Università di Milano–Bicocca
Quaderni di Matematica

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Quaderno n. 12/2007 (arxiv:math/0707.0606)

Stampato nel mese di luglio 2007
presso il Dipartimento di Matematica e Applicazioni,
Università degli Studi di Milano-Bicocca, via R. Cozzi 53, 20125 Milano, ITALIA.

Disponibile in formato elettronico sul sito www.matapp.unimib.it.
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Esemplare fuori commercio per il deposito legale agli effetti della Legge 15 aprile 2004 n.106.

INFINITE HORIZON AND ERGODIC OPTIMAL QUADRATIC CONTROL FOR AN AFFINE EQUATION WITH STOCHASTIC COEFFICIENTS

GIUSEPPINA GUATTERI AND FEDERICA MASIERO

Abstract. We study quadratic optimal stochastic control problems with control dependent noise state equation perturbed by an affine term and with stochastic coefficients. Both infinite horizon case and ergodic case are treated. To this purpose we introduce a Backward Stochastic Riccati Equation and a *dual* backward stochastic equation, both considered in the whole time line. Besides some *stabilizability* conditions we prove existence of a solution for the two previous equations defined as limit of suitable finite horizon approximating problems. This allows to perform the synthesis of the optimal control.

Key words. Linear and affine quadratic optimal stochastic control, random coefficients, infinite horizon, ergodic control, Backward Stochastic Riccati Equation.

AMS subject classifications. 93E20, 49N10, 60H10.

1. Introduction

Backward Stochastic Riccati Equations (BSREs) are naturally linked with stochastic optimal control problems with stochastic coefficients. The first existence and uniqueness result for such a kind of equations has been given by Bismut in [3], but then several works, see [4], [14], [15], [16], [17], [19] and [20], followed as the problem, in its general formulation, turned out to be difficult to handle and challenging. Indeed only very recently Tang in [22] solved the general non singular case corresponding to the linear quadratic problem with random coefficients and control dependent noise.

In his paper the so-called linear quadratic optimal control problem is considered: minimize over $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ the following cost functional

$$J_T(0,x,u) = \mathbb{E}\int_0^T [\langle S_s X_s, X_s\rangle + |u_s|^2]ds + \mathbb{E}\langle P X_T, X_T\rangle \tag{1.1}$$

where $X_s \in \mathbb{R}^n$ is solution of the following linear stochastic system:

$$\begin{cases} dX_s = (A_s X_s + B_s u_s)ds + \sum_{i=1}^d \left(C_s^i X_s + D_s^i u_s\right) dW_s^i \quad s \geq 0 \\ X_0 = x, \end{cases} \tag{1.2}$$

where W is a d dimensional brownian motion and A, B, C, D, S are stochastic processes adapted to its natural filtration completed $\{\mathcal{F}_t\}_{t\geq 0}$ while P is random variable $\mathcal{F}_T$ measurable.

All these results cover the finite horizon case.

In this paper starting from the results of [22], we address the infinite horizon case and the ergodic case. Since our final goal is to address ergodic control, in the state equation we consider a forcing term. Namely, the state equation that describe the system under control is the following affine stochastic equation:

$$\begin{cases} dX_s = (A_s X_s + B_s u_s)ds + \sum_{i=1}^d \left(C_s^i X_s + D_s^i u_s\right) dW_s^i + f_s ds \quad s \geq 0 \\ X_0 = x, \end{cases} \tag{1.3}$$

Our main goal is to minimize with respect to u the infinite horizon cost functional,

$$J_\infty(0,x,u) = \mathbb{E}\int_0^{+\infty} [\langle S_s X_s, X_s\rangle + |u_s|^2]ds \tag{1.4}$$

and the following ergodic cost functional:

$$\liminf_{\alpha\to 0} \alpha J_\alpha(0,x,u) \tag{1.5}$$

where

$$J_\alpha(0,x,u) = \mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\langle S_s X_s, X_s\rangle + |u_s|^2]ds, \tag{1.6}$$

In order to carry on this programme we have first to reconsider the finite horizon case since now the state equation is affine. As it is well known the value function has in the present situation a quadratic term represented in term of the solution of the Backward Stochastic Riccati Equation (BSRE) in $[0,T]$:

$$dP_t = -\left[A_t^* P_t + P_t A_t + S_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t C_t^i + \left(C_t^i\right)^* Q_t + Q_t C_t^i\right)\right] dt + \sum_{i=1}^d Q_t^i dW_t^i +, \tag{1.7}$$

$$\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q^i D_t^i\right)\right]\left[I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i\right]^{-1}\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q_t^i D_t^i\right)\right]^* dt,$$
$$P_T = P,$$

and a linear term involving the so-called *costate equation* (dual equation):

$$\begin{cases} dr_t = -H_t^* r_t dt - P_t f_t dt - \displaystyle\sum_{i=1}^d \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^d g_t^i dW_t^i, \quad t\in[0,T] \\ r_T = 0. \end{cases} \tag{1.8}$$

The coefficients H and K are related with the coefficients of the state equation and the solution to the BSRE in $[0,T]$. In details, if we denote for $t\in[0,T]$

$$f\left(t,P_t,Q_t\right) = -[I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i]^{-1}[P_t B_t + \sum_{i=1}^d (Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i)],^*$$

then we have: $H_t = A_t + B_t f\left(t,P_t,Q_t\right)$ and $K_t^i = C_t^i + D_t^i f\left(t,P_t,Q_t\right)$. The solution (r,g) of this equation together with the solution (P,Q) of the BSRE equation (1.7) allow to describe the optimal control and perform the synthesis of the optimal equation. Equation (1.8) is the generalization of the deterministic equation considered by Bensoussan in [1] and by Da Prato and Ichikawa in [9] and of the stochastic backward equation introduced in [24] for the case without control dependent noise and with deterministic coefficients.

The main difference from the equation considered in [24] is that, being the solution to the Riccati equation a couple of stochastic processes (P,Q) with Q just square integrable, equation (1.8) has stochastic coefficients that are not uniformly bounded. So the usual technique of resolution does not apply directly. When r is one dimensional also the non linear case has been studied in [5] using Girsanov Theorem and properties of BMO martingales. Here being the problem naturally multidimensional we can not apply the Girsanov transformation to get rid of the term $\sum_{i=1}^d \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^d g_t^i dW_t^i$.

Nevertheless we can exploit a duality relation between the dual equation (1.8) and the following equation

$$\begin{cases} dX_s = H_s X_s ds + \displaystyle\sum_{i=1}^d K_s^i X_s dW_s^i \quad s\in[t,T) \\ X_t = x. \end{cases} \tag{1.9}$$

This equation is indeed the closed loop equation related to the linear quadratic problem and can be solved following [11] and its control interpretation allows to gain enough regularity to perform the duality relation with (r,g).

Once we are able to handle the finite horizon case, we can proceed to study the infinite horizon problem. The BSRE corresponding to this problem is, for $t \geq 0$

$$dP_t = -\left[A_t^* P_t + P_t A_t + S_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t C_t^i + \left(C_t^i\right)^* Q_t + Q_t C_t^i\right)\right] dt + \sum_{i=1}^d Q_t^i dW_t^i + \tag{1.10}$$

$$\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q^i D_t^i\right)\right]\left[I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i\right]^{-1}\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q_t^i D_t^i\right)\right]^* dt,$$

note that differently from equation (1.7), the final condition has disappeared since the horizon is infinite. It turns out that, under a suitable finite cost condition, see also [12], there exists a minimal solution $(\overline{P}, \overline{Q})$ and we can perform the synthesis of the optimal control with $f = 0$. More precisely we introduce a sequence (P^N, Q^N) of solutions of the Riccati equation in $[0, N]$ with $P^N(N) = 0$ and we show that for any $t \geq 0$ the sequence of P^N pointwise converge, as N tends to $+\infty$, to a limit denoted by $\overline{P}$. The sequence of Q^N instead only converge weakly in $L^2(\Omega \times [0, T])$ to some process $\overline{Q}$ and this is not enough to pass to the limit in the fundamental relation and then to conclude that the limit $(\overline{P}, \overline{Q})$ is the solution for the infinite horizon Riccati equation (1.10). Therefore, as for the finite horizon case, we have to introduce the stochastic Hamiltonian system to prove the limit $(\overline{P}, \overline{Q})$ solves the BRSE (1.10), see Corollary 3.7. Indeed studying the stochastic Hamiltonian system we can prove that the optimal cost for the approximating problem converge to the optimal cost of the limit problem and this implies that $\overline{P}$ is the solution of the BSRE.

In order to cope with the affine term f we have to introduce an infinite horizon, this time, backward equation

$$dr_t = -H_t^* r_t dt - P_t f_t dt - \sum_{i=1}^d \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^d g_t^i dW_t^i, \qquad t \geq 0. \tag{1.11}$$

Notice that the typical monotonicity assumptions on the coefficients of this infinite horizon BSDE are replaced by the finite cost condition and the Theorem of Datko. As a consequence of this new hypothesis we have that the solution to the closed loop equation considered in the whole positive time line with the coefficients evaluated in $\overline{P}$ and $\overline{Q}$, is exponentially stable.

Hence a solution $(\bar{r}, \bar{g})$ to this equation is obtained as limit of the sequence (r_T, g_T) defined in (1.8), indeed using duality and the exponential stability property of the solution to (1.9), we can prove that the sequence of $r_T's$ and its limit $\bar{r}$ are uniformly bounded. Hence, having both $(\overline{P}, \overline{Q})$ and $(\bar{r}, \bar{g})$, we can express the optimal control and the value function.

Eventually we come up with the ergodic case: first of all we set $X_s^\alpha := e^{-\alpha s} X_s$ and $u_s^\alpha := e^{-\alpha s} u_s$ and we notice that the functional $J_\alpha(0, x, u)$ can be written as an infinite horizon functional in terms of X^α and u^α:

$$J_\alpha(0, x, u) = \mathbb{E} \int_0^{+\infty} [\langle S_s X_s^\alpha, X_s^\alpha \rangle + |u_s^\alpha|^2] ds.$$

This allows us to adapt the previous results on the infinite horizon when $\alpha > 0$ is fixed.
Then, in order to study the limit (1.5), we need to investigate the behaviour of X^α, of the solution P^α of the Riccati equation corresponding to $\overline{J}^\alpha(x) := \inf_u J^\alpha(0, x, u)$ and the solutions (r^α, g^α) of the dual equations corresponding to H^α, K^α and $f_t^\alpha = e^{-\alpha t} f_t$. In the general case it turns out that the ergodic limit has the following form:

$$\underline{\lim}_{\alpha \to 0} \alpha \overline{J}^\alpha(x) = \underline{\lim}_{\alpha \to 0} \alpha \int_0^{+\infty} \langle r_s^\alpha, f_s^\alpha \rangle ds.$$

A better characterization holds if we assume all the coefficients (A, B, C, D) and f to be stationary processes, see definition 6.9. If this is the case we can prove that the stationarity property extends to both $\bar{P}$ and $\bar{r}$, and hence the optimal ergodic cost simplify to:

$$\lim_{\alpha \to 0} \alpha \inf_{u \in \mathcal{U}} J_\alpha(x, u) = \mathbb{E} \langle f(0), \bar{r}(0) \rangle$$

When the coefficients of the state equation are deterministic similar problems have already been treated: we cite [2], [24] and bibliography therein. In [2] in the state equation all the coefficients are deterministic and no control dependent noise is studied, while in [24] only the forcing term f is allowed to be random.

Finally we describe the content of each section: in section 2, after recalling some results of [22], we solve the finite horizon case when the state equation is affine: the key point is the solution of the dual equation (1.8), which is studied in paragraph 2.2; in section 3 we solve the infinite horizon case with $f = 0$, in section 4 we study the infinite horizon equation (1.11), in section 5 we complete the general infinite horizon case, finally in section 6 we study the ergodic case.

Acknowledgments. The authors wish to thank Philippe Briand for the very useful discussions on the role of BMO martingales in the theory of Backward Stochastic Differential Equations.

2. Linear Quadratic optimal control in the finite horizon case

Let $(\Omega, \mathcal{E}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic base verifying the usual conditions. In $(\Omega, \mathcal{E}, \mathbb{P})$ we consider the following stochastic differential equation for $t \geq 0$:

$$\begin{cases} dX_s = (A_s X_s + B_s u_s)ds + \displaystyle\sum_{i=1}^{d} \left(C_s^i X_s + D_s^i u_s\right) dW_s^i + f_s ds \quad s \in [t, T] \\ X_t = x \end{cases} \tag{2.1}$$

where X is a process with values in $\mathbb{R}^n$ and represents the *state* of the system and is our unknown, u is a process with values in $\mathbb{R}^k$ and represents the *control*, $\left\{W_t := \left(W_t^1, ..., W_t^d\right),\ t \geq 0\right\}$ is a d-dimensional standard $\mathcal{F}_t$-Brownian motion and the initial data x belongs to $\mathbb{R}^n$. To stress dependence of the state X on u, t and x we will denote the solution of equation (2.1) by $X^{t,x,u}$ when needed. The norm and the scalar product in any finite dimensional Euclidean space $\mathbb{R}^m$, $m \geq 1$, will be denoted respectively by $|\cdot|$ and $\langle \cdot, \cdot \rangle$.

Our purpose is to minimize with respect to u the cost functional,

$$J(0, x, u) = \mathbb{E} \int_0^T [\langle S_s X_s^{0,x,u}, X_s^{0,x,u} \rangle + |u_s|^2] ds + \mathbb{E} \mid X_T^{0,x,u} \mid^2 . \tag{2.2}$$

We also introduce the following random variables, for $t \in [0, T]$:

$$J(t, x, u) = \mathbb{E}^{\mathcal{F}_t} \int_t^T [\langle S_s X_s^{0,x,u}, X_s^{0,x,u} \rangle + |u_s|^2] ds + \mathbb{E}^{\mathcal{F}_t} \mid X_T^{0,x,u} \mid^2 . \tag{2.3}$$

We make the following assumptions on A, B, C and D.

Hypothesis 2.1.

A1) $A : [0, T] \times \Omega \to \mathbb{R}^{n\times n}$, $B : [0, T] \times \Omega \to \mathbb{R}^{n\times k}$, $C^i : [0, T] \times \Omega \to \mathbb{R}^{n\times n}$, $i = 1, ..., d$ *and* $D^i : [0, T] \times \Omega \to \mathbb{R}^{n\times k}$, $i = 1, ..., d$, *are uniformly bounded processes adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

A2) $S : [0, T] \times \Omega \to \mathbb{R}^{n\times n}$ *is uniformly bounded and adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *and it is almost surely and almost everywhere symmetric and nonnegative.*

A3) $f : [0, T] \times \Omega \to \mathbb{R}^n$ *is adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *and* $f \in L^\infty([0, T] \times \Omega)$.

2.1. **Preliminary results on the unforced case.** Next we recall some results obtained in [22] for the finite horizon case, with $f = 0$ in equation 2.1. In that paper a finite horizon control problem was studied, namely minimize the quadratic cost functional

$$J(0, x, u) = \mathbb{E}\langle P X_T^{0,x,u}, X_T^{0,x,u} \rangle + \mathbb{E} \int_0^T [\langle S_s X_s^{0,x,u}, X_s^{0,x,u} \rangle + |u_s|^2] ds,$$

where P is a random matrix uniformly bounded and almost surely positive and symmetric, $T > 0$ is fixed and $X^{0,x,u}$ is the solution to equation (2.1) with $f = 0$. To this controlled problem, the following (finite horizon) backward stochastic Riccati differential equation (BSRDE in the following) is related:

$$\begin{cases} -dP_t = G\left(A_t, B_t, C_t, D_t; S_t; P_t, Q_t\right) dt + \displaystyle\sum_{i=1}^{d} Q_t^i dW_t^i \\ P_T = P. \end{cases} \tag{2.4}$$

where

$$G\left(A,B,C,D;S;P,Q\right)=A^{*}P+PA+S+\sum_{i=1}^{d}\left(\left(C^{i}\right)^{*}PC^{i}+\left(C^{i}\right)^{*}Q+QC^{i}\right)-G_{1}\left(B,C,D;P,Q\right),$$

and

$$\begin{aligned}G_{1}\left(B,C,D;P,Q\right)=&\left[PB+\sum_{i=1}^{d}\left(\left(C^{i}\right)^{*}PD^{i}+Q^{i}D^{i}\right)\right]*\left[I+\sum_{i=1}^{d}\left(D^{i}\right)^{*}PD^{i}\right]^{-1}*\\&*\left[PB+\sum_{i=1}^{d}\left(\left(C^{i}\right)^{*}PD^{i}+Q^{i}D^{i}\right)\right]^{*}\end{aligned}$$

Definition 2.2. A pair of adapted processes (P,Q) is a *solution* of equation (2.4) if

(1) $\int_0^T |Q_s|^2\,ds<+\infty$, almost surely,

(2)
$$\int_0^T \left|G\left(A_s,B_s,C_s,D_s;S_s;P_s,Q_s\right)\right|ds<+\infty,$$

(3) for all $t\in[0,T]$
$$P_t=P+\int_t^T G\left(A_s,B_s,C_s,D_s;S_s;P_s,Q_s\right)ds-\int_t^T\sum_{i=1}^{d}Q_s^i dW_s^i.$$

Theorem 2.3 ([22], Theorems 3.2 and 5.3)**.** *Assume that A,B,C,D and S verify hypothesis 2.1. Then there exists a unique solution to equation (2.4). Moreover the following fundamental relation holds true, for all $0\le t\le s\le T$, and all $u\in L^2_{\mathcal{P}}\left([0,T]\times\Omega,\mathbb{R}^k\right)$:*

$$\begin{aligned}\langle P_t x,x\rangle=&\,\mathbb{E}^{\mathcal{F}_t}\left\langle PX_T^{t,x,u},X_T^{t,x,u}\right\rangle+\mathbb{E}^{\mathcal{F}_t}\int_t^T[\left\langle S_r X_r^{t,x,u},X_r^{t,x,u}\right\rangle+|u_r|^2]dr\\&-\mathbb{E}^{\mathcal{F}_t}\int_t^T\left|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^{*}P_sD_s^i\right)^{1/2}*\right.\\&\left.*\left[u_s+\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^{*}P_sD_s^i\right)^{-1}\left(P_sB_s+\sum_{i=1}^{d}\left(Q_s^iD_s^i+\left(C_s^i\right)^{*}P_sD_s^i\right)\right)^{*}X_s^{t,x,u}\right]\right|^2ds\end{aligned}\tag{2.5}$$

Then the value function is given by

$$\langle P_0 x,x\rangle=\inf_{u\in L^2_{\mathcal{P}}([0,T]\times\Omega,\mathbb{R}^k)}\mathbb{E}^{\mathcal{F}_t}\left\langle PX_T^{t,x,u},X_T^{t,x,u}\right\rangle+\mathbb{E}^{\mathcal{F}_t}\int_t^T[\left\langle S_r X_r^{t,x,u},X_r^{t,x,u}\right\rangle+|u_r|^2]dr$$

and the unique optimal control has the following closed form:

$$\overline{u}_t=-\left(I+\sum_{i=1}^{d}\left(D_t^i\right)^{*}P_tD_t^i\right)^{-1}\left(P_tB_t+\sum_{i=1}^{d}\left(Q_t^iD_t^i+\left(C_t^i\right)^{*}P_tD_t^i\right)\right)^{*}X_t^{0,x,\bar{u}}.$$

If $\overline{X}$ is the solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{X}$ is the unique solution to the closed loop *equation:*

$$\begin{cases}d\overline{X}_s=(A_s\overline{X}_s-B_s\left(I+\sum_{i=1}^{d}\left(D_t^i\right)^{*}P_tD_t^i\right)^{-1}\left(P_tB_t+\sum_{i=1}^{d}\left(Q_t^iD_t^i+\left(C_t^i\right)^{*}P_tD_t^i\right)\right)^{*}\overline{X}_s)ds+\\\sum_{i=1}^{d}\left(C_s^i\overline{X_s}-D_s^i\left(I+\sum_{i=1}^{d}\left(D_t^i\right)^{*}P_tD_t^i\right)^{-1}\left(P_tB_t+\sum_{i=1}^{d}\left(Q_t^iD_t^i+\left(C_t^i\right)^{*}P_tD_t^i\right)\right)^{*}\overline{X}_s)\right)dW_s^i,\\\overline{X}_t=x\end{cases}\tag{2.6}$$

The optimal cost is therefore given in term of the solution of the Riccati matrix

$$J(0,x,\overline{u}) = \langle P_0 x, x\rangle. \tag{2.7}$$

and also the following identity holds, for all $t\in[0,T]$:

$$J(t,x,\overline{u}) = \langle P_t x, x\rangle. \tag{2.8}$$

For $t\in[0,T]$, we denote by

$$\begin{aligned}
&f\left(t,P_t,Q_t\right) = -\left(I+\sum_{i=1}^{d}\left(D_t^i\right)^* P_t D_t^i\right)^{-1}\left(P_t B_t + \sum_{i=1}^{d}\left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i\right)\right)^*,\\
&H_t = A_t + B_t f\left(t,P_t,Q_t\right),\\
&K_t^i = C_t^i + D_t^i f\left(t,P_t,Q_t\right).
\end{aligned} \tag{2.9}$$

So the closed loop equation (2.6) can be rewritten as

$$\begin{cases} dX_s = H_s ds + \displaystyle\sum_{i=1}^{d} K_s^i X_s dW_s^i \quad s\in[t,T)\\ X_t = x \end{cases} \tag{2.10}$$

It is well known, see e.g. [11], that equation (2.10) admits a solution.

Remark 2.4. f, H and K defined in (2.9) are related to the feedback operator in the solution of the finite horizon optimal control problem with $f=0$. By the boundedness of P and by the fundamental relation (2.5), it turns out for every stopping time $0\le\tau\le T$ a.s.,

$$\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T \left|f\left(t,P_t,Q_t\right)\right|^2 dt \le C,$$

where C is a constant depending on T and x. Since A, B, C and D are bounded, this property holds true also for H and K:

$$\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |H_t|^2 dt + \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |K_t|^2 dt \le C,$$

where now C is a constant depending on T, x, A, B, C and D. In particular, f, H and K are square integrable. In the following we denote by

$$\begin{aligned}
C_H &= \sup_\tau \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |H_t|^2 dt,\\
C_K &= \sup_\tau \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |K_t|^2 dt.
\end{aligned} \tag{2.11}$$

where the supremum is taken over all stopping times τ, $\tau\in[0,T]$ a.s..

2.2. **Costate equation and finite horizon affine control.** In order to solve the optimal control problem related to the nonlinear controlled equation 2.1, we introduce the so called *dual equation*, or *costate equation*,

$$\begin{cases} dr_t = -H_t^* r_t dt - P_t f_t dt - \displaystyle\sum_{i=1}^{d}\left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \quad t\in[0,T]\\ r_T = 0. \end{cases} \tag{2.12}$$

We look for a solution of (2.12), that is a pair of predictable processes (r,g) s.t. $r\in L^\infty_{loc}\left([0,T]\times\Omega,\mathbb{R}^n\right)$ and $g^i\in L^2_{loc}\left([0,T]\times\Omega,\mathbb{R}^n\right)$, for $i=i,...,d$. $L^\infty_{loc}\left([0,T]\times\Omega,\mathbb{R}^n\right)$ is the space of predictable processes r with values in $\mathbb{R}^n$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]}|r_t| < \infty\right) = 1.$$

$L^2_{loc}([0,T]\times\Omega,\mathbb{R}^n)$ is the space of predictable processes g with values in $\mathbb{R}^n$ such that

$$\mathbb{P}\left(\int_0^T |g_s|^2 ds < \infty\right) = 1.$$

Lemma 2.5. *The backward equation* (2.12) *admits a unique solution* (r,g) *that belongs to the space* $L^\infty_{loc}([0,T]\times\Omega,\mathbb{R}^n)\times L^2_{loc}\left([0,T]\times\Omega,\mathbb{R}^{n\times d}\right)$.

Proof. In order to construct a solution to equation (2.12), we essentially follow [25], chapter 7, where linear BSDEs with bounded coefficients are solved directly. Besides equation (2.12) we consider the two following equations with values in $\mathbb{R}^{n\times n}$:

$$\begin{cases} d\Phi_s = -H_s\Phi_s ds + \sum_{i=1}^d \left(K_s^i\right)^* \left(K_s^i\right)^* \Phi_s ds - \sum_{i=1}^d \left(K_s^i\right)^* \Phi_s dW_s^i \quad s\in[0,T] \\ \Phi_0 = I, \end{cases} \tag{2.13}$$

and

$$\begin{cases} d\Psi_s = \Psi_s H_s^* ds + \sum_{i=1}^d \Psi_s \left(K_s^i\right)^* dW_s^i \quad s\in[0,T] \\ \Psi_0 = I. \end{cases} \tag{2.14}$$

By applying Itô formula it turns out that $\Phi_t\Psi_t = I$. By transposing equation (2.14), we obtain the following equation for Ψ^*:

$$\begin{cases} d\Psi_s^* = +H_s\Psi_s^* ds + \sum_{i=1}^d K_s^i \Psi_s^* dW_s^i \quad s\in[0,T] \\ \Psi_0^* = I. \end{cases} \tag{2.15}$$

By [11], equations (2.13), (2.14) and (2.15) admit a unique solution. Moreover, since H and K are related to the feedback operator, see (2.9) where f, H and K are defined, it follows that

$$\mathbb{E}|\Psi_t|^2 \le C|I|^2, t\in[0,T], \tag{2.16}$$

where C is a constant that may depend on T, see also theorem 2.2 in [22], with $\Psi_t^* h = \phi_{0,t}h$, $h\in\mathbb{R}^n$. We set $\theta := -\int_0^T \Psi_s P_s f_s ds$. By boundedness of P and f, and by estimate (2.16) on Ψ, it turns out that $\theta\in L^2(\Omega)$. We define

$$r_t = \Phi_t[\int_0^t \Psi_s P_s f_s ds + \mathbb{E}^{\mathcal{F}_t}\theta],$$

and we want to show that it is solution to equation (2.12). Since $\mathbb{E}^{\mathcal{F}_t}\theta$ is a square integrable martingale, by the representation theorem for martingales, there exists a unique $\eta = \left(\eta^1,...\eta^d\right)\in L^2\left(\Omega\times[0,T],\mathbb{R}^{n\times d}\right)$ such that

$$\mathbb{E}^{\mathcal{F}_t}\theta = \mathbb{E}\theta + \sum_{i=1}^d \int_0^t \eta_s^i dW_s^i.$$

So

$$r_t = \Phi_t\left(\mathbb{E}\theta + \sum_{i=1}^d\int_0^t \eta_s^i dW_s^i + \int_0^t \Psi_s P_s f_s ds\right) := \Phi_t\xi_t,$$

and by this definition we get $r \in L^{\infty}_{loc}\left([0,T]\times\Omega,\mathbb{R}^n\right)$. By applying Ito formula to r we obtain

$$
\begin{aligned}
dr_t &= \left\{ -H_t^*\Phi_t\xi_t + \sum_{i=1}^d \left(K_t^i\right)^* \left(K_t^i\right)^* \Phi_t\xi_t + \Phi_t\Psi_t P_t f_t \right\} dt \\
&\quad - \sum_{i=1}^d \left(K_t^i\right)^* \Phi_t\xi_t dW_t^i + \sum_{i=1}^d \Phi_t\eta_t^i dW_t^i - \sum_{i=1}^d \left(K_t^i\right)^* \Phi_t\eta_t^i dt \\
&= -H_t^*\Phi_t\xi_t dt + \sum_{i=1}^d \left(K_t^i\right)^* g_t^i dt + P_t f_t dt - \sum_{i=1}^d g_t^i dW_t^i,
\end{aligned}
$$

where

$$
g_t^i := \left(K_t^i\right)^* \Phi_t\xi_t - \Phi_t\eta_t^i, i = 1, ..., d.
$$

By this definition it turns out that $g = \left(g^1, ..., g^d\right) \in L^2_{loc}\left([0,T]\times\Omega,\mathbb{R}^{n\times d}\right)$. □

We can prove that the solution (r,g) to equation (2.12) is more regular. To prove this regularity, we need the following *duality relation.*

Lemma 2.6. *Let (r,g) be solution to the equation (2.12), and let $X^{t,x,\eta}$ be solution to the equation*

$$
\begin{cases}
dX_s^{t,x,\eta} = H_s X_s^{t,x,\eta} ds + \sum_{i=1}^d K_s^i X_s^{t,x,\eta} dW_s^i + \eta_s ds, \quad s \in [t,T], \\
X_t^{t,x,\eta} = x,
\end{cases}
\tag{2.17}
$$

where $x \in L^2(\Omega,\mathcal{F}_t)$, η is a predictable process in $L^2(\Omega\times[0,T],\mathbb{R}^n)$. Then the following duality relation holds true:

$$
\mathbb{E}^{\mathcal{F}_t}\left\langle r_T, X_T^{t,x,\eta}\right\rangle - \langle r_t, x\rangle = -\mathbb{E}^{\mathcal{F}_t}\int_t^T \left\langle P_s f_s, X_s^{t,x,\eta}\right\rangle ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle \eta_s, r_s\rangle\, ds. \tag{2.18}
$$

Proof. The proof is an easy application of Itô formula:

$$
\begin{aligned}
&\left\langle r_T, X_T^{t,x,\eta}\right\rangle - \langle r_t, x\rangle \\
&= -\int_t^T \left\langle H_s^* r_s, X_s^{t,x,\eta}\right\rangle ds - \int_t^T \left\langle P_s f_s, X_s^{t,x,\eta}\right\rangle ds - \int_t^T \sum_{i=1}^d \left\langle (K_s^i)^* g_s^i, X_s^{t,x,\eta}\right\rangle ds \\
&+ \int_t^T \sum_{i=1}^d \left\langle g_s^i, X_s^{t,x,\eta}\right\rangle dW_s^i + \int_t^T \left\langle r_s, H_s X_s^{t,x,\eta}\right\rangle ds + \int_t^T \sum_{i=1}^d \left\langle r_s, K_s^i X_s^{t,x,\eta}\right\rangle dW_s^i + \int_t^T \langle r_s, \eta_s\rangle\, ds \\
&+ \int_t^T \left\langle g_s^i, K_s^i X_s^{t,x,\eta}\right\rangle ds
\end{aligned}
$$

By simplifying and by taking conditional expectation on both sides we obtain the desired relation. □

We also need to find a relation between the solution (r,g) of the equation (2.12) and the optimal state $\overline{X}$ corresponding to the optimal control $\overline{u}$. This can be achieved, following e.g. [1], by introducing the so called *stochastic Hamiltonian system*

$$
\begin{cases}
d\overline{X}_s = [A_s\overline{X}_s + B_s\overline{u}_s]ds + \sum_{i=1}^d [C_s^i\overline{X}_s + D_s^i\overline{u}_s]dW_s^i + f_s ds, \\
dy_s = -[A_s^* y_s + \sum_{i=1}^d \left(C_s^i\right)^* z_s^i + S_s\overline{X}_s]ds + \sum_{i=1}^d z_s^i dW_s^i, \qquad t \le s \le T, \\
X_t = x, \\
y_T = \overline{P}_T\overline{X}_T,
\end{cases}
\tag{2.19}
$$

where $y, z^i \in \mathbb{R}^n$, for every $i = 1, ..., d$. By the so called stochastic maximum principle, the optimal control for the finite horizon control problem is given by

$$\overline{u}_s = -\left(B_s^* y_s + \sum_{i=1}^d \left(D_s^i\right)^* z_s^i \right). \tag{2.20}$$

By relation (2.20), equations (2.19) become a fully coupled system of forward backward stochastic differential equations (FBSDE in the following), which admits a unique solution $(\overline{X}, y, z) \in L^2(\Omega \times [0,T], \mathbb{R}^n) \times L^2(\Omega \times [0,T], \mathbb{R}^n) \times L^2\left([0,T] \times \Omega, \mathbb{R}^{n \times d}\right)$, see Theorem 2.6 in [21].

Lemma 2.7. *Let (r,g) be the unique solution to equation (2.12), and let $(\overline{X}, y, z)$ be the unique solution to the FBSDE (2.19). Then the following relation holds true:*

$$y_t = P_t \overline{X}_t + r_t, \qquad 0 \leq t \leq T. \tag{2.21}$$

Proof. We only give a sketch of the proof. For $t = T$ relation (2.21) holds true. By applying Ito formula it turns out that $y_t - P_t\overline{X}_t$ and r_t solve the same BSDE, with the same final datum equal to 0 at the final time T. By uniqueness of the solution of this BSDE, the lemma is proved. □

Remark 2.8. We note that by theorem 2.6 in [21], $y \in L^2(\Omega \times [0,T], \mathbb{R}^n)$. Moreover, by standard calculations, it is easy to check that y admits a continuous version and $y \in L^2(\Omega, C([0,T], \mathbb{R}^n))$. Moreover, if $f = 0$, we get, for every $0 \leq t \leq s \leq T$,

$$\mathbb{E}^{\mathcal{F}_t} |\overline{X}_s|^2 \leq C|x|^2.$$

This estimate can be easily achieved by applying the Gronwall lemma, and by remembering that from (2.8), for the optimal control $\overline{u}$ the following holds:

$$\mathbb{E}^{\mathcal{F}_t} \int_t^T |\overline{u}_s|^2 \leq \langle \overline{P}_t x, x \rangle \leq C|x|^2.$$

As a consequence, if $f \neq 0$, for every $0 \leq t \leq s \leq T$,

$$\mathbb{E}^{\mathcal{F}_t} |\overline{X}_s|^2 \leq C(1 + |x|^2).$$

Since P is bounded, by lemma 2.7, we get that for every $0 \leq t \leq s \leq T$

$$\mathbb{E}^{\mathcal{F}_t} \sup_{t \leq s \leq T} |r_s|^2 \leq C, \tag{2.22}$$

where C is a constant that can depend on T. Moreover, since $\overline{X}$ is continuous an P admits a continuous version, also r admits a continuous version.

We are now ready to prove the following regularity result on (r,g).

Proposition 2.9. *Let (r,g) be the solution to equation (2.12). Then $(r,g) \in L^2(\Omega, C([0,T], \mathbb{R}^n)) \times L^2\left([0,T] \times \Omega, \mathbb{R}^{n \times d}\right)$. Moreover $r \in L^\infty(\Omega \times [0,T])$.*

Proof. Let (r,g) be the solution to equation (2.12) built in lemma 2.5. By the previous remark we know that $r \in L^2(\Omega, C([0,T], \mathbb{R}^n))$, and moreover we have deduced estimates (2.22) on r. By applying Itô formula we get for $0 \leq t \leq w \leq T$,

$$\begin{aligned} |r_t|^2 &= |r_w|^2 + 2\int_t^w \langle H_s^* r_s, r_s \rangle \, ds + 2\int_t^w \langle P_s f_s, r_s \rangle \, ds + 2\int_t^w \sum_{i=1}^d \left\langle \left(K_s^i\right)^* g_s^i, r_s \right\rangle ds \\ &\quad + 2\int_t^w \sum_{i=1}^d \left\langle (g_s^i)^* dW_s^i, r_s \right\rangle - \int_t^w \sum_{i=1}^d |g_s^i|^2 ds. \end{aligned} \tag{2.23}$$

We introduce a sequence of stopping times $(\tau_n)_n$, where $\tau_n = \inf\left\{t\geq 0 : \sup_{0\leq s\leq t}|r_s|\geq n\right\}$. Since $r\in L^\infty([0,T]\times\Omega,\mathbb{R}^n)$, $\tau_n\wedge T\to T$ as $n\to\infty$. By (2.23) and by estimates involving Bulkholder-Davis-Gundy inequality and Young inequality, we get

$$\begin{aligned}
&\mathbb{E}|r_{t\wedge\tau_n}|^2+\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\sum_{i=1}^d|g_s^i|^2ds\\
&\leq \mathbb{E}|r_{T\wedge\tau_n}|^2+2n^2\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}|H_s|^2ds+2n\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}|P_sf_s|ds+4n^2\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\sum_{i=1}^d|\left(K_s^i\right)^*|^2ds\\
&+\frac{1}{4}\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\sum_{i=1}^d|g_s^i|^2ds+\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}|r_s|^2ds+\frac{1}{4}\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\sum_{i=1}^d|g_s^i|^2ds.
\end{aligned}$$

So

$$\mathbb{E}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\sum_{i=1}^d|g_s^i|^2ds\leq C(n,\|f\|_\infty,\|P\|_\infty,C_H,C_K),$$

for the definition of C_H and C_K see (2.11). For every $n\in\mathbb{N}$, we consider the process X_s^n, $s\in[t\wedge\tau_n,T]$, which is solution to the following stochastic differential equation

$$\begin{cases} dX_s^n=H_sX_s^nds+\sum_{i=1}^dK_s^iX_s^ndW_s^i, \quad s\in[t\wedge\tau_n,T], \\ X_{t\wedge\tau_n}^n=r_{t\wedge\tau_n}.\end{cases}\tag{2.24}$$

By remark 2.8, we get for $0\leq t\wedge\tau_n\leq s\leq T$

$$\mathbb{E}^{\mathcal{F}_t}|X_s^n|^2\leq C\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2.$$

By applying the *duality relation* (2.18) to X^n and r we get

$$\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2=\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\langle P_sf_s,X_s^n\rangle\,ds-\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle,$$

and so

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2&\leq|\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle|+|\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\langle P_sf_s,X_s^n\rangle\,ds|\\
&\leq|\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle|+\frac{\mu}{4}\mathbb{E}^{\mathcal{F}_t}\int_0^T|P_sf_s|^2ds+\frac{1}{\mu}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau_n}^{T\wedge\tau_n}|X_s^n|^2ds\\
&\leq|\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle|+\frac{\mu}{4}\mathbb{E}^{\mathcal{F}_t}\int_0^T|P_sf_s|^2ds+\frac{CT}{\mu}\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2.
\end{aligned}$$

By choosing μ such that $\dfrac{CT}{\mu}=\dfrac{1}{2}$, we get

$$\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2\leq|\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle|+C\|P\|_\infty^2\|f\|_\infty^2.$$

Moreover, by similar estimates,

$$|\mathbb{E}^{\mathcal{F}_t}\left\langle r_{T\wedge\tau_n},X_{T\wedge\tau_n}^n\right\rangle|\leq\frac{\mu}{4}\mathbb{E}^{\mathcal{F}_t}|r_{T\wedge\tau_n}|^2+\frac{C}{\mu}\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2.$$

By choosing μ such that $\dfrac{C}{\mu}=\dfrac{1}{2}$, we get

$$\mathbb{E}^{\mathcal{F}_t}|r_{t\wedge\tau_n}|^2\leq\frac{\mu}{4}\mathbb{E}^{\mathcal{F}_t}|r_{T\wedge\tau_n}|^2+C.\tag{2.25}$$

We want to let $n\to\infty$ in the previous relation. By lemma 2.7 and remark 2.8, estimate (2.22), and by the dominated convergence theorem on the right hand side we get that

$$\lim_{n\to\infty}\mathbb{E}^{\mathcal{F}_t}|r_{T\wedge\tau_n}|^2=0.$$

So by taking the limit on both sides in inequality (2.25), and again by dominated convergence theorem applied on the left hand side, we get

$$|r_t| \leq C, \ \ , 0 \leq t \leq T, \tag{2.26}$$

where C is a constant that can depend on T. So $r \in L^\infty(\Omega \times [0,T])$. By applying Itô formula as in (2.23), we get

$$\mathbb{E}^{\mathcal{F}_{t\wedge\tau_n}} |r_{t\wedge\tau_n}|^2 + \mathbb{E}^{\mathcal{F}_{t\wedge\tau_n}} \int_{t\wedge\tau_n}^{T\wedge\tau_n} \sum_{i=1}^d |g_s^i|^2 ds = \mathbb{E}^{\mathcal{F}_{t\wedge\tau_n}} |r_{T\wedge\tau_n}|^2 + 2\mathbb{E}^{\mathcal{F}t\wedge\tau_n} \int_{t\wedge\tau_n}^{T\wedge\tau_n} \langle H_s^* r_s, r_s \rangle \, ds$$
$$+ 2\mathbb{E}^{\mathcal{F}_{t\wedge\tau_n}} \int_{t\wedge\tau_n}^{T\wedge\tau_n} \langle P_s f_s, r_s \rangle \, ds + 2\mathbb{E}^{\mathcal{F}_{t\wedge\tau_n}} \int_{t\wedge\tau_n}^{T\wedge\tau_n} \sum_{i=1}^d \left\langle \left(K_s^i\right)^* g_s^i, r_s \right\rangle ds$$

By estimate (2.26) and by taking $t = 0$, we get

$$\mathbb{E} \int_0^{T\wedge\tau_n} \sum_{i=1}^d |g_s^i|^2 ds \leq C,$$

where C is a constant not depending on n. So by monotone convergence,

$$\mathbb{E} \int_0^T \sum_{i=1}^d |g_s^i|^2 ds \leq C,$$

and the proof is concluded. □

Remark 2.10. The last part of the proof is inspired by arguments used in [5] to prove that, for a one dimensional BSDE, if a solution is bounded then its martingale part is a BMO martingale.

We are ready to prove the main result of this section

Theorem 2.11. *Assume A, B, C, D and f satisfy hypothesis 2.1. Fix $x \in \mathbb{R}^n$, then:*

(1) *there exists a unique optimal control $\overline{u} \in L^2\left(\Omega \times [0,T], \mathbb{R}^k\right)$ such that for every $0 \leq t \leq T$,*
$$J(0,x,\overline{u}) = \inf_{u \in L^2(\Omega\times[0,T],\mathbb{R}^k)} J(0,x,u)$$

(2) *If $\overline{X}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{X}$ is the unique mild solution to the* closed loop *equation:*

$$\begin{cases} d\overline{X}_t = \left[A_t\overline{X}_t - B_t(f(t,P_t,Q_t)\overline{X}_t + (I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i)^{-1}(B_t^* r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i)) \right] dt + \\ \sum_{i=1}^d \left[C_s^i \overline{X}_t - D_s^i(f(t,P_t,Q_t)\overline{X}_t + \left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i \right)^{-1} (B_t^* r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i)) \right] dW_t^i, \\ \overline{X}_0 = x \end{cases} \tag{2.27}$$

(3) *The following feedback law holds $\mathbb{P}$-a.s. for almost every $0 \leq t \leq T$.*

$$\overline{u}_t = -\left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i \right) \right)^* \overline{X}_t + B_t^* r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i. \tag{2.28}$$

(4) *The optimal cost is given by*

$$J(0,x,\overline{u}) = \langle P_0 x, x\rangle + 2\langle r_0, x\rangle - \mathbb{E}\langle P_T \overline{X}_T, \overline{X}_T\rangle + 2\mathbb{E}\int_0^T \langle r_s, f_s\rangle ds$$
$$- \mathbb{E}\int_0^T |(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i)^{-1}(B_t^* r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i|^2) ds.$$

Proof. By computing $d\langle P_t, X_t\rangle + 2\langle r_t, X_t\rangle$, we get the so called fundamental relation

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds \\
&= \langle P_t x, x\rangle + 2\langle r_t, x\rangle - \mathbb{E}^{\mathcal{F}_t}\langle P_T X_T, X_T\rangle + 2\mathbb{E}^{\mathcal{F}_t}\int_t^T \langle r_s, f_s\rangle ds \\
&= \mathbb{E}^{\mathcal{F}_t}\int_t^T |\left(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s D_s^i\right)^{-1}\left(P_s B_s + \sum_{i=1}^d \left(Q_s^i D_s^i + \left(C_s^i\right)^* P_s D_s^i\right)\right)^* X_s + B_s^* r_s + \sum_{i=1}^d \left(D_s^i\right)|^2 ds \\
&- \mathbb{E}^{\mathcal{F}_t}\int_t^T |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s D_s^i)^{-1}(B_s^* r_s + \sum_{i=1}^d \left(D_s^i\right)^* g_s^i)|^2 ds.
\end{aligned}
$$

The theorem now easily follows. □

3. Preliminary results for the infinite horizon case

The next step is to study the optimal control problem in the infinite horizon case and with $f \neq 0$. To this aim we have to study solvability and regularity of the solution of a BSRDE with infinite horizon, in particular we study P. At first we consider the case when $f = 0$. Namely, in this section we consider the following stochastic differential equation where $X^{t,x,u}$ represents the state:

$$
\begin{cases}
dX_s^{t,x,u} = (A_s X_s^{t,x,u} + B_s u_s)ds + \sum_{i=1}^d \left(C_s^i X_s^{t,x,u} + D_s^i u_s\right) dW_s^i \quad s \geq t \\
X_t^{t,x,u} = x
\end{cases}
\tag{3.1}
$$

As a by product of the preliminaries studies, we are able to solve the following stochastic optimal control problem: minimize with respect to every admissible control u the cost functional,

$$J_\infty(0,x,u) = \mathbb{E}\int_0^{+\infty} [\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds. \tag{3.2}$$

We define the set of admissible control

$$\mathcal{U} = \left\{u \in L^2([0,+\infty)) : \mathbb{E}\int_0^{+\infty} \langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2 ds \leq +\infty\right\}. \tag{3.3}$$

We also introduce the following random variables, for $t \in [0,+\infty]$:

$$J_\infty(t,x,u) = \mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty} [\langle S_s X_s^{t,x,u}, X_s^{t,x,u}\rangle + |u_s|^2]ds$$

We will work under the following general assumptions on A, B, C and D that will hold from now on:

Hypothesis 3.1.

A1) $A : [0,+\infty) \times \Omega \to \mathbb{R}^{n\times n}$, $B : [0,+\infty) \times \Omega \to \mathbb{R}^{n\times k}$, $C^i : [0,+\infty) \times \Omega \to \mathbb{R}^{n\times n}$, $i = 1,...,d$ *and* $D^i : [0,+\infty) \times \Omega \to \mathbb{R}^{n\times k}$, $i = 1,...,d$, *are uniformly bounded process adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

A2) $S : [0,+\infty) \times \Omega \to \mathbb{R}^{n\times n}$ *is uniformly bounded and adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *and it is almost surely and almost everywhere symmetric and nonnegative.*

In order to study this control problem in infinite horizon, we consider the following backward stochastic Riccati equation on $[0,+\infty)$:

$$
\begin{aligned}
dP_t = &-\left[A_t^* P_t + P_t A_t + S_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t C_t^i + \left(C_t^i\right)^* Q_t + Q_t C_t^i\right)\right] dt + \sum_{i=1}^d Q_t^i dW_t^i + \qquad (3.4)\\
&\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q^i D_t^i\right)\right]\left[I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i\right]^{-1}\left[P_t B_t + \sum_{i=1}^d \left(\left(C_t^i\right)^* P_t D_t^i + Q_t^i D_t^i\right)\right]^* dt,
\end{aligned}
$$

where we stress that the final condition has disappeared but we ask that the solution can be extended to the whole positive real half-axis.

Definition 3.2. We say that a pair of processes (P, Q) is a solution to equation (3.4) if for every $T > 0$ (P, Q) is a solution to equation (2.4) in the interval time $[0, T]$, with $P_T = P(T)$.

Definition 3.3. We say that (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$ (or $\sqrt{S}$-stabilizable) if there exists a control $u \in L^2_{\mathcal{P}}([0, +\infty) \times \Omega; U)$ such that for all $t \geq 0$ and all $x \in \mathbb{R}^n$

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} [\langle S_s X_s^{t,x,u}, X_s^{t,x,u} \rangle + |u_s|^2] ds < M_{t,x}. \tag{3.5}$$

for some positive constant $M_{t,x}$.

This kind of stabilizability condition has been introduced in [12].

In the following, we consider BSRDEs on the time interval $[0, N]$, with final condition $P_N = 0$. For each integer $N > 0$, let (P^N, Q^N) be the solution of the Riccati equation

$$\begin{cases} -dP_t^N = G\left(A_t, B_t, C_t, D_t; S_t; P_t^N, Q_t^N\right) dt + \sum_{i=1}^{d} Q_t^{N,i} dW_t^i \\ P_N^N = 0. \end{cases}$$

P^N can be defined in the whole $[0, +\infty)$ setting $P_t^N = 0$ for all $t > N$. We prove the following lemma.

Lemma 3.4. *Assume hypothesis 3.1 and that (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$. There exists a random matrix $\overline{P}$ uniformly bounded and almost surely positive and symmetric such that* $\mathbb{P}\left\{\lim_{N\to+\infty} P^N(t)x = \overline{P}(t)x, \ \forall x \in \mathbb{R}^n\right\} = 1$.

Proof. The proof essentially follows the first part of the proof of proposition 3.2 in [12]. For each $t > 0$ fixed the sequence P_t^N is increasing. Indeed by definition

$$\begin{aligned} \langle P_t^{N+1} x, x \rangle &= \inf_{u \in L^2_{\mathcal{P}}([t,N+1]\times\Omega;U)} \mathbb{E}^{\mathcal{F}_t} \int_t^{N+1} (|\sqrt{S_r} X_r^{t,x,u}|^2 + |u_r|^2)\, dr \\ &\geq \inf_{u \in L^2_{\mathcal{P}}([t,N+1]\times\Omega;U)} \mathbb{E}^{\mathcal{F}_t} \int_t^{N} (|\sqrt{S_r} X_r^{t,x,u}|^2 + |u_r|^2)\, dr \\ &\geq \inf_{u \in L^2_{\mathcal{P}}([t,N]\times\Omega;U)} \mathbb{E}^{\mathcal{F}_t} \int_t^{N} (|\sqrt{S_r} X_r^{t,x,u}|^2 + |u_r|^2)\, dr = \langle P_t^N x, x \rangle. \end{aligned}$$

The above implies that for all $t > 0$:

$$\mathbb{P}\left\{\langle P^{N+1}(t)x, x\rangle \geq \langle P^N(t)x, x\rangle \quad \forall N \in \mathbb{N}, \ \forall x \in \mathbb{R}^n\right\} = 1 \tag{3.6}$$

Moreover for each t let $\bar{u}$ be the 'stabilizing' control that exists thank to definition 3.3 then

$$\begin{aligned} \langle P_t^N x, x \rangle = |\sqrt{P_t^N} x|^2 &= \inf_{u \in L^2_{\mathcal{P}}([t,N]\times\Omega;U)} \int_t^N (|\sqrt{S}_r y^{t,x,u}(r)|^2 + |u_r|^2)\, dr \\ &\leq \int_t^N (|\sqrt{S}_r X_r^{t,x,\bar{u}}|^2 + |\bar{u}_r|^2)\, dr \\ &\leq \int_t^{+\infty} (|\sqrt{S}_r X_r^{t,x,\bar{u}}|^2 + |\bar{u}_r|^2)\, dr \leq M_{t,x}, \quad \mathbb{P} - \text{a.s.} \end{aligned} \tag{3.7}$$

for a suitable constant $M_{t,x}$. If we consider the operator $\sqrt{P_t^N}$ as a linear operator from $\mathbb{R}^n$ to $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^n)$ by the Banach-Steinhaus theorem there exists M_t such that

$$|\sqrt{P_t^N}|_{L(\mathbb{R}^n, L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^n))} \leq M_t.$$

Again since $P_t^N \in Mat(n \times n)$- $\mathbb{P}$-a.s. the above implies that

$$\mathbb{P}\left\{\langle P_t^N x, x\rangle \leq M_t |x|^2, \quad \forall N \in \mathbb{N}, \forall x \in \mathbb{R}^n\right\} = 1 \tag{3.8}$$

or equivalently

$$\mathbb{P}\left\{|P_t^N|_{Mat(n\times n)} \leq M_t, \quad \forall N\in\mathbb{N}\right\}=1 \tag{3.9}$$

We finally notice that by construction $\overline{P}x$ is, for all $x\in\mathbb{R}^n$, predictable. We claim that for each $T\geq 0$ there exists a positive constant C_T, eventually depending on T and on known parameters, such that:

$$|\overline{P}_t|_{Mat(n\times n)} \leq C_T \qquad \forall\, 0\leq t\leq T.$$

In order to prove this property for $\bar{P}$ we fix $N>T$ and we write the fundamental relation for P^N corresponding to the control $u=0$:

$$\langle P_t^N x, x\rangle \leq \mathbb{E}^{\mathcal{F}_t}\int_t^T \left|\sqrt{S_s}X_s^{t,x,0}\right|^2 ds \tag{3.10}$$

where $X^{t,x,0}$ is solution to equation (3.1) corresponding to the control $u\equiv 0$. By standard estimates and by Gronwall lemma, there exists a positive constant K_T such that $\sup_{s\in[t,T]}\mathbb{E}^{\mathcal{F}_t}\left|X_s^{t,x,0}\right|^2 \leq K_T\left|x\right|^2$, and so, since S is bounded, we get that

$$\langle P_t^N x, x\rangle \leq C_T\left|x\right|^2, \qquad \mathbb{P}\text{-a.s.}$$

Again the exceptional set does not depend on x, so for all $t\in[0,T]$,

$$\mathbb{P}\left(\langle P_t^N x, x\rangle \leq C_T\left|x\right|^2, \forall N\in\mathbb{N}, \forall x\in\mathbb{R}^n\right)=1,$$

and

$$\mathbb{P}\left(\langle \overline{P}_t x, x\rangle \leq C_T\left|x\right|^2, \forall x\in\mathbb{R}^n\right)=1.$$

□

Remark 3.5. It is clear from the above proof that condition (3.5) is equivalent to the following one:

$$\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}[\langle S_s X_s^{t,x,u}, X_s^{t,x,u}\rangle + |u_s|^2]ds < M|x|^2. \tag{3.11}$$

where the constant M may depend on t.

Next we want to prove that $\overline{P}$ built in the previous lemma is the solution to the BSRDE (3.4). This is achieved through the control meaning of the solution of the Riccati equation. For $T>0$ fixed and for each $N>T$, we consider the following finite horizon stochastic optimal control problem: minimize the cost, over all admissible controls,

$$J(0,x,u)=\mathbb{E}\langle P_T^N X_T^{0,x,u}, X_T^{0,x,u}\rangle + \mathbb{E}\int_0^T [(\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds,$$

where $X^{0,x,u}$ is solution to equation (3.1). Let u^N be the optimal control, and X^N the corresponding optimal state. Let $\widetilde{u}$ be the optimal control, and $\widetilde{X}$ the corresponding optimal state for the following finite horizon optimal control problem: minimize the cost, over all admissible controls,

$$J(0,x,u)=\mathbb{E}\langle \overline{P}_T X_T^{0,x,u}, X_T^{0,x,u}\rangle + \mathbb{E}\int_0^T [(\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds,$$

Let us consider the so called *stochastic Hamiltonian system*

$$\begin{cases} dX_s = [A_s X_s - B_s(B_s^* y_s + \sum_{i=1}^d \left(D_s^i\right)^* z_s^i)]ds + \sum_{i=1}^d [C_s^i X_s + D_s^i(B_s^* y_s + \sum_{k=1}^d (D_s^k)^* z_s^k)]dW_s^i, \\ dy_s = -[A_s^* y_s + \sum_{i=1}^d \left(C_s^i\right)^* z_s^i + S_s X_s]ds + \sum_{i=1}^d z_s^i dW_s^i, \qquad t\leq s\leq T, \\ X_t = x, \\ y_T = \overline{P}_T X_T, \end{cases} \tag{3.12}$$

where $y, z^i \in \mathbb{R}^n$, for every $i = 1, ..., d$. By the so called stochastic maximum principle, the optimal control of the finite horizon control problem is given by

$$u_s = -\left(B_s^* y_s + \sum_{i=1}^d \left(D_s^i\right)^* z_s^i\right).$$

Let us consider the stochastic Hamiltonian systems relative to the optimal control u^N and to the optimal control $\widetilde{u}$, and let us denote by $\left(X^N, y^N, z^N\right)$ and by $\left(\widetilde{X}, \widetilde{y}, \widetilde{z}\right)$ the solutions of the corresponding stochastic Hamiltonian systems.

Lemma 3.6. $\mathbb{E}^{\mathcal{F}_t} \int_t^T [\left|\sqrt{S_s}(\widetilde{X}_s - X_s^N)\right|^2 + |B_s^*(\widetilde{y}_s - y_s^N) + \sum_{i=1}^d \left(D_s^i\right)^* (\widetilde{z}_s^i - z_s^{N,i})|^2] ds \to 0$ *as* $N \to \infty$.

Proof. The proof is based on the application of Itô formula to $\langle \widetilde{y}_t - y_t^N, \widetilde{X}_t - X_t^N \rangle$.

$$\mathbb{E}^{\mathcal{F}_t} \langle \widetilde{y}_T - y_T^N, \widetilde{X}_T - X_T^N \rangle = \mathbb{E}^{\mathcal{F}_t} \int_t^T d\langle \widetilde{y}_s - y_s^N, \widetilde{X}_s - X_s^N \rangle$$

$$= \mathbb{E}^{\mathcal{F}_t} \int_t^T \langle d\left(\widetilde{y}_s - y_s^N\right), \widetilde{X}_s - X_s^N \rangle + \mathbb{E}^{\mathcal{F}_t} \int_t^T \langle \widetilde{y}_s - y_s^N, d\left(\widetilde{X}_s - X_s^N\right) \rangle$$

$$+ \mathbb{E}^{\mathcal{F}_t} \int_t^T \sum_{i=1}^d \langle C_s^i \left(\widetilde{X}_s - X_s^N\right) - D_s^* \left(B_s^* \left(\widetilde{y}_s - y_s^N\right) + \sum_{k=1}^d \left(D_s^k\right)^* \left(\widetilde{z}_s^k - z_s^{N,k}\right)\right), \widetilde{z}_s^i - z_s^{N,i} \rangle ds$$

$$= -\mathbb{E}^{\mathcal{F}_t} \int_t^T [\left|B_s^* \left(\widetilde{y}_s - y_s^N\right)\right|^2 + \left|\sum_{i=1}^d \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right)\right|^2 + 2\sum_{i=1}^d \langle B_s^* \left(\widetilde{y}_s - y_s^N\right), \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right) \rangle] ds$$

$$- \mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|\sqrt{S_s} \left(\widetilde{X}_s - X_s^N\right)\right|^2 ds$$

$$= -\mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|B_s^* \left(\widetilde{y}_s - y_s^N\right) + \sum_{i=1}^d \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right)\right|^2 ds - \mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|\sqrt{S_s} \left(\widetilde{X}_s - X_s^N\right)\right|^2 ds.$$

Since $\widetilde{y}_T = \overline{P}_T \widetilde{X}_T$ and $y_T^N = P_T^N X_T^N$, we finally get

$$\mathbb{E}^{\mathcal{F}_t} \left\langle \overline{P}_T \widetilde{X}_T - P_T^N X_T^N, \widetilde{X}_T - X_T^N \right\rangle$$

$$= -\mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|B_s^* \left(\widetilde{y}_s - y_s^N\right) + \sum_{i=1}^d \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right)\right|^2 ds - \mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|\sqrt{S_s} \left(\widetilde{X}_s - X_s^N\right)\right|^2 ds.$$

By adding and subtracting $\mathbb{E}^{\mathcal{F}_t} \left\langle P_T^N \widetilde{X}_T, \widetilde{X}_T - X_T^N \right\rangle$,

$$\mathbb{E}^{\mathcal{F}_t} \left\langle P_T^N \left(\widetilde{X}_T - X_T^N\right), \widetilde{X}_T - X_T^N \right\rangle + \mathbb{E}^{\mathcal{F}_t} \left\langle \left(\overline{P}_T - P_T^N\right) \widetilde{X}_T, \widetilde{X}_T - X_T^N \right\rangle$$

$$= -\mathbb{E}^{\mathcal{F}_t} \int_t^T \left|B_s^* \left(\widetilde{y}_s - y_s^N\right) + \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right)\right|^2 ds - \mathbb{E}^{\mathcal{F}_t} \sum_{i=1}^d \int_t^T \left|\sqrt{S_s} \left(\widetilde{X}_s - X_s^N\right)\right|^2 ds.$$

Since $\left\langle P_T^N \left(\widetilde{X}_T - X_T^N\right), \widetilde{X}_T - X_T^N \right\rangle \geq 0$, and by definition $\mathbb{E}^{\mathcal{F}_t} \left\langle \left(\widetilde{P}_T - P_T^N\right) \widetilde{X}_T, \widetilde{X}_T - X_T^N \right\rangle \to 0$ for N sufficiently large, also

$$\mathbb{E}^{\mathcal{F}_t} \int_t^T \left|B_s^* \left(\widetilde{y}_s - y_s^N\right) + \sum_{i=1}^d \left(D_s^i\right)^* \left(\widetilde{z}_s^i - z_s^{N,i}\right)\right|^2 ds + \mathbb{E}^{\mathcal{F}_t} \int_t^T \left|\sqrt{S_s} \left(\widetilde{X}_s - X_s^N\right)\right|^2 ds \to 0$$

as $N \to \infty$. In particular this means that $\mathbb{E}^{\mathcal{F}_t} \int_t^T \left|\widetilde{u}_s - u_s^N\right|^2 ds \to 0$ as $N \to \infty$. □

As a consequence of the previous results we deduce the following:

Corollary 3.7. *Assume hypothesis 3.1 and that (A,B,C,D) is stabilizable relatively to $\sqrt{S}$. The process $\overline{P}$ is the minimal solution of the Riccati equation in the sense of definition 3.2.*

Proof. Fix $T<N$ and on $[0,T]$ consider the Riccati equation

$$\begin{cases} -dP_t^N = G\left(A_t, B_t, C_t, D_t; S_t; P_t^N, Q_t^N\right) dt + \sum_{i=1}^{d} Q_t^{i,N} dW_t^i \\ P_T^N = P^N(T). \end{cases} \tag{3.13}$$

Then

$$\langle P_t^N x, x\rangle = \mathbb{E}^{\mathcal{F}_t} \langle P_T^N X_T^N, X_T^N\rangle + \mathbb{E}^{\mathcal{F}_t} \int_t^T [\langle S_r X_r^N, X_r^N\rangle + |u_r^N|^2] dr \tag{3.14}$$

By lemma 3.6 we deduce that

$$\mathbb{E}^{\mathcal{F}_t} \left\langle P_T^N \left(\widetilde{X}_T - X_T^N\right), \widetilde{X}_T - X_T^N \right\rangle \to 0 \qquad \text{as } N\to+\infty.$$

So $\mathbb{E}^{\mathcal{F}_t} \langle P_T^N X_T^N, X_T^N\rangle \to \mathbb{E}^{\mathcal{F}_t} \left\langle \overline{P}_T \widetilde{X}_T, \widetilde{X}_T \right\rangle$ as $N\to+\infty$, since

$$\mathbb{E}^{\mathcal{F}_t} \langle P_T^N X_T^N, X_T^N\rangle = \mathbb{E}^{\mathcal{F}_t} \left\langle P_T^N \left(\widetilde{X}_T - X_T^N\right), \widetilde{X}_T - X_T^N \right\rangle + 2\langle P_T^N \widetilde{X}_T, X_T^N\rangle - \langle P_T^N \widetilde{X}_T, \widetilde{X}_T\rangle.$$

By the construction of $\overline{P}$ and by lemma 3.6, we have that, letting $N\to+\infty$ in (3.14)

$$\langle \overline{P}_t x, x\rangle = \mathbb{E}^{\mathcal{F}_t} \left\langle \overline{P}_T \widetilde{X}_T, \widetilde{X}_T \right\rangle + \mathbb{E}^{\mathcal{F}_t} \int_t^T [\left\langle S_r \widetilde{X}_r, \widetilde{X}_r\right\rangle + |\widetilde{u}_r|^2] dr$$

So $\overline{P}$ is the minimal solution of the Riccati equation in the sense of definition 3.2. □

Remark 3.8. Thanks to its construction it is easy to prove that $(\overline{P}, \overline{Q})$ is the minimal solution, in the sense that if another couple (P,Q) is a solution to the Riccati equation then $P-\overline{P}$ is a non-negative matrix, see also Corollary 3.3 in [12].

By the previous calculations, we can now solve the optimal control problem with infinite horizon, when $f=0$.

Theorem 3.9. *If $A1)-A2)$ hold true and if (A,B,C,D) is stabilizable relatively to S, given $x\in\mathbb{R}^n$, then:*

(1) *there exists a unique optimal control $\overline{u}\in L^2\left(\Omega\times[0,+\infty), \mathbb{R}^k\right)$ such that*

$$J_\infty(0,x,\overline{u}) = \inf_{u\in L^2(\Omega\times[0,+\infty),\mathbb{R}^k)} J_\infty(0,x,u)$$

(2) *The process $\overline{P}$ defined in lemma 3.4 is the minimal solution of the Riccati equation.*

(3) *If $\overline{X}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{X}$ is the unique mild solution to the* closed loop *equation:*

$$\begin{cases} d\overline{X}_t = \left[A\overline{X}_t - B_t \left(I + \sum_{i=1}^d (D_t^i)^* \overline{P}_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + (C_t^i)^* \overline{P}_t D_t^i \right) \right)^* \overline{X}_t \right] dt + \\ \sum_{i=1}^d \left[C_t^i \overline{X}_t - D_t^i \left(I + \sum_{i=1}^d (D_t^i)^* \overline{P}_t D_t^i \right)^{-1} \left(\overline{P}_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + (C_t^i)^* \overline{P}_t D_t^i \right) \right)^* \overline{X}_t \right] dW_t, \\ \overline{X}_0 = x. \end{cases} \tag{3.15}$$

(4) *The following feedback law holds $\mathbb{P}$-a.s. for almost every t:*

$$\overline{u}_t = -\left(I + \sum_{i=1}^d (D_t^i)^* \overline{P}_t D_t^i \right)^{-1} \left(\overline{P}_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + (C_t^i)^* \overline{P}_t D_t^i \right) \right)^* \overline{X}_t. \tag{3.16}$$

(5) *The optimal cost is given by $J_\infty(0,x,\overline{u}) = \langle \overline{P}_0 x, x\rangle$.*

The proof of this theorem is similar, and more immediate, to the proof of theorem 5.1, which is given in detail in section 5. In particular we deduce that

$$\langle \overline{P}_t x, x\rangle = \mathbb{E}^{\mathcal{F}_t} \int_t^\infty [\Big\langle S_r \widetilde{X}_r^{t,x,u}, \widetilde{X}_r^{t,x,u} \Big\rangle + |\widetilde{u}_r|^2] dr. \tag{3.17}$$

4. The infinite horizon dual equation

We first introduce some definitions. We say that a solution P of equation 3.4 is bounded, if there exists a constant $M > 0$ such that for every $t \geq 0$

$$|P_t| \leq M \qquad\qquad \mathbb{P}-a.s.$$

Whenever the constant $M_{t,x}$ that appears in definition 3.3 can be chosen independently of t, then the minimal solution $\overline{P}$ is automatically bounded.

Definition 4.1. Let P be a solution to 3.4. We say that P stabilizes (A, B, C, D) relatively to the identity I if for every $t > 0$ and $x \in \mathbb{R}^n$ there exists a positive constant M, independent of t, such that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |X^{t,x}(r)|^2, dr \leq M \qquad \mathbb{P}-a.s., \tag{4.1}$$

where $X^{t,x}$ is the mild solution to:

$$\begin{cases} d\overline{X}_t = \left[A\overline{X}_t - B_t \left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i \right) \right)^* \overline{X}_t \right] dt + \\ \sum_{i=1}^d \left[C_s^i \overline{X}_t - D_s^i \left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^d \left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i \right) \right)^* \overline{X}_t \right] dW_t, \\ \overline{X}_0 = x \end{cases} \tag{4.2}$$

From now on we assume that the process $\overline{P}$ is bounded and stabilizes (A, B, C, D) with respect to the identity I.

Remark 4.2. It is possible to verify in some concrete situations that (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$ and that $\overline{P}$ stabilizes (A, B, C, D) relatively to the identity I. Here we present the case when, for some $\alpha > 0$, A and C satisfy

$$\langle A_t x, x\rangle + \frac{1}{2}\langle C_t x, C_t x\rangle \leq -\alpha |x|^2, \tag{4.3}$$

for every $t \geq 0$ and $x \in \mathbb{R}^n$, then (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$ uniformly in time. Indeed, by taking the control $u = 0$, applying the Itô formula to the state equation we get, for $0 \leq t \leq s$,

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t} |X_s^{t,x,0}|^2 &\leq |x|^2 + 2\mathbb{E}^{\mathcal{F}_t} \int_t^s \langle A_r X_r^{t,x,0}, X_r^{t,x,0}\rangle dr + \mathbb{E}^{\mathcal{F}_t} \int_t^s \langle C_r X_r^{t,x,0}, C_r X_r^{t,x,0}\rangle dr \\ &\leq |x|^2 - 2\alpha \mathbb{E}^{\mathcal{F}_t} \int_t^s |X_r|^2 dr. \end{aligned}$$

By the Gronwall lemma we get

$$\mathbb{E}^{\mathcal{F}_t} |X_s^{t,x,0}|^2 \leq |x|^2 e^{-2\alpha(s-t)}.$$

So for every $0 \leq t \leq T$

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |\sqrt{S} X_r^{t,x,0}|^2 dr \leq M_x,$$

where M_x is a constant dependent on the initial condition x, but independent on the initial time t. So, according to definition 3.3, (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$, uniformly in time. Moreover, assuming that $S \geq \epsilon I$, for some $\epsilon > 0$, by (4.3), we also get that $\overline{P}$ stabilizes (A, B, C, D) relatively to the identity I. Indeed, by the previous calculations, denoting by $\overline{X}$ and $\overline{u}$ respectively the optimal state and the optimal control for the infinite horizon control problem with $f = 0$, it follows that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} [|\overline{X}_r|^2 + |\overline{u}|^2] dr \leq \mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |X_r^{t,x,0}|^2 dr \leq M_x,$$

which immediately implies (4.1).

Remark 4.3. Here and in the following sections we need to adapt the Dakto Theorem to this case, see for instance the proof of Proposition 4.6 in [12] and also [10] and [13], in order to prove an exponential bound for the process X which solves the following equation

$$\begin{cases} dX_s = -H_s^* X_s ds - \sum_{i=1}^d \left(K_s^i\right)^* X_s dW_s^i, \quad s \geq t \\ X_t = x \end{cases}$$

Indeed, for every $s > t$, we get

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t}|X_s|^2 &\leq C[|x|^2 + \mathbb{E}^{\mathcal{F}_t}|\int_t^s H_r^* X_r dr|^2 + \mathbb{E}^{\mathcal{F}_t}|\int_t^s \sum_{i=1}^d \left(K_r^i\right)^* X_r dW_r^i dr|^2] \\ &\leq C[|x|^2 + \mathbb{E}^{\mathcal{F}_t}\int_t^s |H_r^* X_r|^2 dr + \mathbb{E}^{\mathcal{F}_t}\int_t^s \sum_{i=1}^d |(K_r^i)^* X_r|^2 dr] \\ &\leq C[|x|^2 + \mathbb{E}^{\mathcal{F}_t}\int_t^s |A_r^* X_r|^2 + |B_r \overline{u}_r|^2 dr + \mathbb{E}^{\mathcal{F}_t}\int_t^s |X_r|^2 dr \\ &+ \mathbb{E}^{\mathcal{F}_t}\int_t^s \sum_{i=1}^d |(C_r^i)^* X_r|^2 + \sum_{i=1}^d |(D_r^i)^* \overline{u}_r|^2 dr] \end{aligned}$$

where $\overline{u}$ is the optimal control defined in (3.16), and so, by (3.17), for every $t > 0$,

$$\mathbb{E}^{\mathcal{F}_t}\int_t^s |\overline{u}_r|^2 dr \leq \langle \overline{P}_t x, x\rangle \leq C|x|^2,$$

where the last inequality follows since we are assuming that $\overline{P}$ is bounded. Since C and D are bounded, by applying the Gronwall lemma we get

$$\mathbb{E}^{\mathcal{F}_t}|X_s|^2 \leq M e^{M(s-t)}|x|^2$$

for some positive constant M. By adapting the Datko Theorem , there exist $K, a > 0$ such that for every $s \geq t$:

$$\mathbb{E}^{\mathcal{F}_t}|X_s|^2 \leq K e^{-a(s-t)}|x|^2 \qquad \mathbb{P}-\text{a.s.} \tag{4.4}$$

In order to study the optimal control problem with infinite horizon and with $f \neq 0$, we need to study the BSDE on $[0, \infty)$,

$$dr_t = -H_t^* r_t dt - P_t f_t dt - \sum_{i=1}^d \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^d g_t^i dW_t^i, \qquad t \geq 0 \tag{4.5}$$

where the final condition has disappeared but we ask that the solution can be extended to the whole positive real axis. We make the following assumption on f:

Hypothesis 4.4. *f is a process in $L^2(\Omega \times [0, +\infty), \mathbb{R}^n) \cap L^\infty(\Omega \times [0, +\infty), \mathbb{R}^n)$.*

Proposition 4.5. *Let hypotheses 3.1 and 4.4 hold true and assume that $\overline{P}$ is bounded and stabilizes (A, B, C, D) with respect to the identity I. Then equation (4.5) admits a solution $(\bar{r}, \bar{g}) \in L^2(\Omega \times [0, +\infty), \mathbb{R}^n) \times L^2(\Omega \times [0, T], \mathbb{R}^{n\times d})$, for every $T > 0$.*

Proof. For integer $N > 0$, we consider the BSDEs

$$\begin{cases} dr_t^N = -H_t^* r_t^N dt - P_t f_t dt - \sum_{i=1}^d \left(K_t^i\right)^* g_t^{i,N} dt + \sum_{i=1}^d g_t^{i,N} dW_t^i, \quad t \in [0, T] \\ r_N^N = 0. \end{cases} \tag{4.6}$$

By proposition 2.9, we know that equation (4.6) admits a unique solution $\left(r^N, g^N\right)$ that belongs to $L^2\left(\Omega, C\left([0,N], \mathbb{R}^n\right)\right) \times L^2\left([0,N] \times \Omega, \mathbb{R}^{n\times d}\right)$, for every $N \in \mathbb{N}$. The aim is to write a *duality* relation, see lemma 2.6, between r^N and the process X^N, solution of the following equation

$$\begin{cases} dX_s^N = -H_s^* X_s^N ds + \sum_{i=1}^d \left(K_s^i\right)^* X_s^N dW_s^i, \quad s \in [t,N] \\ X_t^N = r_t^N. \end{cases}$$

By *duality* between r^N and the process X^N, and by estimate (4.4) we get

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t}|r_t^N|^2 &= \mathbb{E}^{\mathcal{F}_t}\int_t^N \langle \overline{P}_s f_s, X_s^N\rangle ds \\ &\leq C\int_t^N \|\overline{P}_s\|_{L^\infty(\Omega)}\|f_s\|_{L^\infty(\Omega)} e^{-\frac{a}{2}(s-t)}(\mathbb{E}^{\mathcal{F}_t}|r_t^N|^2)^{\frac{1}{2}} ds \\ &\leq \frac{C}{\mu}\int_t^N e^{-\frac{a}{2}(s-t)}\|\overline{P}_s\|^2_{L^\infty(\Omega)}\|f_s\|^2_{L^\infty(\Omega)} ds + \mu\frac{2}{a}\mathbb{E}^{\mathcal{F}_t}|r_t^N|^2, \end{aligned}$$

where we can take $\mu > 0$ such that $\mu\dfrac{2}{a} = \dfrac{1}{2}$. So we get

$$|r_t^N|^2 \leq C,$$

where now C is a constant depending on a, $\overline{P}$, f, but C does not depend on N.
So also $\sup_{t\geq 0}\mathbb{E}|r_t^N|^2 \leq C$. By computing $d|r_t^N|^2$, see e.g. relation (2.23) and by the previous estimate we get for every fixed $T > 0$,

$$\mathbb{E}\int_0^T \sum_{i=1}^d |g_s^{i,N}|^2 ds \leq C,$$

where $C > 0$ does not depend on N. Then we can conclude that for every fixed $T > 0$ there exists $\bar{r}$ and $\bar{g}$ such that $r^N \rightharpoonup \bar{r}$ in $L^2(\Omega \times [0,T], \mathbb{R}^n)$ and $g^N \rightharpoonup \bar{g}$ in $L^2(\Omega \times [0,T], \mathbb{R}^{n\times d})$. Moreover, $(\bar{r}, \bar{g})$ satisfy

$$\bar{r}_t = \bar{r}_T + \int_t^T H_s^* \bar{r}_s ds + \int_t^T \overline{P}_s f_s ds + \int_t^T \sum_{i=1}^d (K_s^i)^* g_s^i ds - \int_t^T \sum_{i=1}^d (g_s^i)^* dW_s^i.$$

So the pair $(\bar{r}, \bar{g})$ is a solution to the elliptic dual equation (4.5). Since $T > 0$ is arbitrarily, $(\bar{r}, \bar{g})$ is defined on the whole $[0, +\infty)$. It remains to prove that $\bar{r} \in L^2(\Omega \times [0,+\infty), \mathbb{R}^n)$. We set

$$\eta_t^N = \begin{cases} \bar{r}_t & 0 \leq t \leq N, \\ 0 & t > N. \end{cases}$$

So $\eta_t^N \in L^2(\Omega \times [0,+\infty), \mathbb{R}^n)$. We write a *duality relation*, see (2.18) between $\bar{r}$ and X^{η^N} solution of the following stochastic differential equation

$$\begin{cases} dX_s^{\eta^N} = H_s X_s^{\eta^N} ds + \sum_{i=1}^d K_s^i X_s^{\eta^N} dW_s^i + \eta_s^N ds, \\ X_t^{\eta^N} = 0. \end{cases}$$

By *duality* we get

$$\mathbb{E}\int_0^N |\bar{r}_s|^2 ds = \mathbb{E}\int_0^N \langle \overline{P}_s f_s, X_s^{\eta^N}\rangle ds + \mathbb{E}\langle \bar{r}_N, X_N^{\eta^N}\rangle.$$

Letting $N \to \infty$ on both sides we get on the left hand side

$$\underline{\lim}_{N\to\infty}\mathbb{E}\int_0^N |\bar{r}_s|^2 ds = \lim_{N\to\infty}\mathbb{E}\int_0^N |\bar{r}_s|^2 ds = \mathbb{E}\int_0^{+\infty} |\bar{r}_s|^2 ds,$$

by monotone convergence. On the right hand side, also by remark 4.3, we get

$$
\begin{aligned}
&\underline{\lim}_{N\to\infty}\mathbb{E}\int_0^N\langle P_s f_s, X_s^{\eta^N}\rangle ds+\mathbb{E}\langle\bar{r}_N, X_N^{\eta^N}\rangle\\
&\leq \underline{\lim}_{N\to\infty}\frac{1}{2}\|P\|^2_{L^\infty(\Omega\times[0,+\infty))}\mathbb{E}\int_0^N|f_s|^2ds+\frac{1}{2}\mathbb{E}\int_0^N|X_s^{\eta^N}|^2ds+\frac{1}{2}\mathbb{E}|\bar{r}_N|^2+\frac{1}{2}\mathbb{E}|X_N^{\eta^N}|^2\\
&\leq \frac{1}{2}\|P\|^2_{L^\infty(\Omega\times[0,+\infty))}\|f\|^2_{L^2(\Omega\times[0,+\infty))}ds+\underline{\lim}_{N\to\infty}\frac{1}{2}\int_0^N e^{-2aN}|\eta^N|^2ds+C+e^{-aN}\mathbb{E}|\bar{r}_N|^2\\
&\frac{1}{2}\|P\|^2_{L^\infty(\Omega\times[0,+\infty))}\|f\|^2_{L^2(\Omega\times[0,+\infty))}+\frac{1}{2}\mathbb{E}\int_0^\infty|\bar{r}_s|^2ds+C.
\end{aligned}
$$

Putting together these inequalities we get

$$
\mathbb{E}\int_0^\infty|\bar{r}_s|^2ds\leq\frac{1}{2}\|P\|^2_{L^\infty(\Omega\times[0,+\infty))}\|f\|^2_{L^2(\Omega\times[0,+\infty))}+C,
$$

and this concludes the proof. □

Remark 4.6. As a consequence of the previous proof, we get

$$
\mathbb{E}|\bar{r}_T|^2\to 0 \text{ as } T\to\infty. \tag{4.7}
$$

Remark 4.7. Equation (4.5) has non Lipschitz coefficients and is multidimensional BSDE thus we can not use the Girsanov Theorem, as done in [6], to get rid of the terms involving K. Moreover the typical monotonicity assumptions on the coefficients of this infinite horizon BSDE, see [7], are replaced by the finite cost condition and by the requirement that the minimal solution $(\overline{P},\overline{Q})$ of (1.10) stabilize the coefficients relatively the identity, see definition 4.1.

5. Synthesis of the optimal control in the infinite horizon case

We consider the following stochastic differential equation for $t\geq 0$:

$$
\begin{cases}
dX_s=(A_sX_s+B_su_s)ds+\sum_{i=1}^d\left(C_s^iX_s+D_s^iu_s\right)dW_s^i+f_sds \quad s\geq t\\
X_t=x,
\end{cases} \tag{5.1}
$$

Our purpose is to minimize with respect to u the cost functional,

$$
J_\infty(0,x,u)=\mathbb{E}\int_0^{+\infty}[\langle S_sX_s^{0,x,u},X_s^{0,x,u}\rangle+|u_s|^2]ds \tag{5.2}
$$

We also introduce the following random variables, for $t\in[0,+\infty)$:

$$
J_\infty(t,x,u)=\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}[\langle S_sX_s^{t,x,u},X_s^{t,x,u}\rangle+|u_s|^2]ds
$$

Theorem 5.1. *Let hypotheses 3.1 and 4.4 hold true, let (A,B,C,D) be stabilizable relatively to S, given $x\in\mathbb{R}^n$ and assume that the process $\overline{P}$ is bounded and stabilizes (A,B,C,D) with respect to the identity I, then:*

(1) *there exists a unique optimal control $\overline{u}\in L^2\left(\Omega\times[0,+\infty),\mathbb{R}^k\right)$ such that*

$$
J_\infty(0,x,\overline{u})=\inf_{u\in L^2(\Omega\times[0,+\infty),\mathbb{R}^k)}J_\infty(0,x,u)
$$

(2) *If $\overline{X}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{X}$ is the unique mild solution to the* closed loop *equation for:*

$$\begin{cases} d\overline{X}_t = \left[A_t\overline{X}_t - B_t\left(f(t,\overline{P}_t,\overline{Q}_t)\overline{X}_t + (I+\sum_{i=1}^d \left(D_t^i\right)^*\overline{P}_tD_t^i)^{-1}(B_t^*r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i)\right)\right] dt+ \\ \sum_{i=1}^d \left[C_s^i\overline{X}_t - D_s^i(f(t,\overline{P}_t,\overline{Q}_t)\overline{X}_t + \left(I+\sum_{i=1}^d \left(D_t^i\right)^*\overline{P}_tD_t^i\right)^{-1}(B_t^*r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i))\right] dW_t^i,\ t>0 \\ \overline{X}_0 = x \end{cases} \tag{5.3}$$

(3) *The following feedback law holds $\mathbb{P}$-a.s. for almost every $t \geq 0$.*

$$\overline{u}_t = -\left(I+\sum_{i=1}^d \left(D_t^i\right)^*\overline{P}_tD_t^i\right)^{-1}\left(\overline{P}_tB_t + \sum_{i=1}^d \left(\overline{Q}_t^iD_t^i + \left(C_t^i\right)^*\overline{Q}_tD_t^i\right)\right)^*\overline{X}_t + B_t^*r_t + \sum_{i=1}^d \left((D_t^i)^*g_t^i\right). \tag{5.4}$$

(4) *The optimal cost is given by*

$$\begin{aligned} J(0,x,\overline{u}) &= \langle \overline{P}_0x,x\rangle + 2\langle r_0,x\rangle + 2\mathbb{E}\int_0^\infty \langle r_s,f_s\rangle ds \\ &- \mathbb{E}\int_0^\infty |(I+\sum_{i=1}^d \left(D_t^i\right)^*\overline{P}_tD_t^i)^{-1}(B_t^*r_t + \sum_{i=1}^d \left(D_t^i\right)^* g_t^i)|^2 ds. \end{aligned}$$

Proof. By computing $d\langle {}_sX_s,X_s\rangle + 2\langle \bar{r}_s,X_s\rangle$ we get, for every $T>0$,

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle S_sX_s,X_s\rangle + |u_s|^2]ds = \mathbb{E}\langle \overline{P}_tx,x\rangle - \mathbb{E}^{\mathcal{F}_t}\langle \overline{P}_TX_T,X_T\rangle + 2\langle \bar{r}_t,x\rangle - 2\mathbb{E}^{\mathcal{F}_t}\int_t^T \langle \bar{r}_s,f_s\rangle ds \\ &+ \mathbb{E}^{\mathcal{F}_t}\int_t^T |\left(I+\sum_{i=1}^d \left(D_s^i\right)^*\overline{P}_sD_s^i\right)^{1/2}\left(u_s + (I+\sum_{i=1}^d \left(D_s^i\right)^*\overline{P}_sD_s^i)^{-1} * \right. \\ &* \left.\left(\overline{P}_sB_s + \sum_{i=1}^d \left(\overline{Q}_s^iD_s^i + \left(C_s^i\right)^*\overline{P}_sD_s^i\right)\right)^* X_s + B_s^*\bar{r}_s + \sum_{i=1}^d D_s^i(\bar{g}_s^i)^*\right)|^2 ds \\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^T |\left(I+\sum_{i=1}^d \left(D_s^i\right)^*\overline{P}_sD_s^i\right)^{-1}(B_s^*\bar{r}_s + \sum_{i=1}^d \left(D_s^i\right)^*\bar{g}_s^i)|^2 ds. \end{aligned} \tag{5.5}$$

and so

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle S_sX_s,X_s\rangle + |u_s|^2]ds &\leq \mathbb{E}\langle \overline{P}_tx,x\rangle + 2\langle \bar{r}_t,x\rangle - 2\mathbb{E}^{\mathcal{F}_t}\int_t^T \langle \bar{r}_s,f_s\rangle ds \\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^T \left(I+\sum_{i=1}^d \left(D_s^i\right)^*\overline{P}_sD_s^i\right)^{-1} |B_s^*\bar{r}_s + \sum_{i=1}^d \left(D_s^i\right)^*\bar{g}_s^i|^2 ds. \end{aligned}$$

Since $\overline{P}$ is bounded, and by L^2 estimates on $\bar{r}$ we get for every $T>0$

$$\mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle S_sX_s,X_s\rangle + |u_s|^2]ds \leq C,$$

and so also

$$\mathbb{E}^{\mathcal{F}_t}\int_t^T |\left(I+\sum_{i=1}^d \left(D_s^i\right)^*\overline{P}_sD_s^i\right)^{-1}(B_s^*\bar{r}_s + \sum_{i=1}^d \left(D_s^i\right)^*\bar{g}_s^i)^2 ds \leq C,$$

where C is a constant independent of T and on t. So

$$\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty} [\langle S_sX_s,X_s\rangle + |u_s|^2]ds \leq C,$$

and

$$\mathbb{E}^{\mathcal{F}_t}\int_t^{\infty} |\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^* P_s D_s^i\right)^{-1} (B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2 ds \leq C. \tag{5.6}$$

Since $0\leq \sum_{i=1}^{d}\left(D_s^i\right)^* \overline{P}_s D_s^i \leq C$, where C is a constant independent of s, also

$$\mathbb{E}^{\mathcal{F}_t}\int_t^{\infty} |B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i|^2 ds \leq C.$$

Moreover we obtain, letting $T\to+\infty$ and choosing $t=0$,

$$\begin{aligned}&\mathbb{E}\int_0^{+\infty}[\langle S_sX_s,X_s\rangle+|u_s|^2]ds\leq \mathbb{E}\langle \bar{P}_0x,x\rangle+2\mathbb{E}\langle \bar{r},x\rangle-2\mathbb{E}\int_t^{+\infty}\langle \bar{r}_s,f_s\rangle ds\\&-\mathbb{E}\int_0^{+\infty}\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{P}_sD_s^i\right)^{-1}|B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i|^2ds.\end{aligned}$$

Now we need to prove the opposite inequality. We compute $d\langle P_s^N X_s,X_s\rangle+2\langle \bar{r}_s,X_s\rangle$:

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\int_t^{N}[\langle S_sX_s,X_s\rangle+|u_s|^2]ds=\mathbb{E}\langle P_t^Nx,x\rangle+2\langle \bar{r}_t,x\rangle-2\mathbb{E}^{\mathcal{F}_t}\int_t^{N}\langle \bar{r}_s,f_s\rangle ds\\&+\mathbb{E}^{\mathcal{F}_t}\int_t^{N}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i\right)^{1/2}\left(u_s+(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i)^{-1}*\right.\\&\left.*\left(P_s^NB_s+\sum_{i=1}^{d}\left(Q_s^{i,N}D_s^i+\left(C_s^i\right)^*P_s^ND_s^i\right)\right)^*X_s+B_s^*\bar{r}_s+\sum_{i=1}^{d}D_s^i(\bar{g}_s^i)^*\right)|^2ds\\&-\mathbb{E}^{\mathcal{F}_t}\int_t^{N}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i\right)^{-1}(B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2ds.\end{aligned}\tag{5.7}$$

We observe that,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\int_t^{N}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i\right)^{-1}(B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2ds\\&\to\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{P}_sD_s^i\right)^{-1}|B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i|^2ds,\end{aligned}$$

indeed, by dominated convergence that we can apply thanks to estimate (5.6),

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i\right)^{-1}(B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2ds\\&\to\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{P}_sD_s^i\right)^{-1}(B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2ds\\&\mathbb{E}^{\mathcal{F}_t}\int_N^{+\infty}|\left(I+\sum_{i=1}^{d}\left(D_s^i\right)^*P_s^ND_s^i\right)^{-1}(B_s^*\bar{r}_s+\sum_{i=1}^{d}\left(D_s^i\right)^*\bar{g}_s^i)|^2ds\to 0\end{aligned}$$

So, by (5.7) we get for every admissible control u

$$\mathbb{E}\int_0^{+\infty}[\langle S_sX_s,X_s\rangle+|u_s|^2]ds\geq \mathbb{E}\langle \bar{P}_0x,x\rangle+2\mathbb{E}\langle \bar{r}_,x\rangle-2\mathbb{E}\int_t^{+\infty}\langle \bar{r}_s,f_s\rangle ds$$
$$-\mathbb{E}\int_0^{+\infty}\left(I+\sum_{i=1}^d\left(D_s^i\right)^*\bar{P}_sD_s^i\right)^{-1}|B_s^*\bar{r}_s+\sum_{i=1}^d\left(D_s^i\right)^*\bar{g}_s^i|^2ds.$$

The theorem now easily follows. □

6. Ergodic control

In this section we consider cost functional depending only on the asymptotic behaviour of the state (ergodic control). To do it we first consider discounted cost functional that fit the assumptions of section 5 and then we compute a suitable limit of the discounted cost. Namely, we consider the discounted cost functional

$$J_\alpha(0,x,u)=\mathbb{E}\int_0^{+\infty}e^{-2\alpha s}[\langle S_sX_s^{0,x,u},X_s^{0,x,u}\rangle+|u_s|^2]ds, \tag{6.1}$$

where X is solution to equation (5.1), with A, B, C and D satisfying hypothesis 3.1 and $f\in L^\infty(\Omega\times[0,+\infty))$. When the coefficients are deterministic the problem has been extensively studied, see e.g. [2] and [24].

Throughout this section we assume that

Hypothesis 6.1. *We will make the following assumptions:*

- *$S\geq\epsilon I$, for some $\epsilon>0$.*
- *(A,B,C,D) is stabilizable relatively to S.*
- *The first component of the minimal solution $\overline{P}$ is bounded in time.*

Notice that these conditions implies that $(\overline{P},\overline{Q})$ stabilize (A,B,C,D) relatively the identity.

Our purpose is to minimize the discounted cost functional with respect to every admissible control u. We define the set of admissible controls as

$$\mathcal{U}_\alpha=\left\{u\in L^2(\Omega\times[0,+\infty)):\mathbb{E}\int_0^{+\infty}e^{-2\alpha s}[\langle S_sX_s^{0,x,u},X_s^{0,x,u}\rangle+|u_s|^2]ds<+\infty\right\}.$$

Fixed $\alpha>0$, we define $X_s^\alpha=e^{-\alpha s}X_s$ and $u_s^\alpha=e^{-\alpha s}u_s$: we note that if $u\in\mathcal{U}_\alpha$, then $u^\alpha\in\mathcal{U}$. Moreover we set $A_s^\alpha=A_s-\alpha I$ and $f_s^\alpha=e^{-\alpha s}f_s$, and $f^\alpha\in L^2(\Omega\times[0,+\infty))\cap L^\infty(\Omega\times[0,+\infty))$. X_s^α is solution to equation

$$\begin{cases} dX_s^\alpha=(A_s^\alpha X_s^\alpha+B_sv_s^\alpha)ds+\sum_{i=1}^d\left(C_s^iX_s^\alpha+D_s^iv_s^\alpha\right)dW_s^i+f_s^\alpha ds\quad s\geq 0\\ X_0^\alpha=x, \end{cases} \tag{6.2}$$

By the definition of X^α, we note that if (A,B,C,D) is stabilizable with respect to the identity, then (A^α,B,C,D) also is. We also denote by (P^α,Q^α) the solution of the infinite horizon Riccati equation (3.4), with A^α in the place of A. Since, for $0<\alpha<1$, A^α is uniformly bounded in α, also P^α is uniformly bounded in α. Now we apply theorem 5.1 to the control problem for the discounted cost J_α. Let us denote by (r^α,g^α) the solution of the BSDE obtained by equation (4.5), where f is replaced with f^α, and H and K are replaced respectively by H^α and K^α. H^α and K^α are defined as in (2.9), with A^α and P^α respectively in the place of A and P.

Theorem 6.2. *Let hypotheses 3.1 and 4.4 hold true; assume that $f\in L^\infty(\Omega\times[0,+\infty))$ and that , given $x\in\mathbb{R}^n$, then:*

1. *there exists a unique optimal control $\overline{u}^\alpha\in L^2\left(\Omega\times[0,+\infty),\mathbb{R}^k\right)$ such that*

$$J^\alpha\left(0,x,\overline{u}^\alpha\right)=\inf_{u\in\mathcal{U}_\alpha}J^\alpha\left(0,x,u\right)$$

(2) *The following feedback law holds $\mathbb{P}$-a.s. for almost every $t \geq 0$:*

$$\overline{u}_t^\alpha = -\left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t^\alpha D_t^i\right)^{-1} \left(P_t^\alpha B_t + \sum_{i=1}^d \left(Q_t^{\alpha,i} D_t^i + \left(C_t^i\right)^* P_t^\alpha D_t^i\right)\right)^* \overline{X}_t^\alpha + B_t^* r_t^\alpha + \sum_{i=1}^d D_t^i (g_t^{\alpha,i})^*, \tag{6.3}$$

where $\overline{X}^\alpha$ is the optimal state.

(3) *The optimal cost $J^\alpha(0, x, \overline{u}^\alpha) := \overline{J}^\alpha(x)$ is given by*

$$\begin{aligned}\overline{J}^\alpha(x) &= \langle P_0^\alpha x, x\rangle + 2\langle r_0^\alpha, x\rangle + 2\mathbb{E}\int_0^\infty \langle r_s^\alpha, f_s^\alpha\rangle ds \\ &\quad - \mathbb{E}\int_0^\infty |(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t^\alpha D_t^i)^{-1}(B_t^* r_t^\alpha + \sum_{i=1}^d \left(D_t^i\right)^* g_t^{\alpha,i})|^2 ds. \end{aligned}\tag{6.4}$$

The optimal cost $\overline{J}^\alpha(x) \to +\infty$ as $\alpha \to 0$. We want to compute

$$\lim_{\alpha\to 0} \alpha \overline{J}^\alpha(x).$$

In order to do this, we need some convergence results, the first concerning the Riccati equation. To prove this convergence, we note that, by applying the Datko theorem, we are able to prove estimates independent on α.

Remark 6.3. By the Dakto Theorem, see also remark 4.3, we can prove an exponential bound for the process X^α which solves the following equation

$$\begin{cases} dX_s^\alpha = -(H_s^\alpha)^* X_s^\alpha ds - \sum_{i=1}^d \left(K_s^{\alpha,i}\right)^* X_s^\alpha dW_s^i, \quad s \geq t \\ X_t^\alpha = x \end{cases}$$

We can conclude that there exist $K, a > 0$, independent on α such that for every $s \geq t$:

$$\mathbb{E}^{\mathcal{F}_t}|X_s^\alpha|^2 \leq K e^{-a(s-t)}|x|^2 \qquad \mathbb{P}-\text{a.s.} \tag{6.5}$$

Lemma 6.4. *Assume that hypothesis 3.1 holds true, that $f \in L^\infty(\Omega\times[0,+\infty))$. Then $P_\alpha(t) \to \overline{P}(t)$ as $\alpha \to 0$ for all $t \geq 0$, where $\overline{P}$ is the minimal solution of the BRSE.*

Proof. We can consider the case $t = 0$ without loss of generality.

Since $\langle P_0 x, x\rangle$, respectively $\langle P_0^\alpha x, x\rangle$, is the optimal cost of the linear quadratic control problem with state equation given by (5.1), respectively by (6.2), in the particular case of $f = 0$, and cost functional given by (5.2), respectively by (6.1), we immediately get that

$$P^\alpha \leq \overline{P} \qquad \text{for all } \alpha > 0.$$

Moreover we get that

$$\langle P^\alpha x, x\rangle = \mathbb{E}\int_0^{+\infty} [\langle S_s \widehat{X}^\alpha(s), \widehat{X}^\alpha(s)\rangle + |\widehat{u}^\alpha(s)|^2],$$

where

$$\widehat{u}^\alpha = -\left(I + \sum_{i=1}^d \left(D_t^i\right)^* P_t^\alpha D_t^i\right)^{-1} \left(P_t^\alpha B_t + \sum_{i=1}^d \left(Q_t^{\alpha,i} D_t^i + \left(C_t^i\right)^* P_t^\alpha D_t^i\right)\right)^* \widehat{X}_t^\alpha,$$

and $\widehat{X}^\alpha$ is the state corresponding to the control $\widehat{u}^\alpha$. So the pair $(\widehat{X}^\alpha, \widehat{u}^\alpha)$ is bounded in $L^2(\Omega \times [0,+\infty)) \times L^2(\Omega\times[0,+\infty))$, so there exists a sequence $\alpha_j \to 0$ as $j \to +\infty$ and a pair $(\widehat{X}, \widehat{u})$ such that $(\widehat{X}^{\alpha_j}, \widehat{u}^{\alpha_j}) \rightharpoonup (\widehat{X}, \widehat{u})$ in $L^2(\Omega \times [0,+\infty)) \times L^2(\Omega \times [0,+\infty))$. As a consequence of this convergence, the process $\widehat{X}$ is solution to equation (5.1), with control $\widehat{u}$. So we get

$$\begin{aligned}\langle \overline{P}x, x\rangle &\leq \mathbb{E}\int_0^{+\infty} [\langle S_s\widehat{X}(s), \widehat{X}(s)\rangle + |\widehat{u}(s)|^2] \\ &\leq \underline{\lim}_{j\to+\infty}\int_0^{+\infty} [\langle S_s\widehat{X}^{\alpha_j}(s), \widehat{X}^{\alpha_j}(s)\rangle + |\widehat{u}^{\alpha_j}(s)|^2] \\ &= \underline{\lim}_{j\to+\infty}\langle P^{\alpha_j}x, x\rangle.\end{aligned}$$

□

We remark that we can exploit a sort of separation principle, typical of the linear quadratic case, that allow to estimate separately the quadratic part from the linear part. Next we want to prove that, as $\alpha \to 0$, the optimal pair for the discounted control problem, that we denote by $(\widehat{X}^\alpha, \widehat{u}^\alpha)$ as in the previous proof, converges to the optimal pair $(\overline{X}, \overline{u})$, defined in theorem 5.1.

Lemma 6.5. *Assume that hypothesis 3.1 holds true, that $f \in L^\infty(\Omega\times[0,+\infty))$. Then, for every $T > 0$, $\widehat{X}^\alpha \to \overline{X}$ and $\widehat{u}^\alpha \to \overline{u}$ in $L^2(\Omega\times[0,T])$ as $\alpha \to 0$.*

Proof. We consider the stochastic Hamiltonian system (3.12) and the stochastic Hamiltonian system for the discounted problem

$$\begin{cases} dX_s^\alpha = [A_s^\alpha X_s^\alpha - B_s(B_s^* y_\alpha(s) + \sum_{i=1}^d \left(D_s^i\right)^* z_s^{\alpha,i})]ds + \sum_{i=1}^d [C_s^i X_s^\alpha + D_s^i(B_s^* y_s^\alpha + \sum_{k=1}^d (D_s^k)^* z_s^{\alpha,k})]dW_s^i, \\ dy_s^\alpha = -[(A_s^\alpha)^* y_s^\alpha + \sum_{i=1}^d \left(C_s^i\right)^* z_s^{\alpha,i} + S_s X_s^\alpha]ds + \sum_{i=1}^d z_s^{\alpha,i} dW_s^i, \qquad t \leq s \leq T, \\ X_t^\alpha = x, \\ y_T^\alpha = \overline{P}_T^\alpha X_T^\alpha, \end{cases} \tag{6.6}$$

Proceeding as in lemma 3.6, we get

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\langle y_T^\alpha - y_T, X_T^\alpha - X_T\rangle = \mathbb{E}^{\mathcal{F}_t}\int_t^T \alpha\langle y_s^\alpha, X_s\rangle - \alpha\langle y_s, X_s^\alpha\rangle ds \\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^T |\sqrt{S_s}(X_s^\alpha - X_s)|^2 ds - \mathbb{E}^{\mathcal{F}_t}\int_t^T |B_s^*(X_s^\alpha - X_s) + \sum_{i=1}^d (D_s^i)^*(z_s^{\alpha,i} - z_s^i)|^2 ds,\end{aligned}$$

that is

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\langle P_T^\alpha(X_T^\alpha - X_T), X_T^\alpha - X_T\rangle + \mathbb{E}^{\mathcal{F}_t}\langle (P_T^\alpha - P_T)X_T, X_T^\alpha - X_T\rangle = \mathbb{E}^{\mathcal{F}_t}\int_t^T \alpha\langle y_s^\alpha - y_s, X_s^\alpha\rangle ds \\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^T \alpha\langle y_s^\alpha, X_s^\alpha - X_s\rangle ds - \mathbb{E}^{\mathcal{F}_t}\int_t^T |B_s^*(X_s^\alpha - X_s) + \sum_{i=1}^d (D_s^i)^*(z_s^{\alpha,i} - z_s^i)|^2 ds. \\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^T |\sqrt{S_s}(X_s^\alpha - X_s)|^2 ds\end{aligned}$$

It follows that

$$\mathbb{E}^{\mathcal{F}_t}\int_t^T |\sqrt{S_s}(X_s^\alpha - X_s)|^2 ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T |B_s^*(X_s^\alpha - X_s) + \sum_{i=1}^d (D_s^i)^*(z_s^{\alpha,i} - z_s^i)|^2 ds \to 0$$

as $\alpha \to 0$. □

Finally we need to investigate the convergence of r^α to r, where (r, g) is the solution of equation (4.5).

Lemma 6.6. *For all fixed $T > 0$, $r^\alpha\mid_{[0,T]} \to r\mid_{[0,T]}$ in $L^2(\Omega\times[0,T])$.*

Proof. First we note that f^α is uniformly bounded in α and

$$\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |H_t^\alpha|^2 dt + \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T |K_t^\alpha|^2 dt \le C, \tag{6.7}$$

where C is a constant depending on T, x, A, B, C and D, but not on α. So, see proposition 4.5, equation (4.5), where f is replaced by f^α, and H and K are replaced respectively by H^α and K^α admits a solution $(r^\alpha, g^\alpha) \in L^2(\Omega\times[0,+\infty),\mathbb{R}^n)\times L^2(\Omega\times[0,T],\mathbb{R}^{n\times d})$, for every $T>0$. Now let us consider $\gamma,\eta \in L^2(\Omega\times[0,T])$. γ,η can be defined on $[0,+\infty)$: we set $\gamma_t,\eta_t = 0$ for $t>T$. Let $X^{t,x,\gamma,\eta}$ be the solution of equation (2.17) and let $X^{\alpha,t,x,\gamma,\eta}$ be the solution of an equation obtained by equation (2.17) by replacing H with H^α and K with K^α. By relation (2.18), we get

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle r_s^\alpha,\gamma_s\rangle + \langle\sum_{i=1}^d g_s^{\alpha,i},\eta_s^i\rangle ds\\
&\mathbb{E}\int_0^T \langle P_s^\alpha f_s^\alpha, X_s^{\alpha,0,0,\gamma,\eta}\rangle ds + \mathbb{E}\int_T^{+\infty}\langle P_s^\alpha f_s^\alpha, X_s^{\alpha,T,X_T^{\alpha,0,0,\gamma,\eta}}\rangle ds.
\end{aligned}\tag{6.8}$$

and also

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle \bar r_s,\gamma_s\rangle + \langle\sum_{i=1}^d \bar g_s^{i},\eta_s^i\rangle ds\\
&\mathbb{E}\int_0^T \left\langle P_s f_s, X_s^{0,0,\gamma,\eta}\right\rangle ds + \mathbb{E}\int_T^{+\infty}\left\langle P_s f_s, X_s^{T,X_T^{0,0,\gamma,\eta}}\right\rangle ds.
\end{aligned}\tag{6.9}$$

Take in (6.8) and in (6.9) $\eta=0$. By remark 4.3 and by lemmas 6.4 and 6.5 the right hand side in (6.8) converges to the right hand side of (6.9). So we get that $r^\alpha\mid_{[0,T]}\rightharpoonup r\mid_{[0,T]}$ in $L^2(\Omega\times[0,T])$. In order to get that $r^\alpha\mid_{[0,T]}\to r\mid_{[0,T]}$ in $L^2(\Omega\times[0,T])$, it suffices to prove that $\|r^\alpha\mid_{[0,T]}\|_{L^2(\Omega\times[0,T]}\to\|r\mid_{[0,T]}\|_{L^2(\Omega\times[0,T]}$. We take in (6.8) $\gamma_t = r_t^\alpha$ for $0\le t\le T$, and $\eta=0$. We get

$$\mathbb{E}\int_0^T |r_s^\alpha|^2 ds = \mathbb{E}\int_0^T\langle P_s^\alpha f_s^\alpha, X_s^{\alpha,0,0,r^\alpha,0}\rangle ds + \mathbb{E}\int_0^{+\infty}\langle P_s^\alpha f_s^\alpha, X_s^{\alpha,T,X_T^{\alpha,0,0,r^\alpha,0}}\rangle ds.$$

By remark 4.3 and by lemmas 6.4 and 6.5 the right hand side converges to

$$\mathbb{E}\int_0^T\langle P_s f_s, X_s^{0,0,\bar r,0}\rangle ds + \mathbb{E}\int_0^{+\infty}\langle P_s f_s, X_s^{T,X_T^{0,0,\bar r,0}}\rangle ds,$$

and this concludes the proof. □

Remark 6.7. Following the proof of proposition 4.5, it is easy to check that there exists a constant $C>0$, independent on α such that for every $t>0$, $|r_t^\alpha|\le C$.

We can now study the convergence of $\alpha\overline{J}^\alpha$.

Theorem 6.8. *Assume that hypothesis 3.1 holds true, that $f\in L^\infty(\Omega\times[0,+\infty))$. Then*

$$\underline{\lim}_{\alpha\to0}\alpha\overline{J}^\alpha(x) = \underline{\lim}_{\alpha\to0}\alpha\int_0^{+\infty}\langle r_s^\alpha, f_s^\alpha\rangle ds. \tag{6.10}$$

Proof. For every fixed $T > 0$ we get

$$\mathbb{E}^{\mathcal{F}_t} \int_t^T [\langle S_R S X_s^\alpha, X_s^\alpha\rangle + |u_s^\alpha|^2] ds = \mathbb{E}^{\mathcal{F}_t} \langle P_t^\alpha x, x\rangle + 2\mathbb{E}^{\mathcal{F}_t} \langle r_t^\alpha, x\rangle - \mathbb{E}^{\mathcal{F}_t} \langle P_T^\alpha X_T^\alpha, X_T^\alpha\rangle - 2\mathbb{E}^{\mathcal{F}_t} \langle r_T^\alpha, X_T^\alpha\rangle$$
$$+ 2\mathbb{E}^{\mathcal{F}_t} \int_t^T \langle r_s^\alpha, f_s^\alpha\rangle ds - \mathbb{E}^{\mathcal{F}_t} \int_t^T |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1} (B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^* g_s^{\alpha,i})|^2 ds$$
$$+ \mathbb{E}^{\mathcal{F}_t} \int_t^T |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_B^\alpha S D_s^i)^{1/2} (u_s^\alpha + \left(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i\right)^{-1} *$$
$$* \left(P_s^\alpha B_s + \sum_{i=1}^d \left(Q_s^{\alpha,i} D_s^i + \left(C_s^i\right)^* P_s^\alpha D_s^i\right)\right)^* X_s^\alpha + B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i (g_s^{\alpha,i})^*\right))|.$$

So for every $T > 0$ we get

$$\mathbb{E}^{\mathcal{F}_t} \int_t^T |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1} (B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^* g_s^{\alpha,i})|^2 ds$$
$$\leq \mathbb{E}^{\mathcal{F}_t} \langle P_t^\alpha x, x\rangle + 2\mathbb{E}^{\mathcal{F}_t} \langle r_t^\alpha, x\rangle - 2\mathbb{E}^{\mathcal{F}_t} \langle r_T^\alpha, X_T^\alpha\rangle,$$

so by remark 6.7 and by the Datko Theorem, see remark 6.3, we get that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^T |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1} (B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^* g_s^{\alpha,i})|^2 ds$$

is uniformly bounded in T and α. So, see also relation (6.4),

$$\underline{\lim}_{\alpha\to 0} \alpha \overline{J}^\alpha(x) = \underline{\lim}_{\alpha\to 0} \alpha \langle P_0^\alpha x, x\rangle + 2\underline{\lim}_{\alpha\to 0} \alpha \langle r_0^\alpha, x\rangle + \underline{\lim}_{\alpha\to 0} \alpha \int_0^{+\infty} r_s^\alpha f_s^\alpha ds$$
$$- \underline{\lim}_{\alpha\to 0} \alpha \int_0^{+\infty} |(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1} (B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^* g_s^{\alpha,i})|^2 ds$$
$$= \underline{\lim}_{\alpha\to 0} \alpha \int_0^{+\infty} \langle r_s^\alpha, f_s^\alpha\rangle ds.$$

□

6.1. **Stationary case.** In this paragraph we set a stationary framework, see [8]. $(\Omega, \mathcal{E}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic base verifying the usual conditions. Moreover $\{W_t : t \geq 0\}$ is a Ξ-valued, $(\mathcal{F}_t)_{t\geq 0}$-Wiener process and we assume that $\{W_t : t \geq 0\}$ is independent of $\mathcal{F}_0$ and that $\mathcal{F}_t = \sigma\{\mathcal{F}_0; W_s, s \in [0, t]\}$. Finally we introduce the semigroup $(\theta_t)_{t\geq 0}$ of measurable mappings $\theta_t : (\Omega, \mathcal{E}) \to (\Omega, \mathcal{E})$ verifying

(1) $\theta_0 = \mathrm{Id}$, $\theta_t \circ \theta_s = \theta_{t+s}$, for all $t, s \geq 0$
(2) θ_t is measurable: $(\Omega, \mathcal{F}_t) \to (\Omega, \mathcal{F}_0)$ and $\{\{\theta_t \in A\} : A \in \mathcal{F}_0\} = \mathcal{F}_t$
(3) $\mathbb{P}\{\theta_t \in A\} = \mathbb{P}(A)$ for all $A \in \mathcal{F}_0$
(4) $W_t \circ \theta_s = W_{t+s} - W_s$

According to this framework we introduce the definition of stationary stochastic process.

Definition 6.9. We say that a stochastic process $X : [0, \infty[\times\Omega \to \mathbb{R}^m$, is stationary if for all $s > 0$

$$X_t \circ \theta_s = X_{t+s} \qquad \mathbb{P}\text{-a.s. for a.e. } t \geq 0$$

We consider a particular case in which we assume all the coefficients of hypotheses (3.1) and (4.4) to be stationary stochastic processes. In this case a direct comparison of the integral for equation (6.11) and (6.12) below immediately gives:

Lemma 6.10. *Fix $T > 0$. Let (P, Q) be the solution of the finite horizon BSRE*

$$\begin{cases} -dP_t = G\left(A_t, B_t, C_t, D_t; S_t; P_t, Q_t\right) dt + \sum_{i=1}^{d} Q_t^i dW_t^i, \qquad t \in [0, T] \\ P_T = P_T. \end{cases} \tag{6.11}$$

For fixed $s > 0$ we define $\widehat{P}(t+s) = P(t)\theta_s$, $\widehat{Q}(t+s) = Q(t)\theta_s$ then $(\widehat{P}, \widehat{Q})$ is the unique solution in $[s, T+s]$ of the equation

$$\begin{cases} -d\widehat{P}_t = G\left(A_t, B_t, C_t, D_t; S_t; \widehat{P}_t, \widehat{Q}_t\right) dt + \sum_{i=1}^{d} \widehat{Q}_t^i dW_t^i, \qquad t \in [s, T+s] \\ \widehat{P}_T = P_T \circ \theta_s. \end{cases} \tag{6.12}$$

Proposition 6.11. *Assume Hypothesis 3.1, hypothesis 6.1 and stationarity of the coefficients, then the minimal solution $(\overline{P}, \overline{Q})$ of the infinite horizon stochastic Riccati equation* (3.4) *is stationary.*

Proof. Extending the notation introduced before Lemma 3.4 for all $\rho > 0$ we denote by P^ρ the solution of equation (6.11) in $[0, \rho]$ with final condition $P^\rho(\rho) = 0$. Denoting by $\lfloor \rho \rfloor$ the integer part of ρ, we have, following Lemma 3.4, that for all N for all $t \in [0, \lfloor N+s \rfloor]$, $P^{\lfloor N+s \rfloor}(t) \leq P^{N+s}(t) \leq P^{\lfloor N+s \rfloor + 1}(t)$, $\mathbb{P}$-a.s.. Thus we can conclude noticing that by the previous Lemma

$$P^{N+s}(t+s) = P^N(t) \circ \theta_s.$$

Thus letting $N \to +\infty$ we obtain that for all $t \geq 0$, and $s > 0$:

$$\mathbb{P}\left\{\overline{P}(t+s) = \overline{P}(t) \circ \theta_s\right\} = 1.$$

Now $\overline{P}_{T+s} = \overline{P}_T \circ \theta_s = \overline{P}_T$ so if one consider (6.11) in the intervall $[s, T+s]$ with final data $\overline{P}_{T+s}$ and (6.12) with final data $\overline{P}_T \circ \theta_s$, by the uniqueness of the solution it follows that $Q(r) = \hat{Q}(r)$, $\mathbb{P}-$ a.s. and for all $r \in [s, T+s]$. □

Notice that, thanks to the stationarity assumptions the stabilizability condition can be simplified, see Remark 5.7 of [12]. Hence all the coefficients that appear in equation (2.12) are stationary so exactly as before we deduce that for the solution (r_T, g_T) the following holds:

Lemma 6.12. *Fix $T > 0$ and $r_T \in L^\infty(\Omega, \mathcal{F}_T; \mathbb{R}^n)$. Let (r, g) a solution to equation*

$$\begin{cases} dr_t = -H_t^* r_t dt - P_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \quad t \in [0, T] \\ r_T = r_T. \end{cases} \tag{6.13}$$

For fixed $s > 0$ we define $\widehat{r}(t+s) = r(t)\theta_s$, $\widehat{g}(t+s) = g(t)\theta_s$ then $(\widehat{r}, \widehat{g})$ is the unique solution in $[s, T+s]$ of the equation

$$\begin{cases} d\widehat{r}_t = -H_t^* \widehat{r}_t dt - \bar{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* \widehat{g}_t^i dt + \sum_{i=1}^{d} \widehat{g}_t^i dW_t^i, \quad t \in [s, T+s] \\ \widehat{r}_T = r_T \circ \theta_s. \end{cases} \tag{6.14}$$

Hence arguing as for the first component $\bar{P}$, we get that the first component of the infinite horizon equation $\bar{r}$ has to be stationary:

Proposition 6.13. *Beside Hypothesis 3.1 and stationarity of coefficients assume that (A, B, C, D) is $\sqrt{S}$ stabilizable, we have that first component of the solution of*

$$dr_t = -H_t^* r_t dt - \bar{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \qquad t \in [0, T] \tag{6.15}$$

obtained as the pointwise limit of r_T is stationary.

This is enough to characterize the ergodic limit, since the value function is unique. Indeed we have that:

Theorem 6.14. *Assuming that all the coefficients are stationary processes we get the following characterization of the optimal cost:*

$$\lim_{\alpha\to 0} \alpha \inf_{u\in\mathcal{U}} J_\alpha(x,u) = \mathbb{E}\langle f(0), \bar{r}(0)\rangle$$

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