m} - |s-a|^{1/m}\big) & \text{for } a < s \leq t \end{cases}$$

By Lemma 7

$$\Big| |t-a|^{1/m} - |s-a|^{1/m} \Big| \leq \Big| |t-a| - |s-a| \Big|^{1/m} \leq |t-s|^{1/m}$$

so $I_a(t) - I_a(s) \leq m|t-s|^{1/m}$ in the first and the last case. As for the middle case, namely $s \leq a \leq t$, we have $|s-a| \leq |s-t|$ and $|t-a| \leq |t-s|$, so $|s-a|^{1/m} + |t-a|^{1/m} \leq 2|t-s|^{1/m}$. Therefore in any case $I_a(t) - I_a(s) \leq 2m|t-s|^{1/m}$ and this finally yields (28). This proves (26) with $c_3 = 2mC = 2mc_0c_2$. □

Corollary 4. *Let $\gamma : [0, L] \to X$ be a segment with $\gamma(0) = 0$. Then γ is differentiable at $t = 0$ and the map $\dot\gamma : [0, L] \to \mathbb{C}^n$ is a Hölder continuous of exponent $1/m$.*

Proof. Since shortest paths are injective $\gamma((0, L]) \subset X_{\text{reg}}$. So estimate (26) holds. Therefore $\dot\gamma$ is uniformly continuous on $(0, L]$ and extends continuously for $t = 0$. By the mean value theorem the extension for $t = 0$ is precisely the derivative $\dot\gamma(0)$. □

Lemma 8. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x \in \mathfrak{B}^*(0, \delta)$ and any $v \in T_xX$*

$$(1 - \varepsilon)|v| < |v|_x < (1 + \varepsilon)|v| \tag{29}$$

$$|\pi(v) - v| < \varepsilon |v| \tag{30}$$

$$(1 - \varepsilon)|v| < |\pi(v)| < (1 + \varepsilon)|v|. \tag{31}$$

Proof. (29) holds for x sufficiently close to 0 simply because $g_0 = \langle\ ,\ \rangle$. For the second condition set

$$\delta = \varepsilon^m \Big[(c_1(1 + c_0)(1 + \varepsilon)\Big]^{-m}$$

where c_0 is the constant defined in (11). By Lemma 4 if $x \in \mathfrak{B}(0, \delta)$ and $z = \varphi^{-1}(x) \in \Delta$

$$|z| < c_1 \delta^{1/m} = \frac{\varepsilon}{(1 + c_0)(1 + \varepsilon)}.$$

Therefore

$$|zR(z)| < \frac{\varepsilon}{1 + \varepsilon}.$$

(R is defined in (9).) It follows from (10) that $T_xX = \mathbb{C}\cdot\varphi'(z) = \mathbb{C}\cdot(e_1 + zR(z))$, so any $v \in T_xX$ is of the form $v = \lambda(e_1 + zR(z))$ for some $\lambda \in \mathbb{C}$. Then

$$|v| \geq |\lambda| - |\lambda z R(z)| \geq |\lambda| - \frac{|\varepsilon\lambda|}{1+\varepsilon} = \frac{1}{1+\varepsilon}|\lambda| \qquad |\lambda| \leq (1+\varepsilon)|v|$$

$$|v - \pi(v)| = \min_{w \in \mathbb{C}\times\{0\}} |v - w| \leq |v - \lambda e_1| = |\lambda z R(z)| < |\lambda| \frac{\varepsilon}{1+\varepsilon} \leq \varepsilon |v|$$

$$\Big||\pi(v)| - |v|\Big| \leq |\pi(v) - v| < \varepsilon |v|.$$

□

Lemma 9. *We have*

$$\liminf_{\substack{x,\, y \,\to\, 0 \\ x,\, y \,\in\, X}} \frac{d(x,y)}{|x-y|} \geq 1. \tag{32}$$

Proof. Given $\varepsilon > 0$ let $\delta > 0$ be such that (29) holds for any $x \in \mathfrak{B}^*(0,\delta)$ and any $v \in T_xX$. If $x, y \in \mathfrak{B}^*(0,\delta/3)$ and $\alpha : [0,L] \to X$ is a segment, then $\alpha([0,L]) \subset \mathfrak{B}(0,\delta)$ and the set $J = \{t \in [0,L] : \alpha(t) = 0\}$ contains at most one point. For $t \notin J$ $|\dot\alpha(t)|_{\alpha(t)} \geq (1-\varepsilon)|\dot\alpha(t)|$. Integrating on $[0,L] \setminus J$ yields

$$d(x,y) = L(\alpha) \geq (1-\varepsilon)\int_0^L |\dot\alpha(t)|dt \geq (1-\varepsilon)|x-y|$$

$$\frac{d(x,y)}{|x-y|} \geq 1 - \varepsilon.$$

□

Lemma 10. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x \in \mathfrak{B}^*(0,\delta)$ and any pair of nonzero vectors $v, w \in T_xX$*

$$|\sphericalangle(\pi(v), \pi(w)) - \sphericalangle(v,w)| < \varepsilon.$$

(The angle is computed with respect to $\langle\ ,\ \rangle$.)

Proof. The angle function $\sphericalangle : S^{2m-1} \times S^{2m-1} \to \mathbb{R}$ is the Riemannian distance for the standard metric on unit the sphere. In particular it is Lipschitz continuous with respect to the Euclidean distance. So one can find $\varepsilon_1 > 0$ with the property that that

$$|u_1 - u_2| < \varepsilon_1, |w_1 - w_2| < \varepsilon_1 \Rightarrow |\sphericalangle(u_1, w_1) - \sphericalangle(u_2, w_2)| < \varepsilon. \tag{33}$$

We can assume $\varepsilon_1 < 1$. Choose $\delta > 0$ such that for $x \in \mathfrak{B}^*(0,\delta)$ and $v \in T_xX$

$$|\pi(v) - v| < \frac{\varepsilon_1}{2}|v|.$$

This is possible by Lemma 8. Moreover $v \neq 0 \Rightarrow \pi(v) \neq 0$, because $\varepsilon_1 < 1$. Then for two nonzero vectors $v, w \in T_x X$, $x \in \mathfrak{B}^*(0,\delta)$

$$\left|\frac{v}{|v|} - \frac{\pi(v)}{|\pi(v)|}\right| \leq \frac{2|v-\pi(v)|}{|v|} < \varepsilon_1 \qquad \left|\frac{w}{|w|} - \frac{\pi(w)}{|\pi(w)|}\right| < \varepsilon_1.$$

Together with (33) this yields the result. □

Lemma 11. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any segment $\gamma : [0,L] \to \mathfrak{B}(0,\delta)$ with $\gamma((0,L)) \subset X_{\mathrm{reg}}$ and any $s, s' \in [0,L]$*

$$|\dot\gamma(s) - \dot\gamma(s')| < \varepsilon \qquad \sphericalangle(\dot\gamma(s), \dot\gamma(s')) < \varepsilon. \tag{34}$$

(The angle is computed with respect to $\langle\ ,\ \rangle$.)

Proof. Let $\varepsilon_1 > 0$ be such that

$$|u-w| < \varepsilon_1 \Rightarrow \sphericalangle(u,w) < \varepsilon \qquad \forall u, w \in S^{2m-1}. \tag{35}$$

Choose $\delta > 0$ such that

$$\sqrt[m]{2\delta} < \min\left\{\frac{\varepsilon_1}{2c_0c_3}, \frac{\varepsilon}{c_3}\right\}.$$

If $\gamma : [0,L] \to \mathfrak{B}(0,\delta)$ is a segment with $\gamma((0,L)) \subset X_{\mathrm{reg}}$ at most one of the points $\gamma(0)$ and $\gamma(L)$ coincides with the origin. So the Hölder estimate (26) holds for γ. Then

$$|\dot\gamma(s) - \dot\gamma(s')| \leq c_3 \sqrt[m]{L} \leq c_3 \sqrt[m]{2\delta} < \varepsilon.$$

This proves the first inequality. From (12) it follows that

$$\frac{1}{|\dot\gamma(s)|} \leq c_0$$

so

$$\left|\frac{\dot\gamma(s)}{|\dot\gamma(s)|} - \frac{\dot\gamma(s')}{|\dot\gamma(s')|}\right| \leq \frac{2|\dot\gamma(s) - \dot\gamma(s')|}{|\dot\gamma(s)|} \leq 2c_0c_3 \sqrt[m]{2\delta} < \varepsilon_1.$$

Coupled with (35) this yields the second inequality. □

Lemma 12. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any segment $\gamma : [0,L] \to \mathfrak{B}(0,\delta)$ with $\gamma((0,L)) \subset X_{\mathrm{reg}}$ and any $s, s' \in [0,L]$*

$$\sphericalangle(\pi(\dot\gamma(s)), \pi(\dot\gamma(s'))) < \varepsilon.$$

(The angle is computed with respect to $\langle\ ,\ \rangle$.)

Proof. Let $\varepsilon_1 > 0$ be such that

$$|u - w| < \varepsilon_1 \Rightarrow \sphericalangle(u, w) < \frac{\varepsilon}{3} \qquad \forall u, w \in S^{2m-1}. \tag{36}$$

Let $\delta_1 > 0$ be such that for any $x \in \mathfrak{B}^*(0, \delta_1)$ and any $v \in T_x X$

$$|\pi(v) - v| < \frac{\varepsilon_1 |v|}{c_0}. \tag{37}$$

Such a δ_1 exists by Lemma 8. Next, by Lemma 11, there is $\delta_2 > 0$ such that for any segment $\gamma : [0, L] \to \mathfrak{B}(0, \delta_2)$ with $\gamma((0, L)) \subset X_{\mathrm{reg}}$ and any $s, s' \in [0, L]$

$$\sphericalangle(\dot\gamma(s), \dot\gamma(s')) < \frac{\varepsilon}{3}. \tag{38}$$

Set $\delta = \min\{\delta_1, \delta_2\}$. If $\gamma : [0, L] \to \mathfrak{B}(0, \delta)$ is a segment with $\gamma((0, L)) \subset X_{\mathrm{reg}}$ and $s \in [0, L]$, then by (37) and (12)

$$\left|\pi(\dot\gamma(s)) - \dot\gamma(s)\right| < \frac{\varepsilon_1 |\dot\gamma(s)|}{c_0} \le \varepsilon_1 |\dot\gamma(s)|_{\gamma(s)} = \varepsilon_1$$

so by (36)

$$\sphericalangle\big(\pi(\dot\gamma(s)), \dot\gamma(s)\big) < \frac{\varepsilon}{3}.$$

Then using (38) we get for arbitrary $s, s' \in [0, L]$

$$\sphericalangle(\pi(\dot\gamma(s)), \pi(\dot\gamma(s'))) \le$$
$$\le \sphericalangle\big(\pi(\dot\gamma(s)), \dot\gamma(s)\big) + \sphericalangle(\dot\gamma(s), \dot\gamma(s')) + \sphericalangle\big(\dot\gamma(s'), \pi(\dot\gamma(s'))\big) < \varepsilon$$

as claimed. ☐

4 Tangent vectors in the normalisation

In this section we study the regularity properties of the preimage in Δ of segments in X.

Lemma 13. *If $\gamma : [0, L] \to X$ is a segment with $\gamma(0) = 0$, the path $\varphi^{-1} \circ \gamma : [0, L] \to \Delta$ has finite length.*

Proof. We know from Cor. 4 that $\dot\gamma(0)$ exists. By (17) $\dot\gamma(0) = (\dot\gamma_1(0), 0, \dots, 0)$ and $\dot\gamma_1(0) = e^{i\theta_0}$ for some $\theta_0 \in [0, 2\pi)$. There is $\varepsilon > 0$ such that $\gamma_1((0, \varepsilon])$ is contained in the sector $S(\theta_0, \pi) \subset \Delta$ defined in (14). We can write $\gamma_1(t) = \rho(t) e^{i\theta(t)}$ for appropriate functions $\rho, \theta \in C^1((0, \varepsilon])$. Put $\beta = \varphi^{-1} \circ \gamma$. $\beta((0, t])$ is contained in one of the connected components of $\varphi^{-1} S(\theta_0, \pi)$ hence by (15) there is an integer k, $0 \le k \le m - 1$, such that

$$\beta(t) = u_k^{-1}(\gamma_1(t)) = \rho^{1/m}(t) e^{i\theta(t)/m} \xi_k \tag{39}$$

where $\xi_k = e^{2\pi k/m}$. Then

$$\dot\beta = \frac{1}{m}\rho^{1/m-1}(\rho' + i\theta'\rho)e^{i\theta/m}\xi_k$$
$$|\dot\beta| = \frac{1}{m}\rho^{1/m-1}\sqrt{(\rho')^2 + i(\theta'\rho)^2} \tag{40}$$
$$\lim_{t\to 0}\rho(t) = \lim_{t\to 0}|\gamma_1(t)| = 0$$
$$\lim_{t\to 0}\frac{\rho(t)}{t} = \lim_{t\to 0}\left|\frac{\gamma_1(t)}{t}\right| = |\dot\gamma_1(0)| = 1. \tag{41}$$

If we put $\rho(0) = 0$ and $\rho'(0) = 1$, then $\rho \in C^1([0,\varepsilon])$. Also

$$e^{i\theta_0} = \dot\gamma_1(0) = \lim_{t\to 0}\frac{\gamma_1(t)}{t} = \lim_{t\to 0}\frac{\gamma_1(t)}{\rho(t)} = \lim_{t\to 0} e^{i\theta(t)}$$
$$\lim_{t\to 0}\theta(t) = \theta_0 + 2N\pi \qquad N \in \mathbb{Z}.$$

Change θ by subtracting $2N\pi$ to it and put $\theta(0) = \theta_0$. Then $\theta \in C^0([0,\varepsilon])$.

$$\dot\gamma_1 = \big(\rho' + i\rho\theta'\big)e^{i\theta} \qquad \lim_{t\to 0}\dot\gamma_1(t) = \dot\gamma_1(0) = e^{i\theta(0)}$$
$$\Longrightarrow \quad \lim_{t\to 0}\rho(t)\theta'(t) = 0.$$

Since $\rho'(0) = 1$ and $\rho\theta' \to 0$, we get from (40) that $|\dot\beta| \le C_1\rho^{1/m-1} \le C_2 t^{1/m-1}$. Therefore $L = \int_0^\varepsilon |\dot\beta| < +\infty$ and β has finite length. $\square$

Definition 4. *If $\gamma : [0,L] \to X$ is a unit speed geodesic with $\gamma(0) = 0$ denote by $\underline{\gamma} : [0,\underline{L}] \to \Delta$ the arc-length reparametrisation of the path $\varphi^{-1}\circ\gamma : [0,L] \to \Delta$ (with respect to the Euclidean metric on Δ).*

Proposition 6. *If $\gamma : [0,L] \to X$ is a unit speed geodesic with $\gamma(0) = 0$, then $\underline{\gamma} \in C^1([0,\underline{L}])$ and $\dot\gamma(0) = (\dot{\underline{\gamma}}(0)^m, \dots, 0)$.*

Proof. Put $\beta = \varphi^{-1}\circ\gamma$. By the previous Lemma β has finite length. If we set $h(t) = \int_0^t |\dot\beta(\tau)|d\tau$ then $\underline{\gamma}(s) = \beta(h^{-1}(s))$. It is clear that $\underline{\gamma} \in C^0([0,\underline{L}]) \cap C^1((0,\underline{L}])$, but we have to check that $\underline{\gamma}$ is continuously differentiable at $s = 0$. This is not immediate since $h'(0) = 0$ and h is not a C^1-diffeomorphism at $s = 0$. So we compute the limit:

$$\lim_{s\to 0}\frac{\underline{\gamma}(s)}{s} = \lim_{t\to 0}\frac{\underline{\gamma}(h(t))}{h(t)} = \lim_{t\to 0}\frac{\beta(t)}{h(t)}. \tag{42}$$

Since $\beta(0) = 0$ and $h(0) = 0$ we may apply de L'Hôpital rule:

$$\lim_{s\to 0}\frac{\underline{\gamma}(s)}{s} = \lim_{t\to 0}\frac{\dot\beta(t)}{|\dot\beta(t)|} = \lim_{t\to 0}\frac{\rho' + i\theta'\rho}{\sqrt{(\rho')^2 + i(\theta'\rho)^2}}e^{i\theta/m}\xi_k = e^{i\theta_0/m}\xi_k. \tag{43}$$

(Recall that $\rho'(0) = 1$ and $\theta'\rho \to 0$.) This shows that $\underline{\gamma}$ is C^1 up to $s = 0$. The last assertion is immediate from (43). $\square$

Lemma 14. *Let $\alpha : [0,\varepsilon] \to X$ and $\beta : [0,\varepsilon] \to X$ be segments with $\alpha(0) = \beta(0) = 0$. If $\sphericalangle\big(\underline{\dot\alpha}(0), \underline{\dot\beta}(0)\big) < \pi/m$ then*

$$\lim_{t\to 0} \frac{d(\alpha(t),\beta(t))}{2t} = \sin\frac{\sphericalangle(\dot\alpha(0),\dot\beta(0))}{2} < 1. \tag{44}$$

In particular $\alpha*\beta^0$ is not minimising on any subinterval of $[-\varepsilon,\varepsilon]$ which contains 0 as an interior point.

Proof. By interchanging α and β if necessary, we can assume that $\underline{\dot\alpha}(0) = e^{i\eta_1}$ and $\underline{\dot\beta}(0) = e^{i\eta_2}$ with $0 \le \eta_i < 2\pi$ and $0 \le \eta_2 - \eta_1 < \pi/m$. According to the previous Proposition $\dot\alpha(0) = (\underline{\dot\alpha}^m(0), 0, \ldots, 0)$ and $\dot\beta(0) = (\underline{\dot\beta}^m(0), 0, \ldots, 0)$. Hence we can choose $\theta_i \in \mathbb{R}$ such that

$$\begin{aligned} \dot\alpha(0) &= (e^{i\theta_1}, 0, \ldots, 0) \qquad & 0 \le \theta_1 < 2\pi \\ \dot\beta(0) &= (e^{i\theta_2}, 0, \ldots, 0) \qquad & 0 \le \theta_2 - \theta_1 = \sphericalangle(\dot\alpha(0),\dot\beta(0)) < \pi. \end{aligned}$$

We start by showing that

$$\lim_{t\to 0} \frac{|\beta_1(t) - \alpha_1(t)|}{2t} = \sin\frac{\sphericalangle(\dot\alpha(0),\dot\beta(0))}{2}. \tag{45}$$

We can find continuous function $\rho_\alpha, \rho_\beta, \theta_\alpha, \theta_\beta$ such that

$$\begin{aligned} \alpha_1(t) &= \rho_\alpha(t) e^{i\theta_\alpha(t)} \qquad & \theta_\alpha(0) = \theta_1 \\ \beta_1(t) &= \rho_\beta(t) e^{i\theta_\beta(t)} \qquad & \theta_\beta(0) = \theta_2. \end{aligned}$$

And we know from (41) that $\rho_\alpha(0) = \rho_\beta(0) = 0$, $\rho'_\alpha(0) = \rho'_\beta(0) = 1$. Therefore

$$\begin{aligned} \lim_{t\to 0} \frac{|\beta_1(t) - \alpha_1(t)|}{2t} &= \lim_{t\to 0} \frac{1}{2t}\sqrt{\rho_\alpha^2 + \rho_\beta^2 - 2\rho_\alpha\rho_\beta\cos(\theta_\beta - \theta_\alpha)} = \\ &= \frac{1}{2}\lim_{t\to 0}\sqrt{\Big(\frac{\rho_\alpha}{t}\Big)^2 + \Big(\frac{\rho_\beta}{t}\Big)^2 - 2\frac{\rho_\alpha}{t}\frac{\rho_\beta}{t}\cos(\theta_\beta - \theta_\alpha)} = \\ &= \frac{1}{2}\sqrt{(\rho'_\alpha(0))^2 + (\rho'_\beta(0))^2 - 2\rho'_\alpha(0)\rho'_\beta(0)\cos(\theta_2 - \theta_1)} = \\ &= \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)} = \sqrt{\frac{1 - \cos(\theta_2 - \theta_1)}{2}} = \sin\frac{\theta_2 - \theta_1}{2}. \end{aligned}$$

Thus (45) is proved.

Next set $\theta_0 = (\theta_1 + \theta_2)/2$. Since $\theta_2 - \theta_1 < \pi$ both $\dot\alpha_1(0)$ and $\dot\beta_1(0)$ lie in the sector $S(\theta_0, \pi/2)$. By continuity there is $\delta_1 > 0$ such that

$$\alpha_1((0,\delta_1]) \cup \beta_1((0,\delta_1]) \subset S(\theta_0, \pi/2).$$

For $j = 0, 1, \ldots, m-1$ set

$$S_j := S\Big(\frac{\theta_0 + 2\pi j}{m}, \frac{\pi}{2m}\Big).$$

Then $u^{-1}\big(S(\theta_0,\pi/2)\big) = \sqcup_{j=0}^{m-1} S_j$. Note that $\alpha_1 = \pi\alpha = \pi\varphi\varphi^{-1}\alpha = u\varphi^{-1}\alpha$ and similarly $\beta_1 = u\varphi^{-1}\beta$. Then $\varphi^{-1}\alpha(t) \in u^{-1}S(\theta_0,\pi/2)$ for $t \in (0,\delta_1)$ and by connectedness the image of $\varphi^{-1}\circ\alpha$ must lie inside some component S_j. Similarly the image of $\varphi^{-1}\circ\beta$ is entirely contained in some component S_k. Since the sectors S_j and S_k are convex $\underline{\dot\alpha}(0) \in S_j$ and $\underline{\dot\beta}(0) \in S_k$ as well. But $\eta_2 - \eta_1 \in (0,\pi/m)$ so $\underline{\dot\alpha}(0)$ and $\underline{\dot\beta}(0)$ in the same component of $u^{-1}\big(S(\theta_0,\pi/2)\big)$. Hence $S_k = S_j$. The restriction

$$u_j := u\big|_{S_j} : S_j \to S(\theta_0,\pi/2).$$

is a biholomorphism and

$$\varphi^{-1}\alpha(t) = u_j^{-1}(\alpha_1(t)) \qquad \varphi^{-1}\beta(t) = u_j^{-1}(\beta_1(t)).$$

Fix $t \in (0,\delta_1)$. Since $S_0 \subset \mathbb{C}$ is a convex set the formula

$$\lambda_t(s) = u_j^{-1}\big((1-s)\alpha_1(t) + s\beta_1(t)\big)$$

defines a path $\lambda_t : [0,1] \to \Delta$ and $\mu_t := \varphi\circ\lambda_t : [0,1] \to X$ is a smooth path from $\alpha(t)$ to $\beta(t)$. Hence

$$d(\alpha(t),\beta(t)) \le L(\mu_t) = \int_0^1 \left|\frac{d\mu_t}{ds}(s)\right|_g ds.$$

Differentiating (in s) the identity $u\big(\lambda_t(s)\big) = (1-s)\alpha_1(t) + s\beta_1(t)$ we get

$$\frac{d}{ds}u\big(\lambda_t(s)\big) \equiv \beta_1(t) - \alpha_1(t).$$

On the other hand

$$\frac{d}{ds}u\big(\lambda_t(s)\big) = \frac{d}{ds}\big(\lambda_t(s)\big)^m = m\lambda_t^{m-1}(s)\frac{d\lambda_t}{ds}(s)$$
$$\frac{d\lambda_t}{ds}(s) = \frac{\beta_1(t)-\alpha_1(t)}{m\lambda_t^{m-1}(s)}.$$

Using first (9) and (10) and next (12) and (11) we have

$$\frac{d\mu_t}{ds}(s) = (\beta_1(t)-\alpha_1(t))\big(e_1 + \lambda_t(s)R(\lambda_t(s))\big)$$
$$\left|\frac{d\mu_t}{ds}(s)\right|_g \le \left|\frac{d\mu_t}{ds}(s)\right|(1 + c_0|\mu_t(s)|)$$
$$\le |\beta_1(t)-\alpha_1(t)|(1+c_0|\lambda_t(s)|)(1+c_0|\mu_t(s)|).$$

By (13) $|\mu_t(s)| \le c_0|\lambda_t(s)|$ so

$$\left|\frac{d\mu_t}{ds}(s)\right|_g \le |\beta_1(t)-\alpha_1(t)|\big(1 + (c_0+c_0^2)|\lambda_t(s)| + c_0^3|\lambda_t(s)|\big).$$

Moreover we have

$$\begin{aligned}|\lambda_t(s)|^m = |u(\lambda_t(s))| &= |(1-s)\alpha_1(t) + s\beta_1(s)| \leq \\ &\leq (1-s)|\alpha(t)| + s|\beta(t)| \leq \\ &\leq (1-s)d(0,\alpha(t)) + s\,d(0,\beta(t)) = t \\ |\lambda_t(s)| &\leq t^{1/m}.\end{aligned}$$

So there is a constant $C > 0$ such that

$$\begin{gathered}(c_0 + c_0^2)|\lambda_t(s)| + c_0^3|\lambda_t(s)| \leq Ct^{1/m} \\ \left|\frac{d\mu_t}{ds}(s)\right|_x \leq |\beta_1(t) - \alpha_1(t)|(1 + Ct^{1/m}) \\ d(\alpha(t),\beta(t)) \leq |\beta_1(t) - \alpha_1(t)|(1 + Ct^{1/m}).\end{gathered}$$

This yields the upper bound

$$\limsup_{t\to 0} \frac{d(\alpha(t),\beta(t))}{2t} \leq \lim_{t\to 0} \frac{|\beta_1(t) - \alpha_1(t)|(1 + Ct^{1/m})}{2t} = \lim_{t\to 0} \frac{|\beta_1(t) - \alpha_1(t)|}{2t}.$$

As for the lower bound, using (32) we have

$$\begin{aligned}\liminf_{t\to 0} \frac{d(\alpha(t),\beta(t))}{2t} &\geq \liminf_{t\to 0} \frac{|\alpha(t) - \beta(t)|}{2t} \cdot \liminf_{t\to 0} \frac{d(\alpha(t),\beta(t))}{|\alpha(t) - \beta(t)|} \geq \\ &\geq \liminf_{t\to 0} \frac{|\alpha(t) - \beta(t)|}{2t} \geq \lim_{t\to 0} \frac{|\alpha_1(t) - \beta_1(t)|}{2t}.\end{aligned}$$

Thus using (45) we finally compute the limit

$$\lim_{t\to 0} \frac{d(\alpha(t),\beta(t))}{2t} = \lim_{t\to 0} \frac{|\beta_1(t) - \alpha_1(t)|}{2t} = \sin \frac{\sphericalangle(\dot\alpha(0),\dot\beta(0))}{2}.$$

Since $\sphericalangle(\dot\alpha(0),\dot\beta(0)) < \pi$ this completes the proof. ☐

5 Uniqueness of geodesics

Lemma 15. *Let $\gamma_1 : [0, L_1] \to X$ be a segment between two points $x, y \in X_{\mathrm{reg}}$. If $\gamma_2 : [0, L_2] \to X$ is a unit speed geodesic distinct from γ_1 with $\gamma_2(0) = x$ and $\gamma_2(t_2) = y$ for some $t_2 \in (0, L_2)$, then γ_2 is not minimising beyond t_2, that is $d(\gamma_2(0), \gamma_2(t_t + \varepsilon) < t_2 + \varepsilon$ for any $\varepsilon > 0$.*

Proof. If γ_2 were minimising on $[0, t_2 + \varepsilon]$, then $t_2 = L_1$, the concatenation $\gamma_1 * \gamma_2\big|_{[t_2, t_2+\varepsilon]}$ would be a shortest path from x to $\gamma_2(t_2 + \varepsilon)$ and therefore would be smooth near t_2. This would force $\gamma_1 = \gamma_2$. ☐

In the following we will repeatedly make use of the following celebrated idea of Klingenberg (see [13, Lemma 1] or [14, Lemma 2.1.11(iii)]).

Lemma 16 (Klingenberg)**.** *Let x be a point in a Riemannian manifold (M, g) and let $v_1, v_2 \in U_x M$ be two distinct unit vectors such that γ^{v_1} and γ^{v_2} be defined and minimising on $[0, T]$. Assume that $\gamma_{v_1}(T) = \gamma_{v_2}(T)$, that $\dot\gamma_{v_1}(T) + \dot\gamma_{v_2}(T) \neq 0$ and that $\gamma_{v_i}(T)$ is not a conjugate point of x along γ_{v_i}. Then there are vectors $v_1', v_2' \in U_x X$ arbitrarily close to v_1 and v_2 respectively and such that the geodesics $\gamma_{v_i'}$ are minimising on $[0, T']$ for some $T' < T$ and $\gamma_{v_1'}(T') = \gamma_{v_2'}(T')$.*

Lemma 17. *Let $\gamma_1, \gamma_2 : [0, L] \to X$ be segments with the same endpoints. If $0 \notin \gamma_2([0, L))$ then $\gamma_1 * \gamma_2^0$ is a simple closed curve.*

Proof. Assume by contradiction that there are $t_1, t_2 \in (0, L)$ such that $\gamma_1(t_1) = \gamma_2(t_2)$. Since $x = \gamma_2(0)$ and $y = \gamma_2(t_2)$ are regular points Lemma 15 implies that γ_2 is not minimising on $[0, L]$, contrary to the hypotheses. $\square$

Since X is a topological disc, by the Jordan separation theorem the interior of a simple closed curve contained in X is well defined and is again a topological disc. Fix on X_{reg} the orientation given by the complex structure. If $\alpha : [0, L] \to X_{\text{reg}}$ is a piecewise smooth simple closed path in X we say that it is *positively oriented* if its interior lies on its left [10, p. 268]. If $x \in X_{\text{reg}}$ and $u, v \in T_x X$ are two linearly independent vectors we let $\sphericalangle(u, v)$ denote the *unoriented angle* as before, while $\sphericalangle_{\text{or}}(u, v)$ denotes the *oriented angle*, which is defined by $\sphericalangle_{\text{or}}(u, v) = \sphericalangle(u, v)$ if $\{u, v\}$ is a positive basis of $T_x X$ and by $\sphericalangle_{\text{or}}(u, v) = -\sphericalangle(u, v)$ otherwise. Equivalently, if $v = e^{i\theta} u$ with $\theta \in (-\pi, \pi)$ then $\sphericalangle_{\text{or}}(u, v) = \theta$. If $\alpha : [0, L] \to X_{\text{reg}}$ is a positively oriented piecewise smooth simple closed path and $t \in (0, L)$ is a vertex that is not a cusp, the *external angle* at $\alpha(t)$ is defined as $\theta_{\text{ext}}(t) = \sphericalangle_{\text{or}}(\dot\alpha(t-), \dot\alpha(t+))$, and the *interior angle* as $\theta_{\text{int}}(t) = \pi - \theta_{\text{ext}}$. Note that $\theta_{\text{ext}}(t) \in (-\pi, \pi)$, while $\theta_{\text{int}}(t) \in (0, 2\pi)$ [10, p.266ff].

Lemma 18. *There is $r_1 > 0$ such that for any pair of segments $\gamma_1, \gamma_2 : [0, L] \to \mathfrak{B}^*(0, r_1)$ with the same endpoints $\dot\gamma_1(0) \neq -\dot\gamma_2(0)$ and $\dot\gamma_1(L) \neq -\dot\gamma_2(L)$. Moreover, if $\gamma_1 * \gamma_2^0$ is positively oriented and 0 does not lie in its interior, then the interior angles of $\gamma_1 * \gamma_2^0$ at the two vertices are both smaller than π and*

$$\sphericalangle_{\text{or}}\big(\dot\gamma_1(L), \dot\gamma_2(L)\big) < 0.$$

Proof. Using Lemma 11 we can find a $\delta > 0$ with the following property: for any segment $\gamma : [0, L] \to \mathfrak{B}(0, \delta)$ with $\gamma((0, L)) \subset X_{\text{reg}}$ we have

$$\sphericalangle\big(\dot\gamma(s), \dot\gamma(s')\big) < \frac{\pi}{2} \tag{46}$$

for any $s, s' \in [0, L]$, the angle being computed with respect to the Hermitian product $\langle\ ,\ \rangle$. Let $\kappa \in \mathbb{R}$ be defined as in (7). By Wirtinger theorem [9, p.159] ω is the volume form of $g\big|_{X_{\text{reg}}}$. By Lelong theorem [9, p.173] analytic sets have locally finite mass. Hence there is an $r_1 \in (0, \delta)$ such that

$$\operatorname{vol}\big(\mathfrak{B}(0, c_1^2 r_1^{1/m})\big) = \int_{\mathfrak{B}(0, c_1^2 r_1^{1/m})} \omega < \frac{\pi}{1 + |\kappa|}$$

Here c_1 is the constant in Prop. 4. Let $\gamma_1, \gamma_2 : [0, L] \to \mathfrak{B}^*(0, r_1)$ be a pair of segments with the same endpoints. Assume by contradiction that $\dot\gamma_1(L) = -\dot\gamma_2(L)$. Set $\alpha = \gamma_1 * \gamma_2^0$ and $w = \dot\gamma_1(L) = -\dot\gamma_2(L)$. By (46)

$$\sphericalangle\big(\dot\alpha(s), w\big) < \frac{\pi}{2} \quad \text{i.e.} \quad \langle \dot\alpha(s), w\rangle > 0$$

for any $s \in [0, 2L]$. Hence

$$\langle \alpha(2L), w\rangle - \langle \alpha(0), w\rangle = \int_0^{2L} \langle \dot\alpha(s), w\rangle ds > 0.$$

In particular we would get $\gamma_2(0) = \alpha(2L) \neq \alpha(0) = \gamma_1(0)$ contrary to the hypothesis that the endpoints coincide. This proves that $\dot\gamma_1(L) \neq -\dot\gamma_2(L)$. The same argument of course yields $\dot\gamma_1(0) \neq -\dot\gamma_2(0)$ as well.

Next denote by V be the interior of α and assume that $0 \notin V$ and that α is positively oriented. By (20)

$$\mathfrak{B}(0, r_1) \subset U := \varphi(B(0, c_1 r_1^{1/m})) \subset \mathfrak{B}(0, c_1^2 r_1^{1/m}).$$

Since U is a topological disc and $\partial V \subset U$, also $V \subset U \subset \mathfrak{B}(0, c_1^2 r_1^{1/m})$. Since $\overline{V} \subset X_{\text{reg}}$ Gauss–Bonnet theorem applies and we get

$$\theta_{\text{int}}(0) + \theta_{\text{int}}(L) = \int_V K\omega \leq \kappa \cdot \operatorname{vol}\big(\mathfrak{B}(0, c_1^2 r_1^{1/m})\big) < \pi.$$

Thus $\theta_{\text{int}}(0), \theta_{\text{int}}(L) \in [0, \pi)$. To prove the last assertion set $\theta = \sphericalangle_{\text{or}}\big(\dot\gamma_1(L), \dot\gamma_2(L)\big)$. Since γ_1 and γ_2 are distinct geodesics $\theta \neq 0$. It is easy to check that

$$\theta_{\text{int}}(L) = \begin{cases} 2\pi - \theta & \text{if } \theta \in (0, \pi) \\ -\theta & \text{if } \theta \in (-\pi, 0). \end{cases}$$

Since $\theta_{\text{int}}(L) \in (0, \pi)$, $\theta \in (-\pi, 0)$. □

Let $\delta > 0$ be such that $\mathfrak{B}(0, \delta) \Subset X$. Put

$$r_2 = \frac{1}{2} \min\left\{ \frac{\pi}{\sqrt{\kappa}}, \delta, r_1 \right\}$$

where $\pi/\sqrt{\kappa} = +\infty$ if $\kappa \leq 0$.

Proposition 7. *For any $x \in \mathfrak{B}(0, r_2)$, $B_x\big(0, d(x, 0)\big) \subset \mathscr{D}_x$, $\exp_x$ has no critical points on $B_x(0, r_2) \cap \mathscr{D}_x$ and $\exp_x B_x(0,\, d(x, 0)) = \mathfrak{B}\big(x, d(0, x)\big)$.*

Proof. Let $x \in \mathfrak{B}(0, r_2)$ and $v \in U_x X$. Set $r = d(x, 0)$ and let T_v be as in (5). Assume by contradiction that $T_v < r$ and set $\varepsilon = (r - T_v)/2 > 0$. For any $t \in [0, T_v)$

$$d(\gamma^v(t), 0) \leq d(\gamma^v(t), x) + d(x, 0) \leq t + r < 2r \leq 2r_2 \leq \delta$$
$$d(\gamma^v(t), 0) \geq |d(\gamma(t), x) - d(x, 0)| \geq r - t \geq r - T_v > \varepsilon.$$

So $\gamma^v([0,T_v))$ is contained in $Q := \overline{\mathfrak{B}(0,\delta)} \setminus \mathfrak{B}(0,\varepsilon)$ and $t \mapsto \dot\gamma^v(t)$ is a trajectory of the geodesic flow contained in the compact set $\{(y,w) \in TX_{\mathrm{reg}} : y \in Q, |w| = 1\}$. This contradicts the maximality of T_v. Therefore $T_v \geq r$ and $B_x(0,r) \subset \mathscr{D}_x$. Since $K \leq \kappa$ on X_{reg} and $r_2 \leq \pi/\sqrt{\kappa}$ Rauch theorem [11, p.215] implies that for any $v \in U_xX$ the geodesic γ^v has no conjugate points on $[0, \min\{T_v, r_2\})$. Therefore $\exp_x$ is a local diffeomorphism on $B_x(0,r_2) \cap \mathscr{D}_x$ [11, p.114]. This proves the second claim. The inclusion $\exp_x B_x(0,r) \subset \mathfrak{B}(x,r)$ is obvious. On the other hand if $y \in \mathfrak{B}(x,r)$ let $\gamma : [0, d(x,y)] \to X$ be a segment from x to y. By the triangle inequality γ is contained in X_{reg} so $\gamma(t) = \exp_x tv$ for some $v \in U_xX$. Then $y = \exp_x d(x,y)v \in \exp_x \mathfrak{B}(0,r)$. This proves that $\exp_x B_x(0,r) = \mathfrak{B}(x,r)$. □

For $x \in \mathfrak{B}(0,r_2)$ define $c_x : U_xX \to (0, r_2]$ by

$$c_x(v) = \sup\{t \in (0, \min\{T_v, r_2\}) : \gamma^v \text{ is minimising on } [0,t]\} \tag{47}$$

and put $\check{c}(x) = \inf_{U_xX} c_x$. If γ^v is a segment from x to 0 then $c_x(v) = d(x,0)$, so $\check{c}_x \leq d(x,0)$. In the next two lemmata we adapt to our situation arguments that are classical in the study of the cut locus of a complete Riemannian manifold, see e.g. [21, p.102]. For the reader's convenience we provide all the details.

Lemma 19. *Let $x \in \mathfrak{B}(0,r_2)$, $v \in U_xX$ and $T \in (0, \min\{T_v, r_2\})$. Then $T = c_x(v)$ iff γ^v is minimising on $[0,T]$ and there is another segment $\gamma \neq \gamma^v$ between x and $\gamma^v(T)$. If $d(x,0) + d(0,\gamma^v(T)) > T$ then γ lies entirely in X_{reg}, so $\gamma^u = \gamma$ for some $u \in U_xX$, $u \neq v$. In particular this happens if $T < d(x,0)$.*

Proof. Put $y = \gamma^v(T) \in X_{\mathrm{reg}}$ and assume $T = c_x(v)$. Then γ^v is minimising on $[0,t]$ for any $t < T$, so also on $[0,T]$. Since it is not minimising after T, we may choose a sequence $t_n \searrow T$ such that γ^v is never minimising on $[0,t_n]$. Put $y_n = \gamma^v(t_n)$ and $s_n = d(x,y_n)$. Then $s_n < t_n$ and $s_n \to T$. Let $\gamma_n : [0,s_n] \to X$ be a segment from x to y_n. By Ascoli-Arzelà Theorem and Cor. 1 we can extract a subsequence converging to a segment $\gamma : [0,L] \to X$ from x to y. If $\gamma = \gamma^v$, then γ_n is contained in X_{reg} for large n, so $\gamma_n = \gamma^{v_n}$ for some $v_n \in U_xX$ and $v_n \to v$. But then any neighbourhood of Tv in T_xX contains a pair of distinct points $s_nv_n \neq t_nv$ that are mapped by $\exp_x$ to the same point $y_n \in X_{\mathrm{reg}}$. Since $Tv \in B_x(0,r_2) \cap \mathscr{D}_x$ this contradicts Prop. 7. Therefore $\gamma \neq \gamma^v$. This proves necessity of the condition. Sufficiency follows directly from Lemma 15. The remaining assertions are trivial. □

Lemma 20. *For $x \in \mathfrak{B}(0,r_2)$ the function c_x is lower semicontinuous. In particular the minimum $\check{c}_x$ is attained.*

Proof. Let $v_n \in U_xX$ be a sequence such that $v_n \to v$. Set $T := \liminf_{n\to\infty} c_x(v_n)$. We wish to prove that $c_x(v) \leq T$. If $T = r_2$ this is obvious from the definition (47). Assume instead that $T < r_2$. Passing to a subsequence we can assume that $T_n := c_x(v_n) < r_2$ and $T_n \to T$. By the theorem of Ascoli-Arzelà the segments $\gamma^{v_n}\big|_{[0,T_n]}$ converge to a segment $\alpha : [0,T] \to X$ and $\alpha(t) = \gamma^v(t)$ for $t \in [0,T_v)$. If there is $\tau \in (0,T]$ such that $\alpha(\tau) = 0$ then $c_x(v) \leq T_v \leq \tau \leq T$ and we are done. Otherwise $\alpha([0,T]) \subset X_{\text{reg}}$, so $\gamma^v = \alpha$ is minimising on $[0,T]$ and $T < T_v$. For n large $\gamma^{v_n}([0,T_n] \subset X_{\text{reg}}$ as well. Hence $c_x(v_n) < \min\{T_v, r_2\}$. By the previous lemma there are segments $\gamma_n \neq \gamma^{v_n}$ from x to $\gamma_{v_n}(T_n)$. Again by the theorem of Ascoli-Arzelà we can assume, by passing to a subsequence, that γ_n converge to a segment β from x to $\gamma^v(T)$. If β passes through 0 then $\beta \neq \gamma^v$ and $T = c_x(v)$ by the previous lemma. If β is contained in X_{reg}, the same is true of γ_n for large n. Write $\gamma_n = \gamma^{u_n}$ and extract a subsequence so that $u_n \to u$. Clearly $\beta = \gamma^u$. If $u = v$, any neighbourhood of Tv would contain two distinct vectors $T_n v_n \neq T_n u_n$ with the same image through $\exp_x$. Since $T < r_2$ this possibility is ruled out by Prop. 7. Therefore $u \neq v$ and the previous lemma implies that $c_x(v) = c_x(u) = T$. □

Proposition 8. *If $x \in \mathfrak{B}(0, r_2)$ then $\check{c}_x = \operatorname{inj}_x = d(x, 0)$.*

Proof. Let $x \in \mathfrak{B}(0, r_2)$ and $r = d(x, 0)$. First of all we prove that $\exp_x$ is injective on $B_x(0, \check{c}_x)$. In fact let $w_1, w_2 \in B_x(0, \check{c}_x)$ be such that $\exp_x(w_1) = \exp_x(w_2)$. Write $w_i = t_i v_i$ with $|v_i| = 1$. Since $t_i = |w_i| < \check{c}_x \leq c(v_i)$ the geodesics γ^{v_i} are minimising on $[0, t_i]$. Therefore $t_1 = d(x, \exp_x(w_i)) = t_2$. If $v_1 \neq v_2$, Lemma 19 would imply that $t_1 = c(v_1)$, but this is impossible since $t_1 < \check{c}_x$. So $v_1 = v_2$ and $w_1 = w_2$. This proves that $\exp_x$ is injective on $\mathfrak{B}(0, \check{c}_x)$. Next we prove that $\check{c}_x = r$. We already know that $\check{c}_x \leq r$. Assume by contradiction that $T := \check{c}_x < r$. By Lemma 20 there are $u \neq v \in U_x X$ such that $\gamma^u(T) = \gamma^v(T)$. Since $T < r$ Prop. 7 ensures that $\exp_x$ is a diffeomorphism on appropriate neighbourhoods of Tu and Tv in $T_x X$. By Lemma 16 we conclude that $\dot{\gamma}^u(T) = -\dot{\gamma}^v(T)$. But this is impossible by Lemma 18. Therefore $\check{c}_x = r$ and $\exp_x$ is injective on $B_x(0, r)$. Hence $\exp_x$ is a diffeomorphism of $B_x(0, r)$ onto $\mathfrak{B}(x, r)$. In particular $\operatorname{inj}_x \geq \check{c}_x \geq r$. The reverse inequality is proven in Lemma 3. □

Proposition 9. *There is $r_3 \in (0, r_2/2)$ such that if $\alpha, \beta : [0,T] \to X_{\text{reg}}$ are distinct segments with the same endpoints $x, y \in \mathfrak{B}^*(0, r_3)$, then 0 lies in the interior of $\alpha * \beta^0$.*

Proof. Set

$$r_3 = \left(\frac{r_2}{2c_1^2} \right)^m$$

where c_1 is the constant in (19). By (20)

$$\mathfrak{B}(0, r_3) \subset U := \varphi(B(0, c_1 r_3^{1/m})) \subset \mathfrak{B}(0, c_1^2 r_3^{1/m}) \subset \mathfrak{B}(0, r_2/2) \tag{48}$$

and U is a topological disc. Let α and β be as above and set $x = \alpha(0) = \beta(0), y = \alpha(T) = \beta(T)$. Since $x, y \in \mathfrak{B}(0, r_3) \subset \mathfrak{B}(0, r_2/2)$, α and β lie in $\mathfrak{B}(0, r_2) \subset \mathfrak{B}(0, r_1)$. By Lemma 17 $\alpha * \beta^0$ is a simple closed curve. Denote by V its interior and assume by contradiction that $0 \notin V$. Since $\partial V \subset U$ also $\overline{V} \subset U \subset \mathfrak{B}(0, r_2/2)$ and $\operatorname{diam} \overline{V} < r_2$. In particular $T < r_2$. By interchanging α and β we can assume that V lies on the left of $\alpha * \beta^0$. Set $u_0 = \dot{\alpha}(0) \in U_x X$ and chose $\theta_0 \in (0, 2\pi)$ so that $\dot{\beta}(0) = e^{i\theta_0} u_0$. By hypothesis $\theta_0 > 0$. Denote by E the set of unit vectors $v \in U_x X$ of the form $v = e^{i\theta} u_0$ with $\theta \in [0, \theta_0]$ and by $\operatorname{int} E$ the subset of those with $\theta \in (0, \theta_0)$. By Lemma 20 the function c_x has a minimum on E. We claim that the minimum point lies in $\operatorname{int} E$. By

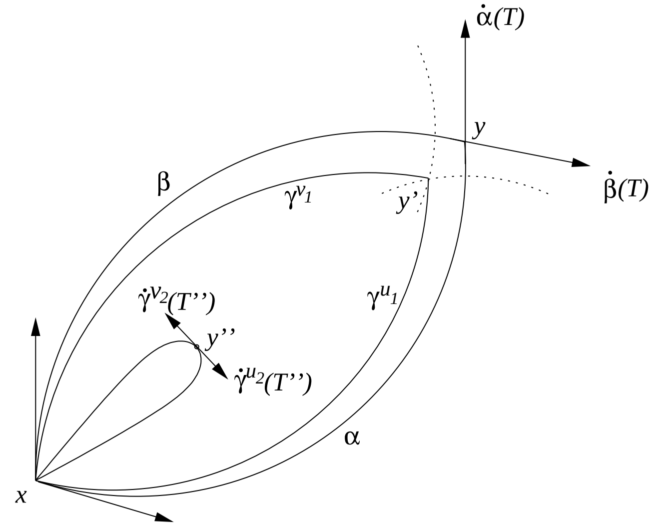


Figure 1:

the hypotheses $c_x(\dot\alpha(0)) = c_x(\dot\beta(0)) = T < r_2$. Lemma 18 implies that $\theta_0 < \pi$ and $\sphericalangle_{\mathrm{or}}(\dot\alpha(T), \dot\beta(T)) \in (-\pi, 0)$ (see Fig. 1). By Klingenberg Lemma 16 there are vectors u_1 and v_1 arbitrarily close to $\dot\alpha(0)$ and $\dot\beta(0)$ respectively, such that $T' = c_x(u_1) = c_x(v_1) < T$ and $\gamma^{u_1}(T') = \gamma^{v_1}(T')$. Since $\sphericalangle_{\mathrm{or}}(\dot\alpha(T), \dot\beta(T)) \in (-\pi, 0)$ the point $y' = \gamma^{u_1}(T') = \gamma^{v_1}(T')$ belongs to V. Therefore $\gamma^{u_1}(t)$ and $\gamma^{v_1}(t)$ lie inside V for any $t \in (0, T']$: otherwise they would meet either α or β at an interior point, which is forbidden by Lemma 15. This shows that $u_1, v_1 \in \operatorname{int} E$ and that $\dot\alpha(0)$ and $\dot\beta(0)$ are not local minima of $c_x\big|_E$ and that the minimum of c_x on E must be attained at some point $u_2 \in \operatorname{int} E$. Set $T'' = c_x(u_2) = \min_E c_x$ and $y'' = \gamma^{u_2}(T'')$. Since $u_2 \in \operatorname{int} E$ and $T'' < T$, the point y'' belongs to V, so $\gamma^{u_2}(t) \in V$ for any $t \in (0, T'']$ (use again Lemma 15). By Lemma 19 there is a segment $\gamma \neq \gamma^{u_2}$ between x and y'' and, again by Lemma 15, it is contained in V as well. So $\gamma = \gamma^{v_2}$ for some $v_2 \in \operatorname{int} E$ and $c_x(v_2) = d(x, y'') = c_x(u_2)$. Since $y'' \in V \subset \mathfrak{B}(0, r_2/2)$ both γ^{u_2} and γ^{v_2} are contained in $\mathfrak{B}^*(0, r_2) \subset \mathfrak{B}^*(0, r_1)$. Hence by Lemma 18 $\gamma^{u_2}(T'') \neq -\gamma^{v_2}(T'')$. But then we can apply again Klingenberg lemma to get a pair of nearby vectors with c_x strictly smaller than T''. Since T'' is the minimum this yields the desired contradiction. □

Theorem 3. *There is $r_4 \in (0, r_3)$ such that for any $x \in \mathfrak{B}(0, r_4)$ there is a unique segment from x to* 0.

Proof. Set

$$r_4 = \Big(\frac{r_3}{c_1^2}\Big)^m$$

where c_1 is the constant in (19). By (20)

$$\mathfrak{B}(0, r_4) \subset U := \varphi(B(0, c_1 r_4^{1/m})) \subset \mathfrak{B}(0, c_1^2 r_4^{1/m}) \subset \mathfrak{B}(0, r_3) \tag{49}$$

and U is a topological disc. Fix $x \in \mathfrak{B}(0, r_4)$ and assume by contradiction that there are two distinct segments $\alpha, \beta : [0, r] \to X$ from x and 0. By Lemma 17 $\alpha * \beta^0$ is a simple closed curve. Let V be the interior of $\alpha * \beta^0$. Since $\partial V \subset U$ also $\overline{V} \subset U \subset \mathfrak{B}(0, r_3) \subset \mathfrak{B}(0, r_2/2)$. Assume that V lies on the left of $\alpha * \beta^0$ and set $u_0 = \dot\alpha(0) \in U_x X$, $\dot\beta(0) = e^{i\theta_0} u_0$ with $\theta_0 \in (0, 2\pi)$. Denote by E be the set of $v \in U_x X$ of the form $v = e^{i\theta} u_0$ with $\theta \in [0, \theta_0]$ and by $\operatorname{int} E$ the subset of those with $\theta \in (0, \theta_0)$. If $v \in \operatorname{int} E$ then $\gamma^v(t) \in V$ for small positive t. Set $r = d(x, 0)$. By Prop. 8 γ^v is defined and minimising on $[0, r)$. Let $[0, T_v)$ be the maximal interval of definition of γ^v. If $\gamma^v((0, T_v))$ were not contained in V, there would be a minimal time t_0 such that $\gamma^v(t_0) \in \alpha((0, L]) \cup \beta((0, L])$. By Lemma 15 this would imply that $t_0 > c_x(v)$, so there would be a point $y = \gamma^v(c_x(v)) \in V$ that is reached by two distinct segments starting from x. But this is impossible because of Prop. 9 because $x, y \in \mathfrak{B}(0, r_3)$. Therefore $\gamma^v((0, T_v))$ has to be contained in V. This implies that $c_x(v) \leq \operatorname{diam} V < r_2$. If $c_x(v) < T_v$ Lemma 19 would give again a pair of distinct segments with the same endpoints $x, \gamma^v(c_x(v)) \in V \subset \mathfrak{B}(0, r_3)$, thus contradicting Prop. 9. So $c_x(v) = T_v$. Now γ^v is minimising hence Lipschitz on $[0, c_x(v))$ and therefore extends continuously to $[0, c_x(v)]$. The only possibility is that $\gamma^v(c_x(v)) = 0$ and $c_x(v) = d(x, 0) = r$. Let S be the set of vectors $v \in T_x X$ of the form $v = \rho e^{i\theta} v_1$ with $\rho \in (0, r)$ and $\theta \in (0, \theta_0)$. We have just proved that the map

$$F : \overline{S} \to \overline{V} \qquad F(w) = \begin{cases} \exp_x(w) & \text{if } |w| < r \\ 0 & \text{if } |w| = r \end{cases}$$

is continuous. Both $\overline{S}$ and $\overline{V}$ are topological discs, $F(\partial S) \subset \partial V$ and $F\big|_{\partial S} : \partial S \to \partial V$ has degree 1 so it is not homotopic to a constant. Therefore F must be onto, $\exp_x(S) = V$ and $V \subset \mathfrak{B}(x, r)$. Now we look at our configuration of geodesics from the point of view of Δ as in §4. Set $\gamma_1 = \alpha^0$ and $\gamma_2 = \beta^0$ and let $\underline{\gamma}_i : [0, L_i] \to \Delta$ be as in Def. 4. By Prop. 6 these $\underline{\gamma}_i$ are C^1 paths on $[0, L_i]$. For small s, each of them intersects the circle $Z_s = \{z \in \Delta : |z| = s\}$ at exactly one point $p_i(s)$. Let $t_i(s) \in (0, r)$ be such that $p_i(s) = \varphi^{-1}(\gamma_i(t_i(s)))$. The functions $t_i : [0, \varepsilon) \to [0, r]$ are continuous, strictly decreasing in a neighbourhood of 0 and such that $t_i(0) = 0$. Since γ_1 and γ_2 do not intersect except at their endpoints $p_1(s) \neq p_2(s)$. Therefore the circle Z_s is cut by $p_1(s)$ and $p_2(s)$ in exactly two arcs. One of them lies in $\varphi^{-1}(V)$ the other outside of it. Denote by $\beta_s : [0, 1] \to \Delta$ a C^1 parametrisation of the former. Then $\alpha_s := \varphi \circ \beta_s$ is a path of length $L(\alpha_s) \leq ||d\varphi||_\infty L(\beta_s) \leq 2\pi c_0 s = Cs$ lying in $\exp_x(B_x(0, r))$ and connecting $\gamma_1(t_1(s)) = \exp_x((r - t_1(s)) u_0)$ to $\gamma_2(t_2(s)) = \exp_x((r - t_2(s)) e^{i\theta_0} u_0)$.

On the other hand

$$r < r_2 \le \frac{\pi}{2\sqrt{\kappa}}$$

$K \le \kappa$ on $\mathfrak{B}(x,r)$ and $\exp_x$ is a diffeomorphism of $B_x(0,r) \subset T_xX$ onto $\mathfrak{B}(x,r)$. Therefore a classical corollary to Rauch theorem [11, Prop. 2.5 p.218] ensures that $L(\alpha_s)$ is bounded from below by some positive constant depending only on θ_0 and κ. This yields the contradiction and shows that the segments α and β coincide. □

Lemma 21. *There is $r_5 \in (0, r_4/3)$ such for any segment $\gamma : [0,L] \to \mathfrak{B}(0, 3r_5)$ with $\gamma((0,L)) \subset X_{\text{reg}}$ and for any $s, s \in [0,L]$*

$$\sphericalangle(\pi(\dot\gamma(s)), \pi(\dot\gamma(s'))) < \frac{\pi}{8}. \tag{50}$$

Proof. By Lemma 12 there is $\delta > 0$ such that (50) holds for any segment $\gamma : [0,L] \to \mathfrak{B}^*(0,\delta)$. Set $r_5 = \min\{\delta, r_4/2\}$. □

If $\gamma : [a,b] \to \mathbb{C}^*$ is a continuous path define its winding number by

$$W(\gamma) = \operatorname{Re} \int_\gamma \frac{dz}{z}. \tag{51}$$

For a non-closed path $W(\gamma) \in \mathbb{R}$. The winding number $W(\gamma)$ depends only on the homotopy class of γ with fixed endpoints. If $\gamma(t) = \rho(t)e^{2\pi i\theta(t)}$ with $\theta \in C^0([a,b])$ then

$$W(\gamma) = \theta(b) - \theta(a). \tag{52}$$

Lemma 22. *If $\gamma : [0,L] \to \mathfrak{B}^*(0,3r_5)$ is a segment then $W(\pi \circ \gamma) < 1$.*

Proof. Set $\alpha = \pi \circ \gamma$ and write $\alpha(t) = \rho(t)e^{i2\pi\theta(t)}$ with $\theta \in C^0([0,L])$. Then $W(\alpha) = \theta(L) - \theta(0)$. Assume by contradiction that $W(\alpha) \ge 1$. Pick $t_0 \in [0,L]$ such that $\theta(t_0) - \theta(0) = 1$ and let $\chi : [0, t_0+1] \to \Delta$ be defined by

$$\chi(t) = \begin{cases} \alpha(t) & t \in [0,t_0] \\ \alpha(t_0) + (t-t_0)(\alpha(0) - \alpha(t_0)) & t \in [t_0, t_0+1]. \end{cases}$$

The second piece of χ is a parametrisation of the segment from $\alpha(t_0)$ to $\alpha(0)$. Since $\theta(t_0) - \theta(0) = 1$, χ is a loop that avoids the origin and has winding number 1, so its homotopy class is a generator of $\pi_1(\Delta^*, \alpha(0))$. Set

$$v = \frac{\alpha(0) - \alpha(t_0)}{|\alpha(0) - \alpha(t_0)|} \qquad \begin{matrix} u_1(t) = \chi(t) \cdot v \\ u_2(t) = \chi(t) \cdot Jv \end{matrix}$$

(J is the complex structure on $\mathbb{C}$.) Since $W(\chi) = 1$ both functions u_1 and u_2 have positive maximum, so their maximum points t_1 and t_2 belong to $(0, t_0)$. Therefore $\dot\alpha(t_2) = \pm v$, $\dot\alpha(t_1) = \pm v$ and $\sphericalangle(\dot\alpha(t_1), \dot\alpha(t_2)) \ge \pi/2$ (see Fig. 2). But α is the projection of the segment γ. This contradicts (50) and proves the lemma. □

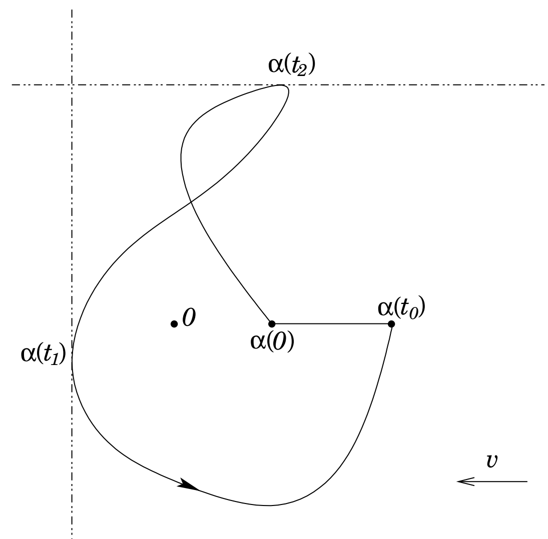


Figure 2:

Theorem 4. *For any $x, y \in \mathfrak{B}^*(0, r_5)$ there is at most one segment from x to y avoiding* 0.

Proof. Let $\gamma_1, \gamma_2 : [0, L] \to X$ be two segments from x to y. Both γ_1 and γ_2 are contained in $\mathfrak{B}(0, 3r_5)$. Assume by contradiction that the two segments are distinct and both lie in X_{reg}. By Lemma 17 $\gamma = \gamma_1 * \gamma_2^0$ is a Jordan curve and by Prop. 9 the origin lies in the interior of γ. Hence $[\gamma]$ is a generator of $\pi_1(X_{\text{reg}}, \gamma(0))$. Set $\alpha_i = \pi \circ \gamma_i$, $\alpha = \pi \circ \gamma = \alpha_1 * \alpha_2^0$. Since $\pi : X_{\text{reg}} \to \Delta^*$ is a degree m unramified covering, the loop α has winding number m. Therefore either α_1 or α_2^0 has winding number at least 1. Nevertheless this is impossible by Lemma 22. □

6 Convexity

For $x \in \mathfrak{B}^*(0, r_5)$ denote by

$$\gamma_x : [0, d(0, x)] \to X$$

the unique segment from 0 to x. Define three maps

$$\begin{aligned} &\mathfrak{F} : \mathfrak{B}^*(0, r_5) \to S^1 \times \{0\} \subset \mathbb{C}^n && \mathfrak{F}(x) = \dot{\gamma}_x(0) \\ &\underline{\mathfrak{F}} : \mathfrak{B}^*(0, r_5) \to S^1 && \underline{\mathfrak{F}}(x) = \underline{\dot{\gamma}}_x(0) \\ &\mathfrak{G} : \mathfrak{B}^*(0, r_5) \times [0, 1] \to X && \mathfrak{G}(x, t) = \gamma_x\big(td(0, x)\big). \end{aligned} \tag{53}$$

$\mathfrak{F}$ takes values in $S^1 \times \{0\}$ because $C_0 X = \mathbb{C} \times \{0\}$ and $g_x = \langle\ ,\ \rangle$.

Proposition 10. *The maps* $\mathfrak{F}$, $\underline{\mathfrak{F}}$ *and* $\mathfrak{G}$ *are continuous and* $\mathfrak{F} = (u \circ \underline{\mathfrak{F}}, 0, \ldots, 0)$.

Proof. Assume $x_n \to x$ and set $\gamma^n = \gamma_{x_n}$. By Theorem 2 $||\dot{\gamma}^n||_{C^{0,1/m}} \leq c_3$. By the Ascoli-Arzelà theorem there is a subsequence, still denoted by γ^n, that converges in the C^1-topology to the unique segment γ_x from 0 to x. In particular $\mathfrak{F}(x_n^*) = \dot{\gamma}^n(0) \to \dot{\gamma}(0) = \mathfrak{F}(x)$. This shows that $\mathfrak{F}$ is continuous. That $\mathfrak{F} = u \circ \underline{\mathfrak{F}}$ was already proved in Prop. 6. If $\dot{\gamma}(0) = (e^{i\theta_0}, \ldots, 0)$, pick $t_0 \in (0, d(x,0))$ sufficiently close to 0 that $\gamma_1(t_0) \in S(\theta_0, \pi/2)$. Denote by $S_1, \ldots, S_m$ the connected components of $u^{-1}(S(\theta_0, \pi/2))$ and assume that $\varphi^{-1}\gamma(t_0) \in S_j$. Then $\varphi^{-1}\gamma((0,t_0])$ is entirely contained in S_j. Since S_j is convex it follows that $\underline{\dot{\gamma}}(0) \in S_j$. As $\gamma^n \to \gamma$ uniformly and φ^{-1} is Hölder (Prop. 4) also $\varphi^{-1}\gamma^n(t_0) \in S_j$ and $\underline{\dot{\gamma}}^n(0) \in S_j$ for large n. The map $u_j = u\big|_{S_j} : S_j \to S$ is a homeomorphism and $u(\underline{\dot{\gamma}}^n(0)) = \dot{\gamma}_1^n(0)$ and $u(\underline{\dot{\gamma}}(0)) = \dot{\gamma}_1(0)$. Therefore $\underline{\dot{\gamma}}^n(0) = u_j^{-1}\dot{\gamma}_1^n(0) \to u_j^{-1}\dot{\gamma}_1(0) = \underline{\dot{\gamma}}(0)$. This proves that $\underline{\mathfrak{F}}$ is continuous. Finally, if $x_n \to x$ and $t_n \to t$, by passing to a subsequence we can assume that $\gamma^n = \gamma_{x_n} \to \gamma_x$ uniformly. Then clearly $\mathfrak{G}(x_n, t_n) = \gamma^n\big(t_n d(x_n, 0)\big) \to \gamma_x\big(td(x,0)\big) = \mathfrak{G}(x,t)$. This proves that the third map $\mathfrak{G}$ is continuous. $\square$

Proposition 11. *For any pair of points* $x, y \in \mathfrak{B}^*(0, r_5)$ *with*

$$\sphericalangle(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)) < \frac{\pi}{m} \tag{54}$$

there is a unique segment $\alpha_{x,y} : [0, d(x,y)] \to X$ *such that* $\alpha_{x,y}(0) = x$ *and* $\alpha_{x,y}(d(x,y)) = y$. *This segment lies entirely in* X_{reg}. *If* $\alpha_{x,y}(t) = \exp_x tv$, *then* $d(x,y) < c_x(v)$. *Finally the map* $(x,y,t) \mapsto \alpha_{x,y}(t)$ *is continuous.*

Proof. Since $\sphericalangle(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)) < \pi/m$ it follows from Lemma 14 that the path $\gamma_x * \gamma_y^0$ is not minimising. If γ_1 and γ_2 were two distinct segments from x to y, by Theorem 4 one of them, say γ_1, would have to pass through 0. But then γ_1 would coincide with $\gamma_x * \gamma_y^0$. This is absurd since γ_1 is a segment. This proves the first two assertions. Since $r_5 \leq r_3 \leq r_2/2$, $x \in \mathfrak{B}(0, r_2)$ and $d(x,y) < 2r_5 < r_2$. Thus the third assertion follows from the first and Lemma 19. The last fact follows from uniqueness by a standard use of Ascoli-Arzelà lemma. $\square$

Proposition 12. *The function* $d(0, \cdot)$ *is* C^1 *on* $\mathfrak{B}^*(0, r_5)$. *If* $x \in \mathfrak{B}^*(0, r_5)$, *the gradient of* $d(0, \cdot)$ *at* x *is* $\dot{\gamma}_x(d(0,x))$.

Proof. For x and γ as above set $L = d(0,x)$ and $v = \dot{\gamma}(L)$. Choose ε such that $0 < \varepsilon < \min\{L, r_5 - L\}$ and extend γ to $[0, L+\varepsilon]$ by setting $\gamma(t) = \exp_x tv$ for $t \in (L, L+\varepsilon]$. Then $L + \varepsilon \leq r_5 \leq r_2$. By Prop. 8 the path $\gamma\big|_{[\delta, L+\varepsilon]}$ is a segment for every $\delta > 0$, hence γ is the unique segment from 0 to $\gamma(L+\varepsilon)$. Set $x_1 = \gamma(L - \varepsilon)$ and $x_2 = \gamma(L+\varepsilon)$. Since $\varepsilon = d(x_1, x) = d(x_2, x) < L = \text{inj}_x$ the functions $d(x_1, \cdot)$ and $d(x_2, \cdot)$ are differentiable at x with gradients v and $-v$ respectively (see e.g. [21, Prop. 4.8, p.108]). So

$$\begin{aligned} d(x_1, \exp_x w) &= d(x_1, x) + g_x(v,w) + o(|w|) \\ d(x_2, \exp_x w) &= d(x_2, x) - g_x(v,w) + o(|w|). \end{aligned}$$

By the triangle inequality

$$\begin{aligned} d(0,\exp_x w) &\leq d(0,x_1) + d(x_1,\exp_x w) = \\ &= d(0,x_1) + d(x_1,x) + g_x(v,w) + o(|w|) = \\ &= d(0,x) + g_x(v,w) + o(|w|) \\ d(0,\exp_x w) &\geq d(0,x_2) - d(x_2,\exp_x w) = \\ &= d(0,x_2) - \big[d(x_2,x) - g_x(v,w) + o(|w|)\big] = \\ &= d(0,x) + g_x(v,w) + o(|w|) \\ |d(0,\exp_x w) &- d(0,x) - g_x(v,w)| = o(|w|). \end{aligned}$$

This proves that $d(0,\cdot)$ is differentiable at x with gradient v. Next we show that the gradient is continuous. Indeed if $\{x_n\}$ is a sequence converging to $x \in X_{\text{reg}}$ and γ_n are segments from 0 to x_n, then by Theorem 2 and the theorem of Ascoli and Arzelà there is a subsequence γ_n^* that converges in the C^1-topology to the unique segment γ from 0 to x. In particular $\dot\gamma_n^*(d(0,x_n)) \to \dot\gamma(d(0,x))$. Therefore the vector field $\nabla d(0,\cdot)$ is continuous on $\mathfrak{B}^*(0,r_5)$. □

We found the above argument for the differentiability of the distance function in [17, Prop. 6].

Lemma 23. *Let (M,g) be a Riemannian manifold with sectional curvature bounded above by $\kappa \in \mathbb{R}$. Let x and y be points of M that are connected by a unique segment $\gamma(t) = \exp_x tv$, $v \in U_xM$ so that $\gamma(t_0) = y$ and*

$$t_0 = d(x,y) < \min\Big\{c_x(v), \frac{\pi}{2\sqrt{\kappa}}\Big\}$$

where as usual $\sqrt{\kappa} = +\infty$ if $\kappa \leq 0$. Then the function $d(x,\cdot)$ is smooth in a neighbourhood of y and its Hessian at y is positive semi-definite.

Proof. This is a classical result in Riemannian geometry following from Rauch comparison theorem. It is commonly stated with stronger (and cleaner) hypotheses, but the usual proof, found e.g. in [21, pp.151-153] goes through without change with the above minimal assumptions. In fact $\exp_x$ is a diffeomorphism in a neighbourhood of $v \in T_xM$, so $d(x,\cdot)$ is smooth and one can compute its derivatives using Jacobi fields. The result then follows from Lemma 4.10 p.109 and Lemma 2.9 p.153 in [21], especially eq. (2.16) p.153. Notice that we are only interested in the first inequality in eq. (2.16) and this only depends on the upper bounds for the sectional curvature of M. □

Proposition 13. *If $\alpha : [0,L] \to \mathfrak{B}^*(0,r_5)$ is a segment, the function $d(0,\alpha(\cdot))$ is convex on $[0,L]$.*

Proof. Pick $s_0 \in [0,L]$ and set $x = \alpha(s_0)$ and $x_n = \gamma_x(1/n)$. Then $\underline{\mathfrak{F}}(x_n) \equiv \underline{\mathfrak{F}}(x)$ since γ_{x_n} is a piece of γ_x. Since $\underline{\mathfrak{F}}$ is continuous, there is an $\varepsilon > 0$ such that for any $s \in J := (s_0 - \varepsilon, s_0 + \varepsilon) \cap [0,L]$

$$\sphericalangle(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(\alpha(s))) < \frac{\pi}{m}.$$

By Prop. 11 for any $s \in J$ there is a unique segment $\alpha_{n,s} : [0, d(x_n, \alpha(s))] \to X_{\mathrm{reg}}$, joining x_n to $\alpha(s)$, it is of the form $\alpha_{n,s}(t) = \exp_{x_n} tv_{n,s}$ and $d(x_n, \alpha(s)) < c_{x_n}(v_{n,s})$. So we can apply Lemma 23 to the effect that the function $u_n = d(x_n, \alpha(\cdot))$ is convex on J. Since $u_n \to d(0, \alpha(\cdot))$ uniformly, also the function $d(0, \alpha(\cdot))$ is convex on J. Since t_0 is arbitrary and convexity is a local condition, this proves convexity on the whole of $[0, L]$ as well. □

Corollary 5. *For any $r \in (0, r_5)$ the ball $\mathfrak{B}(0, r)$ is geodesically convex, that is: any segment whose endpoints lie in $\mathfrak{B}(0, r)$ is contained in $\mathfrak{B}(0, r)$.*

Proof. Let $\alpha : [0, L] \to X$ be a segment with endpoints $x, y \in \mathfrak{B}(0, r)$. If α passes through the origin the assertion is obvious. Otherwise the function $u(t) = d(0, \alpha(t))$ is convex on $[0, L]$ by Proposition 13. Since $x, y \in \mathfrak{B}(0, r)$, $u(0) < r$ and $u(L) < r$ so

$$u(t) \leq \Big(1 - \frac{t}{L}\Big) u(0) + \frac{t}{L} u(L) < r.$$

Therefore $\alpha(t) \in \mathfrak{B}(0, r)$ for any $t \in [0, L]$. □

Now choose a number $r_6 \in (0, r_5)$ and set

$$C = \{x \in X : d(0, x) = r_6\}. \tag{55}$$

It follows from Proposition 12 that C is a smooth 1-dimensional submanifold of X_{reg}. Since it is compact, it is diffeomorphic to S^1. The interior of C is $\mathfrak{B}(0, r_6)$ which is thus a topological disc. Let $\sigma : \mathbb{R} \to C$ be a positively oriented C^1 periodic parametrisation of C of period 1. Since σ is positively oriented the vector $J\dot{\sigma}$ points inside $\mathfrak{B}(0, r_6)$.

Lemma 24. *The maps $\mathfrak{F} \circ \sigma$ and $\underline{\mathfrak{F}} \circ \sigma$ are not constant.*

Proof. Since $\mathfrak{F}(x) = (u(\underline{\mathfrak{F}}(x)), 0, \ldots, 0)$ it is enough to prove that $\mathfrak{F} \circ \sigma$ is not constant. Assume by contradiction that $\mathfrak{F}(x) \equiv v$ for any $x \in C$. Since the range of $\mathfrak{F}$ is contained in $C_0(X)$, $\pi \circ \mathfrak{F} = \mathfrak{F}$. Using (50) we get for $x \in C$, $s \in [0, r_6]$

$$\sphericalangle\big(v, \pi(\dot{\gamma}_x(s)\big) \leq \sphericalangle(v, \mathfrak{F}(x)) + \sphericalangle(\pi(\dot{\gamma}_x(0)), \pi(\dot{\gamma}_x(s))) \leq \frac{\pi}{8}$$
$$\pi(\dot{\gamma}_x(s)) \cdot v > 0$$
$$\pi(x) \cdot v = \pi(\gamma_x(r_6)) \cdot v = \int_0^{r_6} \pi(\dot{\gamma}_x(s)) \cdot v \, ds > 0.$$

Therefore $\pi(C) = \pi\sigma([0, 1])$ would be contained in the half-plane $\{z \in \mathbb{C} : z \cdot v > 0\}$ and $\pi \circ \sigma\big|_{[0,1]}$ would be null-homotopic in Δ^*. This is impossible since $\sigma\big|_{[0,1]}$ generates $\pi_1(X_{\mathrm{reg}}, \sigma(0))$ and $\pi : X_{\mathrm{reg}} \to \Delta^*$ is an $m : 1$ covering. □

Definition 5. *For $t_0, t_1 \in \mathbb{R}$ set*

$$T = \{se^{2\pi it} \in \mathbb{C} : s \in (0,1), t \in (t_0, t_1)\}$$
$$\overline{T} = \{se^{2\pi it} \in \mathbb{C} : s \in [0,1], t \in [t_0, t_1]\}$$
$$\mathfrak{b} : \overline{T} \to X \qquad \mathfrak{b}(se^{it}) = \mathfrak{G}(\sigma(t), s) \tag{56}$$
$$\mathfrak{S}(t_0, t_1) = \mathfrak{b}(T) \qquad \mathfrak{S}[t_0, t_1] = \mathfrak{b}(\overline{T}). \tag{57}$$

Since $d(\sigma(t), 0) = r_6$ for any $t \in \mathbb{R}$

$$\mathfrak{b}(se^{it}) = \gamma_{\sigma(t)}(r_6 s).$$

Lemma 25. *If $t_0 < t_1 < t_0 + 1$ the map $\mathfrak{b}$ is a homeomorphism of $\overline{T}$ onto $\mathfrak{S}[t_0, t_1]$, $\operatorname{int}\mathfrak{S}[t_0, t_1] = \mathfrak{S}(t_0, t_1)$ and*

$$\partial\mathfrak{S}[t_0, t_1] = \sigma([t_0, t_1]) \cup \operatorname{Im}\gamma_{\sigma(t_0)} \cup \operatorname{Im}\gamma_{\sigma(t_1)}. \tag{58}$$

Proof. Continuity of $\mathfrak{b}$ follows from Proposition 10. We prove that it is injective. Let $se^{2\pi it}, s'e^{2\pi it'} \in T$, be such that $\mathfrak{b}(s^{2\pi it}) = \mathfrak{b}(s'e^{2\pi it'}) = y$. If $s = 0$ then

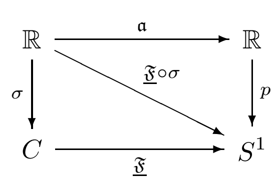
$$y = \gamma_{\sigma(t')}(r_6 s') = 0$$

so $s' = 0$ as well and $se^{2\pi it} = s'e^{2\pi it'} = 0$. If $s, s' > 0$, write $x = \sigma(t)$, $x' = \sigma(t')$. Then

$$\gamma_x(r_6 s) = \gamma_{x'}(r_6 s') = y.$$

So $r_6 s = d(0, y) = r_6 s'$ and $s = s'$. Moreover, from Theorem 3, we get $\gamma_x(t) = \gamma_{x'}(t)$ for $t \in [0, d(y, 0)]$ and by the unique continuation of geodesics also for $t \in [d(y, 0), r_6]$. Hence $x = \gamma_x(r_6) = \gamma_{x'}(r_6) = x'$ and $t = t'$. This shows that $\mathfrak{b}$ is injective and therefore a homeomorphism of T onto its image $\mathfrak{b}(T)$. Since T is homeomorphic to a closed disk, Brouwer theorem on the invariance of the domain and of the boundary (see e.g. [18, p.205f]) implies that $\operatorname{int}\mathfrak{b}(T) = \mathfrak{b}(\operatorname{int} T)$ and $\partial\,\mathfrak{b}(T) = \mathfrak{b}(T) - \mathfrak{b}(\operatorname{int} T) = \mathfrak{b}(\partial T) = \sigma([t_0, t_1]) \cup \operatorname{Im}\gamma_{\sigma(t_0)} \cup \operatorname{Im}\gamma_{\sigma(t_1)}$. ☐

Set $p(t) = e^{2\pi it}$ and let $\mathfrak{a}$ a lifting of $\underline{\mathfrak{F}} \circ \sigma$:



(59)

Lemma 26. *The function $\mathfrak{a}$ is monotone increasing and $\underline{\mathfrak{F}}(C) = S^1$.*

Proof. Mark that we are not saying that $\mathfrak{a}$ is *strictly* increasing. Let $t_0, t_1 \in \mathbb{R}$ be such that $t_0 < t_1 < t_0 + 1$. Set $x_0 = \sigma(t_0)$, $x_1 = \sigma(t_1)$, $R = \varphi^{-1}\big(\mathfrak{S}(t_0, t_1)\big)$, see (56). Since φ^{-1} is an orientation preserving homeomorphism of class C^1

outside the origin, it follows from Lemma 25 and Prop. 6 that R is a region of Δ homeomorphic to a disk with piecewise C^1 boundary

$$\partial R = \varphi^{-1}\sigma([t_0,t_1]) \cup \operatorname{Im} \underline{\gamma}_{\sigma(t_0)} \cup \operatorname{Im} \underline{\gamma}_{\sigma(t_1)}.$$

Moreover R lies on the left of $\underline{\gamma}_{\sigma(t_0)}$ and on the right of $\underline{\gamma}_{\sigma(t_1)}$ and for $t \in (t_0,t_1)$ the path $\underline{\gamma}_{\sigma(t)}$ lies inside R. Accordingly its tangent vector

$$\dot{\underline{\gamma}}_{\sigma(t)}(0) = e^{2\pi i \mathfrak{a}(t)}$$

points inside R. Let $\psi \in [0,1)$ be such that $\underline{\mathfrak{F}}(x_1) = e^{2\pi i\psi}\underline{\mathfrak{F}}(x_0)$. Since $\underline{\gamma}_{\sigma(t_0)}(0) = \underline{\mathfrak{F}}(x_0)$ and $\underline{\gamma}_{\sigma(t_1)}(0) = \underline{\mathfrak{F}}(x_1)$ the unit tangent vectors at 0 pointing inside R are exactly those of the form $e^{i\theta}\underline{\gamma}_{x_0}(0)$ with $\theta \in [0,2\pi\psi]$. So

$$\mathfrak{a}[t_0,t_1] \subset \bigsqcup_{k\in\mathbb{Z}} [\mathfrak{a}(t_0)+k, \mathfrak{a}(t_0)+\psi+k].$$

Since $\mathfrak{a}$ is continuous, we have $\mathfrak{a}[t_0,t_1] = [\mathfrak{a}(t_0), \mathfrak{a}(t_0)+\psi]$ and $\mathfrak{a}(t_1) = \mathfrak{a}(t_0)+\psi$. This proves that $\mathfrak{a}(t_1) \geq \mathfrak{a}(t_0)$. It follows that $\mathfrak{a}$ is increasing on the real line. Since $p\mathfrak{a}(1) = p\mathfrak{a}(0)$, there is $k \in \mathbb{Z}$ such that $\mathfrak{a}(1) = \mathfrak{a}(0)+k$. By the uniqueness of the lifting $\mathfrak{a}(t+1) = \mathfrak{a}(t)+k$ for any $t \in \mathbb{R}$. $k \geq 0$ because $\mathfrak{a}$ is increasing. If $k=0$ then $\mathfrak{a}$ would be constant on $[0,1]$ and so on the whole real line. But this is not the case by Lemma 24. Therefore $k > 0$. It follows that $\underline{\mathfrak{F}}$ is surjective. □

Lemma 27. *If $x \in \mathfrak{B}^*(0,r_6)$ then*

$$\sphericalangle(\pi(x), \mathfrak{F}(x)) < \frac{\pi}{8}.$$

Proof. For $x \in \mathfrak{B}^*(0,r_6)$ set $L = d(0,x)$ and let γ_x be as in (53). The set $E = \{w \in \mathbb{C}^* : \sphericalangle\big(w, \mathfrak{F}(x)\big) < \pi/8\}$ is a convex cone. Since $\mathfrak{F}(x) = \dot\gamma_x(0) = \pi(\dot\gamma_x(0))$, it follows from (50) that $\pi(\dot\gamma_x(s)) \in E$ for any $s \in [0,L]$. Thus

$$\pi(x) = \int_0^L \pi(\dot\gamma_x(s))ds \in E.$$

□

Theorem 5. *Let $t_0, t_1 \in \mathbb{R}$ be such that $t_0 < t_1$ and*

$$\mathfrak{a}(t_1) < \mathfrak{a}(t_0) + \frac{1}{2m}.$$

Then for any $x, y \in \mathfrak{S}[t_0,t_1]$ there is a unique segment joining x to y and it is contained in $\mathfrak{S}[t_0,t_1]$. Therefore $\mathfrak{S}(t_0,t_1)$ and $\mathfrak{S}[t_0,t_1]$ are geodesically convex subsets of (X,d).

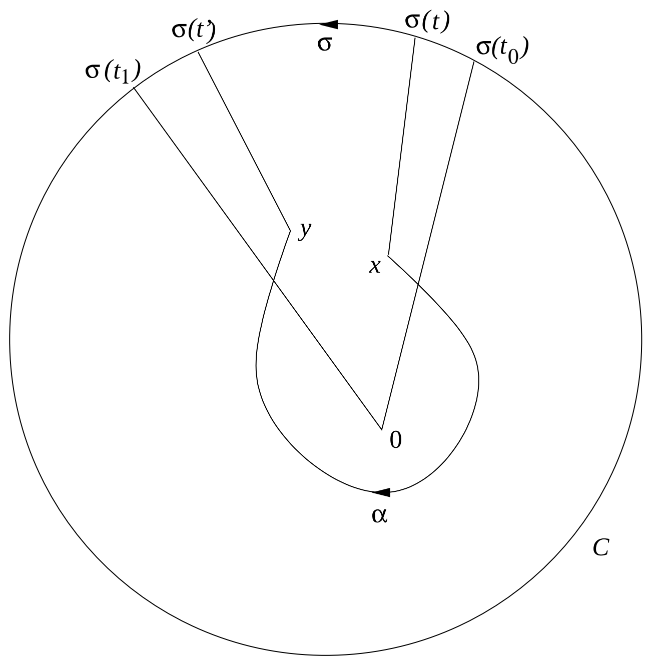


Figure 3:

Proof. Since $\mathfrak{S}(t_0, t_1)$ can be exhausted by sets of the form $\mathfrak{S}[t_0+\delta, t_1-\delta]$ with $(t_1-\delta)-(t_0+\delta) < 1/m$ it is enough to prove the convexity of $\mathfrak{S}[t_0, t_1]$. Let x and y be points in $\mathfrak{S}[t_0, t_1]$. If either $x = 0$ or $y = 0$ the claim is immediate from the definition of $\mathfrak{S}[t_0, t_1]$. Otherwise we can assume that

$$\begin{aligned} x &= \mathfrak{G}(\sigma(t), s) = \gamma_{\sigma(t)}(r_6 s) & \qquad t_0 \le t \le t' \le t_1 \\ y &= \mathfrak{G}(\sigma(t'), s') = \gamma_{\sigma(t')}(r_6 s') & \qquad s, s' \in (0,1]. \end{aligned}$$

Since $\mathfrak{a}$ is monotone $\mathfrak{a}(t_0) \le \mathfrak{a}(t) \le \mathfrak{a}(t') \le \mathfrak{a}(t_1)$ and

$$\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) = 2\pi|\mathfrak{a}(t) - \mathfrak{a}(t')| \le 2\pi\big(\mathfrak{a}(t_1) - \mathfrak{a}(t_0)\big) < \frac{\pi}{m}.$$

It follows from Lemma 2 and Prop. 11 that there is a unique segment $\alpha : [0, L] \to X$ joining x to y (so $L = d(x, y)$). We need to prove that $\alpha([0, L]) \subset \mathfrak{S}[t_0, t_1]$. Assume that it is not. Then α has to cross $\partial\mathfrak{S}[t_0, t_1]$ at least twice. Since $d(0, x) < r_6$ and $d(0, y) < r_6$, we have $\alpha([0, L]) \subset \mathfrak{B}(0, r_6)$ by Corollary 5. It follows from (58) that the set $\operatorname{Im}\alpha \cap \big(\operatorname{Im}\gamma_{\sigma(t_0)} \cup \operatorname{Im}\gamma_{\sigma(t_1)}\big)$ contains at least two points. On the other hand α cannot cross the path $\gamma_{\sigma(t_i)}$ more than once: otherwise by Theorem 4 it would coincide with some prolongation of $\gamma_{\sigma(t_i)}$. This proves that α crosses each of the paths $\gamma_{\sigma(t_i)}$ exactly once. Define $\beta : [0, 3] \to X$

by

$$\beta(\tau) = \begin{cases} \gamma_{\sigma(t)}\big(r_6(s+\tau(1-s))\big) & \text{if } \tau \in [0,1] \\ \sigma(t+(\tau-1)(t'-t) & \text{if } \tau \in [1,2] \\ \gamma_{\sigma(t')}\big(r_6(1+(\tau-2)(s'-1))\big) & \text{if } \tau \in [2,3]. \end{cases}$$

Then $\zeta = \beta * \alpha^0$ is a simple closed curve, $[\zeta]$ is the positive generator of $\pi_1(X_{\text{reg}}, x)$ and $W(\pi \circ \zeta) = m$. By Lemma 22 $W(\pi \circ \alpha) < 1$, so

$$W(\pi \circ \beta) > m - 1 \geq 1. \tag{60}$$

Let $\tau' : [0,3] \to [t,t']$ be the function

$$\tau'(\tau) = \begin{cases} t & \text{if } \tau \in [0,1] \\ t+(\tau-1)(t'-t) & \text{if } \tau \in [1,2] \\ t' & \text{if } \tau \in [2,3]. \end{cases}$$

Clearly $\mathfrak{F}\big(\beta(\tau)\big) = \mathfrak{F}\big(\sigma(\tau'(\tau))\big)$. By Lemma 27

$$\sphericalangle\big(\pi \circ \beta(\tau), \mathfrak{F}(\sigma(\tau'(\tau)))\big) < \frac{\pi}{8}$$

for any $\tau \in [0,3]$. Let $t_2 \in [t,t']$ be such that

$$\mathfrak{a}(t_2) = \frac{\mathfrak{a}(t) + \mathfrak{a}(t')}{2}$$

and set $w = \mathfrak{F}(\sigma(t_2))$. Then for any $\tau' \in [t,t']$

$$|\mathfrak{a}(\tau') - \mathfrak{a}(t_2)| \leq \frac{|\mathfrak{a}(t') - \mathfrak{a}(t)|}{2} \leq \frac{\mathfrak{a}(t_1) - \mathfrak{a}(t_0)}{2} < \frac{1}{4m}$$

so for $\tau \in [0,3]$

$$\sphericalangle\big(\pi \circ \beta(\tau), w\big) \leq \sphericalangle\big(\pi \circ \beta(\tau), \mathfrak{F}(\tau'(\tau))\big) + \sphericalangle\big(\mathfrak{F}(\tau'(\tau)), w\big) < \\ < \frac{\pi}{8} + 2\pi m |\mathfrak{a}(\tau') - \mathfrak{a}(t_2)| < \frac{5\pi}{8}.$$

This shows that $W(\pi \circ \beta) \leq 1$, contradicting (60). Therefore $\alpha([0,L]) \subset \mathfrak{B}(0, r_6)$ as claimed. □

7 Alexandrov curvature

In this section we will finally conclude the proof of Theorem 8. We start by recalling the basic definitions related to upper curvature bounds for a metric space in the sense of A.D.Alexandrov. Next we will come back to the setting considered in §§3–6 and we will prove that $\mathfrak{B}(0, r_6)$ is a CAT(κ)–space (Thm. 7). Theorem 8 follows almost immediately from this.

A thorough treatment of the intrinsic geometry of metric spaces, and especially of curvature bounds in the sense of Alexandr Danilovich Alexandrov can be found in the books [2], [19], [1], [5], [4], [6], [7]. We mostly follow [6].

Let (X, d) denote an arbitrary metric space with intrinsic metric. Given two segments α and β in X with $\alpha(0) = \beta(0) = x$ the *Alexandrov (upper) angle* is defined as

$$\angle_x(\alpha, \beta) = \limsup_{t,t' \to 0} \arccos \frac{t^2 + (t')^2 - d(\alpha(t), \beta(t'))^2}{2tt'}. \tag{61}$$

Fix $\kappa \in \mathbb{R}$. Set $D_\kappa = +\infty$ if $\kappa \leq 0$ and $D_k = \pi/\sqrt{\kappa}$ otherwise. Let M^2_κ denote the complete Riemannian surface with constant curvature κ. A *triangle* $T = \Delta(xyz)$ in X is a triple of points x, y, z together with a choice of three segments connecting them. A *comparison triangle* is a triangle $\overline{T} = \Delta(\bar{x}\bar{y}\bar{z})$ in M^2_κ such that corresponding edges have equal length. We will occasionally let T denote also the union of the edges.

Definition 6. *We say that the* angle condition *holds for a triangle T in a metric space, if the Alexandrov angle between any two edges of T is less or equal than the angle at the corresponding vertex in a comparison triangle $\overline{T}$ in M^2_κ. A metric space (X, d) is called* CAT(κ)–space *if (1) the metric is intrinsic, (2) any pair of points $x, y \in X$ with $d(x, y) < 2D_\kappa$ is connected by a segment and (3) the angle condition holds for any triangle in X. A metric space has* curvature $\leq \kappa$ (in the sense of Alexandrov) *if for every $x \in X$ there is $r_x > 0$ such that the ball of centre x and radius r_x endowed with the induced metric is a CAT(κ)–space.*

Proposition 14. *Let $\kappa \in \mathbb{R}$ and let (X, d) be a D_κ–geodesic metric space (this means that any pair of points a distance less than D_κ apart are connected by a segment). Then (X, d) is a CAT(κ)–space if and only if for any triangle T in X with perimeter less than $2D_\kappa$ the following condition holds: for $x, y \in T$ let $\bar{x}$ and $\bar{y}$ denote the corresponding points on a comparison triangle in M^2_κ; then $d(x, y) \leq d(\bar{x}, \bar{y})$.*

See [6, p.161].

Proposition 15. *Let (M, g) be a Riemannian manifold with sectional curvature bounded above by κ. Then M provided with the Riemannian distance is a metric space of curvature $\leq \kappa$ in the sense of Alexandrov.*

For a proof see e.g. [14, Thm. 2.7.6 p. 219] or [6, Thm. 1A.6 p.173]).

Proposition 16. *Let (X, d) be a CAT(κ)–space and let $\alpha : [0, a] \to X$ and $\beta : [0, b] \to$ be two segments with $\alpha(0) = \beta(0) = x$ and $\angle_x(\alpha, \beta) = \pi$. Then $\alpha^0 * \beta$ is a segment.*

See [7, Prop. 9.1.17(4) p.313].

Lemma 28. *Let (X, d) be a D_κ–geodesic space and let $T = \Delta(xy_1y_2)$ be triangle with perimeter $< 2D_\kappa$ and distinct vertices. Fix a point z on the segment from y_1*

to y_2 *and a segment from* x *to* z*. In this way we get two triangles* $T_1 = \Delta(xy_1z)$ *and* $T_2 = \Delta(xzy_2)$ *with a common edge. If the angle condition holds for both* T_1 *and* T_2 *then it also holds for* T*.*

This is the gluing lemma of [6, p.199].

Proposition 17. *Let* (X,d) *be a metric space of curvature* $\leq \kappa$*. Assume that for every pair of points* $x, y \in X$ *with* $d(x,y) < D_\kappa$ *there is a unique segment* $\alpha_{x,y}$ *which depends continuously on* (x,y)*. Then* X *is a CAT(*κ*)–space.*

See [6, Prop. 4.9 p.199].

Proposition 18. *Let* (X,d) *be a CAT(*κ*)–space and let* α *and* β *be segments with* $\alpha(0) = \beta(0) = x$*. Then the* $\limsup$ *in* (61) *is in fact a limit. Therefore*

$$\angle_x(\alpha,\beta) = \lim_{t\to 0} \arccos \frac{2t^2 - d(\alpha(t),\beta(t))^2}{2t^2} = 2\lim_{t\to 0} \arcsin \frac{d(\alpha(t),\beta(t))}{2t}. \tag{62}$$

If (X,d) is a uniquely geodesic metric space we denote by $[x,y]$ the segment from x to y and by $\angle_x(y,z)$ the Alexandrov angle between the segments $[x,y]$ and $[x,z]$.

Proposition 19. *If* (X,d) *is a CAT(*κ*)–space the function* $(x,y,z) \mapsto \angle_x(y,z)$ *is upper semicontinuous on the set of triples* (x,y,z) *with* $d(x,y), d(x,z) < D_\kappa$*. For fixed* x *the function* $(y,z) \mapsto \angle_x(y,z)$ *is continuous.*

See [6, pp.184-185].

Let us now come back to the setting and the notation of §§3–6. For $t \in \mathbb{R}$ set

$$t^+ = \sup\Big\{\tau > t : \mathfrak{a}(\tau) < \mathfrak{a}(t) + \frac{1}{2m}\Big\}$$
$$t^- = \inf\Big\{\tau < t : \mathfrak{a}(\tau) > \mathfrak{a}(t) - \frac{1}{2m}\Big\}.$$

Clearly

$$\mathfrak{a}(t^\pm) = \mathfrak{a}(t) \pm \frac{1}{2m}. \tag{63}$$

From the monotonicity of $\mathfrak{a}$ it follows that if $t'' < t < t'$ then

$$\begin{aligned} \mathfrak{a}(t') < \mathfrak{a}(t) + \frac{1}{2m} &\iff t' < t^+ \\ \mathfrak{a}(t'') > \mathfrak{a}(t) - \frac{1}{2m} &\iff t'' > t^-. \end{aligned} \tag{64}$$

Proposition 20. *For any* $t \in \mathbb{R}$ *the sectors* $\mathfrak{S}(t,t^+)$ *and* $\mathfrak{S}(t^-,t)$ *are geodesically convex subsets of* (X,d)*. Moreover* $\mathfrak{S}(t,t^+)$*,* $\mathfrak{S}(t^-,t)$*,* $\mathfrak{S}[t,t^+]$ *and* $\mathfrak{S}[t^-,t]$ *provided with the distance induced from* (X,d) *are CAT(*κ*)–spaces.*

Proof. We consider only $\mathfrak{S}(t,t^+)$ and $\mathfrak{S}[t,t^+]$. If $x_0, x_1 \in \mathfrak{S}(t,t^+)$ then by (57) $x_i = \mathfrak{G}(\sigma(t_i), s_i)$ with $t_i \in (t,t^+)$. Assume $t_0 < t_1$ and set

$$t_0' = \frac{t_0 + t}{2} \qquad t_1' = \frac{t_1 + t^+}{2}.$$

Then $t < t_0' < t_0 < t_1 < t_1' < t^+$ and

$$\mathfrak{a}(t_1') < \mathfrak{a}(t) + \frac{1}{2m} \leq \mathfrak{a}(t_0') + \frac{1}{2m}.$$

By Theorem 5 there is a unique segment from x_0 to x_1, and it is contained in $\mathfrak{S}(t_0', t_1') \subset \mathfrak{S}(t,t^+)$. It follows that $\mathfrak{S}(t,t^+)$ is a geodesically convex subset of (X,d). In particular $\mathfrak{S}(t,t^+)$ provided with the distance induced from (X,d) is a geodesic metric space. By continuity the same holds for $\mathfrak{S}[t,t^+] = \overline{\mathfrak{S}(t,t^+)}$: for any $x_0, x_1 \in \mathfrak{S}[t,t^+]$ there is *at least one* segment from x_0 to x_1 that is contained in $\mathfrak{S}[t,t^+]$. Moreover the induced distance on $\mathfrak{S}(t,t^+)$ coincides with the Riemannian distance of the smooth Riemannian surface $(\mathfrak{S}(t,t^+), g\big|_{\mathfrak{S}(t,t^+)})$, whose Gaussian curvature is everywhere $\leq \kappa$. Prop. 15 ensures that $(\mathfrak{S}(t,t^+), d)$ is a metric space with curvature $\leq \kappa$ in the sense of Alexandrov. By Thm. 5 it is uniquely geodesic and by Prop. 11 segments in $\mathfrak{S}(t,t^+)$ depend continuously on the endpoints. Thus Prop. 17 ensures that $(\mathfrak{S}(t,t^+), d)$ is a CAT(κ)–space. Since any triangle in $\mathfrak{S}[t,t^+]$ is a limit of triangles in $\mathfrak{S}(t,t^+)$ a continuity argument applied to the condition in Prop. 14 yields that $\mathfrak{S}[t,t^+]$ is a CAT(κ)–space too. □

Proposition 21. *If* $\mathfrak{a}(t_1) < \mathfrak{a}(t_0) + \frac{1}{2m}$ *then*

$$\angle_0\big(\gamma_{\sigma(t_0)}, \gamma_{\sigma(t_1)}\big) = 2\pi m\big(\mathfrak{a}(t_1) - \mathfrak{a}(t_0)\big). \tag{65}$$

Proof. Both $\gamma_{\sigma(t_0)}$ and $\gamma_{\sigma(t_1)}$ are segments contained in the CAT(κ)–space $\mathfrak{S}[t_0, t_0^+]$. By (62) and (44) their Alexandrov angle is

$$\begin{aligned}\angle_0\big(\gamma_{\sigma(t_0)}, \gamma_{\sigma(t_1)}\big) &= 2\lim_{t\to 0} \arcsin \frac{d(\gamma_{\sigma(t_0)}(t), \gamma_{\sigma(t_1)}(t))}{2t} = \\ &= \sphericalangle\big(\dot\gamma_{\sigma(t_0)}(0), \dot\gamma_{\sigma(t_1)}(0)\big) = \sphericalangle(\mathfrak{F}(\sigma(t_0)), \mathfrak{F}(\sigma(t_1))).\end{aligned}$$

Since $\underline{\mathfrak{F}}(\sigma(t_i)) = e^{2\pi i \mathfrak{a}(t_i)}$, $\mathfrak{F}(\sigma(t_i)) = (e^{2\pi m i \mathfrak{a}(t_i)}, 0, \dots, 0)$ and $2\pi m|\mathfrak{a}(t_0) - \mathfrak{a}(t_1)| < \pi$ we get

$$\sphericalangle(\mathfrak{F}(\sigma(t_0)), \mathfrak{F}(\sigma(t_1))) = 2\pi m\big|\mathfrak{a}(t_0) - \mathfrak{a}(t_1)\big|.$$

□

Proposition 22. *For any* $t \in \mathbb{R}$ *both* $\gamma_{\sigma(t)} * \gamma^0_{\sigma(t^+)}$ *and* $\gamma_{\sigma(t)} * \gamma^0_{\sigma(t^-)}$ *are shortest paths.*

Proof. Consider the first path. Thanks to Props. 20 and 16 it is enough to show that $\angle_0\big(\gamma_{\sigma(t)}, \gamma_{\sigma(t^+)}\big) = \pi$. Indeed by Props. 19 and 21 and (63)

$$\angle_0\big(\gamma_{\sigma(t)}, \gamma_{\sigma(t^+)}\big) = \lim_{\tau < t^+,\ \tau \to t^+} \angle_0\big(\gamma_{\sigma(t)}, \gamma_{\sigma(\tau)}\big) =$$
$$= \lim_{\tau < t^+,\ \tau \to t^+} 2\pi m\big(\mathfrak{a}(\tau) - \mathfrak{a}(t)\big) = 2\pi m\big(\mathfrak{a}(t^+) - \mathfrak{a}(t)\big) = \pi.$$

□

Now we are able to control uniqueness of geodesics in general.

Theorem 6. *For any $x, y \in \mathfrak{B}(0, r_6)$ there is a unique segment $\alpha_{x,y}$ from x to y and it depends continuously on its endpoints. If $x \neq 0$ and $y \neq 0$ the segment $\alpha_{x,y}$ passes through 0 if and only if*

$$\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) \geq \frac{\pi}{m}. \tag{66}$$

Proof. If one of the points is 0 uniqueness is proved in Theorem 3. If x, y are two distinct points in $\mathfrak{B}^*(0, r_6)$ by interchanging them if necessary we can write them as

$$x = \mathfrak{G}(\sigma(t), s) \qquad y = \mathfrak{G}(\sigma(t'), s')$$

with $0 \leq t \leq t' < 1$. We distinguish three cases according to the position of t' with respect to t^-, t, t^+.
1. Assume first that $t' \in (t, t^+)$. Then

$$\mathfrak{a}(t') < \mathfrak{a}(t) + \frac{1}{2m}.$$

It follows from Theorem 5 that the segment is unique, is contained in $\mathfrak{S}[t, t'] \subset$

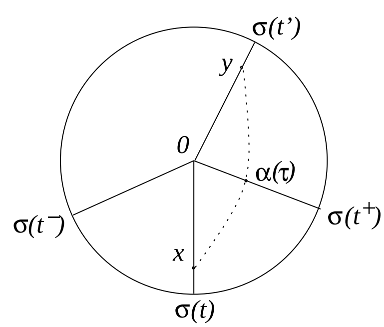


Figure 4:

$\mathfrak{S}[t, t^+]$ and does not pass through 0 because of Lemma 14.
2. If $t' > 1 + t^-$ then

$$\mathfrak{a}(t) < \mathfrak{a}(t' - 1) + \frac{1}{2m}.$$

Since $y = \mathfrak{G}(\sigma(t'-1), s')$ the same argument proves that the segment is unique, is contained in $\mathfrak{S}[t^-, t]$ and does not pass through 0.
3. Finally assume that $t' \in [t^+, 1+t^-]$. This condition is just a restatement of (66). In this case it is enough to prove that any segment α from x to y necessarily passes through the origin: Theorem 3 then yields $\alpha = \gamma_x^0 * \gamma_y$ and in particular uniqueness. Assume by contradiction that α does not pass through 0 (see Fig. 4). Then it has to cross either $\gamma_{\sigma(t^+)}$ or $\gamma_{\sigma(t^-)}$. Assume for example that $\alpha(\tau) = \gamma_{\sigma(t^+)}(\tau')$ for some $\tau \in [0, d(x,y)]$, $\tau' \in [0, r_6]$. By Prop. 22 $\gamma_x^0 * \big(\gamma_{\sigma(t^+)}\big|_{[0,\tau']}\big)$ is a segment. So $\alpha\big|_{[0,\tau]}$ and $\gamma_x^0 * \big(\gamma_{\sigma(t^+)}\big|_{[0,\tau']}\big)$ are two segments contained in $\mathfrak{S}[t, t^+]$ with same endpoints. By Prop. 20 $\mathfrak{S}[t, t^+]$ is a CAT(κ)–space, hence uniquely geodesic. Therefore $\alpha\big|_{[0,\tau]}$ and $\gamma_x^0 * \big(\gamma_{\sigma(t^+)}\big|_{[0,\tau']}\big)$ must coincide, contrary to the assumption that α does not pass through the origin.

Continuous dependence from the endpoints follows from uniqueness by Ascoli-Arzelà theorem. □

Theorem 7. *The ball $\mathfrak{B}(0, r_6)$ provided with the distance induced from (X, d) is a CAT(κ)–space.*

Proof. We will show that any geodesic triangle $T = \Delta(xyz)$ contained in $\mathfrak{B}(0, r_6)$ satisfies the angle condition, Def. 6. We distinguish various cases.
1. Suppose first that the origin is a vertex, say $z = 0$. If $\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) < \pi/m$ by interchanging if necessary x and y we can assume that $x = \mathfrak{G}(\sigma(t), s)$ and $y = \mathfrak{G}(\sigma(t'), s')$ with $t \leq t' < t^+$. Then $T \subset \mathfrak{S}[t, t']$. Since $\mathfrak{S}[t, t']$ is a CAT(κ)–space, the angle condition holds for T.
2. If $z = 0$ and $\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) \geq \pi/m$, then $\alpha_{x,y} = \gamma_x^0 * \gamma_y$ by Theorem 6. So T is degenerate and trivially satisfies the angle condition.
3. Next assume that 0 belongs to some edge but is not a vertex. Say 0 lies on the edge $[x, y]$. By the above both triangles $\Delta(0xz)$ and $\Delta(0yz)$ satisfy the angle condition. By Lemma 28 also $T = \Delta(xyz)$ does.
4. Assume now that 0 lies in the interior of T (of course a non-degenerate triangle is a Jordan curve). Let $\alpha : [0, L] \to X_{\text{reg}}$ be the segment $[x, y]$ and let $\beta_t : [0, 1] \to X$ be a constant speed parametrisation of the segment from z to $\alpha(t)$. Then

$$F : Q = [0,1]^2 \to X \qquad H(t,s) = \beta_t(s)$$

is a continuous map. Since $F\big|_{\partial Q} : \partial Q \to T$ is a degree one map $F(Q)$ must fill the interior of T. In particular there is $t_0 \in (0, L)$ such that β_{t_0} passes through 0. Then we can apply the previous argument to both triangles $\Delta(xz\alpha(t_0))$ and $\Delta(yz\alpha(t_0))$. Applying again Lemma 28 we get that the angle condition holds for T.
5. Finally consider the case in which $0 \notin \overline{R}$, where R is the interior of T. Assume that

$$\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) \geq \max\Big\{\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(z)\big), \sphericalangle\big(\underline{\mathfrak{F}}(z), \underline{\mathfrak{F}}(y)\big)\Big\}. \tag{67}$$

Since 0 does not belong to $\overline{R}$, in particular it does not lie on $[x,y]$. By Theorem 6 this implies

$$\sphericalangle\big(\underline{\mathfrak{F}}(x), \underline{\mathfrak{F}}(y)\big) < \pi/m.$$

Write $x = \mathfrak{G}(\sigma(t), s)$, $y = \mathfrak{G}(\sigma(t'), s')$ and $z = \mathfrak{G}(\sigma(t''), s'')$. Then (67) reads

$$|\mathfrak{a}(t) - \mathfrak{a}(t')| \geq \max\big\{|\mathfrak{a}(t) - \mathfrak{a}(t'')|, |\mathfrak{a}(t'') - \mathfrak{a}(t')|\big\}.$$

By interchanging t and t' (that is x and y) we can then assume that $t \leq t'' \leq t' < t^+$. We claim that $\overline{R} \subset \mathfrak{S}[t,t']$. Indeed if there is a point of $\overline{R}$ outside $\mathfrak{S}[t,t']$, there must be some point $w \in \partial R$ outside of $\mathfrak{S}[t,t']$. But $\partial R = [x,y] \cup [y,z] \cup [x,z]$ and the three segments are contained in $\mathfrak{S}[t,t']$. □

We are now finally ready to prove the main result of the paper.

Theorem 8. *Let (X,ω) be a Kähler curve and let d be the intrinsic distance. If κ is an upper bound for the Gaussian curvature of g on X_{reg}, then (X,d) is a metric space of curvature $\leq \kappa$ in the sense of Alexandrov.*

Proof. We need to prove that for any $x_0 \in X$ there is a geodesic ball centred at x_0 that is a CAT(κ)–space. If $x_0 \in X_{\text{reg}}$ this is well-known (Prop. 15). If x_0 is an analytically irreducible singular point (i.e. a single branch singularity), thanks to Cor. 2 it is enough to consider the situation envisaged in §§3–6. In this case the CAT(κ)–property of sufficiently small balls is what we have just proven (Theorem 7). Finally we have to consider the case in which x_0 is a singular point and X is analytically reducible at x_0. Let U be a neighbourhood of x such that $U = U_1 \cup \cdots \cup U_N$ where U_j are the irreducible components of U, $x_0 \in U_j$ for each j and the singular set of U_j contains at most x_0. Denote by d_j the intrinsic distance of $(U_j, \omega\big|_{U_j})$. For $r > 0$ let $\mathfrak{B}(x_0, r)$ be the geodesic ball in (X,d), as usual, and let $\mathfrak{B}_j(r)$ be the geodesic ball of radius r centred at x_0 in the space (U_j, d_j). By choosing $r > 0$ small enough we can assume that any pair of points in $\mathfrak{B}(x_0, r)$ is joined by a segment in U. This follows from Lemma 2. It is clear that $\mathfrak{B}_j(r) \subset \mathfrak{B}(x_0, r)$. On the other hand if $x \in \mathfrak{B}(x_0, r)$ and $\alpha : [0, L] \to U$ is a segment from x_0 to x then $\alpha(t) \neq x_0$ for $t > 0$. So $\alpha([0, L]) \subset U_j$ for some j. Since $L(\alpha) = d(0, x) < r$ it follows that $x \in \mathfrak{B}_j(r)$ and that $d_j(0, x) = d(0, x)$. This shows that

$$\mathfrak{B}(x_0, r) = \mathfrak{B}_1(r) \cup \cdots \cup \mathfrak{B}_N(r).$$

Moreover if $j \neq k$ any segment joining $x \in \mathfrak{B}_j(r)$ to $y \in \mathfrak{B}_k(r)$ necessarily passes through x_0. Therefore

$$d(x,y) = \begin{cases} d_j(x,y) & \text{if } x, y \in \mathfrak{B}_j(r) \\ d_j(x,0) + d_k(0,y) & \text{if } x \in \mathfrak{B}_j(r), y \in \mathfrak{B}_k(r), j \neq k. \end{cases} \tag{68}$$

Since each $\mathfrak{B}_j(r)$ is either smooth or analytically irreducible, by further decreasing r we can assume that each $\mathfrak{B}_j(r)$ is geodesically convex in (U_j, d_j) and is a

CAT(κ)–space with the distance d_j. It follows from this and (68) that geodesic segments are unique in $\mathfrak{B}(x_0, r)$. Let $T = \Delta(xyz)$ be a triangle in $\mathfrak{B}(x_0, r)$. If the three points lie in the same $\mathfrak{B}_j(r)$ the result follows from the CAT(κ)–space property of $\mathfrak{B}_j(r)$. If $x \in \mathfrak{B}_1(r)$ and $y \in \mathfrak{B}_2(r)$ and $z \in \mathfrak{B}_3(r)$, then the triangle is a tree with three edges. All the angles vanish and the angle condition is trivially satisfied. Finally assume that $x, y \in \mathfrak{B}_1(r)$ and $z \in \mathfrak{B}_2(r)$. Then $x_0 \in [x, z] \cap [y, z]$, so the angle at z vanishes, while the angles at x and y are the same as in $T' = \Delta(xy0)$. Since $T' \subset \mathfrak{B}_1(r)$ the angles in T' are smaller than the ones in the comparison triangle $\overline{T'}$. But $\overline{T}$ is obtained by "straightening" $\overline{T'}$. Thus it follows from Alexandrov lemma [7, Lemma 4.3.3 p.115] that the angle condition holds for T too. □

The argument in the last part of the proof is the same as in Reshetnyak Theorem [7, p. 316]. Our case is the simplest possible one, since the spaces are glued along a set that consists of a single point.

Theorem 9. *If (X, ω) is a Kähler curve and x_0 is a singular point, every geodesic arriving at x_0 branches into a continuum of different segments. In particular as soon as $X_{\text{sing}} \neq \emptyset$, there is no $\kappa \in \mathbb{R}$ such that (X, d) be a metric space of curvature $\geq \kappa$ (in the sense of Alexandrov).*

Proof. Assume that $\mathfrak{B}(x_0, r) = \mathfrak{B}_1(r) \cup \cdots \cup \mathfrak{B}_N(r)$ as above. If $N = 1$ the singularity is analytically irreducible and the claim is already contained in Theorem 6. If $N > 1$ fix $x \in \mathfrak{B}_1(r)$, $x \neq x_0$. For any $y \in \mathfrak{B}_2(r) \setminus \{x_0\}$ the segment from x to y passes through x_0. This proves that there infinitely many segments prolonging the segment from x to x_0. Since segments cannot branch in Alexandrov spaces with curvature bounded below it follows that no such bound can hold on (X, d). □

Remark 2. *The point of the above result is that $\inf_{X_{\text{reg}}} K$ can in fact be finite even when X contains singularities. For example consider $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^n\}$ with the Euclidean metric. A simple computation using* (25) *shows that for $n > 4$ the Gaussian curvature is bounded near* $(0, 0)$. *Nevertheless by Thm. 9 there is no lower bound in the sense of Alexandrov.*

Remark 3. *In the case of an irreducible singularity it would be interesting to understand if different segments starting at the singular point can have the same initial tangent vector. If this were not the case the map $\mathfrak{a}$ in* (59) *would be strictly increasing and $\mathfrak{F}|_C$ would be a homeomorphism of C onto $S^1 \times \{0\} \subset C_0(X)$. Its inverse would share many properties of the exponential map of a Riemannian manifold. We leave the analysis of this problem for the future.*

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