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Pricing of Stochastic Interest Bonds using Affine Term Structure Models: A Comparative Analysis

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Preface

The aim of this work is to use one-factor stochastic term structure models to evaluate stochastic interest bonds, that are bonds bundled together some interest rate derivative, and to compare them with the theoretical value that the issuer indicates in the prospectus for the public offering.

Stochastic interest bonds are a sub-set of the big family of structured bonds, the latter being bonds that present specific algorithms driving coupons computation and payment at maturity, mainly due to the presence of one or more derivative components embedded in their financial structure.

Structured bonds are mainly issued by banks. Over the last two decades the offering of structured bonds to retail investors has consistently increased, with a contextual rise in the variety of the payoff structures.

Chapter 1, after a brief exposure of the evolution of term structure models and their classification, is devoted to analyze several one-factor affine term structure models: the Vasicek model, the Ho-Lee model and the Hull-White model.

Chapter 2 shows how to use the above models to price some typical interest rate derivatives (namely caps and floors) that are often embedded in the structure of stochastic interest bonds like those that will be considered in Chapter 5, which in fact, will include either a cap or a floor or both these two types of interest rate derivatives.

Chapter 3 is devoted to analyze some key concepts about credit risk in order to take into account the impact of this risk factor on the bond value. To this aim, we will illustrate some key results regarding credit derivatives, and, specifically, credit default swaps whose market quotes allow to infer reliable estimates of the cumulative and intertemporal default probabilities of an issuer at various maturities by using the so-called bootstrapping technique. Once these default probabilities are estimated they can be used to derive a general pricing formula for defaultable bonds which will be used to perform the fair evaluation of the ten stochastic interest bonds analyzed in Chapter 5.

Chapter 4 is devoted to study in detail the financial engineering of a specific kind of stochastic interest bonds, namely the so-called *collared floaters*, which are floating-rate coupon bonds whose coupons are subject to both an upper and a lower bound, hence embedding two interest rate derivatives, either a long cap and a short cap or a long floor and a short cap depending on the specific unbundling choice we make.

In particular, the unbundling of a generic *collared floater* into its various elementary components is examined, as it will be useful to the pricing of many bonds included in the set of securities analyzed in Chapter 5.

Chapter 5 is focused on the pricing of ten stochastic interest bonds recently issued by four of the major Italian banks: six of them are pure collared floaters, two of them are mixed fixed-floating coupon bonds, whose floating coupons have the typical structure of collared floaters, one bond is a floating-rate coupon bond embedding a floor, and one bond is a floating-rate coupon bond embedding a floor for the first half of its life and a cap for the second half of its life.

After the illustration of their unbundling, these bonds are priced by means of two alternative pricing methodologies.

The first methodology is based on the unbundling of their financial structure which reveals how these bonds can be seen as the composition of one or more pure bond components and of one or more interest rate derivatives, namely caps and/or floors, whose closed formulas - in the framework of the one-factor affine term structure models of Chapter 1 developed under the risk neutral probability measure - have been presented in Chapter 2.

The second methodology relies instead on Monte Carlo simulations, performed again under the risk neutral probability measure; in this case the fair value of a bond is determined by discounting back at the evaluation date the final value of the security over each simulated trajectory and, then, by averaging these discounted values.

The two pricing methodologies are implemented both in the framework of the Vasicek model and in that of the Hull and White model.

Their results turn out to be consistent and, compared with the theoretical value indicated in the final terms of the prospectus published by the issuers, are a useful instrument to explore the reliability and the accuracy of the informative set included in this document that investors use to take their financial decisions.

Chapter 1

Term structure models

1.1 Introduction

The term structure models, or yield curve models, describe the time evolution of the yield curve, that is the curve that, for each maturity T, evaluated at date t, expresses the yield to maturity (spot rate), $r(t, T)$.

The spot rate is the rate defined at time t for a financial operation that starts at time t and terminates at time T, with $t \leq T$.

From a practical point of view, most of the term structures of the interest rates are computed from the Treasury bond prices, these kind of bonds, in general and in normal market conditions, for the developed countries, being considered risk-less.

Government bonds, like Treasury securities, are financial instruments that provide fixed and certain cash flows, coupons and principal, on a sequence of pre-specified dates.

The return rate corresponding to various maturities of these bonds can be deduced by a specific method called "bootstrapping" from the market price of the most frequently traded coupon-bearing-bonds, sometimes called "benchmark" issues.

This set of yields to maturity associated with different maturities defines the term structure of interest rates. The shape of the term structure changes over time. Most of times it is upward sloping, meaning that the return on long term bonds is greater than the return on short term bonds. The term structure can also be downward sloping; it can depend on macroeconomic state variables.

Unlike Treasury bonds, all structured bonds, like stochastic interest bonds, have payoffs that are neither fixed nor certain.

In the case of structured bonds, these payoffs depend on the future levels of interest rates that are all unknown variables at the time of the evaluation date.

As a consequence, the pricing of these bonds requires specific assumptions on the future evolution of the interest rates, which usually rely on a specific model for the dynamics of interest rates.

1.2 An historical perspective

The first research oriented to the term structure modelling has recognized the importance of the stochastic nature of interest rates and has modelled the spot rate evolution as a random walk.

On 1977 Vasicek introduced a general no arbitrage model for the pricing of zero coupon bonds and he proposed a specific model in which the instantaneous spot rate is described by a mean-reverting Ornstein-Uhlenbeck process.

On 1985 Cox, Ingersoll and Ross have shown how to use the yield curve theory in a realistic economic world and they proposed a model only for positive interest rates, based on a square root process.

In particular, their model (CIR model), as Vasicek model, is a mean reverting model for the instantaneous spot rate but here the variance is not constant. Here the variance of the short rate changes over time proportional to the level of the short rate.

Both of the above mentioned models found out a specific time evolution for the interest rate and they used economic fundamentals to describe that systematic variation of the term structure.

These kinds of models are known in literature as "Equilibrium Models" because they specify the market price of risk and they can be supported by an economic equilibrium model.

In this category there are also the models developed by Brennan and Schwartz (1979), Fong and Vasicek (1991) and Longstaff and Schwartz (1992). All these models are known as multifactor equilibrium models because they assume that the evolution of the term structure of the interest rates depends on the dynamic of more than one factor and that the yield to maturity depends on all these factors too.

Equilibrium models can be calibrated by using historical data on interest rates and bond prices and, then, they can be used to evaluate the price of both plain vanilla and structured bonds and bond options.

Often there is a *mispricing* between the price obtained from this family of models and the market price of a financial product. This problem generated the requirement of term structure models to allow a bond pricing more coherent with the term structure observed on the market.

At this point some authors developed the "no arbitrage models", i.e. models that use a no arbitrage condition to precisely define the relationship between the drift and the diffusion coefficients of the spot rate.

The first contribution in this direction comes from Ho and Lee (1986) and, later, from the models developed respectively by Hull and White (1990) and by the Black, Derman and Toy(1990).

In this "no arbitrage" category there is also the Heath, Jarrow and Morton model (1992).

It is a term structure model that depends on the evolution of the entire forward rate curve, starting from the current interest rate curve observed on the market.

The implementation of these kinds of models for the interest rate derivatives has encountered several difficulties. One of them is that the instantaneous forward rate term structure is not directly observable and then the *Heath-Jarrow-Morton Models* are difficult to apply.

To solve these problems some authors[1] developed new term structure models known as "market models" that study the observable interest rates applying over finite maturities, such as the LIBOR (*London Interbank Offered Rate*) or the swap rates, directly within the *Heath-Jarrow-Morton* framework.

In this work we will analyze in detail the equilibrium model developed by Vasicek and two no arbitrage models, namely those developed by Ho and Lee and by Hull and White.

Equilibrium and no arbitrage models have similar features[2].
The main distinction between the two categories comes from the different input that are used to calibrate the model parameters.
Equilibrium models explicitly specify the market price of risk; the model parameters, assumed constant over time, are estimated statistically from historical data.
No arbitrage models are calibrated to match the observed price on the market and the model price.

We have to point out that some equilibrium models (for example the Vasicek model) and some no arbitrage models (for example the Ho-Lee model and the Hull-White model) belong to a more general class of term structure models known as *Affine Term Structure Models*.

Affine term structure models, introduced in 1996 by Duffie and Kan[3], are models in which the yield to maturity of a zero-coupon bond is a linear function of the underlying variables.

Duffie and Kan described and analyzed a simple multifactor term structure model of the interest rates where the factors are the returns $X_1, X_2, ..., X_n$ of n zero-coupon bonds with different maturity, $T_1, T_2, ..., T_n$.

The models is called "*Affine*" because, for each maturity T, there exists an affine function $Y_T : \mathbb{R}^n \to \mathbb{R}$ such that, for each date t, the return on a zero-coupon bond with maturity T is equal to $Y_T(X_t)$.

1.3 Term structure of interest rates

The base for all the term structure models for the interest rates is the concept of the zero coupon bond.

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[1] See Brace, Gatarek and Musiela (1997), Jamshidian (1997), Miltersen, Sandmnann and Sondermann (1997), and others.

[2] As Black pointed out, this classification may be a misuse of the term, because equilibrium models like the CIR model do not admit arbitrage in the economic environment specified in the model. Moreover, arbitrage models, such as the Hull and White one, are constructed by making time-varying the coefficients of some equilibrium models

[3] See Duffie, D. e Kan, R., (1996), "A Yield-Factor Model of Interest Rates", *Mathematical Finance*, n. 4.

A zero coupon bond or pure discount bond is a bond that entitles its holders to a single certain cash flow of FV (*principal value* or *face value* or *notional amount*) in a certain future date.

Let $P(t,T)$ be the price at time t of a zero coupon bond maturing at time T with face value $FV = 1\$$ and let $r(t,T)$ be the *spot rate* over the period $[t,T]$, then, the relationship between the bond price and its spot rate is:

$$P(t,T) = e^{-r(t,T)(T-t)} \tag{1.1}$$

or, in terms of the spot rate:

$$r(t,T) = -\frac{\ln P(t,T)}{T-t} \tag{1.2}$$

meaning that the price of a zero coupon bond is the discounted value of its future cash flow.

In particular, the spot rate of instantaneous maturity, i.e. the short rate r_t, is simply the limit of $r(t,T)$ when T collapses to t:

$$r_t = \lim_{T\to t} r(t,T) \tag{1.3}$$

The continuously compounded forward rate at time t for the future period between time $T, T \geq t$, and time $T+\tau$, $\tau > 0$, is defined by the following equation for the forward price of the zero coupon bond expiring at time $T+\tau$ and denoted by $P(t,T,T+\tau)$

$$P(t,T,T+\tau) \equiv \frac{P(t,T+\tau)}{P(t,T)} = e^{-(F(t,T,T+\tau))\cdot\tau} \tag{1.4}$$

Therefore, the explicit formula for the forward rate is:

$$F(t,T,T+\tau) = -\frac{\ln P(t,T+\tau) - \ln P(t,T)}{\tau} \tag{1.5}$$

The limit of $F(t,T,T+\tau)$ for $\tau \to 0$ is the *instantaneous forward rate* that we denote with $F(t,T)$. In other terms we have:

$$F(t,T) = \lim_{\tau\to 0} F(t,T,T+\tau) \tag{1.6}$$

and, given equation 1.5, we have:

$$F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \tag{1.7}$$

or, in integral form:

$$P(t,T) = e^{-\int_t^T F(t,s)ds} \tag{1.8}$$

From equations 1.2 and 1.8 we can easily deduce the following equation for the yield to maturity of a zero coupon bond:

$$r(t,T) = \frac{\int_t^T F(t,s)\,ds}{T-t} \tag{1.9}$$

This equation shows how the yield to maturity of a zero coupon bond can be interpreted as an average of the instantaneous forward rates on the time interval corresponding to the time to maturity of the bond.

1.4 Risk neutral valuation and no-arbitrage condition

Derivative securities and structured bonds are financial products whose payoff at one or more future dates corresponds to the state of nature that has occurred at that date(s), which usually depends on the price of the underlying financial instruments or on the evolution of these prices over a given preceding time interval.

In particular the pricing of interest rate derivatives and of stochastic interest bonds depends on the dynamics of the term structure of the interest rates and it is governed by the no-arbitrage condition.

The no-arbitrage condition says that a strategy that has a positive future payoff in at least one state of nature and no negative future payoffs in all the other possible states of nature must have a current value higher than zero.

This condition implies that a contingent claim whose payoff can be replicated by a portfolio of securities should have, under the risk-neutral measure, a price equal to the value of the replicating portfolio. If this equivalence is not satisfied, it will be possible to set up an arbitrage strategy based on the difference between the two prices.

This evaluation principle based on the construction of a replication portfolio led to the Black and Scholes formula (1973) for the European option pricing on stocks and it is also the foundation of the pricing frameworks for interest rate derivatives.

The risk neutral pricing methodology requires to compute the expected value of the discounted future payoffs of a given financial instrument, using a specific probability measure, $\mathbb{P}$, known as *Equivalent Martingale Measure* or *Risk Neutral Measure.*

Intuitively, this means that, under the risk neutral measure, the rate of return on a financial security is equivalent to the istantaneous short rate, r_t.

The price at time t of a zero coupon bond with face value equal to one, i.e. $FV = 1$, and maturity T is equal to the expected value, under the risk neutral probability measure $\mathbb{P}$, of its discounted payoff, i.e.:

$$P(t,T) = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds} \cdot 1 \middle| \mathcal{F}_t \right] \tag{1.10}$$

where r_s is the instantaneous short rate at time s, $s \in [t,T]$.
From equations 1.1 and 1.10 we have the following equality:

$$e^{-r(t,T)(T-t)} = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds} \cdot 1 \middle| \mathcal{F}_t \right]$$

and then:

$$r(t,T) = -\frac{\ln\left\{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}\cdot 1\middle|\mathcal{F}_t\right]\right\}}{T-t} \tag{1.11}$$

Equation 1.11 allows to model the whole term structure of interest rates by using r_t and its risk neutral process.

Under the probability measure $\mathbb{P}$, the term $e^{-\int_t^T r_s ds}$ appearing in equation 1.10 is the discount factor that characterizes the *saving account* or *money market account.* Moreover, under $\mathbb{P}$ the price of any zero-coupon bond is a martingale, meaning that its conditional expected value at some time t, given all the observations up to some earlier time s, is equal to the value of that bond observed at the earlier time s.

1.5 Equilibrium models

Equilibrium models start from precise assumptions about the dynamics of the state variables that describe the economic conditions and model the behavior of the term structure of interest rates into such economic context.

An important aspect of these models is that they explicitly specify the market price of risk, $\lambda(t, r_t)$.

Equilibrium models can be both one factor and multifactor models.

One factor models are based on the assumption that it is sufficient to model only the behavior of one state variable to deduce the whole yield curve.

Multifactor models − as those of Brennan and Schwartz (1979), Fong and Vasicek (1991), and Longstaff and Schwartz (1992) − assume that the evolution of the term structure of the interest rates is governed by the dynamics of more than one factor.

In this work we will focus on one factor models, mainly because the empirical evidence proves that almost the 90% of the variability of the yield curve movements is determined by the movements of first explanatory variable, which is assumed to be the current level of the yield curve. As a consequence, each point on the yield curve can be used as proxy of this level.

Most of one factor models use as proxy the instantaneous short rate, r_t, and for this reason they are also known as *Short Rate Models*.

The main assumption behind *Short Rate Models* is that, under the real-world probability measure $\mathbb{Q}$, the dynamics of the instantaneous short rate follow a Markov diffusive process as:

$$dr_t = \alpha(t, r_t)\, dt + \beta(t, r_t)\, dW_t \tag{1.12}$$

where:

- the drift coefficient, $\alpha(t, r_t)$, and the diffusion coefficient, $\beta(t, r_t)$, are function of two variables, the instantaneous short rate r_t and the time t;
- W_t is a standard Brownian motion under the real-world probability measure $\mathbb{Q}$.

Equation 1.12 shows how the short rate variation can be decomposed in a drift component, $\alpha(t, r_t)$, on the time interval $(t, t+dt)$, and in a random shock component given by the product of the standard Brownian motion increment dW_t and an instantaneous volatility (so-called diffusion coefficient) equal to $\beta(t, r_t)$.

The assumption of only one risk factor is not so restrictive as it could appear. In fact, a one factor model implies that all interest rates will move in the same direction of any small time interval but not that they will move all of the same amount, and hence, the shape of the yield curve can change over time.

In these models the price at time t of a zero coupon bond that pays 1 at time T, i.e. $P(t, T)$, is a function of the instantaneous interest rate and of the time to maturity of the bond. In other words we have:

$$P(t, T) := P(r_t, t, T) \tag{1.13}$$

It is clear that by using equation 1.13 and applying the Itô's formula, we are able to derive the stochastic differential equation that describes the dynamics of the zero coupon bond price starting from the stochastic differential equation that governs the dynamics of the instantaneous short rate (i.e. equation 1.12).

In particular, in order to obtain the stochastic differential equation that describes the dynamics of the zero coupon bond price, we define the following partial derivatives:

$$\begin{aligned} \frac{\partial P(r_t, t, T)}{\partial r_t} &= P_{r_t}(r_t, t, T) \\ \frac{\partial P(r_t, t, T)}{\partial t} &= P_t(r_t, t, T) \\ \frac{\partial^2 P(r_t, t, T)}{\partial r_t^2} &= P_{r_t r_t}(r_t, t, T) \end{aligned}$$

then , by applying Itô's Lemma:

$$\begin{aligned} dP(r_t, t, T) &= \left[\alpha(t, r_t) P_{r_t}(r_t, t, T) + P_t(r_t, t, T) + \frac{1}{2}\beta^2(t, r_t) P_{r_t r_t}(r_t, t, T)\right] dt \\ &\quad + \beta(t, r_t) P_{r_t}(r_t, t, T) dW_t \end{aligned} \tag{1.14}$$

Defining the quantities $P(r_t, t, T)\mu(t, r_t)$ and $P(r_t, t, T)\sigma(t, r_t)$ as follows:

$$P(r_t, t, T)\mu(t, r_t) = \alpha(t, r_t) P_{r_t}(r_t, t, T) + P_t(r_t, t, T) + \frac{1}{2}\beta^2(t, r_t) P_{r_t r_t}(r_t, t, T)$$

$$P(r_t, t, T)\sigma(t, r_t) = \beta(t, r_t) P_{r_t}(r_t, t, T)$$

equation 1.14 can be written as:

$$dP(r_t, t, T) = P(r_t, t, T)\mu(t, r_t) + P(r_t, t, T)\sigma(t, r_t) dW_t \tag{1.15}$$

From equation 1.15 and using a no-arbitrage argument it can be shown that there exists a stochastic process for the market price of risk, $\lambda(t, r_t)$, such that:

$$\frac{\mu(t, r_t) - r_t}{\sigma(t, r_t)} = \lambda(t, r_t) \tag{1.16}$$

for any maturity T.

Equation 1.16 shows that the stochastic process $\lambda(t, r_t)$ depends on the time t and on the short rate r_t, but it doesn't depend on the maturity date T. This stochastic process is different across different models because it depends on the hypothesis about the investors' preferences and on the productivity.

Once the market price of risk is determined, we can find the risk neutral probability measure $\mathbb{P}$ linked with the real-world probability measure $\mathbb{Q}$ by the following conditions:

- $\mathbb{Q}$ and $\mathbb{P}$ are two equivalent probability measures[4];
- the quantity:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\left(\int_0^t \lambda(s,r_s)dW(s) - \frac{1}{2}\int_0^t \lambda^2(s,r_s)ds\right)} \tag{1.17}$$

  is the Radon-Nikodym derivative of the probability measure $\mathbb{P}$ with respect to the probability measure $\mathbb{Q}$.

Given the relationship between $\mathbb{P}$ and $\mathbb{Q}$ expressed in equation 1.17 , the Girsanov theorem allows to state that, if W_t is a standard Brownian motion under $\mathbb{Q}$, then the process:

$$\widetilde{W_t} = W_t - \int_0^t \lambda(s, r_s)\, ds$$

is a standard Brownian motion under $\mathbb{P}$.

At this point we can say that, under the risk neutral probability measure $\mathbb{P}$, the process for r_t evolves according to the following stochastic differential equation:

$$dr_t = \widehat{\alpha}(t, r_t)\, dt + \beta(t, r_t)\, d\widetilde{W_t} \tag{1.18}$$

where:

$$\widehat{\alpha}(t, r_t) = \alpha(t, r_t) - \lambda(t, r_t)\, \beta(t, r_t) \tag{1.19}$$

As a consequence, under the risk neutral probability measure $\mathbb{P}$, the process $P(r_t, t, T)$ is described by the following stochastic differential equation:

$$dP(r_t, t, T) = P(r_t, t, T)\, r_t dt + P(r_t, t, T)\, \sigma(t, r_t)\, d\widetilde{W_t} \tag{1.20}$$

Equation 1.20 shows how, under the risk neutral probability measure, the expected return of any zero coupon bond is equal to the risk free rate r_t.

Using equation 1.18 and applying some stochastic calculus results, namely the Feynman-Kac formula, we can determine the zero coupon bond price as

[4] Two probability measures , $\mathbb{P}$ and $\mathbb{Q}$, are said to be equivalent if:

- are defined on the same measurable space $(\Omega, \mathcal{F})$;
- $\mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \forall A \in \mathcal{F}$

the expected value, under the risk neutral probability measure $\mathbb{P}$, of the future payoff of the bond discounted back from time T to time t, as shown in equation 1.10, i.e.:

$$P(t,T) = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds} \cdot 1 \middle| \mathcal{F}_t \right] \tag{1.10}$$

1.5.1 The Vasicek model

The Vasicek model (1977) was the first term structure model with a mean reverting dynamic for the short rate.

In fact, in this model, under the real-world probability measure $\mathbb{Q}$, the short rate is described by the following Ornstein-Uhlenbeck[5] process:

$$dr_t = a(b - r_t)\,dt + \sigma dW_t \tag{1.21}$$

where:

- $a(b - r_t)$ is the drift of the stochastic process of the short rate and it is mean reverting, a measures the speed of mean reversion and b is the long-run mean to which the short rate is reverting; both these parameters are positive and constant
- W_t is a standard Brownian motion under the probability measure $\mathbb{Q}$;
- σ is the instantaneous volatility of the short rate and it is a positive constant.

The mean reverting property characterizes most of the one factor models based on the instantaneous spot rate dynamic.

Economically the mean reverting property means that, when the interest rates are too high or too low, with respect to their long run level, they will move towards this level.

Equation 1.21 implies that the instantaneous short rate has a conditional Normal probability distribution with mean and variance respectively equal to:

$$E(r_t) = b + (\overline{r} - b)\,e^{-at}$$

$$Var(r_t) = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

In this model the market price of risk is assumed constant – i.e. $\lambda(t, r_t) = \overline{\lambda}$ – and then, applying equation 1.18, we can find the stochastic process for the short rate under the risk neutral probability measure $\mathbb{P}$:

$$dr_t = a(b' - r_t)\,dt + \sigma d\widetilde{W}_t \tag{1.22}$$

[5] In general, an Ornstein-Uhlenbeck, X_t, can be expressed by the following stochastic differential equation:

$$dX_t = -qX_t dt + \sigma dW_t$$

where:
- q and σ are positive parameters
- $dW_t = \varepsilon dt$ e $\varepsilon \sim N(0,1)$ is *white noise*.

where:

$$b' = b - \frac{\overline{\lambda} \cdot \sigma}{a} \tag{1.23}$$

Equations 1.22 and 1.23 show that the instantaneous short rate process under $\mathbb{P}$ is similar to the process under $\mathbb{Q}$, as the only difference consists in the translation of the long run level of the short rate.

Given the pricing formula expressed by equation 1.10 and the evolution of the short rate pointed out by equation 1.22, the price at time t of a zero coupon bond with face value equal to 1 and maturity equal to T is, for $a > 0$:

$$P(t,T) = A(t,T)\, e^{-B(t,T)r_t} \tag{1.24}$$

where[6]:

$$A(t,T) = e^{\frac{[B(t,T)-T+t]\left(a^2 b' - \frac{\sigma^2}{2}\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}} \tag{1.25}$$

and:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{1.26}$$

Substituting the RHS of equation 1.24 into equation 1.2 we have:

$$r(t,T) = -\frac{\ln\left[A(t,T)\, e^{-B(t,T)r_t}\right]}{T-t} \tag{1.27}$$

and then:

$$\begin{aligned} r(t,T) &= -\frac{\left[\ln A(t,T) + \ln e^{-B(t,T)r_t}\right]}{T-t} \\ &= -\frac{\ln\left[A(t,T)\right]}{T-t} + \frac{B(t,T)\, r_t}{T-t} \end{aligned} \tag{1.28}$$

Once we have calibrated the parameters a, b' and σ, we can determine the entire term structure as a function of r_t and we can use the Vasicek model to compute the price of interest rate derivatives as well as to evaluate both plain vanilla and structured bonds, including the stochastic interest bonds that will be analyzed in Chapter 5.

Given equation 1.28, the yield to maturity $r(t,T)$ is a linear function of the instantaneous short rate r_t with intercept equal to $-\frac{\ln[A(t,T)]}{T-t}$ and slope equal to $\frac{B(t,T)}{T-t}$. For this reason the Vasicek model belongs to the family of *Affine term structure models.*

[6] If $a = 0$, the formulas 1.25 and 1.26 become:

$$A(t,T) = e^{\frac{\sigma^2 (T-t)^3}{6}}$$

and:

$$B(t,T) = T - t$$

The Vasicek model is consistent with term structure that can be either upward sloping, downward sloping or humped.

The model can generate negative interest rates, due to the fact that the conditional distribution of the short rate is Gaussian.

This is not necessarily a problem for real interest rates, but it is a problem when modelling nominal rates and pricing interest rate derivatives. However, it can be fixed (at least in first approximation) by imposing some suitable conditions.

1.5.2 The volatility of the short rate in the Vasicek model

For convenience, we rewrite equation 1.22 as follows:

$$dr_s = a\left(b' - r_s\right)ds + \sigma d\tilde{W}_s, \text{ where } a, \sigma > 0 \text{ and } t < s \tag{1.29}$$

In order to compute the solution for equation 1.29, given its initial condition:

$$r_t = r, t < s$$

we define the following Itô's process:

$$Y_s = (b' - r_s)e^{as} \tag{1.30}$$

To obtain the stochastic differential equation for Y_s, we compute the following partial derivatives:

$$\frac{\partial Y_s}{\partial r_s} = -e^{as}$$
$$\frac{\partial Y_s}{\partial s} = a\left(b' - r_s\right)e^{as}$$
$$\frac{\partial^2 Y_s}{\partial r_s^2} = 0$$

and then we apply the Itô's lemma:

$$\begin{aligned} dY_s &= \left[a\left(b' - r_s\right)\frac{\partial Y_s}{\partial r_s} + \frac{\partial Y_s}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 Y_s}{\partial r_s^2}\right]ds + \sigma\frac{\partial Y_s}{\partial r_s}d\tilde{W}_s \\ &= \left[-a\left(b' - r_s\right)e^{as} + a\left(b' - r_s\right)e^{as}\right]ds - \sigma e^{as}d\tilde{W}_s \\ &= -\sigma e^{as}d\tilde{W}_s \end{aligned}$$

or in integral form:

$$Y_s = (b' - r_s)e^{as} = (b' - r_t)e^{at} - \sigma\int_t^s e^{au}d\tilde{W}_u$$

from which we find out the following expression for the instantaneous short rate:

$$r_s = b' - (b' - r)e^{a(t-s)} + \sigma\int_t^s e^{a(u-s)}d\tilde{W}_u \tag{1.31}$$

The term $e^{a(u-s)}$ inside the integrating function in the third term of the RHS of equation 1.31 is a deterministic function and then we can exploit one of the properties of the Itô's integral[7] and say that, given a fixed s, r_s has a conditional variance[8] equal to:

$$\begin{aligned} Var(r_s|r_t) &= Var\left(\sigma\int_t^s e^{a(u-s)}d\tilde{W}_u\right) \\ &= \sigma^2\int_t^s e^{2a(u-s)}du \\ &= \sigma^2\left(\frac{e^{2a(s-s)}}{2a}-\frac{e^{2a(t-s)}}{2a}\right) \\ &= \frac{\sigma^2}{2a}(1-e^{-2a(s-t)}) \end{aligned} \tag{1.32}$$

1.5.3 Valuation of european options on zero-coupon bonds in the Vasicek model

Let us consider an european put option, with strike price K and maturity T, written on a zero-coupon bond with face value equal to 1 and maturity $s > T$.

The price of this option at time $t < T < s$ is:

$$p_t^{zcb} = K\cdot P(t,T)\cdot N(-d_2) - P(t,s)\cdot N(-d_1) \tag{1.33}$$

where:

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right)+\frac{\sigma_p^2}{2}}{\sigma_p}$$

$$d_2 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right)-\frac{\sigma_p^2}{2}}{\sigma_p} = d_1-\sigma_p$$

and where[9]:

$$\sigma_p = \sigma\left(\frac{1-e^{-a(s-T)}}{a}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}} \tag{1.34}$$

In order to prove the validity of equation 1.33, we have to prove that the put option value at time t is equal to the conditional expected value of its payoff at maturity, under the risk neutral probability measure $\mathbb{P}$, discounted at the

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[7] Given a stochastic integral: $I_t = \int_0^t f(u,\omega)dW_u(\omega)$, if $f(\cdot,\omega) = f(\cdot)$ - i.e. if f is a deterministic function - the following relations are true:

$$E(I_t)=0 \;\;;\;\; E\left[(I_t)^2\right] = E\left[\left(\int_0^T f(t,\omega)dW_t(\omega)\right)^2\right] = E\left[\int_0^T f(t,\omega)^2 dt\right] = Var(I_t).$$

See. Øksendal, B., (2003), "Stochastic Differential Equations", *Springer*, pages. 26-29.

[8] I.e. conditional variance to the information set at time t and then to the initial condition $r_t = r$.

[9] See. Appendx A.1 of this Chapter

risk free rate. Because the price of the zero-coupon bond at time T - i.e. the underlying value at maturity - is $P(T,s)$, the price at time t of such option is:

$$p_t^{zcb} = P(t,T) \cdot E_t^{\mathbb{P}}[\max(K - P(T,s), 0)] \tag{1.35}$$

where the value $P(t,T)$ in the RHS of equation 1.35 is the price at time t of a zero-coupon bond with $FV = 1$ and maturity equal to T.

By exploiting the property of the maximum and minimum functions according to which:
$\min(f(x), d(x)) \equiv -\max(-f(x), -d(x))$
equation 1.35 becomes:

$$\begin{aligned} p_t^{zcb} &= P(t,T) \cdot E_t^{\mathbb{P}}[-\min(P(T,s) - K, 0)] \\ &= -P(t,T) \cdot E_t^{\mathbb{P}}[\min(P(T,s) - K, 0)] \end{aligned} \tag{1.36}$$

Let $g(P(T,s))$ be the probability density function of $P(T,s)$. We have that:

$$p_t^{zcb} = -P(t,T) \cdot \int_{-\infty}^{K} (P(T,s) - K) g(P(T,s)) dP(T,s) \tag{1.37}$$

Being $P(T,s)$ a lognormal random variable[10], the variable $\ln P(T,s)$ is conditionally distributed as a normal random variable with standard deviation equal to σ_p, whose value is expressed in equation 1.34. Given the lognormal distribution properties[11], the conditional expected value of $\ln P(T,s)$ is:

$$E_t^{\mathbb{P}}(\ln P(T,s)) = \ln E_t^{\mathbb{P}}(P(T,s)) - \frac{\sigma_p^2}{2} \tag{1.38}$$

[10] Being $P(T,s) = A(T,s)e^{-B(T,s)r_T}$ (see 1.24) and having the instantaneous short rate r_t a conditional normal probability distribution (see. § 1.5.1), we can conclude that $P(T,s)$ has a lognormal distribution.

[11] Let X be a lognormal random variable with density function:

$$f(x) = \begin{cases} \frac{1}{\delta\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \gamma}{\delta}\right)^2} & \text{for } x > 0 \\ 0 & \text{oherwise} \end{cases}$$

whose expected value is $E(X) = e^{\gamma + \frac{\delta^2}{2}}$ and whose variance is $Var(X)$. Then, the random variable $Y = \ln X$ has a normal probability distribution with expected value $E(Y) = E(\ln X) = \ln E(X) - \frac{Var(Y)}{2}$ and variance $Var(Y)$. In fact we have:

$$E(X) = e^{\gamma + \frac{\delta^2}{2}}$$

hence:

$$\ln E(X) = \gamma + \frac{\delta^2}{2}$$

expliciting by γ:

$$\gamma = \ln E(X) - \frac{\delta^2}{2}$$

where $\gamma = E(Y)$ and $\delta^2 = Var(Y)$.

By the martingale property of the zero coupon bond price, the conditional expected value of the spot price $P(T, s)$, evaluated at time t, with $t < T < s$, corresponds to the forward price $P(t, T, s)$, and therefore equation 1.38 becomes:

$$E_t^{\mathbb{P}}(\ln P(T, s)) = \ln P(t, T, s) - \frac{\sigma_p^2}{2} \tag{1.39}$$

Using the definition of the forward price from which[12]:

$$P(t, T, s) = \frac{P(t, s)}{P(t, T)}$$

the expected value of $\ln P(t, T, s)$, shown in equation 1.39, can be expressed as:

$$E_t^{\mathbb{P}}(\ln P(T, s)) = \ln \frac{P(t, s)}{P(t, T)} - \frac{\sigma_p^2}{2} \tag{1.40}$$

We now define a new random variable Q, obtained by standardizing the normal random variable $\ln P(T, s)$:

$$Q = \frac{\ln P(T, s) - E_t^{\mathbb{P}}(\ln P(T, s))}{\sigma_p} \tag{1.41}$$

Then, Q has a standard normal distribution whose probability density function $h(Q)$ is:

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} \tag{1.42}$$

Solving equation 1.41 for $P(T, s)$ we have:

$$P(T, s) = e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} \tag{1.43}$$

Using equations 1.41 and 1.43 to transform the integral in $P(T, s)$ appearing in the RHS of equation 1.37 into an integral in Q, we obtain:

$$\begin{aligned}
p_t^{zcb} &= -P(t, T) \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} \left(e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} - K \right) h(Q) dQ \\
&= -P(t, T) \left(\int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} h(Q) dQ + \right. \\
&\quad \left. -K \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q) dQ \right)
\end{aligned} \tag{1.44}$$

[12] See. equation 1.4, with $T + \tau = s$.

Substituting the value of $h(Q)$ given in equation 1.42, the first term in the RHS of equation 1.44 becomes:

$$\begin{aligned} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} h(Q) &= \frac{1}{\sqrt{2\pi}} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s)) - \frac{Q^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-Q^2 + 2Q\sigma_p + 2E_t^{\mathbb{P}}(\ln P(T,s))}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-(Q-\sigma_p)^2 + 2E_t^{\mathbb{P}}(\ln P(T,s)) + \sigma_p^2}{2}} \\ &= e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-(Q-\sigma_p)^2}{2}} \end{aligned} \tag{1.45}$$

From equation 1.42, we see that the quantity $\frac{1}{\sqrt{2\pi}} e^{\frac{-(Q-\sigma_p)^2}{2}}$ is the probability density function of the random variable $(Q - \sigma_p)$, whose conditional distribution is a normal with parameters $(-\sigma_p, 1)$. Therefore, equation 1.45 can be written as:

$$e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} h(Q) = e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} h(Q - \sigma_p)$$

and, hence, equation 1.44 becomes:

$$\begin{aligned} p_t^{zcb} &= -P(t,T) \left\{ e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q - \sigma_p) dQ + \right. \\ &\quad \left. -K \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q) dQ \right\} \\ &= -P(t,T) \left\{ e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} N\left(\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p} - \sigma_p \right) + \right. \\ &\quad \left. -KN\left(\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p} \right) \right\} \end{aligned} \tag{1.46}$$

where $N(x)$ is a standard normal random variable.
Substituting the value of $E_t^{\mathbb{P}}(\ln P(T,s))$ given from 1.40, equation 1.46 becomes:

$$p_t^{zcb} = -P(t,T) \left(e^{\ln \frac{P(t,s)}{P(t,T)} - \frac{\sigma_p^2}{2} + \frac{\sigma_p^2}{2}} N\left(\frac{\ln K - \ln \frac{P(t,s)}{P(t,T)} + \frac{\sigma_p^2}{2}}{\sigma_p} - \sigma_p \right) + \right.$$
$$\left. -KN\left(\frac{\ln K - \ln \frac{P(t,s)}{P(t,T)} + \frac{\sigma_p^2}{2}}{\sigma_p} \right) \right)$$
$$= -P(t,T) \left(\frac{P(t,s)}{P(t,T)} N\left(\frac{\ln\left(\frac{K \cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} - \sigma_p \right) + \right.$$
$$\left. -KN\left(\frac{\ln\left(\frac{K \cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \right) \right)$$

and finally:

$$p_t^{zcb} = -P(t,s)\cdot N\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p}\right) + K\cdot P(t,T)\cdot N\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}\right) \tag{1.47}$$

Denoting by d_1 and d_2 the quantities:

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}$$

$$d_2 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p} = d_1 - \sigma_p$$

equation 1.47 becomes[13]:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - P(t,s) \cdot N(-d_1) \tag{1.93}$$

We can see that equation 1.93 is similar to the traditional Black-Scholes formula. In fact, under the Black-Scholes formula, the price at time t, under the risk neutral probability measure $\mathbb{P}$, of a put option written on a stock is:

$$p_t^{BS} = K \cdot e^{-r(t,T)(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1) \tag{1.51}$$

where:

$$d_1 = \frac{\ln\frac{S_t}{K} + (r_t + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\frac{S_t}{K} + (r_t - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

and S_t is the price of the underlying asset at time t.

In both cases the price of the underlying asset has a lognormal distribution and σ_p^2 has the same role of $\sigma^2(T-t)$, that represents the conditional variance of the logarithm of the stock price at maturity. The price $P(t,T)$ corresponds

[13] Notice that, if the underlying zero coupon bond has face value different from one, the formula 1.93 to compute p_t^{zcb} changes as follows:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2^*) - FV \cdot P(t,s) \cdot N(-d_1^*) \tag{1.48}$$

where:

$$d_1^* = \frac{\ln\left(\frac{FV\cdot P(t,s)}{K\cdot P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \tag{1.49}$$

$$d_2^* = \frac{\ln\left(\frac{FV\cdot P(t,s)}{K\cdot P(t,T)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p} = d_1^* - \sigma_p \tag{1.50}$$

to the discount factor $e^{-r(t,T)(T-t)}$, in which $r(t,T)$ is the risk neutral interest rate[14].

Using a procedure similar to that one used to determine the price of a put option, we can find out the formula for the evaluation of an european call option with strike price K and maturity T, written on a zero-coupon bond with face value equal to one and maturity $s > T$, starting from the equality:

$$P(t,T) \cdot E_t^{\mathbb{P}}[\max(P(T,s)-K,0)] = P(t,T) \cdot \int_K^{+\infty} (P(T,s)-K) g(P(T,s)) dP(T,s)$$

and obtaining the call price c_t^{zcb} expressed as[15]:

$$c_t^{zcb} = P(t,s) \cdot N(d_1) - K \cdot P(t,T) \cdot N(d_2) \tag{1.53}$$

1.6 No arbitrage models

Equilibrium models may be derived from some equilibrium framework which would preclude the existence of arbitrage in the specified economy.

These models usually calibrated with historical data. This approach is not practical for pricing interest rate derivatives because, it will not guarantee that the model term structure matches the current term structure obtained from market prices.
For this reason, significant researches have been done to make one factor models matching the current yield curve before they are used to price interest rate derivatives.

One way to match the current term structure is to allow to the coefficient in a factor model to vary deterministically over time.
This type of models, known as no arbitrage models, takes the market price of bonds as given and prices interest rate derivatives accordingly.

We proceed to analyze two important models of this category, the Ho-Lee model (1986) and the Hull-White model (1991).

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[14] Recall that, in a risk neutral world, the expected return of any financial asset is equal to the risk-free rate.

[15] As for the put case, notice that, if the underlying zero-coupon bond has a face value different from one, the formula 1.94 to compute c_t^{zcb} changes as follows:

$$c_t^{zcb} = FV \cdot P(t,s) \cdot N(d_1^*) - K \cdot P(t,T) \cdot N(d_2^*) \tag{1.52}$$

where:

$$d_1^* = \frac{\ln\left(\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \tag{1.49}$$

$$d_2^* = \frac{\ln\left(\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p} = d_1^* - \sigma_p \tag{1.50}$$

1.6.1 The Ho and Lee model

In 1986 Ho and Lee published a one factor term structure model for the interest rates where the explanatory factor is, as in the Vasicek model, the instantaneous short rate. Starting from the assumption that the short rate follows a random walk, this model specifies the stochastic process for r_t under the risk neutral probability measure $\mathbb{P}$ as follows:

$$dr_t = \theta(t)\, dt + \sigma d\widetilde{W}_t \tag{1.54}$$

where:

- $\theta(t)$ is the drift of the short rate process and it is a deterministic function of time;
- $\widetilde{W}_t$ is a standard Brownian motion under the risk neutral probability measure $\mathbb{P}$;
- σ is the instantaneous standard deviation of the short rate and it is constant.

In this model $\theta(t)$ is the expected direction of the short rate r_t movement and it doesn't depend on the level of r_t. Equation 1.54 shows that at any time t the expected variation of the interest rates in the immediately following infinitesimal time interval is always the same, no matter if interest rates are high or low.

Computing analytically the variable $\theta(t)$ we find the following equality:

$$\theta(t) = F_t(0,t) + \sigma^2 t \tag{1.55}$$

where $F_t(0,t)$ is the partial derivative with respect to t of the instantaneous forward rate $F(0,t)$ observed at time zero for the maturity t.
In a first approximation $\theta(t)$ is equal to $F_t(0,t)$, meaning that the expected variation of the short rate is approximately equal to the slope of the instantaneous forward rate curve.

Given the pricing formula 1.10 and the short rate dynamic expressed in equation 1.54, the price at time t of a zero-coupon bond with face value equal to 1 and maturity T is:

$$P(t,T) = A(t,T)\, e^{-r_t(T-t)} \tag{1.56}$$

where:

$$\ln[A(t,T)] = \ln\left[\frac{P(0,T)}{P(0,t)}\right] - (T-t)\frac{\partial \ln[P(0,t)]}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2 \tag{1.57}$$

In equations 1.56 and 1.57 , the current time is zero and the times t and T are general future times with $T \geq t > 0$.

Substituting equation 1.57 into equation 1.56 we can find the explicit formula for the price of a bond in the Ho-Lee model, i.e.:

$$P(t,T) = P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot e^{-\left[(T-t)\frac{\partial \ln[P(0,t)]}{\partial t} + \frac{1}{2}\sigma^2 t(T-t)^2 + r_t(T-t)\right]} \qquad (1.58)$$

Given equation 1.58 the zero-coupon bond price, at a given future time t, is a function of the short rate that will be observed at time t, of the istantaneous forward rate $F(0,t)$[16] and of the market prices, at time zero, of the zero-coupon bonds with maturity t and T.

We point out that, substituting the RHS of equation 1.56 into equation 1.2 we have:

$$r(t,T) = -\frac{\ln\left[A(t,T)\,e^{-B(t,T)r_t}\right]}{T-t} \qquad (1.59)$$

and then:

$$\begin{aligned} r(t,T) &= -\frac{\left[\ln A(t,T) + \ln e^{-B(t,T)r_t}\right]}{T-t} \qquad (1.60) \\ &= -\frac{\ln[A(t,T)]}{T-t} + \frac{B(t,T)\,r_t}{T-t} \end{aligned}$$

Given equation 1.60 the yield to maturity $r(t,T)$ is a linear function of the instantaneous short rate r_t with intercept equal to: $-\frac{\ln[A(t,T)]}{T-t}$ and slope equal to: $\frac{B(t,T)}{T-t}$. For this reason the Ho-Lee model belongs to the *Affine term structure Models*.

Through a "discretization" of equation 1.57, we can compute a discrete time pricing formula for a zero-coupon bond.
Let Δt be a very short time interval, for example one day, and let $R(t)$ be the continuously compounded interest rate relative to this time interval.
Then, from equation 1.56 we can derive the following expression[17]:

$$P(t,T) = \widehat{A}(t,T)\,e^{-R(t)(T-t)} \qquad (1.61)$$

where:

$$\ln\left[\widehat{A}(t,T)\right] = \ln\left[\frac{P(0,T)}{P(0,t)}\right] - \frac{(T-t)}{\Delta t}\frac{\ln[P(0,t+\Delta t)]}{P(0,t)} - \frac{1}{2}\sigma^2 t\,(T-t)\,[T-t-(\Delta t)] \qquad (1.62)$$

Equation 1.61 is more used than equation 1.56 because equations 1.61 and 1.62 require only to know the zero-coupon bond prices at time zero.

Moreover, it is obvious that, since the quantity Δt is negligible, the term $\frac{1}{2}\sigma^2 t\,(T-t)\,[T-t-(\Delta t)]$ in the RHS of the 1.62 can be approximated to the quantity: $\frac{1}{2}\sigma^2 t\,(T-t)^2$, and then, equations 1.61 and 1.62 become respectively:

$$P(t,T) = \widehat{A}'(t,T)\,e^{-R(t)(T-t)} \qquad (1.63)$$

[16] See equation 1.7 of section 1.3.

[17] See. Hull, J., (2008), "Options, Futures, and Other Derivatives", *Prentice Hall*, pages. 654-655

and:

$$\ln\left[\widehat{A}'(t,T)\right] = \ln\left[\frac{P(0,T)}{P(0,t)}\right] - \frac{(T-t)}{\Delta t}\ln\frac{[P(0,t+\Delta t)]}{P(0,t)} - \frac{1}{2}\sigma^2 t\,(T-t)^2 \quad (1.64)$$

A drawback of the Ho-Lee model is that it is not a mean-reverting model, since, as shown by equation 1.54, independently of the interest rates level, the mean direction of the istantaneous short rate in the immediately following infinitesimal time interval is always the same.

Another inconvenience of this model is that it allows to represent a reduced set of volatility structures.
In particular:

1. the volatility at time t of a zero-coupon bond with maturity T is a linear function of T;

2. the instantaneous standard deviation at time t of the spot rate of return of a zero-coupon bond with maturity T is constant;

3. the instantaneous standard deviation of the instantaneous forward rate with maturity T is constant.

The Ho and Lee model can generate negative interest rates, due to the fact that the conditional distribution of the short rate is Gaussian.

This is not necessarily a problem for real interest rates, but it is a problem when modelling nominal rates and pricing interest rate derivatives. However, it can be fixed (at least in first approximation) by imposing some suitable conditions.

1.6.2 The volatility of the short rate in the Ho and Lee model

We rewrite the stochastic process of r_t, under the risk neutral probability measure $\mathbb{P}$, for the Ho and Lee model as:

$$dr_s = \theta(s)\,ds + \sigma d\tilde{W}_s \quad (1.65)$$

with the initial condition:

$$r_t = r, t < s$$

The solution of equation 1.65 is:

$$r_s = r_t + \int_t^s \theta(s)\,ds + \sigma\int_t^s d\tilde{W}_u \quad (1.66)$$

The first two terms of equation 1.66 are deterministic functions and, therefore, the conditional variance of the short rate is equal to:

$$\begin{aligned} Var(r_s) &= Var\left(\sigma \int_t^s d\tilde{W}_u\right) \\ &= \sigma^2 Var\left(\int_t^s d\tilde{W}_u\right) \\ &= \sigma^2 (s-t) \end{aligned} \tag{1.67}$$

1.6.3 Valuation of european options on zero-coupon bonds in the Ho-Lee model

The evaluation of european options on zero coupon bonds in the Ho and Lee model[18] is based on a formula quite similar to that derived in Black model [19].

Let us consider an european put option, with strike price K and maturity T, written on a zero-coupon bond with face value equal to 1 and maturity $s > T$.

The price at time t of this put option is denoted with p_t^{zcb} and it is equal to[20]:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - P(t,s)\, N(-d_1) \tag{1.68}$$

where:

- $P(t,T)$ is the price at time t of a zero coupon bond with maturity T, and it is an input required by the model;
- $P(t,s)$ is the price at time t of a zero coupon bond with maturity $s > T$, and it is an input required by the model;
- $N(x)$ is the value in x of the standard normal distribution function;

and the quantities d_1 and d_2 are respectively given by:

$$d_1 = \frac{1}{\sigma_p} \ln\left[\frac{P(t,s)}{K \cdot P(t,T)}\right] + \frac{\sigma_p}{2} \tag{1.69}$$

[18] See. Jamshidian, F., 1989, "An Exact Bond Option Pricing Formula", *Journal of Finance* n. 44.

[19] See Appendix B.1. of this Chapter

[20] If the underlying zero coupon bond has a face value different from one, the 1.68, the 1.69 and the 1.70 changes as follows:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - FV \cdot P(t,s)\, N(-d_1)$$

$$d_1 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right] + \frac{\sigma_p}{2}$$

$$d_2 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right] - \frac{\sigma_p}{2}$$

and:

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{P(t,s)}{K \cdot P(t,T)} \right] - \frac{\sigma_p}{2} \tag{1.70}$$

where:

$$\sigma_p = \sigma(s-T)\sqrt{T-t} \tag{1.71}$$

Analogously, the price at time t of an european call option with strike K and maturity T written on a zero coupon bond with face value equal to 1 and maturity $s, s > T$, is denoted with c_t^{zcb} and it is equal to[21]:

$$c_t^{zcb} = P(t,s) \cdot N(d_1) - K \cdot P(t,T) \cdot N(d_2) \tag{1.72}$$

where:

- $P(t,T)$ is the price at time t of a zero coupon bond with maturity T, and it is an input required by the model;
- $P(t,s)$ is the price at time t of a zero coupon bond with maturity $s > T$, and it is an input required by the model;
- $N(x)$ is the value in x of the standard normal distribution function;

and the quantities d_1 and d_2 are given by the same equations seen for the european put option, i.e. equations 1.69 and 1.70.

1.6.4 The Hull and White model

In 1990 Hull and White published an extension of the Vasicek model in which the short rate process is mean reverting as in the Vasicek model and it is consistent with the initial term structure of interest rates.

In this model, under the risk neutral probability measure, $\mathbb{P}$, the instantaneous short rate dynamics are governed by the following stochastic differential equation:

$$dr_t = [\theta(t) - ar_t]\, dt + \sigma d\widetilde{W}_t \tag{1.73}$$

or:

$$dr_t = a \left[\frac{\theta(t)}{a} - r_t \right] dt + \sigma d\widetilde{W}_t \tag{1.74}$$

where:

[21] If the underlying zero coupon bond has a face value different from one, the 1.72, the 1.69 and the 1.70 changes as follows:

$$c_t^{zcb} = P(t,s) \cdot N(d_1) - K \cdot P(t,T) \cdot N(d_2)$$

$$d_1 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot P(t,s)}{P(t,T) \cdot K} \right] + \frac{\sigma_p}{2}$$

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot P(t,s)}{P(t,T) \cdot K} \right] - \frac{\sigma_p}{2}$$

- $a\left[\frac{\theta(t)}{a}-r_t\right]$ is the drift of the stochastic process of the short rate and it is mean reverting; a is the constant speed of mean reversion and $\frac{\theta(t)}{a}$ is a function representing the long run level of the istantaneous short rate. This means that at the generic time t the instantaneous short rate goes to $\theta(t)/a$ with speed equal to a;

- $\widetilde{W}_t$ is a standard Brownian motion under the risk neutral probability measure $\mathbb{P}$;

- σ is the instantaneous standard deviation of the short rate and it is constant.

We have to observe that, as in the Vasicek model, also in the Hull and White model the short rate drift is mean reverting but in this specification the long run level of the instantaneous short rate is a deterministic function of the time.

Analytically computing the variable $\theta(t)$, under no-arbitrage condition the following equation holds:

$$\theta(t)=F_t(0,t)+aF(0,t)+\frac{\sigma^2}{2a}\left(1-e^{-2at}\right) \tag{1.75}$$

where $F_t(0,t)$ is the partial derivative with respect to t of the instantaneous forward rate $F(0,t)$ observed at time zero for the maturity t.

The first two terms in the RHS of equation 1.75 show that, since $\theta(t)$ is a function of the initial term structure of istantaneous forward rates, the specification of the Hull and White is consistent with the initial term structure observed in the market.
Moreover, the last term in equation 1.75 is negligible, so that the drift of the process r_t at time t is approximately equal to: $F_t(0,t)+aF(0,t)$.

At this point we have that, in average, the short rate follows approximately the slope of the initial instantaneous forward rate curve and, if it is faraway from that level, it will move towards it with a speed equal to a.

Also we can observe that the specification in equation 1.73 include the Ho and Lee model as a particular case when the parameter $a=0$.

The stochastic integral corresponding to the stochastic differential equation 1.73 can be expressed as[22]:

$$r_t=x_t+\alpha_t \tag{1.76}$$

where x_t is a Gaussian stochastic process described by the following stochastic differential equation:

$$dx_t=-ax_tdt+\sigma d\widetilde{W}_t \tag{1.77}$$

and α_t is the following deterministic function:

$$\alpha_t=F(0,t)+\frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2 \tag{1.78}$$

[22] See. Brigo, Mercurio, 2006, "Interest Rate Models - Theory and Practice", *Springer*, pages. 72-74

By substituting the RHS of equation 1.7, i.e.:

$$F(0,t) = -\frac{\partial \ln P(0,t)}{\partial t} \tag{1.7}$$

equation 1.78 can be also written as:

$$\alpha_t = -\frac{\partial \ln P(0,t)}{\partial t} + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 \tag{1.79}$$

The decomposition exhibited in equation 1.76 has two advantages:

- there exist simple discretization formulas to simulate the stochastic process of x_t;
- it allows to get rid of the first derivative of the forward curve appearing in the RHS of equation 1.75.

Given equations 1.10 and 1.73 (or 1.74), the price at time t of a zero-coupon bond with face value 1and maturity T is:

$$P(t,T) = A(t,T)\, e^{-B(t,T)r_t} \tag{1.80}$$

where:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{1.81}$$

and

$$\ln\left[A(t,T)\right] = \ln\left[\frac{P(0,T)}{P(0,t)}\right] - B(t,T)\frac{\partial \ln\left[P(0,t)\right]}{\partial t} - \frac{\sigma^2 \cdot B(t,T)^2 \cdot \left(1 - e^{-2at}\right)}{4a} \tag{1.82}$$

Equations 1.73, 1.75 and 1.80 define the price at time t of a zero coupon bond with maturity T as function of r_t and of the *zero-coupon bond* prices with respectively maturity t and T.

Substituting the 1.81 and the 1.82 into the 1.80 we find the explicit formula for the bond pricing with the Hull and White model, i.e.:

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} \cdot e^{-\left(\frac{1-e^{-a(T-t)}}{a}\right)\frac{\partial \ln[P(0,t)]}{\partial t}} \cdot \\ &\quad \cdot e^{-\frac{\sigma^2\left(\frac{1-e^{-a(T-t)}}{a}\right)^2\left(1-e^{-2at}\right)}{4a} - \left(\frac{1-e^{-a(T-t)}}{a}\right)r_t} \end{aligned} \tag{1.83}$$

Given equation 1.83 the zero-coupon bond price, at a given future time t,is a function of the short rate that will be observed at time t, of the istantaneous forward rate $F(0,t)$[23] and of the market prices, at time zero, of the zero-coupon bonds with maturity t and T.

[23] See equation 1.7 of section 1.3.

Substituting the RHS of equation 1.80 into equation 1.2 we have:

$$r(t,T) = -\frac{\ln\left[A(t,T)\,e^{-B(t,T)r_t}\right]}{T-t} \tag{1.84}$$

and then:

$$\begin{aligned} r(t,T) &= -\frac{\left[\ln A(t,T) + \ln e^{-B(t,T)r_t}\right]}{T-t} \\ &= -\frac{\ln\left[A(t,T)\right]}{T-t} + \frac{B(t,T)\,r_t}{T-t} \end{aligned} \tag{1.85}$$

Given equation 1.85, the yield to maturity $r(t,T)$ is a linear function of the instantaneous short rate r_t with intercept equal to: $-\frac{\ln[A(t,T)]}{T-t}$ and slope equal to:$\frac{B(t,T)}{T-t}$. For this reason the Hull and White model belongs to the family of *Affine term structure Models*.

The volatility structure in the Hull and White model depends both on a and σ. This model allows to represent a large set of volatility structures. In particular:

1. the volatility at time t of a zero-coupon bond with maturity T is:

$$\frac{\sigma}{a}\left[1 - e^{-a(T-t)}\right]$$

2. the instantaneous standard deviation at time t of the interest rate in a zero-coupon bond with maturity T is:

$$\frac{\sigma}{a(T-t)}\left[1 - e^{-a(T-t)}\right] \tag{1.86}$$

3. the instantaneous standard deviation of the instantaneous forward rate with maturity T is:

$$\sigma \cdot e^{-a(T-t)} \tag{1.87}$$

Both in equation 1.86 and in equation 1.87, the parameter a determines the rate at which the standard deviation decreases if the maturity goes up. The greater is a and the faster is the decline.

As the Ho and Lee model, also the Hull-White model can be "discretized" to find a zero-coupon bond pricing formula in discrete time equivalent to equation 1.80.
Let Δt be a very short time interval, for example one day, and let $R(t)$ be the continuously compounded interest rate relative to this time interval. Then, from equation 1.80 it is possible to derive the following expression [24]:

$$P(t,T) = \widehat{A}(t,T)\,e^{-\widehat{B}(t,T)R(t)} \tag{1.88}$$

[24] See. Hull, J., 2008, "Options, Futures, and Other Derivatives", *Prentice Hall*, pages. 656-657

where:

$$\widehat{B}(t,T) = \frac{B(t,T)}{B(t,t+\Delta t)}\Delta t \tag{1.89}$$

and

$$\begin{aligned}\ln\left[\widehat{A}(t,T)\right] &= \ln\left[\frac{P(0,T)}{P(0,t)}\right] - \frac{B(t,T)}{B(t,t+\Delta t)}\ln\frac{[P(0,t+\Delta t)]}{P(0,t)} + \\ &-\frac{\sigma^2}{4a}\left(1-e^{-2at}\right)B(t,T)\left[B(t,T)-B(t,t+\Delta t)\right]\end{aligned} \tag{1.90}$$

Equation 1.88 is more used than equation 1.80 because equations 1.88, 1.89 and 1.90 require only to know the zero-coupon bond prices at time zero.

The Hull and White model can generate negative interest rates, due to the fact that the conditional distribution of the short rate is Gaussian.

This is not necessarily a problem for real interest rates, but it is a problem when modelling nominal rates and pricing interest rate derivatives. However, it can be fixed (at least in first approximation) by imposing some suitable conditions.

1.6.5 The volatility of the short rate in the Hull and White model

For convenience, we rewrite equation 1.74 as follows:

$$dr_s = a\left(\frac{\theta(s)}{a} - r_s\right)ds + \sigma d\tilde{W}_s \quad \text{with } a,\sigma > 0 \tag{1.91}$$

In order to compute the solution of equation 1.91, given its initial condition:

$$r_t = r, t < s$$

we define the following Itô's process:

$$Y_s = \left(\tfrac{\theta(s)}{a} - r_s\right)e^{as} \tag{1.92}$$

To obtain the stochastic differential equation of Y_s, we compute the following partial derivatives:

$$\begin{aligned}\frac{\partial Y_s}{\partial r_s} &= -e^{as}\\ \frac{\partial Y_s}{\partial s} &= a\left(\frac{\theta(s)}{a} - r_s\right)e^{as}\\ \frac{\partial^2 Y_s}{\partial r_s^2} &= 0\end{aligned}$$

and then we apply the Itô's lemma:

$$\begin{aligned} dY_s &= \left[a\left(\frac{\theta(s)}{a}-r_s\right)\frac{\partial Y_s}{\partial r_s}+\frac{\partial Y_s}{\partial s}+\frac{\sigma^2}{2}\frac{\partial^2 Y_s}{\partial r_s^2}\right]ds+\sigma\frac{\partial Y_s}{\partial r_s}d\tilde{W}_s \\ &= \left[-a\left(\frac{\theta(s)}{a}-r_s\right)e^{as}+a\left(\frac{\theta(s)}{a}-r_s\right)e^{as}\right]ds-\sigma e^{as}d\tilde{W}_s \\ &= -\sigma e^{as}d\tilde{W}_s \end{aligned}$$

or, in integral form:

$$Y_s=\left(\tfrac{\theta(s)}{a}-r_s\right)e^{as}=a\left(\tfrac{\theta(s)}{a}-r\right)e^{at}-\sigma\int_t^s e^{au}d\tilde{W}_u \tag{1.93}$$

multiplying both terms of the previous equation by e^{-as} we obtain:

$$\left(\tfrac{\theta(s)}{a}-r_s\right)e^{as}e^{-as}=a\left(\tfrac{\theta(s)}{a}-r\right)e^{at}e^{-as}-\sigma e^{-as}\int_t^s e^{au}d\tilde{W}_u$$

simplifying and solving for r_s we have:

$$r_s=\frac{\theta(s)}{a}-\left(\frac{\theta(s)}{a}-r\right)e^{a(t-s)}+\sigma\int_t^s e^{a(u-s)}d\tilde{W}_u \tag{1.94}$$

The term $e^{a(u-s)}$ inside the integrating function in the third term of the RHS of equation 1.94 is a deterministic function and then we can exploit one of the properties of the Itô's integral[25] and say that, given a fixed s, r_s is conditionally normal distributed with conditional expected value and conditional variance[26], respectively equal to:

$$E(r_s)=\frac{\theta(s)}{a}-\left(\frac{\theta(s)}{a}-r\right)e^{-a(s-t)} \tag{1.95}$$

$$\begin{aligned} Var(r_s) &= Var\left(\sigma\int_t^s e^{a(u-s)}d\tilde{W}_u\right) \\ &= \sigma^2\int_t^s e^{2a(u-s)}du \\ &= \sigma^2\left(\frac{e^{2a(s-s)}}{2a}-\frac{e^{2a(t-s)}}{2a}\right) \\ &= \frac{\sigma^2}{2a}(1-e^{-2a(s-t)}) \end{aligned} \tag{1.96}$$

[25] Given a stochastic integral $I_t=\int_0^t f(u,\omega)dW_u(\omega)$, if $f(\cdot,\omega)=f(\cdot)$ - i.e. if f is a deterministic function - the following relations are true:

$$E(I_t)=0 \quad ; \quad E\left[(I_t)^2\right]=E\left[\left(\int_0^T f(t,\omega)dW_t(\omega)\right)^2\right]=E\left[\int_0^T f(t,\omega)^2dt\right]=Var(I_t).$$

See. Øksendal, B., (2003), "Stochastic Differential Equations", *Springer*, pages. 26-29.

[26] I.e. expected value and variance conditional to the information set at time t and then to the initial condition $r_t=r$.

1.6.6 Valuation of european options on zero-coupon bonds in the Hull-White model

The price at time t, of an european put option with strike K and maturity T written on a zero coupon bond with face value equal to 1 and maturity s, with $s > T$, is denoted by p_t^{zcb}, under the risk neutral probability measure $\mathbb{P}$, it is equal to[27]:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - P(t,s) \cdot N(-d_1) \tag{1.97}$$

where:

$$d_1 = \frac{1}{\sigma_p} \ln \left[\frac{P(t,s)}{K \cdot P(t,T)} \right] + \frac{\sigma_p}{2} \tag{1.98}$$

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{P(t,s)}{K \cdot P(t,T)} \right] - \frac{\sigma_p}{2} \tag{1.99}$$

and:

- $P(t,T)$ is the price at time t of a zero coupon bond with maturity T, and it is an input required by the model;
- $P(t,s)$ is the price at time t of a zero coupon bond with maturity $s > T$, and it is an input required by the model;
- $N(x)$ is the value in x of the standard normal distribution function;
- σ_p is equal to[28]:

$$\sigma_p = \frac{\sigma}{a} \left(1 - e^{-a(s-T)} \right) \sqrt{\frac{(1 - e^{-2a(T-t)})}{2a}} \tag{1.100}$$

In order to prove the validity of equation 1.97, we prove that the put option value at time t is equal to the conditional expected value of its payoff at maturity, under the risk neutral probability measure $\mathbb{P}$, discounted at the risk free rate. The price at time t of the put option written on a zero coupon bond with maturity s, is:

$$p_t^{zcb} = P(t,T) \cdot E_t^{\mathbb{P}}[\max(K - P(T,s), 0)] \tag{1.101}$$

where the value $P(T,s)$ in the RHS of equation 1.101 is the price at time t of a zero-coupon bond with $FV = 1$ and maturity equal to T.

[27] If he underlying zero coupon bond has face value different from one, the 1.97, the 1.98 and the 1.99 changes as follows:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - FV \cdot P(t,s) N(-d_1)$$

$$d_1 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot P(t,s)}{K \cdot P(t,T)} \right] + \frac{\sigma_p}{2}$$

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot P(t,s)}{K \cdot P(t,T)} \right] - \frac{\sigma_p}{2}$$

[28] See. Appendix A.3 of this Chapter

By exploiting the property of maximum an minimum functions according to which:
$\min(f(x), g(x)) \equiv -\max(-f(x), -g(x))$
equation 1.101 becomes:

$$\begin{aligned} p_t^{zcb} &= P(t,T) \cdot E_t^{\mathbb{P}}[-\min(P(T,s)-K,0)] \\ &= -P(t,T) \cdot E_t^{\mathbb{P}}[\min(P(T,s)-K,0)] \end{aligned} \quad (1.102)$$

Let $g(P(T,s))$ be the probability density function of $P(t,T)$. We have that:

$$p_t^{zcb} = -P(t,T) \cdot \int_{-\infty}^{K} (P(T,s)-K) g(P(T,s)) dP(T,s) \quad (1.103)$$

Being $P(T,s)$ a lognormal random variable[29], the variable $\ln P(T,s)$ is conditionally distributed as a normal random variable with standard deviation equal to σ_p, whose value is expressed in equation 1.100. Given the lognormal distribution properties[30], the conditional expected value of $\ln P(T,s)$ is:

$$E_t^{\mathbb{P}}(\ln P(T,s)) = \ln E_t^{\mathbb{P}}(P(T,s)) - \frac{\sigma_p^2}{2} \quad (1.104)$$

By the martingale property of the zero coupon bond price, the conditional expected value of the spot price $P(T,s)$, evaluated at time t, with $t < T < s$, corresponds to the forward price $P(t,T,s)$, and therefore equation 1.104 becomes:

$$E_t^{\mathbb{P}}(\ln P(T,s)) = \ln P(t,T,s) - \frac{\sigma_p^2}{2} \quad (1.105)$$

[29] Being $P(T,s) = A(T,s)e^{-B(T,s)r_T}$ (see 1.24) and having the instantaneous short rate r_t a conditional normal probability distribution (see. § 1.5.1), we can conclude that $P(T,s)$ has a lognormal distribution.

[30] Let X be a lognormal random variable with density function:

$$f(x) = \begin{cases} \frac{1}{\delta\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \gamma}{\delta}\right)^2} & \text{for } x > 0 \\ 0 & \text{oherwise} \end{cases}$$

whose expected value is $E(X) = e^{\gamma + \frac{\delta^2}{2}}$ and whose variance is $Var(X)$. Then, the random variable $Y = \ln X$ has a normal probability distribution with expected value $E(Y) = E(\ln X) = \ln E(X) - \frac{Var(Y)}{2}$ and variance $Var(Y)$. In fact we have:

$$E(X) = e^{\gamma + \frac{\delta^2}{2}}$$

hence:

$$\ln E(X) = \gamma + \frac{\delta^2}{2}$$

expliciting by γ:

$$\gamma = \ln E(X) - \frac{\delta^2}{2}$$

where $\gamma = E(Y)$ and $\delta^2 = Var(Y)$.

Using the definition of the forward price from which[31]:

$$P(t,T,s) = \frac{P(t,s)}{P(t,T)}$$

the expected value of $\ln P(t,T,s)$, shown in equation 1.105, can be expressed as:

$$E_t^{\mathbb{P}}(\ln P(T,s)) = \ln \frac{P(t,s)}{P(t,T)} - \frac{\sigma_p^2}{2} \tag{1.106}$$

We now define a new random variable Q, obtained by standardizing the normal random variable $\ln P(T,s)$:

$$Q = \frac{\ln P(T,s) - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p} \tag{1.107}$$

Then, Q has a standard normal distribution whose probability density function $h(Q)$ is:

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} \tag{1.108}$$

Solving equation 1.107 for $P(T,s)$ we have:

$$P(T,s) = e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} \tag{1.109}$$

Using equation 1.107 and 1.109 to transform the integral in $P(T,s)$ appearing in the RHS of equation 1.103 into an integral in Q, we obtain:

$$\begin{aligned} p_t^{zcb} &= -P(t,T) \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} \left(e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} - K \right) h(Q) dQ \\ &= -P(t,T) \left(\int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} h(Q) dQ + \right. \\ & \left. -K \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q) dQ \right) \end{aligned} \tag{1.110}$$

Substituting the value of $h(Q)$ given from 1.108, the first function in the RHS term of 1.110 becomes:

$$\begin{aligned} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))} h(Q) &= \frac{1}{\sqrt{2\pi}} e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s)) - \frac{Q^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-Q^2 + 2Q\sigma_p + 2E_t^{\mathbb{P}}(\ln P(T,s))}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-(Q-\sigma_p)^2 + 2E_t^{\mathbb{P}}(\ln P(T,s)) + \sigma_p^2}{2}} \\ &= e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(Q-\sigma_p)^2}{2}} \end{aligned} \tag{1.111}$$

[31] See. equation 1.4, with $T + \tau = s$.

From equation 1.108, we see that the quantity $\frac{1}{\sqrt{2\pi}}e^{\frac{-(Q-\sigma_p)^2}{2}}$ is the probability density function of the random variable $(Q-\sigma_p)$, whose conditional distribution is a normal with parameters $(-\sigma_p, 1)$. Therefore, equation 1.111 can be written as:

$$e^{Q\sigma_p + E_t^{\mathbb{P}}(\ln P(T,s))}h(Q) = e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}}h(Q-\sigma_p)$$

and, hence, equation 1.110 becomes:

$$\begin{aligned} p_t^{zcb} &= -P(t,T)\left\{ e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} \int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q-\sigma_p)dQ + \right. \\ &\quad \left. -K\int_{-\infty}^{\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}} h(Q)dQ \right\} \\ &= -P(t,T)\left\{ e^{E_t^{\mathbb{P}}(\ln P(T,s)) + \frac{\sigma_p^2}{2}} N\left(\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p} - \sigma_p\right) + \right. \\ &\quad \left. -KN\left(\frac{\ln K - E_t^{\mathbb{P}}(\ln P(T,s))}{\sigma_p}\right)\right\} \end{aligned} \tag{1.112}$$

where $N(x)$ is a standard normal random variable.
Substituting the value $E_t^{\mathbb{P}}(\ln P(T,s))$ given from 1.106, equation 1.112 becomes:

$$\begin{aligned} p_t^{zcb} &= -P(t,T)\left(e^{\ln\frac{P(t,s)}{P(t,T)} - \frac{\sigma_p^2}{2} + \frac{\sigma_p^2}{2}} N\left(\frac{\ln K - \ln\frac{P(t,s)}{P(t,T)} + \frac{\sigma_p^2}{2}}{\sigma_p} - \sigma_p\right) + \right. \\ &\quad \left. -KN\left(\frac{\ln K - \ln\frac{P(t,s)}{P(t,T)} + \frac{\sigma_p^2}{2}}{\sigma_p}\right)\right) \\ &= -P(t,T)\left(\frac{P(t,s)}{P(t,T)} N\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} - \sigma_p\right) + \right. \\ &\quad \left. -KN\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}\right)\right) \end{aligned}$$

and finally:

$$p_t^{zcb} = -P(t,s)\cdot N\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p}\right) + K\cdot P(t,T)\cdot N\left(\frac{\ln\left(\frac{K\cdot P(t,T)}{P(t,s)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}\right) \tag{1.113}$$

Denoting by d_1 and d_2 the quantities:

$$d_1 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right) + \frac{\sigma_p}{2}}{\sigma_p}$$

$$d_2 = \frac{\ln\left(\frac{P(t,s)}{K\cdot P(t,T)}\right) - \frac{\sigma_p}{2}}{\sigma_p} = d_1 - \sigma_p$$

equation 1.113 becomes[32]:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2) - P(t,s) \cdot N(-d_1) \quad (1.117)$$

We can see that equation 1.117 is similar to the traditional Black-Scholes formula. In fact, under the Black-Scholes formula, the price at time t, under the risk neutral probability measure $\mathbb{P}$, of a put option written on a stock is:

$$p_t^{BS} = K \cdot e^{-r(t,T)(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1) \quad (1.118)$$

where :

$$d_1 = \frac{\ln \frac{S_t}{K} + (r_t + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln \frac{S_t}{K} + (r_t - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

and S_t is the price of the underlying asset at time t.

In both cases the price of the underlying asset has a lognormal distribution and σ_p^2 has the same role of $\sigma^2(T-t)$, that represents the conditional variance of the logarithm of the stock prices at maturity. The price $P(t,T)$ corresponds to the discount factor $e^{-r(t,T)(T-t)}$, in which $r(t,T)$ is the risk neutral interest rate[33].

Using a procedure similar to that one used to determine the price of a put option, we can find out the formula for the evaluation of an european call option with strike price K and maturity T, written on a zero-coupon bond with face value equal to one and maturity $s > T$, starting from the equality:

$$P(t,T) \cdot E_t^{\mathbb{P}}[\max(P(T,s)-K,0)] = P(t,T) \cdot \int_K^{+\infty} (P(T,s)-K) g(P(T,s)) dP(T,s)$$

and obtaining the call price c_t^{zcb} expressed as[34]:

$$c_t^{zcb} = P(t,s) \cdot N(d_1) - K \cdot P(t,T) \cdot N(d_2) \quad (1.120)$$

---

[32] Notice that, if the underlying zero coupon bond has face value different from one, the formula 1.93 to compute p_t^{zcb} changes as follows:

$$p_t^{zcb} = K \cdot P(t,T) \cdot N(-d_2^*) - FV \cdot P(t,s) \cdot N(-d_1^*) \quad (1.114)$$

where:

$$d_1^* = \frac{\ln\left(\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \quad (1.115)$$

$$d_2^* = \frac{\ln\left(\frac{FV \cdot P(t,s)}{K \cdot P(t,T)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p} = d_1^* - \sigma_p \quad (1.116)$$

[33] Recall that, in a risk neutral world, the expected return of any financial asset is equal to the risk free rate.

[34] As for the put case, notice that, if the underlying zero-coupon bond has a face value different from one, the formula 1.94 to compute c_t^{zcb} changes as follows:

$$c_t^{zcb} = FV \cdot P(t,s) \cdot N(d_1^*) - K \cdot P(t,T) \cdot N(d_2^*) \quad (1.119)$$

1.6.7 Appendix A.1 The volatility of the price of a zero coupon bond underlying an european option in the Vasicek model

As shown in section 1.5.2, in the Vasicek model the conditional probability distribution of the short rate is normal. Given the relationship between the instantaneous short rate and the zero coupon bond price expressed in equation 1.24, $P(t,T)$ has a lognormal conditional distribution and consequently the conditional probability distribution of the logarithm of $P(t,T)$ is a normal distribution.

Given equation 1.24 the logarithm of the price of a zero coupon bond with maturity s, evaluated at time T, is equal to:

$$\begin{aligned}\ln P(T,s) &= \ln A(T,s)\, e^{-B(T,s)r_T} \\ &= \ln A(T,s) + \ln e^{-B(T,s)r_T} \\ &= \ln A(T,s) - B(T,s)\, r_T\end{aligned}$$

from which we have that the variance of $\ln P(T,s)$ is given by:

$$Var(\ln P(T,s)) = Var(\ln A(T,s) - B(T,s)\, r_T) \tag{1.121}$$

Since the quantity $\ln A(T,s)$ is not random, equation 1.121 can be rewritten as follows:

$$Var(\ln P(T,s)) = Var(-B(T,s)\, r_T)$$

Using some well-known properties of the variance we obtain:

$$Var(\ln P(T,s)) = B^2(T,s)\, Var(r_T) \tag{1.122}$$

Substituting the expressions for $B^2(T,s)$ and $Var(r_T)$ which can be obtained respectively from equation 1.26 and equation 1.32, equation 1.122 becomes:

$$Var(\ln P(T,s)) = \left(\frac{1-e^{-a(s-T)}}{a}\right)^2 \frac{\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right)$$

Therefore the volatility of the logarithm of the underlying asset price at maturity - which is essential to compute the option price in the Vasicek model - is:

$$\sigma_p = \sigma\left(\frac{1-e^{-a(s-T)}}{a}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}} \tag{1.34}$$

---

where:

$$d_1^* = \frac{\ln\left(\frac{FV\cdot P(t,s)}{K\cdot P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \tag{1.49}$$

$$d_2^* = \frac{\ln\left(\frac{FV\cdot P(t,s)}{K\cdot P(t,T)}\right) - \frac{\sigma_p^2}{2}}{\sigma_p} = d_1^* - \sigma_p \tag{1.50}$$

1.6.8 Appendix A.2 The volatility of the price of a zero coupon bond underlying an european option in the Ho and Lee model

As shown in section 1.6.2, in the Ho and lee model the conditional distribution of the short rate is normal, and therefore, given equation 1.80 the random variable $P(t,T)$ has a lognormal probability distribution, while the conditional distribution of the logarithm of $P(t,T)$ is a normal.

Applying equation 1.80 the logarithm of the price at time t of a zero coupon bond with maturity s is equal to:

$$\begin{aligned}\ln P(t,s) &= \ln\left[A(t,s)\, e^{-B(t,s)r_t}\right] \\ &= \ln A(t,s) + \ln e^{-B(t,s)r_t} \\ &= \ln A(t,s) - B(t,s)\, r_t\end{aligned}$$

from which we have that the variance of $\ln P(t,s)$ is given by:

$$Var(\ln P(t,s)) = Var(\ln A(t,s) - B(t,s)\, r_t) \tag{1.123}$$

Since the quantity $\ln A(t,s)$ is constant, equation 1.123 can be rewritten as follows:

$$Var(\ln P(t,s)) = Var(-B(t,s)\, r_t) \tag{1.124}$$

Using some well-known properties of the variance we obtain:

$$Var(\ln P(t,s)) = B^2(t,s)\, Var(r_t) \tag{1.125}$$

Substituting the expressions for $B^2(t,s)$ and $Var(r_t)$ which can be obtained respectively from equation 1.56 and equation 1.67, equation 1.125 becomes:

$$Var(\ln P(t,s)) = (s-t)^2\, \sigma^2 t \tag{1.126}$$

Since the price at future time T of a zero coupon bond with maturity s is equal to $P(T,s)$ from equation 1.126 follow that the variance of the logarithm of $P(T,s)$ is equal to:

$$Var(\ln P(T,s)) = (s-T)^2\, \sigma^2 (T-t) \tag{1.127}$$

and then the standard deviation of $\ln P(T,s)$ - which is essential to compute the option price in the Ho and Lee model - is:

$$\sigma_p = (s-T)\, \sigma\sqrt{T-t} \tag{1.71}$$

1.6.9 Appendix A.3 The volatility of the price of a zero coupon bond underlying an european option in the Hull and White model

As shown for the Ho and Lee model, also for the Hull and White model it is possible to derive the volatility of the price of a zero coupon bond underlying an european option.
Substituting the expressions for $B^2(t,s)$ and $Var(r_t)$ which can be obtained respectively from equation 1.80 and equation 1.96, equation 1.125 becomes:

$$Var(\ln P(t,s)) = \left(\frac{1-e^{-a(s-t)}}{a}\right)^2 \frac{\sigma^2}{2a}\left(1-e^{-2at}\right) \tag{1.128}$$

Since the price at time T of a zero coupon bond with maturity $s, s > T$, is equal to $P(T,s)$ from equation 1.128 it follows that the variance of the logarithm of $P(T,s)$ is equal to:

$$Var(\ln P(T,s)) = \left(\frac{1-e^{-a(s-T)}}{a}\right)^2 \frac{\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right) \tag{1.129}$$

and therefore the standard deviation of $\ln P(T,s)$ - which is essential to compute the option price in the Hull and White model - is:

$$\sigma_p = \frac{\sigma}{a}\left(1-e^{-a(s-T)}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}} \tag{1.100}$$

Chapter 2

Cap and floor pricing using affine term structure models

In this Chapter we will analyze the payoff structure and the pricing formulas for caps and floors.

Caps and floors are the most widespread interest rate derivatives and they are used both for hedging purposes and for the financial engineering of stochastic interest bonds with various degrees of complexity, as those ones which will be analyzed later in this work.

As we will see in Chapter 5, the products examined have a derivative component that is composed either by a cap or a floor or both. It follows that, in order to compute the fair value of these bonds, we need to know how the derivative component is structured and how to price it under the term structure models illustrated in Chapter 1.

Caps and Floors are options, or more precisely, portfolio of options, that have as underlying asset the value of a monetary market interest rate.

In particular, when these derivatives are embedded in a stochastic interest bond, their evaluation requires to model the term structure of interest rates in order to determine the price of the bond whose coupon payments are linked to an interest rate which is also the underlying rate of the cap and/or the floor included in the bond structure and, consequently, affecting its value.

Therefore, it is obvious that the term structure models examined play a fundamental role for the pricing of this class of interest rate derivatives.

In the next sections we will present the key concepts about caps and floors and their pricing formulas under the three one-factor models of Chapter 1[1]. For

[1] In Appendix B.1. at the end of this Chapter we will also presente the Black formulas to price eupoean call and put options written on the forward interest rate. The utility of these formulas is mainly due to the fact that they are used to find the market value of caps and floors by solving for the forward rate implied volatility.

the sake of simplicity we will assume to be in a risk-less world (i.e. without any credit risk due to the credit worthiness of the issuer of the stochastic interest bonds embedding caps and/or floors). However, when credit risk becomes a material risk factor, it must be taken into account also in the evaluation of these interest rate derivatives, by properly arranging equation 3.23 derived in Chapter 4.

2.1 *Interest rate caps*

Interest rate caps belong to the family of interest rate options.

They are instruments that are mainly traded over the counter, i.e. outside the regulated market, whose finality is to offer to their underwriter a protection against an excessive increase of a floating interest rate at which they are exposed. The most frequent case in which an underwriter of a cap can be exposed to this risk is the case in which he is also the issuer or the seller of a bond that pays periodically a coupon indexed at that floating rate (so-called *floating rate note*, FRN). Therefore, who buys a cap wants to ensure himself against the risk that the interest rate on the FRN whose coupons are index to the same rate underlying the cap could rise over a pre-specified level said the *cap rate*[2].

The cap is a portfolio of call options (*caplets*) written on a reference floating rate and the maturity of each option usually corresponds to the date in which the coupon of a given indexed bond is update to the current market value of the market rate. It follows that the cap price will be equal to the sum of the prices of each caplet in the portfolio.
The time period between two consecutive dates (*reset dates*) of update is called *tenor*.

Example 1 *Let us consider a cap with the following characteristics:*

- *Reference floating rate:* $L_{t_i}\% = 3 - month\ Libor,\ i = 1, 2, ..., n;$
- *Maturity:* $T_{cap} = 5\ years$
- *Cap rate:* $K_{cap}\% = 4\%$
- *Face value:* $FV = 10,000\ Euro$
- *Reset dates:* $t_1, t_2, ..., t^*, ..., t_n$
- *Tenor:* $\delta = 0.25\ years.$

Let us assume that at a particular reset date, t^*, the 3-month Libor will be equal to 4.5%. Since the value of the reference floating rate is higher than the strike price, the option is in-the-money and the cap payment flow at time $t^* + \delta$ will be:

$$FV \cdot \delta \cdot \max(L_{t^*}\% - K_{cap}\%, 0\%) = 10,000 \cdot 0.25 \cdot \max(4.50\% - 4\%, 0\%) = 12.5\ Euro$$

[2] The cap rate is the strike price of all the caplets in a cap.

Suppose now that at a particular reset date, t^*, the 3-month Libor will be equal to 3.85%. Since the value of the reference floating rate is lower than the strike price, the option is in-the-money and the cap payment flow at time $t^* + \delta$ will be:

$$FV \cdot \delta \cdot \max(L_{t^*}\% - K_{cap}\%, 0\%) = 10,000 \cdot 0.25 \cdot \max(3.85\% - 4\%, 0\%) = 0 \ Euro$$

Therefore, if at time t^* the value of the reference floating rate (in this case the 3-month Libor) will be higher than the cap rate (4%), the call option written on the cap reference rate will be in-the money and it will be exercised with a profit. In the opposite case, if at time t^* the value of the reference floating rate will be lower than the cap rate (4%), the call option written on the cap reference rate will be out-of-the money and it will not be exercised.
The same mechanism holds at any reset date.

2.1.1 *Caps* as portfolios of interest rate calls

Let us consider a cap with the following characteristics:

- maturity T_{cap};
- cap rate: $K_{cap}\%$;
- face value: FV;
- reset dates: $t_1, t_2, ., t_i.., t_n$;
- maturity: $T_{cap} = t_{n+1}$;
- $L_i\%$: the reference floating rate at time t_i $(1 \leq i \leq n)$ for the period $\delta_i = t_{i+1} - t_i$

The cap payment at time t_{i+1} $(i = 1, 2, ..., n)$ is equal to:

$$FV \cdot \delta_i \cdot \max(L_i\% - K_{cap}\%, 0) \tag{2.1}$$

Equation 2.1 represents the payoff of an european call option with maturity t_{i+1} and whose underlying asset is $L_i\%$, i.e. the value of the interest rate observed at time t_i. Such option is called *caplet* and the cap is a portfolio of n caplets. Each caplet has a final payoff as in equation 2.1 .
The value of the interest rate underlying the cap is observed at times $t_1, t_2, ..., t_n$ and the payments will occur one period later the observation dates, i.e. at times $t_2, t_3, ..., t_{n+1}$.

2.1.2 *Cap* as portfolios of zero coupon bond puts

Using a specular view with respect to that of the previous section, the cap can be also seen as a portfolio of european put options, in which each option has a maturity equal to the date in which its payoff is computed and has a zero coupon bond as underlying asset.

In fact, the payoff shown in equation 2.1 and paid by the cap at time t_{i+1} $(1 \leq i \leq n)$ is equal to the following payoff paid at time t_i $(1 \leq i \leq n)$:
$\frac{FV \cdot \delta_i}{1+L_i\% \cdot \delta_i} \cdot \max\left(L_i\% - K_{cap}\%, 0\right)$
$= \max\left(\frac{(L_i\% - K_{cap}\%)}{(1+L_i\% \cdot \delta_i)} \cdot FV \cdot \delta_i, 0\right)$
$= \max\left(\frac{L_i\% FV \cdot \delta_i + FV}{1+L_i\% \cdot \delta_i} - \frac{K_{cap}\% \cdot FV \cdot \delta_i + FV}{1+L_i\% \cdot \delta_i}, 0\right)$
$= \max\left(FV \cdot \frac{1+L_i\% \cdot \delta_i}{1+L_i\% \cdot \delta_i} - FV \cdot \frac{1+K_{cap}\% \cdot \delta_i}{1+L_i\% \cdot \delta_i}, 0\right)$

Hence we have:

$$\frac{FV \cdot \delta_i}{1+L_i\% \cdot \delta_i} \cdot \max\left(L_i\% - K_{cap}\%, 0\right) = \max\left(FV - \frac{(FV \cdot (1+K_{cap}\% \cdot \delta_i))}{1+L_i\% \cdot \delta_i}, 0\right) \tag{2.2}$$

where the term $\frac{(FV \cdot (1+K_{cap}\% \cdot \delta_i))}{1+L_i\% \cdot \delta_i}$ in the RHS of equation 2.2 represents the value at time t_i of a zero coupon bond that pays $FV \cdot (1+K_{cap}\% \cdot \delta_i)$ at time t_{i+1}. Therefore in the RHS of the 2.2 we can easily recognize the payoff of an european put option with maturity t_i, strike price FV and whose underlying asset is a zero coupon bond with repayment value equal to $FV \cdot (1+K_{cap}\% \cdot \delta_i)$ at time t_{i+1}, i.e. a period after the put option maturity.
In other words, an interest rate cap can be seen as a portfolio of european put options written on zero coupon bonds.

2.1.3 Pricing of an interest rate cap in the Vasicek model

In section 2.1.2 we have shown how each caplet in a cap can be considered as an european put option written on a zero coupon bond.

In particular, if we have the cap described in the previous section, the payoff of the put option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1+K_{cap}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(FV - \frac{FV \cdot (1+K_{cap}\% \cdot \delta_i)}{1+L_i\% \cdot \delta_i}, 0\right) \tag{2.3}$$

Therefore, using the Vasicek model described in Chapter 1, we can determine the price at time $t < t_1$ of such put option by applying equation 1.48 and obtaining[3]:

$$p_{t,Vas}^{caplet_{t_i,t_{i+1}}} = FV \cdot P(t,t_i) \cdot N(-d_2) - FV \cdot (1+K_{cap}\% \cdot \delta_i) \cdot P(t,t_{i+1}) \cdot N(-d_1) \tag{2.4}$$

where:

$$d_1 = \frac{\ln\left(\frac{FV \cdot (1+K_{cap}\% \cdot \delta_i) \cdot P(t,t_{i+1})}{P(t,t_i) \cdot FV}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}$$

[3] With the notation $p_{t,Vas}^{caplet_{t_i,t_{i+1}}}$ we indicate the price at time t of a put option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Vasicek model.

$$d_2 = d_1 - \sigma_p$$

$$\sigma_p = \sigma \left(\frac{1 - e^{-a(t_{i+1} - t_i)}}{a} \right) \sqrt{\frac{1 - e^{-2a(t_i - t)}}{2a}} \tag{2.5}$$

By using the above formulas, the price at time t of a cap expiring at time T and composed by N caplets, $p_{t,Vas}^{capN,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$p_{t,Vas}^{capN,T}(a, b', \sigma) = \sum_{i=1}^{N-1} p_{t,Vas}^{caplet_{t_i,t_{i+1}}}(a, b', \sigma) \tag{2.6}$$

where we have emphasized that the cap price (as well as the caplets prices) depends on the parameters which characterize the Vasicek model.

2.1.3.1 Pricing of an interest rate cap in the Vasicek model: an example

Let us use the Vasicek model to determine the price at time $t = 0$ of a cap with the following characteristics:

- Maturity (in years): $T_{cap} = 2$
- *Cap rate*: $K_{cap}\% = 11\%$
- Face value: $FV = 100$ *Euro*
- *Reset date* (in years): $t_1 = 1, t_2 = 1.25, t_3 = 1.5, t_4 = 1.75$
- *Tenor* (in years): $\delta = 0.25$

Let also assume that the value of the parameters in equation 1.24 are respectively:

a	$= 0.09$
b'	$= 0.2$
σ	$= 0.05$

and that the value at time 0 of the short rate is: $r_0 = 10\%$.

On the basis of what said in section 2.1.2, this cap is a portfolio of four european put options on zero coupon bonds with the following characteristics:

1. the first put option has maturity $t_1 = 1$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 1.25$ and face value of 102.75 *Euro* (obtained as: $FV \cdot (1 + K_{cap} \cdot \delta) = 100 \cdot (1 + 11\% \cdot 0.25)$); applying equation 2.4, we compute the option price at time 0 as: $p_{0,Vas}^{caplet_{1,1.25}} = 100 \cdot P(0,1) \cdot N(-d_2) - 102.75 \cdot P(0,1.25) \cdot N(-d_1) = 0.421533$ *Euro*

2. the second put option has maturity $t_2 = 1.25$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_3 = 1.5$ and face value of 102.75 *Euro* (obtained as: $FV \cdot (1 + K_{cap} \cdot \delta) = 100 \cdot (1 + 11\% \cdot 0.25)$); applying equation 2.4, we compute the option price at time 0 as:
$p_{0,Vas}^{caplet_{1.25,1.5}} = 100 \cdot P(0, 1.25) \cdot N(-d_2) - 102.75 \cdot P(0, 1.5) \cdot N(-d_1) = 0.468939$ *Euro*

3. the third put option has maturity $t_3 = 1.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_4 = 1.75$ and face value of 102.75 *Euro* (obtained as: $FV \cdot (1 + K_{cap} \cdot \delta) = 100 \cdot (1 + 11\% \cdot 0.25)$); applying equation 2.4, we compute the option price at time 0 as:
$p_{0,Vas}^{caplet_{1.5,1.75}} = 100 \cdot P(0, 1.5) \cdot N(-d_2) - 102.75 \cdot P(0, 1.75) \cdot N(-d_1) = 0.506546$ *Euro*

4. the fourth put option has maturity $t_4 = 1.75$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_5 = 2$ and face value of 102.75 *Euro* (obtained as: $FV \cdot (1 + K_{cap} \cdot \delta) = 100 \cdot (1 + 11\% \cdot 0.25)$); applying the 2.4, we compute the option price at time 0 as:
$p_{0,Vas}^{caplet_{1.75,2}} = 100 \cdot P(0, 1.75) \cdot N(-d_2) - 102.75 \cdot P(0, 2) \cdot N(-d_1) = 0.536199$ *Euro*.

Summing up the prices of the four put options we find out the cap price at time 0, that is equal to 1.933217 *Euro*.

2.1.4 Pricing of an interest rate cap in the Ho and Lee model

In section 2.1.2 we have shown how each caplet in a cap can be considered as an european put option written on a zero coupon bond.

In particular, if we have the cap described in section 2.1.2, the payoff of the put option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1 + K_{cap}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(FV - \frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i)}{(1 + L_i\% \cdot \delta_i)}, 0\right) \tag{2.2}$$

Therefore, using the Ho and Lee model described in Chapter 1, we can determine the price at time t of such put option applying equation 1.68 of section 1.6.3 and obtaining[4]:

$$p_{t,HL}^{caplet_{t_i,t_{i+1}}} = FV \cdot P(t, t_i) \cdot N(-d_2) - FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1}) \cdot N(-d_1) \tag{2.7}$$

[4]With the notation $p_{t,HL}^{caplet_{t_i,t_{i+1}}}$ we indicate the price at time t of a put option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Ho and Lee model.

where:

$$d_1 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1})}{FV \cdot P(t, t_i)} \right] + \frac{\sigma_p}{2} \tag{2.8}$$

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1})}{FV \cdot P(t, t_i)} \right] - \frac{\sigma_p}{2} \tag{2.9}$$

$$\sigma_p = \sigma(t_{i+1} - t_i)\sqrt{t_i - t} \tag{2.10}$$

By using the above formulas, the price at time t of a cap expiring at time T and composed by N caplets, $p_{t,HL}^{capN,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$p_{t,HL}^{capN,T}(\sigma) = \sum_{i=1}^{N-1} p_{t,HL}^{caplet_{t_i,t_{i+1}}}(\sigma) \tag{2.11}$$

where we have emphasized that the cap price (as well as the caplets prices) depends on the parameter which characterizes the Ho and Lee model.

2.1.4.1 Pricing of an interest rate cap in the Ho and Lee model: an example

Let us use the Ho and Lee model to determine the price at time $t = 1$ of a cap with the following characteristics:

- Maturity (in years): $T_{cap} = 4.5$
- *Cap rate*: $K_{cap}\% = 4\%$
- Face value: 100 *Euro*
- *Reset date* (in years): $t_1 = 3; t_2 = 3.5; t_3 = 4$
- *Tenor* (in years): $\delta = 0.5$

Let also assume that the value of the instantaneous volatility of the short rate (i.e. the value of the parameter σ in equation 1.54) is equal to 0.01, and that the expected value of the overnight rate at time 1 is: $R(1) = 3.615\%$.
Moreover, as shown in equation 1.64 of section 1.6.1, the implementation of the model requires as further input the price at time 0 of the zero coupon bonds with maturities 1, 3, 3.5, 4 and 4.5. We assume that such prices are respectively: 0.988, 0.951, 0.934, 0.915 and 0.882.

On the basis of what said in section 2.1.2, this cap is a portfolio of three european put options with the following characteristics:

1. the first put has maturity $t_1 = 3$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 3.5$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$);

2. the second put has maturity $t_2 = 3.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_3 = 4$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$);

3. the third put has maturity $t_3 = 4$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_4 = 4.5$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$).

Applying the formula given in equation 2.7 of section 2.1.4 (suitable modified to consider that the cap evaluation date is: $t = 1$), we compute the price of each put option at time 1:

- $p_{1,HL}^{caplet_{3,3.5}} = 0.65261$ *Euro*
- $p_{1,HL}^{caplet_{3.5,4}} = 0.84667$ *Euro*
- $p_{1,HL}^{caplet_{4,4.5}} = 2.18335$ *Euro*

Summing up the prices of the three put options we find out the cap price at time 1, that is equal to: 3.68263 *Euro*.

2.1.5 Pricing of an interest rate cap in the Hull and White model

In section 2.1.2 we have shown how each caplet in a cap can be considered as an european put option written on a zero coupon bond.

In particular, if we have the cap described in section 2.1.2, the payoff of the put option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1 + K_{cap}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(FV - \frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i)}{(1 + L_i\% \cdot \delta_i)}, 0\%\right) \tag{2.2}$$

Therefore, using the Hull and White model described in Chapter 1, we can determine the price at time t of such put option applying equation 1.97 and obtaining[5]:

$$p_{t,HW}^{caplet_{t_i,t_{i+1}}} = FV \cdot P(t, t_i) \cdot N(-d_2) - FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1}) \cdot N(-d_1) \tag{2.12}$$

where:

$$d_1 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1})}{FV \cdot P(t, t_i)}\right] + \frac{\sigma_p}{2} \tag{2.13}$$

$$d_2 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot (1 + K_{cap}\% \cdot \delta_i) \cdot P(t, t_{i+1})}{FV \cdot P(t, t_i)}\right] - \frac{\sigma_p}{2} \tag{2.14}$$

$$\sigma_p = \frac{\sigma}{a}\left(1 - e^{-a(t_{i+1} - t_i)}\right)\sqrt{\frac{(1 - e^{-2a(t_i - t)})}{2a}} \tag{2.15}$$

[5] With the notation $p_{t,HW}^{caplet_{t_i,t_{i+1}}}$ we indicate the price at time t of a put option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Hull and White model.

By using the above formulas, the price at time t of a cap expiring at time T and composed by N caplets, $p_{t,HW}^{capN,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$p_{t,HW}^{capN,T}(a,\sigma) = \sum_{i=1}^{N-1} p_{t,HW}^{caplet_{t_i,t_{i+1}}}(a,\sigma) \tag{2.16}$$

where we have emphasized that the cap price (as well as the caplets prices) depends on the two parameters which characterize the Hull and White model.

2.1.5.1 Pricing of an interest rate cap in the Hull and White model: an example

Let us use the Hull and White model to determine the price at time $t = 1$ of a cap with the following characteristics:

- Maturity (in years): $T_{cap} = 4.5$
- *Cap rate*: $K_{cap}\% = 4\%$
- Face value: 100 *Euro*
- *Reset date* (in years): $t_1 = 3; t_2 = 3.5; t_3 = 4$
- *Tenor* (in years): $\delta = 0.5$

Let also assume that the mean reverting speed of the short rate (i.e. the parameter a in equation 1.73 of section 1.6.4) is equal to 0.105, the value of the instantaneous volatility of the short rate (i.e. the parameter σ in equation 1.73) is equal to 0.01, and the expected value of the overnight rate at time 1 is: $R(1) = 3.615\%$.
Moreover, as shown in equation 1.90 of section 1.6.4, the implementation of the model require as further input the price at time 0 of the zero coupon bonds with maturities 1, 3, 3.5, 4 and 4.5. We assume that such prices are respectively: 0.988, 0.951, 0.934, 0.915 and 0.882.

On the basis of what said in section 2.1.2, this cap is a portfolio of three european put options with the following characteristics:

1. the first put has maturity $t_1 = 3$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 3.5$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$);
2. the second put has maturity $t_2 = 3.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_3 = 4$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$);
3. the third put has maturity $t_3 = 4$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_4 = 4.5$ and face value of 102 *Euro* (obtained as: $FV \cdot (1 + K_{cap}\% \cdot \delta) = 100 \cdot (1 + 4\% \cdot 0.5)$).

Applying the formula given in equation 2.12 of section 2.1.5, we compute the price of each put option at time 1:
- $p_{1,HW}^{caplet_{3,3.5}} = 0.50595\ Euro$
- $p_{1,HW}^{caplet_{3.5,4}} = 0.66915\ Euro$
- $p_{1,HW}^{caplet_{4,4.5}} = 1.99242\ Euro$

Summing up the prices of the three put options we find out the cap price at time 1, that is equal to: 3.16752 *Euro*.

2.2 *Interest rate floors*

Interest rate floors belong to the family of interest rate options.

As the caps, they are instruments that are mainly traded over the counter, i.e. outside the regulated market, whose finality is to offer to their underwriter a protection against an excessive decrease of a floating interest rate at which they are exposed.
The most frequently case in which an underwriter of a floor can be exposed to this risk is the case in which he is also the issuer or the buyer of a bond that pays periodically a coupon indexed at that floating rate (so-called *floating rate note*, FRN). Therefore, who buys a floor wants to ensure himself against the risk that the interest rate on the FRN whose coupons are indexed to the same rate underlying the floor could fall over a pre-specified level said the *floor rate*[6].

The floor is a portfolio of put options (*floorlets*) written on a reference floating rate and the maturity of each option usually corresponds to the date in which the coupon of a given indexed bond is update to the current value of the market rate. It follows that the floor price will be equal to the sum of the prices of each option of the portfolio.
The time period between two consecutive dates (*reset dates*) of update is called *tenor*.

Example 2 *Let us consider a floor with the following characteristics:*

- *Reference floating rate:* $L_{t_i}\% = 3 - month\ Libor, i = 1, 2, ..., n;$
- *Maturity:* $T_{floor} = 4\ years$
- *Floor rate:* $K_{floor}\% = 3.5\%$
- *Face value:* $FV = 10,000\ Euro$
- *Reset dates:* $t_1, t_2, ..., t^*, ..., t_n$
- *Tenor:* $\delta = 0.25\ years.$

[6] The floor rate is the strike price of all the floorlets in a floor.

Let us assume that a particular reset date, t^*, the 3-month Libor will be equal to 3%. *Since the value of the reference floating rate is lower than the strike price, the option is in-the-money and the floor payment flow at time $t^* + \delta$ will be:*

$$FV \cdot \delta \cdot \max(K_{floor}\% - L_{t^*}\%, 0\%) = 10,000 \cdot 0.25 \cdot \max(3.5\% - 3\%, 0\%) = 12.5 Euro$$

Suppose now that at a particular reset date, t^*, the 3-month Libor will be equal to 3.7%. *Since the value of the reference floating rate is higher than the strike price, the option is out-of-the-money and the floor payment flow at time $t^* + \delta$ will be:*

$$FV \cdot \delta \cdot \max(K_{floor}\% - L_{t^*}\%, 0\%) = 10,000 \cdot 0.25 \cdot \max(3.5\% - 3.7\%, 0\%) = 0 \; Euro$$

*Therefore, if at time t^* the value of the reference floating rate (in this case the 3-month Libor) will be lower then the floor rate (*3.5%*), the put option written on the floor reference rate will be in-the money and it will be exercised with a profit. In the opposite case, if at time t^*, the value of the reference floating rate will be higher than the floor rate (*3.5%*), the put option written on the floor reference rate will be out-of-the money and it will not be exercised.*
This mechanism works in each floor reset date.

2.2.1 *Floors* as portfolios of interest rate puts

Let us consider a floor with the following characteristics:

- Maturity T_{floor};
- *Floor rate*: $K_{floor}\%$;
- Face value: FV;
- *Reset dates*: $t_1, t_2, ..., t_i, ...t_n$;
- Maturity: $T_{floor} = t_n$;
- $L_i\%$: the reference floating rate at time t_i $(1 \leq i \leq n)$ for the period $\delta_i = t_{i+1} - t_i$.

The floor payment at time t_{i+1} $(i = 1, 2, ..., n)$ is equal to:

$$FV \cdot \delta_i \cdot \max\left(K_{floor}\% - L_i\%, 0\%\right) \tag{2.17}$$

Equation 2.17 represents the payoff of an european put option with maturity t_{i+1} and whose underlying asset is $L_i\%$, i.e. the value of the interest rate observed at time t_i. Such option is called *floorlet* and the floor is a portfolio of n floorlets. Each floorlet has a final payoff as in equation 2.17.
The value of the interest rate underlying the floor is observed at times $t_1, t_2, ..., t_n$ and the payments will occur a period later the observation dates, i.e. at times $t_2, t_3, ..., t_{n+1}$.

2.2.2 *Floors* as portfolios of zero coupon bond calls

Using a specular view with respect to that of the previous section, the floor can be also seen as a portfolio of european call options, in which each option has maturity equal to the date in which its payoff is computed and has a zero coupon bond as underlying asset.

In fact, the payoff shown in equation 2.17 and paid by the floor at time t_{i+1} $(1 \leq i \leq n)$ is equal to the following payoff paid at time t_i $(1 \leq i \leq n)$:
$\frac{FV \cdot \delta_i}{1+L_i\% \cdot \delta_i} \cdot \max\left(K_{floor}\% - L_i\%, 0\right)$
$= \max\left(\frac{(K_{floor}\% - L_i\%)}{(1+L_i\% \cdot \delta_i)} \cdot FV \cdot \delta_i, 0\right)$
$= \max\left(\frac{K_{floor}\% \cdot FV \cdot \delta_i + FV}{1+L_i\% \cdot \delta_i} - \frac{L_i\% \cdot FV \cdot \delta_i + FV}{1+L_i\% \cdot \delta_i}, 0\right)$
$= \max\left(FV \cdot \frac{1+K_{floor}\% \cdot \delta_i}{1+L_i\% \cdot \delta_i} - FV \cdot \frac{1+L_i\% \cdot \delta_i}{1+L_i\% \cdot \delta_i}, 0\right)$

Hence we have:

$$\frac{FV \cdot \delta_i}{1+L_i\% \cdot \delta_i} \cdot \max\left(K_{floor}\% - L_i\%, 0\right) = \max\left(\frac{FV \cdot (1+K_{floor}\% \cdot \delta_i)}{1+L_i\% \cdot \delta_i} - FV, 0\right) \tag{2.18}$$

where the term $\frac{(FV \cdot (1+K_{floor}\% \cdot \delta_i))}{1+L_i\% \cdot \delta_i}$ in the RHS of equation 2.18 represents the value at time t_i of a zero coupon bond that pays $FV \cdot (1 + K_{floor}\% \cdot \delta_i)$ at time t_{i+1}.
Therefore in the RHS of equation 2.18 we can easily recognize the payoff of an european call option with maturity t_i, strike price FV and whose underlying asset is a zero coupon bond with repayment value equal to $FV \cdot (1 + K_{floor}\% \cdot \delta_i)$ at time t_{i+1}, i.e. a period after the call option maturity.
In other words, an interest rate floor can be seen as a portfolio of european call options written on zero coupon bonds.

2.2.3 Pricing of an interest rate floor in the Vasicek model

In section 2.2.2 we have shown how each floorlet in a floor can be considered as an european call option written on a zero coupon bond.

In particular, if we have the floor described in the previous section, the payoff of the call option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1 + K_{floor}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(\frac{FV \cdot (1+K_{floor}\% \cdot \delta_i)}{1+L_i\% \cdot \delta_i} - FV, 0\right) \tag{2.19}$$

Therefore, using the Vasicek model described in Chapter 1, we can determine the price at time $t < t_1$ of such call option applying equation 1.52 and obtaining[7]:

$$c_{t,Vas}^{floorlet_{t_i,t_{i+1}}} = FV \cdot (1 + K_{floor} \cdot \delta_i) \cdot P\left(t, t_{i+1}\right) \cdot N\left(d_1\right) - FV \cdot P\left(t, t_i\right) \cdot N\left(d_2\right) \tag{2.20}$$

[7] With the notation $c_{t,Vas}^{floorlet_{t_i,t_{i+1}}}$ we indicate the price at time t of a call option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Vasicek model.

where:

$$d_1 = \frac{\ln\left(\frac{FV\cdot(1+K_{cap}\%\cdot\delta_i)\cdot P(t,t_{i+1})}{P(t,t_i)\cdot FV}\right) + \frac{\sigma_p^2}{2}}{\sigma_p}$$

$$d_2 = d_1 - \sigma_p$$

$$\sigma_p = \sigma\left(\frac{1-e^{-a(t_{i+1}-t_i)}}{a}\right)\sqrt{\frac{1-e^{-2a(t_i-t)}}{2a}} \tag{2.5}$$

By using the above formulas, the price at time t of a floor expiring at time T and composed by N floorlets, $c_{t,Vas}^{floor N,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$c_{t,Vas}^{floor N,T}(a,b',\sigma) = \sum_{i=1}^{N-1} c_{t,Vas}^{floorlet_{t_i,t_{i+1}}}(a,b',\sigma) \tag{2.21}$$

where we have emphasized that the floor price (as well as the floorlets prices) depends on the parameters which characterize the Vasicek model.

2.2.3.1 Pricing of an interest rate floor in the Vasicek model: an example

Let us use the Vasicek model to determine the price at time $t = 0$ of a floor with the following characteristics:

- Maturity (in years): $T_{floor} = 2$
- *Floor rate*: $K_{floor} = 9\%$
- Face value: 100 *Euro*
- *Reset date* (in years): $t_1 = 1, t_2 = 1.25, t_3 = 1.5, t_4 = 1.75$
- *Tenor* (in years): $\delta = 0.25$

Let also assume that the mean reverting speed of the short rate (i.e. the parameter a in equation 1.24 of section 1.5.1) is equal to 0.09, the value of the the long-run mean (i.e. the parameter b' in equation 1.24) is equal to 0.2, the value of the instantaneous volatility of the short rate (i.e. the parameter σ in equation 1.24) is equal to 0.05, and the value at time 0 of the short rate is: $r_0 = 10\%$.

On the basis of what said in section 2.2.2 this floor is a portfolio of four european call options on zero coupon bonds with the following characteristics:

1. the first call option has maturity $t_1 = 1$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 1.25$ and face value of 102.25 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta) = 100 \cdot (1 + 9\% \cdot 0.25)$); applying equation 2.20, we compute the option price at time 0 as:
$c_{0,Vas}^{floorlet_{1,1.25}} = 102.25 \cdot P(0,1.25) \cdot N(d_1) - 100 \cdot P(0,1) \cdot N(d_2) = 0.242898$ *Euro*

2. the second call option has maturity $t_2 = 1.25$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_3 = 1.5$ and face value of 102.25 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta) = 100 \cdot (1 + 9\% \cdot 0.25)$); applying equation 2.20, we compute the option price at time 0 as:
$c_{0,Vas}^{floorlet_{1.25,1.5}} = 102.25 \cdot P(0, 1.5) \cdot N(d_1) - 100 \cdot P(0, 1.25) \cdot N(d_2) =$ 0.266564 *Euro*

3. the third call option has maturity $t_3 = 1.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_4 = 1.75$ and face value of 102.25 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta) = 100 \cdot (1 + 9\% \cdot 0.25)$); applying equation 2.20, we compute the option price at time 0 as:
$c_{0,Vas}^{floorlet_{1.5,1.75}} = 102.25 \cdot P(0, 1.75) \cdot N(d_1) - 100 \cdot P(0, 1.5) \cdot N(d_2) =$ 0.284879 *Euro*

4. the fourth call option has maturity $t_4 = 1.75$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_5 = 2$ and face value of 102.25 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta) = 100 \cdot (1 + 9\% \cdot 0.25)$); applying equation 2.20, we compute the option price at time 0 as:
$c_{0,Vas}^{floorlet_{1.75,2}} = 102.25 \cdot P(0, 2) \cdot N(d_1) - 100 \cdot P(0, 1.75) \cdot N(d_2) = 0.299235$ *Euro*

Summing up the prices of the four call options we find out the floor price at time 0, that is equal to: 1.093576 *Euro*.

2.2.4 Pricing of an interest rate floor in the Ho and Lee model

In section 2.2.2 we have shown how each floorlet in a floor can be considered as an european call option written on a zero coupon bond.

In particular, if we have the floor described in section 2.2.2, the payoff of the call option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1 + K_{floor}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(\frac{FV \cdot (1 + K_{floor}\% \cdot \delta_i)}{1 + L_i\% \cdot \delta_i} - FV, 0\right) \tag{2.18}$$

Therefore, using the Ho and Lee model described in Chapter 1, we can determine the price at time t of such call option applying equation 1.72 of section 1.6.3 and obtaining[8]:

$$c_{t,HL}^{floorlet_{t_i,t_{i+1}}} = FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1}) \cdot N(d_1) - FV \cdot P(0, t_i) \cdot N(d_2) \tag{2.22}$$

[8] With the notation $c_{t,HL}^{floorlet_{t_i,t_{i+1}}}$ we indicate the price at time t of a put option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Ho and Lee model.

where:

$$d_1 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1})}{P(0, t_i) \cdot FV} \right] + \frac{\sigma_p}{2} \tag{2.8}$$

$$d_2 = \frac{1}{\sigma_p} \ln \left[\frac{FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1})}{P(0, t_i) \cdot FV} \right] - \frac{\sigma_p}{2} \tag{2.9}$$

$$\sigma_p = \sigma(t_{i+1} - t_i)\sqrt{t_i - t} \tag{2.10}$$

By using the above formulas, the price at time t of a floor expiring at time T and composed by N floorlets, $c_{t,HL}^{floor N,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$c_{t,HL}^{floor N,T}(\sigma) = \sum_{i=1}^{N-1} c_{t,HL}^{floorlet_{t_i,t_{i+1}}}(\sigma) \tag{2.23}$$

where we have emphasized that the floor price (as well as the floorlets prices) depends on the parameters which characterize the Ho and Lee model.

2.2.4.1 Pricing of an interest rate floor in the Ho and Lee model: an example

Let us use the Ho and Lee model to determine the price at time $t = 1$ of a floor with the following characteristics:

- Maturity (in years): $T_{floor} = 4.5$
- *Cap rate*: $K_{floor}\% = 3\%$
- Face value: 100 *Euro*
- *Reset date* (in year): $t_1 = 3; t_2 = 3.5; t_3 = 4$
- *Tenor* (in year): $\delta = 0.5$

Let also assume that the value of the instantaneous volatility of the short rate (i.e. the value of the parameter σ in equation 1.54) is equal to 0.02, and that the expected value of the overnight rate at time 1 is: $R(1) = 3.615\%$. Moreover, as shown in equation 1.64 of section 1.6.1, the implementation of the model requires as further input the price at time 0 of the zero coupon bonds with maturities 1, 3, 3.5, 4 and 4.5. We assume that such prices are respectively: 0.988, 0.951, 0.934, 0.915 and 0.882.

On the basis of what we said in section 2.1.2, this floor is a portfolio of three european call options with the following characteristics:

1. the first call has maturity $t_1 = 3$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 3.5$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0.5)$);

2. the second call has maturity $t_1 = 3.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 4$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0.5)$);

3. the third call has maturity $t_1 = 4$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 4.5$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0,5)$).

Applying the formula given in equation 2.22 of section 2.2.4 (suitable modified to consider that the floor evaluation date is: $t = 1$), we compute the price of each call option at time 1:
- $c_{1,HL}^{floorlet_{3,3.5}} = 0.15284$ *Euro*
- $c_{1,HL}^{floorlet_{3.5,4}} = 0.14412$ *Euro*
- $c_{1,HL}^{floorlet_{4,4.5}} = 0.02385$ *Euro*

Summing up the prices of the three put options we find out the cap price at time 1, that is equal to: 0.3208 *Euro*.

2.2.5 Pricing of an interest rate floor in the Hull and White model

In section 2.2.2 we have shown how each floorlet in a floor can be considered as an european call option written on a zero coupon bond.

In particular, if we have the floor described in section 2.2.2, the payoff of the call option with maturity t_i, strike price FV and underlying asset a zero coupon bond with face value equal to $FV \cdot (1 + K_{floor}\% \cdot \delta_i)$ and maturity t_{i+1} is:

$$\max\left(\frac{(FV \cdot (1 + K_{floor}\% \cdot \delta_i))}{1 + L_i \cdot \delta_i} - FV, 0\right) \tag{2.18}$$

Therefore, using the Hull and White model described in section 1.6.4, we can determine the price at time t of such call option applying equation 1.120 of section 1.6.6 and obtaining[9]:

$$c_{t,HW}^{floorlet_{t_i,t_{i+1}}} = FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1}) \cdot N(d_1) - FV \cdot P(0, t_i) \cdot N(d_2) \tag{2.24}$$

where:

$$d_1 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1})}{P(0, t_i) \cdot FV}\right] + \frac{\sigma_p}{2} \tag{2.13}$$

$$d_1 = \frac{1}{\sigma_p} \ln\left[\frac{FV \cdot (1 + K_{floor}\% \cdot \delta_i) \cdot P(0, t_{i+1})}{P(0, t_i) \cdot FV}\right] - \frac{\sigma_p}{2} \tag{2.14}$$

$$\sigma_p = \frac{\sigma}{a}\left(1 - e^{-a(t_{i+1} - t_i)}\right)\sqrt{\frac{(1 - e^{-2a(t_i - t)})}{2a}} \tag{2.15}$$

[9] With the notation $c_{t,HW}^{floorlet_{t_i,t_{i+1}}}$ we indicate the price at time t of a put option expiring at time t_i and written on a zero coupon bond with maturity t_{i+1} in the context of the Hull and Whie model.

By using the above formulas, the price at time t of a floor expiring at time T and composed by N floorlets, $c_{t,HW}^{floorletN,T}$, can be easily obtained as the sum of the prices of all the options included in this portfolio, i.e.:

$$c_{t,HW}^{floorN,T}(a,\sigma) = \sum_{i=1}^{N-1} c_{t,HW}^{floor_{t_i,t_{i+1}}}(a,\sigma) \tag{2.25}$$

where we have emphasized that the floor price (as well as the floorlets prices) depends on the parameters which characterize the Hull and White model.

2.2.5.1 Pricing of an interest rate floor in the Hull and White model: an example

Let us use the Hull and White model to determine the price at time $t = 1$ of a floor with the following characteristics:

- Maturity (in years): $T_{floor} = 4.5$
- *Cap rate*: $K_{floor}\% = 3\%$
- Face value: 100 *Euro*
- *Reset date* (in year): $t_1 = 3; t_2 = 3.5; t_3 = 4$
- *Tenor* (in year): $\delta = 0.5$

Let also assume that the mean reverting speed of the short rate (i.e. the parameter a in equation 1.73 of section 1.6.4) is equal to 0.105, the value of the instantaneous volatility of the short rate (i.e. the parameter σ in equation 1.73) is equal to 0.02, and the expected value of the overnight rate at time 1 is: $R(1) = 3.615\%$.
Moreover, as shown in equation 1.90 of section 1.6.4, the implementation of the model requires as further input the price at time 0 of the zero coupon bonds with maturities 1, 3, 3.5, 4 and 4,.5. We assume that such prices are respectively: 0.988, 0.951, 0.934, 0.915 and 0.882.

On the basis of what said in section 2.1.2, this floor is a portfolio of three european call options with the following characteristics:

1. the first call has maturity $t_1 = 3$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 3.5$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0.5)$);
2. the second call has maturity $t_1 = 3.5$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 4$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0.5)$);
3. the third call has maturity $t_1 = 4$, strike price of 100 *Euro* and as underlying asset a zero coupon bond with maturity $t_2 = 4.5$ and face value of 101.5 *Euro* (obtained as: $FV \cdot (1 + K_{floor}\% \cdot \delta_i) = 100 \cdot (1 + 3\% \cdot 0.5)$).

Applying the formula given in equation 2.24 of section 2.2.5, we compute the price of each call option at time 1:
- $c_{1,HW}^{floorlet_{3,3.5}} = 0.06104$ *Euro*
- $c_{1,HW}^{floorlet_{3.5,4}} = 0.0402$ *Euro*
- $c_{1,HW}^{floorlet_{4,4.5}} = 00004$ *Euro*

Summing up the prices of the three put options we find out the cap price at time 1, that is equal to: 0.10165 *Euro*.

2.2.6 Appendix B.1 Black Formula

The Black model provides the price of an european option on a forward rate, as the Libor or the Euribor. The formula is derived under the assumption that the forward rate process is governed by a geometric Brownian motion.

The Black formula is similar to the Black-Scholes formula used for the evaluation of the stock option but, instead of the spot value of the underlying asset, it considers the corresponding forward value.

By equation 1.5 of section 1.3, we have that the forward rate is defined as:

$$F(t,T,T+\tau) = -\frac{\ln P(t,T+\tau) - \ln P(t,T)}{\tau}, \text{ with } t < T < T+\tau \tag{1.5}$$

The Black formula for the price at time t of an European call option having strike K, maturity T and underlying asset the forward rate $F(t,T,T+\tau)$ is:

$$c_t^{Bl_{T,T+\tau}} = P(t,T)\left[F(t,T,T+\tau)\cdot N\left(d_1^{Bl}\right) - K\cdot N\left(d_2^{Bl}\right)\right] \tag{2.26}$$

where:

$$\begin{aligned} d_1^{Bl} &= \frac{\ln\frac{F(t,T,T+\tau)}{K} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2^{Bl} &= \frac{\ln\frac{F(t,T,T+\tau)}{K} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1^{Bl} - \sigma\sqrt{T-t} \end{aligned} \tag{2.27}$$

In the case of a European put option with same underlying, strike and maturity, the price at time t is:

$$p_t^{Bl_{T,T+\tau}} = P(t,T)\left[K\cdot N\left(-d_2^{Bl}\right) - F(t,T,T+\tau)\cdot N\left(-d_1^{Bl}\right)\right] \tag{2.28}$$

The Black formulas are useful to price interest rate derivatives like caps and floors.

In practice the market prices of any cap are expressed in terms of flat volatility. Considering that, as seen in section 2.1.1, any cap is simply a portfolio of caplets, the Black formula of equation 2.26 can be used to obtain the market price of any cap. In fact, we just have to substitute the caps volatilities quoted by the market into this formula to obtain the market prices of all the caplets in a given cap.

Therefore the market price at time t of a cap expiring at time T, composed by N caplets and denoted by $cap^{N,T}_{t,market}$, is the sum of the prices of all these caplets, i.e.:

$$cap^{N,T}_{t,market} = \sum_{i=1}^{N-1} c_t^{Bl_{t_i,t_{i+1}}} \tag{2.29}$$

where $c_t^{Bl_{t_i,t_{i+1}}}$ is the price at time t of the generic caplet in the cap, namely a caplet with reset date t_i and maturity date t_{i+1}.

By substituting the RHS of equation 2.26 in the RHS of equation 2.29 we obtain:

$$cap^{N,T}_{t,market} = \sum_{i=1}^{N-1} P(t,t_i)\left[F\left(t,t_i,t_{i+1}\right)\cdot N\left(d_1^{Bl_{t_i,t_{i+1}}}\right) - K\cdot N\left(d_2^{Bl_{t_i,t_{i+1}}}\right)\right] \tag{2.30}$$

In the same way, we can compute the market price of a floor.

As in the caps case, the market prices of any floor are expressed in terms of flat volatility. Considering that, as seen in section 2.2.1, any floor is simply a portfolio of floorlets, the Black formula of equation 2.28 can be used to obtain the market price of any floor. In fact, we just have to substitute the floors volatilities quoted by the market into this formula to obtain the market prices of all the floorlets in a given floor.

Therefore the market price at time t of a floor expiring at time T, composed by N floorlets and denoted by $floor^{N,T}_{t,market}$, is the sum of the prices of all these floorlets, i.e.:

$$floor^{N,T}_{t,market} = \sum_{i=1}^{N-1} p_t^{Bl_{t_i,t_{i+1}}} \tag{2.31}$$

where $p_t^{Bl_{t_i,t_{i+1}}}$ is the price at time t of the generic floorlet in the floor, namely a floorlet with reset date t_i and maturity date t_{i+1}.

By substituting the RHS of equation 2.28 in the RHS of equation 2.31 we obtain:

$$floor^{N,T}_{t,market} = \sum_{i=1}^{N-1} P(t,t_i)\left[K\cdot N\left(-d_2^{Bl_{t_i,t_{i+1}}}\right) - F\left(t,t_i,t_{i+1}\right)\cdot N\left(-d_1^{Bl_{t_i,t_{i+1}}}\right)\right] \tag{2.32}$$

Chapter 3

Credit risk and defaultable bonds valuation

3.1 Introduction

The pricing procedure of defaultable bonds, like those that will be analyzed in Chapter 5, has to consider the credit risk of the bond issuer in order to take into account its impact on the bond value.

Credit risk is the risk of loss due to a debtor's non-payment (of a part or of the whole amount), of a loan or other line of credit, either the principal or coupon or both, due to the occurrence of some credit events, such as a delay in repayments, restructuring of borrower repayments, and bankruptcy. The likelihood of these events depends on the credit standing of a given subject. Indeed, the worsening of the credit worthiness implies a higher probability of a credit event. It follows that for existing bonds, this worsening reduces, *ceteris paribus*, their market value, while for new bonds issues it usually raises the cost of funding of a given offeror, since investors demand a higher premium (in terms of bonds' returns) to balance off the risk of a credit event which would depress the value of the securities subscribed.

Credit risk measurement in the framework of bonds fair evaluation requires to estimate the probability that, starting from a certain point in time, the issuer will be insolvent to bondholders and into the inclusion of this probability in the bond pricing along with a proxy of the amount recovered in the case of default.

Market expectations on the default probabilities of an issuer are implicitly embedded in market prices of credit derivatives, which are derivative instruments enabling participants in financial markets to trade in credit as an asset, as they isolate and transfer risk.

In fact, a credit derivative can be considered like an insurance against the credit risk of a reference entity or a reference asset or basket of assets.

More specifically, they are over the counter financial contracts designed to replicate credit exposure by exhibiting a payoff profile that is linked to the occurrence of credit events.

Any credit derivative involves two counterparts: a subject who buys protection against the credit risk (so-called protection buyer), and another one who sells this protection in exchange of a given premium (so-called protection buyer).

The occurrence of a specified credit event will trigger the termination of the credit derivative contract, and transfer of the default payment from the protection seller to the protection buyer.

In this Chapter we will focus on the most widespread credit derivative, the credit default swap (hereafter also CDS). In a credit default swap the protection buyer typically pays a stream of periodic premiums (so-called premium leg) in front of the obligation taken by the protection seller to pay him a given amount (so-called protection leg) in the occurrence of a specified credit event.

We will describe in detail the contractual terms of a CDS by defining the key variables which characterize this kind of derivative instrument. We will illustrate in technical terms the meaning of default probabilities and the relationship between cumulative and intertemporal default probabilities. In this framework we will present the evaluation formulas for the two legs of a CDS: intuitively the value of each leg is obtained as the discounted expected payoff of its cashflows calculated under the risk neutral probability measure.

We will then introduce the concepts of survival probabilities and hazard rates and we will show the relationship existing between these quantities and default probabilities. Exploiting this relationship we will show how to derive the default probabilities from the market quotes of CDS contracts referred to an increasing set of maturities through the so-called "bootstrapping" technique. This technique is based on an iterative procedure which allows to extract the term structure of default probabilities from the term structure of the CDS market quotes.

In the last section of this Chapter we illustrate how, by making use of the default probabilities bootstrapped from CDS market quotes , it can be obtained a general pricing formula for defaultable coupon bonds, which will be concretely used for the evaluation of the stochastic interest bonds analyzed in Chapter 5.

3.2 Credit default swaps

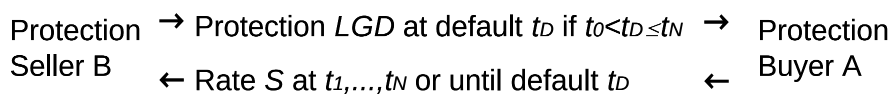
Credit default swaps are the most common credit derivatives.

A CDS is a bilateral contract that provides protection on the par value of a given reference asset (like a bond) or basket of assets. The protection buyer usually pays to the protection seller a periodic fixed premium, the so-called credit swap spread or swap rate S, that is quoted as a basis point multiplier on the nominal value of the contract and it is usually paid quarterly until default or maturity. In return for the premium the protection seller will make a payment corresponding to the loss given default (hereafter also LGD) on the occurrence of a specific credit event. In percentage terms the loss given default will be equal to $(1 - R)$ where R denotes the recovery rate, that is the percentage value which is expected to be paid at default by the issuer of the reference entity.

As already said, the payoff of a CDS is composed by two legs, the premium leg and the protection leg. The premium leg is referred to the amounts paid by the protection buyer and it is equal to the sum of the spreads (multiplied by the nominal value of the contract) at the CDS payment dates t_i, $i = 1, 2, ..., N$, hence starting from the first payment date t_1 which follows the subscription date t_0 and ending at the CDS maturity date t_N or at the random default time t_D, whichever occurs earlier. The protection leg is represented by the amount of the loss given default LGD paid by the protection buyer at default time t_D.

The basic structure of a CDS is summarized in Figure 3.1 here below:

Figure 3.1: Credit default swap structure



3.3 Credit default swaps pricing

In this section we will present the pricing of a CDS by defining the default probabilities and by using them to compute the value of the two legs of the contract at the subscription date $t_0 = 0$[1].

The CDS pricing procedure consists into determine that value of the spread which makes equal the discounted cash flows of the two legs.

Formally, we have to find out the spread value S^* such that, under the risk neutral probability measure $\mathbb{P}$, the discounted expected values of the two legs are the same.

Let us denote by:

- PD_{0,t_i} the cumulative default probability until time t_i, that is the probability that the default occurs between time 0 and time t_i;

- PD_{t_{i-1},t_i} the intertemporal default probability between time t_{i-1} and time t_i.

Then, the following equality holds:

$$PD_{t_{i-1},t_i} = PD_{0,t_i} - PD_{0,t_{i-1}} \tag{3.1}$$

which means that the intertemporal default probability is simply the difference between two consecutive cumulative default probabilities.

As described in previous section, the CDS contract is composed by two legs, one premium and one protection leg. The premium leg is composed by the sum of all the discounted amounts paid by the protection buyer at each payment

[1] For the sake of simplicity, in this Chapter we will assume that $t_0 = 0$.

date until the CDS maturity or until the random default time, t_D, if it is before, and the protection leg is given by the discounted value of the amount paid by the protection seller at default, that is the loss given default.

Assuming that the nominal value of the contract is equal to 1, the expected discounted value of the premium leg, under the risk neutral probability measure $\mathbb{P}$, is:

$$PREM\_Leg\,(0, t_N) = \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} S \cdot e^{-\int_0^{t_i} r_s ds}\mathbf{I}_{[t_D > t_i]}\right] \tag{3.2}$$

by assuming independence between the default event and the interest rate dynamics and considering that S is constant, we can rewrite equation 3.2 as follows:

$$PREM\_Leg\,(0, t_N) = S\sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^{t_i} r_s ds}\right]\mathbb{E}^{\mathbb{P}}\left[\mathbf{I}_{[t_D > t_i]}\right] \tag{3.3}$$

From equation 1.10 in Chapter 1, we know that the term $\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^{t_i} r_s ds}\right]$, $i = 1, 2, ..., N$, appearing in the RHS of equation 3.3 is the price at time 0 of a zero coupon bond with face value 1 and maturity t_i, so that we have:
$PREM\_Leg\,(0, t_N) = S\sum_{i=1}^{N} P\,(0, t_i)\,\mathbb{E}^{\mathbb{P}}\left[\mathbf{I}_{[t_D > t_i]}\right]$
by using a well-known property of the expected value of the indicator function:
$PREM\_Leg\,(0, t_N) = S\sum_{i=1}^{N} P\,(0, t_i)\Pr\,(t_D > t_i)$
and finally, being $\Pr\,(t_D > t_i)$ the probability to have no default before time t_i (that is $\Pr\,(t_D > t_i) = 1 - PD_{0,t_i}$)):

$$PREM\_Leg\,(0, t_N) = S\sum_{i=1}^{N} P\,(0, t_i)\left(1 - PD_{0,t_i}\right) \tag{3.4}$$

With regard to the expected discounted value of the protection leg we have:

$$PROT\_Leg\,(0, t_N) = \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} LGD \cdot e^{-\int_0^{t_i} r_s ds}\mathbf{I}_{[t_{i-1} < t_D < t_i]}\right] \tag{3.5}$$

or, in terms of recovery rate:

$$PROT\_Leg\,(0, t_N) = \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N} (1 - R) \cdot e^{-\int_0^{t_i} r_s ds}\mathbf{I}_{[t_{i-1} < t_D < t_i]}\right] \tag{3.6}$$

By assuming independence between the default event and the interest rate dynamics and that the recovery rate is deterministic, we can rewrite equation 3.6 as follows:

$$PROT\_Leg\,(0, t_N) = (1 - R)\sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^{t_i} r_s ds}\right]\mathbb{E}^{\mathbb{P}}\left[\mathbf{I}_{[t_{i-1} < t_D < t_i]}\right] \tag{3.7}$$

and, by using the same arguments seen for the premium leg:
$PROT\_Leg\,(0, t_N) = (1 - R)\sum_{i=1}^{N} P\,(0, t_i)\Pr\,(t_{i-1} < t_D < t_i)$ recognizing in

$\Pr(t_{i-1} < t_D < t_i)$ the intertemporal default probability between time t_{i-1} and time t_i :

$$PROT\_Leg(0, t_N) = (1 - R) \sum_{i=1}^{N} P(0, t_i) PD_{t_{i-1}, t_i} \tag{3.8}$$

Given the equations 3.4 and 3.8, the value of the CDS spread S^* is determined by solving the following equality with respect to the unknown constant S:

$$S \sum_{i=1}^{N} P(0, t_i) \left(1 - PD_{0, t_i}\right) = (1 - R) \sum_{i=1}^{N} P(0, t_i) PD_{t_{i-1}, t_i} \tag{3.9}$$

and hence:

$$S^* = \frac{(1 - R) \sum_{i=1}^{N} P(0, t_i) PD_{t_{i-1}, t_i}}{\sum_{i=1}^{N} P(0, t_i) \left(1 - PD_{0, t_i}\right)} \tag{3.10}$$

3.4 Bootstrapping default probabilities from CDS spreads

In this section we describe how to perform the bootstrapping technique to infer default probabilities of an issuer from the term structure of the CDS spreads referred to him.

To this aim, we firstly introduce the concept of survival probability, that is the probability that no default occurs before a given date.

Formally, let us denote by PS_{0,t_i} the cumulative survival probability until time t_i, that is the probability that no default occurs between time 0 and time t_i. This probability is simply one's complement of the probability of default in the same time interval, which is given by the cumulative default probability until time t_i, i.e.:

$$PS_{0,t_i} = 1 - PD_{0,t_i} \tag{3.11}$$

From equation 3.11 we can also derive the relationship between survival probabilities and intertemporal default probabilities.
In fact, the difference between two consecutive survival probabilities is equal to:
$PS_{0,t_{i-1}} - PS_{0,t_i} = \left(1 - PD_{0,t_{i-1}}\right) - \left(1 - PD_{0,t_i}\right)$
simplifying:
$PS_{0,t_{i-1}} - PS_{0,t_i} = PD_{0,t_i} - PD_{0,t_{i-1}}$
using equation 3.1:

$$PS_{0,t_{i-1}} - PS_{0,t_i} = PD_{t_{i-1},t_i} \tag{3.12}$$

meaning that the difference between two consecutive cumulative survival probabilities is equal to the intertemporal default probability.

Using the above equalities, we can rewrite equation 3.9 in terms of survival probabilities:

$$S \sum_{i=1}^{N} P(0, t_i) PS_{0,t_i} = (1 - R) \sum_{i=1}^{N} P(0, t_i) \left(PS_{0,t_{i-1}} - PS_{0,t_i}\right) \tag{3.13}$$

Assuming that the occurrence of the default is described by a Poisson process, we can introduce the cumulative hazard rate parameter λ_{0,t_i} to express the cumulative survival probability until time t_i as:

$$PS_{0,t_i} = e^{-\lambda_{0,t_i}} \tag{3.14}$$

or, equivalently, being:
$\lambda_{0,t_i} = \sum_{j=1}^{i} \lambda_{t_{j-1},t_j}$,
where λ_{t_{j-1},t_j}, $j = 1, 2, ..., i$, is the intertemporal hazard rate:

$$PS_{0,t_i} = e^{-\sum_{j=1}^{i} \lambda_{t_{j-1},t_j}} \tag{3.15}$$

By using equation 3.15, equation 3.13 can be written in term of hazard rates as follows:

$$\begin{aligned} & S\sum_{i=1}^{N} P(0,t_i)\, e^{-\sum_{j=1}^{i} \lambda_{t_{j-1},t_j}} = \\ & = (1-R)\sum_{i=1}^{N} P(0,t_i)\left(e^{-\sum_{j=1}^{i-1} \lambda_{t_{j-1},t_j}} - e^{-\sum_{j=1}^{i} \lambda_{t_{j-1},t_j}}\right) \end{aligned} \tag{3.16}$$

where for $i = 1$ we have: $e^{-\sum_{j=1}^{0} \lambda_{t_{j-1},t_j}} = 1$.

The LHS of equation 3.16 is the expected discounted value of the CDS premium leg, i.e.:

$$PREM\_Leg\,(0,t_N) = S\sum_{i=1}^{N} P(0,t_i)\, e^{-\sum_{j=1}^{i} \lambda_{t_{j-1},t_j}}$$

and the RHS is the expected discounted value of the CDS protection leg, i.e.:

$$PROT\_Leg\,(0,t_N) = (1-R)\sum_{i=1}^{N} P(0,t_i)\left(e^{-\sum_{j=1}^{i-1} \lambda_{t_{j-1},t_j}} - e^{-\sum_{j=1}^{i} \lambda_{t_{j-1},t_j}}\right)$$

Knowing that the market quotes CDS spread values over an increasing set of maturities, we can use equation 3.16 to apply the bootstrapping technique described hereafter which allows to derive the implicit intertemporal hazard rates and then the survival probabilities and the default probabilities (both cumulative and intertemporal) for different maturities.

Let us assume that in the market are available CDS spreads S_1, $S_2, ..., S_m$ associated with the annual maturities $T_y, y = 1, ..., m$.

The bootstrapping technique is an iterative procedure that, starting from the quotes of CDS spreads associated with the lowest maturity, allows to obtain all the probabilities we are looking for by using at any step the results of the previous one.

The first step provides an estimate of λ_{0,T_1} from the quote, S_{1y}, of the one-year by solving the following equation (which is nothing more than a particular case of equation 3.16):

$$S_{T_1} \cdot P(0,T_1) \cdot e^{-\lambda_{0,T_1}} = (1-R) \cdot P(0,T_1) \cdot \left(1 - e^{-\lambda_{0,T_1}}\right) \tag{3.17}$$

where $e^{-\lambda_{0,T_1}}$ is the cumulative survival probability between time 0 and time T_1 and $\left(1-e^{-\lambda_{0,T_1}}\right)$ is the cumulative default probability between 0 and 1. From this results and applying equation 3.14 we can determine the cumulative (and actually also intertemporal) survival probability time T_1 (that is PS_{0,T_1}), and, by using equation 3.11, also the cumulative (and actually also intertemporal) default probability until time T_1, (that is PD_{0,T_1}).

The second step consists into find out the second intertemporal hazard rate, i.e. λ_{T_1,T_2}. This can be done by using as input the two-year maturity CDS spread S_{T_2} and the estimate of the first intertemporal hazard rate resulting from the previous step, i.e. λ_{0,T_1}, in order to solve the following equation for λ_{T_1,T_2}:

$$\begin{aligned} & S_{T_2} \cdot \left[P\left(0,T_1\right) \cdot e^{-\lambda_{0,T_1}} + P\left(0,T_2\right) \cdot e^{-\left(\lambda_{0,T_1}+\lambda_{T_1,T_2}\right)}\right] = \\ & = (1-R) \cdot \left[P\left(0,T_1\right) \cdot \left(1-e^{-\lambda_{0,T_1}}\right) + P\left(0,T_2\right) \cdot \left(e^{-\lambda_{0,T_1}} - e^{-\left(\lambda_{0,T_1}+\lambda_{T_1,T_2}\right)}\right)\right] \end{aligned} \tag{3.18}$$

We have to repeat this procedure for all maturities, using each time the obtained results from the previous years, until the largest maturity T_m.

In general, for any maturity T_y, given the y-year CDS quote S_{T_y} and all the intertemporal hazard rates obtained for all the shortest maturities – i.e. λ_{0,T_1}, λ_{T_1,T_2}, ..., $\lambda_{T_{y-2},T_{y-1}}$ – we can determine the intertemporal hazard rate λ_{T_{y-1},T_y} by solving the following equation:

$$\begin{aligned} & S_{T_y} \cdot \sum_{i=1}^{y} P\left(0,T_i\right) \cdot e^{-\sum_{j=1}^{i} \lambda_{T_{j-1},T_j}} = \\ & = (1-R) \cdot \sum_{i=1}^{y} P\left(0,T_i\right) \cdot \left(e^{-\sum_{j=1}^{i-1} \lambda_{T_{j-1},T_j}} - e^{-\sum_{j=1}^{i} \lambda_{T_{j-1},T_j}}\right) \end{aligned} \tag{3.19}$$

At this point we have all the values of the hazard rates and we are able to compute all the survival and the default probabilities for a given set of maturities and, then, we have all the elements that we need to derive the pricing formula for a defaultable coupon bond as described in next section.

3.5 Pricing of a defaultable coupon bond

In this section we will present the impact of the credit risk on the price of a defaultable coupon bond. As we said before, credit risk has a negative impact on the bond value, namely the higher is the credit risk and lower is the value of the bond.

Let us assume we want to price a coupon bond expiring at time T with the following characteristics:

- Payment dates: t_1, t_2, ..., $t_M = T$;
- Cash flows: X_1, X_2, ..., X_M, where the first $M-1$ cash flows are coupons paid at the corresponding payment dates and the last cash flow X_M includes both the last coupon and the principal amount.

We firstly observe that the above cash flows embed the credit risk of the issuer.

The price at time 0 of this bond is equal to the expected discounted value of all bond payments (both coupons and principal at maturity T) under the risk neutral probability measure $\mathbb{P}$, i.e.:

$$Bond\_Value\,(0,T) = \sum_{i=1}^{M} \mathbb{E}^{\mathbb{P}} \left[e^{-\int_0^{t_i} r_s ds} X_i \right] \tag{3.20}$$

In presence of credit risk, each cash flow X_i can be expressed as the sum of two random variables:

$$X_i = Y_i \cdot \mathbf{I}_{[t_D > t_i]} + R \cdot \mathbf{I}_{[t_{i-1} < t_D < t_i]} \tag{3.21a}$$

where:

- Y_i is the payment if no default occurs until time t_i;
- R is the recovery amount that will be paid by the issuer in case of default between time t_{i-1} and time t_i.

Substituting equation 3.21a in equation 3.20 we have that the price at time 0 of the considered bond is:
$Bond\_Value\,(0,T) = \sum_{i=1}^{M} \mathbb{E}^{\mathbb{P}} \left[e^{-\int_0^{t_i} r_s ds} \left(Y_i \cdot \mathbf{I}_{[t_D > t_i]} + R \cdot \mathbf{I}_{[t_{i-1} < t_D < t_i]} \right) \right]$
that is:

$$\begin{aligned} &Bond\_Value\,(0,T) = \\ &= \textstyle\sum_{i=1}^{M} \left[\mathbb{E}^{\mathbb{P}} \left(e^{-\int_0^{t_i} r_s ds} Y_i \cdot \mathbf{I}_{[t_D > t_i]} \right) + \mathbb{E}^{\mathbb{P}} \left(e^{-\int_0^{t_i} r_s ds} R \cdot \mathbf{I}_{[t_{i-1} < t_D < t_i]} \right) \right] \end{aligned} \tag{3.22}$$

By assuming that there is independence between the occurrence of the default and the interest rate dynamics we can rewrite equation 3.22 as:
$Bond\_Value\,(0,T) = \sum_{i=1}^{M} \left[\mathbb{E}^{\mathbb{P}} \left(e^{-\int_0^{t_i} r_s ds} Y_i \right) \mathbb{E}^{\mathbb{P}} \left(\mathbf{I}_{[t_D > t_i]} \right) + \right.$
$\left. + R\mathbb{E}^{\mathbb{P}} \left(e^{-\int_0^{t_i} r_s ds} \right) \mathbb{E}^{\mathbb{P}} \left(\mathbf{I}_{[t_{i-1} < t_D < t_i]} \right) \right]$
and therefore:

$$Bond\_Value\,(0,T) = \sum_{i=1}^{M} \left[\mathbb{E}^{\mathbb{P}} \left(e^{-\int_0^{t_i} r_s ds} Y_i \right) \cdot PS_{0,t_i} + R \cdot P\,(0,t_i) \cdot PD_{t_{i-1},t_i} \right] \tag{3.23}$$

Given equation 3.23 we can determine the price at time 0 of any defaultable coupon bond. To compute this price we need to determine the survival probabilities and the default probabilities of the issuer at each payment date through the bootstrapping technique described in the previous section[2].

After that we have to compute the discounted expected value of the payments without default and then we have to substitute all these values into the pricing formula given in equation 3.23 in order to obtain the fair value of the bond.

[2]Notice that this formula is so general that it holds clearly also for zero-coupon bonds

It is important to observe that equation 3.23 is a quite general formula which can be applied to defaultable bonds having whatever coupons structure, hence including those bonds whose coupons are indexed to a monetary market interest rate with the presence of an upper bound (cap) and/or a lower bound (floor).

As a consequence the above formula can be used in the evaluation of the stochastic interest bonds that we will analyze in Chapter 5.

More in detail, the formula in equation 3.23 will be used to determine the fair value of the examined bonds according to the unbundling methodology, to price the pure-bond component of those securities; a similar formula (in the sense of a weighted risk-neutral discounted expectation) will be also used to price the derivative components present in those bonds.

With regard to the pricing procedure based on Monte Carlo simulation, we shall not use the above formula. On the contrary, in order to take into account for the credit risk, at any payment date we will assume that in a certain number of trajectories, determined proportionally to intertemporal estimated default probabilities, the cash flow of the bond is equal to the recovery amount R, which will be assumed equal to the 40% of the bond face value.

Chapter 4

The collared floaters

4.1 Introduction

The aim of this Chapter is the study of a specific kind of stochastic interest bonds called *collared floaters* which have been quite frequent in the recent bond issues made by Italian banks, and represent the majority of the specific bonds that will be analyzed in Chapter 5.

Collared floaters are structured bonds with variable coupons indexed to a floating market rate such as the Libor or the Euribor and bounded both up and down by two fixed rates.

In next section we will discuss the payoff structure of a generic collared floater by providing a description of its main features, and in particular of its coupon structure and of the derivatives embedded in it.

Then, we will proceed to the unbundling of a generic collared floater by showing how its payoff can be decomposed into the sum of elementary payoffs, namely either as the sum of several zero coupon bonds, a long cap and a short cap or as the sum of a floating-rate coupon bond, a long floor and a short cap.

4.2 General features and risk profile

The coupon structure of a collared floater can be written as follows[1]:

$$cpn\ rate_t = \min\left[\max(E_{t-1}\%; k_1\%); k_2\%\right] \tag{4.1}$$

where:

- $cpn\ rate_t$ is the coupon rate associated with the generic time t;
- $k_1\%$ is minimum fixed coupon rate;

[1] It is worth noticing that $k_1\%$ and $k_2\%$ can vary over the bond life, as in some the stochastic interest bonds analized in Chapter 5.

- $k_2\%$ is maximum fixed coupon rate;
- $E_{t-1}\%$ is the value at time $t-1$ of the floating market rate.

From equation 4.1 we can deduce the following relationship:

$$k_1\% > k_2\% \geqslant 0 \tag{4.2}$$

The presence of a minimum and a maximum fixed coupon rate guarantees the bond-holder against the risk of an excessive fall in the underlying floating rate and the issuer against the risk of an excessive rise in the same rate.

4.3 Unbundling of a generic collared floater

The unbundling of a structured bond consists into its decomposition in elementary parts: one or more bond-like components and one or more derivative components.

To this aim we consider a collared floater with the following general features:

- issue date: $t = 0$
- maturity date: T_{CF}
- face value or repayment value: FV
- frequency of coupon payments (*tenor*): δ
- coupon payment dates: $t_1, ..., t_n = T_{CF}$ (with $0 < t_1 < ... < t_n$)
- coupon structure as described in the previous section.

As shown hereafter, the collared floater can be decomposed as follows:

1. a bond component that is the sum of a zero-coupon bond with face value FV and maturity date T_{CF} that replicates the repayment of the capital invested in the collared floater at maturity;
2. a derivative component that replicates the variable coupons paid by the bond from the date t_1 until the maturity T_{CF} according to the coupon rate expressed in equation 4.1.

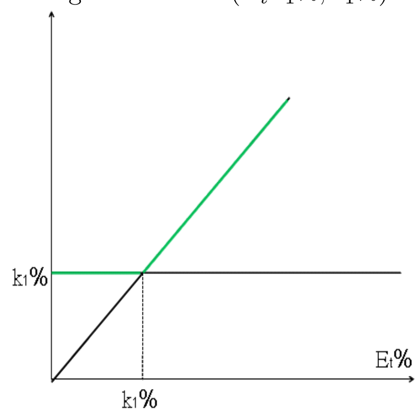
By analyzing the RHS of equation 4.1, we can better understand the coupon structure of the collared floater.

The first step of the unbundling procedure is to find:

$$\max(E_{t-1}\%, k_1\%)$$

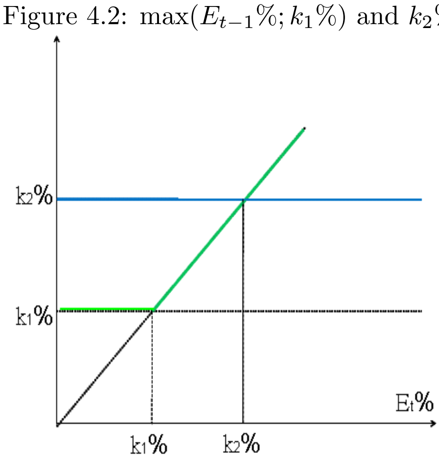
that is the maximum between the bisector of the first and third quadrant and a straight line parallel to the abscissa axis, with intercept equal to $k_1\%$ (see Figure 4.1).

Figure 4.1: $\max(E_{t-1}\%; k_1\%)$



The second step is the research of the minimum between the above maximum and the function $k_2\%$, that is a straight line parallel to the abscissa axis (see Figure 4.2).

Figure 4.2: $\max(E_{t-1}\%; k_1\%)$ and $k_2\%$



The coupon rate for the generic payment date t, denoted by $cpn\ rate_t$, is the red line in Figure 4.3.

The red line represents the payoff of a bull spread composed as follows:

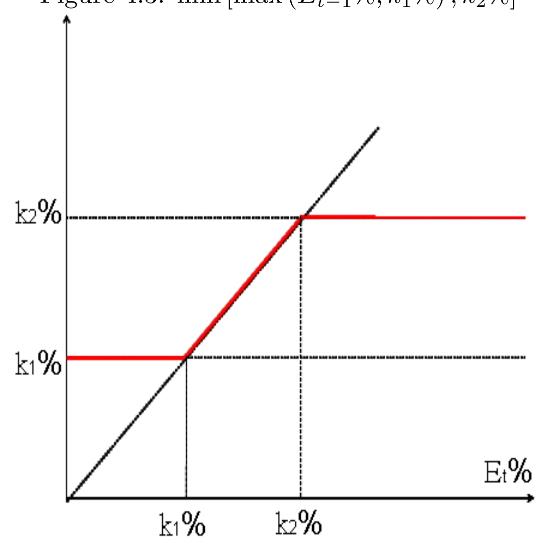
1. a long call on the interest rate $E_{t-1}\%$ with strike price equal to $K_{long}\% = k_1\%$ and maturity T, purchased by paying a premium whose value at maturity is equal to $C_{long}\%$;

2. a short call on the interest rate $E_{t-1}\%$ with strike price equal to $K_{short}\% = k_2\%$ and maturity T, sold by receiving a premium whose value at maturity is equal to $C_{short}\%$;

3. a positive component $H\%$ defined as follows:

$$H\% = k_1\% + C_{long}\% - C_{short}\%$$

Figure 4.3: $\min\left[\max\left(E_{t-1}\%, k_1\%\right), k_2\%\right]$



The combination of buying and selling the two call options is shown in Figure 4.4, where:

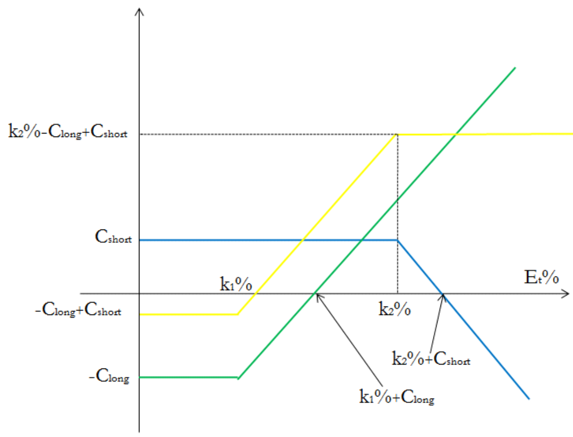
- the green line represents the long call payoff, i.e.:

$$-C_{long}\% + \max(E_{t-1}\% - k_1\%, 0)$$

- the blue line represents the short call payoff, i.e.:

$$C_{short}\% - \max(E_{t-1}\% - k_2\%, 0)$$

Figure 4.4: Payoff of a bull spread obtained as combination of two call options



- the yellow line represents the bull spread payoff as sum of the two above payoffs:

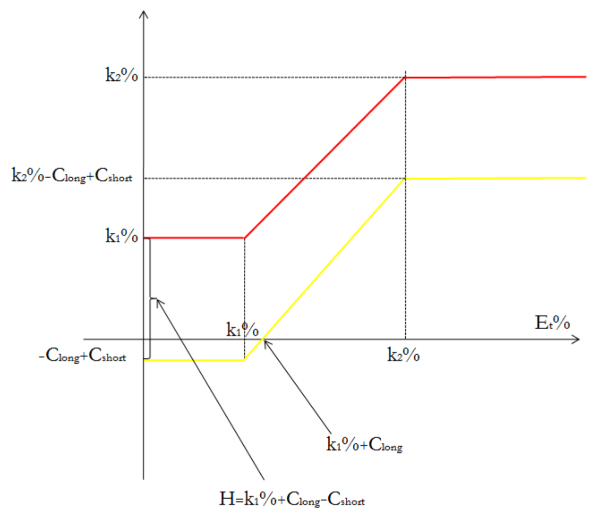
$$\begin{aligned} & -C_{long}\% + \max(E_{t-1}\% - k_1\%, 0) + \\ & +C_{short}\% - \max(E_{t-1}\% - k_2\%, 0) \end{aligned} \tag{4.3}$$

As a bull spread holder hopes that the underlying price will go up, the holder of a collared floater hopes that the market rate $E_{t-1}\%$ will go up to obtain the maximum coupon rate.

From an analytical point of view, by adding $H\% = k_1\% + C_{long}\% - C_{short}\%$ to the bull spread payoff expressed in equation 4.3 we obtain the coupon rate of the collared floater at the payment date t (see Figure 4.5):

$$\begin{aligned} cpn\ rate_t &= -C_{long}\% + \max(E_{t-1}\% - k_1\%, 0) + \\ & \quad +C_{short}\% - \max(E_{t-1}\% - k_2\%, 0) + k_1\% + C_{long}\% - C_{short}\% \\ &= k_1\% + \max(E_{t-1}\% - k_1\%, 0) - \max(E_{t-1}\% - k_2\%, 0) \end{aligned} \tag{4.4}$$

Figure 4.5: Coupon rate $cpn\ rate_t$ as a vertical upward traslated bull spread by $H\%$



In the RHS of equation 4.4 we recognize the sum of a fixed coupon rate $k_1\%$, a long caplet written on the floating interest rate behind the collared floater and whose cap rate is equal to $k_1\%$, and a short caplet written on the same floating interest rate and whose cap rate is equal to $k_2\%$.

From a graphical point of view, adding $H\% = k_1\% + C_{long}\% - C_{short}\%$ to the bull spread payoff we have an upward translation of the yellow line (see Figure 4.4) and we obtain the coupon rate of the collared floater as appearing in the red line of Figure 4.3 (see Figure 4.5).

From Figures 4.4 and 4.5 we can see that:

- if $E_{t-1}\% < k_1\%$:

  - the short call payoff is equal to $C_{short}\%$;

  - the long call payoff is equal to $-C_{long}\%$;

  and then:

$$\begin{aligned} cpn\ rate_t &= C_{short}\% - C_{long}\% + H\% = \\ &= C_{short}\% - C_{long}\% + k_1\% + C_{long}\% - C_{short}\% = \\ &= k_1\% \end{aligned}$$

- if $k_1\% \leqslant E_{t-1}\% \leqslant k_2\%$:

  - the short call payoff is equal to $C_{short}\%$;

  - the long call payoff is equal to $-C_{long}\% + (E_{t-1}\% - k_1\%)$;

and then:

$$\begin{aligned} \textit{cpn rate}_t &= C_{short}\% - C_{long}\% + (E_{t-1}\% - k_1\%) + H\% = \\ &= C_{short}\% - C_{long}\% + (E_{t-1}\% - k_1\%) + \\ &+ k_1\% + C_{long}\% - C_{short}\% = \\ &= E_{t-1}\% \end{aligned}$$

- if $E_{t-1}\% > k_2\%$:

  - the short call payoff is equal to $C_{short}\% - (E_{t-1}\% - k_2\%)$;

  - the long call payoff is equal to $-C_{long}\% + (E_{t-1}\% - k_1\%)$;

  and then:

$$\begin{aligned} \textit{cpn rate}_t &= C_{short}\% - (E_{t-1}\% - k_2\%) - C_{long}\% + (E_{t-1}\% - k_1\%) + H\% = \\ &= C_{short}\% - (E_{t-1}\% - k_2\%) - C_{long}\% + (E_{t-1}\% - k_1\%) + \\ &+ k_1\% + C_{long}\% - C_{short}\% = \\ &= -(E_{t-1}\% - k_2\%) + (E_{t-1}\% - k_1\%) + k_1\% = \\ &= k_2\% \end{aligned}$$

In the light of the above analysis it is clear that the payoff of the generic collared floater is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value FV and maturity date $t_n = T_{CF}$. This component replicates the repayment of the capital invested in the collared floater at maturity;

   b) n zero-coupon bonds with face value equal to $k_1\% \cdot FV \cdot \delta$, and maturity dates $t_1, ..., t_n = T_{CF}$. These n zero coupon bonds replicate the fixed part of each coupon;

2. a derivative component that, in accordance with equation 4.4, is structured as follows:

   a) a long cap with face value FV that:
   - has maturity equal to $t_n = T_{CF}$;
   - is composed by n caplets with maturities respectively equal to $t_1, ..., t_n = T_{CF}$;
   - has a tenor equal to δ;
   - has a cap rate equal to $k_1\%$;
   - has as underlying the floating interest rate $E_{t-1}\%$ at which are indexed the coupons;

   b) a short cap with face value FV that:
   - has maturity equal to $t_n = T_{CF}$;
   - is composed by n caplets with maturities respectively equal to $t_1, ..., t_n = T_{CF}$;
   - has a tenor equal to δ;
   - has a cap rate equal to $k_2\%$;
   - has as underlying the floating interest rate $E_{t-1}\%$ at which are indexed the coupons.

With reference to the two caps listed at points sub **2.a)** and sub **2.b)** we point out that, on the basis of the results presented in Chapter 2:

- the long cap is equal to a long portfolio of n European put options with maturities $t_0, ..., t_{n-1}$, strike equal to FV and underlying assets n zero-coupon bonds with maturities $t_1, ..., t_n = T_{CF}$, each of them with face value equal to $FV \cdot (1 + k_1\% \cdot \delta)$;
- the short cap is equal to a short portfolio of n European put options with maturities $t_0, ..., t_{n-1}$, strike equal to FV and underlying assets n zero-coupon bonds with maturities $t_1, ..., t_n = T_{CF}$, each of them with face value equal to $FV \cdot (1 + k_2\% \cdot \delta)$.

We also need to precise that other unbundlings of collared floater can be done, which are equivalent to that one presented here above.

In particular, starting from the formula in equation 4.1:

$$cpn\ rate_t = \min\left[\max(E_{t-1}\%; k_1\%); k_2\%\right] \tag{4.1}$$

the coupon rate can be also expressed as follows:

$$cpn\ rate_t = E_{t-1}\% + \max\left(k_1\% - E_{t-1}\%, 0\right) - \max\left(E_{t-1}\% - k_2\%, 0\right) \tag{4.5}$$

that is as the sum of a pure floater bond, a long floor with floor rate equal to $k_1\%$ and a short cap with cap rate equal $k_2\%$.

After a little algebra, this alternative unbundling can be led back to that one presented in equation 4.4:

$$\begin{aligned} cpn\ rate_t &= E_{t-1}\% + \max\left(k_1\% - E_{t-1}\%, 0\right) - \max\left(E_{t-1}\% - k_2\%, 0\right) \\ &= \max\left(k_1\%; E_{t-1}\%\right) - \max\left(E_{t-1}\% - k_2\%, 0\right) \\ &= k_1\% + \max\left(0, E_{t-1}\% - k_1\%\right) - \max\left(E_{t-1}\% - k_2\%, 0\right) \quad (4.4) \end{aligned}$$

As final remarks, we underline that in some cases the coupon structure represented in equation 4.1 applies only to a sub-set of the coupons paid by the stochastic interest bond. Moreover it is possible that the bond pays a floating rate increased by a given spread, say spr. The presence of the spread modifies equation 4.1 as follows:

$$cpn\ rate_t = \min\left[\max(E_{t-1}\% + spr; k_1\%); k_2\%\right] \tag{4.6}$$

and, as a consequence equation 4.4 becomes:

$$cpn\ rate_t = k_1\% + \max\left(0, E_{t-1}\% - k_1^*\%\right) - \max\left(E_{t-1}\% - k_2^*\%, 0\right) \tag{4.7}$$

where: $k_1^* = k_1 - spr$ and $k_2^* = k_2 - spr$.

Chapter 5

Pricing of some stochastic interest bonds

5.1 Introduction

In this Chapter we will use the equilibrium model developed by Vasicek and the no-arbitrage model developed by Hull and White (see Chapter 1) to price some stochastic interest bonds issued by four of the major Italian banks in the first semester of 2010. We will consider 10 recently issued bonds whose characteristics are summarized in Table 1 hereafter.

Table 1. Stochastic Interest Bonds

BOND IDENTIFICATION NUMBER	ISSUER	ISIN	ISSUE DATE	MATURITY	COUPON RATE	FLOOR	CAP
BNL_1	BNL	IT0004589484	31/03/2010	31/03/2015	eur3m+0.10%	2%	3.75%
BNL_2	BNL	IT0004596414	30/04/2010	30/04/2015	2%(2years); eur3m+0.10%	2.1%	4%
BNL_3	BNL	IT0004606411	31/05/2010	31/05/2015	2%(2years); eur3m+0.10%	2.1%	4%
POPOLARE_1	POPOLARE	IT0004605009	31/05/2010	31/05/2015	eur6m	2.8%	3.65%
POPOLARE_2	POPOLARE	IT0004593874	30/04/2010	30/04/2015	eur6m+0.40%	3%	4%
UNICREDIT_1	UNICREDIT	IT0004607302	31/05/2010	31/05/2016	eur3m	2.1% first 3 years	4% last 3 years
UNICREDIT_2	UNICREDIT	IT0004587496	31/03/2010	31/03/2016	eur3m	2%	4.1%
UNICREDIT_3	UNICREDIT	IT0004591456	15/04/2010	15/04/2016	eur3m	2%(years1,2) 2.5%(years3,4) 2.83%(years5,6)	3.5%(years1,2) 4%(years3,4) 4.5%(years5,6)
UNICREDIT_4	UNICREDIT	IT0004566193	29/01/2010	29/01/2016	eur3m	2.3%	4.9%
INTESA_1	INTESA	IT0004594658	19/04/2010	19/04/2016	eur6m	2.7%	-

As indicated by Table 1 bonds BNL_1, Popolare_1, Popolare_2, Unicredit_2, Unicredit_3 and Unicredit_4 are pure collared floaters; in particular the bond Unicredit_3 is a collared floater where the strikes $k_1\%$ and $k_2\%$ vary over time. Bonds BNL_2 and BNL_3 are mixed fixed-floating rate bonds, where the coupon structure of the floating coupon is again collared floater-like. The bond Unicredit_1 is a six-year stochastic interest security whose coupon structure embeds a long floor for the first three years and a short cap for the remaining three years, and the bond Intesa_1 is a stochastic interest security embedding a floor with the same maturity of the bond.

In order to compute the fair value of these bonds we will use two alternative methodologies.

The first one will rely on the results obtained from the unbundling of the bond performed consistently with the results of Chapters 2 and 4 and it will use them to evaluate the stochastic interest bonds as the sum of the prices of each elementary component bundled in their structure.

The second pricing technique will skip any consideration on the specific components of the financial structure of the bonds and it will rely on Monte Carlo simulations.

In both methodologies we will take into account the credit risk of the issuer and its impact on the bond value.

In particular, we will exploit the results of Chapter 3, and when using the unbundling technique we will apply the general pricing formula for defaultable coupon bonds of equation 3.23 assuming a recovery rate $R = 40\%$. Similarly, when using the Monte Carlo simulations methodology, at each coupon payment date we will cut off a number of trajectories linearly proportional to the intertemporal default probability estimated (from the CDS spreads quoted on the market for any issuer) for the time interval going from the previous coupon payment date to the considered coupon payment date. Indeed, for these trajectories we will assume the occurrence of the default and the payment of an amount equal to the recovery rate $R = 40\%$ multiplied by the face value of the bond.

The two pricing methodologies provide consistent results, hence representing two valid alternatives to evaluate stochastic interest bonds.

The fair values of the bonds obtained via the two above mentioned methodologies (and for each of the two affine term structure models considered) will be also compared with their theoretical values provided by the issuer inside the final terms of the prospectus, when available, in order to explore the reliability and the accuracy of the informative set included in the documents that investors use to take their financial decisions.

5.2 Concrete examples

5.2.1 Description and unbundling of the bond BNL_1

The main characteristics of the bond BNL_1 are summarized in Table 2 hereafter:

Table 2. Characteristics of the bond BNL_1

Denomination of the financial instrument	BNL protected yield 2010/2015. Five years bonds with quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2% and maximum rate (cap) 3.75%
ISIN	IT0004589484
Total amount and currency	75,000,000.00 Euro
Face value	1000.00 Euro
Issue date	31/03/2010
Maturity date	31/03/2015
Repayment date	31/03/2015
Issue price	100% of face value
Return	Quarterly floating coupon from 30/06/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci = min[max(3mEuribor+0.10%;2%);3.75%] i = 1,...,20
Coupon payment dates	31/03, 30/06, 30/09, 31/12 from 30/06/2010 to 31/03/2015

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater, namely:

1. issue date: 31/03/2010;
2. maturity date: 31/03/2015;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): quarterly;
5. coupon payment dates:

30/06/2010	30/09/2011	31/12/2012	31/03/2014
30/09/2010	31/12/2011	31/03/2013	30/06/2014
31/12/2010	31/03/2012	30/06/2013	30/09/2014
31/03/2011	30/06/2012	30/09/2013	31/12/2014
30/06/2011	30/09/2012	31/12/2013	31/03/2015

6. each coupon is indexed to the 3-month Euribor plus a spread of 10 basis points according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2\%; 3mEuribor_{t_{i-1}} + 0.10\%\right), 3.75\%\right] \qquad (5.1)$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 1, ..., 20$;
- $3mEuribor_{t_{i-1}}$ is the three months Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.7, the coupon rate of equation 5.1 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} \quad &= 2\% + \max\big(3mEuribor_{t_{i-1}} - 1.9\%, 0\big) + \\ &- \max\big(3mEuribor_{t_{i-1}} - 3.65\%, 0\big) \end{aligned} \tag{5.2}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 31/03/2015. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 20 zero-coupon bonds with face value 5 *Euro* (i.e.: $1000 \cdot (2\% \cdot 0.25)$) and with maturities equal to the coupon payment dates;

2. a derivative component that, in accordance with equation 5.2, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 31/03/2015;
   - is composed by 20 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 1.9%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

   b) a short cap with value of 1000 *Euro* that:
   - has maturity 31/03/2015;
   - is composed by 20 *caplets* with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 3.65%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 20 European put options with maturities:

31/03/2010	30/06/2011	30/09/2012	31/12/2013
30/06/2010	30/09/2011	31/12/2012	31/03/2014
30/09/2010	31/12/2011	31/03/2013	30/06/2014
31/12/2010	31/03/2012	30/06/2013	30/09/2014
31/03/2011	30/06/2012	30/09/2013	31/12/2014

whose underlying securities are 20 zero-coupon bonds with maturities:

30/06/2010	30/09/2011	31/12/2012	31/03/2014
30/09/2010	31/12/2011	31/03/2013	30/06/2014
31/12/2010	31/03/2012	30/06/2013	30/09/2014
31/03/2011	30/06/2012	30/09/2013	31/12/2014
30/06/2011	30/09/2012	31/12/2013	31/03/2015

and face value of 1004.75 (i.e.: $1000 \cdot (1 + 1.9\% \cdot 0.25)$).

Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 20 European put options with maturities:

31/03/2010	30/06/2011	30/09/2012	31/12/2013
30/06/2010	30/09/2011	31/12/2012	31/03/2014
30/09/2010	31/12/2011	31/03/2013	30/06/2014
31/12/2010	31/03/2012	30/06/2013	30/09/2014
31/03/2011	30/06/2012	30/09/2013	31/12/2014

whose underlying securities are 20 zero-coupon bonds with maturities:

30/06/2010	30/09/2011	31/12/2012	31/03/2014
30/09/2010	31/12/2011	31/03/2013	30/06/2014
31/12/2010	31/03/2012	30/06/2013	30/09/2014
31/03/2011	30/06/2012	30/09/2013	31/12/2014
30/06/2011	30/09/2012	31/12/2013	31/03/2015

and with face value 1009.125 (i.e.: $1000 \cdot (1 + 3.65\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.2 Description and unbundling of the bond BNL_2

The main characteristics of the bond BNL_2 are summarized in Table 3 hereafter:

Table 3. Characteristics of the bond BNL_2

Denomination of the financial instrument	BNL mixed rate with cap and floor 2010/2015. Five years bonds with quarterly fixed coupon at 2% for the first two years and quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2.10% and maximum rate (cap) 4%
ISIN	IT0004596414
Total amount and currency	150,000,000.00 Euro
Face value	1000.00 Euro
Issue date	30/04/2010
Maturity date	30/04/2015
Repayment date	30/04/2015
Issue price	100% of face value
Return	Quarterly floating coupon from 30/07/2010 to maturity
Coupon type	Fixed plus floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci =2% i = 1,...,8 Ci = min[max(3mEuribor+0.10%;2.10%);4%] i = 9,...,20
Coupon payment dates	30/01, 30/04, 30/07, 30/10 from 30/07/2010 to 30/04/2015

From the above table it is possible to identify the key information required for the unbundling and the pricing of this bond which combines a fixed coupon rate with a collared floater structure, namely:

1. issue date: 30/04/2010;
2. maturity date: 30/04/2015;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): quarterly;
5. coupon payment dates:

30/07/2010	30/10/2011	30/01/2012	30/04/2014
30/10/2010	30/01/2011	30/04/2013	30/07/2014
30/01/2010	30/04/2012	30/10/2013	30/10/2014
30/04/2011	30/07/2012	30/07/2013	30/01/2014
30/07/2011	30/10/2012	30/01/2013	30/04/2015

6. 8 fixed coupons at a coupon rate of 2% for the first two years;

7. 12 floating coupons for the last three years, each one of them indexed to the 3 months Euribor plus a spread of 10 bps according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.1\%; \left(3mEuribor_{t_{i-1}} + 0.10\%\right)\right), 4\%\right] \quad (5.3)$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 9, ..., 20$;
- $3mEuribor_{t_{i-1}}$ is the three months Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.7, the coupon rate of equation 5.3 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.1\% + \max\left(3mEuribor_{t_{i-1}} - 2\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 3.9\%, 0\right) \end{aligned} \quad (5.4)$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 30/04/2015. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 8 zero coupon bonds with face value equal to 5 *Euro* (i.e.: $1000 \cdot (2\% \cdot 0.25)$) and with maturities:

30/07/2010	30/07/2011
30/10/2010	30/10/2011
30/01/2010	30/01/2011
30/04/2011	30/04/2012

   c) 12 zero-coupon bonds with face value 5.25 *Euro* (i.e.: $1000 \cdot (2.1\% \cdot 0.25)$) and with maturities:

30/07/2012	30/01/2013
30/10/2012	30/04/2014
30/01/2012	30/07/2014
30/04/2013	30/10/2014
30/07/2013	30/01/2014
30/10/2013	30/04/2015

2. a derivative component that, in accordance with equation 5.4, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 30/04/2015;

- is composed by 12 caplets each with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 2%;
- has as underlying the $3-month$ Euribor ($3mEuribor$) at which are indexed the coupons;

b) a short cap with value of 1000 *Euro* that:
- has maturity 30/04/2015;
- is composed by 12 caplets with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 3.9%;
- has as underlying the $3-month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 12 European put options with maturities:

30/04/2012	30/10/2013
30/07/2012	30/01/2013
30/10/2012	30/04/2014
30/01/2012	30/07/2014
30/04/2013	30/10/2014
30/07/2013	30/01/2014

  whose underlying securities are 12 zero-coupon bonds with maturities:

30/07/2012	30/01/2013
30/10/2012	30/04/2014
30/01/2012	30/07/2014
30/04/2013	30/10/2014
30/07/2013	30/01/2014
30/10/2013	30/04/2015

  and face value of 1005 (i.e.: $1000 \cdot (1 + 2\% \cdot 0.25)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 12 European put options with maturities:

30/04/2012	30/10/2013
30/07/2012	30/01/2013
30/10/2012	30/04/2014
30/01/2012	30/07/2014
30/04/2013	30/10/2014
30/07/2013	30/01/2014

whose underlying securities are 12 zero-coupon bonds with maturities:

30/07/2012	30/01/2013
30/10/2012	30/04/2014
30/01/2012	30/07/2014
30/04/2013	30/10/2014
30/07/2013	30/01/2014
30/10/2013	30/04/2015

and with face value 1009.75 (i.e.: $1000 \cdot (1 + 3.9\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.3 Description and unbundling of the bond BNL_3

The main characteristics of the bond BNL_3 are summarized in Table 4 hereafter:

Table 4. Characteristics of the bond BNL_3

Denomination of the financial instrument	BNL mixed rate with cap and floor 2010/2015. Five years bonds with quarterly fixed coupon at 2% for the first two years and quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2.10% and maximum rate (cap) 4%
ISIN	IT0004606411
Total amount and currency	150,000,000.00 Euro
Face value	1000.00 Euro
Issue date	31/05/2010
Maturity date	31/05/2015
Repayment date	31/05/2015
Issue price	100% of face value
Return	Quarterly floating coupon from 31/08/2010 to maturity
Coupon type	Fixed plus floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci =2% i = 1,...,8 Ci = min[max(3mEuribor+0.10%;2.10%);4%] i = 9,...,20
Coupon payment dates	28/02, 31/05, 31/08, 30/11 from 31/08/2010 to 31/05/2015

From the above table it is possible to identify the key information required for the unbundling and the pricing of this bond which combines a fixed coupon rate with a collared floater structure, namely:

1. issue date: 31/05/2010;
2. maturity date: 31/05/2015;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): quarterly;

5. coupon payment dates:

31/08/2010	30/11/2011	28/02/2013	31/05/2014
30/11/2010	28/02/2012	31/05/2013	31/08/2014
28/02/2011	31/05/2012	31/08/2013	30/11/2014
31/05/2011	31/08/2012	30/11/2013	28/02/2015
31/08/2011	30/11/2012	28/02/2014	31/05/2015

6. 8 fixed coupons at a coupon rate of 2% for the first two years;

7. 12 floating coupons for the last three years, each one of them indexed to the 3 months Euribor plus a spread of 10 bps according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.1\%; \left(3mEuribor_{t_{i-1}} + 0.10\%\right)\right), 4\%\right] \tag{5.5}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 9, ..., 20$;
- $3mEuribor_{t_{i-1}}$ is the three months Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.7, the coupon rate of equation 5.5 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.1\% + \max\left(3mEuribor_{t_{i-1}} - 2\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 3.9\%, 0\right) \end{aligned} \tag{5.6}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

a) a zero-coupon bond with face value 1000 *Euro* and maturity date 31/05/2015. This component replicates the repayment of the capital invested in the bond at maturity;

b) 8 zero coupon bonds with face value equal to 5 *Euro* (i.e.: $1000 \cdot (2\% \cdot 0.25)$) and with maturities:

31/08/2010	31/08/2011
30/11/2010	30/11/2011
28/02/2011	28/02/2012
31/05/2011	31/05/2012

c)12 zero-coupon bonds with face value 5.25 *Euro* (i.e.: $1000 \cdot (2.1\% \cdot 0.25)$) and with maturities equal:

31/08/2012	28/02/2014
30/11/2012	31/05/2014
28/02/2013	31/08/2014
31/05/2013	30/11/2014
31/08/2013	28/02/2015
30/11/2013	31/05/2015

2. a derivative component that, in accordance with equation 5.6, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 31/05/2015;
   - is composed by 12 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 2%;
   - has as underlying the $3month$ Euribor ($3mEuribor$) at which are indexed the coupons;

   b) a short cap with value of 1000 *Euro* that:
   - has maturity 31/05/2015;
   - is composed by 12 caplets with maturities equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 3.9%;
   - has as underlying the $3month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 12 European put options with maturities:

31/05/2012	30/11/2013
31/08/2012	28/02/2014
30/11/2012	31/05/2014
28/02/2013	31/08/2014
31/05/2013	30/11/2014
31/08/2013	28/02/2015

  whose underlying securities are 12 zero-coupon bonds with maturities:

31/08/2012	28/02/2014
30/11/2012	31/05/2014
28/02/2013	31/08/2014
31/05/2013	30/11/2014
31/08/2013	28/02/2015
30/11/2013	31/05/2015

  and face value of 1005 (i.e.: $1000 \cdot (1 + 2\% \cdot 0.25)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 12 European put options with maturities:

31/05/2012	30/11/2013
31/08/2012	28/02/2014
30/11/2012	31/05/2014
28/02/2013	31/08/2014
31/05/2013	30/11/2014
31/08/2013	28/02/2015

whose underlying securities are 12 zero-coupon bonds with maturities equal to:

31/08/2012	28/02/2014
30/11/2012	31/05/2014
28/02/2013	31/08/2014
31/05/2013	30/11/2014
31/08/2013	28/02/2015
30/11/2013	31/05/2015

and with face value 1009.75 (i.e.: $1000 \cdot (1 + 3.9\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.4 Description and unbundling of the bond Popolare_1

The main characteristics of the bond Popolare_1 are summarized in Table 5 hereafter:

Table 5. Characteristics of the bond Popolare_1

Denomination of the financial instrument	Banco Popolare S.C. serie 159. Five years bonds with six-monthly floating coupon indexed to the 6 months Euribor, with minimum rate (floor) 2.80% and maximum rate (cap) 3.65%
ISIN	IT0004605009
Total amount and currency	250,000,000.00 Euro
Face value	1000.00 Euro
Issue date	31/05/2010
Maturity date	31/05/2015
Repayment date	31/05/2015
Issue price	100% of face value
Return	Six-monthly floating coupon from 30/11/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	6 months Euribor
Coupon formula	Ci = min[max(6mEuribor;2.80%);3.65%] i = 1,...,10
Coupon payment dates	31/05, 30/11 from 30/11/2010 to 31/05/2015

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater, namely:

1. issue date: 31/05/2010;

2. maturity date: 31/05/2015;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): semiannual;
5. coupon payment dates:

30/11/2010	31/05/2013
31/05/2011	30/11/2013
30/11/2011	31/05/2014
31/05/2012	30/11/2014
30/11/2012	31/05/2015

6. each coupon is indexed to the 6-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.8\%; 6mEuribor_{t_{i-1}}\right), 3.65\%\right] \tag{5.7}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable every six months at the date t_i;
- $i = 1, ..., 10$;
- $6mEuribor_{t_{i-1}}$ is the six-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

Given equation 4.4, the coupon rate of equation 5.7 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.8\% + \max\left(6mEuribor_{t_{i-1}} - 2.8\%, 0\right) + \\ &- \max\left(6mEuribor_{t_{i-1}} - 3.65\%, 0\right) \end{aligned} \tag{5.8}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 31/05/2015. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 10 zero-coupon bonds with face value 14 *Euro* (i.e.: $1000 \cdot (2.8\% \cdot 0.5)$) and with maturities equal to the coupon payment dates;

2. a derivative component that, in accordance with equation 5.8, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 31/05/2015;
   - is composed by 10 *caplets* each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.5 years;

- has a cap rate equal to 2.8%;
- has as underlying the $6 - month$ Euribor ($6mEuribor$) at which are indexed the coupons;

b) a short cap with value of 1000 *Euro* that:
- has maturity 31/05/2015;
- is composed by 10 caplets with maturities equal to the coupon payment dates;
- has a tenor equal to 0.5 years;
- has a cap rate equal to 3.65%;
- has as underlying the $6month$ Euribor rate ($6mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 10 European put options with maturities:

31/05/2010	30/11/2012
30/11/2010	31/05/2013
31/05/2011	30/11/2013
30/11/2011	31/05/2014
31/05/2012	30/11/2014

  whose underlying securities are 10 zero-coupon bonds with maturities:

30/11/2010	31/05/2013
31/05/2011	30/11/2013
30/11/2011	31/05/2014
31/05/2012	30/11/2014
30/11/2012	31/05/2015

  and face value of 1014 (i.e.: $1000 \cdot (1 + 2.8\% \cdot 0.5)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 10 European put options with maturities:

31/05/2010	30/11/2012
30/11/2010	31/05/2013
31/05/2011	30/11/2013
30/11/2011	31/05/2014
31/05/2012	30/11/2014

  whose underlying securities are 10 zero-coupon bonds with maturities:

30/11/2010	31/05/2013
31/05/2011	30/11/2013
30/11/2011	31/05/2014
31/05/2012	30/11/2014
30/11/2012	31/05/2015

and with face value 1018.25 (i.e.: $1000 \cdot (1 + 3.65\% \cdot 0.5)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.5 Description and unbundling of the bond Popolare_2

The main characteristics of the bond Popolare_2 are summarized in Table 6 hereafter:

Table 6. Characteristics of the bond Popolare_2

Denomination of the financial instrument	Banco Popolare S.C. serie 156. Five years bonds with six-monthly floating coupon indexed to the 6 months Euribor, with minimum rate (floor) 3% and maximum rate (cap) 4%
ISIN	IT0004593874
Total amount and currency	350,000,000.00 Euro
Face value	1000.00 Euro
Issue date	30/04/2010
Maturity date	30/04/2015
Repayment date	30/04/2015
Issue price	100% of face value
Return	Six-monthly floating coupon from 30/10/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	6 months Euribor
Coupon formula	Ci = min[max(6mEuribor+0.40%;3%);4%] i = 1,...,10
Coupon payment dates	30/04, 30/10 from 30/10/2010 to 30/04/2015

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater, namely:

1. issue date: 30/04/2010;

2. maturity date: 30/04/2015;

3. face value: 1000 *Euro*;

4. frequency of payment (i.e. *tenor*): semiannual;

5. coupon payment dates:

30/10/2010	30/04/2013
30/04/2011	30/10/2013
30/10/2011	30/04/2014
30/04/2012	30/10/2014
30/10/2012	30/04/2015

6. each coupon is indexed to the 6-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(3\%; 6mEuribor_{t_{i-1}} + 0.40\%\right), 4\%\right] \tag{5.9a}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable every six months at the date t_i;
- $i = 1, ..., 10$;
- $6mEuribor_{t_{i-1}}$ is the six-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.7, the coupon rate of equation 5.9a can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 3\% + \max\left(6mEuribor_{t_{i-1}} - 2.6\%, 0\right) + \\ &- \max\left(6mEuribor_{t_{i-1}} - 3.6\%, 0\right) \end{aligned} \tag{5.10}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 30/04/2015. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 10 zero-coupon bonds with face value 15 *Euro* (i.e.: $1000 \cdot (3\% \cdot 0.5)$) and with maturities equal to the coupon payment dates;

2. a derivative component that, in accordance with equation 5.10, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 30/04/2015;
   - is composed by 10 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.5 years;
   - has a cap rate equal to 2.6%;
   - has as underlying the $6 - month$ Euribor ($6mEuribor$) at which are indexed the coupons;

   b) a short cap with value of 1000 *Euro* that:
   - has maturity 30/04/2015;
   - is composed by 10 caplets with respectively maturity equal to the coupon payment dates;
   - has a tenor equal to 0.5 years;
   - has a cap rate equal to 3.6%;
   - has as underlying the $6 - month$ Euribor ($6mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 10 European put options with maturities:

30/04/2010	30/10/2012
30/10/2010	30/04/2013
30/04/2011	30/10/2013
30/10/2011	30/04/2014
30/04/2012	30/10/2014

  whose underlying securities are 10 zero-coupon bonds with maturities:

30/10/2010	30/04/2013
30/04/2011	30/10/2013
30/10/2011	30/04/2014
30/04/2012	30/10/2014
30/10/2012	30/04/2015

  and face value of 1013 (i.e.: $1000 \cdot (1 + 2.6\% \cdot 0.5)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 10 European put options with maturities:

30/04/2010	30/10/2012
30/10/2010	30/04/2013
30/04/2011	30/10/2013
30/10/2011	30/04/2014
30/04/2012	30/10/2014

  whose underlying securities are 10 zero-coupon bonds with maturities:

30/10/2010	30/04/2013
30/04/2011	30/10/2013
30/10/2011	30/04/2014
30/04/2012	30/10/2014
30/10/2012	30/04/2015

  and with face value 1018 (i.e.: $1000 \cdot (1 + 3.6\% \cdot 0.5)$).
  Each of these put options has strike price equal to: $K = 1000$.

5.2.6 Description and unbundling of the bond Unicredit_1

The main characteristics of the bond Unicredit_1 are summarized in Table 7 hereafter:

Table 7. Characteristics of the bond Unicredit_1

Denomination of the financial instrument	Unicredit S.p.A 2010-2016 serie 12/10. Six years bonds with quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2.1% for the first three years and maximum rate (cap) 4% for the last three years.
ISIN	IT0004607302
Total amount and currency	1,030,000,000.00 Euro
Face value	1000.00 Euro
Issue date	31/05/2010
Maturity date	31/05/2016
Repayment date	31/05/2016
Issue price	100% of face value
Return	Quarterly floating coupon from 31/08/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci = max(3mEuribor;2.1%) i = 1,...,12 Ci = min(3mEuribor;4%) i = 13,...,24
Coupon payment dates	28/02, 31/05, 31/08, 30/11 from 31/08/2010 to 31/05/2016

From the above table it is possible to identify the key information required for the unbundling and the pricing of this bond whose coupon structure embeds a long floor for the first three years and a short cap for the remaining three years, namely:

1. issue date: 31/05/2010;
2. maturity date: 31/05/2016;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): quarterly;
5. coupon payment dates:

31/08/2010	31/08/2012	31/08/2014
30/11/2010	30/11/2012	30/11/2014
28/02/2011	28/02/2013	28/02/2015
31/05/2011	31/05/2013	31/05/2015
31/08/2011	31/08/2013	31/08/2015
30/11/2011	30/11/2013	30/11/2015
28/02/2012	28/02/2014	28/02/2016
31/05/2012	31/05/2014	31/05/2016

6. each coupon in the first three years is indexed to the 3-month Euribor subject to a 2.1% floor according to the following formula:

$$cpn\ rate_{t_i} = 3mEuribor_{t_{i-1}} + \max\left(2.1\% - 3mEuribor_{t_{i-1}}, 0\right) \quad (5.11)$$

   where:
   - $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
   - $i = 1, ..., 12$;
   - $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

7. each coupon in the last three years is indexed to the 3-month Euribor subject to a 4% cap according to the following formula:

$$cpn\ rate_{t_i} = 3mEuribor_{t_{i-1}} - \max\left(3mEuribor_{t_{i-1}} - 4\%, 0\right) \quad (5.12)$$

   where:
   - $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
   - $i = 13, ..., 24$;
   - $3mEuribor_{t_{i-1}}$ is the three months Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a floating-rate quarterly coupon bond which is indexed to the $3 - month$ Euribor ($3mEuribor$) and which at the maturity date 31/05/2015 repays a face value of 1000 *Euro*;

2. a derivative component that is structured as follows:

   a) a long floor with face value of 1000 *Euro* that:
   - has maturity 31/05/2013;
   - is composed by 12 floorlets, with the following maturities:

31/08/2010	28/02/2012
30/11/2010	31/05/2012
28/02/2011	31/08/2012
31/05/2011	30/11/2012
31/08/2011	28/02/2013
30/11/2011	31/05/2013

   - has a tenor equal to 0.25 years;
   - has a floor rate equal to 2.1%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

   b) a short cap with value of 1000 *Euro* that:
   - has maturity 31/05/2016;

- is composed by 12 caplets with the following maturities;

31/08/2013	28/02/2015
30/11/2013	31/05/2015
28/02/2014	31/08/2015
31/05/2014	30/11/2015
31/08/2014	28/02/2016
30/11/2014	31/05/2016

- has a tenor equal to 0.25 years;
- has a cap rate equal to 4%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps and floors, we can say that:

- the long floor of the previous point sub **2.a)** is equivalent to a long portfolio of 12 European call options with maturities:

31/05/2010	30/11/2011
31/08/2010	28/02/2012
30/11/2010	31/05/2012
28/02/2011	31/08/2012
31/05/2011	30/11/2012
31/08/2011	28/02/2013

  whose underlying securities are 12 zero-coupon bonds with maturities:

31/08/2010	28/02/2012
30/11/2010	31/05/2012
28/02/2011	31/08/2012
31/05/2011	30/11/2012
31/08/2011	28/02/2013
30/11/2011	31/05/2013

  and face value of 1005.25 (i.e.: $1000 \cdot (1 + 2.1\% \cdot 0.25)$).

  Each of these call options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 12 European put options with maturities:

31/05/2013	30/11/2014
31/08/2013	28/02/2015
30/11/2013	31/05/2015
28/02/2014	31/08/2015
31/05/2014	30/11/2015
31/08/2014	28/02/2016

  whose underlying securities are 12 zero-coupon bonds with maturities:

31/08/2013	28/02/2015
30/11/2013	31/05/2015
28/02/2014	31/08/2015
31/05/2014	30/11/2015
31/08/2014	28/02/2016
30/11/2014	31/05/2016

and with face value 1010.25 (i.e.: $1000 \cdot (1 + 4.1\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.7 Description and unbundling of the bond Unicredit_2

The main characteristics of the bond Unicredit_2 are summarized in Table 8 hereafter:

Table 8. Characteristics of the bond Unicredit_2

Denomination of the financial instrument	Unicredit S.p.A 2010-2016 serie 06/10. Six years bonds with quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2% and maximum rate (cap) 4.1%
ISIN	IT0004587496
Total amount and currency	1,050,000,000.00 Euro
Face value	1000.00 Euro
Issue date	31/03/2010
Maturity date	31/03/2016
Repayment date	31/03/2016
Issue price	100% of face value
Return	Quarterly floating coupon from 30/06/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci = min[max(3mEuribor;2%);4.1%] i = 1,...,24
Coupon payment dates	31/03, 30/06, 30/09, 31/12 from 30/06/2010 to 31/03/2016

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater, namely:

1. issue date: 31/03/2010;

2. maturity date: 31/03/2016;

3. face value: 1000 *Euro*;

4. frequency of payment (i.e. *tenor*): quarterly;

5. coupon payment dates:

30/06/2010	30/06/2012	30/06/2014
30/09/2010	30/09/2012	30/09/2014
31/12/2010	31/12/2012	31/12/2014
31/03/2011	31/03/2013	31/03/2015
30/06/2011	30/06/2013	30/06/2015
30/09/2011	30/09/2013	30/09/2015
31/12/2011	31/12/2013	31/12/2015
31/03/2012	31/03/2014	31/03/2016

6. each coupon is indexed to the 3-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2\%; 3mEuribor_{t_{i-1}}\right), 4.1\%\right] \tag{5.13}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 1, ..., 24$;
- $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

Given equation 4.4, the coupon rate of equation 5.13 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2\% + \max\left(3mEuribor_{t_{i-1}} - 2\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 4.1\%, 0\right) \end{aligned} \tag{5.14}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 31/03/2016. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 24 zero-coupon bonds with face value 5 *Euro* (i.e.: $1000 \cdot (2\% \cdot 0.25)$) and with maturities equal to the coupon payment dates;

2. a derivative component that, in accordance with equation 5.14, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 31/03/2016;
   - is composed by 24 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 2%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

b) a short cap with value of 1000 *Euro* that:
- has maturity 31/03/2016;
- is composed by 24 caplets with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 4.1%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 24 European put options with maturities:

31/03/2010	31/03/2012	31/03/2014
30/06/2010	30/06/2012	30/06/2014
30/09/2010	30/09/2012	30/09/2014
31/12/2010	31/12/2012	31/12/2014
31/03/2011	31/03/2013	31/03/2015
30/06/2011	30/06/2013	30/06/2015
30/09/2011	30/09/2013	30/09/2015
31/12/2011	31/12/2013	31/12/2015

  whose underlying securities are 24 zero-coupon bonds with maturities:

30/06/2010	30/06/2012	30/06/2014
30/09/2010	30/09/2012	30/09/2014
31/12/2010	31/12/2012	31/12/2014
31/03/2011	31/03/2013	31/03/2015
30/06/2011	30/06/2013	30/06/2015
30/09/2011	30/09/2013	30/09/2015
31/12/2011	31/12/2013	31/12/2015
31/03/2012	31/03/2014	31/03/2016

  and face value of 1005 (i.e.: $1000 \cdot (1 + 2\% \cdot 0.25)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 24 European put options with maturities:

31/03/2010	31/03/2012	31/03/2014
30/06/2010	30/06/2012	30/06/2014
30/09/2010	30/09/2012	30/09/2014
31/12/2010	31/12/2012	31/12/2014
31/03/2011	31/03/2013	31/03/2015
30/06/2011	30/06/2013	30/06/2015
30/09/2011	30/09/2013	30/09/2015
31/12/2011	31/12/2013	31/12/2015

whose underlying securities are 24 zero-coupon bonds with maturities:

30/06/2010	30/06/2012	30/06/2014
30/09/2010	30/09/2012	30/09/2014
31/12/2010	31/12/2012	31/12/2014
31/03/2011	31/03/2013	31/03/2015
30/06/2011	30/06/2013	30/06/2015
30/09/2011	30/09/2013	30/09/2015
31/12/2011	31/12/2013	31/12/2015
31/03/2012	31/03/2014	31/03/2016

and with face value 1010.25 (i.e.: $1000 \cdot (1 + 4.1\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.8 Description and unbundling of the bond Unicredit_3

The main characteristics of the bond Unicredit_3 are summarized in Table 9 hereafter:

Table 9. Characteristics of the bond Unicredit_3

Denomination of the financial instrument	Unicredit S.p.A 2010-2016 serie 07/10. Six years bonds with quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2% and maximum rate (cap) 3.5% for the first and the second year, minimum rate (floor) 2.5% and maximum rate (cap) 4% for the third and the fourth year, minimum rate (floor) 2.83% and maximum rate (cap) 4.5% for the fifth and the sixth year.
ISIN	IT0004591456
Total amount and currency	175,000,000.00 Euro
Face value	1000.00 Euro
Issue date	15/04/2010
Maturity date	15/04/2016
Repayment date	15/04/2016
Issue price	100% of face value
Return	Quarterly floating coupon from 15/07/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci = min[max(3mEuribor;2%);3.5%] i = 1,...,8 Ci = min[max(3mEuribor;2.5%);4%] i = 9,...,16 Ci = min[max(3mEuribor;2.83%);4.5%] i = 17,...,24
Coupon payment dates	15/01, 15/04, 15/07, 15/10 from 15/07/2010 to 15/04/2016

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater where the cap rate and the floor rate are varying over time, namely:

1. issue date: 15/04/2010;
2. maturity date: 15/04/2016;
3. face value: 1000 $Euro$;

4. frequency of payment (i.e. *tenor*): quarterly;

5. coupon payment dates:

15/07/2010	15/07/2012	15/07/2014
15/10/2010	15/10/2012	15/10/2014
15/01/2011	15/01/2013	15/01/2015
15/04/2011	15/04/2013	15/04/2015
15/07/2011	15/07/2013	15/07/2015
15/10/2011	15/10/2013	15/10/2015
15/01/2012	15/01/2014	15/01/2016
15/04/2012	15/04/2014	15/04/2016

6. each coupon of the first two years is indexed to the 3-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2\%; 3mEuribor_{t_{i-1}}\right), 3.5\%\right] \tag{5.15}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 1, ..., 8$;
- $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

Given equation 4.4, the coupon rate of equation 5.15 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2\% + \max\left(3mEuribor_{t_{i-1}} - 2\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 3.5\%, 0\right) \end{aligned} \tag{5.16}$$

7. each coupon of the third and the fourth year is indexed to the 3 months Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.5\%; 3mEuribor_{t_{i-1}}\right), 4\%\right] \tag{5.17}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 9, ..., 16$;
- $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

Given equation 4.4, the coupon rate of equation 5.17 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.5\% + \max\left(3mEuribor_{t_{i-1}} - 2.5\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 4\%, 0\right) \end{aligned} \tag{5.18}$$

8. each coupon of the last two years is indexed to the 3-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.83\%; 3mEuribor_{t_{i-1}}\right), 4.5\%\right] \tag{5.19}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 17, ..., 24$;
- $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.4, the coupon rate of equation 5.19 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.83\% + \max\left(3mEuribor_{t_{i-1}} - 2.83\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 4.5\%, 0\right) \end{aligned} \tag{5.20}$$

From what stated above, it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 31/03/2015. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 8 zero-coupon bonds with face value 5 *Euro* (i.e.: $1000 \cdot (2\% \cdot 0.25)$) and with maturities:

15/07/2010	15/07/2011
15/10/2010	15/10/2011
15/01/2011	15/01/2012
15/04/2011	15/04/2012

   c) 8 zero-coupon bonds with face value 6.25 *Euro* (i.e.: $1000 \cdot (2.5\% \cdot 0.25)$) and with maturities:

15/07/2012	15/07/2013
15/10/2012	15/10/2013
15/01/2013	15/01/2014
15/04/2013	15/04/2014

   d) 8 zero-coupon bonds with face value 7.075 *Euro* (i.e.: $1000 \cdot (2.83\% \cdot 0.25)$) and with maturities:

15/07/2014	15/07/2015
15/10/2014	15/10/2015
15/01/2015	15/01/2016
15/04/2015	15/04/2016

2. a derivative component that is structured as follows:

   a) i) a long cap with face value of 1000 *Euro* that:
   - has maturity 15/04/2012;
   - is composed by 8 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;

- has a cap rate equal to 2%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

ii) a short cap with value of 1000 $Euro$ that:
- has maturity 15/04/2012;
- is composed by 8 caplets with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 3.5%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

b) i) a long cap with face value of 1000 $Euro$ that:
- has maturity 15/04/2014;
- is composed by 8 caplets each with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 2.5%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

ii) a short cap with value of 1000 $Euro$ that:
- has maturity 15/04/2014;
- is composed by 8 caplets with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 4%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

c) i) a long cap with face value of 1000 $Euro$ that:
- has maturity 15/04/2016;
- is composed by 8 caplets each with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 2.83%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

ii) a short cap with value of 1000 $Euro$ that:
- has maturity 15/04/2016;
- is composed by 8 caplets with maturity equal to the coupon payment dates;
- has a tenor equal to 0.25 years;
- has a cap rate equal to 4.5%;
- has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a) i)** is equivalent to a long portfolio of 8 European put options with maturities:

15/04/2010	15/04/2011
15/07/2010	15/07/2011
15/10/2010	15/10/2011
15/01/2011	15/01/2012

  whose underlying securities are 20 zero-coupon bonds with maturities:

15/07/2010	15/07/2011
15/10/2010	15/10/2011
15/01/2011	15/01/2012
15/04/2011	15/04/2012

  and face value of 1005 (i.e.: $1000 \cdot (1 + 2\% \cdot 0.25)$).

  Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.a) ii)** is equivalent to a short portfolio of 8 European put options with maturities:

15/04/2010	15/04/2011
15/07/2010	15/07/2011
15/10/2010	15/10/2011
15/01/2011	15/01/2012

  whose underlying securities are 8 zero-coupon bonds with maturities:

15/07/2010	15/07/2011
15/10/2010	15/10/2011
15/01/2011	15/01/2012
15/04/2011	15/04/2012

  and with face value 1008.75 (i.e.: $1000 \cdot (1 + 3.5\% \cdot 0.25)$).
  Each of these put options has strike price equal to: $K = 1000$.

- the long cap of the previous point sub **2.b) i)** is equivalent to a long portfolio of 8 European put options with maturities:

15/04/2012	15/04/2013
15/07/2012	15/07/2013
15/10/2012	15/10/2013
15/01/2013	15/01/2014

  whose underlying securities are 8 zero-coupon bonds with maturities:

15/07/2012	15/07/2013
15/10/2012	15/10/2013
15/01/2013	15/01/2014
15/04/2013	15/04/2014

and face value of 1006.25 (i.e.: $1000 \cdot (1 + 2.5\% \cdot 0.25)$).

Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b) ii)** is equivalent to a short portfolio of 8 European put options with maturities:

15/04/2012	15/04/2013
15/07/2012	15/07/2013
15/10/2012	15/10/2013
15/01/2013	15/01/2014

whose underlying securities are 8 zero-coupon bonds with maturities:

15/07/2012	15/07/2013
15/10/2012	15/10/2013
15/01/2013	15/01/2014
15/04/2013	15/04/2014

and with face value 1010 (i.e.: $1000 \cdot (1 + 4\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

- the long cap of the previous point sub **2.c) i)** is equivalent to a long portfolio of 8 European put options with maturities:

15/04/2014	15/04/2015
15/07/2014	15/07/2015
15/10/2014	15/10/2015
15/01/2015	15/01/2016

whose underlying securities are 8 zero-coupon bonds with maturities:

15/07/2014	15/07/2015
15/10/2014	15/10/2015
15/01/2015	15/01/2016
15/04/2015	15/04/2016

and face value of 1007.075 (i.e.: $1000 \cdot (1 + 2.83\% \cdot 0.25)$).

Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.c) ii)** is equivalent to a short portfolio of 8 European put options with maturities:

15/04/2014	15/04/2015
15/07/2014	15/07/2015
15/10/2014	15/10/2015
15/01/2015	15/01/2016

whose underlying securities are 8 zero-coupon bonds with maturities:

15/07/2014	15/07/2015
15/10/2014	15/10/2015
15/01/2015	15/01/2016
15/04/2015	15/04/2016

and with face value 1011.25 (i.e.: $1000 \cdot (1 + 4.5\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.9 Description and unbundling of the bond Unicredit_4

The main characteristics of the bond Unicredit_4 are summarized in Table 10 hereafter:

Table 10. Characteristics of the bond Unicredit_4

Denomination of the financial instrument	Unicredit S.p.A 2010-2016. Six years bonds with quarterly floating coupon indexed to the 3 months Euribor, with minimum rate (floor) 2.3% and maximum rate (cap) 4.9%
ISIN	IT0004566193
Total amount and currency	150,000,000.00 Euro
Face value	1000.00 Euro
Issue date	29/01/2010
Maturity date	29/01/2016
Repayment date	29/01/2016
Issue price	100% of face value
Return	Quarterly floating coupon from 29/04/2010 to maturity
Coupon type	Floating coupon with cap and floor
Coupon frequency	Quarterly
Underlying	3 months Euribor
Coupon formula	Ci = min[max(3mEuribor;2.3%);4.9%] i = 1,...,24
Coupon payment dates	29/01, 29/04, 29/07, 29/10 from 29/04/2010 to 29/01/2016

From the above table it is possible to identify the key information required for the unbundling and the pricing of this collared floater, namely:

1. issue date: 29/01/2010;
2. maturity date: 29/01/2016;
3. face value: 1000 *Euro*;
4. frequency of payment (i.e. *tenor*): quarterly;
5. coupon payment dates:

29/04/2010	29/04/2012	29/04/2014
29/07/2010	29/07/2012	29/07/2014
29/10/2010	29/10/2012	29/10/2014
29/01/2011	29/01/2013	29/01/2015
29/04/2011	29/04/2013	29/04/2015
29/07/2011	29/07/2013	29/07/2015
29/10/2011	29/10/2013	29/10/2015
29/01/2012	29/01/2014	29/01/2016

6. each coupon is indexed to the 3-month Euribor according to the following formula:

$$cpn\ rate_{t_i} = \min\left[\max\left(2.3\%; 3mEuribor_{t_{i-1}}\right), 4.9\%\right] \tag{5.21}$$

where:
- $cpn\ rate_{t_i}$ is the annual percentage value of the coupon payable quarterly at the date t_i;
- $i = 1, ..., 24$;
- $3mEuribor_{t_{i-1}}$ is the three-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).
Given equation 4.4, the coupon rate of equation 5.21 can be equivalently expressed as:

$$\begin{aligned} cpn\ rate_{t_i} &= 2.3\% + \max\left(3mEuribor_{t_{i-1}} - 2.3\%, 0\right) + \\ &- \max\left(3mEuribor_{t_{i-1}} - 4.9\%, 0\right) \end{aligned} \tag{5.22}$$

From what stated above it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a bond component composed as follows:

   a) a zero-coupon bond with face value 1000 *Euro* and maturity date 29/01/2016. This component replicates the repayment of the capital invested in the bond at maturity;

   b) 24 zero-coupon bonds with face value 5.75 *Euro* (i.e.: $1000 \cdot (2.3\% \cdot 0.25)$) and with maturities equal to the coupon payment dates;

2. a derivative component that, in accordance with equation 5.22, is structured as follows:

   a) a long cap with face value of 1000 *Euro* that:
   - has maturity 29/01/2016;
   - is composed by 24 caplets each with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 2.3%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons;

   b) a short cap with value of 1000 *Euro* that:
   - has maturity 29/01/2016;
   - is composed by 24 caplets with maturity equal to the coupon payment dates;
   - has a tenor equal to 0.25 years;
   - has a cap rate equal to 4.9%;
   - has as underlying the $3 - month$ Euribor ($3mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate caps, we can say that:

- the long cap of the previous point sub **2.a)** is equivalent to a long portfolio of 24 European put options with maturities:

29/01/2010	29/01/2012	29/01/2014
29/04/2010	29/04/2012	29/04/2014
29/07/2010	29/07/2012	29/07/2014
29/10/2010	29/10/2012	29/10/2014
29/01/2011	29/01/2013	29/01/2015
29/04/2011	29/04/2013	29/04/2015
29/07/2011	29/07/2013	29/07/2015
29/10/2011	29/10/2013	29/10/2015

whose underlying securities are 24 zero-coupon bonds with maturities:

29/04/2010	29/04/2012	29/04/2014
29/07/2010	29/07/2012	29/07/2014
29/10/2010	29/10/2012	29/10/2014
29/01/2011	29/01/2013	29/01/2015
29/04/2011	29/04/2013	29/04/2015
29/07/2011	29/07/2013	29/07/2015
29/10/2011	29/10/2013	29/10/2015
29/01/2012	29/01/2014	29/01/2016

and face value of 1005.75 (i.e.: $1000 \cdot (1 + 2.3\% \cdot 0.25)$).

Each of these put options has strike price equal to: $K = 1000$;

- the short cap of the previous point sub **2.b)** is equivalent to a short portfolio of 24 European put options with maturities:

29/01/2010	29/01/2012	29/01/2014
29/04/2010	29/04/2012	29/04/2014
29/07/2010	29/07/2012	29/07/2014
29/10/2010	29/10/2012	29/10/2014
29/01/2011	29/01/2013	29/01/2015
29/04/2011	29/04/2013	29/04/2015
29/07/2011	29/07/2013	29/07/2015
29/10/2011	29/10/2013	29/10/2015

whose underlying securities are 24 zero-coupon bonds with maturities:

29/04/2010	29/04/2012	29/04/2014
29/07/2010	29/07/2012	29/07/2014
29/10/2010	29/10/2012	29/10/2014
29/01/2011	29/01/2013	29/01/2015
29/04/2011	29/04/2013	29/04/2015
29/07/2011	29/07/2013	29/07/2015
29/10/2011	29/10/2013	29/10/2015
29/01/2012	29/01/2014	29/01/2016

and with face value 1012.25 (i.e.: $1000 \cdot (1 + 4.9\% \cdot 0.25)$).
Each of these put options has strike price equal to: $K = 1000$.

5.2.10 Description and unbundling of the bond Intesa_1

The main characteristics of the bond Intesa_1 are summarized in Table 11 hereafter:

Table 11. Characteristics of the bond Intesa_1

Denomination of the financial instrument	INTESA. Six years bonds with annual floating coupon indexed to the 6 months Euribor, with minimum rate (floor) 2.7%
ISIN	IT0004594658
Total amount and currency	250,000,000.00 Euro
Face value	1000.00 Euro
Issue date	19/04/2010
Maturity date	19/04/2016
Repayment date	19/04/2016
Issue price	100% of face value
Return	Annual floating coupon from 19/04/2011 to maturity
Coupon type	Floating coupon with floor
Coupon frequency	Annual
Underlying	6 months Euribor
Coupon formula	Ci = max(6mEuribor;2.7%) i = 1,...,6
Coupon payment dates	19/04/2011, 19/04/2012, 19/04/2013, 19/04/2014, 19/04/2015, 19/04/2016

From the above table it is possible to identify the key information required for the unbundling and the pricing of this bond, whose coupon structure embeds a long floor, namely:

1. issue date: 19/04/2010;
2. maturity date: 19/04/2016;
3. face value: 1000 $Euro$;
4. frequency of payment (i.e. *tenor*): annual;
5. coupon payment dates:

19/04/2011	19/04/2014
19/04/2012	19/04/2015
19/04/2013	19/04/2016

6. each coupon is indexed to the 6 months Euribor by the following formula:

$$cpn\ rate_{t_i} = 6mEuribor_{t_{i-1}} + \max\left(2.7\% - 6mEuribor_{t_{i-1}}, 0\right) \quad (5.23)$$

where:
- $cpn\ rate_{t_i}$ is the percentage value of the annual coupon payable at the

date t_i;
- $i = 1, ..., 6$;
- $6mEuribor_{t_{i-1}}$ is the six-month Euribor observed at time t_{i-1} (where $t_{i-1} = t_i - \delta$).

From what stated above it follows that the payoff of this bond is equal to the sum of two payoffs respectively associated with:

1. a floating-rate annual coupon bond which is indexed to the $6 - month$ Euribor ($6mEuribor$) and which at the maturity date 19/04/2016 repays a face value of 1000 *Euro*;

2. a derivative component that, in accordance with equation 5.23, embeds a long floor with notional value of 1000 *Euro* that:
- has maturity 19/04/2016;
- is composed by 6 floorlets, each with maturity equal to the coupon payment dates;
- has a tenor equal to 1 year;
- has a floor rate equal to 2.7%;
- has as underlying the $6 - month$ Euribor ($6mEuribor$) at which are indexed the coupons.

In particular, by exploiting the formulas derived in Chapter 2 to price interest rate floors, we can say that this long floor is equivalent to a long portfolio of 6 European call options with maturities:

19/04/2011	19/04/2014
19/04/2012	19/04/2015
19/04/2013	19/04/2016

whose underlying securities are 6 zero-coupon bonds with maturities:

19/04/2010	19/04/2013
19/04/2011	19/04/2014
19/04/2012	19/04/2015

and face value of 1027 (i.e.: $1000 \cdot (1 + 2.7\%)$).
Each of these call options has strike price equal to: $K = 1000$.

5.3 Calibration of the Vasicek model

The first step to price interest rate derivatives in the framework of the Vasicek model is the calibration of the parameters, that is finding those values of a, b' and σ which allow for the best fit of the observed market data.

In section 1.5.1 we have seen that:

- a represents the speed of adjustment of the short rate to its long run mean;
- b' represents the long-run mean of the short rate;
- σ represents the volatility of the short rate.

We have also shown that the analytical expression for the price at time t of a zero coupon bond maturing at time T in the Vasicek model depends on the three parameters listed above.

The calibration procedure consists into retrieving the market quotes for a set of zero coupon bonds representing the interest rate term structure we want to model and into finding the parameters for the Vasicek model which provide the best matching between market prices and model prices.

In order to price the ten stochastic interest bonds listed in Table 1 of this Chapter, we have to calibrate the Vasicek model on the Euribor zero curve, since the coupon payments are indexed to the Euribor.

Our calibration procedure uses, therefore, as input, the market quotes of the zero coupon bonds implied in the so-called "Euro vs Euribor zero curve" available on Bloomberg.

Hence, taking these quotes as input, the calibration of the Vasicek model in continuous time was performed according to the following steps:

1. the prices of the zero coupon bonds were evaluated at the issue date by using the Vasicek model (namely, equations 1.24, 1.25 and 1.26 of section 1.5.1;

2. it was computed the sum, $\chi(a, \sigma, b')$, of the squares of the differences between the market prices of the zero coupon bonds and their theoretical values determined at the previous step, i.e.:

$$\chi(a, b', \sigma) = \sum_{i=1}^{N_{zcb}} \left(P_{t,market}^{zcb\ i} - P_{t,Vas}^{zcb\ i}(a, b', \sigma)\right)^2 \tag{5.24}$$

   where:

   - N_{zcb} is the number of zero coupon bonds used for the calibration;
   - $P_{t,market}^{zcb\ i}$ is the market price of the i^{th} zero coupon bond at the calibration date t;
   - $P_{t,Vas}^{zcb\ i}(a, b', \sigma)$ is the price of the same zero coupon bond evaluated at the same date t according to the Vasicek model;

3. the quantity $\chi(a, b', \sigma)$ was minimized with respect to the values of the parameters a, b' and σ in order to find their optimal values denoted by a_*, b'_* and σ_*.

The above minimization procedure was performed numerically through suitable optimization algorithms.

Clearly, the results of the calibration varied depending on the calibration date, and, hence, of the issue date of each of the ten bonds.

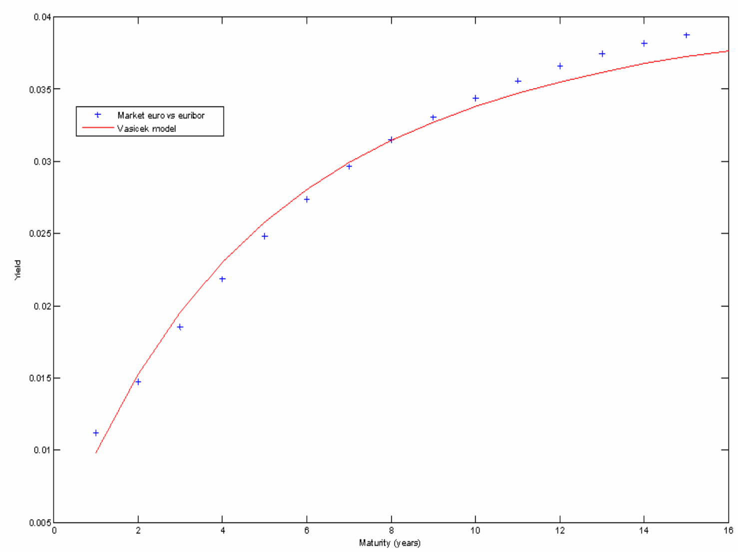
For example, in the case of the bond Unicredit_2, the calibration results were:

a_*	0.378782
b'_*	0.044136
σ_*	0.011036

and in Figure 5.1 we compared the zero curve quoted on the market and the zero curve obtained in the Vasicek model by using these parameters.

As we can see from the Figure, the Vasicek model - despite to the fact that it is a one-factor Gaussian model with constant parameters and without a no-arbitrage condition - exhibits a good fitting of the Euribor zero curve observed on the market.

Figure 5.1: Comparison between market zero curve and Vasicek zero curve



5.4 Pricing with the Vasicek model

This section describes the two alternative methodologies we used to price the ten stochastic interest bonds in the Vasicek model.

The first methodology takes advantage of the closed formulas for caps and floors prices illustrated in Chapter 2 to evaluate the derivative components of these bonds. More in detail we used the parameters estimated through the calibration procedure described in the previous section to price the interest rate derivatives embedded in the financial structure of the examined bonds according to the formulas 2.6 and 2.21 of Chapter 2, suitably modified in order to take into account the credit risk to which bond-holders are exposed.

The theoretical values of the pure bond components of the ten bonds were obtained by applying the general formula for the evaluation of a defaultable coupon bond derived in Chapter 3 (see equation 3.23 of section 3.5 of that Chapter[1]). It is worth pointing out that the generic term $P(0, t_i)$ appearing in the RHS of the mentioned formula has been determined by using the formula for the zero coupon bond price in the Vasicek model (see equations 1.24, 1.26 and 1.25 in section 1.5.1 of Chapter 1).

The second pricing methodology required to simulate the trajectories of the short rate r_t consistently with equation 1.22 in section 1.5.1 for a period equal to the length of the bonds.

For any given trajectory of r_t the associated Euribor trajectory was calculated and, then, compared at intervals corresponding to the coupon payment dates with the strikes of the caps and/or floors embedded in the bond analyzed[2].

The sum of the face value of the bond at the maturity date and of all its coupons (discounted back to the evaluation date along the simulated values of r_t over a given simulated trajectory) returned the total value of the bond over that trajectory.

By iterating this procedure 50000 times and taking the mean of the discounted values associated with all the simulated trajectories we obtained the theoretical bond values according to the second pricing technique.

Also in this methodology the credit risk of the issuer was taken into account, since at any payment date we assumed that in a certain number of trajectories, determined proportionally to intertemporal estimated default probabilities, a bond cash flow equal to the recovery rate $R = 40\%$ multiplied by the nominal value of the payment scheduled for that date.

Table 12 hereafter reports the results of the two pricing techniques in the framework of the Vasicek model, showing that they provide consistent results. The small difference between the prices obtained with the two methods is due to the statistical error associated with the Monte Carlo approach and to the discretization of the Vasicek stochastic differential equation we adopted in the simulation.

[1] Notice that this formula is so general that it holds clearly also for zero-coupon bonds

[2] Notice that, since the Vasicek model belongs to the family of Gaussian short rate models, the simulation needs a further correction to avoid the possibility to get negative rates.

Table 12. Stochastic Interest Bonds Pricing with the Vasicek Model

BOND IDENTIFICATION NUMBER	PRICING THROUGH THE UNBUNDLING TECHNIQUE					PRICING THROUGH THE MONTE CARLO SIMULATION
	VALUE OF THE BOND COMPONENT	VALUE OF THE LONG CAP(S)	VALUE OF THE SHORT CAP(S)	VALUE OF THE LONG FLOOR(S)	THEORETICAL VALUE	
BNL_1	94,85%	3,94%	-0,75%		98,04%	98,17%
BNL_2	95,12%	3,28%	-0,54%		97,86%	97,81%
BNL_3	94,69%	3,26%	-0,53%		97,42%	97,38%
POPOLARE_1	96,07%	1,74%	-0,72%		97,09%	97,16%
POPOLARE_2	97,18%	1,43%	-0,46%		98,15%	98,08%
UNICREDIT_1	94,60%		-0,75%	1,70%	95,56%	96,01%
UNICREDIT_2	91,48%	5,18%	-0,74%		95,92%	95,71%
UNICREDIT_3	93,72%	3,39%	-0,57%		96,55%	96,63%
UNICREDIT_4	93,09%	4,21%	-0,24%		97,06%	96,89%
INTESA_1	95,42%			3,34%	98,76%	98,92%

In section 5.7 of this Chapter we will present a comparison between these results, those obtained from the Hull and White model (see section 5.6 below) and the theoretical values of the bonds reported in the final terms of the prospectus published by the issuers, when available.

5.5 Calibration of the Hull and White Model

The first step to price interest rate derivatives in the framework of Hull and White model is the calibration of the parameters, that is finding those values of a and σ which allow for the best fit of the observed market data.

In section 1.6.4 we have seen that:

- a represents the speed of adjustment of the short rate to its long run mean;
- σ represents the volatility of the short rate.

We have also shown that the function $\theta(t)$ (i.e. the long run mean term in equation 1.75) can be chosen so as to reproduce exactly the actual forward rate curve observed in the market and that the analytical expression for the price at time t of a zero coupon bond maturing at time T in the Hull and White model depends on the two parameters listed above and on $\theta(t)$.

The calibration procedure consists into retrieving the market quotes for a set of caps or floors and into finding the parameters for the Hull and White model which provide the best matching between the market prices and the model prices of these interest rate derivatives.

In order to price the ten stochastic interest bonds listed in Table 1 of this Chapter, we have to calibrate the Hull and White model on the market volatilities of caps or floors having the Euribor as underlying interest rate, since the coupon payments are indexed to the Euribor.

In particular, we chose to use as input of the calibration procedure market caps volatilities available on Bloomberg.

Hence, taking these market volatilities as input, the calibration of the Hull and White model in continuous time was performed according to the following steps:

1. the prices of the caps were evaluated at the issue date by using the Hull and White model (namely, equations 2.12, 2.13, 2.14, 2.15 and 2.16 of section 2.1.5);

2. it was computed the sum, $\chi(a,\sigma)$, of the squares of the differences between the market prices of the caps and their theoretical values determined at the previous step, i.e.:

$$\chi(a,\sigma) = \sum_{i=1}^{N_{caps}} \left(cap^{i}_{t,market} - p^{cap_i}_{t,HW}(a,\sigma) \right)^2 \tag{5.25}$$

   where:

   - N_{caps} is the number of caps used for the calibration;
   - $cap^{i}_{t,market}$ is the market price of the i^{th} cap at the calibration date t. This price was calculated through the Black formula of equation 2.30 in Appendix B.1 of Chapter 2;
   - $p^{cap_i}_{t,HW}(a,\sigma)$ is the price of the same cap evaluated at the same date t according to the Hull-White model (see equation 2.16 of section 2.1.5);

3. the quantity of the previous step $\chi(a,\sigma)$ was minimized with respect to the values of the parameters a and σ in order to identify their optimal values denoted by a_* and σ_*.

The above minimization procedure was performed numerically through suitable optimization algorithms.

Clearly, the results of the calibration varied depending on the calibration date, and, hence, of the issue date of each of the ten bonds.

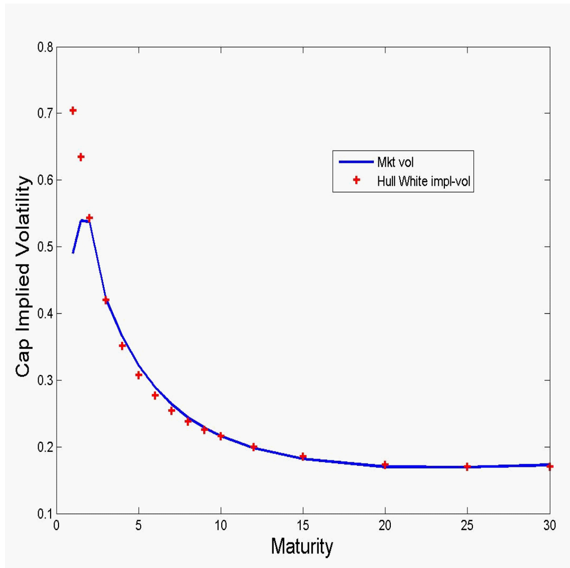
For example, in the case of the bond Unicredit_2 the calibration results have been:

a_*	0.0578
σ_*	0.0092

and in Figure 5.2 we compared the cap volatilities quoted on the market and the cap volatilities obtained in the Hull and White model by using these parameters.

As we can see from the Figure, even if it does not catch the first part of the term structure of cap volatilities (mainly due to the constant parameters), however the Hull and White model exhibits overall a good fitting of the market caps volatilities.

Figure 5.2: Comparison between market cap volatilities and Hull and White implied volatilities



5.6 Pricing with the Hull and White Model

This section describes the two alternative methodologies we used to price the ten stochastic interest bonds in the Hull and White model.

The first methodology takes advantage of the closed formulas for caps and floors prices illustrated in Chapter 2 to evaluate the derivative components of these bonds. More in detail we used the parameters estimated through the calibration procedure described in the previous section to price the interest rate derivatives embedded in the financial structure of the examined bonds according to the formulas 2.16 and 2.25 of Chapter 2, suitably modified in order to take into account the credit risk to which bond-holders are exposed.

The theoretical values of the pure bond components of the ten bonds were obtained by applying the general formula for the evaluation of a defaultable coupon bond derived in Chapter 3 (see equation 3.23 of section 3.5 of that Chapter[3]). It is worth pointing out that the generic term $P(0, t_i)$ appearing in the RHS of the mentioned formula has been determined by using the formula for the zero coupon bond price in the Vasicek model (see equation 1.83 in section 1.6.4 of Chapter 1).

The second pricing methodology required to simulate the trajectories of the short rate r_t consistently with equation 1.76 in section 1.6.4 for a period equal to the length of the bonds. As shown in Chapter 1, the following equality holds:

$$r_t = x_t + \alpha_t \tag{1.76}$$

[3]Notice that this formula is so general that it holds clearly also for zero-coupon bonds

and, as a consequence, for this model the simulation procedure does not involve to calculate the first derivative of the forward rate curve[4].

For any given trajectory of r_t the associated Euribor trajectory was calculated and, then, compared at intervals corresponding to the coupon payment dates with the strikes of the caps and/or floors embedded in the bond analyzed.

The sum of the face value of the bond at the maturity date and of all its coupons (discounted back to the evaluation date along the simulated values of r_t over a given simulated trajectory) returned the total value of the bond over that trajectory.

By iterating this procedure 50000 times and taking the mean of the discounted values associated with all the simulated trajectories we obtained the theoretical bond values according to the second pricing technique.

Also in this methodology the credit risk of the issuer was taken into account, since at any payment date we assumed that in a certain number of trajectories, determined proportionally to intertemporal estimated default probabilities, a bond cash flow equal to the recovery rate $R = 40\%$ multiplied by the nominal value of the payment scheduled for that date.

Table 13 hereafter reports the results of the two pricing techniques in the framework of the Hull and White model, showing that they provide consistent results. The small difference between the prices obtained with the two methods is due to the statistical error associated with the Monte Carlo approach and to the discretization step we adopted in the simulation.

Table 13. Stochastic Interest Bonds Pricing with the Hull and White Model

BOND IDENTIFICATION NUMBER	PRICING THROUGH THE UNBUNDLING TECHNIQUE					PRICING THROUGH THE MONTE CARLO SIMULATION
	VALUE OF THE BOND COMPONENT	VALUE OF THE LONG CAP(S)	VALUE OF THE SHORT CAP(S)	VALUE OF THE LONG FLOOR(S)	THEORETICAL VALUE	
BNL_1	94,85%	4,18%	-1,10%		97,93%	98,06%
BNL_2	95,12%	3,53%	-0,86%		97,79%	97,73%
BNL_3	94,69%	3,51%	-0,86%		97,34%	97,30%
POPOLARE_1	96,07%	2,11%	-1,06%		97,12%	97,19%
POPOLARE_2	97,18%	1,81%	-0,77%		98,23%	98,16%
UNICREDIT_1	94,60%		-1,23%	1,80%	95,17%	95,62%
UNICREDIT_2	91,48%	5,52%	-1,24%		95,76%	95,56%
UNICREDIT_3	93,72%	3,89%	-1,05%		96,56%	96,64%
UNICREDIT_4	93,09%	4,62%	-0,59%		97,12%	96,95%
INTESA_1	95,42%			3,86%	99,27%	99,43%

[4]Notice that, since the Hull-White model belongs to the family of Gaussian short rate models, the simulation needs a further correction to avoid the possibility to get negative rates.

In next section we will present a comparison between these results, those of section 5.4 and the theoretical values of the bonds reported in the final terms of the prospectus published by the issuers, when available.

5.7 Comparison with the prices published in the prospectus

In this section we will compare between the theoretical value of the ten bonds reported by issuers in the prospectuses, when available, with the theoretical values computed using two affine term structure models (Vasicek and Hull and White) and the two different pricing techniques described in the previous sections.

Table 14 hereafter summarizes these quantities.

Table 14. Theoretical bond values: models VS prospectus

BOND IDENTIFICATION NUMBER	PRICING THROUGH AFFINE TERM STRUCTURE MODELS				PRICING FROM THE PROSPECTUS
	Vasicek (Unbundling)	Vasicek (Monte Carlo)	Hull and White (Unbundling)	Hull and White (Monte Carlo)	
BNL_1	98,04%	98,17%	97,93%	98,06%	99,51%
BNL_2	97,86%	97,81%	97,79%	97,73%	99,63%
BNL_3	97,42%	97,38%	97,34%	97,30%	99,55%
POPOLARE_1	97,09%	97,16%	97,12%	97,19%	98,81%
POPOLARE_2	98,15%	98,08%	98,23%	98,16%	98,83%
UNICREDIT_1	95,56%	96,01%	95,17%	95,62%	96,37%
UNICREDIT_2	95,92%	95,71%	95,76%	95,56%	96,35%
UNICREDIT_3	96,55%	96,63%	96,56%	96,64%	97,85%
UNICREDIT_4	97,06%	96,89%	97,12%	96,95%	97,82%
INTESA_1	98,76%	98,92%	99,27%	99,43%	100%*

*Issue price (no prospectus published)

As we can see from Table 14, the theoretical values obtained in sections 5.4 and 5.6 with the two different models are consistent. On the contrary, the theoretical values resulting from the prospectuses are usually higher than the theoretical values determined according to the two term structure models considered in this Chapter.

In percentage terms, this difference is equal on average to about 1.22% and it provides an important signal about the reliability and the accuracy of the informative set included in the document that investors use to take their financial decisions

Chapter 6

Conclusions

The aim of this work is to use one-factor stochastic term structure models to evaluate stochastic interest bonds, that is bonds bundled together some interest rate derivative, and to compare them with the theoretical that the issuer indicates in the prospectus for the public offering.

Stochastic interest bonds are a sub-set of the big family of structured bonds, the latter being bonds that present specific algorithms driving coupons computation and payment at maturity, mainly due to the presence of one or more derivative components embedded in their financial structure.

Structured bonds are mainly issued by banks. Over the last two decades the offering of structured bonds to retail investors has consistently increased, with a contextual rise in the variety of the payoff structures.

In Chapter 1, after a brief exposure of the evolution of term structure models and their classification, we analyzed several one-factor affine term structure models: the Vasicek model, the Ho-Lee model and the Hull-White model.

In Chapter 2 we showed how to use the above models to price some typical interest rate derivatives (namely caps and floors) that are often embedded in the structure of stochastic interest bonds.

In Chapter 3 we presented some key concepts about credit risk in order to take into account the impact of this risk factor on the bond value. To this aim, we illustrated some key results regarding credit derivatives, and, specifically, credit default swaps whose market quotes allow to infer reliable estimates of the cumulative and intertemporal default probabilities of an issuer at various maturities by using the so-called bootstrapping technique. Once these default probabilities are estimated they can be used to derive a general pricing formula for defaultable bonds which will be used to perform the fair evaluation of the ten stochastic interest bonds analyzed in Chapter 5.

In Chapter 4 we studied in detail the financial engineering of a specific kind of stochastic interest bonds, the so-called *collared floaters*, which are floating-rate coupon bonds whose coupons are subject to both an upper and a lower bound, hence embedding two interest rate derivatives, either a long cap and a short cap or a long floor and a short cap depending on the specific unbundling choice we make.

In particular, the unbundling of a generic *collared floater* into its various elementary components was examined.

In Chapter 5 we dealt with the pricing of ten stochastic interest bonds recently issued by four of the major Italian banks: six of them were pure collared floaters, two of them were mixed fixed-floating coupon bonds, whose floating coupons had the typical structure of collared floaters, one bond was a floating-rate coupon bond embedding a floor, and one bond was a floating-rate coupon bond embedding a floor for the first half of its life and a cap for the second half of its life.

After the illustration of their unbundling, these bonds were priced by means of two alternative pricing methodologies.

The first methodology was based on the unbundling of their financial structure: this technique relies on the fact that stochastic interest bonds can be seen as the composition of one or more pure bond components and of one or more interest rate derivatives, namely caps and/or floors, whose closed formulas - in the framework of the one-factor affine term structure models of Chapter 1 developed under the risk neutral probability measure - were presented in Chapter 2.

The second methodology relies instead on Monte Carlo simulations, performed again under the risk neutral probability measure; in this case the fair value of a bond was determined by discounting back at the evaluation date the final value of the security over each simulated trajectory and, then, by averaging these discounted values.

The two pricing methodologies were implemented both in the framework of the Vasicek model and in that of the Hull and White model.

Their results turn out to be consistent and, compared with the theoretical value indicated in the final terms of the prospectus published by the issuers, they resulted a useful instrument to explore the reliability and the accuracy of the informative set included in this document that investors use to take their financial decisions.

Indeed, as shown by Table 14 of Chapter 5, the theoretical values displayed in the prospectus proved to be usually higher than those calculated according the two above mentioned affine term structure models, with an average percentage difference of about 1.22%.

We hope that the adopted approach shown in detail in this work for the pricing of stochastic interest bonds could help for the pricing of this type of products and could support the analysis of other structures.

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