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V-cycle optimal convergence for DCT-III matrices

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Dedicated to Georg Heinig

Abstract. The paper analyzes a two-grid and a multigrid method for matrices belonging to the DCT-III algebra and generated by a polynomial symbol. The aim is to prove that the convergence rate of the considered multigrid method (V-cycle) is constant independent of the size of the given matrix. Numerical examples from differential and integral equations are considered to illustrate the claimed convergence properties.

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1. Introduction

In the last two decades, an intensive work has concerned the numerical solution of structured linear systems of large dimensions [6, 14, 16]. Many problems have been solved mainly by the use of (preconditioned) iterative solvers. However, in the multilevel setting, it has been proved that the most popular matrix algebra preconditioners cannot work in general (see [23, 26, 20] and references therein). On the other hand, the multilevel structures often are the most interesting in practical applications. Therefore, quite recently, more attention has been focused (see [1, 2, 7, 5, 27, 9, 12, 10, 13, 22, 25, 19]) on the multigrid solution of multilevel structured (Toeplitz, circulants, Hartley, sine (τ class) and cosine algebras) linear systems in which the coefficient matrix is banded in a multilevel sense and positive definite. The reason is due to the fact that these techniques are very efficient, the total cost for reaching the solution within a preassigned accuracy being linear as the dimensions of the involved linear systems.

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In this paper we deal with the case of matrices generated by a polynomial symbol and belonging to the DCT-III algebra. This kind of matrices appears in the solution of differential equations and integral equations, see for instance [4, 18, 24]. In particular, they directly arise in certain image restoration problems or can be used as preconditioners for more complicated problems in the same field of application [17, 18].

In [7] a Two-Grid (TGM)/Multi-Grid (MGM) Method has been proposed and the theoretical analysis of the TGM has been performed in terms of the algebraic multigrid theory developed by Ruge and Stüben [21].
Here, the aim is to provide general conditions under which the proposed MGM results to be optimally convergent with a convergence rate independent of the dimension and to perform the corresponding theoretical analysis.
More precisely, for MGM we mean the simplest (and less expensive) version of the large family of multigrid methods, i.e. the V-cycle procedure. For a brief description of the TGM and of the MGM (standard V-cycle) we refer to §2. An extensive treatment can be found in [11], and especially in [28].
In all the considered cases the MGM results to be optimal in the sense of Definition 1.1, i.e. the problem of solving a linear system with coefficient matrix A_m is asymptotically of the same cost as the direct problem of multiplying A_m by a vector.

Definition 1.1. [3] Let $\{A_m x_m = b_m\}$ be a given sequence of linear systems of increasing dimensions. An iterative method is *optimal* if

1. the arithmetic cost of each iteration is at most proportional to the complexity of a matrix vector product with matrix A_m,
2. the number of iterations for reaching the solution within a fixed accuracy can be bounded from above by a constant independent of m.

In fact, the total cost of the proposed MGM will be of $O(m)$ operations since for any coarse level s we can find a projection operator P^s_{s+1} such that

- the matrix vector product involving P^s_{s+1} costs $O(m_s)$ operations where $m_s = m/2^s$;
- the coarse grid matrix $A_{m_{s+1}} = P^s_{s+1} A_{m_s} (P^s_{s+1})^T$ is also a matrix in the DCT III algebra generated by a polynomial symbol and can be formed within $O(m_s)$ operations;
- the convergence rate of the MGM is independent of m.

The paper is organized as follows. In §2 we briefly report the main tools regarding to the convergence theory of algebraic multigrid methods [21]. In §3 we consider the TGM for matrices belonging to DCT-III algebra with reference to some optimal convergence properties, while §4 is devoted to the convergence analysis of its natural extension as V-cycle. In §5 numerical evidences of the claimed results are discussed and §6 deals with complexity issues and conclusions.

2. Two-grid and Multi-grid methods

In this section we briefly report the main results pertaining to the convergence theory of algebraic multigrid methods.
Let us consider the generic linear system $A_m x_m = b_m$, where $A_m \in \mathbb{C}^{m\times m}$ is a Hermitian positive definite matrix and $x_m, b_m \in \mathbb{C}^m$. Let $m_0 = m > m_1 > \ldots > m_s > \ldots > m_{s_{\min}}$ and let $P_{s+1}^s \in \mathbb{C}^{m_{s+1}\times m_s}$ be a given full-rank matrix for any s. Lastly, let us denote by $\mathcal{V}_s$ a class of iterative methods for linear systems of dimension m_s.
According to [11], the algebraic Two-Grid Method (TGM) is an iterative method whose generic step is defined as follow.

$$x_s^{\text{out}} = \mathcal{TGM}(s, x_s^{\text{in}}, b_s)$$

$x_s^{\text{pre}} = \mathcal{V}_{s,\text{pre}}^{\nu_{\text{pre}}}(x_s^{\text{in}})$	Pre-smoothing iterations
$r_s = A_s x_s^{\text{pre}} - b_s$ $r_{s+1} = P_{s+1}^s r_s$ $A_{s+1} = P_{s+1}^s A_s (P_{s+1}^s)^H$ Solve $A_{s+1} y_{s+1} = r_{s+1}$ $\hat{x}_s = x_s^{\text{pre}} - (P_{s+1}^s)^H y_{s+1}$	Exact Coarse Grid Correction
$x_s^{\text{out}} = \mathcal{V}_{s,\text{post}}^{\nu_{\text{post}}}(\hat{x}_s)$	Post-smoothing iterations

where the dimension m_s is denoted in short by the subscript s.
In the first and last steps a *pre-smoothing iteration* and a *post-smoothing iteration* are respectively applied ν_{pre} times and ν_{post} times, according to the chosen iterative method in the class $\mathcal{V}_s$. Moreover, the intermediate steps define the so called *exact coarse grid correction operator*, that depends on the considered projector operator P_{s+1}^s. The global iteration matrix of the TGM is then given by

$$\begin{aligned} TGM_s &= V_{s,\text{post}}^{\nu_{\text{post}}} CGC_s V_{s,\text{pre}}^{\nu_{\text{pre}}}, && (2.1)\\ CGC_s &= I_s - (P_{s+1}^s)^H A_{s+1}^{-1} P_{s+1}^s A_s \quad A_{s+1} = P_{s+1}^s A_s (P_{s+1}^s)^H, && (2.2)\end{aligned}$$

where $V_{s,\text{pre}}$ and $V_{s,\text{post}}$ respectively denote the pre-smoothing and post-smoothing iteration matrices.

By means of a recursive procedure, the TGM gives rise to a Multi-Grid Method (MGM): the standard V-cycle is defined as follows.

$$x_s^{\text{out}} = \mathcal{MGM}(s, x_s^{\text{in}}, b_s)$$

`if` $s \le s_{\min}$ `then`

$\boxed{\text{Solve } A_s x_s^{\text{out}} = b_s}$ Exact solution

`else`

$\boxed{x_s^{\text{pre}} = \mathcal{V}_{s,\text{pre}}^{\nu_{\text{pre}}}(x_s^{\text{in}})}$ Pre-smoothing iterations

$$\boxed{\begin{array}{l} r_s = A_s x_s^{\text{pre}} - b_s \\ r_{s+1} = P_{s+1}^s r_s \\ y_{s+1} = \mathcal{MGM}(s+1, \mathbf{0}_{s+1}, r_{s+1}) \\ \hat{x}_s = x_s^{\text{pre}} - (P_{s+1}^s)^H y_{s+1} \end{array}}$$ Coarse Grid Correction

$\boxed{x_s^{\text{out}} = \mathcal{V}_{s,\text{post}}^{\nu_{\text{post}}}(\hat{x}_s)}$ Post-smoothing iterations

Notice that in MGM the matrices $A_{s+1} = P_{s+1}^s A_s (P_{s+1}^s)^H$ are more profitably formed in the so called *setup phase* in order to reduce the computational costs.
The global iteration matrix of the MGM can be recursively defined as

$$\begin{aligned} MGM_{s_{\min}} &= O \in \mathbb{C}^{s_{\min} \times s_{\min}}, \\ MGM_s &= V_{s,\text{post}}^{\nu_{\text{post}}} \left[I_s - (P_{s+1}^s)^H \left(I_{s+1} - MGM_{s+1} \right) A_{s+1}^{-1} P_{s+1}^s A_s \right] V_{s,\text{pre}}^{\nu_{\text{pre}}}, \\ & \qquad s = s_{\min} - 1, \ldots, 0. \end{aligned}$$

Some general conditions that ensure the convergence of an algebraic TGM and MGM are due to Ruge and Stüben [21].
Hereafter, by $\|\cdot\|_2$ we denote the Euclidean norm on $\mathbb{C}^m$ and the associated induced matrix norm over $\mathbb{C}^{m \times m}$. If X is positive definite, $\|\cdot\|_X = \|X^{1/2} \cdot\|_2$ denotes the Euclidean norm weighted by X on $\mathbb{C}^m$ and the associated induced matrix norm. Finally, if X and Y are Hermitian matrices, then the notation $X \le Y$ means that $Y - X$ is nonnegative definite.

Theorem 2.1 (TGM convergence [21]**).** *Let m_0, m_1 be integers such that $m_0 > m_1 > 0$, let $A \in \mathbb{C}^{m_0 \times m_0}$ be a positive definite matrix. Let $\mathcal{V}_0$ be a class of iterative methods for linear systems of dimension m_0 and let $P_1^0 \in \mathbb{C}^{m_1 \times m_0}$ be a given full-rank matrix. Suppose that there exist $\alpha_{\text{pre}} > 0$ and $\alpha_{\text{post}} > 0$ independent of m_0*

such that

$$\|V_{0,\mathrm{pre}}\, x\|_A^2 \leq \|x\|_A^2 - \alpha_{\mathrm{pre}} \|V_{0,\mathrm{pre}}\, x\|_{AD^{-1}A}^2 \quad \text{for any } x \in \mathbb{C}^{m_0} \tag{2.3a}$$

$$\|V_{0,\mathrm{post}}\, x\|_A^2 \leq \|x\|_A^2 - \alpha_{\mathrm{post}}\, \|x\|_{AD^{-1}A}^2 \quad \text{for any } x \in \mathbb{C}^{m_0} \tag{2.3b}$$

(where D denotes the main diagonal of A) and that there exists $\gamma > 0$ independent of m_0 such that

$$\min_{y \in \mathbb{C}^{m_1}} \|x - (P_1^0)^H y\|_D^2 \leq \gamma \|x\|_A^2 \quad \text{for any } x \in \mathbb{C}^{m_0}. \tag{2.4}$$

Then, $\gamma \geq \alpha_{\mathrm{post}}$ and

$$\|TGM_0\|_A \leq \sqrt{\frac{1 - \alpha_{\mathrm{post}}/\gamma}{1 + \alpha_{\mathrm{pre}}/\gamma}}. \tag{2.5}$$

It is worth stressing that in Theorem 2.1 the matrix $D \in \mathbb{C}^{m_0 \times m_0}$ can be substituted by any Hermitian positive definite matrix X: clearly the choice $X = I$ can give rise to valuable simplifications [1].

At first sight, the MGM convergence requirements are more severe since the smoothing and CGC iteration matrices are linked in the same inequalities as stated below.

Theorem 2.2 (MGM convergence [21]**).** *Let $m_0 = m > m_1 > m_2 > \ldots > m_s > \ldots > m_{s_{\min}}$ and let $A \in \mathbb{C}^{m \times m}$ be a positive definite matrix. Let $P_{s+1}^s \in \mathbb{C}^{m_{s+1} \times m_s}$ be full-rank matrices for any level s. Suppose that there exist $\delta_{\mathrm{pre}} > 0$ and $\delta_{\mathrm{post}} > 0$ such that*

$$\|V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}} x\|_{A_s}^2 \leq \|x\|_{A_s}^2 - \delta_{\mathrm{pre}}\, \|CGC_s V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}} x\|_{A_s}^2 \quad \text{for any } x \in \mathbb{C}^{m_s} \tag{2.6a}$$

$$\|V_{s,\mathrm{posr}}^{\nu_{\mathrm{post}}} x\|_{A_s}^2 \leq \|x\|_{A_s}^2 - \delta_{\mathrm{post}}\, \|CGC_s x\|_{A_s}^2 \quad \text{for any } x \in \mathbb{C}^{m_s} \tag{2.6b}$$

both for each $s = 0, \ldots, s_{\min} - 1$, then $\delta_{\mathrm{post}} \leq 1$ and

$$\|MGM_0\|_A \leqslant \sqrt{\frac{1 - \delta_{\mathrm{post}}}{1 + \delta_{\mathrm{pre}}}} < 1. \tag{2.7}$$

By virtue of Theorem 2.2, the sequence $\{x_m^{(k)}\}_{k \in \mathbb{N}}$ will converge to the solution of the linear system $A_m x_m = b_m$ and within a constant error reduction not depending on m and $s_{\min}$ if at least one between δ_{pre} and δ_{post} is independent of m and $s_{\min}$.

Nevertheless, as also suggested in [21], the inequalities (2.6a) and (2.6b) can be respectively splitted as

$$\begin{cases} \|V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}} x\|_{A_s}^2 & \leq \|x\|_{A_s}^2 - \alpha\, \|V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}} x\|_{A_s D_s^{-1} A_s} \\ \|CGC_s x\|_{A_s}^2 & \leq \gamma\, \|x\|_{A_s D_s^{-1} A_s}^2 \\ \delta_{\mathrm{pre}} = \alpha/\gamma \end{cases} \tag{2.8}$$

and

$$\begin{cases} \|V_{s,\mathrm{post}}^{\nu_{\mathrm{post}}} x\|_{A_s}^2 & \leq \|x\|_{A_s}^2 - \beta\, \|x\|_{A_s D_s^{-1} A_s}^2 \\ \|CGC_s\, x\|_{A_s}^2 & \leq \gamma\, \|x\|_{A_s D_s^{-1} A_s}^2 \\ \delta_{\mathrm{post}} = \beta/\gamma \end{cases} \tag{2.9}$$

where D_s is the diagonal part of A_s (again, the $AD^{-1}A$-norm is not compulsory [1] and the A^2-norm will be considered in the following) and where, more importantly, the coefficients α, β and γ can differ in each recursion level s since the step from (2.8) to (2.6a) and from (2.9) to (2.6b) are purely algebraic and do not affect the proof of Theorem 2.2.
Therefore, in order to prove the V-cycle optimal convergence, it is possible to consider the inequalities

$$\|V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}}x\|_{A_s}^2 \le \|x\|_{A_s}^2 - \alpha_s \|V_{s,\mathrm{pre}}^{\nu_{\mathrm{pre}}}x\|_{A_s^2}^2 \qquad \text{for any } x \in \mathbb{C}^{m_s} \tag{2.10a}$$

$$\|V_{s,\mathrm{post}}^{\nu_{\mathrm{post}}}x\|_{A_s}^2 \le \|x\|_{A_s}^2 - \beta_s \|x\|_{A_s^2}^2 \qquad \text{for any } x \in \mathbb{C}^{m_s} \tag{2.10b}$$

$$\|CGC_s x\|_{A_s}^2 \le \gamma_s \|x\|_{A_s^2}^2 \qquad \text{for any } x \in \mathbb{C}^{m_s}. \tag{2.10c}$$

where it is required that α_s, β_s, $\gamma_s \ge 0$ for each $s = 0, \dots, s_{\min} - 1$ and

$$\delta_{\mathrm{pre}} = \min_{0 \le s < s_{\min}} \frac{\alpha_s}{\gamma_s}, \qquad \delta_{\mathrm{post}} = \min_{0 \le s < s_{\min}} \frac{\beta_s}{\gamma_s}. \tag{2.11}$$

We refer to (2.10a) as the *pre-smoothing property*, (2.10b) as the *post-smoothing property* and (2.10c) as the *approximation property* (see [21]).
An evident benefit in considering the inequalities (2.10a)-(2.10c) relies on to the fact that the analysis of the smoothing iterations is distinguished from the more difficult analysis of the projector operator.
Moreover, the MGM smoothing properties (2.10a) and (2.10b) are nothing more than the TGM smoothing properties (2.3a) and (2.3b) with D substituted by I, in accordance with the previous reasoning (see [1]).

3. Two-grid and Multi-grid methods for DCT III matrices

Let $\mathcal{C}_m = \{C_m \in \mathbb{R}^{m \times m} | C_m = Q_m D_m Q_m^T\}$ the unilevel DCT-III cosine matrix algebra, i.e. the algebra of matrices that are simultaneously diagonalized by the orthogonal transform

$$Q_m = \left[\sqrt{\frac{2 - \delta_{j,1}}{m}} \cos\left\{ \frac{(i-1)(j-1/2)\pi}{m} \right\} \right]_{i,j=1}^{m} \tag{3.1}$$

with $\delta_{i,j}$ denoting the Kronecker symbol.
Let f be a real-valued even trigonometric polynomial of degree k and period 2π. Then, the DCT III matrix of order m generated by f is defined as

$$C_m(f) = Q_m D_m(f) Q_m^T, \quad D_m(f) = \mathrm{diag}_{1 \le j \le m} f\left(x_j^{[m]}\right), \quad x_j^{[m]} = \frac{(j-1)\pi}{m}.$$

Clearly, $C_m(f)$ is a symmetric band matrix of bandwidth $2k+1$. In the following, we denote in short with $C_s = C_{m_s}(g_s)$ the DCT III matrix of size m_s generated by the function g_s.
An algebraic TGM/MGM method for (multilevel) DCT III matrices generated by a real-valued even trigonometric polynomial has been proposed in [7]. Here, we

briefly report the relevant results with respect to TGM convergence analysis, the aim being to prove in §4 the V-cycle optimal convergence under suitable conditions. Indeed, the projector operator P^s_{s+1} is chosen as

$$P^s_{s+1} = T^s_{s+1} C_s(p_s)$$

where $T^s_{s+1} \in \mathbb{R}^{m_{s+1}\times m_s}$, $m_{s+1} = m_s/2$, is the cutting operator defined as

$$\left[T^s_{s+1}\right]_{i,j} = \begin{cases} 1/\sqrt{2} & \text{for } \ j \in \{2i-1, 2i\}, \ i = 1, \ldots, m_{s+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

and $C_s(p_s)$ is the DCT-III cosine matrix of size m_s generated by a suitable even trigonometric polynomial p_s.
Here, the scaling by a factor $1/\sqrt{2}$ is introduced in order to normalize the matrix T^s_{s+1} with respect to the Euclidean norm. From the point of view of an algebraic multigrid this is a natural choice, while in a geometric multigrid it is more natural to consider just a scaling by $1/2$ in the projector, to obtain an average value.
The cutting operator plays a leading role in preserving both the structural and spectral properties of the projected matrix C_{s+1}: in fact, it ensures a spectral link between the space of the frequencies of size m_s and the corresponding space of frequencies of size m_{s+1}, according to the following Lemma.

Lemma 3.1. [7] *Let $Q_s \in \mathbb{R}^{m_s\times m_s}$ and $T^s_{s+1} \in \mathbb{R}^{m_{s+1}\times m_s}$ be given as in* (3.1) *and* (3.2) *respectively. Then*

$$T^s_{s+1} Q_s = Q_{s+1}[\Phi_{s+1}, \Theta_{s+1}\Pi_{s+1}], \tag{3.3}$$

where

$$\Phi_{s+1} = \mathrm{diag}_{j=1,\ldots,m_{s+1}} \left[\cos\left(\frac{1}{2}\left(\frac{x_j^{[m_s]}}{2}\right)\right)\right], \quad x_j^{[m_s]} = \frac{(j-1)\pi}{m_s}, \tag{3.4a}$$

$$\Theta_{s+1} = \mathrm{diag}_{j=1,\ldots,m_{s+1}} \left[-\cos\left(\frac{1}{2}\left(\frac{x_j^{[m_s]}}{2} + \frac{\pi}{2}\right)\right)\right], \tag{3.4b}$$

and $\Pi_{s+1} \in \mathbb{R}^{m_{s+1}\times m_{s+1}}$ is the permutation matrix

$$(1, 2, \ldots, m_{s+1}) \mapsto (1, m_{s+1}, m_{s+1} - 2, \ldots, 2).$$

As a consequence, let $A_s = C_s(f_s)$ be the DCT-III matrix generated by f_s, then

$$A_{s+1} = P^s_{s+1} A_s (P^s_{s+1})^T = C_{s+1}(f_{s+1})$$

where

$$\begin{aligned} f_{s+1}(x) \quad = \quad & \cos^2\left(\frac{x/2}{2}\right) f_s\left(\frac{x}{2}\right) p_s^2\left(\frac{x}{2}\right) \\ & + \cos^2\left(\frac{\pi - x/2}{2}\right) f_s\left(\pi - \frac{x}{2}\right) p_s^2\left(\pi - \frac{x}{2}\right), \quad x \in [0, \pi]. \end{aligned} \tag{3.5}$$

On the other side, the convergence of proposed TGM at size m_s is ensured by choosing the polynomial as follows.

Definition 3.2. Let $x^0 \in [0, \pi)$ a zero of the generating function f_s. The polynomial p_s is chosen so that

$$\lim_{x \to x^0} \frac{p_s^2(\pi - x)}{f_s(x)} < +\infty, \tag{3.6a}$$

$$p_s^2(x) + p_s^2(\pi - x) > 0. \tag{3.6b}$$

In the special case $x^0 = \pi$, the requirement (3.6a) is replaced by

$$\lim_{x \to x^0 = \pi} \frac{p_s^2(\pi - x)}{\cos^2\left(\frac{x}{2}\right) \, f_s(x)} < +\infty. \tag{3.7a}$$

If f_s has more than one zero in $[0, \pi]$, then p_s will be the product of the polynomials satisfying the condition (3.6a) (or (3.7a)) for every single zero and globally the condition (3.6b).

It is evident from the quoted definition that the polynomial p_s must have zeros of proper order in any mirror point $\hat{x}^0 = \pi - x^0$, where x^0 is a zeros of f_s. It is worth stressing that conditions (3.6a) and (3.6b) are in perfect agreement with the case of other structures such as τ, symmetric Toeplitz and circulant matrices (see e.g. [22, 25]), while the condition (3.7a) is proper of the DCT III algebra and it corresponds to a worsening of the convergence requirements.
Moreover, as just suggested in [7], in the case $x^0 = 0$ the condition (3.6a) can also be weakened as

$$\lim_{x \to x^0 = 0} \frac{\cos^2\left(\frac{\pi - x}{2}\right) \, p_s^2(\pi - x)}{f_s(x)} < +\infty. \tag{3.8a}$$

We note that if f_s is a trigonometric polynomial of degree k, then f_s can have a zero of order at most $2k$. If none of the root of f_s are at π, then by (3.6a) the degree of p_s has to be less than or equal to $\lceil k/2 \rceil$. If π is one of the roots of f_s, then the degree of p_s is less than or equal to $\lceil (k+1)/2 \rceil$.
Notice also that from (3.5), it is easy to obtain the Fourier coefficients of f_{s+1} and hence the nonzero entries of $A_{s+1} = C_{s+1}(f_{s+1})$. In addition, we can obtain the roots of f_{s+1} and their orders by knowing the roots of f_s and their orders.

Lemma 3.3. [7] *If $0 \leq x^0 \leq \pi/2$ is a zero of f_s, then by* (3.6a)*, $p_s(\pi - x^0) = 0$ and hence by* (3.5)*, $f_{s+1}(2x^0) = 0$, i.e. $y^0 = 2x^0$ is a zero of f_{s+1}. Furthermore, because $p_s(\pi - x^0) = 0$, by* (3.6b)*, $p_s(x^0) > 0$ and hence the orders of x^0 and y^0 are the same. Similarly, $\pi/2 \leq x^0 < \pi$, then $y^0 = 2(\pi - x^0)$ is a root of f_{s+1} with the same order as x^0. Finally, if $x^0 = \pi$, then $y^0 = 0$ with order equals to the order of x^0 plus two.*

In [7] the Richardson method has be considered as the most natural choice for the smoothing iteration, since the corresponding iteration matrix $V_m := I_m - \omega A_m \in \mathbb{C}^{m \times m}$ belongs to the DCT-III algebra, too. Further remarks about such a type of smoothing iterations and the tuning of the parameter ω are reported in [25, 2].

Theorem 3.4. [7] *Let $A_{m_0} = C_{m_0}(f_0)$ with f_0 being a nonnegative trigonometric polynomial and let $V_{m_0} = I_{m_0} - \omega A_{m_0}$ with $\omega = 2/\|f_0\|_\infty$ and $\omega = 1/\|f_0\|_\infty$, respectively for the pre-smoothing and the post-smoothing iteration. Then, under the quoted assumptions and definitions the inequalities* (2.3a), (2.3b), *and* (2.4) *hold true and the proposed TGM converges linearly.*

Here, it could be interesting to come back to some key steps in the proof of the quoted Theorem 3.4 in order to highlight the structure with respect to any point and its mirror point according to the considered notations.
By referring to a proof technique developed in [22], the claimed thesis is obtained by proving that the right-hand sides in the inequalities

$$\gamma \geq \frac{1}{d_s(x)}\left[\cos^2\left(\frac{\pi-x}{2}\right)\frac{p_s^2(\pi-x)}{f_s(x)}\right], \tag{3.9a}$$

$$\gamma \geq \frac{1}{d_s(x)}\left[\cos^2\left(\frac{\pi-x}{2}\right)\frac{p_s^2(\pi-x)}{f_s(x)} + \cos^2\left(\frac{x}{2}\right)\frac{p_s^2(x)}{f_s(\pi-x)}\right], \tag{3.9b}$$

$$d_s(x) = \cos^2\left(\frac{x}{2}\right)p_s^2(x) + \cos^2\left(\frac{\pi-x}{2}\right)p_s^2(\pi-x) \tag{3.9c}$$

are uniformly bounded on the whole domain so that γ is an universal constant.
It is evident that (3.9a) is implied by (3.9b). Moreover, both the two terms in (3.9b) and in $d_s(x)$ can be exchanged each other, up to the change of variable $y = \pi - x$.
Therefore, if $x^0 \neq \pi$ it is evident that Definition 3.2 ensures the required uniform boundedness since the condition $p_s^2(x) + p_s^2(\pi-x) > 0$ implies $d_s(x) > 0$.
In the case $x^0 = \pi$, the inequality (3.9b) can be rewritten as

$$\gamma \geq \frac{1}{\dfrac{p_s^2(x)}{\cos^2\left(\frac{\pi-x}{2}\right)} + \dfrac{p_s^2(\pi-x)}{\cos^2\left(\frac{x}{2}\right)}}\left[\frac{p_s^2(\pi-x)}{\cos^2\left(\frac{x}{2}\right)\, f_s(x)} + \frac{p_s^2(x)}{\cos^2\left(\frac{\pi-x}{2}\right)\, f_s(\pi-x)}\right] \tag{3.10}$$

so motivating the special case in Definition 3.2.

4. V-cycle optimal convergence

In this section we propose a suitable modification of Definition 3.2 with respect to the choice of the polynomial involved into the projector, that allows us to prove the V-cycle optimal convergence according to the verification of the inequalities (2.10a)-(2.10c) and the requirement (2.11).
It is worth stressing that the MGM smoothing properties do not require a true verification, since (2.10a) and (2.10b) are exactly the TGM smoothing properties (2.3a) and (2.3b) (with $D = I$).

Proposition 4.1. *Let $A_s = C_{m_s}(f_s)$ for any $s = 0, \ldots, s_{\min}$, with $f_s \geq 0$, and let ω_s be such that $0 < \omega_s \leq 2/\|f_s\|_\infty$. If we choose α_s and β_s such that $\alpha_s \leq$*

$\omega_s \min\left\{2, (2-\omega_s\|f_s\|_\infty)/(1-\omega_s\|f_s\|_\infty)^2\right\}$ and $\beta_s \leqslant \omega_s(2-\omega_s\|f_s\|_\infty)$ then for any $x \in \mathbb{C}^m$ the inequalities

$$\|V_{s,\mathrm{pre}}\, x\|_{A_s}^2 \;\leq\; \|x\|_{A_s}^2 - \alpha_s\, \|V_{s,\mathrm{pre}}\, x\|_{A_s}^2 \tag{4.1}$$
$$\|V_{s,\mathrm{post}}\, x\|_{A_s}^2 \;\leq\; \|x\|_{A_s}^2 - \beta_s\, \|x\|_{A_s}^2 \tag{4.2}$$

hold true.

Notice, for instance, that the best bound to β_s is given by $1/\|f_s\|_\infty$ and it is obtained by taking $\omega_s = 1/\|f_s\|_\infty$ [25, 2].

Concerning the analysis of the approximation condition (2.10c) we consider here the case of a generating function f_0 with a single zero at x^0. In such a case, the choice of the polynomial in the projector is more severe with respect to the case of TGM.

Definition 4.2. Let $x^0 \in [0, \pi)$ a zero of the generating function f_s. The polynomial p_s is chosen in such a way that

$$\lim_{x\to x^0} \frac{p_s(\pi - x)}{f_s(x)} < +\infty, \tag{4.3a}$$
$$p_s^2(x) + p_s^2(\pi - x) > 0. \tag{4.3b}$$

In the special case $x^0 = \pi$, the requirement (4.3a) is replaced by

$$\lim_{x\to x^0=\pi} \frac{p_s(\pi - x)}{\cos\left(\frac{x}{2}\right)\, f_s(x)} < +\infty. \tag{4.4a}$$

Notice also that in the special case $x^0 = 0$ the requirement (4.3a) can be weakened as

$$\lim_{x\to x^0=0} \frac{\cos\left(\frac{\pi-x}{2}\right)\, p_s(\pi - x)}{f_s(x)} < +\infty. \tag{4.5a}$$

Proposition 4.3. *Let $A_s = C_{m_s}(f_s)$ for any $s = 0, \ldots, s_{\min}$, with $f_s \geq 0$. Let $P_{s+1}^s = T_{s+1}^s C_s(p_s)$, where $p_s(x)$ is fulfilling* (4.3a) (*or* (4.4a)) *and* (4.3b). *Then, for any $s = 0, \ldots, s_{\min} - 1$, there exists $\gamma_s > 0$ independent of m_s such that*

$$\|CGC_s x\|_{A_s}^2 \leq \gamma_s\, \|x\|_{A_s^2}^2 \quad \textit{for any } x \in \mathbb{C}^{m_s}, \tag{4.6}$$

where CGC_s is defined as in (2.2).

Proof. Since

$$CGC_s = I_s - (P_{s+1}^s)^T (P_{s+1}^s A_s (P_{s+1}^s)^T)^{-1} P_{s+1}^s A_s$$

is an unitary projector, it holds that $CGC_s^T\, A_s\, CGC_s = A_s\, CGC_s$. Therefore, the target inequality (4.6) can be simplified and symmetrized, giving rise to the matrix inequality

$$\widetilde{CGC}_s = I_s - A_s^{1/2} (P_{s+1}^s)^T (P_{s+1}^s A_s (P_{s+1}^s)^T)^{-1} P_{s+1}^s A_s^{1/2} \leq \gamma_s A_s. \tag{4.7}$$

Hence, by invoking Lemma 3.1, $Q_s^T \widetilde{CGC_s} Q_s$ can be permuted into a 2×2 block diagonal matrix whose jth block, $j = 1, \ldots, m_{s+1}$, is given by the rank-1 matrix (see [8] for the analogous τ case)

$$I_2 - \frac{1}{c_j^2 + s_j^2} \begin{bmatrix} c_j^2 & c_j s_j \\ c_j s_j & s_j^2 \end{bmatrix},$$

where

$$c_j = \cos\left(\frac{x_j^{[m_s]}}{2}\right) p^2 f(x_j^{[m_s]}) \quad s_j = -\cos\left(\frac{\pi - x_j^{[m_s]}}{2}\right) p^2 f(\pi - x_j^{[m_s]}).$$

As in the proof of the TGM convergence, due to the continuity of f_s and p_s, (4.7) is proven if the right-hand sides in the inequalities

$$\gamma_s \geq \frac{1}{\widetilde{d}_s(x)} \left[\cos^2\left(\frac{\pi - x}{2}\right) \frac{p_s^2 f_s(\pi - x)}{f_s(x)}\right] \tag{4.8a}$$

$$\gamma_s \geq \frac{1}{\widetilde{d}_s(x)} \left[\cos^2\left(\frac{\pi - x}{2}\right) \frac{p_s^2 f_s(\pi - x)}{f_s(x)} + \cos^2\left(\frac{x}{2}\right) \frac{p_s^2 f_s(x)}{f_s(\pi - x)}\right] \tag{4.8b}$$

$$\widetilde{d}_s(x) = \cos^2\left(\frac{x}{2}\right) p_s^2 f_s(x) + \cos^2\left(\frac{\pi - x}{2}\right) p_s^2 f(\pi - x) \tag{4.8c}$$

are uniformly bounded on the whole domain so that γ_s are universal constants. Once again, it is evident that (4.8a) is implied by (4.8b). Moreover, both the terms in (4.8b) and in $\widetilde{d}_s(x)$ can be exchanged each other, up to the change of variable $y = \pi - x$.
Therefore, if $x^0 \neq \pi$, (4.8b) can be rewritten as

$$\gamma_s \geq \frac{1}{\hat{d}_s(x)} \left[\cos^2\left(\frac{\pi - x}{2}\right) \frac{p_s^2(\pi - x)}{f_s^2(x)} + \cos^2\left(\frac{x}{2}\right) \frac{p_s^2(x)}{f_s^2(\pi - x)}\right] \tag{4.9}$$

where

$$\hat{d}_s(x) = \cos^2\left(\frac{x}{2}\right) \frac{p_s^2(x)}{f_s(\pi - x)} + \cos^2\left(\frac{\pi - x}{2}\right) \frac{p_s^2(\pi - x)}{f_s(x)},$$

so that Definition 4.2 ensures the required uniform boundedness.
In the case $x^0 = \pi$, the inequality (4.8b) can be rewritten as

$$\begin{aligned} \gamma_s \quad \geq \quad & \frac{1}{\dfrac{p_s^2(x)}{\cos^2\left(\frac{\pi - x}{2}\right) f_s(\pi - x)} + \dfrac{p_s^2(\pi - x)}{\cos^2\left(\frac{x}{2}\right) f_s(x)}} \left[\frac{p_s^2(\pi - x)}{\cos^2\left(\frac{x}{2}\right) f_s^2(x)}\right. \\ & \left. + \frac{p_s^2(x)}{\cos^2\left(\frac{\pi - x}{2}\right) f_s^2(\pi - x)}\right] \end{aligned} \tag{4.10}$$

so motivating the special case in Definition 4.2. $\square$

Remark 4.4. Notice that in the case of pre-smoothing iterations and under the assumption $V_{s,\text{pre}}$ nonsingular, the approximation condition

$$\|CGC_s V_{s,\text{pre}}^{\nu_{\text{pre}}} x\|_{A_s}^2 \leq \gamma_s \|V_{s,\text{pre}}^{\nu_{\text{pre}}} x\|_{A_s^2}^2 \text{ for any } x \in \mathbb{C}^{m_s}, \tag{4.11}$$

is equivalent to the condition, in matrix form, $\widetilde{CGC_s} \leq \gamma_s A_s$ obtained in Proposition 4.3.

In Propositions 4.1 and 4.3 we have obtained that for every s (independent of $m = m_0$) the constants α_s, β_s, and γ_s are absolute values not depending on $m = m_0$, but only depending on the functions f_s and p_s. Nevertheless, in order to prove the MGM optimal convergence according to Theorem 2.2, we should verify at least one between the following inf–min conditions [1]:

$$\delta_{\text{pre}} = \inf_{m_0} \min_{0 \leq s \leq \log_2(m_0)} \frac{\alpha_s}{\gamma_s} > 0, \quad \delta_{\text{post}} = \inf_{m_0} \min_{0 \leq s \leq \log_2(m_0)} \frac{\beta_s}{\gamma_s} > 0. \tag{4.12}$$

First, we consider the inf-min requirement (4.12) by analyzing the case of a generating function $\tilde{f}_0$ with a single zero at $x^0 = 0$.
It is worth stressing that in such a case the DCT-III matrix $\tilde{A}_{m_0} = C_{m_0}(\tilde{f}_0)$ is singular since 0 belongs to the set of grid points $x_j^{[m_0]} = (j-1)\pi/m_0$, $j = 1, \ldots, m_0$. Thus, the matrix $\tilde{A}_{m_0}$ is replaced by

$$A_{m_0} = C_{m_0}(f_0) = C_{m_0}(\tilde{f}_0) + \tilde{f}_0\left(x_2^{[m_0]}\right) \cdot \frac{ee^T}{m_0}$$

with $e = [1, \ldots, 1]^T \in \mathbb{R}^{m_0}$ and where the rank-1 additional term is known as Strang correction [29]. Equivalently, $\tilde{f}_0 \geq 0$ is replaced by the generating function

$$f_0 = \tilde{f}_0 + \tilde{f}_0\left(x_2^{[m_0]}\right) \chi_{w_1^{[m_s]} + 2\pi\mathbb{Z}} > 0, \tag{4.13}$$

where χ_X is the characteristic function of the set X and $w_1^{[m_0]} = x^0 = 0$.
In Lemma 4.5 is reported the law to which the generating functions are subjected at the coarser levels. With respect to this target, it is useful to consider the following factorization result: let $f \geq 0$ be a trigonometric polynomial with a single zero at x^0 of order $2q$. Then, there exists a positive trigonometric polynomial ψ such that

$$f(x) = [1 - \cos(x - x_0)]^q \ \psi(x). \tag{4.14}$$

Notice also that, according to Lemma 3.3, the location of the zero is never shifted at the subsequent levels.

Lemma 4.5. *Let $f_0(x) = \tilde{f}_0(x) + c_0\chi_{2\pi\mathbb{Z}}(x)$, with $\tilde{f}_0(x) = [1 - \cos(x)]^q \psi_0(x)$, q being a positive integer and ψ_0 being a positive trigonometric polynomial and with $c_0 = \tilde{f}_0\left(x_2^{[m_0]}\right)$. Let $p_s(x) = [1 + \cos(x)]^q$ for any $s = 0, \ldots, s_{\min} - 1$. Then, under the same assumptions of Lemma* 3.1*, each generating function f_s is given by*

$$f_s(x) = \tilde{f}_s(x) + c_s\chi_{2\pi\mathbb{Z}}(x), \quad \tilde{f}_s(x) = [1 - \cos(x)]^q \psi_s(x).$$

The sequences $\{\psi_s\}$ *and* $\{c_s\}$ *are defined as*

$$\psi_{s+1} = \Phi_{q,p_s}(\psi_s), \qquad c_{s+1} = c_s p_s^2(0), \qquad s = 0, \ldots, s_{\min} - 1,$$

where $\Phi_{q,p}$ *is an operator such that*

$$\left[\Phi_{q,p}(\psi)\right](x) = \frac{1}{2^{q+1}} \left[(\varphi\, p\, \psi) \left(\frac{x}{2}\right) + (\varphi\, p\, \psi) \left(\pi - \frac{x}{2}\right) \right], \tag{4.15}$$

with $\varphi(x) = 1 + \cos(x)$. *Moreover, each* $\tilde{f}_s$ *is a trigonometric polynomial that vanishes only at* $2\pi\mathbb{Z}$ *with the same order* $2q$ *as* $\tilde{f}_0$.

Proof. The claim is a direct consequence of Lemma 3.1. Moreover, since the function ψ_0 is positive by assumption, the same holds true for each function ψ_s. $\square$

Hereafter, we make use of the following notations: for a given function f, we will write $M_f = \sup_x |f|$, $m_f = \inf_x |f|$ and $\mu_\infty(f) = M_f/m_f$.
Now, if $x \in (0, 2\pi)$ we can give an upper bound for the left-hand side $R(x)$ in (4.9), since it holds that

$$\begin{aligned}
R(x) &= \frac{\dfrac{\cos^2\left(\frac{x}{2}\right) p_s^2(x)}{f_s^2(\pi - x)} + \dfrac{\cos^2\left(\frac{\pi - x}{2}\right) p_s^2(\pi - x)}{f_s^2(x)}}{\dfrac{\cos^2\left(\frac{x}{2}\right) p_s^2(x)}{f_s(\pi - x)} + \dfrac{\cos^2\left(\frac{\pi - x}{2}\right) p_s^2(\pi - x)}{f_s(x)}} \\
&= \frac{\dfrac{\cos^2\left(\frac{x}{2}\right)}{\psi_s^2(\pi - x)} + \dfrac{\cos^2\left(\frac{\pi - x}{2}\right)}{\psi_s^2(x)}}{\dfrac{\cos^2\left(\frac{\pi - x}{2}\right) p_s(x)}{\psi_s(\pi - x)} + \dfrac{\cos^2\left(\frac{\pi - x}{2}\right) p_s(\pi - x)}{\psi_s(x)}} \\
&\le \frac{M_{\psi_s}}{m_{\psi_s}^2} \frac{1}{\cos^2\left(\frac{x}{2}\right) p_s(x) + \cos^2\left(\frac{\pi - x}{2}\right) p_s(\pi - x)} \\
&\le \frac{M_{\psi_s}}{m_{\psi_s}^2},
\end{aligned}$$

we can consider $\gamma_s = M_{\psi_s}/m_{\psi_s}^2$. In the case $x = 0$, since $p_s(0) = 0$, it holds $R(0) = 1/f_s(\pi)$, so that we have also to require $1/f_s(\pi) \le \gamma_s$. However, since $1/f_s(\pi) \le M_{\psi_s}/m_{\psi_s}^2$, we take $\gamma_s^* = M_{\psi_s}/m_{\psi_s}^2$ as the best value.
In (2.9), by choosing $\omega_s^* = \|f_s\|_\infty^{-1}$, we simply find $\beta_s^* = \|f_s\|_\infty^{-1} \ge 1/(2^q M_{\psi_s})$ and as a consequence, we obtain

$$\frac{\beta_s^*}{\gamma_s^*} \ge \frac{1}{2^q M_{\psi_s}} \cdot \frac{m_{\psi_s}^2}{M_{\psi_s}} = \frac{1}{2^q \mu_\infty^2(\psi_s)}. \tag{4.16}$$

A similar relation can be found in the case of a pre-smoothing iteration. Nevertheless, since it is enough to prove one between the inf-min conditions, we focus our attention on condition (4.16). So, to enforce the inf–min condition (4.12), it is enough to prove the existence of an absolute constant L such that $\mu_\infty(\psi_s) \leqslant L < +\infty$ uniformly in order to deduce that $\|MGM_0\|_{A_0} \leqslant \sqrt{1 - (2^q L^2)^{-1}} < 1$.

Proposition 4.6. *Under the same assumptions of Lemma* 4.5*, let us define* $\psi_s = [\Phi_{p_s,q}]^s(\psi)$ *for every* $s \in \mathbb{N}$*, where* $\Phi_{p,q}$ *is the linear operator defined as in* (4.15)*. Then, there exists a positive polynomial* ψ_∞ *of degree* q *such that* ψ_s *uniformly converges to* ψ_∞*, and moreover there exists a positive real number* L *such that* $\mu_\infty(\psi_s) \leqslant L$ *for any* $s \in \mathbb{N}$*.*

Proof. Due to the periodicity and to the cosine expansions of all the involved functions, the operator $\Phi_{q,p}$ in (4.15) can be rewritten as

$$[\Phi_{q,p}(\psi)]\,(x) = \frac{1}{2^{q+1}}\left[(\varphi\, p\, \psi)\left(\frac{x}{2}\right) + (\varphi\, p\, \psi)\left(\pi + \frac{x}{2}\right)\right]. \tag{4.17}$$

The representation of $\Phi_{q,p}$ in the Fourier basis (see Proposition 4.8 in [1]) leads to an operator from $\mathbb{R}^{m(q)}$ to $\mathbb{R}^{m(q)}$, $m(q)$ proper constant depending only on q, which is identical to the irreducible nonnegative matrix $\bar{\Phi}_q$ in equation (4.14) of [1], with $q+1$ in place of q.
As a consequence, the claimed thesis follows by referring to the Perron–Frobenius theorem [15, 30] according to the very same proof technique considered in [1]. □

Lastly, by taking into account all the previous results, we can claim the optimality of the proposed MGM.

Theorem 4.7. *Let* $\tilde{f}_0$ *be a even nonnegative trigonometric polynomial vanishing at* 0 *with order* $2q$*. Let* $m_0 = m > m_1 > \ldots > m_s > \ldots > m_{s_{\min}}$*,* $m_{s+1} = m_s/2$*. For any* $s = 0, \ldots, m_{s_{\min}}-1$*, let* P^s_{s+1} *be as in Proposition* 4.3 *with* $p_s(x) = [1+\cos(x)]^q$*, and let* $V_{s,\mathrm{post}} = I_{m_s} - A_{m_s}/\|f_s\|_\infty$*. If we set* $A_{m_0} = C_{m_0}(\tilde{f}_0 + c_0\chi_{2\pi\mathbb{Z}})$ *with* $c_0 = \tilde{f}_0(w_2^{[m_0]})$ *and we consider* $b \in \mathbb{C}^{m_0}$*, then the MGM (standard V-cycle) converges to the solution of* $A_{m_0}x = b$ *and is optimal* (*in the sense of Definition* 1.1)*.*

Proof. Under the quoted assumptions it holds that $\tilde{f}_0(x) = [1-\cos(x)]^q\,\psi_0(x)$ for some positive polynomial $\psi_0(x)$. Therefore, it is enough to observe that the optimal convergence of MGM as stated in Theorem 2.2 is implied by the inf–min condition (4.12). Thanks to (4.16), the latter is guaranteed if the quantities $\mu_\infty(\psi_s)$ are uniformly bounded and this holds true according to Proposition 4.6. □

Now, we consider the case of a generating function f_0 with a unique zero at $x^0 = \pi$, this being particularly important in applications since the discretization of certain integral equations leads to matrices belonging to this class. For instance, the signal restoration leads to the case of $f_0(\pi) = 0$, while for the super-resolution problem and image restoration $f_0(\pi, \pi) = 0$ is found [5].
By virtue of Lemma 3.3 we simply have that the generating function f_1 related to the first projected matrix uniquely vanishes at 0, i.e. at the first level the MGM projects a discretized integral problem, into another which is spectrally and structurally equivalent to a discretized differential problem.
With respect to the optimal convergence, we have that Theorem 2.2 holds true with $\delta = \min\{\delta_0, \bar{\delta}\}$ since δ results to be a constant and independent of m_0.

More precisely, δ_0 is directly related to the finest level and $\bar{\delta}$ is given by the inf-min condition of the differential problem obtained at the coarser levels. The latter constant value has been previously shown, while the former can be proven as follows: we are dealing with $f_0(x) = (1+\cos(x))^q \psi_0(x)$ and according to Definition 4.2 we choose $\tilde{p}_0(x) = p_0(x) + d_0\chi_{2\pi\mathbb{Z}}$ with $p_0(x) = (1+\cos(x))^{q+1}$ and $d_0 = p_0(w_2^{[m_0]})$.
Therefore, an upper bound for the left-hand side $\tilde{R}(x)$ in (4.10) is obtained as

$$\tilde{R}(x) \leq \frac{M_{\psi_0}}{m_{\psi_0}^2},$$

i.e. we can consider $\gamma_0 = M_{\psi_0}/m_{\psi_0}^2$ and so that a value δ_0 independent of m_0 is found.

5. Numerical experiments

Hereafter, we give numerical evidence of the convergence properties claimed in the previous sections, both in the case of proposed TGM and MGM (standard V-cycle), for two types of DCT-III systems with generating functions having zero at 0 (differential like problems) and at π (integral like problems).
The projectors P_{s+1}^s are chosen as described in §3 in §4 and the Richardson smoothing iterations are used twice in each iteration with $\omega = 2/\|f\|$ and $\omega = 1/\|f\|$ respectively. The iterative procedure is performed until the Euclidean norm of the relative residual at dimension m_0 is greater than 10^{-7}. Moreover, in the V-cycle, the exact solution of the system is found by a direct solver when the coarse grid dimension equals to 16 (16^2 in the additional two-level tests).

5.1. Case $\mathbf{x^0 = 0}$ (differential like problems)

First, we consider the case $A_m = C_m(f_0)$ with $f_0(x) = [2 - 2\cos(x)]^q$, i.e. with a unique zero at $x^0 = 0$ of order $2q$.
As previously outlined, the matrix $C_m(f_0)$ is singular, so that the solution of the rank-1 corrected system is considered, whose matrix is given by $C_m(f_0) + (f_0(\pi/m)/m)ee^T$, with $e = [1, \ldots, 1]^T$. Since the position of the zero $x^0 = 0$ at the coarser levels is never shifted, then the function $p_s(x) = [2 - 2\cos(\pi - x)]^r$ in the projectors is the same at all the subsequent levels s.
To test TGM/MGM linear convergence with rate independent of the size m_0 we tried for different r: according to (3.6a), we must choose r at least equal to 1 if $q = 1$ and at least equal to 2 if $q = 2, 3$, while according to (4.3a) we must always choose r equal to q. The results are reported in Table 1.
By using tensor arguments, the previous results plainly extend to the multilevel case. In Table 2 we consider the case of generating function $f_0(x, y) = f_0(x) + f_0(y)$, that arises in the uniform finite difference discretization of elliptic constant coefficient differential equations on a square with Neumann boundary conditions, see e.g [24].

TABLE 1. Twogrid/Multigrid - $1D$ Case: $f_0(x) = [2 - 2\cos(x)]^q$ and $p(x) = [2 - 2\cos(\pi - x)]^r$.

	TGM					MGM					
	$q=1$	$q=2$		$q=3$			$q=1$	$q=2$		$q=3$	
m_0	$r=1$	$r=1$	$r=2$	$r=2$	$r=3$	m_0	$r=1$	$r=1$	$r=2$	$r=2$	$r=3$
16	7	15	13	34	32	16	1	1	1	1	1
32	7	16	15	35	34	32	7	16	15	34	32
64	7	16	16	35	35	64	7	17	16	35	34
128	7	16	16	35	35	128	7	18	16	35	35
256	7	16	16	35	35	256	7	18	16	35	35
512	7	16	16	35	35	512	7	18	16	35	35

TABLE 2. Twogrid/Multigrid - $2D$ Case: $f_0(x, y) = [2 - 2\cos(x)]^q + [2 - 2\cos(y)]^q$ and $p(x, y) = [2 - 2\cos(\pi - x)]^r + [2 - 2\cos(\pi - y)]^r$.

	TGM					MGM					
	$q=1$	$q=2$		$q=3$			$q=1$	$q=2$		$q=3$	
m_0	$r=1$	$r=1$	$r=2$	$r=2$	$r=3$	m_0	$r=1$	$r=1$	$r=2$	$r=2$	$r=3$
16^2	15	34	30	-	-	16^2	1	1	1	1	1
32^2	16	36	35	71	67	32^2	16	36	35	71	67
64^2	16	36	36	74	73	64^2	16	36	36	74	73
128^2	16	36	36	74	73	128^2	16	36	36	74	73
256^2	16	36	36	74	73	256^2	16	37	36	74	73
512^2	16	36	36	74	73	512^2	16	37	36	74	73

5.2. Case $\mathbf{x^0} = \pi$ (integral like problems)

DCT III matrices $A_{m_0} = C_{m_0}(f_0)$ whose generating function shows a unique zero at $x^0 = \pi$ are encountered in solving integral equations, for instance in image restoration problems with Neumann (reflecting) boundary conditions [18].
According to Lemma 3.3, if $x^0 = \pi$, then the generating function f_1 of the coarser matrix $A_{m_1} = C_{m_1}(f_1)$, $m_1 = m_0/2$ has a unique zero at 0, whose order equals the order of $x^0 = \pi$ with respect to f_0 plus two.
It is worth stressing that in such a case the projector at the first level is singular so that its rank-1 Strang correction is considered. This choice gives rise in a natural way to the rank-1 correction considered in §5.1. Moreover, starting from the second coarser level, the new location of the zero is never shifted from 0.
In Table 3 are reported the numerical results both in the unilevel and two-level case.

6. Computational costs and conclusions

Some remarks about the computational costs are required in order to highlight the optimality of the proposed procedure.

TABLE 3. Twogrid/Multigrid - $1D$ Case: $f_0(x) = 2+2\cos(x)$ and $p_0(x) = 2 - 2\cos(\pi - x)$ and $2D$ Case: $f_0(x, y) = 4 + 2\cos(x) + 2\cos(y)$ and $p_0(x, y) = 4 - 2\cos(\pi - x) - 2\cos(\pi - y)$.

$1D$	TGM	MGM
16	15	1
32	14	14
64	12	13
128	11	13
256	10	12
512	8	10

$2D$	TGM	MGM
16^2	7	1
32^2	7	7
64^2	7	7
128^2	7	6
256^2	7	6
512^2	7	6

Since the matrix $C_{m_s}(p)$ appearing in the definition of P^s_{s+1} is banded, the cost of a matrix vector product involving P^s_{s+1} is $O(m_s)$. Therefore, the first condition in Definition 1.1 is satisfied. In addition, notice that the matrices at every level (except for the coarsest) are never formed since we need only to store the $O(1)$ nonzero Fourier coefficients of the related generating function at each level for matrix-vector multiplications. Thus, the memory requirements are also very low. With respect to the second condition in Definition 1.1 we stress that the representation of $A_{m_{s+1}} = C_{m_{s+1}}(f_{s+1})$ can be obtained formally in $O(1)$ operations by virtue of (3.5). In addition, the roots of f_{s+1} and their orders are obtained according to Lemma 3.3 by knowing the roots of f_s and their orders. Finally, each iteration of TGM costs $O(m_0)$ operations as A_{m_0} is banded. In conclusion, each iteration of the proposed TGM requires $O(m_0)$ operations.
With regard to MGM, optimality is reached since we have proven that there exists δ is independent from both m and $s_{\min}$ so that the number of required iterations results uniformly bounded by a constant irrespective of the problem size. In addition, since each iteration has a computational cost proportional to matrix-vector product, Definition 1.1 states that such a kind of MGM is *optimal*.

As a conclusion, we observe that the reported numerical tests in §5 show that the requirements on the order of zero in the projector could be weakened. Future works will deals with this topic and with the extension of the convergence analysis in the case of a general location of the zeros of the generating function.

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