Università di Milano–Bicocca
Quaderni di Matematica

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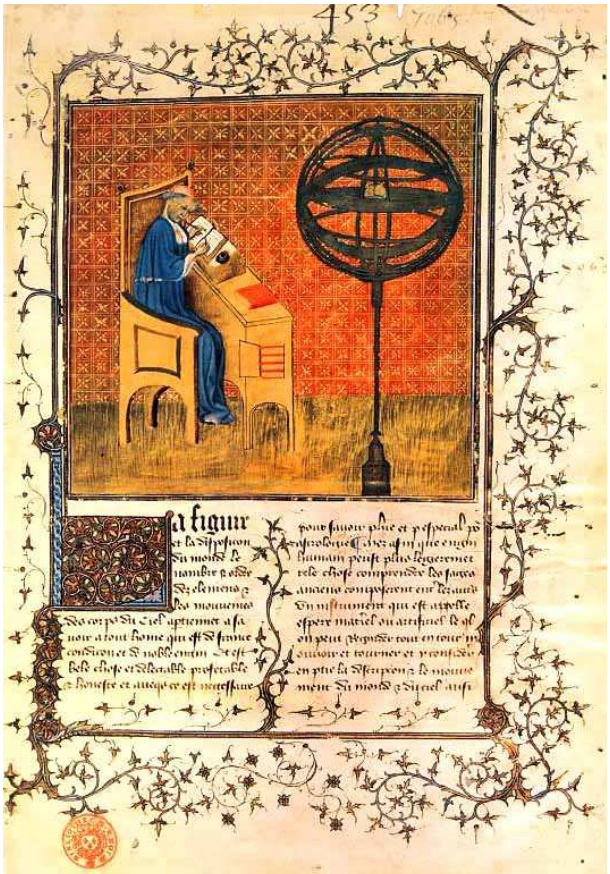
M. A. Pellegrini, M. C. Tamburini

Quaderno n. 1/2010 (arXiv:1001.5125v1)



Stampato nel mese di febbraio 2010
presso il Dipartimento di Matematica e Applicazioni,
Università degli Studi di Milano-Bicocca, via R. Cozzi 53, 20125 Milano, ITALIA.

Disponibile in formato elettronico sul sito www.matapp.unimib.it.
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Hurwitz generation of the universal covering of $\mathrm{Alt}(n)$

M. A. Pellegrini

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi, 53, 20125 Milano Italy
E-mail: marco.pellegrini@unimib.it

M. C. Tamburini

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, via Musei, 41, 25121 Brescia Italy
E-mail: c.tamburini@dmf.unicatt.it

DEDICATED TO JOHN S. WILSON ON THE OCCASION OF HIS 65-TH BIRTHDAY

We prove that the universal covering of an alternating group $\mathrm{Alt}(n)$ which is Hurwitz is still Hurwitz, with 31 exceptions, 30 of which are detectable by the genus formula.

1. INTRODUCTION

A finite group is Hurwitz if it can be generated by two elements of respective orders 2 and 3, whose product has order 7. In [2] M. Conder has constructed a $(2,3,7)$-generating triple of the alternating group $\mathrm{Alt}(n)$, for all $n > 167$, and has indicated the exact values of $n \leq 167$ for which $\mathrm{Alt}(n)$ is Hurwitz: they are displayed in Table 1 below. This result was a key step in the field, and allowed further progress, e.g., the discovery that very many linear groups over f.g. rings are $(2,3,7)$-generated (see [5], [6] and [11], for example). In this paper, applying Conder's method, we prove the following:

THEOREM 1.1. *The universal covering* $\widetilde{\mathrm{Alt}(n)}$ *of an alternating group* $\mathrm{Alt}(n)$ *which is Hurwitz, is still Hurwitz, except for the following values of* n*:*

15	21	22	29	37	45	52	71	79	86	87
94	101	102	109	116	117	124	132	143	151	158
159	166	173	174	181	188	215	223	230		

Apart from the case $n = 21$, these exceptions are due to the failure of inequality (3). This inequality follows from Scott's formula [8] or, equivalently, from the genus formula [3, Corollary page 82]:

$$n = 84(g-1) + 21r + 28s + 36t \tag{1}$$

where $g \geq 0$, and r, s, t are the numbers of fixed points of a Hurwitz generating triple of a transitive group of degree n. Hence these formulas essentially discriminate the alternating group from its universal covering.
For each degree n with positive answer, we exhibit a $(2,3,7)$-generating triple of $\widetilde{\mathrm{Alt}(n)}$, up to an element of its center. Comparison of our generators with those of Conder provides further evidence that the same alternating group may well admit non-conjugate Hurwitz generators.
Our proofs are computer independent, but the algebraic softwares Magma and GAP have been of invaluable help.

Acknowledgements We are very grateful to A. Zalesskii, who suggested this problem, predicting its answer in connection to Scott's formula. We are also indebted to M. Conder who provided Hurwitz generators for Alt(96).

2. PRELIMINARY RESULTS

For the definition and general properties of universal coverings we refer to the book of M. Aschbacher [1, Section 33]. To our purposes it is enough to recall that, for $n \geq 8$, $\widetilde{\mathrm{Alt}(n)}$ is perfect, its center $\widetilde{Z}$ has order 2 and the factor group $\widetilde{\mathrm{Alt}(n)}/\widetilde{Z}$ is isomorphic to $\mathrm{Alt}(n)$.

THEOREM 2.1. *Let $\widetilde{x}$ be a 2-element in the universal covering $\widetilde{\mathrm{Alt}(n)}$, whose image x in $\mathrm{Alt}(n)$ is an involution. Then $\widetilde{x}$ has order 2 if x is the product of $4k$ cycles, has order 4 if x is the product of $4k+2$ cycles.*

The previous result, which is part of Proposition 33.15 in [1], reduces our problem to $\mathrm{Alt}(n)$. Indeed:

COROLLARY 2.1. *$\widetilde{\mathrm{Alt}(n)}$ is Hurwitz if and only if $\mathrm{Alt}(n)$ admits a $(2,3,7)$-generating triple (x, y, xy) in which x is the product of $4k$ cycles.*

Proof. Let x, y be as in the statement. Any preimage $\tilde{x}$ of x in $\widetilde{\mathrm{Alt}(n)}$ has order 2 by Theorem 2.1. Clearly y has a preimage $\tilde{y}$ of order 3. If $(\tilde{x}\tilde{y})^7 = -I$, the central involution of $\widetilde{\mathrm{Alt}(n)}$, we substitute $\tilde{x}$ with $-\tilde{x}$ so that $(-\tilde{x}\tilde{y})^7 = I$. As both groups $\langle \tilde{x}, \tilde{y} \rangle$ and $\langle -\tilde{x}, \tilde{y} \rangle$ map onto $\mathrm{Alt}(n)$, each of them coincides with the group $\widetilde{\mathrm{Alt}(n)}$, as it is perfect . We conclude that this group admits a $(2,3,7)$–generating triple. The converse is obvious. ▌

Table 1 shows the values of $n < 168$ for which $\mathrm{Alt}(n)$ is Hurwitz: this classification appears in [2] (mention of 139, which does not satisfy (2), is omitted there).

Table 1

	15						21	22					
28	29						35	36	37				
42	43		45				49	50	51	52			
56	57	58					63	64	65	66			
70	71	72	73				77	78	79	80	81		
84	85	86	87	88			91	92	93	94		96	
98	99	100	101	102			105	106	107	108	109		
112	113	114	115	116	117		119	120	121	122	123	124	
126	127	128	129	130		132	133	134	135	136	137	138	
140	141	142	143	144	145		147	148	149	150	151	152	153
154	155	156	157	158	159	160	161	162	163	164	165	166	

3. NEGATIVE RESULTS

Assume that $\mathrm{Alt}(n)$ is Hurwitz. It follows that

$$2\left[\frac{n}{4}\right]+2\left[\frac{n}{3}\right]+6\left[\frac{n}{7}\right]\geq 2n-2. \qquad (2)$$

Similarly, if $\widetilde{\mathrm{Alt}(n)}$ is Hurwitz, then

$$4\left[\frac{n}{8}\right]+2\left[\frac{n}{3}\right]+6\left[\frac{n}{7}\right]\geq 2n-2. \qquad (3)$$

These inequalities follow almost immediately from (1), but also from Scott's formula (details can be found in [9, page 399]). To this respect, it is useful to note that an involution $x \in \mathrm{Alt}(n)$ is the product of $\ell \leq 2[\frac{n}{4}]$ disjoint 2-cycles and, if x is the image an involution of $\widetilde{\mathrm{Alt}(n)}$, then $\ell \leq 4[\frac{n}{8}]$.
The values of n for which $\mathrm{Alt}(n)$ is Hurwitz, but do not satisfy (3) are:

15	22	29	37	45	52	71	79	86	87	94	101	102	109	116
117	124	132	143	151	158	159	166	173	174	181	188	215	223	230

Lemma 3.1. *The covering* $\widetilde{\mathrm{Alt}(21)}$ *is not Hurwitz.*

Proof. By contradiction let (x, y, xy) be the image in $\mathrm{Alt}(n)$ of a $(2,3,7)$-generating triple of $\widetilde{\mathrm{Alt}(21)}$. It follows that x fixes at least 5 points (actually 5 by the genus formula). Let $\mathbb{C}^{21}$ be the natural permutational module for $\mathrm{Alt}(n)$ and V its irreducible 20-dimensional component. Consider the diagonal action of $H = \langle x, y\rangle$ on the symmetric square S of V and, for $h \in H$, denote by d_S^h the dimension of the space of points fixed by h. Then Scott's formula gives:

$$d_S^x + d_S^y + d_S^{xy} \leq \frac{20 \cdot 21}{2} + 2.$$

Again, for details see [10, Lemma 2.2] or [9, page 400]. On the other hand we have $d_S^x \geq 114$, $d_S^y \geq 70$ and $d_S^z \geq 30$, whence the contradiction $114 + 70 + 30 = 214 \leq 212$. ∎

4. PROOF OF THE RESULT FOR ALMOST ALL DEGREES

Let $T(2,3,7) = \left\langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1\right\rangle$ be the infinite triangle group. In Conder's paper a permutation representation $\mu : T(2,3,7) \to \mathrm{Alt}(m)$ is depicted by a diagram M, say, with m vertices. It will be convenient to say that $x = \mu(X)$ and $y = \mu(Y)$ are defined by M. Assume that two vertices $j \neq k$ of M form an (i)-handle, for some $i \leq 6$. This means that j and k are fixed by x and that $(xy)^i$ takes j to k. The following property is used repeatedly. Let $\mu' : T(2,3,7) \to \mathrm{Alt}(m')$ be another representation, depicted by M'. If M' has an (i)-handle j', k', one obtains a new representation $T(2,3,7) \to \mathrm{Alt}(m + m')$ by extending the action:

$$X \mapsto \mu(x)\mu'(x)(j, j')(k, k'), \quad Y \mapsto \mu(y)\mu'(y).$$

The diagram which depicts this representation is denoted by $M(i)M'$.
So the starting point of [2] is a list of basic diagrams. The corresponding transitive permutation representations that will be used here are given explicitly in [9, Appendix A] and [11, Appendix A].

Lemma 4.1. *In the notation of* [9]*, let* x, y *be defined by diagram* G*, with vertices* $\{1, \ldots, 42\}$ *and* (1)*-handles* $\{2, 3\}$, $\{14, 15\}$, $\{32, 33\}$*. Set*

$$x' = x\,(14, 32)(15, 33).$$

Then the product $x'y$ *and the commutator* (x', y) *are respectively conjugate to* xy *and* (x, y)*.*

Proof. Direct calculation shows that $(15, 33)$ conjugates xy to $x'y$, and $(35, 17, 31, 32, 34, 16, 37, 28, 30, 21, 20, 8, 18, 25, 10, 27, 23, 24, 41)$ conjugates (x, y) to (x', y). We note that the cycle structure of both commutators (x, y) and (x', y) are $(2, \ldots, 1, \ldots)(14, \ldots, 13, \ldots)(32, \ldots, 31, \ldots)1^3$, where each non-trivial cycle has length 13. ▌

As $x'y$ has order 7, we may denote by G' the diagram which depicts the representation $X \mapsto x'$, $Y \mapsto y$ of the previous Lemma.

In Table 2 we list each basic diagram that will be needed, with its degree, and the number m of 2-cycles of the corresponding involution x. When a suitable power of the commutator (x, y) is a cycle of prime length p, we indicate explicitly this prime (also called "useful"). We use the notation of [9, Appendix A] for the diagrams called G, A, E, H_i $(0 \leq i \leq 13)$, the notation of [2] for B, C, D, J and that of [11, Appendix A] for the remaining ones.

Table 2

Diagram	deg	m	p	Diagram	deg	m	p
A	14	6		C	21	8	
B	15	6		E	28	12	
D	22	10		G'	42	20	
G	42	18		H_2	142	68	23
H_0	42	18	17	H_3	115	56	17
H_1	57	26	5	H_5	187	92	43
H_4	144	70	17	H_7	77	36	17
H_6	216	106	5	H_8	36	16	5
H_{10}	136	66	5	H_9	135	64	19
J	72	34		H_{11}	165	80	19
O	7	2		H_{12}	180	88	47
P	15	6		H_{13}	195	96	23
R	22	10		Q	21	8	
				S	36	16	
				T	66	32	

For each H_i in Table 2, we define three composite diagrams, namely:

$$\Omega_0^i = H_i(1)G, \quad \Omega_1^i = {}^{H_i(1)}_{A(1)}G, \quad \Omega_2^i = {}^{H_i(1)}_{E(1)}G.$$

Here we mean that Ω_1^i is obtained by two joins. The first is done via the (1)-handle $\{2,3\}$ of G and the (1)-handle of H_i. The second via the (1)-handle $\{14,15\}$ of G and the (1)-handle of A. Similarly for Ω_2^i.
We are now ready to prove Theorem 1.1 for all values of n of shapes:

$$\begin{aligned} &n = 42 + d, \\ &n = 42r + 14s + d, \;\; r \geq 2, \;\; s = 0,1,2, \end{aligned} \tag{4}$$

where d is the degree of the unique diagram H_i such that $n \equiv i \pmod{14}$. For any such n, there exists a composite diagram of degree n: e.g. one of the diagrams Ω_0^i or

$$\Omega_j^i(1)\underbrace{G(1)\ldots(1)G}_{r-1 \text{ times}}, \quad j := 0,1,2.$$

If x, y are defined by this diagram, then $\langle x, y\rangle$ is a primitive subgroup of $\mathrm{Alt}(n)$ and a power of the commutator (x, y) is a p-cycle of prime length $p \leq n-3$ (see [2]). By a result of Jordan [7], $\langle x, y\rangle = \mathrm{Alt}(n)$.
Let m be the number of 2-cycles of x. If $m \equiv 0 \pmod 4$, by Corollary 2.1 the group $\widetilde{\mathrm{Alt}(n)}$ is Hurwitz. Otherwise we may consider the diagram obtained substituting the last copy of G by G'. The number of 2-cycles of the involution x' defined by this modified diagram is $m+2 \equiv 0 \pmod 4$. It follows from Lemma 4.1 that the cycle structure of (x', y) is the same of (x, y). This allows to conclude that $\langle x', y\rangle = \mathrm{Alt}(n)$ by the same argument used for $\langle x, y\rangle$.
Note that every $n \geq 300$ and the values listed below have shape (4).

78	84	99	119	120	126	134	140	141	148	154	155	157	161
162	168	169	175	176	177	178	182	183	184	186	189	190	196
197	199	203	204	207	210	211	213	217	218	219	220	222	224
225	226	227	228	229	231	232	233	234	237	238	239	240	241
242	245	246	247	248	249	252	253	254	255	256	258	259	260
261	262	263	264	266	267	268	269	270	271	273	274	275	276
277	278	279	280	281	282	283	284	285	287	288	289	290	291
292	293	294	295	296	297	298	299						

5. THE REMAINING CASES

In this Section, for each remaining degree n, we give a diagram which defines a $(2,3,7)$ triple (x, y, xy) such that x is the product of $m \equiv 0 \pmod 4$ disjoint 2-cycles, $\langle x, y\rangle$ is a primitive subgroup of $\mathrm{Alt}(n)$ which

contains a p-cycle (called useful) of prime length $p \leq n-3$. As above we conclude that the group $\mathrm{Alt}(n)$ is Hurwitz.
We first consider diagrams which involve an H_i, for some i with $0 \leq i \leq 13$. In accordance with Table 2, it is convenient to split this interval in the two subsets:

$$I_1 = \{0, 1, 4, 6, 10\}, \quad I_2 = \{2, 3, 5, 7, 8, 9, 11, 12, 13\}.$$

So, for the number m of 2-cycles defined by an H_i, we have $m \equiv 2 \pmod 4$ if $i \in I_1$ and $m \equiv 0 \pmod 4$ if $i \in I_2$.
Hence, for each $i_1 \in I_1$ and each $i_2 \in I_2$, we consider the diagrams:

$$H_{i_1}(1)E, \quad H_{i_2}, \quad O(1)H_{i_2}, \quad A(1)H_{i_2}, \quad R(1)H_{i_2}$$

of respective degrees $\deg(H_{i_1}) + 28$, $\deg(H_{i_2}) + k$ $(k = 0, 7, 14, 22)$.
They provide the following values of n (omitting those already obtained):

$$\begin{array}{cccccccccccccc} 36 & 43 & 50 & 58 & 70 & 77 & 85 & 91 & 115 & 122 & 129 & 135 & 137 & 142 \\ 149 & 156 & 164 & 165 & 172 & 179 & 180 & 187 & 194 & 195 & 201 & 202 & 209 & 244. \end{array}$$

Similarly, for $i_1 \in I_1$ and $i_2 \in I_2$, the diagrams

$$\substack{H_{i_1}(1)\\ E(1)}G, \quad \substack{H_{i_2}(1)\\ A(1)}G, \quad P(1)G(1)H_{i_2}, \quad \substack{A(1)\\ A(1)}G(1)H_{i_2}$$

of respective degrees $\deg(H_{i_1}) + 42 + 28$, $\deg(H_{i_2}) + 42 + k$ $(k = 14, 15, 28)$ give the new values:

$$\begin{array}{ccccccccccccc} 92 & 93 & 106 & 112 & 127 & 133 & 147 & 171 & 185 & 191 & 192 & 198 & 205 \\ 206 & 212 & 214 & 221 & 235 & 236 & 243 & 250 & 251 & 257 & 265 & 286. & \end{array}$$

Moreover the diagrams

$$P(1)H_3, \;\; P(1)H_9, \;\; \substack{R(1)\\ H_8(1)}G, \;\; \substack{A(1)\\ P(1)}G(1)H_8, \;\; \substack{A(1)\\ R(1)}G(1)H_8, \;\; \substack{P(1)\\ P(1)}G(1)H_8,$$

give:

$$100 \;\; 107 \;\; 108 \;\; 114 \;\; 130 \;\; 150.$$

For each of these diagrams, a suitable power of the commutator (x, y) is the p-cycle listed in Table 2, associated to the H_i involved by the diagram.
Next we consider the diagrams listed in Table 3, where the p-cycle is the word described in the fourth column.

Table 3

n	Diagram	p	p-cycle
66	T	47	$(xy^2xyxy^2xyxy)^{44}$
28	$O(1)Q$	13	$(xy^2xyxyxy^2)^{24}$
35	$O(1)E$	17	$(xy^2xyxy^2xy^2xy^2xyxy)^{77}$
42	$A(1)E$	11	$(xy^2xyxy^2xyxy^2xyxy)^{60}$
49	$O(1)G'$	19	$(x,y)^{13}$
51	$P(1)H_8$	11	$(x,y)^{100}$
57	$P(1)G'$	23	$(xy^2xy^2xy^2xyxyxy^2xy^2xyxy)^{70}$
63	$G(1)C$	13	$(xy^2xyxy^2xyxy^2xyxy)^{210}$
64	$R(1)G'$	17	$(xyxy^2xyxyxy^2xyxy^2)^{30}$
73	$O(1)T$	47	$(xyxy^2xy^2xy^2xy^2xyxyxy^2xyxy^2xyxy^2xy^2)^{84}$
80	$A(1)T$	23	$(xy^2xyxyxy^2xyxy^2xyxy^2xyxy^2xy)^{168}$
81	$P(1)T$	67	$(xyxy^2xy^2xyxy^2xyxy^2xy^2xyxyxy^2xy)^{7}$
88	$R(1)T$	71	$(xyxy^2xy^2xyxyxy^2xyxy^2xy)^{12}$
123	$H_1(1)T$	23	$(xyxy^2xy^2xyxy^2xy^2xy)^{1872}$
138	$J(1)T$	13	$(xyxy^2xy^2xy)^{228}$
105	$C(1)G(1)G$	19	$(xyxy^2xy^2xy^2xyxy^2xy^2xyxy)^{210}$
121	$O(1)J(1)G'$	17	$(x,y)^{17160}$
128	$A(1)J(1)G'$	17	$(xyxy^2xy^2xyxyxy^2xyxy^2xy)^{390}$
136	$R(1)J(1)G'$	23	$(xyxy^2xyxy^2xyxy^2xy)^{11970}$
145	$O(1)J(1)T$	17	$(x,y)^{8360}$
152	$A(1)J(1)T$	83	$(xyxy^2xy^2xyxyxy^2xy^2xyxy^2xyxy^2)^{828}$
153	$P(1)J(1)T$	53	$(xyxy^2xy^2xyxy^2xyxy^2)^{690}$
160	$R(1)J(1)T$	11	$(xyxy^2xy^2xyxyxy^2xyxy^2xy)^{9300}$
163	$A(1)J(1)H_7$	17	$(x,y)^{3960}$
98	$\substack{A(1)\\A(1)}G(1)E$	19	$(xyxy^2xy^2xyxy^2xy^2xyxy)^{660}$
113	$\substack{A(1)\\P(1)}G(1)G'$	23	$(xyxy^2xy^2xy^2xyxy^2xyxy^2xyxyxy)^{70}$
144	$\substack{A(1)\\R(1)}G(1)T$	61	$(xyxy^2xyxy^2xyxy^2xy)^{690}$
170	$\substack{A(1)\\J(1)}G(1)G'$	23	$(xyxy^2xyxy^2xyxy)^{5460}$
193	$\substack{A(1)\\R(1)}G(1)H_3$	29	$(xyxy^2xyxy^2xyxy)^{2520}$
200	$\substack{R(1)\\R(1)}G(1)J(1)G'$	47	$(xyxy^2xy^2xyxy^2xyxy^2)^{6930}$
208	$\substack{A(1)\\A(1)}G(1)J(1)T$	7	$(xyxy^2xyxy^2xyxyxy^2)^{150}$
216	$\substack{A(1)\\R(1)}G(1)J(1)T$	7	$(xyxy^2xyxy^2xyxy^2xy)^{330}$
272	$\substack{R(1)\\R(1)}G(1)J(1)J(1)G'$	17	$(xyxy^2xyxyxy^2xyxy^2xy^2xy^2xy)^{155610}$

Using the (2)-handles of the diagrams B, D and S we solve the cases $n = 65, 72$. Indeed, we construct the diagrams $B(2)S(1)A$ and $D(2)S(1)A$: it turns out that in the first case the word

$$(xyxy^2xy^2xyxyxy^2xy^2xyxy^2xy)^3$$

is a 59-cycle and in the second case the word

$$(xyxy^2xy^2xyxy^2xy^2xyxy^2xyxy^2xyxy)^{140}$$

is a 41-cycle. We conclude with the last two cases $n = 56, 96$, giving explicitly the generators.

$\underline{n = 56:}$

$$\begin{aligned} x &= (1,52)(2,6)(3,7)(4,53)(5,9)(8,12)(10,15)(11,13)(14,18)(16,21)\\ &\quad (17,22)(19,24)(20,34)(23,27)(25,30)(26,32)(28,33)(29,41)(31,36)\\ &\quad (35,54)(37,42)(38,40)(39,45)(43,48)(44,49)(46,51)(47,56)(50,55).\\ y &= \prod_{i=0}^{16}(3i+1, 3i+2, 3i+3). \end{aligned}$$

Useful prime: $p = 41$; p-cycle: $(xyxyxy^2xy^2xyxy^2xyxy^2xy^2)^{13}$.

$\underline{n = 96}$ ([4]):

$$\begin{aligned} x &= (1,2)(3,4)(5,7)(6,10)(8,13)(9,16)(11,19)(12,14)(15,22)(17,25)\\ &\quad (18,28)(20,23)(21,31)(24,30)(26,34)(27,37)(29,35)(32,33)(36,40)\\ &\quad (38,43)(39,46)(41,48)(42,49)(44,52)(45,55)(47,58)(50,56)(51,53)\\ &\quad (54,61)(57,64)(59,67)(60,70)(62,63)(65,72)(66,68)(69,73)(71,76)\\ &\quad (74,79)(75,82)(77,85)(78,88)(80,90)(81,91)(83,89)(84,86)(87,94)\\ &\quad (92,93)(95,96).\\ y &= \prod_{i=0}^{31}(3i+1, 3i+2, 3i+3). \end{aligned}$$

Useful prime: $p = 59$; p-cycle: $(xyxy^2xyxyxy^2xyxy^2)^{420}$.

REFERENCES

1. M. Aschbacher, *Finite group theory* (Cambridge University Press 1986).
2. M. D. E. Conder. Generators for alternating and symmetric groups. *J. London Math. Soc.* (2) **22** (1980), 75–86.

3. M. D. E. Conder. Some results on quotients of triangle groups, *Bull. Austral. Math. Soc.* (29) **22** (1984), 73–90.

4. M. D. E. Conder. Private communication.

5. A. Lucchini and M. C. Tamburini, *Classical groups of large rank as Hurwitz groups*, J. Algebra **219** (1999), 531–546.

6. A. Lucchini, M. C. Tamburini and J. S. Wilson, *Hurwitz groups of large rank*, J. London Math. Soc. (2) **61** (2000), 81-92.

7. C. Jordan. Sur la limite de transitivité des groupes non alternés. *Bull. Soc. Math. France* **1** (1872/73), 40–71.

8. L. L. Scott. Matrices and cohomology. *Ann. Math.* **105** (1977), 473-492.

9. M. C. Tamburini and M. Vsemirnov. Hurwitz groups and Hurwitz generation, Invited Chapter. *Handbook of Algebra*, vol. 4, edited by M. Hazewinkel (Elsevier 2006), 385–426.

10. R. Vincent and A. E. Zalesski. Non-Hurwitz classical groups, *LMS J. Comput. Math.* **10** (2007), 21 – 82.

11. M. Vsemirnov. Hurwitz groups of intermediate rank. *LMS J. Comput. Math.* **7** (2004), 300–336 .