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DISCRETE WEIGHTED LAPLACIAN

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A Note on Algebraic Multigrid Methods for the Discrete Weighted Laplacian

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Abstract

In recent contributions, algebraic multigrid methods have been designed and studied from the viewpoint of the spectral complementarity. In this note we focus our efforts on specific applications and, more precisely, on large linear systems arising from the approximation of weighted Laplacian with various boundary conditions. We adapt the multigrid idea to this specific setting and we present and critically discuss a wide numerical experimentation showing the potentiality of the considered approach.

1 Introduction

In the present note we test a specific application of a previously proposed algebraic multigrid procedure [18]. In that manuscript, we posed and partially answered the following question: having at our disposal an optimal multigrid procedure for $A_n x = b$, $\{A_n\}$ being a given sequence of Hermitian positive definite matrices of increasing dimension, which are the minimal changes (if any) to the procedure for maintaining the optimality for $B_n y = c$, $\{B_n\}$ new sequence of matrices with $B_n = A_n + R_n$?

Of course if there is no relation between $\{A_n\}$ and $\{B_n\}$ nothing can be said. However, under the mild assumption that there exists a value ϑ independent of n such that $A_n \leq \vartheta B_n$ and $B_n \leq M I_n$ with M again independent of n , it has been clearly shown that the smoothers can be simply adapted and the prolongation and restriction operators can be substantially kept unchanged. The aim of this paper is to show the effectiveness of this approach in a specific

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setting. More precisely, we consider linear systems $A_n(a)u = b$ arising from Finite Difference (FD) approximations of

$$-\nabla(a(x)\nabla u(x)) = f(x), \quad x \in \Omega = (0, 1)^d, \quad d \geq 1,$$

where $a(x) \geq a_0 > 0$, $f(x)$ are given bounded functions and with Dirichlet boundary conditions (BCs). Some remarks about the case of periodic or reflective BCs are also considered (for a discussion on this topic see [12,17]).

We recall that in the case $a(x) \equiv 1$, the matrix $A_n(1)$ is structured, positive definite, ill-conditioned, and an optimal algebraic multigrid method is already available (see [1,2,20,8,9,11,6,16,19,7,21]) according to different BCs.

Hereafter, owing to the spectral equivalence between the matrix sequences $\{A_n(a)\}$ and $\{A_n(1)\}$, the key idea is that the multigrid procedure just devised for $\{A_n(1)\}$ can be successfully applied to $\{A_n(a)\}$ too.

More in general in [18], we treated the case of structured-plus-banded uniformly bounded Hermitian positive definite linear systems, where the banded part R_n which is added to the structured coefficient matrix A_n is not necessarily definite and not necessarily structured. In our setting $A_n = A_n(1)$ is the structured part (it is Toeplitz, circulant etc, according to BCs) and $R_n = A_n(a - 1)$ is the non-structured, non necessarily definite contribution.

However, while a theoretical analysis of the Two-Grid Method (TGM) for structured+banded uniformly bounded Hermitian positive definite linear systems has been given in [18], in terms of the algebraic multigrid theory by Ruge and Stüben [14], the corresponding analysis for the multigrid method (MGM) is not complete and deserves further attention. Here, for MGM algorithm, we mean the simplest (and less expensive) version of the large family of multigrid methods, i.e., the V-cycle procedure: for a brief description of the TGM and of the V-cycle algorithms we refer to Section 2, while an extensive treatment can be found in [10], and especially in [22].

Indeed, the numerics in this note suggest that the MGM is optimal in the sense that (see [4]) the cost of solving the linear system (inverse problem) is proportional, by a pure constant not depending on n , to the cost of the matrix-vector product (direct problem): in our case more details can be given and in fact:

- a.** the observed number of iterations is bounded by a constant independent of the size of the algebraic problem;
- b.** the cost per iteration (in terms of arithmetic operations) is just linear as the size of the algebraic problem.

Furthermore, given the spectral equivalence between $\{A_n(a)\}$, $a(x) \geq a_0 > 0$, and $\{A_n(1)\}$, a simpler numerical strategy could be used: use $A_n(1)$ as preconditioner for $A_n(a)$ in a PCG method and solve the linear systems with coefficient matrix $A_n(1)$ by MGM. Of course, this approach is simpler to implement, but since several linear systems have to be solved by MGM, the flop count can be more favorable in applying the MGM directly instead of using it as solver for the preconditioner.

The paper is organized as follows. In Section 2 we report the standard TGM and MGM algorithms, together with the reference theoretical results on the TGM optimal rate of convergence, under some general and weak assumptions. In Section 3 the proposed approach is applied to the discrete weighted Laplacian and several numerical experiments are considered, by varying the diffusion function $a(x)$ with respect to its analytical features. Finally, Section 4 deals with further considerations concerning future work and perspectives.

2 Two-grid and Multigrid Method

We carefully report the TGM and MGM algorithms and we describe the theoretical ground on which we base our proposal. We start with the simpler TGM and then we describe the MGM and its interpretation as stationary or multi-iterative method, see [15].

2.1 Algorithm definition

Let n_0 be a positive d -index, $d \geq 1$, and let $N(\cdot)$ be an increasing function with respect to n_0 . In devising a TGM and a MGM for the linear system $A_{n_0}x_{n_0} = b_{n_0}$, where $A_{n_0} \in \mathbb{C}^{N(n_0) \times N(n_0)}$ and $x_{n_0}, b_{n_0} \in \mathbb{C}^{N(n_0)}$, the ingredients below must be considered.

Let $n_1 < n_0$ (componentwise) and let $p_{n_0}^{n_1} \in \mathbb{C}^{N(n_0) \times N(n_1)}$ be a given full-rank matrix. In order to simplify the notation, in the following we will refer to any multi-index n_s by means of its subscript s , so that, e.g. $A_s := A_{n_s}$, $b_s := b_{n_s}$, $p_s^{s+1} := p_{n_s}^{n_{s+1}}$, etc.

With these notations, a class of stationary iterative methods of the form $x_s^{(j+1)} = V_s x_s^{(j)} + \tilde{b}_s$ is also considered in such a way that $\mathcal{S}mooth(x_s^{(j)}, b_s, V_s, \nu_s)$ denotes the application of this rule ν_s times, with ν_s positive integer number, at the dimension corresponding to the index s .

Thus, the solution of the linear system $A_{n_0}x_{n_0} = b_{n_0}$ is obtained by applying repeatedly the TGM iteration, where the j^{th} iteration

$$x_0^{(j+1)} = \mathcal{TGM}(x_0^{(j)}, b_0, A_0, V_{0,\text{pre}}, \nu_{0,\text{pre}}, V_{0,\text{post}}, \nu_{0,\text{post}})$$

is defined by the following algorithm [10]:

$y_0 := \mathcal{TGM}(x_0, b_0, A_0, V_{0,\text{pre}}, \nu_{0,\text{pre}}, V_{0,\text{post}}, \nu_{0,\text{post}})$	
$\tilde{x}_0 := \mathcal{Smooth}(x_0, b_0, V_{0,\text{pre}}, \nu_{0,\text{pre}})$	<i>Pre-smoothing iterations</i>
$r_0 := b_0 - A_0 \tilde{x}_0$ $r_1 := (p_0^1)^H r_0$ Solve $A_1 y_1 = r_1$, with $A_1 := (p_0^1)^H A_0 p_0^1$ $\tilde{y}_0 := \tilde{x}_0 + p_0^1 y_1$	<i>Exact Coarse Grid Correction</i>
$y_0 := \mathcal{Smooth}(\tilde{y}_0, b_0, V_{0,\text{post}}, \nu_{0,\text{post}})$	<i>Post-smoothing iterations</i>

The first and last steps concern the application of $\nu_{0,\text{pre}}$ steps of the *pre-smoothing* (or *intermediate*) iteration and of $\nu_{0,\text{post}}$ steps of the *post-smoothing* iteration, respectively. Moreover, the intermediate steps define the so called *coarse grid correction*, that depends on the projection operator $(p_0^1)^H$. In such a way, the TGM iteration represents a classical stationary iterative method whose iteration matrix is given by

$$\mathcal{TGM}_0 = V_{0,\text{post}}^{\nu_{0,\text{post}}} \mathcal{CGC}_0 V_{0,\text{pre}}^{\nu_{0,\text{pre}}}, \quad (2.1)$$

where $\mathcal{CGC}_0 = I_0 - p_0^1 [(p_0^1)^H A_0 p_0^1]^{-1} (p_0^1)^H A_0$ denotes the coarse grid correction iteration matrix.

The names *intermediate* and *smoothing iteration* used above refer to the multi-iterative terminology [15]: we say that a method is multi-iterative if it is composed by at least two distinct iterations. The idea is that these basic components should have complementary spectral behaviors so that the whole procedure is quickly convergent (for details see [15] and Sections 7.2 and 7.3 in [19]). Notice that in the setting of Hermitian positive definite and uniformly bounded sequences, the subspace where A_0 is ill-conditioned corresponds to the subspace in which A_0 has small eigenvalues.

Starting from the TGM, the MGM can be introduced as follows: instead of solving directly the linear system with coefficient matrix A_1 , the projection strategy is recursively applied, so obtaining a *multigrid method*.

Let us use the Galerkin formulation and let $n_0 > n_1 > \dots > n_l > 0$, with l being the maximal number of recursive calls and with $N(n_s)$ being the corresponding matrix sizes.

The corresponding MGM generates the j^{th} iteration

$$x_0^{(j+1)} = \mathcal{MGM}(0, x_0^{(j)}, b_0, A_0, V_{0,\text{pre}}, \nu_{0,\text{pre}}, V_{0,\text{post}}, \nu_{0,\text{post}})$$

according to the following algorithm:

$$y_s := \mathcal{MGM}(s, x_s, b_s, A_s, V_{s,\text{pre}}, \nu_{s,\text{pre}}, V_{s,\text{post}}, \nu_{s,\text{post}})$$

if $s = l$ then

$$\text{Solve}(A_s y_s = b_s)$$

Exact solution

else

$$\tilde{x}_s := \text{Smooth}(x_s, b_s, V_{s,\text{pre}}, \nu_{s,\text{pre}})$$

Pre-smoothing iterations

$$r_s := b_s - A_s \tilde{x}_s$$

Coarse Grid Correction

$$r_{s+1} := (p_s^{s+1})^H r_s$$

$$y_{s+1} := \mathcal{MGM}(s+1, 0_{s+1}, b_{s+1}, A_{s+1}, V_{s+1,\text{pre}}, \nu_{s+1,\text{pre}}, V_{s+1,\text{post}}, \nu_{s+1,\text{post}})$$

$$\tilde{y}_s := \tilde{x}_s + p_s^{s+1} y_{s+1}$$

$$y_s := \text{Smooth}(\tilde{y}_s, b_s, V_{s,\text{post}}, \nu_{s,\text{post}})$$

Post-smoothing iterations

where the matrix $A_{s+1} := (p_s^{s+1})^H A_s p_s^{s+1}$ is more profitably computed in the so called *pre-computing phase*.

Since the MGM is again a linear fixed-point method, the j^{th} iteration $x_0^{(j+1)}$ can be expressed as $MGM_0 x_0^{(j)} + (I_0 - MGM_0) A_0^{-1} b_0$, where the iteration matrix MGM_0 is recursively defined according to the following rule (see [22]):

$$MGM_l = O,$$

$$MGM_s = V_{s,\text{post}}^{\nu_{s,\text{post}}} \left[I_s - p_s^{s+1} (I_{s+1} - MGM_{s+1}) A_{s+1}^{-1} (p_s^{s+1})^H A_s \right] V_{s,\text{pre}}^{\nu_{s,\text{pre}}}, \quad (2.2)$$

$$s = 0, \dots, l-1,$$

and with MGM_s and MGM_{s+1} denoting the iteration matrices of the multi-grid procedures at two subsequent levels.

At the last recursion level l , the linear system is solved by a direct method and hence it can be interpreted as an iterative method converging in a single step: this motivates the chosen initial condition $MGM_l = O$.

By comparing the TGM and MGM, we observe that the coarse grid correction operator CGC_s is replaced by an approximation, since the matrix A_{s+1}^{-1} is approximated by $(I_{s+1} - MGM_{s+1}) A_{s+1}^{-1}$ as implicitly described in (2.2) for $s = 0, \dots, l-1$. In this way step 4., at the highest level $s = 0$, represents an approximation of the exact solution of step 4. displayed in the TGM algorithm (for the matrix analog compare (2.2) and (2.1)). Finally, for $l = 1$ the MGM reduces to the TGM if $\text{Solve}(A_1 y_1 = b_1)$ is $y_1 = A_1^{-1} b_1$.

2.2 Some theoretical results on TGM convergence and optimality

In this paper we refer to the multigrid solution of special linear systems of the form

$$B_n x = b, \quad B_n \in \mathbb{C}^{N(n) \times N(n)}, \quad x, b \in \mathbb{C}^{N(n)} \quad (2.3)$$

with $\{B_n\}$ Hermitian positive definite uniformly bounded matrix sequence, n being a positive d -index, $d \geq 1$ and $N(\cdot)$ an increasing function with respect to it. More precisely, we assume that there exists $\{A_n\}$ Hermitian positive definite matrix sequence such that some order relation is linking $\{A_n\}$ and $\{B_n\}$, for n large enough and we suppose that an optimal algebraic multigrid method is available for the solution of the systems

$$A_n x = b, \quad A_n \in \mathbb{C}^{N(n) \times N(n)}, \quad x, b \in \mathbb{C}^{N(n)}. \quad (2.4)$$

The underlying idea is to apply for the systems (2.3) the some algebraic TGM and MGM considered for the systems (2.4), i.e., when considering the very same projectors. In fact, the quoted choice will give rise to a relevant simplification, since it is well-known that a very crucial role in MGM is played by the choice of projector operator.

In the algebraic multigrid theory some relevant convergence results are due to Ruge and Stüben [14], to which we referred in order to prove our convergence results.

Hereafter, by $\|\cdot\|_2$ we denote the Euclidean norm on \mathbb{C}^m and the associated induced matrix norm over $\mathbb{C}^{m \times m}$. If X is Hermitian positive definite, then its square root obtained via the Schur decomposition is well defined and positive definite. As a consequence we can set $\|\cdot\|_X = \|X^{1/2} \cdot\|_2$ the Euclidean norm weighted by X on \mathbb{C}^m , and the associated induced matrix norm. In addition, the notation $X \leq Y$, with X and Y Hermitian matrices, means that $Y - X$ is nonnegative definite. In addition the sequence $\{X_n\}$, with X_n Hermitian positive definite matrices, is a uniformly bounded matrix sequence if there exists $M > 0$ independent of n such that $\|X_n\|_2 \leq M$, for n large enough.

Theorem 2.1 [14] *Let A_0 be a Hermitian positive definite matrix of size $N(n_0)$, let $p_0^1 \in \mathbb{C}^{N(n_0) \times N(n_1)}$, $n_0 > n_1$, be a given full-rank matrix and let $V_{0,\text{post}}$ be the post-smoothing iteration matrix. Suppose that there exists $\alpha_{\text{post}} > 0$, independent of n_0 , such that for all $x \in \mathbb{C}^{N(n_0)}$*

$$\|V_{0,\text{post}} x\|_{A_0}^2 \leq \|x\|_{A_0}^2 - \alpha_{\text{post}} \|x\|_{A_0 D_0^{-1} A_0}^2, \quad (2.5)$$

where D_0 is the diagonal matrix formed by the diagonal entries of A_0 .

Assume, also, that there exists $\beta > 0$, independent of n_0 , such that for all $x \in \mathbb{C}^{N(n_0)}$

$$\min_{y \in \mathbb{C}^{N(n_1)}} \|x - p_0^1 y\|_{D_0}^2 \leq \beta \|x\|_{A_0}^2. \quad (2.6)$$

Then, $\beta \geq \alpha_{\text{post}}$ and $\|TGM_0\|_{A_0} \leq \sqrt{1 - \alpha_{\text{post}}/\beta} < 1$.

Notice that all the constants α_{post} and β are required to be independent of the actual dimension in order to ensure a TGM convergence rate independent of the size of the algebraic problem.

It is worth stressing that Theorem 2.1 still holds if the diagonal matrix D_0 is replaced by any Hermitian positive matrix X_0 (see e.g. [2]). Thus, $X_0 = I$ could be a proper choice for its simplicity.

Thus, by referring to the problem in 2.3 we can claim the following results.

Proposition 2.2 [18] *Let $\{A_n\}$ be a matrix sequence with A_n Hermitian positive definite matrices and let $p_0^1 \in \mathbb{C}^{N(n_0) \times N(n_1)}$ be a given full-rank matrix for any $n_0 > 0$ such that there exists $\beta_A > 0$ independent of n_0 so that for all $x \in \mathbb{C}^{N(n_0)}$*

$$\min_{y \in \mathbb{C}^{N(n_1)}} \|x - p_0^1 y\|_2^2 \leq \beta_A \|x\|_{A_0}^2. \quad (2.7)$$

Let $\{B_n\}$ be another matrix sequence, with B_n Hermitian positive definite matrices, such that $A_n \leq \vartheta B_n$, for n large enough, with $\vartheta > 0$ absolute constant. Then, for all $x \in \mathbb{C}^{N(n_0)}$ and n_0 large enough, it also holds $\beta_B = \beta_A \vartheta$ and

$$\min_{y \in \mathbb{C}^{N(n_1)}} \|x - p_0^1 y\|_2^2 \leq \beta_B \|x\|_{B_0}^2. \quad (2.8)$$

Therefore, the convergence result in Theorem 2.1 holds true also for the matrix sequence $\{B_n\}$, if the validity of condition (2.5) it is also guaranteed. It is worth stressing that in the case of Richardson smoothers such topic is not related to any partial ordering relation connecting the Hermitian matrix sequences $\{A_n\}$ and $\{B_n\}$, i.e. inequalities (2.5), and the corresponding for the pre-smoother case, with $\{B_n\}$ instead of $\{A_n\}$, have to be proved independently.

Proposition 2.3 [18] *Let $\{B_n\}$ be an uniformly bounded matrix sequence, with B_n Hermitian positive definite matrices. For any $n_0 > 0$, let $V_{n,\text{pre}} = I_n - \omega_{\text{pre}} B_n$, $V_{n,\text{post}} = I_n - \omega_{\text{post}} B_n$ be the pre-smoothing and post-smoothing iteration matrices, respectively considered in the TGM algorithm. Then, there exist $\alpha_{B,\text{pre}}, \alpha_{B,\text{post}} > 0$ independent of n_0 such that for all $x \in \mathbb{C}^{N(n_0)}$*

$$\|V_{0,\text{pre}} x\|_{B_0}^2 \leq \|x\|_{B_0}^2 - \alpha_{B,\text{pre}} \|V_{0,\text{pre}} x\|_{B_0^2}^2, \quad (2.9)$$

$$\|V_{0,\text{post}} x\|_{B_0}^2 \leq \|x\|_{B_0}^2 - \alpha_{B,\text{post}} \|x\|_{B_0^2}^2. \quad (2.10)$$

See Proposition 3 in [1] for the analogous claim in the case of $\nu_{\text{pre}}, \nu_{\text{post}} > 0$.

In this way, according to the Ruge and Stüben algebraic theory, we have proved the TGM optimality, that is its convergence rate independent of the size $N(n)$ of the involved algebraic problem.

Theorem 2.4 [18] *Let $\{B_n\}$ be an uniformly bounded matrix sequence, with B_n Hermitian positive definite matrices. Under the same assumptions of Propo-*

sitions 2.2 and 2.3 the TGM with only one step of post-smoothing converges to the solution of $B_n x = b$ and its convergence rate is independent of $N(n)$.

Clearly, as just discussed in [18], the TGM iteration with both pre-smoothing and post-smoothing is never worse than the TGM iteration with only post-smoothing. Therefore Theorem 2.4 implies that the TGM with both post-smoothing and pre-smoothing has a convergence rate independent of the dimension for systems with matrices B_n under the same assumptions as in Theorem 2.4.

Furthermore, the same issues as before, but in connection with the MGM, deserve to be discussed. First of all, we expect that a more severe assumption between $\{A_n\}$ and $\{B_n\}$ has to be fulfilled in order to infer the MGM optimality for $\{B_n\}$ starting from the MGM optimality for $\{A_n\}$. The reason is that the TGM is just a special instance of the MGM when setting $l = 1$.

In the TGM setting we have assumed a one side ordering relation: here the most natural step is to consider a two side ordering relation, that is to assume that there exist positive constants ϑ_1, ϑ_2 independent of n such that $\vartheta_1 B_n \leq A_n \leq \vartheta_2 B_n$, for every n large enough. The above relationships simply represent the spectral equivalence condition for sequences of Hermitian positive definite matrices, which is plainly fulfilled in our setting whenever the weight function is positive, well separated from zero, and bounded.

In the context of the preconditioned conjugate gradient method (see [3]), it is well known that if $\{P_n\}$ is a given sequence of optimal (i.e., spectrally equivalent) preconditioners for $\{A_n\}$, then $\{P_n\}$ is also a sequence of optimal preconditioners for $\{B_n\}$ (see e.g. [13]). The latter fact just follows from the observation that the spectral equivalence is an *equivalence* relation and hence is transitive.

In summary, we have enough heuristic motivations in order to conjecture that the spectral equivalence is the correct, sufficient assumption and, in reality, the numerical experiments reported in Section 3 give a support to the latter statement. Refer to [18] for some further remark about this topic.

3 Numerical Examples

Hereafter, the aim relies in testing our TGM and MGM (standard V-cycle according to Section 2) applied to standard FD approximations to

$$-\nabla(a(x)\nabla u(x)) = f(x), \quad x \in \Omega = (0, 1)^d, \quad d \geq 1, \quad (3.1)$$

with assigned BCs and for several choices examples of the diffusion coefficient $a(x) \geq a_0 > 0$.

The projectors are properly chosen according to the nature of structured part, that depends on the imposed BCs. For instance, in the case of Dirichlet BCs

we split the arising FD matrix $A_n(a)$ as

$$A_n(a) = a_{\min}\tau_n(A_n(1)) + R_n(a), \quad R_n(a) = A_n(a) - a_{\min}\tau_n(A_n(1)),$$

where $\tau_n(A_n(1))$ denotes the FD matrix belonging to the τ (or DST-I) algebra [5] obtained in the case of $a(x) \equiv 1$ and a_{\min} equals the minimum of $a(x)$ on $\bar{\Omega}$ in order to guarantee the positivity of $R_n(a)$.

On the other hand, we will use, in general as first choice, the Richardson smoothing/intermediate iteration step twice in each iteration, before and after the coarse grid correction, with different values of the parameter ω . In some cases better results are obtained by considering the Gauss-Seidel method for the pre-smoothing iteration.

According to the algorithm in Section 2, when considering the TGM, the exact solution of the system is obtained by using a direct solver in the immediately subsequent coarse grid dimension, while, when considering the MGM, the exact solution of the system is computed by the same direct solver, when the coarse grid dimension equals 15^d (where $d = 1$ for the one-level case and $d = 2$ for the two-level case).

In all tables we report the numbers of iterations required for the TGM or MGM convergence, assumed to be reached when the Euclidean norm of the relative residual becomes less than 10^{-7} . We point out that the CPU times are consistent with the iteration counts.

Finally, we stress that at every level (except for the coarsest) the structured matrix parts are never formed since we need only to store the nonzero Fourier coefficients of the generating function at every level for matrix-vector multiplications. Thus, besides the $O(N(n))$ operations complexity of the proposed MGM both with respect to the structured part and clearly with respect to the non-structured one, the memory requirements of the structured part are also very low since there are only $O(1)$ nonzero Fourier coefficients of the generating function at every level. On the other hand, the projections of the initial matrix correction $R_n(a)$ are stored at each level according to standard sparse matrix techniques during the pre-computing phase.

3.1 Dirichlet BCs

We begin by considering the FD approximation of (3.1) with Dirichlet BCs in the one-level setting. As already outlined, in this case the arising matrix sequence $\{A_n(a)\}$ can be split as

$$A_n(a) = a_{\min}\tau_n(A_n(1)) + R_n(a), \quad R_n(a) = A_n(a) - a_{\min}\tau_n(A_n(1)),$$

where $\tau_n(A_n(1))$ and a_{\min} are defined as before. More precisely, $\{\tau_n(A_n(1))\}$ is the τ /Toeplitz matrix sequence generated by the function $f(t) = 2 - 2\cos(t)$, $t \in (0, 2\pi]$ and a_{\min} equals the minimum of $a(x)$ on $\bar{\Omega}$.

Let us consider $A_0(a) \in \mathbb{R}^{n_0 \times n_0}$, with 1-index $n_0 > 0$ (according to the notation introduced in Section 2, we refer to any multi-index n_s by means of its subscript s). Following [8,16], we denote by $T_0^1 \in \mathbb{R}^{n_0 \times n_1}$, $n_0 = 2n_1 + 1$, the operator such that

$$(T_0^1)_{i,j} = \begin{cases} 1 & \text{for } i = 2j, \quad j = 1, \dots, n_1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

and we define a projector $(p_0^1)^H$, $p_0^1 \in \mathbb{R}^{n_0 \times n_1}$ as

$$p_0^1 = \frac{1}{\sqrt{2}} P_0 T_0^1, \quad P_0 = \text{tridiag}_0 [1, 2, 1] = \tau_0(\tilde{f}), \quad \tilde{f}(t) = 2 + 2 \cos(t). \quad (3.3)$$

On the other hand, for the smoothing/intermediate Richardson iterations, the parameters ω are chosen as

$$\omega_{\text{pre}} = 2/(\|f\|_\infty + \|R_n(a)\|_\infty)$$

$$\omega_{\text{post}} = 1/(\|f\|_\infty + \|R_n(a)\|_\infty),$$

and we set $\nu_{\text{pre}} = \nu_{\text{post}} = 1$.

The first set of numerical tests refer to the following settings: $a(x) \equiv 1$, $a(x) = e^x$, $a(x) = e^x + 1$, (denoted in short as a1, a2, a3 respectively).

In Table 1 we report the numbers of iterations required for the TGM convergence, both in the case of the Richardson pair, and of the Richardson + Gauss-Seidel pair. All these results confirm the optimality of the proposed TGM in the sense that the number of iterations is uniformly bounded by a constant not depending on the size $N(n)$ indicated in the first column.

In Table 2 we report the some results, but with respect to the V-cycle application. The numerics seems allow to claim the optimality convergence property can be extended to the MGM.

It is worth stressing that the difference in considering Richardson or Gauss-Seidel in the pre-smoothing iterations is quite negligible in the MGM case.

In Table 3 we report a deeper analysis of the TGM superlinear behavior in the a2 setting. More precisely, we consider the test functions $a(x) = e^x + 10^k$ with k ranging from 0 to 6. The convergence behavior is unaltered in the case of the Richardson + Gauss-Seidel pair, while for increasing k we observe that the number of required iterations by considering the Richardson+Richardson pair progressively approaches the reference a1 case. In fact as $k \rightarrow \infty$ the function $a(x)$ after a proper scaling converges to the constant 1.

The projector definition plainly extends to the two-level setting by using tensor arguments: $(p_0^1)^H$ is constructed in such a way that

$$p_0^1 = P_0 U_0^1 \quad (3.4)$$

$$P_0 = \text{tridiag}_{n_0^{(1)}} [1, 2, 1] \otimes \text{tridiag}_{n_0^{(2)}} [1, 2, 1], \quad (3.5)$$

$$U_0^1 = T_0^1(n_0^{(1)}) \otimes T_0^1(n_0^{(2)}) \quad (3.6)$$

Table 1

Number of iterations required by TGM - one-level case with Dirichlet BCs

Richardson+Richardson				Richardson+Gauss-Seidel			
$N(n)$	a1	a2	a3	$N(n)$	a1	a2	a3
31	2	8	5	31	8	8	8
63	2	6	4	63	8	8	8
127	2	5	4	127	8	8	8
255	2	4	4	255	8	8	8
511	2	4	3	511	8	8	8

Table 2

Number of iterations required by MGM - one-level case with Dirichlet BCs

Richardson+Richardson				Richardson+Gauss-Seidel			
$N(n)$	a1	a2	a3	$N(n)$	a1	a2	a3
15	1	1	1	15	1	1	1
31	2	8	5	31	8	8	8
63	7	7	7	63	9	9	9
127	8	8	8	127	9	9	9
255	8	8	8	255	9	9	9
511	8	8	8	511	9	9	9

Table 3

Number of iterations required by TGM - one-level case with Dirichlet BCs

Richardson+Richardson								Richardson+Gauss-Seidel							
		$a(x) = e^x + 10^k$								$a(x) = e^x + 10^k$					
		k								k					
$N(n)$	a1	0	1	2	3	4	5	$N(n)$	a1	0	1	2	3	4	5
31	2	5	4	3	3	3	2	31	8	8	8	8	8	8	8
63	2	4	4	3	3	3	2	63	8	8	8	8	8	8	8
127	2	4	4	3	3	3	2	127	8	8	8	8	8	8	8
255	2	4	3	3	3	3	2	255	8	8	8	8	8	8	8
511	2	3	3	3	3	2	2	511	8	8	8	8	8	8	8

with $n_0^{(r)} = 2n_1^{(r)} + 1$ and where $T_0^1(n_0^{(r)}) \in \mathbb{R}^{n_0^{(r)} \times n_1^{(r)}}$ is the one-level matrix given in (3.2).

The quoted choice represents the most trivial extension of the one-level projector to the two-level setting and is also the less expensive from a computational point of view: in fact, $p_0^1 = \tau_0((2 + 2 \cos(t_1))(2 + 2 \cos(t_2)))U_0^1$ equals $[\tau_{n_0^{(1)}}(p(2 + 2 \cos(t_1)))T_0^1(n_0^{(1)})] \otimes [\tau_{n_0^{(2)}}(p(2 + 2 \cos(t_2)))T_0^1(n_0^{(2)})]$.

Tables 4 and 5 report the number of iterations with the same notation as before and where we are considering the following function tests: $a(x) \equiv 1$, $a(x) = e^{x_1+x_2}$, $a(x) = e^{x_1+x_2} + 2$, (denoted in short as a1, a2, a3, respectively). Though the convergence behavior in the case of the Richardson+Richardson pair is quite slow, we can observe that the number of MGM iterations required to achieve the convergence is essentially the same as in the TGM. This phenomenon is probably due to some inefficiency in considering the approximation $\|R_n(a)\|_\infty$ in the tuning of the parameter ω_{pre} and ω_{post} . In fact, it

Table 4

Number of iterations required by TGM - two-level case with Dirichlet BCs

Richardson+Richardson				Richardson+Gauss-Seidel			
$N(n)$	a1	a2	a3	$N(n)$	a1	a2	a3
31^2	16	73	38	31^2	13	14	14
63^2	16	82	41	63^2	13	15	14
127^2	16	86	43	127^2	13	15	14
255^2	16	89	44	255^2	13	15	14

Table 5

Number of iterations required by MGM - two-level case with Dirichlet BCs

Richardson+Richardson				Richardson+Gauss-Seidel			
$N(n)$	a1	a2	a3	$N(n)$	a1	a2	a3
15^2	1	1	1	15^2	1	1	1
31^2	16	73	38	31^2	13	14	14
63^2	16	83	42	63^2	13	15	15
127^2	16	88	43	127^2	13	15	15
255^2	16	90	44	255^2	13	15	15

is enough to substitute, for instance, the pre-smoother with the Gauss-Seidel method in order to preserve the optimality both in the TGM and the MGM case.

Finally, in Table 6, we report the number of iterations required by MGM, in the case of some other test functions. More precisely, we are considering the \mathcal{C}^1 function $a(x, y) = e^{x+|y-1/2|^{3/2}}$, the \mathcal{C}^0 function $a(x, y) = e^{x+|y-1/2|}$, and the piecewise constant function $a(x, y) = 1$ if $x, y < 1/2$, δ otherwise, with $\delta = 10, 100, 1000$ (denoted in short as a4, a5, a6, a7, and a8, respectively). Taking into account the previous remarks, our smoothing choice is represented by the Richardson+Gauss-Seidel pair. Moreover, the CG choice is also investigated, both in connection to the Richardson or the Gauss-Seidel smoother.

The MGM optimality is again observed, according to a proper choice of the smoother pair.

In conclusion for keeping a proper optimal convergence, we can claim that Gauss-Seidel is necessary and the best pair is with conjugate gradient. The explanation of this behavior is again possible in terms of multi-iterative procedures and spectral complementarity: in fact while Richardson is effective essentially only in the high frequencies space, both Gauss-Seidel and CG are able to reduce the error also in the middle frequencies and in addition they are robust with respect to the scaling produced by the weight function a .

3.2 Periodic and Reflective BCs

Hereafter, we briefly address the case of periodic or reflective BCs. In particular we focus on the structured part of the splitting related to the FD

Table 6

Number of iterations required by MGM - two-level case with Dirichlet BCs (\dagger = more than $N(n)$ iterations required for convergence)

Richardson+Gauss-Seidel					
$N(n)$	a4	a5	a6	a7	a8
15^2	1	1	1	1	1
31^2	14	14	13	13	13
63^2	15	15	13	13	13
127^2	15	15	14	14	14
255^2	15	15	14	14	14

Richardson+CG					
$N(n)$	a4	a5	a6	a7	a8
15^2	1	1	1	1	1
31^2	21	24	46	1472	\dagger
63^2	26	28	59	1990	\dagger
127^2	26	30	64	1783	\dagger
255^2	27	31	60	1973	\dagger

Gauss-Seidel+CG					
$N(n)$	a4	a5	a6	a7	a8
15^2	1	1	1	1	1
31^2	12	12	11	10	10
63^2	12	12	11	10	10
127^2	12	12	11	10	10
255^2	12	12	11	10	10

discretization with respect to $a(x) \equiv 1$, since our multigrid strategy is tuned just with respect to it.

In the case of periodic BCs the obtained matrix sequence is the one-level circulant matrix sequence $\{S_n(f)\}$ generated by the function $f(t) = 2 - 2 \cos(t)$, $t \in (0, 2\pi]$. Following [19], we consider the operator $T_0^1 \in \mathbb{R}^{n_0 \times n_1}$, $n_0 = 2n_1$, such that

$$(T_0^1)_{i,j} = \begin{cases} 1 & \text{for } i = 2j - 1, \quad j = 1, \dots, n_1, \\ 0 & \text{otherwise,} \end{cases}$$

and we define a projector $(p_0^1)^H$, $p_0^1 \in \mathbb{R}^{n_0 \times n_1}$, as $p_0^1 = P_0 T_0^1$, $P_0 = S_0(p)$, $p(t) = 2 + 2 \cos(t)$. Clearly, the arising matrices are singular, so that we consider, for instance, the classical Strang correction [23]

$$\tilde{S}_{n_0}(f) = S_{n_0}(f) + f \left(\frac{2\pi}{N(n_0)} \right) \frac{ee^t}{N(n_0)},$$

where e is the vector of all ones.

By using tensor arguments, our approach plainly extend to the two-level setting.

When dealing with reflective BCs, the obtained matrix sequence is the one-

level DCT III matrix sequence $C_n(f)_n$ generated by the function $f(t) = 2 - 2 \cos(t)$, $t \in (0, 2\pi]$. Following [7], we consider the operator $T_0^1 \in \mathbb{R}^{n_0 \times n_1}$, $n_0 = 2n_1$, such that

$$(T_0^1)_{i,j} = \begin{cases} 1 & \text{for } i \in \{2j - 1, 2j\}, \quad j = 1, \dots, n_1, \\ 0 & \text{otherwise,} \end{cases}$$

and we define a projector $(p_0^1)^H$, $p_0^1 \in \mathbb{R}^{n_0 \times n_1}$, as $p_0^1 = P_0 T_0^1$, $P_0 = C_0(p)$, $p(t) = 2 + 2 \cos(t)$. Clearly, due to the singularity, we consider, for instance,

$$\tilde{C}_{n_0}(f) = C_{n_0}(f) + f \left(\frac{\pi}{N(n_0)} \right) \frac{ee^t}{N(n_0)}.$$

Again, the two-level setting is treated by using tensor arguments.

The numerical tests performed in the case of periodic or reflective BCs have the same flavor as those previously reported in the case of Dirichlet BCs and hence we do not report them since the observed numerical behavior gives the same information as in the case of Dirichlet BCs.

4 Concluding Remarks

We have presented a wide numerical experimentation concerning a multigrid technique for the discrete weighted Laplacian with various BCs. In accordance with the theoretical study in [18], the choice of the smoothers can be done taking into account the spectral complementarity, typical of any multi-iterative procedure. In particular, we have noticed that when the weight function a adds further difficulties in the middle frequencies (e.g., when a is discontinuous), the use of pure smoothers like Richardson, reducing the error only the high frequencies, is not sufficient. Conversely, both CG and Gauss-Seidel work also reasonably well in the middle frequencies (what is called the intermediate space in a multi-iterative method) and in fact, in some cases, their use is mandatory if we want to keep the optimality of the method, i.e., a convergence within a given accuracy and within a number of iterations not depending on the size of the considered algebraic problem.

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