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On the Stability Functional for Conservation Laws

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Abstract

This note is devoted to the explicit construction of a functional defined on all pairs of $\mathbf{L^1}$ functions with small total variation, which is equivalent to the $\mathbf{L^1}$ distance and non increasing along the trajectories of a given system of conservation laws. The present definition of this functional does not need any construction of approximate solutions.

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1 Introduction

Let the smooth map $f\colon \Omega \mapsto \mathbb{R}^n$ define the strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0 \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}$ and $u \in \Omega$, with $\Omega \subseteq \mathbb{R}^n$ being an open set.

Most functional theoretic methods fail to tackle these equations, essentially due to the appearance of shock waves. Since 1965, the Glimm functional [14] has been a major tool in any existence proof for (1.1) and related equations. More recently, an analogous role in the proofs of continuous dependence has been played by the stability functional Φ introduced in [8, 21, 22], see also [4]. The functional Φ has been widely used to prove the $\mathbf{L^1}$–Lipschitz dependence of solutions to (1.1) (and related problems) from initial data having small total variation, see for example [1, 2, 11, 12, 16, 17]. Special cases comprising data with large total variation are considered in [9, 15, 18, 19, 20]. Nevertheless, the use of Φ is hindered by the necessity of introducing specific approximate solutions, namely the ones based either on Glimm scheme [14] or on the wave front tracking algorithm [4, 13]. The present paper makes the use of the stability functional Φ *independent* from any kind of approximate solutions.

Namely, we extend the stability functional to all $\mathbf{L^1}$ functions with sufficiently small total variation. Moreover, we define it in terms of general piecewise constant functions, making it completely independent from the construction of any sort of approximate solutions. Furthermore, using the present definition, we prove its lower semicontinuity.

Taking advantage of the machinery presented below, we also extend the classical Glimm functionals [4, 14] to general $\mathbf{L^1}$ functions with small total variation and prove their lower semicontinuity, recovering some of the results in [3], but with a shorter proof.

As a byproduct, the present functional allows to simplify several parts of the cited papers, where the presentation of the stability functional needs to be preceded by the introduction of all the machinery related to Glimm's scheme or wave front tracking approximations, see for instance [10].

A further expression of the stability functional in terms of the *wave measures* introduced in [4, § 10.1] is easily available and does not rely on piecewise constant approximations at all. However, with such expression, the proof of the lower semicontinuity is far less direct. Furthermore, any application of this functional is based on approximating the functional evaluating it on piecewise constant functions and on the lower semicontinuity to pass to the limit. The construction below allows this approach.

The next section introduces the basic notation. Section 3 is devoted to the Glimm functional. The main result is presented in Section 4. The final Appendix contains a technical proof added for the sake of completeness, but not necessary for Theorem 4.1.

2 Notation

Our general reference for the basic definitions related to systems of conservation laws is [4]. We assume throughout that $0 \in \Omega$ and that the flux f satisfies

(F) $f \in \mathbf{C^4}(\Omega; \mathbb{R}^n)$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

Let $\lambda_1(u), \ldots, \lambda_n(u)$ be the n real distinct eigenvalues of $Df(u)$, indexed so that $\lambda_j(u) < \lambda_{j+1}(u)$ for all j and u. The j-th right eigenvector is denoted $r_j(u)$.

Let $\sigma \mapsto R_j(\sigma)(u)$, respectively $\sigma \mapsto S_j(\sigma)(u)$, be the rarefaction curve, respectively the shock curve, exiting u. If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [4, Definition 5.2], we choose σ so that

$$\frac{\partial \lambda_j}{\partial \sigma}\big(R_j(\sigma)(u)\big) = k_j \quad \text{and} \quad \frac{\partial \lambda_j}{\partial \sigma}\big(S_j(\sigma)(u)\big) = k_j\,,$$

where $k_1, \ldots, k_n$ can be arbitrary positive fixed numbers. In [4] the choice $k_j = 1$ for all $j = 1, \ldots, n$ was used, while in [2] another choice was made to cope with diagonal dominant sources. Introduce the j-Lax curve

$$\sigma \mapsto \psi_j(\sigma)(u) = \left\{ \begin{array}{lcrcl} R_j(\sigma)(u) & \quad & \text{if} & \sigma & \geq & 0 \\ S_j(\sigma)(u) & \quad & \text{if} & \sigma & < & 0 \end{array} \right.$$

and for $\sigma \equiv (\sigma_1, \ldots, \sigma_n)$, define the map

$$\Psi(\sigma)(u^-) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)(u^-) .$$

By [4, § 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to 0, there exists a map E such that

$$(\sigma_1, \ldots, \sigma_n) = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = \Psi(\sigma)(u^-) . \tag{2.1}$$

Similarly, let the map $\mathbf{S}$ and the vector q be defined by

$$u^+ = \mathbf{S}(q)(u^-) = S_n(q_n) \circ \ldots \circ S_1(q_1)(u^-) \tag{2.2}$$

as the gluing of the Rankine - Hugoniot curves.

Let u be piecewise constant with finitely many jumps and assume that $\mathrm{TV}(u)$ is sufficiently small. Call $\mathcal{I}(u)$ the finite set of points where u has a jump. Let $\sigma_{x,i}$ be the strength of the i-th wave in the solution of the Riemann problem for (1.1) with data $u(x-)$ and $u(x+)$, i.e. $(\sigma_{x,1}, \ldots, \sigma_{x,n}) = E\big(u(x-), u(x+)\big)$. Obviously if $x \notin \mathcal{I}(u)$ then $\sigma_{x,i} = 0$, for all $i = 1, \ldots, n$. As in [4, § 7.7], $\mathcal{A}(u)$ denotes the set of approaching waves in u:

$$\mathcal{A}(u) = \left\{ \begin{array}{c} \big((x,i),(y,j)\big) \in \big(\mathcal{I}(u) \times \{1, \ldots, n\}\big)^2 : \\ x < y \text{ and either } i > j \text{ or } i = j, \text{ the } i\text{-th field} \\ \text{is genuinely non linear, } \min\left\{\sigma_{x,i}, \sigma_{y,j}\right\} < 0 \end{array} \right\}$$

while the linear and the interaction potential, following [14] see also [4, formula (7.99)], are

$$\mathbf{V}(u) = \sum_{x \in I(u)} \sum_{i=1}^{n} \left|\sigma_{x,i}\right| \quad \text{and} \quad \mathbf{Q}(u) = \sum_{\big((x,i),(y,j)\big) \in \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| .$$

Moreover, let

$$\mathbf{\Upsilon}(u) = \mathbf{V}(u) + C_0 \cdot \mathbf{Q}(u) \tag{2.3}$$

where $C_0 > 0$ is the constant appearing in the functional of the wave–front tracking algorithm, see [4, Proposition 7.1]. Recall that C_0 depends only on the flow f and the upper bound of the total variation of initial data.

Remark 2.1 *The maps defined on Ω^N with values in $\mathbb{R}^+$ by*

$$\begin{array}{rcl}(u_1, \ldots, u_N) & \mapsto & \mathbf{V}\left(\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha, x_{\alpha+1}[}\right) \\ (u_1, \ldots, u_N) & \mapsto & \mathbf{Q}\left(\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha, x_{\alpha+1}[}\right)\end{array}$$

for fixed $x_1 < \ldots < x_{N+1}$, are Lipschitz continuous. Moreover, the Lipschitz constant of the maps

$$u_{\bar\alpha} \mapsto \mathbf{V}\left(\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha, x_{\alpha+1}[}\right) \qquad u_{\bar\alpha} \mapsto \mathbf{Q}\left(\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha, x_{\alpha+1}[}\right)$$

is bounded uniformly in N, $\bar\alpha$ and u_α for $\alpha \neq \bar\alpha$.

Finally we define

$$\mathcal{D}_\delta^* = \left\{ v \in \mathbf{L^1}(\mathbb{R}, \Omega) : v \text{ is piecewise constant and } \mathbf{\Upsilon}(u) < \delta \right\} \tag{2.4}$$

and

$$\mathcal{D}_\delta = \mathrm{cl}\left\{\mathcal{D}_\delta^*\right\}$$

where the closure is in the strong $\mathbf{L^1}$–topology. Observe that $\mathcal{D}_\delta$ contains all $\mathbf{L^1}$ functions with sufficiently small total variation.

For later use, for $u \in \mathcal{D}_\delta$ and $\eta > 0$, introduce the set

$$B_\eta(u) = \left\{ v \in \mathbf{L^1}(\mathbb{R}; \Omega) \colon v \in \mathcal{D}_\delta^* \text{ and } \|v - u\|_{\mathbf{L^1}} < \eta\,. \right\}\,. \tag{2.5}$$

Note that, by the definition of $\mathcal{D}_\delta$, $B_\eta(u)$ is not empty and if $\eta_1 < \eta_2$, then $B_{\eta_1}(u) \subseteq B_{\eta_2}(u)$. Recall the following fundamental result, proved in [7]:

Theorem 2.2 *Let f satisfy **(F)**. Then, there exists a positive δ_o such that the equation* (1.1) *generates for all $\delta \in \,]0, \delta_o[$ a Standard Riemann Semigroup (SRS) $S\colon [0, +\infty[\, \times \mathcal{D}_\delta \mapsto \mathcal{D}_\delta$, with Lipschitz constant L.*

We refer to [4, Chapters 7 and 8] for the proof of the above result as well as for the definition and further properties of the SRS.

3 The Glimm Functionals

Extend the Glimm functionals to all $u \in \mathcal{D}_\delta$ as follows:

$$\bar{\mathbf{Q}}(u) = \lim_{\eta \to 0+} \inf_{v \in B_\eta(u)} \mathbf{Q}(v) \quad \text{ and } \quad \bar{\mathbf{\Upsilon}}(u) = \lim_{\eta \to 0+} \inf_{v \in B_\eta(u)} \mathbf{\Upsilon}(v)\,. \tag{3.1}$$

The maps $\eta \to \inf_{v \in B_\eta(v)} \mathbf{Q}(v)$ and $\eta \to \inf_{v \in B_\eta(v)} \mathbf{\Upsilon}(v)$ are non increasing. Thus the limits above exist and

$$\bar{\mathbf{Q}}(u) = \sup_{\eta > 0} \inf_{v \in B_\eta(u)} \mathbf{Q}(v) \quad \text{ and } \quad \bar{\mathbf{\Upsilon}}(u) = \sup_{\eta > 0} \inf_{v \in B_\eta(u)} \mathbf{\Upsilon}(v)\,.$$

We prove in Proposition 3.4 below that $\bar{\mathbf{Q}}$, respectively $\bar{\mathbf{\Upsilon}}$, coincides with $\mathbf{Q}$, respectively $\mathbf{\Upsilon}$, when evaluated on piecewise constant functions. Moreover, $\bar{\mathbf{Q}}$ also coincides with the functional defined in [6, formula (1.15)], see also [4, formula (10.10)]. Preliminarily, we exploit the formulation (3.1) to prove the lower semicontinuity of $\mathbf{Q}$ and $\mathbf{\Upsilon}$ more directly than in [4, Theorem 10.1, p.203–208], see also [3].

Proposition 3.1 *The functionals $\bar{\mathbf{Q}}$ and $\bar{\mathbf{\Upsilon}}$ are lower semicontinuous with respect to the $\mathbf{L^1}$ norm.*

Proof. We prove the lower semicontinuity of $\bar{\mathbf{\Upsilon}}$, the case of $\bar{\mathbf{Q}}$ is analogous.

Fix u in $\mathcal{D}_\delta$. Let u_ν be a sequence in $\mathcal{D}_\delta$ converging to u in $\mathbf{L^1}$. Define $\varepsilon_\nu = \|u_\nu - u\|_{\mathbf{L^1}} + 1/\nu$. Fix $v_\nu \in B_{\varepsilon_\nu}(u_\nu)$ so that

$$\mathbf{\Upsilon}(v_\nu) \leq \inf_{v \in B_{\varepsilon_\nu}(u)} \mathbf{\Upsilon}(v) + \varepsilon_\nu \leq \bar{\mathbf{\Upsilon}}(u_\nu) + \varepsilon_\nu \,.$$

Since

$$\|v_\nu - u\|_{\mathbf{L^1}} \leq \|v_\nu - u_\nu\|_{\mathbf{L^1}} + \|u_\nu - u\|_{\mathbf{L^1}} < 2\varepsilon_\nu \,,$$

we deduce that $v_\nu \in B_{2\varepsilon_\nu}(u)$ and

$$\begin{array}{rcccl} \displaystyle\inf_{v \in B_{2\varepsilon_\nu}(u)} \mathbf{\Upsilon}(v) & \leq & \mathbf{\Upsilon}(v_\nu) & \leq & \bar{\mathbf{\Upsilon}}(u_\nu) + \varepsilon_\nu \,; \\ \bar{\mathbf{\Upsilon}}(u) & = & \displaystyle\lim_{\nu \to +\infty} \inf_{v \in B_{2\varepsilon_\nu}(u)} \mathbf{\Upsilon}(v) & \leq & \displaystyle\liminf_{\nu \to +\infty} \bar{\mathbf{\Upsilon}}(u_\nu) \end{array}$$

completing the proof. □

The next proposition contains in essence the reason why the Glimm functionals $\mathbf{Q}$ and $\mathbf{\Upsilon}$ decrease. Compute them on a piecewise constant function u and *"remove"* one (or more) of the values attained by u, then the values of both $\mathbf{Q}$ and $\mathbf{\Upsilon}$ decrease.

Let $u = \sum_{\alpha \in I} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}$ be a piecewise constant function, with $u_\alpha \in \Omega$, $x_1 < x_2 < \ldots < x_{N+1}$ and I be a finite set of integers. Then, we say that $u_1, u_2, \ldots, u_N$ is the *ordered sequence* of the values attained by u and, with a slight abuse of notation, we denote it by $(u_\alpha \colon \alpha \in I)$.

Proposition 3.2 *Let u and $\check{u}$ be piecewise constant functions attaining values in Ω. Assume that the ordered sequence of the values attained by u is $(u_\alpha \colon \alpha \in I)$, while the ordered sequence of the values attained by $\check{u}$ is $(u_\alpha \colon \alpha \in J)$, with $J \subseteq I$. Then,*

$$\mathbf{Q}(\check{u}) \leq \mathbf{Q}(u) \quad \text{ and } \quad \mathbf{\Upsilon}(\check{u}) \leq \mathbf{\Upsilon}(u) \,.$$

Proof. Consider the case $\sharp I = \sharp J + 1$, see also [4, Step 1, Lemma 10.2]. Then, the above inequalities follow from the usual Glimm interaction estimates [14], see Figure 1.

The general case follows recursively. □

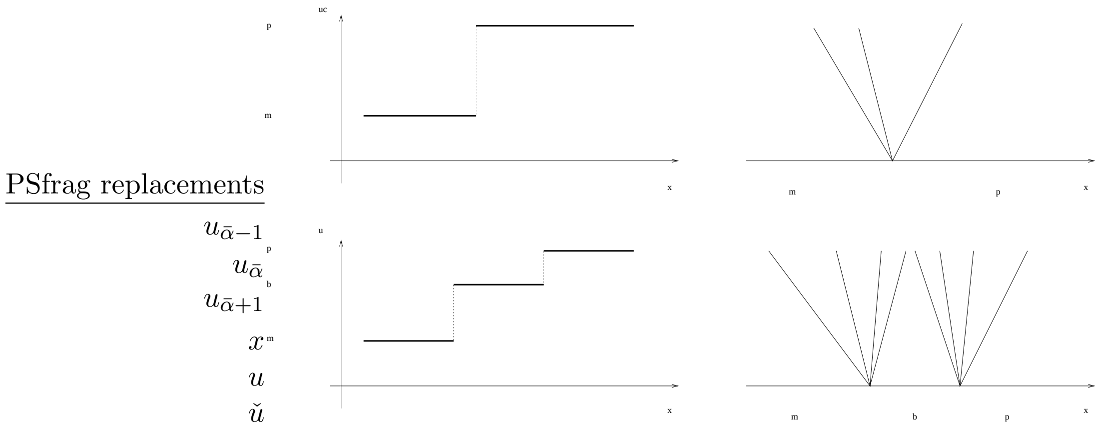

Figure 1: Proof of Proposition 3.2: $u_{\bar{\alpha}}$ is attained by u and not by $\check{u}$.

The next lemma is a particular case of [4, Theorem 10.1]. However, the present construction allows to consider only the case of piecewise constant functions, allowing a much simpler proof.

Lemma 3.3 *The functionals* $\mathbf{Q}$ *and* $\mathbf{\Upsilon}$*, defined on* $\mathcal{D}^*_\delta$*, are lower semicontinuous with respect to the* $\mathbf{L^1}$ *norm.*

Proof. We consider only $\mathbf{\Upsilon}$, the case of $\mathbf{Q}$ being similar.

Let u_ν be a sequence in $\mathcal{D}^*_\delta$ converging in $\mathbf{L^1}$ to $u = \sum_\alpha u_\alpha \chi_{[x_\alpha, x_{\alpha+1}[} \in \mathcal{D}^*_\delta$ as $\nu \to +\infty$. By possibly passing to a subsequence, we may assume that $\mathbf{\Upsilon}(u_\nu)$ converges to $\liminf_{\nu\to+\infty} \mathbf{\Upsilon}(u_\nu)$ and that u_ν converges a.e. to u. Therefore, for all $\alpha = 1, \ldots, N$, we can select points $y_\alpha \in]x_\alpha, x_{\alpha+1}[$ so that $\lim_{\nu\to+\infty} u_\nu(y_\alpha) = u(y_\alpha) = u_\alpha$. Define

$$\check{u}_\nu = \sum_\alpha u_\nu(y_\alpha)\, \chi_{[x_\alpha, x_{\alpha+1}[}\,.$$

By Proposition 3.2, $\mathbf{\Upsilon}(\check{u}_\nu) \leq \mathbf{\Upsilon}(u_\nu)$. The convergence $u_\nu(y_\alpha) \to u_\alpha$ for all α and Remark 2.1 allow to complete the proof. □

Proposition 3.4 *Let* $u \in \mathcal{D}^*_\delta$*. Then* $\bar{\mathbf{Q}}(u) = \mathbf{Q}(u)$ *and* $\bar{\mathbf{\Upsilon}}(u) = \mathbf{\Upsilon}(u)$*.*

Proof. We consider only $\mathbf{\Upsilon}$, the case of $\mathbf{Q}$ being similar.

Since $u \in \mathcal{D}^*_\delta$, we have that $u \in B_\eta(u)$ for all $\eta > 0$ and $\bar{\mathbf{\Upsilon}}(u) \leq \mathbf{\Upsilon}(u)$.

To prove the other inequality, recall that by the definition (3.1) of $\bar{\mathbf{\Upsilon}}$, there exists a sequence v_ν of piecewise constant functions in $\mathcal{D}^*_\delta$ such that $v_\nu \to u$ in $\mathbf{L^1}$ and $\mathbf{\Upsilon}(v_\nu) \to \bar{\mathbf{\Upsilon}}(u)$ as $\nu \to +\infty$. By Lemma 3.3,

$$\mathbf{\Upsilon}(u) \leq \liminf_{\nu\to+\infty} \mathbf{\Upsilon}(v_\nu) \leq \bar{\mathbf{\Upsilon}}(u)$$

completing the proof. □

Therefore, in the sequel we write $\mathbf{Q}$ for $\bar{\mathbf{Q}}$ and $\boldsymbol{\Upsilon}$ for $\bar{\boldsymbol{\Upsilon}}$.

For the sake of completeness, our next step consists in showing that the functionals $\mathbf{Q}$ and $\boldsymbol{\Upsilon}$ coincide with the analogous quantities in [4, Section 7.7], see also [3, 5]. To this aim, we temporarily denote by $\mathbf{V}^B$ and $\mathbf{Q}^B$ the functionals defined therein, moreover we set $\boldsymbol{\Upsilon}^B = \mathbf{V}^B + C_0 \cdot \mathbf{Q}^B$.

Proposition 3.5 $\mathbf{Q}^B = \mathbf{Q}$ *and* $\boldsymbol{\Upsilon}^B = \boldsymbol{\Upsilon}$.

Proof. We consider only $\boldsymbol{\Upsilon}$, the case of $\mathbf{Q}$ being similar.

Note first that if u is piecewise constant, then clearly $\boldsymbol{\Upsilon}^B(u) = \boldsymbol{\Upsilon}(u)$. By the definition (3.1) of $\boldsymbol{\Upsilon}$, there exists a sequence v_ν of functions in $\mathcal{D}_\delta^*$ converging to u in $\mathbf{L^1}$ and such that $\boldsymbol{\Upsilon}(v_\nu) \to \boldsymbol{\Upsilon}(u)$ as $\nu \to +\infty$. By the lower semicontinuity of $\boldsymbol{\Upsilon}^B$, see [4, Theorem 10.1], we obtain

$$\boldsymbol{\Upsilon}^B(u) \leq \liminf_{\nu\to+\infty} \boldsymbol{\Upsilon}^B(v_\nu) = \liminf_{\nu\to+\infty} \boldsymbol{\Upsilon}(v_\nu) = \lim_{\nu\to+\infty} \boldsymbol{\Upsilon}(v_\nu) = \boldsymbol{\Upsilon}(u)\,.$$

Analogously, following [4, Step 3, Theorem 10.1], we may take a sequence v_ν of functions in $\mathcal{D}_\delta^*$ such that $v_\nu \to u$ in $\mathbf{L^1}$, $\mathbf{V}^B(v_\nu) \to \mathbf{V}^B(u)$ and $\mathbf{Q}^B(v_\nu) \to \mathbf{Q}^B(u)$ as $\nu \to +\infty$. Hence, also $\boldsymbol{\Upsilon}^B(v_\nu) \to \boldsymbol{\Upsilon}^B(u)$. Therefore, along this particular sequence, we may repeat the estimates as above:

$$\boldsymbol{\Upsilon}(u) \leq \liminf_{\nu\to+\infty} \boldsymbol{\Upsilon}(v_\nu) = \liminf_{\nu\to+\infty} \boldsymbol{\Upsilon}^B(v_\nu) = \lim_{\nu\to+\infty} \boldsymbol{\Upsilon}^B(v_\nu) = \boldsymbol{\Upsilon}^B(u)$$

where we applied also Proposition 3.1. □

4 The Stability Functional

If $\delta \in]0, \delta_o[$, choose $v, \tilde{v}$ piecewise constant in $\mathcal{D}_\delta^*$. Now, as a first step, we slightly modify the construction of the stability functional, see [8, 21, 22] and also [4, Section 8.1]. Namely, we construct a similar functional defined on all piecewise constant functions and without any reference to both ε–approximate front tracking solutions and non physical waves.

Define implicitly the function $q(x) \equiv \big(q_1(x), \ldots q_n(x)\big)$ by

$$\tilde{v}(x) = \mathbf{S}\left(q(x)\right)\left(v(x)\right)$$

with $\mathbf{S}$ as in (2.2). We now consider the functional

$$\boldsymbol{\Phi}\left(v, \tilde{v}\right) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \left|q_i(x)\right| \mathbf{W}_i(x)\, dx \tag{4.1}$$

where the weights $\mathbf{W}_i$ are defined by

$$\mathbf{W}_i(x) = 1 + \kappa_1 \mathbf{A}_i(x) + \kappa_1\kappa_2 \left(\mathbf{Q}(v) + \mathbf{Q}(\tilde{v})\right)\,, \tag{4.2}$$

the constants κ_1 and κ_2 being defined in [4, Chapter 8]. Denote by $\sigma_{x,i}$, respectively $\tilde\sigma_{x,i}$, the size of the i–wave in the solution of the Riemann Problem with data $v(x-)$ and $v(x+)$, respectively $\tilde v(x-)$ and $\tilde v(x+)$. If the i–th characteristic field is linearly degenerate, then $\mathbf{A}_i$ is defined as

$$\begin{aligned}\mathbf{A}_i(x) \;=\; & \sum\left\{\left|\sigma_{y,j}\right| + \left|\tilde\sigma_{y,j}\right| : (y,j) \text{ such that } y \le x \text{ and } i < j \le n\right\} \\ & + \sum\left\{\left|\sigma_{y,j}\right| + \left|\tilde\sigma_{y,j}\right| : (y,j) \text{ such that } y > x \text{ and } 1 \le j < i\right\}.\end{aligned}$$

In the genuinely nonlinear case, let

$$\begin{aligned}\mathbf{A}_i(x) \;=\; & \sum\left\{\left|\sigma_{y,j}\right| + \left|\tilde\sigma_{y,j}\right| : (y,j) \text{ such that } y \le x \text{ and } i < j \le n\right\} \\ & + \sum\left\{\left|\sigma_{y,j}\right| + \left|\tilde\sigma_{y,j}\right| : (y,j) \text{ such that } y > x \text{ and } 1 \le j < i\right\} \\ & + \begin{cases} \sum\left\{\left|\sigma_{y,i}\right| : y \le x\right\} + \sum\left\{\left|\tilde\sigma_{y,i}\right| : y > x\right\} & \text{if } q_i(x) < 0, \\ \sum\left\{\left|\sigma_{y,i}\right| : y > x\right\} + \sum\left\{\left|\tilde\sigma_{y,i}\right| : y \le x\right\} & \text{if } q_i(x) \ge 0. \end{cases}\end{aligned}$$

We stress that $\mathbf{\Phi}$ is different from the functional Φ introduced in [21, 22] and defined in [4, formula (8.6)]. Indeed, here *all* jumps in v or in $\tilde v$ are considered. There, on the contrary, exploiting the structure of ε-approximate front tracking solutions, see [4, Definition 7.1], in the definition of Φ the jumps due to non physical waves are neglected when defining the weights A_i and are considered as belonging to a fictitious $n+1$-th family in the definition [4, formula (7.54)] of Q. To stress this dependence, in the sequel we denote by Φ^ε the stability functional as presented in [4, Chapter 8].

We now move towards the extension of $\mathbf{\Phi}$ to $\mathcal{D}_\delta$. Define

$$\mathbf{\Xi}_\eta(u,\tilde u) = \inf\left\{\mathbf{\Phi}\,(v,\tilde v) : v \in B_\eta(u) \text{ and } \tilde v \in B_\eta(\tilde u)\right\}$$

The map $\eta \to \mathbf{\Xi}_\eta(u,\tilde u)$ is non increasing. Thus, we may finally define

$$\mathbf{\Xi}(u,\tilde u) = \lim_{\eta\to 0+} \mathbf{\Xi}_\eta(u,\tilde u) = \sup_{\eta > 0} \mathbf{\Xi}_\eta(u,\tilde u)\,. \tag{4.3}$$

We are now ready to state the main result of this paper.

Theorem 4.1 *The functional* $\mathbf{\Xi}\colon \mathcal{D}_\delta \times \mathcal{D}_\delta \mapsto [0,+\infty[$ *defined in* (4.3) *enjoys the following properties:*

(i) $\mathbf{\Xi}$ *is equivalent to the* $\mathbf{L^1}$ *distance, i.e. there exists a* $C > 0$ *such that for all* $u, \tilde u \in \mathcal{D}_\delta$

$$\frac{1}{C}\cdot \left\|u - \tilde u\right\|_{\mathbf{L^1}} \le \mathbf{\Xi}(u,\tilde u) \le C \cdot \left\|u - \tilde u\right\|_{\mathbf{L^1}}\,.$$

(ii) Ξ *is non increasing along the semigroup trajectories of Theorem 2.2, i.e. for all* $u, \tilde{u} \in \mathcal{D}_\delta$ *and for all* $t \geq 0$

$$\Xi(S_t u, S_t \tilde{u}) \leq \Xi(u, \tilde{u}) .$$

(iii) Ξ *is lower semicontinuous with respect to the* $\mathbf{L^1}$ *norm.*

Here and in what follows, we denote by C positive constants dependent only on f and δ_0. We split the proof of the above theorem in several steps.

Lemma 4.2 *For all* $u, \tilde{u} \in \mathcal{D}^*_\delta$*, one has* $\Xi(u, \tilde{u}) \leq \mathbf{\Phi}\,(u, \tilde{u})$*.*

We remark that actually Ξ coincides with $\mathbf{\Phi}$ on all piecewise constant functions. However, the other inequality is rather technical and not necessary for the proof of Theorem 4.1. Therefore, we postpone it to the Appendix.

Proof of Lemma 4.2. By the definition (2.5) we have $u \in B_\eta(u)$ and $\tilde{u} \in B_\eta(\tilde{u})$ for all $\eta > 0$, hence $\Xi_\eta(u, \tilde{u}) \leq \mathbf{\Phi}\,(u, \tilde{u})$ for all positive η. The lemma is proved passing to the limit $\eta \to 0+$. $\square$

Proposition 4.3 *The functional* $\Xi \colon \mathcal{D}_\delta \mapsto \mathbb{R}$ *is lower semicontinuous with respect to the* $\mathbf{L^1}$ *norm.*

Proof. Fix u and $\tilde{u}$ in $\mathcal{D}_\delta$. Let u_ν, respectively $\tilde{u}_\nu$, be a sequence in $\mathcal{D}_\delta$ converging to u, respectively $\tilde{u}$. Define $\varepsilon_\nu = \|u_\nu - u\|_{\mathbf{L^1}} + \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L^1}} + 1/\nu$. Then, for each ν, there exist piecewise constant $v_\nu \in B_{\varepsilon_\nu}(u_\nu)$, respectively $\tilde{v}_\nu \in B_{\varepsilon_\nu}(\tilde{u}_\nu)$, such that

$$\mathbf{\Phi}\,(v_\nu, \tilde{v}_\nu) \leq \Xi_{\varepsilon_\nu}(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu \leq \Xi(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu \,. \tag{4.4}$$

Moreover

$$\begin{array}{rcccc} \|v_\nu - u\|_{\mathbf{L^1}} & \leq & \|v_\nu - u_\nu\|_{\mathbf{L^1}} + \|u_\nu - u\|_{\mathbf{L^1}} & < & 2\varepsilon_\nu \\ \|\tilde{v}_\nu - \tilde{u}\|_{\mathbf{L^1}} & \leq & \|\tilde{v}_\nu - \tilde{u}_\nu\|_{\mathbf{L^1}} + \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L^1}} & < & 2\varepsilon_\nu \end{array}$$

so that $v_\nu \in B_{2\varepsilon_\nu}(u)$ and $\tilde{v}_\nu \in B_{2\varepsilon_\nu}(\tilde{u})$. Hence, $\Xi_{2\varepsilon_\nu}(u, \tilde{u}) \leq \mathbf{\Phi}\,(v_\nu, \tilde{v}_\nu)$. Using (4.4), we obtain $\Xi_{2\varepsilon_\nu}(u, \tilde{u}) \leq \Xi(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu$. Finally, passing to the liminf for $\nu \to +\infty$, we have $\Xi(u, \tilde{u}) \leq \liminf_{\nu \to +\infty} \Xi(u_\nu, \tilde{u}_\nu)$. $\square$

In the next proposition, we compare the functional $\mathbf{\Phi}$ defined in (4.1) with the stability functional Φ^ε as defined in [4, formula (8.6)]

Proposition 4.4 *Let $\delta > 0$. Then, there exists a positive C such that for all $\varepsilon > 0$ sufficiently small and for all ε-approximate front tracking solutions $w(t,x), \tilde{w}(t,x)$ of* (1.1)

$$\left|\mathbf{\Phi}\left(w(t,\cdot),\tilde{w}(t,\cdot)\right) - \Phi^\varepsilon(w,\tilde{w})(t)\right| \leq C \cdot \varepsilon \cdot \left\|w(t,\cdot) - \tilde{w}(t,\cdot)\right\|_{\mathbf{L^1}} .$$

Proof. Setting $\tilde{w}(t,x) = \mathbf{S}\left(q(t,x)\right)\left(w(t,x)\right)$ and omitting the explicit time dependence in the integrand, we have:

$$\left|\mathbf{\Phi}\left(w(t,\cdot),\tilde{w}(t,\cdot)\right) - \Phi^\varepsilon(w,\tilde{w})(t)\right| \leq \int_{\mathbb{R}} \sum_{i=1}^n \left|q_i(x)\right|\left|\mathbf{W}_i(x) - W_i(x)\right| dx .$$

We are thus lead to estimate

$$\begin{aligned} \left|\mathbf{W}_i(x) - W_i(x)\right| \quad \leq \quad & \kappa_1\left|\mathbf{A}_i(x) - A_i(x)\right| + \\ & + \kappa_1\kappa_2\left|\mathbf{Q}(w) - Q(w)\right| + \kappa_1\kappa_2\left|\mathbf{Q}(\tilde{w}) - Q(\tilde{w})\right| . \end{aligned}$$

The second and third summands are each bounded as in [4, formula (7.100)] by $C\,\varepsilon$. Concerning the former one, recall that A_i and $\mathbf{A_i}$ differ only in the absence of non physical waves in A_i. In other words, physical jumps are counted in the same way in both A_i and $\mathbf{A_i}$ while non physical waves appear in $\mathbf{A}_i$ but not in A_i. Therefore, $\left|\mathbf{A}_i(x) - A_i(x)\right|$ is bounded by the sum of the strengths of all non physical waves, i.e. $\left|\mathbf{A}_i(x) - A_i(x)\right| \leq C\,\varepsilon$ by [4, formula (7.11)]. Finally, using [4, formula (8.5)]:

$$\frac{1}{C} \cdot \left\|v(x) - \tilde{v}(x)\right\| \leq \sum_{i=1}^n \left|q_i(x)\right| \leq C \cdot \left\|v(x) - \tilde{v}(x)\right\| \tag{4.5}$$

we obtain

$$\left|\mathbf{\Phi}\left(v(t,\cdot),\tilde{v}(t,\cdot)\right) - \Phi^\varepsilon(v,\tilde{v})(t)\right| \leq C\,\varepsilon \int_{\mathbb{R}} \sum_{i=1}^n \left|q_i(x)\right| dx \leq C\,\varepsilon \|v - \tilde{v}\|_{\mathbf{L^1}}$$

completing the proof. □

Proof of Theorem 4.1. The estimates (4.5) show that $\mathbf{\Phi}$ is equivalent to the $\mathbf{L^1}$ distance between functions in $\mathcal{D}^*_\delta$. Indeed, if δ is sufficiently small, then $\mathbf{W}_i(x) \in [1,2]$ for all $i = 1, \ldots, n$ and all $x \in \mathbb{R}$, so that

$$\frac{1}{C} \cdot \|v - \tilde{v}\|_{\mathbf{L^1}} \leq \mathbf{\Phi}\left(v,\tilde{v}\right) \leq 2C \cdot \|v - \tilde{v}\|_{\mathbf{L^1}} . \tag{4.6}$$

To prove *(i)*, fix $u, \tilde{u} \in \mathcal{D}_\delta$ and choose $v \in B_\eta(u)$, $\tilde{v} \in B_\eta(\tilde{u})$. By (4.6),

$$
\begin{array}{rcccl}
\frac{1}{C} \cdot \left(\|u-\tilde{u}\|_{\mathbf{L^1}} - 2\eta\right) & \leq & \mathbf{\Phi}\left(v, \tilde{v}\right) & \leq & 2C \cdot \left(\|u-\tilde{u}\|_{\mathbf{L^1}} + 2\eta\right) \\
\frac{1}{C} \cdot \left(\|u-\tilde{u}\|_{\mathbf{L^1}} - 2\eta\right) & \leq & \mathbf{\Xi}_\eta(u, \tilde{u}) & \leq & 2C \cdot \left(\|u-\tilde{u}\|_{\mathbf{L^1}} + 2\eta\right) .
\end{array}
$$

The proof of *(i)* is completed passing to the limit $\eta \to 0+$.

To prove *(ii)*, fix $u, \tilde{u} \in \mathcal{D}_\delta$ and $\eta > 0$. Correspondingly, choose $v_\eta \in B_\eta(u)$ and $\tilde{v}_\eta \in B_\eta(\tilde{u})$ satisfying

$$
\mathbf{\Xi}(u, \tilde{u}) \geq \mathbf{\Xi}_\eta(u, \tilde{u}) \geq \Phi(v_\eta, \tilde{v}_\eta) - \eta \,. \tag{4.7}
$$

Let now $\varepsilon > 0$ and introduce the ε-approximate solutions v_η^ε and $\tilde{v}_\eta^\varepsilon$ with initial data $v_\eta^\varepsilon(0, \cdot) = v_\eta$ and $\tilde{v}_\eta^\varepsilon(0, \cdot) = \tilde{v}_\eta$. Note that for ε sufficiently small

$$
\begin{array}{rcccccl}
\mathbf{\Upsilon}\left(v_\eta^\varepsilon(t)\right) & \leq & \Upsilon^\varepsilon\left(v_\eta^\varepsilon\right)(t) + C\varepsilon & \leq & \Upsilon^\varepsilon(v_\eta^\varepsilon)(0) + C\varepsilon & & \\
& \leq & \mathbf{\Upsilon}(v_\eta) + C\varepsilon & < & \delta + C\varepsilon & < & \delta
\end{array}
$$

and an analogous inequality holds for $\tilde{v}_\eta^\varepsilon$. Therefore $v_\eta^\varepsilon(t)$, $\tilde{v}_\eta^\varepsilon(t) \in \mathcal{D}_\delta^*$. Here we denoted with Υ^ε the sum $V + C_0 Q$ defined on ε–approximate wave front tracking solutions (see [4, formulæ (7.53), (7.54)]). We may thus apply Lemma 4.2, Proposition 4.4 and the main result in [4, Chapter 8], that is [4, Theorem 8.2], to obtain

$$
\begin{array}{rl}
& \mathbf{\Xi}\left(v_\eta^\varepsilon(t), \tilde{v}_\eta^\varepsilon(t)\right) \\
\leq & \mathbf{\Phi}\left(v_\eta^\varepsilon(t), \tilde{v}_\eta^\varepsilon(t)\right) \\
\leq & \Phi^\varepsilon\left(v_\eta^\varepsilon, \tilde{v}_\eta^\varepsilon\right)(t) + C\varepsilon\left\|v_\eta^\varepsilon(t) - \tilde{v}_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} \\
\leq & \Phi^\varepsilon\left(v_\eta^\varepsilon, \tilde{v}_\eta^\varepsilon\right)(0) + C\varepsilon t + C\varepsilon\left\|v_\eta^\varepsilon(t) - \tilde{v}_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} \\
\leq & \mathbf{\Phi}\left(v_\eta, \tilde{v}_\eta\right) + C\varepsilon t + C\varepsilon\left\|v_\eta^\varepsilon(t) - \tilde{v}_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} + C\varepsilon\left\|v_\eta - \tilde{v}_\eta\right\|_{\mathbf{L^1}} .
\end{array}
$$

Recall that as $\varepsilon \to 0$ by [4, Theorem 8.1] $v_\eta^\varepsilon(t) \to S_t v_\eta$ and $\tilde{v}_\eta^\varepsilon(t) \to S_t \tilde{v}_\eta$. Hence, Proposition 4.3 and (4.7) ensure that

$$
\mathbf{\Xi}(S_t v_\eta, S_t \tilde{v}_\eta) \leq \liminf_{\varepsilon \to 0+} \mathbf{\Xi}\left(v_\eta^\varepsilon(t), \tilde{v}_\eta^\varepsilon(t)\right) \leq \mathbf{\Phi}\left(v_\eta, \tilde{v}_\eta\right) \leq \mathbf{\Xi}(u, \tilde{u}) + \eta \,.
$$

By the choice of v_η and $\tilde{v}_\eta$, we have that $v_\eta \to u$ and $\tilde{v}_\eta \to \tilde{u}$ in $\mathbf{L^1}$ as $\eta \to 0+$. Therefore, using the continuity of the SRS in $\mathbf{L^1}$ and applying again Proposition 4.3, we may conclude that

$$
\mathbf{\Xi}(S_t u, S_t \tilde{u}) \leq \liminf_{\eta \to 0+} \mathbf{\Xi}(S_t v_\eta, S_t \tilde{v}_\eta) \leq \mathbf{\Xi}(u, \tilde{u}) \,,
$$

completing the proof of *(ii)*. The latter item *(iii)* follows from Proposition 4.3. $\square$

5 Appendix

Proposition 5.1 *For all u, $\tilde u$ in $\mathcal{D}^*_\delta$, one has $\Xi(u,\tilde u) = \Phi(u,\tilde u)$.*

Lemma 4.2 provides the first inequality. The proof of the other one follows from the next two lemmas.

Lemma 5.2 *The functional $\mathbf{\Phi}$, defined on all piecewise constant functions in $\mathcal{D}^*_\delta$, is lower semicontinuous with respect to the $\mathbf{L^1}$ norm.*

Proof. Fix $u, \tilde u$ piecewise constant in $\mathcal{D}^*_\delta$. Choose two sequences of piecewise constant maps $u_\nu, \tilde u_\nu$ in $\mathcal{D}^*_\delta$ converging to $u, \tilde u$ in $\mathbf{L^1}$. We want to show that $\mathbf{\Phi}(u,\tilde u) \le \liminf_{\nu\to+\infty} \mathbf{\Phi}(u_\nu, \tilde u_\nu)$. Call $l = \liminf_{\nu\to+\infty} \mathbf{\Phi}(u_\nu,\tilde u_\nu)$ and note that, up to subsequences, we may assume that $\lim_{\nu\to+\infty} \mathbf{\Phi}(u_\nu,\tilde u_\nu) = l$. By possibly selecting a further subsequence, we also have that both u_ν and $\tilde u_\nu$ converge a.e. to u and $\tilde u$.

Introduce the functions $q = (q_1,\ldots,q_n)$ and $q^\nu = (q_1^\nu,\ldots,q_n^\nu)$ by

$$\tilde u(x) = \mathbf{S}\left(q(x)\right)\left(u(x)\right) \qquad \text{and} \qquad \tilde u_\nu(x) = \mathbf{S}\left(q^\nu(x)\right)\left(u_\nu(x)\right)\,.$$

with $\mathbf{S}$ defined in (2.2). For the computations below, we need the following more explicit notation: fix $\bar v(x) \in \mathcal{D}^*_\delta$ and $\bar q$ piecewise constant, and define $\mathbb{A}_1(\bar v,\bar q),\ldots,\mathbb{A}_n(\bar v,\bar q)$ through

$$\begin{aligned}\left(\mathbb{A}_i(\bar v,\bar q)\right)(x) &= \sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y \le x \text{ and } j > i\right\}\\ &\quad+\sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y > x \text{ and } j < i\right\}\end{aligned}$$

for the linearly degenerate case, while for the genuinely nonlinear case:

$$\begin{aligned}\left(\mathbb{A}_i(\bar v,\bar q)\right)(x) &= \sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y \le x \text{ and } j > i\right\}\\ &\quad+\sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y > x \text{ and } j < i\right\}\\ &\quad+\left\{\begin{array}{ll}\sum\left\{\left|\bar\sigma_{y,i}\right| : (y,i) \text{ such that } y \le x\right\} & \text{if } \bar q_i(x) < 0\\ \sum\left\{\left|\bar\sigma_{y,i}\right| : (y,i) \text{ such that } y > x\right\} & \text{if } \bar q_i(x) \ge 0\end{array}\right.\\ \left(\tilde{\mathbb{A}}_i(\bar v,\bar q)\right)(x) &= \sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y \le x \text{ and } j > i\right\}\\ &\quad+\sum\left\{\left|\bar\sigma_{y,j}\right| : (y,j) \text{ such that } y > x \text{ and } j < i\right\}\\ &\quad+\left\{\begin{array}{ll}\sum\left\{\left|\bar\sigma_{y,i}\right| : (y,i) \text{ such that } y \le x\right\} & \text{if } \bar q_i(x) > 0\\ \sum\left\{\left|\bar\sigma_{y,i}\right| : (y,i) \text{ such that } y > x\right\} & \text{if } \bar q_i(x) \le 0\end{array}\right.\end{aligned}$$

where, using E as in (2.1),

$$(\bar\sigma_{x,1},\ldots,\bar\sigma_{x,n}) = E\left(\bar v(x-), \bar v(x+)\right)\,.$$

Remark that $\left(\mathbb{A}_i(\bar v,\bar q)\right)(x)$ and $\left(\tilde{\mathbb{A}}_i(\bar v,\bar q)\right)(x)$ are Lipschitz function of the values assumed by $\bar v$ (for fixed shock positions). Finally introduce also

$$\begin{aligned}\left(\mathbb{B}_i(\bar v,\bar q)\right)(x) &= \left(\mathbb{A}_i(\bar v,\bar q)\right)(x) + \kappa_2 \mathbf{Q}(\bar v)\\ \left(\tilde{\mathbb{B}}_i(\bar v,\bar q)\right)(x) &= \left(\tilde{\mathbb{A}}_i(\bar v,\bar q)\right)(x) + \kappa_2 \mathbf{Q}(\bar v)\end{aligned}$$

And therefore one has:

$$\begin{aligned}\mathbf{\Phi}(u,\tilde u) &= \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \times\\ &\qquad \times \left[1+\kappa_1\left(\left(\mathbb{B}_i(u,q)\right)(x) + \left(\tilde{\mathbb{B}}_i(\tilde u,q)\right)(x)\right)\right] dx\\ \mathbf{\Phi}(u_\nu,\tilde u_\nu) &= \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i^\nu(x)\right| \times\\ &\qquad \times \left[1+\kappa_1\left(\left(\mathbb{B}_i(u_\nu,q^\nu)\right)(x) + \left(\tilde{\mathbb{B}}_i(\tilde u_\nu,q^\nu)\right)(x)\right)\right] dx\end{aligned}$$

Let $\{x_1,\ldots,x_{N+1}\}$ be the set of the jump points in u and $\tilde u$ and write

$$u=\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha,x_{\alpha+1}[}, \qquad \tilde u=\sum_{\alpha=1}^{N} \tilde u_\alpha\, \chi_{[x_\alpha,x_{\alpha+1}[}\,.$$

For all α, select $y_\alpha \in \,]x_\alpha,x_{\alpha+1}[$ so that as $\nu\to+\infty$, the sequence $u_\nu(y_\alpha)$ converges to $u(y_\alpha)=u_\alpha$ and $\tilde u_\nu(y_\alpha)$ to $\tilde u(y_\alpha)=\tilde u_\alpha$. Introduce the piecewise constant function $\check u_\nu = \sum_\alpha u_\nu(y_\alpha)\,\chi_{[x_\alpha,x_{\alpha+1}[}$. Let $\check{\tilde u}_\nu$ be defined analogously.

Because of the Lipschitz dependence (for fixed jump positions) of $\mathbb{A}_i(\bar v)$, $\tilde{\mathbb{A}}_i(\bar v)$ and $\mathbf{Q}(\bar v)$ on the states attained by $\bar v$ we obtain the pointwise limits

$$\lim_{\nu\to+\infty} \mathbb{B}_i(\check u_\nu,q) = \mathbb{B}_i(u,q) \quad \text{and} \quad \lim_{\nu\to+\infty} \tilde{\mathbb{B}}_i(\check{\tilde u}_\nu,q) = \tilde{\mathbb{B}}_i(\tilde u,q)\,. \tag{5.1}$$

Claim: there exists a uniformly bounded sequence of positive maps ω_ν with

$$\lim_{\nu\to+\infty}\omega_\nu(x)=0 \quad \text{a.e. in } x$$

such that, with the notation above, the following inequality holds:

$$\left(\mathbb{B}_i(\check u_\nu,q)\right)(x) \le \left(\mathbb{B}_i(u_\nu,q)\right)(x) + \omega_\nu(x)$$

and a similar inequality holds for $\tilde{\mathbb{B}}_i$.

Proof of the claim. Consider only $\mathbb{B}_i$ since the case with $\tilde{\mathbb{B}}_i$ is similar. Fix $\bar x\in\mathbb{R}$ and prove the above inequality passing from $\check u_\nu$ to u_ν recursively applying 3 elementary operations:

PSfrag replacements
t
$w_{\bar\alpha-1}$
$w_{\bar\alpha}$
$\xi_{\bar\alpha}$
$\xi_{\bar\alpha+1}$
$\bar x$
$w_{\bar\alpha+1}$
x
w
w'

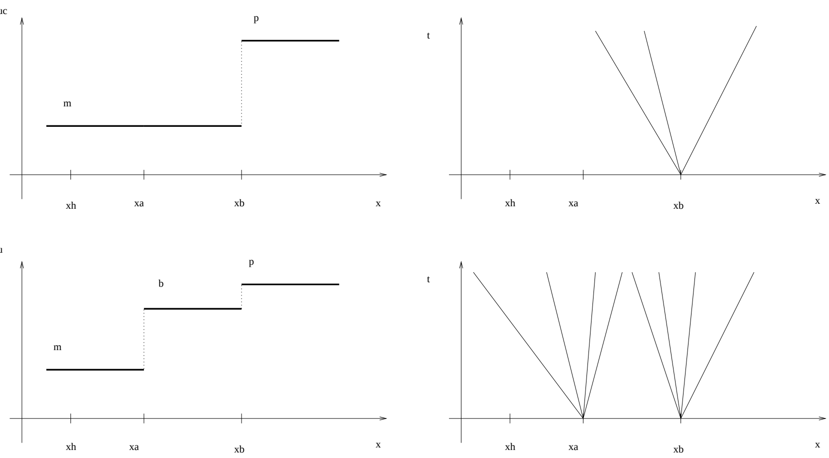

Figure 2: Exemplification of point 2.

1. w' is obtained from w only shifting the position of the points of jump but without letting any point of jump cross $\bar x$. More formally, if $w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[}$ with $\xi_\alpha < \xi_{\alpha+1}$, $w' = \sum_\alpha w_\alpha \chi_{[\xi'_\alpha,\xi'_{\alpha+1}[}$ with $\xi'_\alpha < \xi'_{\alpha+1}$ and moreover $\bar x \in \left]\xi_\alpha, \xi_{\alpha+1}\right[\cap \left]\xi'_\alpha, \xi'_{\alpha+1}\right[$, then

$$\big(\mathbb{B}_i(w',q)\big)(\bar x) = \big(\mathbb{B}_i(w,q)\big)(\bar x)\,.$$

Indeed, if all the jumps stay unchanged and no shocks crosses $\bar x$, then nothing changes in the definition of $\mathbb{A}_i$ and $\mathbf{Q}$.

2. w' is obtained from w removing a value attained by w on an interval not containing $\bar x$, see Figure 2. More formally, if

$$w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[} \qquad \text{with} \qquad \xi_\alpha < \xi_{\alpha+1}$$

and $\bar x \notin \left[\xi_{\bar\alpha}, \xi_{\bar\alpha+1}\right[$, then

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha\, \chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w_{\bar\alpha-1}\, \chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[}$$

or

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha\, \chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w_{\bar\alpha+1}\, \chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[}\,.$$

In this case

$$\big(\mathbb{A}_i(w',q)\big)(\bar x) + \kappa_2 Q(w') \leq \big(\mathbb{A}_i(w,q)\big)(\bar x) + \kappa_2 Q(w)\,.$$

Indeed, consider for example the case in Figure 2. The two jumps at the points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ in w are substituted by a single jump in w' at the point $\xi_{\bar\alpha+1}$. The points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ are both to the right of $\bar x$, therefore the waves in w' at the point $\xi_{\bar\alpha+1}$ which appear in $\left(\mathbb{A}_i(w',q)\right)(\bar x)$ are of the same families of the waves in w at the points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ which appear in $\left(\mathbb{A}_i(w,q)\right)(\bar x)$. Since all the other waves in $\mathbb{A}_i$ are left unchanged we have

$$\left(\mathbb{A}_i(w',q)\right)(\bar x) - \left(\mathbb{A}_i(w,q)\right)(\bar x) \leq \sum_{j=1}^{n} \left|\sigma'_{\xi_{\bar\alpha+1},j} - \sigma_{\xi_{\bar\alpha},j} - \sigma_{\xi_{\bar\alpha+1},j}\right|$$

where

$$\begin{array}{rclcl} \sigma'_{\xi_{\bar\alpha+1},j} & = & E_j\left(w'\left(\xi_{\bar\alpha+1}-\right), w'\left(\xi_{\bar\alpha+1}+\right)\right) & = & E_j\left(w_{\bar\alpha-1}, w_{\bar\alpha+1}\right) \\ \sigma_{\xi_{\bar\alpha+1},j} & = & E_j\left(w\left(\xi_{\bar\alpha+1}-\right), w\left(\xi_{\bar\alpha+1}+\right)\right) & = & E_j\left(w_{\bar\alpha}, w_{\bar\alpha+1}\right) \\ \sigma_{\xi_{\bar\alpha+1},j} & = & E_j\left(w\left(\xi_{\bar\alpha}-\right), w\left(\xi_{\bar\alpha}+\right)\right) & = & E_j\left(w_{\bar\alpha-1}, w_{\bar\alpha}\right) . \end{array}$$

Therefore, the increase in $\mathbb{A}_i$ evaluated at $\bar x$ is bounded by the interaction potential between the waves at $\xi_{\bar\alpha}$ and those at $\xi_{\bar\alpha+1}$ and is compensated by the decrease in $\kappa_2\mathbf{Q}$, as in the standard Glimm interaction estimates.

3. w' is obtained from w changing the value assumed by w in the interval containing $\bar x$. More formally, if

$$w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[} \qquad \text{with} \qquad \xi_\alpha < \xi_{\alpha+1}$$

and $\bar x \in [\xi_{\bar\alpha}, \xi_{\bar\alpha+1}[$, then

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w'_{\bar\alpha}\chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[} .$$

In this case

$$\left(\mathbb{B}_i(w',q)\right)(\bar x)+ \leq \left(\mathbb{B}_i(w,q)\right)(\bar x) + C\left|w_{\bar\alpha} - w'_{\bar\alpha}\right| .$$

Indeed, this inequality directly follows from the Lipschitz dependence of $\mathbb{A}_i(w,q)(\bar x)$ and of $\mathbf{Q}(w)$ on the values attained by w for fixed jump positions.

For $\bar x \in [x_{\bar\alpha}, x_{\bar\alpha+1}[$ we can pass from u_ν to to the function w_ν defined by

$$w_\nu = \sum_{\alpha\neq\bar\alpha} u_\nu(y_\alpha)\chi_{[x_\alpha,x_{\alpha+1}[} + u_\nu(\bar x)\chi_{[x_{\bar\alpha},x_{\bar\alpha+1}[}$$

applying the first two steps a certain number of times. And we obtain the estimate

$$\left(\mathbb{B}_i(w_\nu,q)\right)(\bar x) \leq \left(\mathbb{B}_i(u_\nu,q)\right)(\bar x) .$$

Finally with the third step we go from w_ν to $\check u_\nu$ obtaining the estimate:

$$\begin{array}{rcl} \left(\mathbb{B}_i(\check u_\nu,q)\right)(\bar x) & \leq & \left(\mathbb{B}_i(w_\nu,q)\right)(\bar x) + C\left|u_\nu(\bar x) - u_\nu(y_{\bar\alpha})\right| \\ & \leq & \left(\mathbb{B}_i(u_\nu,q)\right)(\bar x) + C\left|u_\nu(\bar x) - u_\nu(y_{\bar\alpha})\right| \\ & \leq & \left(\mathbb{B}_i(u_\nu,q)\right)(\bar x) + \omega_\nu(\bar x) . \end{array}$$

with

$$\omega_\nu(x) = C \sum_{\alpha=1}^{N} \left|u_\nu(x) - u_\nu(y_\alpha)\right| \chi_{[x_\alpha, x_{\alpha+1}[}(x)\,.$$

But a.e. $x \in \mathbb{R}$ one has

$$\sum_{\alpha=1}^{N} \left|u_\nu(x) - u_\nu(y_\alpha)\right| \chi_{[x_\alpha, x_{\alpha+1}[}(x) \to \sum_{\alpha=1}^{N} \left|u_\alpha - u_\alpha\right| \chi_{[x_\alpha, x_{\alpha+1}[}(x) = 0$$

proving the claim.

Now we write:

$$\mathbf{\Phi}\left(u, \tilde{u}\right) = \chi_{1,\nu} + \chi_{2,\nu} + \chi_{3,\nu} + \chi_{4,\nu} + \mathbf{\Phi}\left(u_\nu, \tilde{u}_\nu\right)\,,$$

with

$$\begin{aligned}
\chi_{1,\nu} &= \mathbf{\Phi}\left(u, \tilde{u}\right) \\
&\quad - \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \left[1 + \kappa_1 \left(\left(\mathbb{B}_i(\check{u}_\nu, q)\right)(x) + \left(\tilde{\mathbb{B}}_i(\check{\tilde{u}}_\nu, q)\right)(x) \right)\right] dx \\
\chi_{2,\nu} &= \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \kappa_1 \Big[\left(\mathbb{B}_i(\check{u}_\nu, q)\right)(x) - \left(\mathbb{B}_i(u_\nu, q)\right)(x) \\
&\qquad\qquad + \left(\tilde{\mathbb{B}}_i(\check{\tilde{u}}_\nu, q)\right)(x) - \left(\tilde{\mathbb{B}}_i(\tilde{u}_\nu, q)\right)(x) \Big] dx \\
\chi_{3,\nu} &= \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \kappa_1 \Big[\left(\mathbb{B}_i(u_\nu, q)\right)(x) - \left(\mathbb{B}_i(u_\nu, q^\nu)\right)(x) \\
&\qquad\qquad + \left(\tilde{\mathbb{B}}_i(\tilde{u}_\nu, q)\right)(x) - \left(\tilde{\mathbb{B}}_i(\tilde{u}_\nu, q^\nu)\right)(x) \Big] dx \\
\chi_{4,\nu} &= \int_{\mathbb{R}} \sum_{i=1}^{n} \left(\left|q_i(x)\right| - \left|q_i^\nu(x)\right| \right) \cdot \\
&\qquad\qquad \cdot \left[1 + \kappa_1 \left(\left(\mathbb{B}_i(u_\nu, q^\nu)\right)(x) + \left(\tilde{\mathbb{B}}_i(\tilde{u}_\nu, q^\nu)\right)(x) \right)\right] dx\,.
\end{aligned}$$

By the pointwise convergence (5.1) of the integrand, $\lim_{\nu \to +\infty} \chi_{1,\nu} = 0$.

Passing to the next summand, note that the claim implies that

$$\chi_{2,\nu} \leq 2C \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \kappa_1 \omega_\nu(x)\, dx$$

and the Dominated Convergence Theorem implies $\liminf_{\nu \to +\infty} \chi_{2,\nu} \leq 0$.

Concerning $\chi_{3,\nu}$, observe that q_i^ν converges a.e. to $q(x)$, therefore for a.e. $x \in \mathbb{R}$ such that $q_i(x) \neq 0$, $q_i^\nu(x)$ has the same sign as $q(x)$ for ν

sufficiently large and this implies $\left(\mathbb{B}_i(u^\nu, q)\right)(x) = \left(\mathbb{B}_i(u^\nu, q^\nu)\right)(x)$. Hence the integrand in $\chi_{3,\nu}$ converges a.e. to zero and we have $\lim_{\nu\to+\infty} \chi_{3,\nu} = 0$.

Finally the L^1 convergence of q_i^ν to q_i implies $\lim_{\nu\to+\infty} \chi_{4,\nu} = 0$, concluding the proof since $\mathbf{\Phi}(u_\nu, \tilde{u}_\nu) \to l$. □

Lemma 5.3 *For all piecewise constant $u, \tilde{u} \in \mathcal{D}^*_\delta$, $\mathbf{\Xi}(u,\tilde{u}) \geq \mathbf{\Phi}(u,\tilde{u})$.*

Proof. By the definition (4.3) of $\mathbf{\Xi}$, for all $u, \tilde{u} \in \mathcal{D}^*_\delta$, there exist sequences $v_\nu, \tilde{v}_\nu$ of piecewise constant functions such that for $\nu \to +\infty$ we have $v_\nu \to u$, $\tilde{v}_\nu \to u$ in $\mathbf{L^1}$ and $\mathbf{\Phi}(v_\nu, \tilde{v}_\nu) \to \mathbf{\Xi}(u,\tilde{u})$. Hence, by Lemma 5.2

$$\mathbf{\Phi}(u,\tilde{u}) \leq \liminf_{\nu\to+\infty} \mathbf{\Phi}(v_\nu, \tilde{v}_\nu) = \Xi(u,\tilde{u}),$$

completing the proof. □

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