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Hyperbolic Balance Laws with a Non Local Source

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Abstract

This paper is devoted to hyperbolic systems of balance laws with non local source terms. The existence, uniqueness and Lipschitz dependence proved here comprise previous results in the literature and can be applied to physical models, such as Euler system for a radiating gas and Rosenau regularization of the Chapman-Enskog expansion.

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1 Introduction

This paper is devoted to systems of conservation laws with non local sources, i.e. to equations of the form

$$\partial_t u + \partial_x f(u) = G(u) \tag{1.1}$$

where f is the flow of a nonlinear hyperbolic system of conservation laws and $G\colon \mathbf{L^1} \mapsto \mathbf{L^1}$ is a (possibly) *non local* operator. As examples, we consider below the case $G(u) = g(u) + Q*u$ that enters a classical radiating gas model, see [24], as well as Rosenau regularization of Chapman-Enskog expansion of the Boltzmann equation, see [20, 21]. We establish that (1.1) is well posed in $\mathbf{L^1}$, locally in time, for data having sufficiently small total variation. To this aim, we require on (1.1) those assumptions that separately guarantee the well posedness of the convective part

$$\partial_t u + \partial_x f(u) = 0 \tag{1.2}$$

and of the source part

$$\partial_t u = G(u)\,. \tag{1.3}$$

These two equations generate two semigroups of solutions, say S and Σ. To obtain our results we exploit the techniques in [2, 9], essentially based on the *fractional step* algorithm, see [9, 10, 13, 23]. Its core idea is to get a solution of the original equation as a limit of approximations obtained suitably merging S and Σ.

On the two semigroups we require the following two key conditions (see [9, **(S2)** and **(C*)**]):

i) A Grönwall type estimate with respect to a suitable metric $d(\cdot,\cdot)$: i.e. for a positive C

$$\begin{array}{l} d(S_t u, S_t v) \leq e^{Ct} d(u,v) \\ d(\Sigma_t u, \Sigma_t v) \leq e^{Ct} d(u,v) \end{array} \quad \text{for all } t \geq 0. \tag{1.4}$$

ii) A commutativity relation

$$d(\Sigma_t S_t u, S_t \Sigma_t u) \leq K\, t^2 \text{ as } t \to 0. \tag{1.5}$$

The former assumption is used to prove the uniformly continuous dependence of the approximations from the initial data, also called *stability condition* in the framework of Lie-Trotter formula, see [14, Corollary 5.8, Chapter 3]. The latter condition yields the convergence of the approximations and ensures the uniqueness of the limit.

Assume that *i)* holds and d is a reasonable (in the sense of Proposition 3.18) metric equivalent to the $\mathbf{L^1}$ distance. Then, the invariance under the hyperbolic rescaling $(t,x) \to (\lambda t, \lambda x)$ of solutions to system of conservation laws, implies that $C = 0$, see Proposition 3.18. Hence, to apply the operator splitting techniques, we need a contractive metric for the conservation law (1.2). This role is naturally played by the well known functional Φ in [18, 19]. Note, however, that this functional is *not* a metric, for it may lack to satisfy the triangle inequality. The proof, then, consists in showing that the semigroup generated by the source part (1.3) satisfies (1.4) with respect to Liu & Yang functional and commutes with the semigroup generated by the conservation law in the sense of (1.5).

More precisely, let Ω be a open subset of $\mathbb{R}^n$ with $0 \in \Omega$. For all positive δ, define

$$\mathcal{U}_\delta = \left\{ u \in \mathbf{L^1}(\mathbb{R};\Omega) \colon \mathrm{TV}(u) \leq \delta \right\}.$$

As a general reference on conservation laws we refer to [6]. On the convective and on the source parts we assume throughout that

(F) $f \in \mathbf{C^4}(\Omega;\mathbb{R}^n)$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

(G) For a positive δ_o, $G: \mathcal{U}_{\delta_o} \mapsto \mathbf{L^1}(\mathbb{R}, \mathbb{R}^n)$ is such that for suitable positive L_1, L_2, L_3

$$\begin{array}{lrcl} \forall\, u, w \in \mathcal{U}_{\delta_o} & \left\|G(u) - G(w)\right\|_{\mathbf{L^1}} & \leq & L_1 \cdot \|u - w\|_{\mathbf{L^1}} \\ \forall\, u \in \mathcal{U}_{\delta_o} & \mathrm{TV}\left(G(u)\right) & \leq & L_2 \cdot \mathrm{TV}(u) + L_3 \,. \end{array}$$

Note that **(F)**, respectively **(G)**, ensures the local in time well posedness of (1.2), respectively (1.3). A class of functions satisfying **(G)** is provided by the following proposition.

Proposition 1.1 *Let $g, h: \Omega \mapsto \mathbb{R}^n$ be locally Lipschitz and $Q \in \mathbf{L^1}(\mathbb{R}; \mathbb{R}^{n\times n})$. Then, the operator $G(u) = g(u) + Q * h(u)$ satisfies **(G)** with $L_3 = 0$.*

The proof is deferred to Section 3. We are now ready to state the main result of this work.

Theorem 1.2 *Let f satisfy **(F)** and G satisfy **(G)**. Then, there exist positive T, $\tilde{\delta}$, $\mathcal{L}$, closed domains $\mathcal{D}_t$ and processes*

$$F_t: \mathcal{D}_{T-t} \mapsto \mathcal{D}_T \qquad \forall\, t \in [0, T]$$

with the properties:

(1) for $t, s \in [0, T]$ with $t < s$, $\mathcal{U}_{\tilde{\delta}} \subseteq \mathcal{D}_t \subseteq \mathcal{D}_s \subseteq \mathcal{U}_{\delta_o}$;

(2) for u in $\mathcal{D}_T$, $F_0 u = u$; for $t, s \in [0, T]$ with $t + s \in [0, T]$, $F_s \mathcal{D}_t \subseteq \mathcal{D}_{t+s}$ and for $u \in \mathcal{D}_{T-t-s}$, $F_t F_s u = F_{t+s} u$;

(3) for $\bar{t} \in [0, T]$ and $u \in \mathcal{D}_{\bar{t}}$, the map $t \mapsto F_t u$ is a weak entropy solution to (1.1) for $t \in [0, T - \bar{t}]$;

(4) if S is the SRS generated by (1.2), then for $\bar{t} \in [0, T[$ and $u \in \mathcal{D}_{\bar{t}}$,

$$\lim_{t\to 0} \frac{1}{t} \left\| F_t u - \left(S_t u + t\, G(u)\right) \right\|_{\mathbf{L^1}} = 0\,;$$

(5) for $t, s \in [0, T]$, $u, w \in \mathcal{D}_{T-t}$ and $s < t$, then

$$\begin{array}{rcl} \|F_t u - F_t w\|_{\mathbf{L^1}} & \leq & \mathcal{L} \cdot \|u - w\|_{\mathbf{L^1}} \\ \|F_t u - F_s u\|_{\mathbf{L^1}} & \leq & \mathcal{L} \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \cdot |t - s|\,; \end{array} \tag{1.6}$$

(6) for $\bar{t} \in [0, T[$, $u \in \mathcal{D}_{\bar{t}}$ and $\tau \in [0, T - \bar{t}\,]$ the map $u(\tau) = F_\tau u$ satisfies

(6a) for $\xi \in \mathbb{R}$, $\displaystyle\lim_{\vartheta\to 0+} \frac{1}{\vartheta} \int_{\xi-\vartheta\hat{\lambda}}^{\xi+\vartheta\hat{\lambda}} \left\| \left(F_\vartheta u(\tau)\right)(x) - U^\sharp_{(u(\tau),\xi)}(\vartheta, x) \right\| dx = 0,$

(6b) there exists a positive C such that for all a, b with $-\infty \leq a < \xi < b \leq +\infty$

$$\begin{aligned} \limsup_{\vartheta\to 0+} \frac{1}{\vartheta} \int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda} \left\| \left(F_\vartheta u(\tau)\right)(x) - U^\flat_{(u(\tau),\xi)}(\vartheta, x) \right\| dx \quad & \leq \\ \leq \quad C \cdot \left(\mathrm{TV}\left(u(\tau);]a, b[\right)\right)^2 , & \end{aligned}$$

where $U^\sharp_{(u(\tau),\xi)}$ solves (3.20) and $U^\flat_{(u(\tau),\xi)}$ solves (3.21);

(7) for $\bar t \in [0, T[$, if a Lipschitz map $u \colon [0, T - \bar t] \mapsto \mathcal{D}_T$ is such that $u(t) \in \mathcal{D}_{\bar t + t}$ and for $\tau \in [0, T - \bar t]$

(7a) for $\xi \in \mathbb{R}$, $\displaystyle \lim_{\vartheta\to 0+} \frac{1}{\vartheta} \int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda} \left\| \left(F_\vartheta u(\tau)\right)(x) - U^\sharp_{(u(\tau),\xi)}(\vartheta, x) \right\| dx = 0,$

(7b) there exists a finite measure μ_τ such that for all a, b with $-\infty \leq a < \xi < b \leq +\infty$

$$\limsup_{\vartheta\to 0+} \frac{1}{\vartheta} \int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda} \left\| \left(F_\vartheta u(\tau)\right)(x) - U^\flat_{(u(\tau),\xi)}(\vartheta, x) \right\| dx \leq \left(\mu_\tau\left(]a, b[\right)\right)^2$$

where $U^\sharp_{(u(\tau),\xi)}$ solves (3.20) and $U^\flat_{(u(\tau),\xi)}$ solves (3.21), then $u(t) = F_t u(0)$.

*Moreover, if f_1, f_2 both satisfy **(F)** and G_1, G_2 both satisfy **(G)**, then, denoting by F^i the process generated by f_i and G_i, for all $t \in [0, T]$ and $u \in \mathcal{D}_0$*

$$\begin{aligned} \left\| F^1_t u - F^2_t u \right\|_{\mathbf{L^1}} \quad \leq \quad & \mathcal{L} \cdot \left\| Df_1 - Df_2 \right\|_{\mathbf{C^0}(\Omega, \mathbb{R}^{n\times n})} \cdot t \\ & + \mathcal{L} \cdot \left\| G_1 - G_2 \right\|_{\mathbf{C^0}(\mathcal{U}_{\delta_o}; \mathbf{L^1}(\mathbb{R};\mathbb{R}^n))} \cdot t \,. \end{aligned} \tag{1.7}$$

For the definition and properties of the SRS, refer to [6]. Point 4. characterizes the tangent vector to $t \mapsto F_t u$ in the sense of [5, § 5]. It is through this characterization that the integral inequalities 6 and 7 are proved. The proof of Theorem 1.2 follows from the results presented in Section 3 below.

The first part of next section is devoted to the application of the above result to the Euler system for a radiating gas and to Rosenau regularization of the Chapman-Enskog expansion. The framework of local sources is recovered in the subsequent paragraph and, finally, we quickly comprise also the case of a non autonomous source.

Remark that in the estimate (1.6) the presence of the term $\|u\|_{\mathbf{L^1}}$ is mandatory, as the example $\partial_t u = u$ shows. Indeed, the domains $\mathcal{U}_\delta$ is unbounded in $\mathbf{L^1}$. Moreover, note that the analogous estimate in [2, 8, 11] should be understood with a time Lipschitz constant dependent on the $\mathbf{L^1}$ norm of the initial datum.

We stress that the estimate (6b) is sharper than [1, formula (5.18)] thanks to the finite total variation of the source term, ensured by **(G)**.

2 Applications and Extensions

2.1 Euler System for a Radiating Gas

The following model for a radiating polytropic gas was considered in [24, Chapter XXII, § 6], see also [16, formula (1.2)]:

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v) = 0 \\ \partial_t (\rho\, v) + \partial_x \left(\rho\, v^2 + p\right) = 0 \\ \partial_t \left(\rho\, e + \frac{1}{2} \rho\, v^2\right) + \partial_x \left(v \left(\rho\, e + \frac{1}{2}\rho\, v^2 + p\right) + q\right) = 0 \\ -\partial^2_{xx} q + a\, q + b\, \partial_x \vartheta^4 = 0 \end{cases}$$

Here, as usual, ρ is the gas density, v its speed, e the internal energy, p the pressure, $\vartheta = e/c_v$ the temperature and q is the radiative heat flux. The system is closed by means of the equation of state and specifying the values of the characteristic constants a and b.

Solving the latter equation in q we have $q = -\frac{b}{\sqrt{a}}\, Q_a * \left(\frac{d}{dx} \vartheta^4\right)$, where $Q_a(x) = \frac{1}{2} \exp\left(-\sqrt{a}\, |x|\right)$ and we are lead to consider the system

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v) = 0 \\ \partial_t (\rho\, v) + \partial_x \left(\rho\, v^2 + p\right) = 0 \\ \partial_t \left(\rho\, e + \frac{1}{2} \rho\, v^2\right) + \partial_x \left(v \left(\rho\, e + \frac{1}{2}\rho\, v^2 + p\right)\right) = b\, \left(-\vartheta^4 + \sqrt{a}\, Q_a * \vartheta^4\right). \end{cases} \tag{2.1}$$

It is well known that Euler system satisfies **(F)**. Condition **(G)** holds by Proposition 1.1. Hence, Theorem 1.2 applies and we obtain the local in time well posedness of (2.1). Note that this result also ensures the local Lipschitz dependence of the solutions to (2.1) from the parameters a and b.

2.2 Rosenau Regularization of the Chapman-Enskog Expansion

In his classical work [20], Rosenau proposed a system of balance laws that provides a regularized version of the Chapman-Enskog expansion for hydrodynamics in a linearized framework. The 1D version is the following:

$$\begin{cases} \partial_t \rho + \partial_x v = 0 \\ \partial_t v + \partial_x p = \mu_* * \partial^2_{xx} v \\ \partial_t \left(\frac{3}{2}\vartheta\right) + \partial_x v = \lambda_* * \partial^2_{xx} \vartheta \end{cases}$$

where ρ is the fluid density, v is its speed and ϑ is the temperature. μ_*, respectively λ_*, is a convolution kernel related to viscosity, respectively to thermal conductivity. This linear system motivated analytical results, see for instance [15, 17, 21], mostly related to the quasilinear scalar equation

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2\right) = -u + Q * u$$

since the source term $-u + Q * u$ is equal to $Q * \partial^2_{xx} u$, provided $Q(x) = \frac{1}{2} \exp\left(-|x|\right)$. Therefore, it is natural to consider the following Euler system with Rosenau-type sources

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v) = 0 \\ \partial_t (\rho\, v) + \partial_x \left(\rho\, v^2 + p\right) = \mu_* * \partial^2_{xx} v \\ \partial_t \left(\rho\, e + \frac{1}{2} \rho\, v^2\right) + \partial_x \left(v \left(\rho\, e + \frac{1}{2} \rho\, v^2 + p\right) + q\right) = \lambda_* * \partial^2_{xx} \vartheta\,. \end{cases} \tag{2.2}$$

Rosenau kernels, see [20, formulæ (4a) and (6)] read

$$\mu_*(x) = \frac{\mu}{2\, m\, \varepsilon} \exp\left(-|x|/\varepsilon\right) \qquad \text{and} \qquad \lambda_*(x) = \frac{\lambda}{2\, s\, \varepsilon} \exp\left(-|x|/\varepsilon\right)$$

for suitable positive parameters $\mu, \lambda, m, s, \varepsilon$. With the above choices, the sources in the last two equations in (2.2) can be rewritten as

$$\mu_* * \partial^2_{xx} v = \frac{1}{\varepsilon^2} \left(-\frac{\mu}{m}\, v + \mu_* * v\right) \qquad \text{and} \qquad \lambda_* * \partial^2_{xx} \vartheta = \frac{1}{\varepsilon^2} \left(-\frac{\lambda}{s}\, \vartheta + \lambda_* * \vartheta\right).$$

By Proposition 1.1, system (2.2) falls within the scope of Theorem 1.2. Thus, we prove the local in time well posedness of (2.2) as well as the local Lipschitz dependence of the solutions to (2.1) from the parameters $\mu, \lambda, m, s, \varepsilon$.

2.3 Local Inhomogeneous Source

Theorem 1.2 can be applied also in the standard case of a *local* source. Indeed, it is immediate to see that **(G)** is implied by the following conditions **(g1)** and **(g2)**.

Proposition 2.1 *Let $g \colon \mathbb{R} \times \Omega \mapsto \mathbb{R}^n$ be such that*

(g1) *there exists an $L_1 > 0$ such that for $u_1, u_2 \in \Omega$, $\left\|g(x, u_2) - g(x, u_1)\right\| \le L_1 \cdot \|u_2 - u_1\|$;*

(g2) *there exists a finite measure μ on $\mathbb{R}$ such that for $u \in \Omega$ and $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, $\left\|g(x_2, u) - g(x_1, u)\right\| \le \mu\left(\left]x_1, x_2\right]\right)$.*

*and assume that f satisfies **(F)**. Then, setting $\left(G(u)\right)(x) = g(x, u)$, Theorem 1.2 applies.*

Note that the integral estimates 6 and 7 in Theorem 1.2 ensure that the solution constructed here coincide with those in [1]. Similarly, the characterization 4 of the tangent vector imply that the present solutions coincide with those in [2].

2.4 The Non Autonomous Case

Theorem 1.2 can be extended to the non autonomous balance law

$$\partial_t u + \partial_x f(u) = G(t,u)$$

provided f satisfies **(F)**, $G\colon [0,T_o] \times \mathbf{L^1} \mapsto \mathbf{L^1}$ satisfies

(G') For positive δ_o, T_o, the map $G\colon [0,T_o] \times \mathcal{U}_{\delta_o} \mapsto \mathbf{L^1}(\mathbb{R},\mathbb{R}^n)$ admits suitable positive L_1, L_2, L_3 such that for all $u,w \in \mathcal{U}_{\delta_o}$ and for all $t,s \in [0,T_o]$

$$\begin{array}{rcl} \left\|G(t,u) - G(s,w)\right\|_{\mathbf{L^1}} & \leq & L_1 \cdot \left(\|u-w\|_{\mathbf{L^1}} + |t-s|\right) \\ \mathrm{TV}\left(G(t,u)\right) & \leq & L_2 \cdot \mathrm{TV}(u) + L_3\,. \end{array}$$

Indeed, let $\hat\lambda$ be an upper bound for all moduli of characteristic speeds, i.e. $\hat\lambda > \sup_{\|u\| \leq \delta_o} \max_{i=1,\ldots,n} \left|\lambda_i(u)\right|$, and define

$$\tilde f(u,w) = \left(f(u), \hat\lambda\, w\right) \qquad \tilde G(u,w) = \left(G\left(\textstyle\int_{\mathbb{R}} w, u\right), \chi_{[0,1]}\right)$$

here, $\chi_{[0,1]}$ is the characteristic function of the real interval $[0,1]$. Then, $\tilde f$ satisfies **(F)** and $\tilde G$ satisfies **(G)**, so that Theorem 1.2 applies and the balance law $\partial_t(u,w) + \partial_x \tilde f(u,w) = \tilde G(u,w)$ generates the operator $\tilde F$. The Cauchy problem

$$\left\{\begin{array}{l} \partial_t u + \partial_x f(u) = G(t,u) \\ u(0,x) = u_o(x) \end{array}\right.$$

is solved by $t \mapsto F_t(u_o,0)$ where $F_t(u_o,0)$ is given by the first n component of $\tilde F_t(u_o,0)$.

Again, the integral estimates 6 and 7 in Theorem 1.2 ensure that the solutions constructed here coincide with those in [11].

3 Technical Proofs

Proof of Proposition 1.1. The Lipschitz property is immediate. To prove the bound on the total variation, call $\mathbf{Lip}\,(h)$ the Lipschitz constant of h. Then, it is sufficient to compute:

$$\begin{array}{rcl} \mathrm{TV}\left(Q * h(u)\right) & \leq & \sup \displaystyle\sum_i \int_{\mathbb{R}} \left\|Q(y)\right\| \left\|h\left(u(x_i - y)\right) - h\left(u(x_{i-1} - y)\right)\right\| dy \\ & \leq & \mathbf{Lip}\,(h) \sup \displaystyle\sum_i \int_{\mathbb{R}} \left\|Q(y)\right\| \left\|u(x_i - y) - u(x_{i-1} - y)\right\| dy \\ & \leq & \mathbf{Lip}\,(h) \sup \displaystyle\int_{\mathbb{R}} \left\|Q(y)\right\| \sum_i \left\|u(x_i - y) - u(x_{i-1} - y)\right\| dy \end{array}$$

$$
\begin{aligned}
&\leq \mathbf{Lip}\,(h)\,\sup \int_{\mathbb{R}} \left\|Q(y)\right\| \mathrm{TV}(u)\,dy \\
&\leq \mathbf{Lip}\,(h)\,\|Q\|_{\mathbf{L}^1}\,\mathrm{TV}(u)
\end{aligned}
$$

□

3.1 Convective part

Let $\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)$ be the n real distinct eigenvalues of $Df(u)$, indexed so that $\lambda_j < \lambda_{j+1}$ for all j and u. The j-th right eigenvector is $r_j(u)$ and we assume that $\left\|r_j(0)\right\| = 1$.

Let $\sigma \mapsto R_j(\sigma)(u)$ and $\sigma \mapsto S_j(\sigma)(u)$ be respectively the rarefaction and the shock curve exiting u. If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [6, Definition 5.2], we choose σ so that for a suitable constant $k_j > 0$

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\lambda_j\left(R_j(\sigma)(u)\right) &= k_j \quad \text{and} \quad & \frac{\partial R_j}{\partial\sigma}(0)(0) &= r_j(0) \\
\frac{\partial}{\partial\sigma}\lambda_j\left(S_j(\sigma)(u)\right) &= k_j, \quad \text{and} \quad & \frac{\partial S_j}{\partial\sigma}(0)(0) &= r_j(0)\,.
\end{aligned}
$$

The above choices were introduced in [2, § 2], see also [6, 7].

Introduce the j-Lax curve

$$
\sigma \mapsto \psi_j(\sigma)(u) = \begin{cases} R_j(\sigma)(u) & \text{if} \quad \sigma \geq 0 \\ S_j(\sigma)(u) & \text{if} \quad \sigma < 0 \end{cases}
$$

and define the map

$$
\Psi(\sigma)(u^-) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)(u^-)\,.
$$

By [6, § 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to 0, there exists a vector $(\sigma_1, \ldots, \sigma_n) = E(u^-, u^+)$ such that $u^+ = \Psi(\sigma)(u^-)$.

Similarly, let $\mathbf{S}$ be defined by

$$
u^+ = \mathbf{S}(\sigma)(u^-) = S_n(\sigma_n) \circ \ldots \circ S_1(\sigma_1)(u^-)
$$

as the gluing of the Rankine - Hugoniot curves.

For a sufficiently small δ_o, let $u \in \mathcal{U}_{\delta_o}$ be piecewise constant with finitely may jumps sited in a finite set of points denoted by $\mathcal{I}(u)$. Let $\sigma_{x,i}$ be the strength of the i-th wave in the solution of the Riemann problem for (1.2) with data $u(x-)$ and $u(x+)$. i.e. $(\sigma_{x,1}, \ldots, \sigma_{x,n}) = E\left(u(x-), u(x+)\right)$. Obviously if $x \notin \mathcal{I}(u)$ then $\sigma_{x,i} = 0$, for all $i = 1, \ldots, n$. As in [6, § 7.7], $\mathcal{A}(u)$ denotes the set of approaching waves in u:

$$
\mathcal{A}(u) = \left\{ \begin{array}{c} \left((x,i),(y,j)\right) \in \left(\mathcal{I}(u) \times \{1, \ldots, n\}\right)^2 : \\ x < y \text{ and either } i > j \text{ or } i = j, \text{ the } i\text{-th field} \\ \text{is genuinely non linear, } \min\left\{\sigma_{x,i}, \sigma_{y,j}\right\} < 0 \end{array} \right\}
$$

while the linear and the interaction potential are

$$V(u) = \sum_{x \in I(u)} \sum_{i=1}^{n} \left|\sigma_{x,i}\right|, \qquad Q(u) = \sum_{\left((x,i),(y,j)\right) \in \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right|.$$

Moreover, let

$$\Upsilon(u) = V(u) + C_0 Q(u) \tag{3.1}$$

where $C_0 > 0$ is the constant appearing in the functional of the wave–front tracking algorithm, see [6, Proposition 7.1]. Finally we define

$$\overline{\mathcal{D}}_\delta = \mathrm{cl}\left\{u \in \mathbf{L^1}\left(\mathbb{R}, \mathbb{R}^n\right) : u \text{ is piecewise constant and } \Upsilon(u) < \delta\right\} \tag{3.2}$$

where the closure is in the strong $\mathbf{L^1}$–topology. We remark for later use that there exists a positive constant $c = c(\delta_o)$ with $c \in \left]0, 1\right[$, such that for all $\delta \in [0, \delta_o]$

$$\mathcal{U}_\delta \supseteq \overline{\mathcal{D}}_{c\delta} \supseteq \mathcal{U}_{c^2\delta}.$$

Proposition 3.1 *Let f satisfy **(F)**. Then, there exists a positive and suitably small $\bar\delta_o$ such that* (1.2) *generates a Standard Riemann Semigroup (SRS), with Lipschitz constant L, defined on the domain $\overline{\mathcal{D}}_\delta$, for all $\delta \in \left]0, \bar\delta_o\right[$.*

We refer to [6, Chapters 7 and 8] for the proof of the above result as well as for the definition and further properties of the SRS.

Lemma 3.2 *Let f satisfy **(F)**, Ω be a sufficiently small neighborhood of the origin; $a, b \in \mathbb{R}^n$ and $s \in \left[0, +\infty\right[$ be sufficiently small. Choose $u^-, v^- \in \Omega$ and define*

$$u^+ = u^- + sa, \quad v^+ = v^- + sb.$$

Then, there exist σ^-, σ^+ such that $v^- = \Psi(\sigma^-)(u^-)$ and $v^+ = \Psi(\sigma^+)(u^+)$. Moreover,

$$\sum_{i=1}^{n} \left|\sigma_i^+ - \sigma_i^-\right| \le \mathcal{O}(1) \cdot \left(\|a - b\| + \sum_{i=1}^{n} \left|\sigma_i^-\right|\right) \cdot s. \tag{3.3}$$

An entirely analogous result holds with Ψ replaced by $\mathbf{S}$, i.e. $v^- = \mathbf{S}(\sigma^-)(u^-)$ and $v^+ = \mathbf{S}(\sigma^+)(u^+)$.

The proof is an extension of [2, Lemma 2.1] and, hence, omitted.

3.2 Source part

Concerning the source term we have the following results.

Proposition 3.3 *Let G satisfy **(G)**. Then, for any $\delta \in \left]0, \delta_o\right[$ and $u \in \mathcal{U}_\delta$, the Cauchy Problem* (1.3) *with initial data u admits a solution $\Sigma_t u$ defined for $t \in [0, \widetilde{T}]$ where $\widetilde{T} = \min\left\{\frac{\delta_o - \delta}{\delta_o L_2 + L_3}, \frac{1}{L_1+1}\right\}$. Moreover the trajectory $\Sigma_t u$ has the following properties for all $t \in [0, \widetilde{T}]$ and $u, v \in \mathcal{U}_\delta$:*

$$\begin{aligned} \left\|\Sigma_t u - \Sigma_t w\right\|_{\mathbf{L^1}} &\leq \left\|u - w\right\|_{\mathbf{L^1}} e^{L_1 t} \\ \mathrm{TV}\left(\Sigma_t u\right) &\leq \delta + \left(\delta_o\ L_2 + L_3\right) t. \end{aligned}$$

The existence of $\Sigma_t u$ is a standard application of Banach Fixed Point Theorem and the estimates follow from Grönwall Lemma.

Since the computations on the convective part is mainly done on piecewise constant approximate solutions, we need to approximate the source term with piecewise constant functions.

Let $\mathbf{PC}(\mathbb{R}; \mathbb{R}^n)$ be the set of piecewise constant functions in $\mathbf{L^1}$. For any $N \in \mathbb{N}$, define the operator $\Pi_N \colon \mathbf{L^1}(\mathbb{R}; \mathbb{R}^n) \mapsto \mathbf{PC}(\mathbb{R}; \mathbb{R}^n)$ by

$$\Pi_N(u) = N \sum_{k=-1-N^2}^{-1+N^2} \int_{k/N}^{(k+1)/N} u(\xi)\, d\xi\ \chi_{\left]k/N, (k+1)/N\right]}.$$

Lemma 3.4 Π_N *is a linear operator with norm* 1*. Moreover,* $\mathrm{TV}\left(\Pi_N u\right) \leq 2\mathrm{TV}(u)$ *and for all $u \in \mathbf{L^1}(\mathbb{R}; \mathbb{R}^n) \cap \mathbf{BV}(\mathbb{R}; \mathbb{R}^n)$, $\Pi_N u \to u$ in $\mathbf{L^1}(\mathbb{R}; \mathbb{R}^n)$.*

Proof. Linearity and the estimate on the norm are immediate.

$$\begin{aligned} \mathrm{TV}(\Pi_N u) &= \left\|(\Pi_N u)(-N)\right\| + \left\|(\Pi_N u)(N)\right\| \\ &\quad + \sum_{k=-N^2}^{N^2-1} \left\|(\Pi_N u)\left((k+1)/N\right) - (\Pi_N u)(k/N)\right\| \\ &\leq N \int_{-N-1/N}^{-N} \left\|u(\xi)\right\| d\xi + N \int_{N-1/N}^{N} \left\|u(\xi)\right\| d\xi \\ &\quad + N \sum_{k=-N^2}^{N^2-1} \left\| \int_{k/N}^{(k+1)/N} u(\xi)\, d\xi - \int_{(k-1)/N}^{k/N} u(\xi)\, d\xi \right\| \\ &\leq \mathrm{TV}\left(u; \left]-\infty, -N\right[\right) + \mathrm{TV}\left(u; \left]N - 1/N, +\infty\right[\right) \\ &\quad + N \sum_{k=-N^2}^{N^2-1} \int_{k/N}^{(k+1)/N} \left\|u(\xi) - u(\xi - 1/N)\right\| d\xi \\ &\leq \mathrm{TV}\left(u; \left]-\infty, -N\right[\right) + \mathrm{TV}\left(u; \left]N - 1/N, +\infty\right[\right) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k=-N^2}^{N^2-1} \mathrm{TV}\left(u; \left](k-1)/N, (k+1)/N\right[\right) \\
\leq \; & 2\,\mathrm{TV}(u)\,.
\end{aligned}
$$

Concerning the pointwise convergence $\Pi_N \to \mathrm{Id}$:

$$
\begin{aligned}
\left\|\Pi_N u - u\right\|_{\mathbf{L^1}} \leq \; & \int_{-\infty}^{-N-1/N} \left\|u(\xi)\right\| d\xi + \int_N^{+\infty} \left\|u(\xi)\right\| d\xi \\
& + N \sum_{k=-N^2-1}^{N^2-1} \int_{k/N}^{(k+1)/N} \int_{k/N}^{(k+1)/N} \left\|u(\xi) - u(x)\right\| d\xi\, dx \\
\leq \; & \int_{-\infty}^{-N-1/N} \left\|u(\xi\right\| d\xi + \int_N^{+\infty} \left\|u(\xi)\right\| d\xi + \frac{1}{N}\,\mathrm{TV}(u)
\end{aligned}
$$

So that $\lim_{N\to+\infty} \left\|\Pi_N u - u\right\|_{\mathbf{L^1}} = 0$. □

Corollary 3.5 *Let G satisfy **(G)**. Then, for any N, also $\Pi_N \circ G$ satisfies **(G)** with the same Lipschitz constant L_1 and with L_2, L_3 replaced by $2L_2$ and $2L_3$.*

To simplify the operator splitting algorithm, we substitute the semigroup generated by (1.3) with the Euler approximation $P_t \colon \mathcal{U}_{\delta_o} \mapsto \mathbf{L^1}(\mathbb{R};\mathbb{R}^n)$ defined by

$$
P_t u = u + t\, G(u)\,.
$$

It is immediate to prove that P_t is $\mathbf{L^1}$-Lipschitz with constant $1 + tL_1$. We often use below the estimate $1+L_1 t \leq e^{L_1 t}$. On the other hand, observe that, despite the notation (useful in the sequel), P does not satisfy the semigroup composition law. Indeed we can only say

$$
\left\|P_s P_t u - P_{s+t} u\right\|_{\mathbf{L^1}} \leq \mathcal{O}(1) \cdot s \cdot t \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right)\,.
$$

3.3 Operator Splitting

Lemma 3.6 *Let G attain values in $\mathbf{PC}(\mathbb{R};\mathbb{R}^n)$. Then, there exist positive $\bar\delta_o$ and T_o such that for all $s \in [0, T_o]$ and for all piecewise constant $u \in \mathcal{D}_{\bar\delta_o}$*

$$
\begin{aligned}
V\left(P_s u\right) &\leq V(u) + \mathcal{O}(1)\, s \left[L_3 + V(u)\right] && (3.4) \\
Q\left(P_s u\right) &\leq Q(u) + \mathcal{O}(1)\, s \left[L_3 + V^2(u)\right] && (3.5) \\
\Upsilon\left(P_s u\right) &\leq \Upsilon(u) + \mathcal{O}(1)\, s \left[L_3 + V(u)\right] && (3.6)
\end{aligned}
$$

Proof. Let $u' = P_s u$, so that $u' = u + s\,G(u)$. Call $\sigma'_{x,i}$ (resp. $\sigma_{x,i}$) the size of the i-wave in u' (resp. u) at the point x, observe that $\sigma'_{x,i}$ (resp. $\sigma_{x,i}$) vanishes whenever $x \notin \mathcal{I}(u')$ (resp. $x \notin \mathcal{I}(u)$). Then, by Lemma 3.2 and **(G)**

$$\begin{aligned}
V(u') - V(u) &\leq \sum_{x \in \mathcal{I}(u') \cup \mathcal{I}(u)} \sum_{i=1}^{n} \left| \sigma_{x,i} - \sigma'_{x,i} \right| \\
&\leq \mathcal{O}(1) \cdot s \cdot \sum_{x \in \mathcal{I}(u') \cup \mathcal{I}(u)} \left(\sum_{i=1}^{n} \left|\sigma_{x,i}\right| + \left\| \Delta \left(G(u) \right) (x) \right\| \right) \\
&\leq \mathcal{O}(1) \cdot s \cdot \Big(V(u) + \mathrm{TV} \left(G(u) \right) \Big) \\
&\leq \mathcal{O}(1) \cdot s \cdot \left(L_3 + V(u) \right) .
\end{aligned}$$

To derive (3.5), we observe that if $\big((x,i),(y,j)\big) \in \mathcal{A}(u') \setminus \mathcal{A}(u)$, then either $\sigma'_{x,i}\sigma_{x,i} \leq 0$ or $\sigma'_{y,j}\sigma_{y,j} \leq 0$. Suppose, for instance, that $\sigma'_{x,i}\sigma_{x,i} \leq 0$ (the other case being similar) then by (3.3) we obtain

$$\left|\sigma'_{x,i}\right| + \left|\sigma_{x,i}\right| = \left|\sigma'_{x,i} - \sigma_{x,i}\right| \leq \mathcal{O}(1)\, s \left(\sum_{j=1}^{n} \left|\sigma_{x,j}\right| + \left\| \Delta G(u)(x) \right\| \right)$$

and therefore

$$\sum_{\big((x,i),(y,j)\big) \in \mathcal{A}(u') \setminus \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| \leq \mathcal{O}(1)\, s \left(V(u) + L_3 \right) V(u) . \tag{3.7}$$

Applying (3.7), (3.4) and since $s, V(u), V(u') << 1$ we finally get

$$\begin{aligned}
Q(u') &= \sum_{\mathcal{A}(u')} \left|\sigma'_{x,i}\sigma'_{y,j}\right| \\
&\leq \sum_{\mathcal{A}(u')} \Big(\left|\sigma'_{x,i} - \sigma_{x,i}\right| \left|\sigma'_{y,j}\right| + \left|\sigma_{x,i}\right| \left|\sigma'_{y,j} - \sigma_{y,j}\right| \Big) + \sum_{\mathcal{A}(u')} \left|\sigma_{x,i}\sigma_{y,j}\right| \\
&\leq \mathcal{O}(1)\, s \Big\{ \left[V(u) + L_3 \right] V(u') + V(u) \left[V(u) + L_3 \right] \Big\} \\
&\quad + \sum_{\mathcal{A}(u') \setminus \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| + \sum_{\mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| \\
&\leq \mathcal{O}(1) \cdot s \cdot \Big(V(u)^2 + L_3 \Big) + Q(u) .
\end{aligned}$$

Since L_3 is a possibly null constant which depends only on the system, we can say $L_3 + V(u) = \mathcal{O}(1)$. Finally, the latter estimate follows combining the previous results. □

Corollary 3.7 *Let $\delta \in \left]0, \bar{\delta}_o\right[$ and assume that G satisfies **(G)**. Then we have*

$$P_s \overline{\mathcal{D}}_\delta \subseteq \overline{\mathcal{D}}_{\delta+\mathcal{O}(1)s(2L_3+\delta)} \qquad \text{and} \qquad P_s \mathcal{U}_\delta \subseteq \mathcal{U}_{\delta+s(L_2\delta+L_3)}\,.$$

In particular take a constant $C \geq \mathcal{O}(1)\left(2L_3 + \bar{\delta}_o\right)$, a number $\delta \in \left]0, \bar{\delta}_o\right[$ and time $T \in \left]0, T_o\right]$ such that $\delta + C\,T \leq \bar{\delta}_o$ then for any $t \in [0,T]$, $s \in [0, T-t]$ we have

$$P_s \overline{\mathcal{D}}_{\delta+tC} \subseteq \overline{\mathcal{D}}_{\delta+C(t+s)} \tag{3.8}$$

Proof. Fix $u \in \overline{\mathcal{D}}_\delta$ and an approximating sequence of piecewise constant function u_k with $\Upsilon(u_k) < \delta$. The previous Lemma shows that

$$\begin{aligned} \Upsilon\left(u_k + s(\Pi_N \circ G)(u_k)\right) &< \Upsilon(u_k) + \mathcal{O}(1)\, s\left[2L_3 + V(u_k)\right] \\ &< \delta + \mathcal{O}(1)\, s\left[2L_3 + \delta\right]\,. \end{aligned}$$

But $u_k + s\Pi_N \circ G(u_k)$ converges to $P_s u$ as $k, N \to +\infty$ and so $P_s u \in \overline{\mathcal{D}}_{\delta+\mathcal{O}(1)s(2L_3+\delta)}$. The proofs of the other inclusions are straightforward. □

Corollary 3.7 allows to define the domains appearing in Theorem 1.2 as

$$\mathcal{D}_t = \overline{\mathcal{D}}_{\delta+Ct} \qquad \forall\, t \in [0,T]\,.$$

Let $h \in \mathbb{N}$ and define

$$F_t^s u = S_{t-hs}(P_s \circ S_s)^h\, u \qquad t \in \left[hs, (h+1)s\right[\,. \tag{3.9}$$

In other words, in any interval $\left]hs, (h+1)s\right[$, we apply the semigroup S. In turn, at the times $t = hs$, P_s is applied.

If a time T and a $\delta \in \left]0, \bar{\delta}_o\right[$ are chosen as in Corollary 3.7, then $F_t^s u$ is defined up to the time $T - \bar{t}$ for any $u \in \mathcal{D}_{\bar{t}}$ and $\bar{t} \in [0,T]$.

Observe that for $t', t'' \in [0,T]$ with $t' + t'' \in [0,T]$, the following inclusion holds:

$$F_{t'}^s \mathcal{D}_{t''} \subseteq \mathcal{D}_{t'+t''}\,. \tag{3.10}$$

Note that F_t^s is $\mathbf{L^1}$-Lipschitz with Lipschitz constant bounded by $L \cdot L^{t/s} e^{L_1 t}$, with L as in Proposition 3.1 and L_1 as in **(G)**. The following theorem shows that the Lipschitz constant of F_t^s actually is bounded from above by a quantity independent from s.

Theorem 3.8 *Let f satisfy **(F)** and G satisfy **(G)**. If $\bar{\delta}_o$ is chosen sufficiently small and δ, T are chosen as in Corollary 3.7, then there exists a constant $\mathcal{L}$ such that for all $\bar{t} \in [0,T]$, $u, w \in \mathcal{D}_{\bar{t}}$ and $t \in [0, T-\bar{t}]$, we have*

$$\left\| F_t^s u - F_t^s w \right\|_{\mathbf{L^1}} \leq \mathcal{L} \cdot \left\| u - w \right\|_{\mathbf{L^1}}\,. \tag{3.11}$$

Proof. The proof can be carried out following essentially the same line used in [2, Theorem 4.2]. Therefore we only outline it. The key ingredient is the Liu & Yang functional which, unfortunately, is defined only on piecewise constant ε–approximations of the trajectories of the convective part (see [6]). Therefore we first suppose that the source $G(u)$ attains values in $\mathbf{PC}(\mathbb{R}, \mathbb{R}^n)$. For this kind of sources one can prove (3.11) following almost exactly the proof of [2, Theorem 4.2]. The only difference is the *global* nature of our source term. But the non locality of $G(u)$ can easily be tackled using Lemma 3.2, integrating and summing up the pointwise estimates obtained in [2, Theorem 4.2] and finally using **(G)** similarly to what we have done in a detailed way in the proof of Lemma 3.6.

Once we obtained (3.11) for piecewise constant source terms, we apply it to the source term $\Pi_N \circ G(u)$. If we denote by $F_t^{s,N} u$ the trajectory defined by (3.9) with $\Pi_N \circ G(u)$ in place of $G(u)$, it is quite easy to see that Lemma 3.4 implies the strong convergence of $F_t^{s,N} u$ to $F_t^s u$ proving the theorem for a general source G. □

We need below the following estimates concerning the dependence of F^s on time.

Lemma 3.9 *Let T and δ be as in corollary 3.7. Then, for all $u \in \mathcal{D}_{\bar t}$ and all $t \in [0, T - \bar t]$,*

$$\begin{aligned}
\left\|F_t^s u\right\|_{\mathbf{L^1}} &\leq \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) && (3.12)\\
\left\|(P_s)^k u\right\|_{\mathbf{L^1}} &\leq \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) && (3.13)\\
\left\|F_t^s u - u\right\|_{\mathbf{L^1}} &\leq \mathcal{O}(1) \cdot t \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) && (3.14)\\
\left\|(P_s)^k u - u\right\|_{\mathbf{L^1}} &\leq \mathcal{O}(1) \cdot k \cdot s \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) . && (3.15)
\end{aligned}$$

Proof. Consider first (3.13) with $k = 1$. By **(G)** and the properties of S

$$\begin{aligned}
\|P_s u\|_{\mathbf{L^1}} &= \left\|u + s\,G(u) - s\,G(0) + s\,G(0)\right\|_{\mathbf{L^1}}\\
&\leq \|u\|_{\mathbf{L^1}} + s\,L_1 \|u\|_{\mathbf{L^1}} + s\left\|G(0)\right\|_{\mathbf{L^1}}\\
&\leq e^{sL_1}\left(\tilde c\, s + \|u\|_{\mathbf{L^1}}\right)\\
\|S_s u\|_{\mathbf{L^1}} &\leq \|S_s u - u\|_{\mathbf{L^1}} + \|u\|_{\mathbf{L^1}}\\
&\leq L\, s + \|u\|_{\mathbf{L^1}}\\
&\leq e^{Ls}\left(\tilde c\, s + \|u\|_{\mathbf{L^1}}\right)
\end{aligned}$$

where $\tilde c = \max\{L, \left\|G(0)\right\|_{\mathbf{L^1}}\}$ and L is as in Proposition 3.1. Proceed now by induction on k and, for $k\,s \in \left[0, T - \bar t\right[$,

$$\begin{aligned}
\left\|F_{ks}^s u\right\|_{\mathbf{L^1}} &\leq e^{(L+L_1)ks}\left(2\,\tilde c\, k\, s + \|u\|_{\mathbf{L^1}}\right)\\
&\leq e^{(L+L_1)ks}\left(2\,\tilde c\, T + \|u\|_{\mathbf{L^1}}\right)\\
&\leq \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) .
\end{aligned}$$

Therefore, for $t \in \left[0, T-\bar{t}\right[$ and $\bar{k} = \left[\frac{t}{s}\right]$, we have

$$\begin{aligned}\left\|F_t^s u\right\|_{\mathbf{L^1}} &= \left\|S_{t-ks} F_{ks}^s u\right\|_{\mathbf{L^1}} \\ &\leq L\, s + \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \\ &\leq \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) .\end{aligned}$$

The estimate (3.13) is obtained similarly. Passing to the Lipschitz estimates, for $(k+1)s \in \left[0, T-\bar{t}\right[$, by (3.12)

$$\begin{aligned}\left\|F_{(k+1)s}^s u - u\right\|_{\mathbf{L^1}} &= \left\|F_s^s\, F_{ks}^s u - F_{ks}^s u\right\|_{\mathbf{L^1}} + \left\|F_{ks}^s u - u\right\|_{\mathbf{L^1}} \\ &\leq \left\|S_s F_{ks}^s u + sG(S_s F_{ks}^s u) - F_{ks}^s u\right\|_{\mathbf{L^1}} + \left\|F_{ks}^s u - u\right\|_{\mathbf{L^1}} \\ &\leq s \cdot \left(L + L_1 \left\|S_s F_{ks}^s u\right\|_{\mathbf{L^1}} + \left\|G(0)\right\|_{\mathbf{L^1}}\right) + \left\|F_{ks}^s u - u\right\|_{\mathbf{L^1}} \\ &\leq \mathcal{O}(1) \cdot s \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) + \left\|F_{ks}^s u - u\right\|_{\mathbf{L^1}} .\end{aligned}$$

By induction, $\left\|F_{ks}^s u - u\right\|_{\mathbf{L^1}} \leq \mathcal{O}(1) \cdot k \cdot s \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right)$.

We are left with the case $\frac{t}{s} \notin \mathbb{N}$. If $t \in [0, s[$ we have

$$\left\|F_t^s u - u\right\|_{\mathbf{L^1}} = \|S_t u - u\|_{\mathbf{L^1}} \leq \mathcal{O}(1) \cdot t\,,$$

while if $t \geq s$, so that $\bar{k} = \left[\frac{t}{s}\right] \geq 1$,

$$\begin{aligned}\left\|F_t^s u - u\right\|_{\mathbf{L^1}} &\leq L\, s + \left\|F_{\bar{k}s}^s u - u\right\|_{\mathbf{L^1}} \\ &\leq L\, s + \mathcal{O}(1) \cdot \bar{k} \cdot s \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \\ &\leq \mathcal{O}(1) \cdot t \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) .\end{aligned}$$

The proof of (3.15) is entirely similar. □

The next step consists in showing the convergence of F^s as s tends to zero. This result will be obtained with the help of the commutation relation (1.5) which we will show to be true for S_t and P_t. We need the following result that is an easy consequence of [2, Remark 4.1].

Proposition 3.10 *Take $u, v, \omega \in \mathcal{D}_T$, then, for any $t_1 < t_2$ one has the estimate*

$$\begin{aligned}\left\|S_{t_2} w - S_{t_2} u - \omega\right\|_{\mathbf{L^1}} &\leq L \cdot \left\|S_{t_1} w - S_{t_1} u - \omega\right\|_{\mathbf{L^1}} \\ &\quad + \mathcal{O}(1) \cdot (t_2 - t_1) \cdot \mathrm{TV}(\omega)\,.\end{aligned} \tag{3.16}$$

Proposition 3.10 implies the following commutation result:

Theorem 3.11 *Let δ and T be in Corollary 3.7. For any $\bar{t} \in [0, T[$, $u \in \mathcal{D}_{\bar{t}}$ and $t \in \left[0, T-\bar{t}\right]$, we have the estimate*

$$\|S_t P_t u - P_t S_t u\|_{\mathbf{L^1}} \leq \mathcal{O}(1) \cdot t^2\,. \tag{3.17}$$

Proof. Since $u+tG(u)$, $tG(S_tu)$ and u all belong to $\mathcal{D}_T$, we can apply (3.16) with $w = u + t\,G(u)$ and $\omega = t\,G\,(S_tu)$ to obtain:

$$\begin{aligned}
\|S_tP_tu - P_tS_tu\|_{\mathbf{L^1}} &= \Big\|S_t\left[u + t\,G(u)\right] - S_tu - t\,G\,(S_tu)\Big\|_{\mathbf{L^1}} \\
&\leq L\cdot\Big\|S_0\left[u + t\,G(u)\right] - S_0u - t\,G\,(S_tu)\Big\|_{\mathbf{L^1}} \\
&\quad + \mathcal{O}(1)\cdot t\cdot \mathrm{TV}\left[t\,G\,(S_tu)\right] \\
&\leq L\cdot t\cdot \big\|G(u) - G\,(S_tu)\big\|_{\mathbf{L^1}} + \mathcal{O}(1)\cdot t^2 \\
&\leq L^2\cdot L_1\cdot t^2 + \mathcal{O}(1)\cdot t^2\,.
\end{aligned}$$

□

Now we show that (3.17) and the uniform Lipschitz property (3.11) of the approximations imply the existence of a "tangent vector" and the strong convergence of the approximations. We will show that these two conditions are enough and that there is no need to use again the almost decreasing functional as was done in [2, Lemma 5.1].

Proposition 3.12 *Let δ and T be as in Corollary 3.7. For any $\bar t \in [0,T[$, $u \in \mathcal{D}_{\bar t}$, $t \in \left[0, T-\bar t\right]$ and $s, s' \in \left]0, t^2\right]$, we have*

$$\begin{aligned}
\big\|F_t^su - S_tP_tu\big\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot\left(1 + \|u\|_{\mathbf{L^1}}\right)\cdot t^2 \\
\big\|F_t^su - P_tS_tu\big\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot\left(1 + \|u\|_{\mathbf{L^1}}\right)\cdot t^2 \qquad (3.18)\\
\Big\|F_t^su - F_t^{s'}u\Big\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot\left(1 + \|u\|_{\mathbf{L^1}}\right)\cdot t^2\,.
\end{aligned}$$

Proof. We prove only the first inequality in (3.18), the other two inequalities being consequences of this one and of Theorem 3.11. For any integer $k \in [1, T/s]$ define

$$\rho_k(s) = \sup_{u\in\mathcal{D}_{T-ks}} \Big\|(P_s)^kS_su - S_s(P_s)^ku\Big\|_{\mathbf{L^1}}.$$

Now, for $u \in \mathcal{D}_{T-(k+1)s}$ (and hence $P_su \in \mathcal{D}_{T-ks}$), we can compute

$$\begin{aligned}
&\Big\|(P_s)^{k+1}S_su - S_s(P_s)^{k+1}u\Big\|_{\mathbf{L^1}} \leq \\
\leq\;& \Big\|(P_s)^kP_sS_su - (P_s)^kS_sP_su\Big\|_{\mathbf{L^1}} + \Big\|(P_s)^kS_sP_su - S_s(P_s)^kP_su\Big\|_{\mathbf{L^1}} \\
\leq\;& e^{TL_1}\|P_sS_su - S_sP_su\|_{\mathbf{L^1}} + \sup_{w\in\mathcal{D}_{T-ks}}\Big\|(P_s)^kS_sw - S_s(P_s)^kw\Big\|_{\mathbf{L^1}} \\
\leq\;& e^{TL_1}\rho_1(s) + \rho_k(s)\,.
\end{aligned}$$

And hence $\rho_{k+1}(s) \leq e^{TL_1}\rho_1(s) + \rho_k(s)$ that, by induction, gives

$$\rho_k(s) \leq e^{TL_1}k\rho_1(s)\,.$$

Now define

$$\bar\rho_k(s) = \sup_{u\in\mathcal{D}_{T-ks}} \left\| F^s_{ks}u - S_{ks}(P_s)^k u \right\|_{\mathbf{L^1}}.$$

Again we can compute for $u \in \mathcal{D}_{T-(k+1)s}$

$$\begin{aligned}
& \left\| F^s_{(k+1)s}u - S_{(k+1)s}(P_s)^{k+1}u \right\|_{\mathbf{L^1}} \leq \\
\leq\ & \left\| F^s_{ks}P_sS_su - F^s_{ks}S_sP_su \right\|_{\mathbf{L^1}} + \left\| F^s_{ks}S_sP_su - S_{ks}(P_s)^kS_sP_su \right\|_{\mathbf{L^1}} \\
& + \left\| S_{ks}(P_s)^kS_sP_su - S_{ks}S_s(P_s)^kP_su \right\|_{\mathbf{L^1}} \\
\leq\ & \mathcal{L}\rho_1(s) + \bar\rho_k(s) + L\left\| (P_s)^kS_sP_su - S_s(P_s)^kP_su \right\|_{\mathbf{L^1}} \\
\leq\ & \mathcal{L}\rho_1(s) + \bar\rho_k(s) + L\rho_k(s) \\
\leq\ & \left(\mathcal{L} + Le^{L_1T}k\right)\rho_1(s) + \bar\rho_k(s)\,.
\end{aligned}$$

Therefore we have $\bar\rho_{k+1}(s) \leq \left(\mathcal{L} + L\,e^{L_1T}k\right)\rho_1(s) + \bar\rho_k(s)$ which gives, together to $\rho_1(s) = \bar\rho_1(s)$, by induction

$$\bar\rho_k(s) \leq \left(\mathcal{L} + L\,e^{L_1T}k\right)\,k\,\rho_1(s).$$

Fix now $t \in [0, T-\bar t]$, take $s \in \left]0, t^2\right[$ and define $\hat k = \left[\frac{t}{s}\right]$. We have for all $u \in \mathcal{D}_{\bar t} \subset \mathcal{D}_{T-\hat k s}$

$$\begin{aligned}
\left\| F^s_t u - S_t(P_s)^{\hat k}u \right\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot(t-\hat k\,s) + \left\| F^s_{\hat k s}u - S_{\hat k s}(P_s)^{\hat k}u \right\|_{\mathbf{L^1}} \\
&\leq \mathcal{O}(1)\cdot(t-\hat k s) + \left(\mathcal{L} + L\,e^{L_1T}\frac{t}{s}\right)\frac{t}{s}\,\rho_1(s) \\
&\leq \mathcal{O}(1)\cdot s + \mathcal{O}(1)\cdot\left(t\,s + t^2\right)\frac{\rho_1(s)}{s^2} \\
&\leq \mathcal{O}(1)\cdot t^2\,,
\end{aligned}$$

where the last inequality is a consequence to the fact that $\frac{\rho_1(s)}{s^2}$ is bounded because of (3.17).

We are left to prove that

$$\left\| (P_s)^{\hat k}u - P_tu \right\|_{\mathbf{L^1}} \leq \mathcal{O}(1)\cdot\left(1 + \|u\|_{\mathbf{L^1}}\right)\cdot t^2\,.$$

By (3.15),

$$\begin{aligned}
\left\| (P_s)^{k+1}u - P_{(k+1)s}u \right\|_{\mathbf{L^1}} &= \left\| P_s(P_s)^ku - P_{ks}u - s\,G(u) \right\|_{\mathbf{L^1}} \\
&= \left\| (P_s)^ku - P_{ks}u + s\,G\left((P_s)^ku\right) - s\,G(u) \right\|_{\mathbf{L^1}} \\
&\leq \left\| (P_s)^ku - P_{ks}u \right\|_{\mathbf{L^1}} + \mathcal{O}(1)\cdot\left(1 + \|u\|_{\mathbf{L^1}}\right)\cdot k\,s^2
\end{aligned}$$

We thus recursively obtain

$$\left\|(P_s)^k - P_{ks}u\right\|_{\mathbf{L^1}} \leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot k^2\, s^2 \,.$$

Therefore

$$\begin{aligned}
\left\|(P_s)^{\hat{k}}u - P_t u\right\|_{\mathbf{L^1}} &\leq \left\|(P_s)^{\hat{k}}u - P_{\hat{k}s}u\right\|_{\mathbf{L^1}} + \left\|P_{\hat{k}s}u - P_t u\right\|_{\mathbf{L^1}} \\
&\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot \hat{k}^2\, s^2 + \mathcal{O}(1)\cdot s \\
&\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot t^2 \,.
\end{aligned}$$

□

Now we prove the convergence of the approximations and the characterization of the tangent vector, i.e. 4. in Theorem 1.2.

Theorem 3.13 *Let δ and T be as in Corollary 3.7. For any $\bar{t} \in [0, T[$, $u \in \mathcal{D}_{\bar{t}}$ and $t \in \left[0, T-\bar{t}\right]$ the sequence $F_t^s u$ converges in $\mathbf{L^1}$ as $s \to 0$ to a limit trajectory $F_t u$ which satisfies the tangency conditions*

$$\begin{aligned}
\|F_t u - S_t P_t u\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot t^2 \\
\|F_t u - P_t S_t u\|_{\mathbf{L^1}} &\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot t^2 \,.
\end{aligned} \tag{3.19}$$

Proof. Because of (3.18), we need only to show that $s \to F_t^s u$ is a Cauchy sequence in $\mathbf{L^1}$ as $s \to 0$. Fix $\varepsilon > 0$ arbitrary. Then choose $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = t$ so that $t_i - t_{i-1} < \varepsilon$ for $i = 1, \ldots, N$. Then observe that Definition (3.9), Theorem 3.8 and (3.12) imply that F^s satisfies an approximated semigroup condition:

$$\left\|F_{t_1}^s F_{t_2}^s u - F_{t_1+t_2}^s u\right\|_{\mathbf{L^1}} = \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L^1}}\right)\cdot s \,.$$

Therefore, for any $0 < s, s' < \min_{i=1\ldots N}\left\{(t_i - t_{i-1})^2\right\}$, we can compute

$$\begin{aligned}
\left\|F_t^{s'}u - F_t^s u\right\|_{\mathbf{L^1}} &\leq \sum_{i=1}^N \left\|F_{t-t_i}^s F_{t_i}^{s'} u - F_{t-t_{i-1}}^s F_{t_{i-1}}^{s'} u\right\|_{\mathbf{L^1}} \\
&\leq \sum_{i=1}^N \left\|F_{t-t_i}^s F_{t_i}^{s'} u - F_{t-t_i}^s F_{t_i-t_{i-1}}^s F_{t_{i-1}}^{s'} u\right\|_{\mathbf{L^1}} \\
&\quad + \mathcal{O}(1)\left(1+\|u\|_{\mathbf{L^1}}\right) N s \\
&\leq \mathcal{L}\sum_{i=1}^N \left\|F_{t_i}^{s'} u - F_{t_i-t_{i-1}}^s F_{t_{i-1}}^{s'} u\right\|_{\mathbf{L^1}} + \mathcal{O}(1)\left(1+\|u\|_{\mathbf{L^1}}\right) N s \\
&\leq \mathcal{L}\sum_{i=1}^N \left\|F_{t_i-t_{i-1}}^{s'} F_{t_{i-1}}^{s'} u - F_{t_i-t_{i-1}}^s F_{t_{i-1}}^{s'} u\right\|_{\mathbf{L^1}}
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}(1)\left(1+\|u\|_{\mathbf{L}^1}\right)\,N\,(s+s')\\
\leq\;& \mathcal{O}(1)\left(1+\|u\|_{\mathbf{L}^1}\right)\left(\sum_{i=1}^{N}(t_i-t_{i-1})^2+N\,(s+s')\right)\\
\leq\;& \mathcal{O}(1)\left(1+\|u\|_{\mathbf{L}^1}\right)\left(\varepsilon\, t+N\,(s+s')\right)\,.
\end{aligned}
$$

And, finally, as $s, s' \to 0$ we get

$$
\limsup_{s,s'\to 0}\left\|F_t^{s'}u-F_t^s u\right\|_{\mathbf{L}^1}\leq \mathcal{O}(1)\;\left(1+\|u\|_{\mathbf{L}^1}\right)\,\varepsilon\, t
$$

which proves the Theorem because of the arbitrariness of ε. □

The limit trajectory thus obtained satisfies (2) in Theorem 1.2, as can be seen passing to the limit $s \to 0$ in (3.10) and in the approximate semigroup condition

$$
\left\|F_{t_1}^s F_{t_2}^s u-F_{t_1+t_2}^s u\right\|_{\mathbf{L}^1}\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L}^1}\right)\cdot s\,.
$$

Taking the same limit in (3.11), we prove the former inequality in (1.6). To prove the latter estimate, observe that by (3.14)

$$
\|F_t u-u\|_{\mathbf{L}^1}\leq \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L}^1}\right)\cdot t
$$

while, for $t_2 > t_1$, the semigroup property implies

$$
\begin{aligned}
\left\|F_{t_2}u-F_{t_1}u\right\|_{\mathbf{L}^1} \;\leq\;& \left\|F_{t_2-t_1}F_{t_1}u-F_{t_1}u\right\|_{\mathbf{L}^1}\\
\leq\;& \mathcal{O}(1)\cdot\left(1+\left\|F_{t_1}u\right\|_{\mathbf{L}^1}\right)\cdot(t_2-t_1)\\
\leq\;& \mathcal{O}(1)\cdot\left(1+\|u\|_{\mathbf{L}^1}\right)\cdot(t_2-t_1)\,.
\end{aligned}
$$

Assertion (4) follows from (3.19).

We pass now to (3). The trajectory $t \mapsto F_t u$ is a weak entropic solution of (1.1). This can be proved using the properties of the approximate solutions constructed above, as in [1, 2, 11]. Here, we prefer to exploit the tangent vector provided by theorem 3.13

Corollary 3.14 *Let δ and T be as in Corollary 3.7. For any $\bar t \in [0, T[$, $u \in \mathcal{D}_{\bar t}$ and $t \in [0, T - \bar t\,]$ the trajectory $t \mapsto F_t u$ is a weak entropic solution of* (1.1).

For the definition of weak entropic solutions of a balance law, refer to [12], [13, (2.16) and (2.19)] or [11, § 6].

Proof of Corollary 3.14. We show below only the entropy inequality, since the proof that $t \mapsto F_t u$ is a weak solution is entirely similar.

Observe that, by (3.19)

$$\begin{aligned} F_t u &= S_t u + \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \cdot t \\ F_t u &= S_t u + t\, G(u) + \mathcal{O}(1) \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \cdot t^2 \end{aligned}$$

in $\mathbf{L^1}$. Let (η, q) be an entropy-entropy flux pair and $\varphi \in \mathbf{C^1_c}$ be a non negative test function. Fix a positive ε and denote $I_i = \left[i\varepsilon, (i+1)\varepsilon\right[\times \mathbb{R}$ for $i \in \mathbb{N}$. By the properties of S, $\partial_t \eta(S_t u) + \partial_x q(S_t u) \le 0$ in the sense of distribution, using the Divergence Theorem we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(\eta(F_t u)\partial_t \varphi + q(F_t u)\partial_x \varphi\right)\, dx\, dt \quad = \\
= \;& \sum_i \iint_{I_i} \left(\eta(F_t u)\partial_t \varphi + q(F_t u)\partial_x \varphi\right)\, dx\, dt \\
= \;& \sum_i \left(\iint_{I_i} \left(\eta(S_{t-i\varepsilon} F_{i\varepsilon} u)\partial_t \varphi + q(S_{t-i\varepsilon} F_{i\varepsilon} u)\partial_x \varphi\right)\, dx\, dt \right) \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
\ge \;& \sum_i \left(\int_{\mathbb{R}} \Big(\eta(S_\varepsilon F_{i\varepsilon} u)\, \varphi\left((i+1)\varepsilon, x\right) - \eta(F_{i\varepsilon} u)\, \varphi(i\varepsilon, x) \Big)\, dx \right) \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \sum_i \left(\int_{\mathbb{R}} \Big(\eta(S_\varepsilon F_{i\varepsilon} u) - \eta(F_{(i+1)\varepsilon} u) \Big)\, \varphi\left((i+1)\varepsilon, x\right)\, dx \right) \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \sum_i \left(\int_{\mathbb{R}} \left(\eta(S_\varepsilon F_{i\varepsilon} u) - \eta(F_\varepsilon F_{i\varepsilon} u) \right)\, \varphi\left((i+1)\varepsilon, x\right)\, dx \right) \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \sum_i \int_{\mathbb{R}} \Big(\eta(S_\varepsilon F_{i\varepsilon} u) - \eta\left(S_\varepsilon F_{i\varepsilon} u + \varepsilon G(F_{i\varepsilon} u)\right) \Big)\, \varphi\left((i+1)\varepsilon, x\right)\, dx \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \sum_i \int_{\mathbb{R}} \Big(-D\eta(F_{i\varepsilon} u)\, \varepsilon\, G(F_{i\varepsilon} u)\, \varphi\left((i+1)\varepsilon, x\right) \Big)\, dx \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \sum_i \iint_{I_i} \left(-D\eta(F_t u)\, G(F_t u)\, \varphi(t, x) \right)\, dx\, dt \\
& + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon \\
= \;& \int_0^T \int_{\mathbb{R}} -D\eta(F_t u)\, G(F_t u)\, \varphi(t, x)\, dx\, dt + \mathcal{O}(1) \left(1 + \|u\|_{\mathbf{L^1}}\right) \varepsilon
\end{aligned}$$

By the arbitrariness of ε, we conclude with the distributional inequality

$$\partial_t \eta(F_t u) + \partial_x q(F_t u) - D\eta(F_t u)\, G(F_t u) \leq 0\,.$$

□

Now we show a result on the dependence of the solution with respect to the source term.

Theorem 3.15 *Let f_1, f_2 satisfy **(F)** and G_1, G_2 satisfy **(G)**. Call F^1, F^2 the corresponding semigroups and assume they are defined on a common family of domains $\mathcal{D}_t$. Then, for any $\bar{t} \in [0, T[$, $u \in \mathcal{D}_{\bar{t}}$ and $t \in \left[0, T - \bar{t}\right]$, the Lipschitz estimate* (1.7) *holds.*

*Moreover, fix a flux f satisfying **(F)** and a sequence of source terms G_k satisfying **(G)**. If G_k converges pointwise to G, then the corresponding semigroups F^k converge pointwise to the semigroup F generated by G.*

Proof. We apply the well known integral estimate, see [6, Theorem 2.9]:

$$\left\| F_t^1 u - F_t^2 u \right\|_{\mathbf{L^1}} \leq \mathcal{L} \int_0^t \liminf_{\vartheta \to 0} \frac{\left\| F^2_{\vartheta+\tau} u - F^1_\vartheta F^2_\tau u \right\|_{\mathbf{L^1}}}{\vartheta}\, d\tau$$

By (3.19) and using the stability result [4, Corollary 2.5], with obvious notation we have

$$\begin{aligned}
& \liminf_{\vartheta \to 0} \frac{1}{\vartheta} \left\| F^2_\vartheta F^2_\tau u - F^1_\vartheta F^2_\tau u \right\|_{\mathbf{L^1}} \leq \\
\leq\; & \liminf_{\vartheta \to 0} \frac{1}{\vartheta} \left\| S^2_\vartheta F^2_\tau u + \vartheta G_2(F^2_\tau u) - S^1_\vartheta F^2_\tau u - \vartheta\, G_1(F^2_\tau u) \right\|_{\mathbf{L^1}} \\
\leq\; & \liminf_{\vartheta \to 0} \frac{1}{\vartheta} \left\| S^2_\vartheta F^2_\tau u - S^1_\vartheta F^2_\tau u \right\|_{\mathbf{L^1}} + \left\| G_2(F^2_\tau u) - G_1(F^2_\tau u) \right\|_{\mathbf{L^1}} \\
\leq\; & \mathcal{O}(1) \cdot \left(\| Df_1 - Df_2 \|_{\mathbf{C^0}(\Omega, \mathbb{R}^{n\times n})} + \| G_1 - G_2 \|_{\mathbf{C^0}(\mathcal{U}_{\delta_o}; \mathbf{L^1}(\mathbb{R};\mathbb{R}^n))} \right)\,.
\end{aligned}$$

Concerning the pointwise convergence, note that

$$\left\| F_t^k u - F_t u \right\|_{\mathbf{L^1}} \leq \mathcal{L} \int_0^t \left\| G(F_\tau u) - G_k(F_\tau u) \right\|_{\mathbf{L^1}}\, d\tau$$

and the proof is concluded trough Lebesgue convergence theorem. □

3.4 Integral Characterization

Following [6, § 9.2], [1, § 5.2] and [3, Definition 15.1], let $v \in \mathcal{D}_T$. For all $\xi \in \mathbb{R}$, define $U^\sharp_{(v,\xi)}$ as the solution to the homogeneous Riemann problem

$$\left\{ \begin{array}{l} \partial_t w + \partial_x f(w) = 0 \\ w(0,x) = \left\{ \begin{array}{lccc} \lim\limits_{y \to \xi-} v(y) & \text{if} & x < \xi \\ \lim\limits_{y \to \xi+} v(y) & \text{if} & x > \xi \,. \end{array} \right. \end{array} \right. \tag{3.20}$$

Define $U^\flat_{(v,\xi)}$ as the broad solution (see [6, § 3.1]) to the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t w + Df\left(v(\xi)\right) \partial_x w = G(v) \\ w(0,x) = v(x)\,, \end{array} \right. \tag{3.21}$$

so that denoting $l_i^\xi = l_i\left(v(\xi)\right)$, $r_i^\xi = r_i\left(v(\xi)\right)$ and $\lambda_i^\xi = \lambda_i\left(v(\xi)\right)$,

$$\begin{aligned} U^\flat_{(v,\xi)}(t,x) &= U^{\flat,1}_{(v,\xi)}(t,x) + U^{\flat,2}_{(v,\xi)}(t,x) \\ U^{\flat,1}_{(v,\xi)}(t,x) &= \sum_{i=1}^n \left(l_i^\xi \cdot v\left(x - \lambda_i^\xi t\right)\right) r_i^\xi \\ U^{\flat,2}_{(v,\xi)}(t,x) &= \sum_{i=1}^n \int_0^t \left(l_i^\xi \cdot G(v)(x - \lambda_i^\xi s)\right) r_i^\xi \, ds\,. \end{aligned}$$

We are now ready to prove the first part of the characterization stated in Theorem 1.2.

Theorem 3.16 *Let F be the map constructed in Theorem 3.13, let $\hat\lambda$ be an upper bound for all characteristic speeds. Then, for all $\bar t \in [0,T[$, $u \in \mathcal{D}_{\bar t}$ and all $\tau \in \left[0, T-\bar t\right]$, $F_\tau u$ satisfies 6 in Theorem 1.2.*

Proof. To obtain (6a), compute:

$$\begin{aligned} &\frac{1}{\vartheta}\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda} \left\| \left(F_\vartheta u(\tau)\right)(x) - U^\sharp_{(v,\xi)}(\vartheta,x)\right\| dx \;\leq \\ \leq\; &\frac{1}{\vartheta}\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda} \left\| \left(F_\vartheta u(\tau)\right)(x) - \left(S_\vartheta u(\tau)\right)(x) - \vartheta\left(G\left(u(\tau)\right)\right)(x)\right\| dx \\ &+\frac{1}{\vartheta}\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda} \left\| \left(S_\vartheta u(\tau)\right)(x) - U^\sharp_{(v,\xi)}(\vartheta,x)\right\| dx \\ &+\frac{1}{\vartheta}\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda} \left\| \vartheta\left(G\left(u(\tau)\right)\right)(x)\right\| dx \\ \leq\; &\frac{1}{\vartheta}\left\| F_\vartheta u(\tau) - S_\vartheta u(\tau) - \vartheta\, G\left(u(\tau)\right)\right\|_{\mathbf{L^1}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\vartheta}\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda}\left\|\big(S_\vartheta u(\tau)\big)(x)-U^\sharp_{(v,\xi)}(\vartheta,x)\right\|dx\\
&+\int_{\xi-\vartheta\hat\lambda}^{\xi+\vartheta\hat\lambda}\left\|\Big(G\big(u(\tau)\big)\Big)(x)\right\|dx
\end{aligned}
$$

As $\vartheta \to 0$, the first summand above vanishes by (4) in Theorem 1.2, the second by [6, (9.16), § 9.2] and the latter one by $G\big(u(\tau)\big) \in \mathbf{L^1}$.

Similarly, to obtain (6b), we exploit [6, § 9.2]:

$$
\begin{aligned}
&\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\big(F_\vartheta u(\tau)\big)(x)-U^\flat_{(v,\xi)}(\vartheta,x)\right\|dx \quad\le\\
\le\;&\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\big(F_\vartheta u(\tau)\big)(x)-\big(S_\vartheta u(\tau)\big)(x)-\vartheta\,\Big(G\big(u(\tau)\big)\Big)(x)\right\|dx\\
&+\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\big(S_\vartheta u(\tau)\big)(x)-U^\flat_{(v,\xi)}(\vartheta,x)+\vartheta\,\Big(G\big(u(\tau)\big)\Big)(x)\right\|dx\\
\le\;&\frac{1}{\vartheta}\left\|F_\vartheta u(\tau)-S_\vartheta u(\tau)-\vartheta\,G\big(u(\tau)\big)\right\|_{\mathbf{L^1}}\\
&+\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\big(S_\vartheta u(\tau)\big)(x)-U^{\flat,1}_{(v,\xi)}(\vartheta,x)\right\|dx\\
&+\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\vartheta\,\Big(G\big(u(\tau)\big)\Big)(x)-U^{\flat,2}_{(v,\xi)}(\vartheta,x)\right\|dx\\
\le\;&\frac{1}{\vartheta}\left\|F_\vartheta u(\tau)-S_\vartheta u(\tau)-\vartheta\,G\big(u(\tau)\big)\right\|_{\mathbf{L^1}}\\
&+\mathcal{O}(1)\,\Big(\mathrm{TV}\,\big(u(\tau);\,]a,b[\big)\Big)^2\\
&+\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\vartheta\,\Big(G\big(u(\tau)\big)\Big)(x)-U^{\flat,2}_{(v,\xi)}(\vartheta,x)\right\|dx
\end{aligned}
$$

As $\vartheta \to 0$, the first summand above vanishes by (4) in Theorem 1.2, while the latter vanishes as $\vartheta \to 0$. Indeed, using [6, Lemma 2.3],

$$
\begin{aligned}
&\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\vartheta\,\Big(G\big(u(\tau)\big)\Big)(x)-U^{\flat,2}_{(v,\xi)}(\vartheta,x)\right\|dx\le\\
\le\;&\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\int_0^\vartheta\sum_{i=1}^n l_i^{(\tau,\xi)}\cdot\Big(G\big(u(\tau)\big)(x)\Big)\,r_i^{(\tau,\xi)}\,ds\right.\\
&\qquad\left.-\sum_{i=1}^n\int_0^\vartheta l_i^{(\tau,\xi)}\cdot\Big(G\big(u(\tau)\big)\Big)(x-\lambda_i^{(\tau,\xi)}s)r_i^{(\tau,\xi)}\,ds\right\|dx\\
\le\;&\frac{\mathcal{O}(1)}{\vartheta}\sum_{i=1}^n\int_0^\vartheta\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda}\left\|\Big(G\big(u(\tau)\big)\Big)(x)-\Big(G\big(u(\tau)\big)\Big)(x-\lambda_i^{(\tau,\xi)}s)\right\|dx\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\mathcal{O}(1)}{\vartheta} \sum_{i=1}^{n} \int_0^{\vartheta} \mathrm{TV}\left(G\left(u(\tau)\right);]a,b[\right) \left|\lambda_i^{(\tau,\xi)}\right| s\, ds \\
&\leq \mathcal{O}(1) \cdot \vartheta \cdot \mathrm{TV}\left(G\left(u(\tau)\right);]a,b[\right) .
\end{aligned}
$$

completing the proof of (6b). □

Theorem 3.17 *Let F be the map constructed in Theorem 3.13 and $\hat{\lambda}$ be an upper bound for all characteristic speeds. Assume $\bar{t} \in [0,T[$, $u: [0, T-\bar{t}] \mapsto \mathcal{D}_T$ is Lipschitz, satisfies $u(t) \in \mathcal{D}_{\bar{t}+t}$ and both* (7a) *and* (7b) *in Theorem 1.2 hold. Then, $u(t) = F_t u$ for all $t \in [0, T-\bar{t}]$.*

We omit this proof, since it is a slight modification of [6, Part 2 of Theorem 9.2].

3.5 Consequences of the Hyperbolic Rescaling

Given a function $v: \mathbb{R} \mapsto \mathbb{R}^n$ and $\lambda > 0$, we denote by v_λ the function obtained by applying a dilatation to v, i.e. $v_\lambda(x) = v(\lambda x)$. Obviously $v \in \overline{\mathcal{D}}_\delta$ implies $v_\lambda \in \overline{\mathcal{D}}_\delta$. We have the following Proposition (see also [22, Corollary 1]).

Proposition 3.18 *Let $S: [0,T] \times \overline{\mathcal{D}}_\delta \mapsto \overline{\mathcal{D}}_\delta$ be the semigroup generated by a system of conservation laws and let $d: \overline{\mathcal{D}}_\delta \times \overline{\mathcal{D}}_\delta \mapsto \mathbb{R}^+$ be a distance satisfying*

$$
d(u_\lambda, v_\lambda) = \frac{1}{\lambda}\, d(u,v) \text{ for all } u,v \in \overline{\mathcal{D}}_\delta, \text{ and } \lambda > 0,
$$

and the Grönwall estimate for a positive C:

$$
d(S_t u, S_t v) \leq e^{Ct}\, d(u,v) \text{ for all } u,v \in \overline{\mathcal{D}}_\delta \text{ and } t \in [0,T].
$$

Then $C = 0$, i.e. d is non expansive with respect to S.

Proof. If $u(t,x) = (S_t u)(x)$ is a semigroup trajectory, then also $u(\lambda t, \lambda x) = (S_t u_\lambda)(x)$ is a semigroup trajectory. Therefore we have the equality

$$
(S_t u)_\lambda = S_{\frac{t}{\lambda}} u_\lambda \text{ for all } u \in \overline{\mathcal{D}}_\delta,\ t \in [0,T] \text{ and } \lambda > 0\,.
$$

Hence we can compute for all $u, v \in \overline{\mathcal{D}}_\delta$ and $\lambda > 0$

$$
\begin{aligned}
d\left(S_t u, S_t v\right) &= \lambda\, d\left((S_t u)_\lambda, (S_t v)_\lambda\right) = \lambda\, d\left(S_{\frac{t}{\lambda}} u_\lambda, S_{\frac{t}{\lambda}} v_\lambda\right) \\
&\leq \lambda\, e^{C\frac{t}{\lambda}} d\left(u_\lambda, v_\lambda\right) = e^{C\frac{t}{\lambda}} d\left(u,v\right)
\end{aligned}
$$

Now, letting λ tend to infinity, we get the non expansive property

$$
d\left(S_t u, S_t v\right) \leq d\left(u,v\right) \text{ for all } u,v \in \overline{\mathcal{D}}_\delta \text{ and } t \in [0,T].
$$

□

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