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The small world effect on the coalescing time of random walks

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Abstract

A small world is obtained from the d-dimensional torus of size $2L$ adding randomly chosen connections between sites, in a way such that each site has exactly one random neighbour in addition to its deterministic neighbours. We study the asymptotic behaviour of the meeting time T_L of two random walks moving on this small world and compare it with the result on the torus. On the torus, in order to have convergence, we have to rescale T_L by a factor C_1L^2 if $d=1$, by $C_2L^2\log L$ if $d=2$ and C_dL^d if $d\geq 3$. We prove that on the small world the rescaling factor is C'_dL^d and identify the constant C'_d, proving that the walks always meet faster on the small world than on the torus if $d\leq 2$, while if $d\geq 3$ this depends on the probability of moving along the random connection. As an application, we obtain results on the hitting time to the origin of a single walk and on the convergence of coalescing random walk systems on the small world.

Key words: small world, random walk, coalescing random walk.

AMS 2010 subject classification: Primary 60K37; Secondary 60J26, 60J10.

1 Introduction

Graphs provide a mathematical model in many scientific areas, from physics (magnetization properties of metals, evolution of gases) to biology (neural networks, disease spreading) and sociology (social networks, opinion spreading). Individuals (atoms, molecules, neurons, animals) are identified with vertices and an edge drawn between two vertices identifies a relation as proximity or existence of some sort of contact. When a part or the totality of the edges are subject to some randomness, it is natural to deal with random graphs (see [6] for a survey). One can construct a random graph starting from a deterministic graph either by adding random connections, or by removing some connections randomly, as in percolation. A particular class of random graphs of the first type are *small world graphs*, constructed starting from a d-dimensional (discrete) torus, whose edges are called *short range connections*, with some random connections, called *long range connections*.

Bollobas and Chung [5] first noted that adding a random matching in a cycle (i.e. $d=1$), the average distance between sites is considerably smaller than on the deterministic graph. Watts and Strogatz [17] introduced, as a model for biological applications, the random graph obtained in $d=1$ with each site connected to the ones at Euclidean distance smaller than m and long range connections constructed by taking the deterministic ones and by moving with probability p one of the end sites to a new one chosen at random. Another possible construction was introduced by Newmann and Watts [14]: they took the same deterministic short range connections of Watts and Strogatz, but they added a density p of long range connections between randomly chosen sites. Average distance

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between sites and clustering coefficient of small world graphs have been well investigated ([1],[2], [17]). See [11] for a historical introduction of small world graphs and main results.

Recently some authors have been focusing on processes taking place on random graphs. Durrett and Jung [12] have studied the contact process on the small world. Their version of the small world is a generalization of the Bollobas-Chung model: they take the d-dimensional torus $\Lambda^d(L) = \mathbb{Z}^d \mod 2L$ with short range connections between each pair of vertices at Euclidean distance smaller than m. The long range connections are drawn choosing at random a partition of the $(2L)^d$ vertices in pairs and connecting each pair of the partition (see Section 2.1 for more details about the construction). Note that all sites have exactly one long range neighbour and the degree of the graph is constant: being this small world graph homogeneous makes it easier to study processes on it. The main advantage of such a costruction is that we can associate to the random graph a non-random translation invariant graph $\mathcal{B}$, called *big world*: see [12] where the big world was first introduced and Section 2.2 for more details on this deterministic graph and on its relationship with the small world.

One expects that if the distance between sites plays an important role (as for random walks, coalescing random walk or the contact process), a process taking place on a small world will behave differently from the same one on the torus. We consider random walks on the small world and, under some assumptions on the starting sites, we study the asymptotic behaviour of three sequences of random times: the time W_L after which a single random walk first hits the origin, the time T_L after which two random walks first meet and the coalescing time τ_L of a coalescing random walk starting from a fixed number of particles. Recall that coalescing random walk on a graph is a Markov process in which n particles perform independent random walks subject to the rule that when one particle jumps onto an already occupied site, the two particles coalesce to one. The time when we have only one particle left is called *coalescing time*.

It is natural to compare our results with the corresponding results on the torus: for the simple symmetric continuous time random walk on the d-dimensional torus, Cox [7, Theorem 4] proved (under some assumptions on the initial position) that for $d = 2$ $W_L/C_2(2L)^2 \log(2L)$, with $C_2 = 2/\pi$, and for $d \geq 3$, $W_L/C_d(2L)^d)$, with C_d equal to the expected number of visits to the origin of a discrete time simple symmetric random walk, converge to an exponential of mean 1. One can also get the same result for the random walk starting from the stationary distribution (this was proved in [13, Theorem 6.1] in discrete time) as a corollary.

Cox and Durrett (see [8, Theorem 2]) proved a result in the 2-dimensional case under more general conditions on the starting point and on the transition matrix for a discrete time random walk. The case $d = 1$ is slightly different: Flatto, Odlyzko and Wales [13, Theorem 6.1] proved that for the discrete time random walk starting from the uniform distribution W_L/L^2 converges to a certain law. It is possible to show that these results also hold in continuous time.

Note that, by the symmetry of the walks on the torus, it is easy to show that the meeting time T_L of two independent random walks X_t and Y_t on the torus, conditioned to $X_0 = x$ and $Y_0 = y$, coincides with the law of $2W_L$ conditioned to $W_0 = x - y$. Therefore Theorems [7, Theorem 4], [8, Theorem 2]) and [13, Theorem 6.1] give also the asymptotic behaviors of the meeting time of two particles.

To explain the results on the small world more clearly, let S^L be one of the possible realizations of the small world graph $\mathcal{S}^L$ and let $\mathcal{S}^L(\widetilde{\Omega})$ be the set of all possible realizations of $\mathcal{S}^L$. We introduce a transition matrix P_{S^L} depending on S^L: when the random walk moves, with probability β it takes the long range connection and with probability $1 - \beta$ it

chooses a short range one uniformly. We denote by $\mathbb{P}^{\mu,\nu}_{S^L}$ the law of the two independent continuous time random walks X_t and Y_t ($\widetilde{X}_n$ and $\widetilde{Y}_n$ in discrete time) ruled by P_{S^L} on S^L starting from the probability distributions μ and ν on $\Lambda^d(L)$ (see Section 2.1). Since the walkers are on a random graph, first of all we have to fix a possible graph and then move the process on it. We can look for results for each graph S in a set of large probability ("*quenched*" point of view) or average results on all realizations S of the graph ("*annealed*" point of view). We study random walks both in discrete and continuous time. In Section 2.1 we give the formal definitions.
Durrett [11, Chapter 6] studied the coalescing random walk on a one dimensional version of BC small world. He proved, for a large class of random graphs with N vertices, that for each sequence $\{S^N\}_N$ in a set of large probability the rescaled meeting time T of two particles starting from the stationary distribution converges to the exponential distribution: the rescaling factor is given by the number of vertices N times a constant $C > 0$ (depending on the local structure of the graph). In particular such a result holds for the 1 dimensional BC small world with $N = 2L$.
We use the Laplace transform technique to prove more accurate results under more general initial conditions on the meeting time T_L, which will be useful for getting results on coalescing random walk. The exact estimation of the rescaling constants for any dimension allows the comparison with the results on the d-dimensional torus for $d \geq 3$, which would not be possible otherwise.
A fundamental tool consists in constructing a random map from the deterministic big world graph onto the small world random graph. As we explain in Section 2.2, through this map we can associate to each site $x \in \Lambda^d(L)$ a particular site $+(x)$ in the big world. Moreover we associate to the random walk on the small world, a random walk on the big world, and we denote its law by $\mathbb{P}_{\mathcal{B}}$ ($\widetilde{\mathbb{P}}_{\mathcal{B}}$ in discrete time).

We denote by

$$\widetilde{G}^{ev}_{\mathcal{B}}(+(x)) := \sum_{n=0}^{\infty} \widetilde{\mathbb{P}}^{+(x)}_{\mathcal{B}}(\widetilde{X}_{2n} = 0); \qquad G^{ev}_{\mathcal{B}}(+(x)) := \int_0^{\infty} \mathbb{P}^{+(x)}_{\mathcal{B}}(X_{2t} = 0)dt, \tag{1.1}$$

Note that such constants, which are not necessarily equal, depend both on the geometry of the graph and on the transition probability of the random walk on it: in particular it depends on the probability β to take a shortcut in the small world graph.
The following result, stated in continuous time, holds also in discrete case with the corresponding constant and involves the meeting time of two particles starting from 0 and x_L. We give the limit law of $T_L/(2L)^d$ as L goes to infinity both for starting points $x_L \in \Lambda^d(L)$ such that $|x_L|$ goes to infinity and for $x_L = x$ which does not depend on L.

Theorem 1.1 *Denote by* 0 *the origin of the big world* $+(0)$ *and by* $d_S(0, x)$ *the length of the shortest path connecting* x *to* 0 *in the small world* S.

1. *Let* $x_L \in \Lambda(L)$ *for all* L *such that* $x_L = x$ *for all* L *sufficiently large. Then if* $x \neq 0$, *uniformly in* $t \geq 0$

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \left(1 - \frac{G^{ev}_{\mathcal{B}}(+(x))}{G^{ev}_{\mathcal{B}}(0)}\right) \exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right). \tag{1.2}$$

If $x = 0$ *then uniformly in* $t \geq 0$

$$\mathbb{P}^{0,0}\left(\frac{T_L}{(2L)^d} > t\right) \overset{L\to\infty}{\to} \frac{1}{G^{ev}_{\mathcal{B}}(0)} \exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right). \tag{1.3}$$

2. *If $\alpha_L \geq (\log\log L)^2$, then uniformly in $t \geq 0$ and x_L such that $|x_L| \geq \alpha_L$,*

$$\mathbb{P}^{x_L,0}\left(\frac{T_L}{(2L)^d} > t\right) \to \exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right). \tag{1.4}$$

3. *Let $x_L \in \Lambda(L)$ for all L such that $x_L = x \neq 0$ for all L sufficiently large. For all $\varepsilon > 0$*

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}^{x_L,0}_S\left(\frac{T_L}{(2L)^d} > t\right) - \left(1 - \frac{G_{\mathcal{B}}(+(x))}{G^{ev}_{\mathcal{B}}(0)}\right)\exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right)\right| < \varepsilon, \forall t \geq 0\right) \overset{L\to\infty}{\to} 1. \tag{1.5}$$

If $x = 0$ then

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \left|\mathbb{P}^{0,0}_S\left(\frac{T_L}{(2L)^d} > t\right) - \frac{1}{G^{ev}_{\mathcal{B}}(0)}\exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right)\right| < \varepsilon, \forall t \geq 0\right) \overset{L\to\infty}{\to} 1. \tag{1.6}$$

4. *Choose $\alpha_L \geq (\log\log L)^2$. For all $\varepsilon > 0$*

$$\mathbf{P}\left(S \in \mathcal{S}^L(\widetilde{\Omega}) : \sup_{\{x_L : d_S(0,x_L) \geq \alpha_L\}}\left|\mathbb{P}^{x_L,0}_S\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right)\right| < \varepsilon, \forall t \geq 0\right) \overset{L\to\infty}{\to} 1. \tag{1.7}$$

Using similar techniques we get results about the return time of a single particle to the origin (see Section 4.1, Theorems 4.2). As a corollary one can get the law of the meeting time of two random walks and the law of the hitting time to the origin of a single walker starting from the uniform distribution.
Since we do not work with a reversible Markov chain on a translation invariant graph we cannot easily get results on meeting time of two random walks from the hitting time of a single one: the key in the proofs is that in most sites the local structure can be mapped through a bijection into the big world.

Theorem 1.1 states that in order to have convergence of the rescaled meeting time in the small world, we need to rescale with $(2L)^d$. Comparing with the results on the torus, if $d \leq 2$ the small world effect is clear (convergence has a faster rate); if $d \geq 3$ the small world effect is not so evident, since the convergence has the same rate and we need to know more about the rescaling constant. As we can expect, the effect of shortcuts is larger for lower dimension and less evident when d is large.
The table gives a comparison between the meeting time of two simple symmetric continuous time random walks on the torus $\Lambda(L)$ and on the BC small world when the initial distance converges to infinity: $G^{ev}_{\mathbb{Z}^d}(0)$, if $d \geq 3$, is the constant corresponding to (1.1) on $\mathbb{Z}^d$, which is a half of the expected number of returns to the origin of a discrete time random walk on $\mathbb{Z}^d$.

As already observed, if $d = 1$ then $T_L/(C(\Lambda(L))R(\Lambda(L)))$ and $T_L/C(\mathcal{S}^L)R(\mathcal{S}^L))$ converge to different laws: a comparison is possible since the rescaling factors are different.

Starting points	d	Rescaling Factor R $R(\Lambda(L))$	$R(\mathcal{S}^L)$	Constant C $C(\Lambda(L))$	$C(\mathcal{S}^L)$
Uniform distribution	1	L^2	L	$1/12$	$G^{ev}_{\mathcal{B}}(0)$
$(\log^+ \lvert x_L\rvert)/\log L \to \delta$ or					
$\lvert x_L\rvert \geq \alpha_L$, $\alpha_L \frac{\sqrt{\log L}}{L} \to \infty$	2	$L^2 \log L$	L^2	$1/\pi$	$G^{ev}_{\mathcal{B}}(0)$
$\lvert x_L\rvert \geq (\log\log L)^2$	≥ 3	L^d	L^d	$G^{ev}_{\mathbb{Z}^d}(0)$	$G^{ev}_{\mathcal{B}}(0)$

Comparison Table

If $d \geq 3$ the limit laws and the rescaling factors are identical in both cases, thus we need to compare the constants $G^{ev}_{\mathbb{Z}^d}(0)$ and $G^{ev}_{\mathcal{B}}(0)$. The relative order depends on the probability β to move across a long range connection.
If β is small then $G^{ev}_{\mathbb{Z}^d}(0) < G^{ev}_{\mathcal{B}}(0)$, the small world effect still persists and the two particles meet faster; this is not the case if β is close to 1, where the opposite inequality holds (see Appendix A). Moreover the proof of Proposition $A.1$ states that $G^{ev}_{\mathcal{B}}(0) \stackrel{\beta\to 0}{\to} G^{ev}_{\mathbb{Z}^d}(0)$ and $G^{ev}_{\mathcal{B}}(0) \stackrel{\beta\to 1}{\to} \infty$. Therefore the function $G^{ev}_{\mathcal{B}}(0)$ is not monotone in β.

In [11, Chapter 6], the author sketches a proof that the number of particles of a normalized n-coalescing random walk (that is with n particles at time 0) starting from the stationary distribution on one dimensional BC nearest neighbors small world converges to the Kingman's coalescent. Briefly, the Kingman's coalescent is a Markov process starting from N individuals without spatial structure: each couple has an exponential clock with mean 1 after which the two particles coalesce (see [7], [9] and [16]).
We use Theorem 1.1 to get new information about the number of particles $(|\xi_t(A)|)_{t\geq 0}$ of the coalescing random walk $(\xi_t(A))_{t\geq 0}$ starting from $A = \{x_1, \dots, x_n\}$, $x_i \in \Lambda^d(L)$ for $1 \leq i \leq n$ in continuous time, extending the previous result to d-dimensional BC small world with general transition probabilities and more general initial distance between particles. The result is

Theorem 1.2 *Let $h_L \geq (\log\log L)^2$ such that $\lim_{L\to\infty} M^{h_L}/(2L)^d = 0$, then for each $A = \{x_1, \dots, x_n\} \subset \Lambda^d(L)$ with $|x_i - x_j| \geq h_L$ for $i \neq j$, $T > 0$ there exists a sequence of sets $\{H^L\}_L$ of small world graphs such that $\mathbf{P}(H^L) \stackrel{L\to\infty}{\to} 1$ and for each sequence $\{S^L\}_L$, $S^L \in H^L$, uniformly in $0 \leq t \leq T$*

$$\left|\mathbb{P}^A_{S^L}\left(|\xi_{s_L t}(A)| < k\right) - P_n\Big(D_t < k\Big)\right| \stackrel{L\to\infty}{\to} 0, \qquad k = 2, \dots, n. \tag{1.8}$$

where $s_L = (2L)^d G^{ev}_{\mathcal{B}}(0)$, M is a constant depending on the number of short range connections per site and D_t is the number of particles in a Kingman's coalescent at time $t \geq 0$.

We worked on graphs with a single long range connection per site. One can show the same results for random graphs with fixed $K > 1$ (not depending on L) random long range connections in the same way starting either from the d-dimensional torus or from a translation invariant finite graph. The exponential limit will have a different parameter which we guess would be the return time on a different big world structure.

We proceed as follows: in Section 2.1 we give the definitions needed in the sequel and we construct the random graph. In order to get results on the meeting time of two particles, we largely use the Laplace transform technique and we treat the law of the walkers in different ways for small or large time. For small time, the random graph structure is similar to the big world. In Section 2.2 we introduce the map from the big world graph onto the small world: Proposition 2.7 states that they do not differ when L is large in a ball with radius smaller than $(\log\log L)^2$. When time is large the law of the random walk is close to the stationary distribution. In Section 2.4 we remind some well known estimations for the speed of convergence to equilibrium of a random walk, involving the isoperimetric constant. A useful estimation of the isoperimetric constant for a set with large probability of small world graph is given by Theorem 2.10, Section 2.3.
In Section 3 we use the comparison with the big world for small time and the convergence to equilibrium for large time to prove the main lemmas involving the Laplace transform of the meeting time of two particles. We detail the proofs in continuous time case.
The main result on the meeting time of two particles is proved in Section 4.1. Here we also give a similar result for the hitting time of a single random walk. In Section 5 we introduce the coalescing random walk and we prove the convergence theorem to Kingman's coalescent. Finally in Appendix A we prove a proposition which allows to compare our results with the ones of the meeting time on the d-dimensional torus for $d \geq 3$.

2 Preliminaries

2.1 BC small world

The vertices of the random graph are the ones of the d-dimensional torus, which we denote by

$$\Lambda(L) = \Lambda(L, d) = (\mathbb{Z} \mod 2L)^d,$$

when there is no ambiguity, we will omit the dependence on d.

The set of edges $\mathcal{E}^L$ of the graph is partly deterministic (short range connections) and partly random (long range connections). Note that we consider nonoriented edges, that is, if $(x, y) \in \mathcal{E}^L$ then also $(y, x) \in \mathcal{E}^L$ (thus we identify edges with subsets of order two).

We will consider two kinds of short range connections, one between neighbours (i.e. vertices x, y such that $\|x-y\|_1 = 1$), and the other between vertices x, y such that $\|x-y\|_\infty \leq m$: the corresponding neighbourhoods are

$$\mathcal{N}(x) = \{y \in \Lambda(L) : \|x - y\|_1 = 1\}, \qquad x \in \Lambda(L),$$

$$\mathcal{N}_m^\infty(x) = \{y \in \Lambda(L) : \|x - y\|_\infty \leq m\}, \quad x \in \Lambda(L).$$

For all $x, y \in \Lambda(L)$ we denote by $d(x, y)$ the graph distance between x and y. Let $\widetilde{\Omega}$ be the set of all partitions of the set of $\Lambda(L)$ into $(2L)^d/2$ subsets of cardinality two. Let $\mathbf{P}$ be the uniform probability on $\mathcal{P}(\widetilde{\Omega})$: the random choice of $\widetilde{\omega} \in \widetilde{\Omega}$ represents the choice of the set of long range connections (some of which may coincide with short range ones). Note that both $\widetilde{\Omega}$ and $\mathbf{P}$ depend on L.

Definition 2.1 *Let $\mathcal{G}^L$ be the family of all graphs with set of vertices $\Lambda(L)$. The small world $\mathcal{S}^L$ is a random variable $\mathcal{S}^L(\widetilde{\omega}) : \widetilde{\Omega} \to \mathcal{G}^L$ such that $\mathcal{S}^L(\widetilde{\omega}) = \big(\Lambda(L), \mathcal{E}^L(\widetilde{\omega})\big)$, where*

$$\mathcal{E}^L(\widetilde{\omega}) = \widetilde{\omega} \cup \{\{x, y\} : x \in \Lambda(L), y \in \mathcal{N}(x)\}.$$

The set of edges of the small world $\mathcal{S}^L(\widetilde{\Omega})_m$ is defined as

$$\mathcal{E}^L_m(\widetilde{\omega}) = \widetilde{\omega} \cup \{\{x, y\} : x \in \Lambda(L), y \in \mathcal{N}^\infty_m(x)\}.$$

We denote by $\mathcal{S}^L(\widetilde{\Omega}) = \{\mathcal{S}^L(\widetilde{\omega}) : \widetilde{\omega} \in \widetilde{\Omega}\}$ and by $\mathcal{S}^L_m(\widetilde{\Omega}) = \{\mathcal{S}^L_m(\widetilde{\omega}) : \widetilde{\omega} \in \widetilde{\Omega}\}$.
For any fixed $\widetilde{\omega}$, given two short range neighbours x and y, we write $x \sim y$; if they are long range neighbours we write $x \frown y$ (it may happen that $x \sim y$ and $x \frown y$ at the same time).

Note that $\mathbf{P}$ clearly defines a probability measure on $\mathcal{G}^L$: with a slight abuse of notation we denote this measure with $\mathbf{P}$ as well. Given $\widetilde{\omega}$, we will also call "small world" the graph $\mathcal{S}^L(\widetilde{\omega})$. For the sake of simplicity we will focus here on the case $\mathcal{S}^L$, but our proofs can be adapted to $\mathcal{S}^L_m$. Moreover, when there is no ambiguity, we will write $\mathcal{S}$ and $\mathcal{S}_m$ instead of $\mathcal{S}^L$ and $\mathcal{S}^L_m$.

Remark 2.2 *We note that the small world could be defined imposing that we consider as probability space $\Theta \subset \widetilde{\Omega}$, the family of partitions where no couple is a short range connection (thus the random graph has fixed degree), instead of $\widetilde{\Omega}$.*

Given a small world, we consider a random walk on it. We assume that the discrete time random walk is assigned through a family of adapted (i.e. transition from x to y may occur only if they are connected by an edge), symmetric transition matrices $\{P_S = (p_S(x, y))_{x,y\in\Lambda(L)}\}_{S\in\mathcal{S}^L(\widetilde{\Omega})}$, with the property that $p_S(0, y) = p_S(x, x + y)$ whenever y and $x+y$ are short range neighbours of 0 and x respectively (which implies that the probability from a site towards its long range neighbour is fixed as well), with the assumption that the transition probabilities towards a short range neighbour which is also a long range neighbour is the sum of the two corresponding probabilities.

The transition matrix P_S we will consider is given by

$$p_S(x, y) = \begin{cases} 1 - 2dp - \beta & \text{if } x = y, \\ p & \text{if } x \sim y, \text{ and } x \not\frown y \\ \beta & \text{if } x \frown y, \text{ and } x \not\sim y \\ p + \beta & \text{if } x \sim y, \text{ and } x \frown y \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where $p \in (0, 1/2d)$ and $\beta \in (0, 1 - 2dp]$ (on $\mathcal{S}_m$ substitute $|\mathcal{N}^\infty_m(0)|$ for $2d$). Nevertheless our results hold also for transition matrices with a different distribution among short range neighbours (we only need symmetry and translation invariance).

Definition 2.3 *Given a probability measure μ on $\Lambda(L)$, a small world $S = \mathcal{S}(\widetilde{\omega})$ and a transition matrix P_S, we denote by $\widetilde{\mathbb{P}}^\mu_S$ the law of the discrete time random walk on S with initial probability μ and transitions ruled by P_S. If $\mu = \delta_{x_0}$ we write $\widetilde{\mathbb{P}}^{x_0}_S$.*

Definition 2.4 *Given a probability measure μ on $\Lambda(L)$, and a family of transition matrices $\{P_S\}_{S\in\mathcal{S}^L(\widetilde{\Omega})}$, we denote by $\widetilde{\mathbb{P}}^\mu$ the product of $\mathbf{P}$ and $\widetilde{\mathbb{P}}^\mu_S$, that is*

$$\widetilde{\mathbb{P}}^\mu(A, \mathcal{C}(x_0, \ldots, x_n)) = \sum_{S\in A} \mathbf{P}(S)\mu(x_0)p_S(x_0, x_1)\cdots p_S(x_{n-1}, x_n),$$

where $A \subset \mathcal{G}^L$ and $\mathcal{C}(x_0, \ldots, x_n)$ is the cylinder with base $(x_0, \ldots, x_n)$.

We construct the continuous time version X_t of the random walk $\widetilde{X}_t$ by continuization. In other words we define $X_t \overset{d}{:=} \widetilde{X}_{N_t}$ where N_t has Poisson distribution with mean t independent of X_t: the law of X_t on S starting from a probability measure μ on $\Lambda(L)$ is given by

$$\mathbb{P}^{\mu}_{S}(X_t = y) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} \widetilde{\mathbb{P}}^{\mu}_{S}(\widetilde{X}_k = y). \tag{2.2}$$

From now on the family of transition matrices $\{P_S\}_{S \in \mathcal{S}^L(\widetilde{\Omega})}$ is considered fixed.

2.2 The mapping into the big world

The small worlds $\mathcal{S}^L$ and $\mathcal{S}^L_m$ (which are random graphs) can be mapped into a non-random graph, the *big worlds* $\mathcal{B}^L$ and $\mathcal{B}^L_m$ respectively, as in [12]. We recall here its construction. The sites are all vectors $\pm(z_1, \dots, z_n)$, with $n \geq 1$ components, $z_j \in \mathbb{Z}^d$ and $z_j \neq 0$ for $j < n$. The edges in $\mathcal{B}^L$ are drawn between $+(z_1, \dots, z_n)$ and $+(z_1, \dots, z_n + y)$ if and only if $y \in \mathcal{N}(0)$; for $\mathcal{B}^L_m$ we consider $y \in \mathcal{N}^{\infty}_m(0)$ (these edges correspond to the short range connections). The same is done for vectors with a minus sign.

Moreover $+(z_1, \dots, z_n)$ has a long range neighbour, namely

$$\begin{array}{ll} +(z_1, \dots, z_n, 0) & \text{if } z_n \neq 0, \\ +(z_1, \dots, z_{n-1}) & \text{if } z_n = 0, \\ -(0) & \text{if } z_n = 0, n = 1. \end{array}$$

Analogously one defines the long range neighbour of $-(z_1, \dots, z_n)$. Note that the big world is a vertex transitive graph (i.e. the automorphism group acts transitively). We denote by $|x|$ the graph distance on the big world from x to $+(0)$ and we also write 0 instead of $+(0)$. To each small world we associate a map onto it, from the big world.

Definition 2.5 *Given a small world $\mathcal{S}$ and $x \in \Lambda(L)$, let $LR_{\mathcal{S}}(x)$ be the long range neighbour of x. The map $\phi : \widetilde{\Omega} \to \Lambda(L)^{\mathcal{B}^L}$ is recursively defined as follows:*

$$\begin{array}{ll} \phi(\widetilde{\omega})(+(z)) & = z \mod (2L), \\ \phi(\widetilde{\omega})(-(z)) & = LR_{\mathcal{S}(\widetilde{\omega})}(0) + z \mod (2L), \\ \phi(\widetilde{\omega})(\pm(z_1, \dots, z_n)) & = LR_{\mathcal{S}(\widetilde{\omega})}\left(\phi(\widetilde{\omega})(\pm(z_1, \dots, z_{n-1})) + z_n \mod (2L)\right). \end{array}$$

Note that the transition matrix $P = (p(x, y))_{x, y \in \mathcal{B}^L}$, defined by

$$p(x, y) = \begin{cases} 1 - 2dp - \beta & \text{if } x = y, \\ p & \text{if } x \text{ and } y \text{ are short range neighbours} \\ \beta & \text{if } x \text{ and } y \text{ are long range neighbours} \\ p + \beta & \text{if } x \text{ and } y \text{ are short and long range neighbours} \\ 0 & \text{otherwise,} \end{cases}$$

naturally induces the transition matrix in (2.1) on the small world S. Analogously one can proceed on $\mathcal{B}^L_m$ if the neighbouhood relation used in $\mathbb{Z}^d$ is the one given by $\mathcal{N}^{\infty}_m$.
We will compare the random walk on the small world with the associated random walk on the big world, whose law we denote by $\mathbb{P}_{\mathcal{B}}$ (resp. $\widetilde{\mathbb{P}}_{\mathcal{B}}$ in discrete time).
The random walk $(\mathcal{B}^L, P)$ is symmetric and translation invariant; moreover the discrete version is aperiodic if $\beta \in (0, 1 - 2dp)$, with period 2 if $\beta = 1 - 2dp$ and one can prove, by using Cauchy-Schwarz's inequality, the symmetry and the translational invariance of the walk, that for all $x \in \mathcal{B}^L$ and $n \in \mathbb{N}$,

$$\widetilde{\mathbb{P}}^x_{\mathcal{B}}(\widetilde{X}_{2n} = 0) \leq \widetilde{\mathbb{P}}^0_{\mathcal{B}}(\widetilde{X}_{2n} = 0); \qquad \widetilde{\mathbb{P}}^x_{\mathcal{B}}(\widetilde{X}_{2n+1} = 0) \leq \widetilde{\mathbb{P}}^0_{\mathcal{B}}(\widetilde{X}_{2n} = 0). \tag{2.3}$$

Using (2.2) we get on the continuous time version that for each $t \geq 0$

$$\mathbb{P}^x_{\mathcal{B}}(X_{2t} = 0) \quad \leq \mathbb{P}^0_{\mathcal{B}}(X_{2t} = 0). \tag{2.4}$$

We denote by $\widetilde{G}_{\mathcal{B}}(x) := \sum_{n=0}^{\infty} \widetilde{\mathbb{P}}^x_{\mathcal{B}}(\widetilde{X}_n = 0)$ the expected number of visits to 0 of the discrete time random walk on the big world (starting from x and associated to $\{P_S\}_S$) and let $G^{ev}_{\mathcal{B}}(x)$ and $\widetilde{G}_{\mathcal{B}}(x)$ as in (1.1). We can prove, starting from (2.2), that

$$\widetilde{G}_{\mathcal{B}}(x) = \int_0^{\infty} \mathbb{P}^x_{\mathcal{B}}(X_t = 0)dt =: G_{\mathcal{B}}(x); \qquad \frac{1}{2} G_{\mathcal{B}}(x) = \int_0^{\infty} \mathbb{P}^x_{\mathcal{B}}(X_t = Y_t)dt.$$

Clearly $\widetilde{G}_{\mathcal{B}}(x) \leq G_{\mathcal{B}}(x)$ and they coincide if the random walk has period 2 (in which case they are nonzero only if $|x|$ is even). Note that if $m = 1$ the big world is the Cayley graph of $\mathbb{Z}^d * \mathbb{Z}_2$ and the random walk on it is transient and $G_{\mathcal{B}}(x)$ is finite. If $m \geq 2$ the big world is the Cayley graph of $\widetilde{\mathbb{Z}}^d * \mathbb{Z}_2$, where $\widetilde{\mathbb{Z}}^d$ has the m-neighbourhood relation, and one has the same result (this can be proven via the flow criterion, see [18]).
Moreover, by (2.4), $G^{ev}_{\mathcal{B}}(x) \leq G^{ev}_{\mathcal{B}}(0)$, and a similar remark holds in discrete time.

We are interested in the event where *locally the small world does not differ from the big world.*

Definition 2.6 *If $x \in \Lambda(L)$ and $t > 0$, we denote by $I(x,t)$ the event in $\widetilde{\Omega}$*

$$I(x,t) := \{\phi_{|B_{\mathcal{B}}(x,t)} \textit{ is injective}\},$$

where $B_{\mathcal{B}}(x,t)$ is the ball of radius t centered at x in the big world.

Clearly $\mathbf{P}(I(x,t))$ does not depend on x.

Proposition 2.7 *If $t \leq (\log\log L)^2$ then*

$$\mathbf{P}(I(x,t)) \geq 1 - \frac{CM^{3t}}{L^d} \overset{L\to\infty}{\to} 1,$$

where C and M are positive constants depending on the neighbourhood structure we consider.

Proof. Denote by K_t the number of long range connections in $B_{\mathcal{B}}(0,t)$, and by J_t the total number of sites in $B_{\mathcal{B}}(0,t)$. Clearly $K_t \leq J_t$ and $|\{x \in \Lambda(L) : d(0,x) \leq t\}| \leq J_t$. Since each site has M neighbours ($M = (2m+1)^d$ in $\mathcal{B}^L_m$ and $M = 2d+1$ in $\mathcal{B}^L$), then $J_t \leq CM^t$. Enumerate the long range connections in $B_{\mathcal{B}}(0,t)$ from 1 to K_t and construct the mapping ϕ. Note that $I(0,t)$ contains the set A of $\widetilde{\omega}$ such that the long range connections in the image of $B_{\mathcal{B}}(0,t)$ in the small world $\mathcal{S}$ are all sites at distance at least $2t$ on $\Lambda(L)$. Thus $\mathbf{P}(I(0,t)) \geq \mathbf{P}(A)$ and

$$\begin{aligned}\mathbf{P}(A) &\geq \frac{(2L)^d - J_{2t}}{(2L)^d} \frac{(2L)^d - 2J_{2t}}{(2L)^d} \cdots \frac{(2L)^d - (K_t - 1)J_{2t}}{(2L)^d} \\ &= \prod_{i=1}^{K_t - 1} \left(1 - \frac{iJ_{2t}}{(2L)^d}\right) = \exp\left(\sum_{i=1}^{K_t-1} \log\left(1 - \frac{iJ_{2t}}{(2L)^d}\right)\right).\end{aligned}$$

Pick $\varepsilon > 0$ and note that $\log(1-x) \geq -(1+\varepsilon)x$ if $x \in (0, \bar{x}_\varepsilon)$. Choosing $t \leq (\log\log L)^2$ we get

$$\frac{iJ_{2t}}{(2L)^d} \leq \frac{K_t J_{2t}}{(2L)^d} \overset{L\to\infty}{\to} 0,$$

thus for L sufficiently large we have

$$\mathbf{P}(A) \geq \exp\left(-(1+\varepsilon)\frac{J_{2t}K_t^2}{(2L)^d}\right) \geq \exp\left(-\frac{CM^{3t}}{L^d}\right) \overset{L\to\infty}{\to} 1.$$

□

By $d_{\mathcal{S}}(x,y)$ we denote the (random) graph distance between x and y. Depending on $\widetilde{\omega}$, x and y, it may happen that $d_{\mathcal{S}}(x,y) = d(x,y)$ or $d_{\mathcal{S}}(x,y) < d(x,y)$. The following proposition provides probability estimates of these events.

Proposition 2.8 *a. If $d(0,x) \leq t$, then*

$$\mathbf{P}\big(d_{\mathcal{S}}(0,x) = d(0,x)\big) \geq 1 - \frac{CM^{3t}}{L^d}. \tag{2.5}$$

b. If $d(0,x) > t$, then

$$\mathbf{P}\big(d_{\mathcal{S}}(0,x) > t\big) \geq 1 - \frac{CM^{3t}}{L^d}. \tag{2.6}$$

Proof.

a. It suffices to note that the event $(d_{\mathcal{S}}(0,x) = d(0,x))$ contains the event A of the previous proposition.

b. We note that the event $(d_{\mathcal{S}}(0,x) > t)$ contains C_x which is the event that all the $2K_t$ long range connections in $B_{\mathcal{B}}(0,t/2)$ and $B_{\mathcal{B}}(x,t/2)$ are mapped by ϕ into vertices of $\Lambda(L)$ at distance at least t from each other and from the balls of radius t centered at 0 and at x in $\Lambda(L)$. We work as in the previous proposition to estimate

$$\mathbf{P}(C_x) \geq \frac{(2L)^d - 2J_t}{(2L)^d}\frac{(2L)^d - 3J_t}{(2L)^d}\cdots\frac{(2L)^d - (2K_{t/2}-1)J_t}{(2L)^d}.$$

and we proceed in a similar way to get the thesis.

□

2.3 Isoperimetric constant

Estimates of the distance between the random walk and the equilibrium measure involve the isoperimetric constant. Thus we will get bounds for the *edge isoperimetric constant*

$$\iota = \min_{|V|\leq n/2} \frac{e(V,V^{\complement})}{|V|}, \tag{2.7}$$

where n is the total number of vertices in the graph and $e(V,V^{\complement})$ is the total number of edges between V and $V^{\complement}$.

The following result is essentially Theorem 6.3.2 of [11]: there it was stated that there is a lower bound for ι on the complement of a set whose probability is $o(1)$. We slightly modify the proof in order to get a "bad set" of probability which is $o(n^{-l})$ for l positive integer.

Proposition 2.9 *Consider a random regular graph of n vertices and degree r. Then for all $l \in \mathbb{N}$ there exists $\alpha_l > 0$ independent of n and r (one may choose $\alpha_l = 1/10l$) such that $\mathbb{P}(\iota \leq \alpha_l) = o(n^{-l})$ ($\mathbb{P}$ being the probability associated to the choice of the graph).*

Proof. Let $P(u,s)$ be the probability that there exists a subset of vertices U such that $|U| = u$ and $e(U, U^{\complement}) = s$. Note that

$$\mathbb{P}(\iota \leq \alpha_l) \leq \sum_{\substack{u \leq n/2 \\ s \leq \alpha_l u}} P(u,s) \leq C\alpha_l n^2 \sup_{\substack{1/\alpha_l \leq u \leq n/2 \\ 1 \leq s \leq \alpha_l u}} P(u,s).$$

By equations (6.3.2), (6.3.3), (6.3.4) and (6.3.5) in [11] we have the upper bound

$$P(u,s) \leq C\sqrt{n}e^u \left(\frac{e^2 ru}{s}\right)^s \left(\frac{u}{n}\right)^{\gamma ru} \left(1 - \frac{ru+s}{rn}\right)^{(rn-ru-s)/2},$$

where $\gamma = \frac{1}{2}\left(1 - \frac{s}{ru}\right) - \frac{1}{r}$. Let

$$G_s(u) = e^u \left(\frac{e^2 ru}{s}\right)^s \left(\frac{u}{n}\right)^{\gamma ru} \left(1 - \frac{ru+s}{rn}\right)^{(rn-ru-s)/2},$$

it suffices to prove that there exists $\alpha_l > 0$ such that for all $1 \leq s \leq \alpha_l u$

$$\sup_{1/\alpha_l \leq u \leq n/2} G_s(u) = o(n^{-l-5/2}).$$

First we write $G_s(u)$ as a function of $\alpha = s/ru$ (note that $2/rn \leq \alpha \leq \alpha_l/r$):

$$G_s(u) = e^u \left(\frac{e^2}{\alpha}\right)^{\alpha ur} \left(\frac{u}{n}\right)^{ru\left(\frac{1-\alpha}{2} - \frac{1}{r}\right)} \left(1 - \frac{u}{n}(1+\alpha)\right)^{\frac{rn-ru}{2}(1+\alpha)}.$$

In [11] it is shown that G_s is convex, so it is enough to estimate it in $n/2$ and $1/\alpha_l$. It is easy to show that for some $C \in (0,1)$

$$G_s(n/2) \leq C^n,$$

while

$$G_s(1/\alpha_l) \leq Cn^{1 - \frac{r}{\alpha_l}(1/2 - 1/r)}.$$

Choosing $\alpha_l < 1/(7+2l)$ (for instance $\alpha_l = 1/10l$) we get the thesis. □

Now we use this result to prove the analog for the BC small world. The ideas are taken from Theorem 6.3.4 of [11].

Proposition 2.10 *Consider the small world $\mathcal{S}^L$ and its (random) edge isoperimetric constant ι. Then for all $l \in \mathbb{N}$ if $\alpha_l = 1/10l$ then $\mathbf{P}(\iota \leq \alpha_l) = o(L^{-dl})$.*

Proof. First, we partition the set $\Lambda(L)$ in $n = \lfloor (2L)^d/3 \rfloor$ subsets of cardinality three (let $\{I_j\}_{j=1}^n$ be their collection) plus eventually one subset of cardinality one or two. We associate to $\mathcal{S}^L$ the random regular graph of degree three and n vertices: join j with k whenever there exist $x \in I_j$ and $y \in I_k$ such that x is the long range neighbour of y.

Given $A \subset \Lambda(L)$ define J_A as the family of indices j such that $I_j \cap A \neq \emptyset$. Note that $A \subset \bigcup_{j \in J_A} I_j$, $|J_A| \geq |A|/3$ and that if there is an edge between J_A and $J_A^{\complement}$ in the random regular graph then there is a long range connection between A and $A^{\complement}$.

Suppose that $|J_A| \le n/2$, then by Proposition 2.9, outside a set of probability $o(n^{-l}) = o(L^{-dl})$ we have

$$\frac{e(A, A^{\complement})}{|A|} \ge \frac{e(J_A, J_A^{\complement})}{|A|} \ge \frac{3\alpha_l |J_A|}{|J_A|}.$$

In the case that $|J_A| > n/2$ we exchange J_A with $J_A^{\complement}$ and we are done. $\square$

2.4 Convergence to equilibrium

Note by symmetry that the reversible distribution of the walk on $\mathcal{S}^L$ is the uniform probability.

We recall that given a discrete time random walk on a finite set, with transition matrix P and reversible measure the uniform measure π, a result of Sinclair and Jerrum [15] gives an estimate of the speed of convergence to equilibrium. Indeed in this case P has all real eigenvalues, namely $1 = \lambda_0 > \lambda_1 \ge \cdots \ge \lambda_{n-1}$. Let $\lambda = \max\{|\lambda_i| : i = 1, \ldots, n-1\}$. It is well known that $\lambda < 1$. Then for all $t \in \mathbb{N}_0 := \{n \in \mathbb{Z} : n \ge 0\}$

$$\max_{x,y} \left| p^{(t)}(x, y) - \pi(y) \right| \le \lambda^t \le \exp(-t(1-\lambda)),$$

where $p^{(t)}(x, y)$ is a t-step probability of the walk.

We are interested in estimates for λ. If $\lambda = \lambda_1$ then the following (which is known as Cheeger's inequality (see [11, Theorem 6.2.1]), is useful

$$\frac{1}{2}\iota^2 \left(\min_{x,y:p(x,y)>0} p(x, y) \right)^2 \le 1 - \lambda_1 \le 2\iota. \tag{2.8}$$

A sufficient condition for $\lambda = \lambda_1$ is that all the eigenvalues are positive, which for instance holds when we consider a lazy random walk, that is one which stays put with probability 1/2.

It is thus clear that for any small world S such that $\iota(S) > \alpha$, a random walk $\widetilde{X}_t$ on S with symmetric transition matrix P_S such that $\lambda = \lambda_1$,

$$\max_{x,y} \left| \widetilde{\mathbb{P}}_S^x(\widetilde{X}_t = y) - \pi(y) \right| \le \exp(-c\alpha^2 t), \tag{2.9}$$

where c depends only on $\min\limits_{x,y:p_S(x,y)>0} p_S(x, y)$. Moreover by Proposition 2.9 with $\alpha_l = k/l$

$$\begin{aligned} \max_{x,y} \left| \widetilde{\mathbb{P}}^x(\widetilde{X}_t = y) - \pi(y) \right| &\le \sum_S \mathbf{P}(S) \max_{x,y} \left| \widetilde{\mathbb{P}}_S^x(\widetilde{X}_t = y) - \pi(y) \right| \\ &\le \exp(-c(k/l)^2 t) \mathbf{P}(\iota > k/l) + 2\mathbf{P}(\iota \le k/l) \\ &\le \exp(-c(k/l)^2 t) + o(L^{-dl}). \end{aligned} \tag{2.10}$$

It is easy starting from (2.2) to prove that (2.9) and (2.10) still hold in continuous time with a different constant in the exponential. Namely, one has to replace $c\alpha_l^2$ with

$$\gamma := 1 - e^{-c\alpha_l^2}. \tag{2.11}$$

in order to get

$$\max_{x,y} |\mathbb{P}_S^x(X_t = y) - \pi(y)| \le e^{-\gamma t} \tag{2.12}$$

$$\max_{x,y} |\mathbb{P}^x(X_t = y) - \pi(y)| \le e^{-\gamma t} + o(L^{-dl}). \tag{2.13}$$

Remark 2.11 *We are able to prove that* (2.9) *and* (2.10) *hold for any random walk (not just for the lazy one) with different constants. The key consists in coupling $\widetilde{X}$ with a random walk $\widetilde{Y} = \{\widetilde{Y}_t\}_{t\geq 0}$ with transition matrix Q such that $q(x,x) = (1+p(x,x))/2$, $q(x,y) = p(x,y)/2$: the process $\widetilde{Y}$ moves with $\widetilde{X}$ when a Bernoulli random variable with parameter $1/2$ equals 1, otherwise it stay put.*

3 Laplace transform estimates

Let $T_L = \inf\{s > 0 : X_s = Y_s\}$ (resp. $\widetilde{T}_L$) be the first time, after time 0, that two independent continuous (resp. discrete) time random walks X_t and Y_t on the random graph $\mathcal{S}$ meet. Clearly the law of T_L (with respect to either $\mathbb{P}_S$ or $\mathbb{P}$) depends on the starting sites of the walkers. Without loss of generality, we assume that $Y_0 = 0$ and $X_0 = x$ (if we need to stress the dependence on L, we write $X_0 = x_L$).
We introduce the following (annealed) Laplace transforms in continuous time,

$$\begin{aligned} G^L(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\mathbb{P}^{x,0}(X_t = Y_t)dt = \int_0^\infty e^{-\lambda t}\mathbb{P}^x(X_{2t} = 0)dt, \\ F^L(x,\lambda) &:= \int_0^\infty e^{-\lambda t}\mathbb{P}^{x,0}(T_L \in dt), \end{aligned}$$

where $\mathbb{P}^{x,0}$ denotes the product law of the two walkers. The corresponding quenched transforms are, given $S \in \mathcal{S}^L(\widetilde{\Omega})$,

$$G_S^L(x,\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{P}_S^{x,0}(X_t = Y_t)dt, \qquad F_S^L(x,\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{P}_S^{x,0}(T_L \in dt).$$

We are interested in the asymptotic behaviour, as $L \to \infty$, of $T_L/(2L)^d$, thus we study the previous Laplace transforms with parameter $\lambda/(2L)^d$.
The discrete time version of such Laplace transforms are defined in a similar way, but the integrals are replaced by sums. With a slight abuse of notation we omit the superscript $\sim$ on the discrete time random walk when not necessary and we use X_t, Y_t, T_L, $\mathbb{P}_S^\mu$, $G_S^L(x,\lambda)$ and $F_S^L(x,\lambda)$ both in discrete and continuos time version of the process: since the proofs are similar, we detail the latter one and we only point out the differences.

3.1 Estimates for G

We first note that the evaluation of the limit of the annealed transforms can be done considering only small worlds with large isoperimetric constants.
Given $\alpha > 0$, we define

$$Q_\alpha^L := (S \in \mathcal{S}^L(\widetilde{\Omega}) : \iota(S) > \alpha) \tag{3.1}$$

Let $\mathcal{K} := \{K \subset \mathbb{R} : \inf K > 0\}$.

Lemma 3.1 *There exists $\alpha > 0$ such that $\mathbf{P}(Q_\alpha^L) \overset{L\to\infty}{\to} 1$. Moreover if*

$$\begin{aligned} g_L &:= \sum_{S \in (Q_\alpha^L)^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^x(X_{2t} = 0)dt, \\ f_L &:= \sum_{S \in (Q_\alpha^L)^c} \mathbf{P}(S) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^{x,0}(T_L \in dt), \end{aligned}$$

then $g_L \overset{L\to\infty}{\to} 0$ and $f_L \overset{L\to\infty}{\to} 0$ (for each $K \in \mathcal{K}$, uniformly for $\lambda \in K$).

Proof. We choose $\alpha = 1/20$: by Proposition 2.10 we have $\mathbf{P}(\iota \le \alpha) = o(L^{-2d})$. Moreover

$$0 \le f_L \le g_L \le \mathbf{P}((Q^L_\alpha)^c) \int_0^\infty e^{-\frac{\lambda t}{(2L)^d}} dt = \mathbf{P}((Q^L_\alpha)^c) \frac{(2L)^d}{\lambda} \overset{L\to\infty}{\to} 0.$$

□

The limit of the sum defining G, from $\log\log L$ ($\lfloor \log\log L \rfloor$ in discrete time) to infinity does not depend on the sequence of small worlds, provided that they are chosen with large isoperimetric constant. From now on, if not otherwise stated, we write $t_L = \log\log L$ and $Q^L = (S \in \mathcal{S}^L : \iota(S) > 1/20)$.

Lemma 3.2 *If for all L we choose $S \in Q^L$ and $x_L \in \Lambda(L)$, then for all $\lambda > 0$*

$$\lim_{L\to\infty} \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) dt = \frac{1}{\lambda}.$$

Moreover, the convergence is uniform with respect to the choice of the sequences $S \in Q^L$, $x_L \in \Lambda(L)$ (and of λ).

Proof. Note that

$$\begin{aligned} &\int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t} = 0) dt \\ &= \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \frac{1}{(2L)^d} dt + \int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} \left(\mathbb{P}_S^{x_L}(X_{2t} = 0) - \frac{1}{(2L)^d} \right) dt. \end{aligned} \tag{3.2}$$

The limit of the first term is uniform in λ and it converges to $1/\lambda$. Since S is chosen in Q^L, by (2.12) there exists a constant $\gamma = 1 - \exp(-c\alpha^2) > 0$ such that the second sum on the right hand side of (3.2) is smaller or equal to

$$\int_{t_L}^\infty e^{-\frac{\lambda t}{(2L)^d}} e^{-\gamma 2t} dt = \frac{e^{-(\lambda/(2L)^d + 2\gamma)t_L}}{\lambda/(2L)^d + 2\gamma},$$

which tends to 0 as L goes to infinity (uniformly with respect to all the choices of the statement). □

Recall that, given a vertex $x \in \Lambda(L)$, there is a unique vertex $+(x)$ in the big world. If L is sufficiently large, for a wide choice of S (i.e. S in a set with $\mathbf{P}$-probability which tends to 1 as L increases to infinity), we have that $G_S^L(x_L, \lambda/(2L)^d)$ is close to $1/\lambda + G_{\mathcal{B}}^{ev}(+(x_L))$.

Theorem 3.3 *Let*

$$h_S^L(\lambda) = \left| G_S^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{\mathcal{B}}^{ev}(+(x_L)) \right|.$$

For all $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all λ, x_L and $L \ge \widetilde{L}$ we have that $Q^L \cap I(0, t_L^2) \subset (S : h_S^L(\lambda) \le \varepsilon)$.
If $d_S(0, x_L) > t_L^2$ then $Q^L \subset (S : h_S^L(\lambda) \le \varepsilon)$.

Proof. Note that

$$G_{\mathcal{B}}^{ev}(+(x_L)) = \int_0^\infty \mathbb{P}_{\mathcal{B}}^{+(x_L)}(X_{2t} = 0) dt, \tag{3.3}$$

thus

$$h_S^L(\lambda) \le \Big| \int_{t_L}^{\infty} e^{-\frac{\lambda t}{(2L)^d}} \mathbb{P}_S^{x_L}(X_{2t}=0)dt - \frac{1}{\lambda}\Big|$$
$$+ \Big| \int_0^{t_L} e^{-\frac{\lambda t}{(2L)^d}} \big(\mathbb{P}_S^{x_L}(X_{2t}=0)dt - \mathbb{P}_{\mathcal{B}}^{+(x_L)}(X_{2t}=0)\big)dt \Big|$$
$$+ \int_{t_L}^{\infty} \mathbb{P}_{\mathcal{B}}^{+(x_L)}(X_{2t}=0)dt + \int_0^{t_L} (1-e^{-\frac{\lambda t}{(2L)^d}})\mathbb{P}_{\mathcal{B}}^{+(x_L)}(X_{2t}=0)dt.$$

Since either $S \in I(0, t_L^2)$ or $d_S(0, x_L) > t_L^2$, the probabilities of a meeting before time t_L on S and on the big world differ only if the number of exponential clocks $Z(t_L)$ before time t_L is at least t_L^2: by Chebyshev's inequality the second term of the right hand side is smaller than

$$2t_L \mathbb{P}(Z(t_L) \ge t_L^2) \le \frac{2t_L^2}{(t_L^2 - t_L)^2} \le \varepsilon/4$$

if L is large enough.
By Lemma 3.2, if $S \in Q^L$, the first term is smaller than $\varepsilon/4$ and by the Dominated Convergence Theorem and (2.3), the last two terms are both smaller than $\varepsilon/4$ if L is sufficiently large. □

Theorem 3.4 *For all $K \in \mathcal{K}$, $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$, $x_L \in \Lambda(L)$, and $\lambda \in K$,*

$$\left| G^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{\mathcal{B}}^{ev}(+(x_L)) \right| \le \varepsilon.$$

Proof. Recall that

$$G^L(x_L, \lambda/(2L)^d) = \sum_S \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d).$$

By Theorem 3.3 there exists $\widetilde{L}$ such that for all $L \ge \widetilde{L}$,

$$\left| \sum_{S \in Q^L \cap I(0, t_L^2)} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) - \frac{1}{\lambda} - G_{\mathcal{B}}^{ev}(+(x_L)) \right| \le \varepsilon/3.$$

Thus, since $\mathbf{P}((Q^L)^c)$ and $\mathbf{P}(I(0, t_L^2)^c)$ are both small if L is large, we may choose $\widetilde{L}$ such that for all $\lambda \in K$

$$\sum_{S \in (Q^L)^c \cup (I(0, t_L^2))^c} \mathbf{P}(S) \left(\frac{1}{\lambda} + G_{\mathcal{B}}^{ev}(+(x_L)) \right) \le \varepsilon/3.$$

Now we only need to prove that

$$\sum_{S \in (Q^L)^c \cup I(0, t_L^2)^c} \mathbf{P}(S) G_S^L(x_L, \lambda/(2L)^d) \le \varepsilon/3.$$

By Lemma 3.1 we know that $\sum_{S\in(Q^L)^c}\mathbf{P}(S)G_S^L(x_L,\lambda/(2L)^d)\le\varepsilon/6$ for all $L\ge\widetilde{L}$ and $\lambda>0$. Finally, by Proposition 2.7 and Lemma 3.2, for some $C>0$ and L sufficiently large

$$
\begin{aligned}
&\sum_{S\in Q^L\cap I(0,t_L^2)^c}\mathbf{P}(S)G_S^L(x_L,\lambda/(2L)^d)\\
&=\sum_{S\in Q^L\cap I(0,t_L^2)^c}\mathbf{P}(S)\left(\int_0^{t_L}e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^{x_L}(X_{2t}=0)+\int_{t_L}^{\infty}e^{-\frac{\lambda t}{(2L)^d}}\mathbb{P}_S^{x_L}(X_{2t}=0)\right)\\
&\le\left(t_L+\frac{1}{\lambda}+C\right)\mathbf{P}(I(0,t_L^2)^c)\le\varepsilon/6.
\end{aligned}
$$

□

3.2 From G to F

We note that if $x_L\neq 0$ then $G_S^L(x_L,\lambda/(2L)^d)$ may be written as

$$
\sum_z\int_0^{\infty}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_0^{\infty}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z). \tag{3.4}
$$

while $G_S^L\left(0,\frac{\lambda}{(2L)^d}\right)$ is equal to

$$
1+\sum_z\int_0^{\infty}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_0^{\infty}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{0,0}(T_L\in ds,X_s=z).
$$

Define H_1, H_2 and H_3 (which depend on S, x_L and L) by

$$
\begin{aligned}
H_1&:=\sum_z\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_0^{t_L}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z)\\
H_2&:=\sum_z\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_{t_L}^{\infty}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z)\\
H_3&:=\sum_z\int_{t_L}^{\infty}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_0^{\infty}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z).
\end{aligned}
$$

By Lemma 3.1, for all L sufficiently large and if the limit exists,

$$
\lim_{L\to\infty}G^L\left(x_L,\frac{\lambda}{(2L)^d}\right)=\lim_{L\to\infty}\sum_{S\in Q^L}\mathbf{P}(S)(H_1+H_2+H_3). \tag{3.5}
$$

Clearly if $x_L=0$ for all L sufficiently large we only need to add 1 to the previous limit. The same equality holds in discrete time, replacing the integral with the sum.
We now study each of the three summands separately, in order to obtain the limit of F^L as a function of the limit of G^L.

Lemma 3.5 *If $S\in I(0,t_L^2)$ and $x_L\in\Lambda(L)$, for each $\varepsilon>0$ there exists $\widetilde{L}$ such that for each $L>\widetilde{L}$ then*

$$
\left|H_1-\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)dq\int_0^{t_L}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds)\right|<\varepsilon. \tag{3.6}
$$

This inequality also holds whenever $d_S(0,x_L) > t_L^2$. Moreover, uniformly with respect to the choice of the sequence $\{x_L\}_L$ and of λ,

$$\sum_{S\in(I(0,t_L^2))^c} \mathbf{P}(S)\sum_z \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds, X_s=z) \overset{L\to\infty}{\to} 0. \tag{3.7}$$

Proof. Since in H_1, z is the site where the two random walks X_t and Y_t meet at a time $s\in[0,t_L]$. Since $S\in I(0,t_L^2)$ or $d_S(0,x_L) > t_L^2$, $\int_0^{t_L}\mathbb{P}_S^z(X_{2q}=z)dq$ differs from $\int_0^{t_L}\mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)dq$ only if the number of the exponential clocks $Z(2t_L)$ before time $2t_L$ is larger than t_L^2; by Chebyshev's inequality

$$\int_0^{t_L}\left|\mathbb{P}_S^z(X_{2q}=z)-\mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)\right|dq \le \frac{4t_L^2}{(t_L^2-2t_L)^2}. \tag{3.8}$$

which proves (3.6) since $\sum_z\int_0^{t_L} e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L=s,X_s=z)ds\le 1$.
Note that for some $C>0$

$$\begin{aligned}\sum_{S\in I(0,t_L^2)^c}\mathbf{P}(S)\sum_z\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)\int_0^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L=s,X_s=z)\\ \le Ct_L\,\mathbf{P}(I(0,t_L^2)^c)\,F^L(x_L,\lambda/(2L)^d),\end{aligned} \tag{3.9}$$

which, by Proposition 2.7 and since $F^L(x,\lambda)\le 1$ for all λ and x, goes to 0, uniformly in x_L and λ, as L goes to infinity. This proves (3.7). □

Lemma 3.6 *For all $K\in\mathcal{K}$, $\varepsilon>0$ there exists $\widetilde{L}$ such that for all $L\ge\widetilde{L}$, x_L and $\lambda\in K$,*

$$\left|\sum_{S\in Q^L}\mathbf{P}(S)H_2-\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)dq\sum_{S\in Q^L}\mathbf{P}(S)\int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds)\right|\le\varepsilon. \tag{3.10}$$

Proof. Note that $\sum_{S\in Q^L}\mathbf{P}(S)H_2$ can be written as

$$\begin{aligned}&\sum_z\sum_{S\in Q^L\cap I(z,t_L^2)}\mathbf{P}(S)\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z)\\ &+\sum_z\sum_{S\in Q^L\cap I(z,t_L^2)^c}\mathbf{P}(S)\int_0^{t_L}e^{-\frac{\lambda q}{(2L)^d}}\mathbb{P}_S^z(X_{2q}=z)dq\int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(T_L\in ds,X_s=z)\\ &=H_{2,1}+H_{2,2}.\end{aligned}$$

We prove that $H_{2,2}\to0$, indeed since $\exp(-\lambda q/(2L)^d)\mathbb{P}_S^z(X_{2q}=z)\le1$ then

$$\begin{aligned}H_{2,2}\le t_L\sum_z\sum_{S\in Q^L\cap I(z,t_L^2)^c}\mathbf{P}(S)\Big\{\int_{t_L}^{\log L}e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(X_s=Y_s=z)ds\\ +\int_{\log L}^\infty e^{-\frac{\lambda s}{(2L)^d}}\mathbb{P}_S^{x_L,0}(X_s=Y_s=z)ds\Big\} =: H_{2,2,1}+H_{2,2,2}.\end{aligned}$$

Note that $H_{2,2,1}$ is smaller or equal to

$$t_L \sum_{S\in Q^L} \mathbf{P}(S) \int_{t_L}^{\log L} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^0(X_{2s} = x_L) ds.$$

We write $\mathbb{P}_S^0(X_{2s} = x_L) \le \left|\mathbb{P}_S^0(X_{2s} = x_L) - 1/(2L)^d\right| + 1/(2L)^d$, which by (2.12) is smaller or equal to $e^{-\gamma s} + 1/(2L)^d$. It is thus only a matter of computation to show that $H_{2,2,1}$ goes to zero (uniformly in x_L and λ) as L goes to infinity.
Now we consider $H_{2,2,2}$. Note that $\mathbb{P}_S^{x_L,0}(X_s = Y_s = z) = \mathbb{P}_S^z(X_s = 0)\mathbb{P}_S^z(Y_s = x_L)$. Write

$$\mathbb{P}_S^z(X_s = 0) = \mathbb{P}_S^z(X_s = 0) - \frac{1}{(2L)^d} + \frac{1}{(2L)^d}$$

and do the same for $\mathbb{P}_S^z(Y_s = x_L)$. Using (2.12) and Proposition 2.7, we have that $H_{2,2,2}$ is smaller or equal to

$$\begin{aligned}
&t_L \sum_z \sum_{S\in Q^L\cap I(z,t_L^2)^c} \mathbf{P}(S) \int_{\log L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left(e^{-\gamma s} + \frac{1}{(2L)^d}\right)^2 ds \\
&\le C t_L L^d \frac{M^{t_L^2}}{L^d} \int_{\log L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \left(e^{-2\gamma s} + \frac{1}{(2L)^{2d}} + \frac{2e^{-\gamma s}}{(2L)^d}\right) ds
\end{aligned}$$

which goes to 0 (uniformly in x_L and $\lambda \in K$ for each $K \in \mathcal{K}$) as $L \to \infty$.
We now consider $H_{2,1}$. We split

$$\begin{aligned}
&\sum_z \sum_{S\in Q^L\cap I(z,t_L^2)} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds, X_s = z) \\
&= \sum_{S\in Q^L} \sum_z \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds, X_s = z) \\
&- \sum_z \sum_{S\in Q^L\cap I(z,t_L^2)^c} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds, X_s = z).
\end{aligned}$$

the second summand converges to 0 by the same arguments we used to prove that $H_{2,2}$ converges to 0; we replace the first one in $H_{2,1}$ and we get

$$\int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \sum_{S\in Q^L} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds).$$

If $S \in I(z, t_L^2)$, by (3.8) then $\int_0^{t_L} |\mathbb{P}_S^z(X_{2q} = z) - \mathbb{P}_{\mathcal{B}}^0(X_{2q} = 0)| dq \le C/t_L^2$. Thus for some constant $C > 0$

$$\begin{aligned}
&\Big| \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_S^z(X_{2q} = z) dq \sum_{S\in Q^L} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds) \\
&- \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{\mathcal{B}}^0(X_{2q} = 0) dq \sum_{S\in Q^L} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds)\Big| \\
&\le \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \left|\mathbb{P}_S^z(X_{2q} = z) dq - \mathbb{P}_{\mathcal{B}}^0(X_{2q} = 0)\right| dq \sum_{S\in Q^L} \mathbf{P}(S) \int_{t_L}^{\infty} e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds) \\
&\le \frac{C}{t_L^2} F^L(x_L, \lambda/(2L)^d)
\end{aligned}$$

which can be taken as small as we want if L is large. □

Lemma 3.7 *Let*

$$a_\lambda^L(S) := H_2 - \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)dq \int_{t_L}^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L \in ds). \tag{3.11}$$

Then $a_\lambda^L > 0$ and $a_\lambda^L \to 0$ in probability (for each $K \in \mathcal{K}$, uniformly in x_L and $\lambda \in K$), that is for all $K \in \mathcal{K}$, $\varepsilon > 0$ and $\delta > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$ and x_L

$$\mathbf{P}(A_\varepsilon^L(K)) := \mathbf{P}(S : a_\lambda^L(S) \leq \varepsilon, \forall \lambda \in K) \geq 1-\delta.$$

Proof. We first note that for all z and S, $\mathbb{P}_S^z(X_{2q}=z) \geq \mathbb{P}_{\mathcal{B}}^0(X_{2q}=0)$, hence $a_\lambda^L(S) \geq 0$. Suppose by contradiction that there exist K, $\varepsilon > 0$ and $\delta > 0$ such that $\mathbf{P}(A_\varepsilon^L(K)) \leq 1-\delta$ infinitely often. Then infinitely often

$$\sum_S \mathbf{P}(S) a_\lambda^L(S) > \delta\varepsilon.$$

By Lemmas 3.6 and 3.1, there exists $\widetilde{L}$ such that $\sum_S \mathbf{P}(S) a_\lambda^L(S) < \delta\epsilon$ for each $L \geq \widetilde{L}$, x_L, $\lambda \in K$, whence the contradiction. □

Lemma 3.8 *For all $K \in \mathcal{K}$ and $\varepsilon > 0$ there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$, $S \in Q^L$, x_L and $\lambda \in K$,*

$$\left| H_3 - \frac{1}{\lambda} F_S^L(x_L, \lambda/(2L)^d) \right| \leq \varepsilon. \tag{3.12}$$

Proof. Note that

$$\begin{aligned} H_3 = &\sum_z \int_{t_L}^\infty \left(\mathbb{P}_S^z(X_{2q}=z) - \frac{1}{(2L)^d} \right) e^{-\frac{\lambda q}{(2L)^d}} dq \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s, X_s = z) ds \\ &+ \int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d}} \frac{1}{(2L)^d} F_S^L(x_L, \lambda/(2L)^d) dq, \end{aligned}$$

whose modulus of the first member does not exceed

$$\int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d} - \gamma q} \int_0^\infty e^{-\frac{\lambda s}{(2L)^d}} \mathbb{P}_S^{x_L,0}(T_L = s) ds \leq C \exp(-\gamma t_L)$$

by (2.12). The claim follows since for L sufficiently large

$$\left| \frac{1}{(2L)^d} \int_{t_L}^\infty e^{-\frac{\lambda q}{(2L)^d}} dq - \frac{1}{\lambda} \right| < \varepsilon/2.$$

□

Theorem 3.9 *Let*

$$b_\lambda^L(S) := \left| F_S^L\left(x_L, \frac{\lambda}{(2L)^d}\right) - \frac{G_{\mathcal{B}}^{ev}(+(x_L)) + \frac{1}{\lambda} - \mathbb{1}_{\{0\}}(+(x_L))}{G_{\mathcal{B}}^{ev}(0) + \frac{1}{\lambda}} \right|.$$

Then $b_\lambda^L \to 0$ in probability, for each $K \in \mathcal{K}$ uniformly in $x_L \in \Lambda(L)$ and $\lambda \in K$, namely for all $\varepsilon > 0$ $(S : b_\lambda^L(S) \leq \varepsilon, \forall \lambda \in K) \supset Q^L \cap I(0, t_L^2) \cap A_{\varepsilon/2}^L(K)$ for all L sufficiently large. Moreover for all $\varepsilon > 0$, $(S : b_\lambda^L(S) \leq \varepsilon, \forall \lambda \in K) \supset Q^L \cap (S : d_S(0, x_L) > t_L^2) \cap A_{\varepsilon/2}^L(K)$ for all L sufficiently large.

Proof. Note that by Dominated Convergence Theorem and a change of variables, for each $\varepsilon > 0$ and $L > \widetilde{L}$ large enough

$$\left| \int_0^{t_L} e^{-\frac{\lambda q}{(2L)^d}} \mathbb{P}^0_{\mathcal{B}}(X_{2q} = 0) dq - G^{ev}_{\mathcal{B}}(0) \right| < \varepsilon. \tag{3.13}$$

Consider

$$G^L_S(x_L, \lambda/(2L)^d) - 1\!\!1_{\{0\}}(+(x_L)) - \left(G^{ev}_{\mathcal{B}}(0) + \frac{1}{\lambda} \right) F^L_S(x_L, \lambda/(2L)^d).$$

Writing $G^L_S(x_L, \lambda/(2L)^d) = 1\!\!1_{\{0\}}(+(x_L)) + H_1 + H_2 + H_3$, using Lemmas 3.5, 3.7, 3.8 and (3.13) follows that the previous difference is smaller than ε when L is sufficiently large and $S \in Q^L \cap I(0, t_L^2) \cap A^L_{\varepsilon/2}(K)$ or $S \in Q^L \cap (S : d_S(0, x_L) > t_L^2) \cap A^L_{\varepsilon/2}(K)$. By Theorem 3.3 we have the conclusion. □

Theorem 3.10 *For each $\varepsilon > 0$, $K \in \mathcal{K}$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $\lambda \in K$ and for each sequence $\{x_L\}_L$ such that $x_L \in \Lambda(L)$,*

$$\left| F^L \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G^{ev}_{\mathcal{B}}(+(x_L)) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{G^{ev}_{\mathcal{B}}(0) + \frac{1}{\lambda}} \right| \leq \varepsilon.$$

Proof. To keep notation simple we deal only with the case $+(x_L) \neq 0$ (the case $+(x_L) = 0$ is completely analogous). Let

$$Q^L_{\varepsilon,\lambda} = \left\{ S : b^L_\lambda \leq \varepsilon \right\}, \tag{3.14}$$

(b^L_λ was defined in Theorem 3.9). By Theorem 3.9 there exists $\widetilde{L}$ such that for all $L \geq \widetilde{L}$ we have $\mathbf{P}(Q^L_{\varepsilon,\lambda}) > 1 - \varepsilon$.
Then since both $F^L_S(x_L, \lambda/(2L)^d)$ and $(G^{ev}_{\mathcal{B}}(+(x_L)) + 1/\lambda)/(G^{ev}_{\mathcal{B}}(0) + 1/\lambda)$ are in $[0, 1]$, for all $L \geq \widetilde{L}$

$$\sum_S \mathbf{P}(S) \left| F^L_S \left(x_L, \frac{\lambda}{(2L)^d} \right) - \frac{G^{ev}_{\mathcal{B}}(+(x_L)) + \frac{1}{\lambda}}{G^{ev}_{\mathcal{B}}(0) + \frac{1}{\lambda}} \right| \leq 2\mathbf{P}((Q^L_{\varepsilon,\lambda})^c) + \varepsilon \leq 3\varepsilon.$$

□

Remark 3.11 *Clearly if*

$$\frac{G^{ev}_{\mathcal{B}}(+(x_L)) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L))}{G^{ev}_{\mathcal{B}}(0) + \frac{1}{\lambda}}$$

has a limit $f(\lambda)$ then we have that $F^L \left(x_L, \frac{\lambda}{(2L)^d} \right)$ has limit $f(\lambda)$. Regarding F^L_S we can have existence of the limit provided that the sequence of small worlds S is chosen wisely. Indeed let $K_n = [1/n, +\infty)$. For all n we know that there exists L_n such that $\mathbf{P}(S : b^L_\lambda(S) \leq 1/n, \forall \lambda \in K_n)$ for all $L \geq L_n$. Thus if for all $L \in [L_n, L_n + 1)$ we choose $S \in (S : b^L_\lambda(S) \leq 1/n, \forall \lambda \in K_n)$ we get that also $F^L_S \left(x_L, \frac{\lambda}{(2L)^d} \right)$ has limit $f(\lambda)$ for all $\lambda > 0$ (uniformly with respect to x_L if $(G^{ev}_{\mathcal{B}}(+(x_L)) + \frac{1}{\lambda} - \mathbf{1}_{\{0\}}(+(x_L)))/(G^{ev}_{\mathcal{B}}(0) + \frac{1}{\lambda})$ converges uniformly with respect to x_L).

Remark 3.12 *In discrete time one can show the same results with $2t_L$ instead of t_L^2 and constant $\widetilde{G}_{\mathcal{B}}(+(x_L))$. Since at each time each random walk moves once, they cannot meet in a time smaller than a half of the initial distance, and the proofs are similar but easier.*

4 Hitting time of random walks

4.1 The limit in law of $T_L/(2L)^d$

It is clear that, if $G^{ev}_{\mathcal{B}}(+(x_L))$ has a limit as L goes to infinity, then Theorems 3.9 and 3.10 provide the results for the limit of $F^L_S(x_L, \lambda/(2L)^d)$ and $F^L(x_L, \lambda/(2L)^d)$. The limit of $G^{ev}_{\mathcal{B}}(+(x_L))$ exists for instance in two particular cases: $x_L = x$ for all L sufficiently large, or $|x_L| \to \infty$. In the first case clearly $\lim_{L\to\infty} G^{ev}_{\mathcal{B}}(+(x_L)) = G^{ev}_{\mathcal{B}}(+(x))$.
In the second case, $G^{ev}_{\mathcal{B}}(+(x_L))$ converges to 0 by the Dominated Convergence Theorem. Similar remarks hold in discrete time case.
We are now ready to prove Theorem 1.1.

Proof. We prove the claim in continuous time. The proof in discrete time works in a similar way.

1. By Theorem 3.10 we know that for all $\lambda > 0$

$$F^L\left(x_L, \frac{\lambda}{(2L)^d}\right) \overset{L\to\infty}{\to} \frac{\lambda G^{ev}_{\mathcal{B}}(+(x)) + 1 - \lambda \mathbf{1}_{\{0\}}(+(x))}{\lambda G^{ev}_{\mathcal{B}}(0) + 1}. \tag{4.1}$$

   Since for each L, F^L is a monotone function of λ and so is the right hand side of (4.1), which is also continuous in λ, it follows that (4.1) holds uniformly in $\lambda \geq 0$.

   Thus, if $x \neq 0$, $T_L/(2L)^d$ converges in law (with respect to $\mathbb{P}^{x,0}$) to

$$\frac{G^{ev}_{\mathcal{B}}(+(x))}{G^{ev}_{\mathcal{B}}(0)}\delta_0 + \left(1 - \frac{G^{ev}_{\mathcal{B}}(+(x))}{G^{ev}_{\mathcal{B}}(0)}\right) \exp\left(\frac{1}{G^{ev}_{\mathcal{B}}(0)}\right),$$

   while if $x = 0$ then it converges to

$$\left(\frac{1}{G^{ev}_{\mathcal{B}}(0)}\right)\delta_0 + \frac{1}{G^{ev}_{\mathcal{B}}(0)} \exp\left(\frac{1}{G^{ev}_{\mathcal{B}}(0)}\right).$$

   Then (1.2) and (1.3) hold, and by monotonicity they hold uniformly in $t \geq 0$.

2. It follows as in the previous step using the fact that $G^{ev}_{\mathcal{B}}(+(x_L)) \to 0$ uniformly in $\{x_L\}_L$ such that $|x_L| \geq \alpha_L$. Indeed $G^{ev}_{\mathcal{B}}(+(x_L)) = \int_0^\infty \mathbb{P}^0_{\mathcal{B}}(X_{2t} = +(x_L))dt$ goes to 0 by the Dominated Convergence Theorem since $\mathbb{P}^0_{\mathcal{B}}(X_{2t} = +(x_L)) \leq \mathbb{P}^0_{\mathcal{B}}(X_{2t} = 0)$ and $\int_0^\infty \mathbb{P}^0_{\mathcal{B}}(X_{2t} = 0)dt \leq G_{\mathcal{B}}(0) < \infty$.

3. As said in Remark 3.11, choosing for all $L \in [L_n, L_{n+1})$ the corresponding set of small worlds $S \in H^L = (b^L_\lambda(S) \leq 1/n, \forall \lambda \in [1/n, \infty))$, we have that for all $\lambda > 0$

$$F^L_S\left(x_L, \frac{\lambda}{(2L)^d}\right) \overset{L\to\infty}{\to} \frac{\lambda G^{ev}_{\mathcal{B}}(+(x)) + 1}{\lambda G^{ev}_{\mathcal{B}}(0) + 1}. \tag{4.2}$$

   This, by an argument as in step 1, proves that

$$\mathbb{P}^{x_L,0}_S\Big(\frac{T_L}{(2L)^d} > t\Big) \overset{L\to\infty}{\to} \Big(1 - \frac{G^{ev}_{\mathcal{B}}(+(x))}{G^{ev}_{\mathcal{B}}(0)}\Big) \exp\Big(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\Big),$$

   uniformly in $t \geq 0$. Thus if $L \in [L_n, L_{n+1})$, the event in (1.5) contains H^L and $\mathbf{P}(H^L) \overset{n\to\infty}{\to} 1$ implies the assertion.

4. Choosing S as in previous step, uniformly with respect to $\{x_L\}$ such that either $|x_L| \geq \alpha_L$ or $d_S(0, x_L) \geq \alpha_L$ we get

$$\mathbb{P}_S^{x_L,0}\Big(\frac{T_L}{(2L)^d} > t\Big) \stackrel{L\to\infty}{\to} \exp\left(-\frac{t}{G_{\mathcal{B}}^{ev}(0)}\right),$$

uniformly in $t \geq 0$. This proves the claim.

□

Remark 4.1 *Theorem 1.1.4 holds if we fix $0 \in \Lambda(L)$ and we consider the supremum over all possible $x_L \in \Lambda(L)$ such that $d_S(x_L, 0) \geq \alpha_L$. We can repeat the same proof to show that the result still holds if we take the supremum over all possible pairs $(x_L, y_L) \in \Lambda(L) \times \Lambda(L)$ such that $d_S(x_L, y_L) \geq \alpha_L$. Namely, let $\alpha_L > t_L^2$ and $t \geq 0$, then for all $\varepsilon > 0$*

$$\mathbf{P}\Big(S \in \mathcal{S}^L(\widetilde{\Omega}) : \sup_{(x_L,y_L)\in\Lambda(L)^2 : d_S(x_L,y_L)\geq\alpha_L} \Big|\mathbb{P}_S^{x_L,y_L}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G_{\mathcal{B}}^{ev}(0)}\right)\Big| > \varepsilon\Big) \stackrel{L\to\infty}{\to} 0, \quad (4.3)$$

We observe that the same technique we employed to determine the asymptotic behaviour of the first encounter time of two random walkers, one starting at x_L and the other at 0, may be used to obtain similar results for the first time that a single random walker starting at x_L hits 0.

Theorem 4.2 *Let W_L be the first time that a random walk starting at x_L hits 0 either in discrete or in continuous time. Then Theorem 1.1 still holds with constant $G_{\mathcal{B}}(x)$ instead of $G_{\mathcal{B}}^{ev}(x)$.*

Proof. (Discrete time) The proof is analogous to the one of Theorem 1.1 but easier, since we consider the return time of one single walk. Notice that the constant is the expected number of visits to 0 of the discrete time random walk on the big world starting at 0.
(Continuous time) A standard approach (for instance use Slutsky theorems) allows to get the result starting from the one in discrete time. □

Remark 4.3 *As a corollary of Theorems* 1.1 *and* 4.2 *one can get a similar convergence result for random walkers starting from the stationary distribution π. The key is that the initial distance between the random walk and the origin (resp. between two random walks) is larger than t_L^2 with probability which converges to* 1 *as L goes to infinity, so that we are under hypothesis of Theorem 1.1 either* 2)*, in the annealed case, or* 4) *in the quenched one.*

5 Coalescing random walk on small world

The goal of this section is to prove a convergence result for coalescing random walk of n particles on BC small world. From now on we work on the continuous time process.

Let $\mathcal{I}(n) = \{\{x_1, \dots, x_n\} : x_i \in \Lambda(L), x_i \neq x_j\}$. Given $A \in \mathcal{I}(n)$, let $\{(X_t^S(x_i))_{t\geq 0}\}_{x_i\in A}$ be a family of independent random walks on small world $S \in \mathcal{S}^L$ such that $X_0^S(x_i) = x_i$. We define for each $(x_i, x_j) \in \Lambda(L) \times \Lambda(L)$ and $S \in \mathcal{S}^L$

$$\tau_S(i,j) := \inf\{s > 0 : X_s^S(x_i) = X_s^S(x_j)\} \tag{5.1}$$

and for each $A \in \mathcal{I}(n)$

$$\tau_S(A) := \inf_{\{x_i,x_j\}\subseteq A}\{\tau_S(i,j)\}. \tag{5.2}$$

Let $\{\xi_t^S(A)\}_{t\geq 0}$ be the coalescing random walk starting from $A \in \mathcal{I}(n)$ on $S \in \mathcal{S}^L$, that is the process of n independent random walks subjected to the rule that when two particles reach the same site they coalesce to one particle. Let $|\xi_t^S(A)|$ be the number of particles of $\xi_t^S(A)$ at time t. When not necessary we omit the dependence on S and we simply write $\{\xi_t(A)\}_{t\geq 0}$, $X_t(x_i)$, $\tau(i,j)$ and $\tau(A)$.
The Kingman's coalescent is a Markov process $(D_t)_{t\geq 0}$ on $\{0, 1, \dots, n\}$ with transition mechanism

$$n \to n-1 \text{ at rate } \binom{n}{2}.$$

The law $P_n(D_t = k) = q_{n,k}(t)$ is given by

$$q_{n,k}(t) = \sum_{j=k}^{n} \frac{(-1)^{j+k}(2j-1)(j+k-2)!\binom{n}{j}}{k!(k-1)!(j-k)!\binom{n+j-1}{j}} \exp\left(-t\binom{j}{2}\right);$$

$$q_{\infty,k}(t) = \sum_{j=k}^{\infty} \frac{(-1)^{j+k}(2j-1)(j+k-2)!}{k!(k-1)!(j-k)!} \exp\left(-t\binom{j}{2}\right).$$

see for instance [7], [16].
We define

$$\mathcal{A}^L(h,n) := \Big\{A \in \mathcal{I}_n : d(x_i, x_j) > h, \text{ for all } i \neq j\Big\} \tag{5.3}$$

$$\mathcal{A}_S^L(h,n) := \Big\{A \in \mathcal{I}_n : d_S(x_i, x_j) > h, \text{ for all } i \neq j\Big\} \tag{5.4}$$

the set of n-uples with distance larger than h respectively on $\Lambda(L)$ and on a fixed small world S. Notice that $\mathcal{A}_S^L(h,n) \subseteq \mathcal{A}^L(h,n)$ for all $S \in \mathcal{S}^L$. Given $A \in \mathcal{A}^L(h,n)$, we introduce

$$\mathcal{D}(A) := \Big\{S \in \mathcal{E}^L : A \in \mathcal{A}^L(h,n) \setminus \mathcal{A}_S^L(h,n)\Big\}. \tag{5.5}$$

Remember that we focus on the nearest neighbor case, but all results can be extended to the case with neihbourhood structure given by $\mathcal{N}_m^\infty$.
Given a probability measure μ on $\Lambda(L)^n$, we denote by $\mathbb{P}_S^\mu$ the law of the coalescing random walk on S with initial probability μ and transitions ruled by P_S. If $\mu = \delta_A$ with $|A| = n$, we write $\mathbb{P}_S^A$.
We begin from n particles in $A \in \mathcal{A}^L(h,n)$. We prove that by taking a particular $h := h_L$ and L large we get that $A \in \mathcal{A}_S^L(h,n)$ with large probability. We assume

$$i) \quad h_L \geq t_L^2 \qquad ii) \lim_{L\to\infty} \frac{M^{h_L}}{(2L)^d} = 0 \tag{5.6}$$

where $M = (2m+1)^d$ or $M = 2d+1$ depending on the neighbourhood structure we work with. Note that hypothesis (5.6) are satisfied if $h_L = t_L^2$.

Lemma 5.1 *If (5.6) holds, for each $n < \infty$, $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$ and $A \in \mathcal{A}^L(h_L, n)$,*

$$\mathbf{P}\big(\mathcal{D}(A)\big) < \epsilon. \tag{5.7}$$

Proof. Let $A \in \mathcal{A}^L(h_L, n)$. If $S \in \mathcal{D}(A)$, then there exists at least one pair of elements $(x_i, x_j) \in A \times A$, $i \neq j$, such that $d_S(x_i, x_j) < h_L$. By (5.5) and (2.6)

$$\mathbf{P}\big(\mathcal{D}(A)\big) = \mathbf{P}\big(S \in \mathcal{E}^L : \exists (x_i, x_j) \in A \times A : d_S(x_i, x_j) \leq h_L\big) \leq n^2 \frac{CM^{h_L}}{L^d}.$$

Since n is fixed, the claim follows by (5.6) (ii). $\square$

Therefore given $A \in \mathcal{A}^L(h_L, n)$ with large probability $A \in \mathcal{A}^L_S(h_L, n)$.
By Remark 4.1, if $\alpha_L \geq t_L^2$, there exists a sequence $\{\widetilde{H}^L\}_L$ with $\widetilde{H}^L \subseteq \mathcal{S}^L$ such that $\mathbf{P}(\widetilde{H}_L) \overset{L \to \infty}{\to} 1$ and for each sequence $\{S^L\}_L$ with $S^L \in \widetilde{H}^L$

$$\sup_{(x_L, y_L): d_S(x_L, y_L) \geq \alpha_L} \left| \mathbb{P}_S^{x_L, y_L}\left(\frac{T_L}{(2L)^d} > t\right) - \exp\left(-\frac{t}{G^{ev}_{\mathcal{B}}(0)}\right)\right| \overset{L \to \infty}{\to} 0 \tag{5.8}$$

Note that (5.8) still holds for the sequence $\{Q^L \cap \widetilde{H}^L\}_L$ and $\mathbf{P}(Q^L \cap \widetilde{H}_L) \overset{L \to \infty}{\to} 1$. Let $H^L := \widetilde{H}^L \cap Q^L$.

The following lemma states that, starting from 4 particles in a set of small world with large probability, when two particles meet the others are distant.

Lemma 5.2 *Assume (5.6). For each $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $S \in H^L$ and $A \in \mathcal{A}^L_S(h_L, 4)$,*

$$\int_0^\infty \mathbb{P}_S^A\Big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\Big) < \epsilon, \tag{5.9}$$

$$\int_0^\infty \mathbb{P}_S^A\Big(\tau(1,2) \in ds, d_S(X_s(x_3), X_s(x_4)) \leq h_L\Big) < \epsilon. \tag{5.10}$$

Proof. We prove (5.9); (5.10) can be proved in a similar way. We split the integral in two parts. By Theorem 1.1.4 and by (5.6) (ii), for each $\varepsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$

$$\begin{aligned}
\int_0^{\frac{d}{\gamma}\log(2L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\big) &\leq \int_0^{\frac{d}{\gamma}\log(2L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds\big) \\
= 1 - \exp\Big(-\frac{d\log(2L)}{\gamma G^{ev}_{\mathcal{B}}(0)(2L)^d}\Big) + \epsilon/6 < \epsilon/3. &\qquad (5.11)
\end{aligned}$$

The second part is

$$\begin{aligned}
&\int_{\frac{d}{\gamma}\log(2L)}^\infty \mathbb{P}_S^A\big(\tau(1,2) \in ds, d_S(X_s(x_1), X_s(x_3)) \leq h_L\big) \\
&\leq \int_{\frac{d}{\gamma}\log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z: d_S(y,z) \leq h_L} \left|\mathbb{P}_S^{x_3}(X_s = z) - \frac{1}{(2L)^d}\right| \\
&+ \int_{\frac{d}{\gamma}\log(2L)}^\infty \sum_{y \in \Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z: d_S(y,z) \leq h_L} \frac{1}{(2L)^d} := I(1) + I(2).
\end{aligned}$$

Since the number of sites z such that $d_S(y,z) \le h_L$ is at most M^{h_L} for each $y \in \Lambda(L)$, for each L large enough we get

$$I(2) \le \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \mathbb{P}_S^A\big(\tau(1,2) \in ds\big)\frac{M^{h_L}}{(2L)^d} = \mathbb{P}_S^{x_1,x_2}\big(T_L > \frac{d}{\gamma}\log(2L)\big)\frac{M^{h_L}}{(2L)^d} \le \epsilon/3 \quad (5.12)$$

by (5.6) (ii). Note that if $s \ge \frac{d}{\gamma}\log(2L)$ then $e^{-\gamma s} \le \frac{1}{(2L)^d}$; therefore by (2.12) then $I(1)$ is smaller or equal to

$$\int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big) \sum_{z:d_S(y,z)\le h_L} e^{-\gamma s}$$

$$\le \int_{\frac{d}{\gamma}\log(2L)}^{\infty} \sum_{y\in\Lambda(L)} \mathbb{P}_S^A\big(\tau(1,2) \in ds, X_s(x_1) = y\big)\frac{M^{h_L}}{(2L)^d} < \epsilon/3 \quad (5.13)$$

and the claim follows by (5.11), (5.12) and (5.13). □

Remark 5.3 *Since $S \in H^L$, Lemma 5.2 still holds if for all $A \in \mathcal{A}^L(h_L, n)$ we choose $S \in H^L \cap \mathcal{D}(A)^c$. Moreover by Lemma 5.1 and (5.8) such a set has probability which converges to 1 as L goes to infinity.*

We prove that the number of particles in the rescaled coalescing random walk converges in law to the number of particles of a Kingman's coalescent. A similar approach has been used for [7, Theorem 5] and in [9].
We work by induction on the number of particles n. If $n = 2$, the induction basis is given by Theorem 1.1.4. The following lemma shows that the assertion is true before the first collision of two particles.

Lemma 5.4 *Assume (5.6). For each $n \in \mathbb{N}$, $T > 0$, $A \in \mathcal{A}^L(h_L, n)$, and $\epsilon > 0$ there exists $\widetilde{L}$ such that for each $L > \widetilde{L}$, $S \in H^L \cap \mathcal{D}(A)^c$ and $0 \le t \le T$,*

$$\Big|\mathbb{P}_S\big(|\xi_{s_L t}(A)| = n\big) - \exp\Big(-\binom{n}{2}t\Big)\Big| < \epsilon$$

where $s_L := (2L)^d G_{\mathcal{B}}^{ev}(0)$.

Proof. Note that $\mathbb{P}_S(|\xi_{s_L t}(A)| = n)$ and $\exp\Big(-\binom{n}{2}t\Big)$ are non-increasing monotone t functions. We define, for each pair $\{i,j\} \subseteq \{1, 2, \ldots, n\}$,

$$H_t(i,j) := \{\tau(i,j) \le s_L t\}; \qquad q_t = q_t(A) := \mathbb{P}(\tau(A) \le s_L t).$$

For all $S \in H^L \cap \mathcal{D}(A)^c$,

$$\mathbb{P}_S^A\big(H_t(i,j)\big) = \mathbb{P}_S^A\big(\tau = \tau(i,j) \le s_L t\big) + \sum_{\{k,l\}\ne\{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, \tau(i,j) \le s_L t\big) \quad (5.14)$$

Each term of the sum on the right hand side is equal to

$$\int_0^{s_L t} \sum_{y,z} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i,j) \le s_L t\big).$$

By Lemma 5.2 for all L sufficiently large

$$\begin{aligned}
&\int_0^{s_L t} \sum_y \sum_{z: d_S(y,z) \le h_L} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z, \tau(i,j) \le s_L t\big) \\
&\le \int_0^\infty \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, d_S(X_s(x_i), X_s(x_j)) \le h_L\big) \le \varepsilon/(8n^4)
\end{aligned}$$

for all choices of $S \in H^L \cap \mathcal{D}(A)^c$, $\{i,j\} \subseteq \{1, \ldots n\}$ and $t \ge 0$.
We are left with evaluating

$$\int_0^{s_L t} \sum_y \sum_{z: d_S(y,z) > h_L} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z\big) \mathbb{P}_S^{y,z}\big(T_L \le s_L t - s\big).$$

By Remark 4.1, $|\mathbb{P}_S^{y,z}(T_L \le s_L t - s) - 1 + \exp(-t + s/s_L)| < \varepsilon/(8n^4)$ for all L sufficiently large and for all choices of $S \in H^L \cap \mathcal{D}(A)^c$, y and z such that $d_S(y,z) \ge h_L$, $0 \le s \le t$. Then evaluate the remaining part of the integral, it does not differ by more than $\varepsilon/(8n^4)$ from

$$\begin{aligned}
&\int_0^{s_L t} \sum_{y,z} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, X_s(x_i) = y, X_s(x_j) = z\big) \Big(1 - \exp(-t + s/s_L)\Big) \\
&= \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds\big) \Big(1 - \exp(-t + s/s_L)\Big). \qquad (5.15)
\end{aligned}$$

Integrating by parts and changing variables, we get

$$\begin{aligned}
&\int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds\big) \Big(1 - \exp(-t + s/s_L)\Big) \\
&= \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \le s\big) \frac{1}{s_L} \exp\Big(-t + s/s_L\Big) ds \\
&= \int_0^t \mathbb{P}_S^A\big(\tau = \tau(k,l) \le s_L u\big) \exp\Big(-(t-u)\Big) du. \qquad (5.16)
\end{aligned}$$

For all L sufficiently large $|\mathbb{P}_S^A\big(H_t(i,j) \le t\big) - (1 - e^{-t})| \le \varepsilon/(4n^2)$ for all $S \in H^L \cap \mathcal{D}(A)^c$, $(i,j) \subseteq \{1, \ldots n\}$ and $t \ge 0$. Summing over all pairs of i and j on (5.14) and using (5.16)

$$\begin{aligned}
q(t) &= \sum_{\{i,j\}} \mathbb{P}_S^A\big(\tau = \tau(i,j) \le s_L t\big) \\
&= \sum_{i,j} \mathbb{P}_S^A\big(H_t(i,j)\big) - \sum_{\{i,j\}} \sum_{\{k,l\} \ne \{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A\big(\tau = \tau(k,l) \in ds, \tau(i,j) \le s_L t\big) \\
&= \binom{n}{2}(1 - e^{-t}) - \Big(\binom{n}{2} - 1\Big) e^{-t} \int_0^t q(s) e^s ds + R
\end{aligned}$$

where the modulus of R, for all L sufficiently large for all choices of $S \in H^L \cap \mathcal{D}(A)^c$, y and z such that $d_S(y,z) \ge h_L$ and for all $0 \le t \le T$ is smaller than $\varepsilon/2$. We know (see [9, Lemma 2]) that if

$$u^L(t) = \binom{n}{2}(1 - e^{-t}) - \Big(\binom{n}{2} - 1\Big) e^{-t} \int_0^t u^L(s) e^s ds + R$$

then for L large enough $u^L(t)$ does not differ by more than $\varepsilon/2$ from $u(t)$, the solution of

$$u(t) = \binom{n}{2}(1-e^{-t}) - \Big(\binom{n}{2} - 1\Big)e^{-t}\int_0^t u(s)e^s ds$$

which is

$$u(t) = 1 - \exp\Big(-\binom{n}{2}t\Big)$$

and the claim follows. □

We are now ready to prove the final result.

Proof of Theorem 1.2.
We fix $A \in \mathcal{A}^L(h_L, n)$ and we show (1.8) by induction in n. Note that if $k = n$ the proof is given by Lemma 5.4.
Theorem 1.1 gives the result when $n = 2$ for all k (that is $k = 2$) and Lemma 5.4 gives the result for n and $k = n$.
Suppose the result holds for $n-1$ for all k. We have to prove it for n and $k < n$.

$$\begin{aligned}\mathbb{P}_S^A(|\xi_{s_L t}(A)| < k) &= \int_0^{s_L t} \mathbb{P}_S^A(\tau \in ds, |\xi_{s_L t}(A)| < k) \\ &= \int_0^{s_L t} \sum_{B \in I(n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B)\mathbb{P}_S^B(|\xi_{s_L t - s}(B)| < k). \qquad (5.17)\end{aligned}$$

Using Lemma 5.2, if $B \notin \mathcal{A}_S^L(h_L, n-1)$, for all L sufficiently large

$$\begin{aligned}&\int_0^{s_L t} \sum_{B \notin \mathcal{A}_S^L(h_L, n-1)} \mathbb{P}_S^A(\tau \in ds, \xi_s(A) = B)\mathbb{P}_S^B(|\xi_{s_L t - s}(B)| < k) \\ &\leq \sum_{\{i,j\}} \sum_{\{k,l\} \neq \{i,j\}} \int_0^{s_L t} \mathbb{P}_S^A(\tau(i,j) \in ds, d_S(X_s(x_k), X_s(x_l) \leq h_L) < \varepsilon/3\end{aligned}$$

since n is fixed, for each $S \in H^L \cap \mathcal{D}(A)^c$, $t \geq 0$.
Changing variables, setting $s = s_L v$, then (5.17) is equal to

$$\int_0^t \sum_{B \in \mathcal{A}_S^L(h_L, n-1)} \mathbb{P}_S^A(\tau \in s_L dv, \xi_{s_L v}(A) = B)\mathbb{P}_S^B(|\xi_{s_L(t-v)}(B)| < k) + R.$$

where the modulus of R is smaller than $\varepsilon/3$ for all L sufficiently large for all choices of $A \in \mathcal{A}^L(h_L, n)$, $S \in H^L \cap \mathcal{D}(A)^c$, $0 \leq t \leq T$. By induction hypothesis, for all L sufficiently large

$$\big|\mathbb{P}_S^B(|\xi_{s_L(t-s)}(B)| < k) - P_{n-1}(D_{t-s} < k)\big| < \varepsilon/3$$

for $B \in \mathcal{A}_S^L(h_L, n-1)$ and for each $S \in H^L \cap \mathcal{D}(A)^c$ and $0 \leq s \leq t$. Thus the last term of the previous integral differs at most by ε from

$$\int_0^t \mathbb{P}_S^A\left(\frac{\tau}{s_L} \in dv\right) P_{n-1}(D_{t-v} < k) = -\int_0^t \mathbb{P}_S^A\Big(\frac{\tau}{s_L} \leq v\Big)\frac{d}{dv}P_{n-1}(D_{t-v} < k)dv$$

after an integration by parts. Note that $v \to P_{n-1}(D_{t-v}=k)$ is a continuous function; therefore by definition of Kingman's coalescent and because the right hand side $P_n(D_t<k)$ is finite, we get (see [7])

$$\begin{aligned}\mathbb{P}_S^A(|\xi_{s_L t}(A)|<k) =& \sum_{i=1}^{k-1}\int_0^t \binom{n}{2}\exp\Big(-\binom{n}{2}v\Big)P_{n-1}(D_{t-v}=k)dv + R\\ =&\sum_{i=1}^{k-1}P_n(D_t=k)+R = P_n(D_t<k)+R\end{aligned}$$

where the modulus of R, for all L sufficiently large, for all choices of $S\in H^L\cap\mathcal{D}(A)^c$ and $0\le t\le T$ is smaller than ε. □

Remark 5.5 *In Theorem* 1.2 *we fix* $A\in\mathcal{A}^L(h_L,n)$ *and the result holds in a sequence of small world graphs depending on A. One can prove that the same result holds for the sequence* $(H^L)_L$ *uniformly in* $\mathcal{A}_S^L(h_L,n)$ *and* $S\in H^L$.

Remark 5.6 *By summing over all realizations of the small world graph, one can get the annealed result as a corollary of Theorem 1.2.*

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Appendix

A Comparison with the d-dimensional torus

As observed in the introduction, the rescaling factor is $(2L)^d$ both on the d-dimensional torus and on the small world if $d \geq 3$. In order to understand if two particles meet faster, in continuous time we need to compare $G^{ev}_{\mathbb{Z}^d}(0)$ with $G^{ev}_{\mathcal{B}}(0)$. Proposition A.1 gives some information in this direction.

Proposition A.1 *Suppose $d \geq 3$.*
i) There exists $\beta_1 > 0$ such that $G^{ev}_{\mathbb{Z}^d}(0) < G^{ev}_{\mathcal{B}}(0)$ for each $\beta \in [\beta_1, 1]$.
ii) There exists $\beta_2 > 0$ such that $G^{ev}_{\mathbb{Z}^d}(0) > G^{ev}_{\mathcal{B}}(0)$ for each $\beta \in (0, \beta_2]$.

Proof. Since $G^{ev}_{\mathcal{B}}(0) = G_{\mathcal{B}}(0)/2$ and $G^{ev}_{\mathbb{Z}^d}(0) = G_{\mathbb{Z}^d}(0)/2$, we prove that $G_{\mathbb{Z}^d}(0)$ is smaller (resp. larger) than $G_{\mathcal{B}}(0)$ for β large (resp. small) enough.
i) Since $\mathbb{P}^0_{\mathcal{B}}(X_{2n} = 0) \geq \beta^{2n}$ we get

$$G_{\mathcal{B}}(0) \geq \sum_{n=0}^{\infty} \beta^{2n} = \frac{1}{1-\beta^2}.$$

and the claim follows by taking β close to 1 since $G_{\mathbb{Z}^d}(0) < \infty$ if $d \geq 3$.
ii) Given an irreducibile Markov chain (Y, Q), let

$$\widehat{G}(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0) z^n; \qquad \widehat{F}(z) = \sum_{n=0}^{\infty} \mathbb{P}^0(Y_n = 0, Y_s \neq 0 \text{ for all } s < n) z^n.$$

By [18, Proposition 9.10], there exists $r > 0$ and a function $\Phi(\cdot)$ such that

$$\widehat{G}(z) = \Phi(z\widehat{G}(z)), \qquad z \in [0, r). \tag{A.1}$$

Moreover there exists Φ' and Φ'' and $\Phi(\cdot)$ is strictly increasing and strictly convex.
Let P be the transition matrix on the big world defined in Section 2.2 in nearest neighbor case. We denote by $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}$, $\Phi_{\mathbb{Z}^d}$ and $\Phi_{\mathbb{Z}_2}$ the functions which satisfy (A.1) respectively for the Markov chain (X, P) on the big world, for the simple random walk on $\mathbb{Z}^d$ and for the simple random walk on $\mathbb{Z}_2$.
The function $\Phi_{\mathbb{Z}_2}(t)$ can be computed explicitly,

$$\Phi_{\mathbb{Z}_2}(t) = \frac{1}{2}(1 + \sqrt{1 + 4t^2}).$$

By [18, Theorem 9.19]

$$\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}(t) = \frac{1}{2}(1 + \sqrt{1 + 4\beta^2 t^2}) + \Phi_{\mathbb{Z}^d}((1-\beta)t) - 1.$$

By choosing $t = \widehat{G}_{\mathbb{Z}^d * \mathbb{Z}_2}(1)$ we get $\Phi_{\mathbb{Z}^d * \mathbb{Z}_2}(t) = t$ by $(A.1)$. We denote by $\widehat{G}_\beta = \widehat{G}_{\mathbb{Z}^d * \mathbb{Z}_2}(1) = G_{\mathcal{B}}(0)$, then

$$\widehat{G}_\beta = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\beta^2 \widehat{G}_\beta^2} + \Phi_{\mathbb{Z}^d}((1-\beta)\widehat{G}_\beta).$$

Let $\widehat{G} = \widehat{G}_{\mathbb{Z}^d}(1) = G_{\mathbb{Z}^d}(0)$. By $(A.1)$, $\widehat{G}$ is a fixed point of $\Phi_{\mathbb{Z}^d}$, then $\lim_{\beta \to 0} \widehat{G}_\beta = \widehat{G}$. Notice that as $\beta \to 0$

$$\sqrt{1 + 4\beta^2 \widehat{G}_\beta^2} = 1 + 2\beta^2 \widehat{G}_\beta^2 + o(\beta^3 \widehat{G}_\beta^3).$$

and by Taylor series of $\Phi_{\mathbb{Z}^d}$ centered at $\widehat{G}$ with Lagrange form of the remainder:

$$\Phi_{\mathbb{Z}^d}((1-\beta)\widehat{G}_\beta) = \Phi_{\mathbb{Z}^d}(\widehat{G}) + \Phi'_{\mathbb{Z}^d}(\widehat{G}) \left[(1-\beta)\widehat{G}_\beta - \widehat{G}\right] + \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y - \widehat{G})^2,$$

where y is between $\widehat{G}$ and $(1-\beta)\widehat{G}_\beta$. Two useful formulas for Φ' and Φ'' can be found in [18, p.99]:

$$\Phi'(t) = 1/(z + \widehat{G}(z)/\widehat{G}'(z)), \qquad \Phi''(t) = (\widehat{G}(z)/(\widehat{G}(z) + z\widehat{G}'(z)))^3 \widehat{F}''(z),$$

where z is such that $t = z\widehat{G}(z)$. If $t = \widehat{G}$ then

$$\Phi'_{\mathbb{Z}^d}(\widehat{G}) = \frac{\widehat{G}'}{\widehat{G}' + \widehat{G}},$$

where $\widehat{G}' = \frac{d}{dz}\widehat{G}_{\mathbb{Z}^d}(z)|_{z=1^-}$. By convexity $\Phi'' > 0$. Moreover

$$\Phi''_{\mathbb{Z}^d}(\widehat{G}) = \Big(\frac{\widehat{G}}{\widehat{G}+\widehat{G}'}\Big)^3 \widehat{F}''(1).$$

Therefore

$$\widehat{G}_\beta = \beta^2\widehat{G}_\beta^2 + \widehat{G} + \frac{\widehat{G}'}{\widehat{G}'+\widehat{G}}\Big[\widehat{G}_\beta - \widehat{G} - \beta\widehat{G}_\beta\Big] + \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y-\widehat{G})^2 + o(\beta^3\widehat{G}_\beta^3)$$

Since $(y-\widehat{G})^2 \le (\widehat{G}_\beta - \widehat{G} - \beta\widehat{G}_\beta)^2$ we get

$$(\widehat{G}_\beta - \widehat{G})\frac{\widehat{G}}{\widehat{G}'+\widehat{G}} \le \beta^2\widehat{G}_\beta^2 - \beta\widehat{G}_\beta\frac{\widehat{G}'}{\widehat{G}'+\widehat{G}} + C_\beta\Big[(\widehat{G}_\beta-\widehat{G})^2 + \beta^2\widehat{G}_\beta^2 - 2\beta\widehat{G}_\beta(\widehat{G}_\beta-\widehat{G}) + o(\beta^3\widehat{G}_\beta^3)\Big]$$

where $C_\beta = \frac{1}{2}\Phi''_{\mathbb{Z}^d}(y)(y-\widehat{G})^2$. Thus

$$(\widehat{G}_\beta - \widehat{G})\Big[\frac{\widehat{G}}{\widehat{G}'+\widehat{G}} + \Phi''_{\mathbb{Z}^d}(y)\beta\widehat{G}_\beta\Big] \le -\beta\widehat{G}_\beta\frac{\widehat{G}'}{\widehat{G}'+\widehat{G}} + C_\beta(\widehat{G}_\beta-\widehat{G})^2 + \beta^2\widehat{G}_\beta^2(1+C_\beta+o(\beta\widehat{G}_\beta)$$

$$(\widehat{G}_\beta - \widehat{G})\Big[\frac{\widehat{G}}{\widehat{G}'+\widehat{G}} + \Phi''_{\mathbb{Z}^d}(y)\beta\widehat{G}_\beta - C_\beta(\widehat{G}_\beta-\widehat{G})\Big] \le -\beta\widehat{G}_\beta\Big[\frac{\widehat{G}'}{\widehat{G}'+\widehat{G}} + \beta\widehat{G}_\beta(1+C_\beta+o(\beta\widehat{G}_\beta)\Big]$$

By continuity of $\Phi''_{\mathbb{Z}^d}$ we get that $\Phi''_{\mathbb{Z}^d}(y) \overset{\beta\to 0}{\to} \Phi''_{\mathbb{Z}^d}(1)$, then the coefficient of $(\widehat{G}_\beta - \widehat{G})$ on the left hand side as $\beta \to 0$ is asymptotically $\widehat{G}/(\widehat{G}+\widehat{G}')$, which is strictly positive; the coefficient of $\beta\widehat{G}_\beta$ on the right hand side is asymptotically $-\widehat{G}'/(\widehat{G}+\widehat{G}')$, which is strictly negative and the claim follows. $\square$