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# A SELF-REGULATING AND PATCH SUBDIVIDED POPULATION

LAMIA BELHADJI, DANIELA BERTACCHI, AND FABIO ZUCCA

**ABSTRACT.** We consider an interacting particle system on a graph which, from a macroscopic point of view, looks like  $\mathbb{Z}^d$  and, at a microscopic level, is a complete graph of degree  $N$  (called a patch). There are two birth rates: an inter-patch one  $\lambda$  and an intra-patch one  $\phi$ . Once a site is occupied, there is no breeding from outside the patch and the probability  $c(i)$  of success of an intra-patch breeding decreases with the size  $i$  of the population in the site. We prove the existence of a critical value  $\lambda_{cr}(\phi, c, N)$  and a critical value  $\phi_{cr}(\lambda, c, N)$ . We consider a sequence of processes generated by the families of control functions  $\{c_i\}_{i \in \mathbb{N}}$  and degrees  $\{N_i\}_{i \in \mathbb{N}}$ ; we prove, under mild assumptions, the existence of a critical value  $i_{cr}(\lambda, \phi, c)$ . Roughly speaking we show that, in the limit, these processes behave as the branching random walk on  $\mathbb{Z}^d$  with intra-neighbor birth rate  $\lambda$  and on-site birth rate  $\phi$ . Some examples of models that can be seen as particular cases are given.

**Keywords:** contact process, restrained branching random walk, epidemic model, phase transition, critical parameters.

**AMS subject classification:** 60K35.

## 1. INTRODUCTION

Stochastic models for the demographic expansion (or contraction) of a biological population have attracted the attention of many researchers since Galton and Watson introduced the branching process as a very simple and non-spatial model to study the survival of surnames. After the appearance of interacting particle systems and of the paper of Harris [10] on the contact process it has become increasingly clear that the spatial structure of the environment and the interaction between individuals bring a desirable complexity in the models one can define (see for instance [9] for a discussion on spatial models).

We want to introduce and study a model for a biological population where individuals breed and die living on a patchy habitat. Clearly, if we substitute the words *infect* and *recover* for *breed* and *die*, our model also serves as a picture for the spreading of an infectious disease.

The main idea is that the environment is subdivided in patches and the patches are centered at the vertices of the  $d$ -dimensional lattice  $\mathbb{Z}^d$  (see Figure 1, where each square is a patch and  $d = 2$ ). Individuals in the same patch are strongly connected to each other (indeed the patch may be thought as a complete graph with an intra-patch breeding rate  $\phi$ ) and are also related to individuals in neighboring patches (the inter-patch breeding rate being  $\lambda$ ).

Each patch has the same fixed number of sites and a site accepts individuals from neighboring patches if it is empty. This kind of biparametric interaction is similar to the one introduced in [11] and later modified in [3]. In both papers the population is divided in equally sized clusters, one cluster per vertex of  $\mathbb{Z}^d$ ; in [11] individuals in the same clusters die at the same time, while

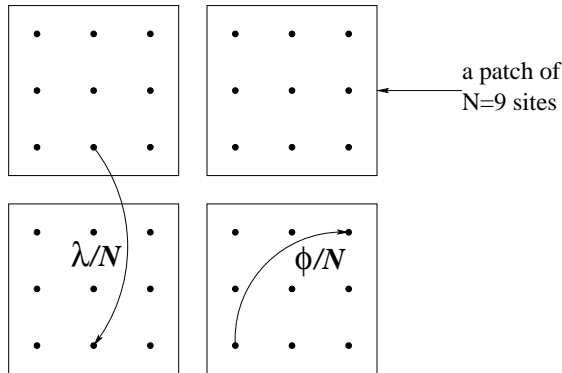


FIGURE 1. The habitat.

in [3] they die one at the time. In their model the cluster has no inner structure, that is, the only important thing is whether it is empty or not (if not, immigration is not allowed). In our model instead of clusters, we have patches and the process depends not only on the total number of particles in the patch, but also on their position among the sites within the patch. For instance let there be  $n$  particles in the patch. If they are on the same site, then all sites in the patch but this accept immigration (that is, reproductions from neighboring patches) while intra-patch reproductions are, a priori, always allowed. If they are scattered in  $n$  different sites all this sites will block immigrants.

To strengthen the dependence of the process on the number of individuals per site, we admit a self-regulating mechanism, namely there exists a control function  $c \in [0, 1]^{\mathbb{N}}$  such that if a site is already occupied by  $j$  individuals, newborns are accepted only with probability  $c(j)$ . This kind of regulating mechanism has been introduced in [5] as a general model of which the branching random walk and the contact process are particular cases (when  $c \equiv 1$  and  $c = \delta_0$  respectively). Note that the presence of this function  $c$  mimics the fact that when an ecosystem is already exploited by many individuals, the younglings may not have enough food to live and to reach sexual maturity, thus many of them will not contribute to the breeding process. On the other hand many animals feed on their younglings if food is scarce.

We observe that the idea of a patchy environment has already been introduced in [4]. The main difference is that in [4] the inter-patch interaction is allowed only between the centers of neighboring patches, while in our model each site may receive immigrants from the neighboring patches.

In the present work we mainly study the effect of the parameters on the probability of survival of the entire population. In particular we proved if  $\lambda$  and  $\phi$  are both small then there is almost sure extinction (Theorem 3.1). This result is in agreement with the conclusion of the mean field model in a particular case (see Subsection 2.4). If  $\phi$  is small then survival is guaranteed only by  $\lambda$  sufficiently large; while  $\phi$  large and a “mild” control function  $c$  implies survival even when  $\lambda = 0$  (the precise statements are Theorem 3.2 and Corollary 3.3 where we give explicit conditions on  $\phi$

and  $c$ ). Analogously a large  $\lambda$  guarantees survival even when  $\phi = 0$ , while for small  $\lambda$  a large  $\phi$  (and a suitable  $c$ ) is needed to ensure survival (see Theorem 3.4).

To investigate the impact of different environment structures on the process we let the patch dimension go to infinity. We obtain Theorem 3.5(1) which agrees with the intuition that the limiting process is a branching random walk with intra-site breeding rate  $\phi$  and inter-site rate  $\lambda$ . The reader may want to compare this convergence to the one studied in [8], where a long range contact process is proved to converge to a branching random walk. We also study the effect of a progressive removing of the control function: we obtain survival when  $\phi > 1$  and  $c$  is close to 1 (Theorem 3.5(2)). Finally we study the asymptotic behavior of the critical parameter of the general process and of some particular cases when the size of the patch goes to infinity (Proposition 3.7 and Corollary 3.9) or when the control function is progressively removed (Proposition 3.8).

Here is a brief outline of the paper. In Section 2 we introduce our process and discuss some models that can be seen as particular cases. Subsection 2.4 is devoted to the study of the mean field model. In Section 3 we discuss our main results; the proofs can be found in Section 4. Section 5 is devoted to the discussion of open questions.

## 2. THE MODEL

The patches of the habitat are complete graphs  $\mathbb{K}_N$  with  $N$  vertices (that is any vertex is a neighbor of any other vertex, including itself). In each single patch the sites are enumerated from 1 to  $N$  and in each site there might be a priori any number of particles. The patches are located at the vertices of the lattice  $\mathbb{Z}^d$ . Formally, the state space is  $\mathbb{N}^{\mathcal{E}_N^d}$  where the environment is  $\mathcal{E}_N^d := \mathbb{Z}^d \times \mathbb{K}_N$  and the graph structure of  $\mathcal{E}_N^d$  is as follows:  $(x, r) \sim (y, r_1)$  if and only if  $|x - y|_{\mathbb{Z}^d} \leq 1$  (where  $|\cdot|_{\mathbb{Z}^d}$  denotes the usual distance in the graph  $\mathbb{Z}^d$ ). On the other hand, if  $x, y \in \mathbb{Z}^d$  then we write  $x \sim y$  meaning that  $|x - y|_{\mathbb{Z}^d} = 1$ . With this notation, the *patch* at  $x \in \mathbb{Z}^d$  is the set  $\{(x, r) : r \in \mathbb{K}_N\}$ .

The dynamics of the process is the following. Each particle dies at rate 1; when a site is vacant it receives births from each particle in the same patch at rate  $\phi/N$  and from each particle in the neighboring patches at rate  $\lambda/N$ . When a site  $x$  is occupied by  $i \geq 1$  particles, then it allows birth attempts only from particles in the same patch, but an attempt succeeds only with probability  $c(i)$  (the function  $c : \mathbb{N} \rightarrow [0, 1]$  represents a self-regulating mechanism as in [5]). We suppose that  $c(0) = 1$  (when a site is vacant the first reproduction trial is almost surely successful) and that  $c(j) \geq c(j + 1)$  for all  $j \geq 0$  (the more individuals are present, the less likely it is to produce new offspring). Clearly, if  $c(j) = 0$  for all  $j > i_0$  and  $c(i_0) > 0$ , then in each site there might be at most  $i_0 + 1$  particles.

An equivalent description of the breeding mechanism is that each particle breeds at rate  $\phi$  inside its patch and at rate  $\lambda$  towards each neighboring patch. Newborns choose the target sites inside the destination patch at random. The reproduction from a different patch is successful only if the target site is vacant, while inside the patch the reproduction is regulated by the function  $c$ .

Formally, the transition rates at  $(x, r)$  for the process  $\eta_t$  are

$$\begin{aligned}
0 \rightarrow 1 & \text{ at rate } \frac{\lambda}{N} \sum_{(z, r_1): (z, r_1) \sim (x, r)} \eta(z, r_1) + \frac{\phi}{N} \sum_{r_1 \in \mathbb{K}_N} \eta(x, r_1); \\
i \rightarrow i + 1 & \text{ at rate } \frac{\phi}{N} c(\eta(x, r)) \sum_{r_1 \in \mathbb{K}_N} \eta(x, r_1), \text{ for } 1 \leq i; \\
i \rightarrow i - 1 & \text{ at rate } i.
\end{aligned} \tag{2.1}$$

In the sequel we refer to the process with these rates as the *Patchy Restrained Process* or briefly PRP. When we need to stress the dependence on the parameters, we write  $\text{PRP}(\lambda, \phi, c, N)$ . We observe that many explicit models are recovered as particular cases of this patchy habitat model.

In particular, by choosing  $\phi = 0$  we obtain a process which does not depend on  $c$  and allows at most one particle per site. We call this process the *contact process* on  $\mathcal{E}_N^d$ , briefly CP. It is not difficult to prove that this process has a critical parameter  $\lambda_{CP}(\mathcal{E}_N^d) \in [\frac{1}{2d}, \lambda_{CP}(\mathbb{Z}^d)]$  (namely the population dies out for  $\lambda$  below it and survives with positive probability for  $\lambda$  above). Indeed  $\lambda_{CP}(\mathcal{E}_N^d) \leq \lambda_{CP}(\mathcal{E}_1^d) = \lambda_{CP}(\mathbb{Z}^d)$  since  $\lambda_{CP}(\mathcal{E}_N^d)$  is nonincreasing with respect to  $N$  and it is possible to prove that  $\lambda_{CP}(\mathcal{E}_N^d) \downarrow 1/2d$  as  $N \rightarrow \infty$  (see Proposition 3.7(1)).

On the other hand, if we choose  $c = \delta_0$ , we obtain a process that we call *biparametric contact process* (or  $\text{BCP}(\lambda, \phi)$ ) on  $\mathcal{E}_N^d$  which coincides with the contact process when  $\phi = 0$  (this is in general a process which allows at most one particle per site and has two reproduction rates – the inter-patch  $\lambda$  and the intra-patch  $\phi$ ).

The *individual recovery process* (briefly IRP) introduced in [3] can be seen as a particular case of the PRP, namely by setting  $N = 1$  and  $c = \mathbb{1}_{\{0, \dots, \kappa-1\}}$ .

It is natural to study sequences of PRPs (see Subsection 2.4 and Section 3) by considering a sequence  $\{N_i\}_{i \in \mathbb{N}}$  and a corresponding sequence of controlling functions  $\{c_i\}_{i \in \mathbb{N}}$  (we keep  $\lambda$  and  $\phi$  fixed). We are interested in the behavior of the process as  $i \rightarrow \infty$ . We assume that  $c_i \leq c_{i+1}$  (the regulation gets weaker as  $i$  grows). Note that  $c_i(j)$  may be positive for all  $j$ , thus there could be no *a priori* bounds on the site carrying capacity. We denote by  $c_\infty(i) := \lim_{n \rightarrow \infty} c_n(i)$  for all  $i$ .

We conclude this section by listing some particular cases of the PRP which are modifications of the IRP.

**2.1. Logistic IRP.** If we choose  $N = 1$  and  $c(i) = \max\{0, 1 - i/\kappa\}$  (for some  $\kappa \in \mathbb{N} \setminus \{0\}$ ) then we get a cluster of size  $\kappa$  at each site in  $\mathbb{Z}^d$  and an interaction between sites which is allowed only when the target is empty. This kind of interaction was introduced by Schinazi [11] with a cluster recovery clearing mechanism (or mass extinction), and modified by Belhadji and Lanchier with individual recoveries in [3]. Note nevertheless that this model differs from the IRP in the fact that the breeding inside the cluster becomes increasingly difficult as we approach the full carrying

capacity  $\kappa$ . For this process  $\eta_t$ , that we call the *logistic* IRP, the transition rates at site  $x$  are:

$$\begin{aligned}
0 \rightarrow 1 & \text{ at rate } \lambda \sum_{z \sim x} \eta(z); \\
i \rightarrow i+1 & \text{ at rate } i\phi \left(1 - \frac{i}{\kappa}\right), \text{ for } 1 \leq i \leq \kappa - 1; \\
i \rightarrow i-1 & \text{ at rate } i, \text{ for } i \geq 1.
\end{aligned} \tag{2.2}$$

Clearly the presence of the logistic factor  $1 - i/\kappa$  in the breeding rate may be interpreted as a self-regulating mechanism of the population which slows down the reproductions when the patch is almost completely exploited.

**2.2. Self-regulating IRP.** If  $N = 1$  in general we get a process  $\eta_t$  that we call *self-regulating* IRP. The process has the following rates at site  $x$ :

$$\begin{aligned}
0 \rightarrow 1 & \text{ at rate } \lambda \sum_{z \sim x} \eta(z); \\
i \rightarrow i+1 & \text{ at rate } i\phi c(i); \\
i \rightarrow i-1 & \text{ at rate } i.
\end{aligned} \tag{2.3}$$

**2.3. Logistic IRP with persistent inter-patches reproduction.** Given a general PRP, we may identify all the sites in the same patch, meaning that we consider the process  $\xi_t(x) = \sum_{r \in \mathbb{K}_N} \eta_t(x, r)$ . We refer to  $\xi_t$  as the projection on  $\mathbb{Z}^d$  of  $\eta_t$ . The projection is a Markov process only when  $c \equiv \delta_0$ . In this case it has the following transition rates at site  $x$ :

$$\begin{aligned}
i \rightarrow i+1 & \text{ at rate } \left(i\phi + \lambda \sum_{z \sim x} \xi(z)\right) \left(1 - \frac{i}{N}\right), \text{ for } 0 \leq i \leq N - 1; \\
i \rightarrow i-1 & \text{ at rate } i, \text{ for } i \geq 1.
\end{aligned}$$

We note that this process is similar to the IRP of [3] with cluster size equal to  $N$ . Nevertheless it has two main differences: the interaction between clusters is always active (while in the IRP it vanishes once the cluster is nonempty) and it gets more difficult to increase the number of individuals in the cluster if the cluster is crowded.

**2.4. The mean field equations for the self-regulating IRP.** The usual approach, before actually studying the spatial stochastic model, is to derive a non-spatial deterministic version called the mean field. We consider the differential evolution equations for the concentrations  $\{u_i\}_{i \geq 0}$  where  $u_i$  represents the fraction of sites having  $i$  individuals (clearly  $\sum_i u_i = 1$ ). We compute the stationary solutions of the system of equations: a stationary solution with  $u_0 < 1$  corresponds to survival, while  $u_0 \geq 1$  suggests almost sure extinction. To avoid unnecessary complications, in this preliminary study, we take  $N = 1$ , that is, there is no difference between patches and sites.

Let us discuss briefly the mean field equation for the logistic IRP. The mean field equations are

$$\begin{cases} u'_0 = u_1 - \lambda u_0 \sum_{i=1}^{\kappa} i u_i \\ u'_1 = 2u_2 + \lambda u_0 \sum_{i=1}^{\kappa} i u_i - u_1(1 + \phi(1 - \frac{1}{\kappa})), \\ u'_i = (i+1)u_{i+1} + (i-1)(1 - \frac{i-1}{\kappa})\phi u_{i-1} - i(1 + (1 - \frac{i}{\kappa})\phi)u_i, \quad 1 \leq i \leq \kappa - 1, \\ u'_\kappa = -\kappa u_\kappa + (\kappa - 1)(1 - \frac{\kappa-1}{\kappa})\phi u_{\kappa-1}. \end{cases}$$

Put  $u'_i = 0$ , for  $i = 1, \dots, \kappa$  and sum equations from  $i$  to  $\kappa$ . Then the solutions, for  $i = 1, \dots, \kappa$ , is

$$u_i = \frac{(\kappa - 1)!}{(\kappa - i)! i} \left(\frac{\phi}{\kappa}\right)^{i-1} u_1.$$

We plug it into the first equation and require  $u'_0 = 0$  to get

$$\begin{aligned} u_0 &= \frac{1}{\lambda \sum_{i=0}^{\kappa-1} \frac{(\kappa-1)!}{(\kappa-i-1)!} \left(\frac{\phi}{\kappa}\right)^i} \\ &= \frac{1}{\lambda} \left(1 + \sum_{i=1}^{\kappa-1} \phi^i \prod_{j=1}^i \left(1 - \frac{j}{\kappa}\right)\right)^{-1} \end{aligned}$$

From this we get that  $u_0 \rightarrow 0$  as  $\kappa \rightarrow \infty$  (thus indicating the possibility of an endemic state for  $\kappa$  large) when  $\phi \geq 1$  ( $\phi > 1$  implies exponential convergence, when  $\phi = 1$  then  $u_0 \sim C/\lambda\sqrt{\kappa}$ ).

More precisely, Monotone Convergence Theorem implies that  $u_0 \downarrow \min(0, \frac{1-\phi}{\lambda})$  as  $\kappa \rightarrow \infty$ . Hence, if  $\phi + \lambda \leq 1$  then  $u_0 \geq 1$  for all  $\kappa$ , while if  $\phi + \lambda > 1$  then  $u_0 < 1$  for  $\kappa$  sufficiently large.

If we consider the self-regulating IRP the mean field equations are

$$\begin{cases} u'_0 = u_1 - \lambda u_0 \sum_{i=1}^{\infty} i u_i \\ u'_1 = 2u_2 + \lambda u_0 \sum_{i=1}^{\infty} i u_i - u_1(1 + \phi c_n(1)), \\ u'_i = (i+1)u_{i+1} + (i-1)c_n(i-1)\phi u_{i-1} - i u_i(1 + \phi c_n(i)), \quad 1 \leq i. \end{cases}$$

We can write the solution of the system  $u'_i = 0$ , for  $i \geq 1$  as

$$u_i = \frac{\phi^{i-1}}{i} c_n!(i-1) u_1,$$

where  $c_n!(i) := \prod_{l=0}^i c_n(l)$ . We plug it into the first equation and require  $u'_0 = 0$  to get

$$u_0 = \frac{1}{\lambda \sum_{i=0}^{\infty} \phi^i c_n!(i)}.$$

Since  $c_n \uparrow c_\infty$  as  $n \rightarrow \infty$  then, by Monotone Convergence Theorem,

$$u_0 \downarrow \frac{1}{\lambda \sum_{i=0}^{\infty} \phi^i c_\infty!(i)} \geq \min\left(0, \frac{1-\phi}{\lambda}\right)$$

and the equality holds if and only if  $c_\infty \equiv 1$ . In this case, as before for the logistic IRP,  $\phi + \lambda \leq 1$  implies  $u_0 \geq 1$  for all  $c_n$ , while  $\phi + \lambda > 1$  implies  $u_0 < 1$  for  $n$  sufficiently large.

Thus the mean field model for the self-regulating IRP suggests that, in the spatial case, if  $\phi + 2d\lambda \leq 1$  there is extinction for all controlling functions  $c_n$  (see Theorem 3.1) and, if  $c_\infty \equiv 1$ , survival is implied by  $\phi + 2d\lambda > 1$  provided that  $n$  is sufficiently large (we prove a slightly different result, see Theorem 3.5).



### 3. MAIN RESULTS AND DISCUSSION

In this section we discuss the effect of the parameters  $\lambda$ ,  $\phi$ ,  $c$  and  $N$  on the behavior of the PRP. Part of the arguments are done using monotonicity and coupling with known processes. Indeed for any fixed  $N$ , the PRP is attractive with respect to  $\lambda, \phi, c$ . Moreover we may couple a PRP on  $\mathcal{E}_N^d$  with another one on  $\mathcal{E}_1^d \equiv \mathbb{Z}^d$  in the following natural way. Consider two PRPs:  $\{\eta_t\}_{t \geq 0}$  on  $\mathcal{E}_N^d$  (with parameters  $\lambda, \phi$  and function  $c$ ) and  $\{\xi_t\}_{t \geq 0}$  on  $\mathbb{Z}^d$  (with parameters  $\lambda_1, \phi_1$  and function  $c_1$ ). If  $\lambda \geq \lambda_1$ ,  $\phi \geq \phi_1$ ,  $c \geq c_1$  then the projection of  $\eta_t$  on  $\mathbb{Z}^d$  (namely  $\sum_{r \in \mathbb{K}_N} \eta_t(\cdot, r)$ ) dominates  $\xi_t(\cdot)$ .

This kind of coupling is the key of the proof of the following result, which states that if the breeding parameters  $\lambda$  and  $\phi$  are sufficiently small then we have almost sure extinction.

**Theorem 3.1.** *For all functions  $c$ , if  $\phi + 2d\lambda \leq 1$  then there is a.s. extinction.*

By monotonicity it is clear that, if  $N$ ,  $c$  and  $\phi$  are fixed then there exists a critical  $\lambda_{cr} = \lambda_{cr}(\phi, c, N)$  such that given  $\lambda > \lambda_{cr}$  the PRP survives with positive probability, while  $\lambda < \lambda_{cr}$  implies almost sure extinction. By stochastic domination  $\lambda_{cr} \leq \lambda_{CP}(\mathcal{E}_N^d)$  thus if  $\lambda > \lambda_{CP}(\mathcal{E}_N^d)$  then PRP survives with positive probability for all choices of  $\phi$  and  $c$ .

The following theorem describes the dependence of  $\lambda_{cr}$  on  $\phi$  and  $c$ . Since  $c(i)$  can be equal to 0 for some  $i$ , we identify  $1/0$  with  $+\infty$ .

**Theorem 3.2.** (1) *Given any  $\phi, c$  such that  $\sum_{n=0}^{\infty} (\phi^n \prod_{i=0}^n c(i))^{-1} < +\infty$  then for every  $\lambda \geq 0$  the PRP survives with positive probability, hence  $\lambda_{cr}(\phi, c, N) = 0$ .*

(2) *Given any  $\phi, c$  such that  $\sum_{n=0}^{\infty} \phi^n \prod_{i=0}^n c(i) < +\infty$  then  $\lambda_{cr}(\phi, c, N) \in (0, \lambda_{CP}(\mathcal{E}_N^d))$ .*

In particular from the previous theorem we deduce that the value of  $\lim_{i \rightarrow \infty} c(i)$  (which exists by the assumption of monotonicity that we made) tells us whether  $\lambda_{cr} = 0$  or not in almost every case.

**Corollary 3.3.** *Let  $c(\infty) = \lim_{i \rightarrow \infty} c(i)$ . If  $\phi \cdot c(\infty) > 1$  then  $\lambda_{cr}(\phi, c, N) = 0$ . If  $\phi \cdot c(\infty) < 1$  then  $\lambda_{cr}(\phi, c, N) > 0$ . As a particular case, for all functions  $c$ ,  $\phi < 1$  implies  $\lambda_{cr} > 0$ .*

In the case  $\phi \cdot c(\infty) = 1$  the behavior depends on the speed of convergence of  $c(n)$  to  $c(\infty)$ . For instance if  $c(n) := \frac{(n+3)^2}{2(n+2)^2}$  and  $\phi = 2$  then Theorem 3.2(1) applies and  $\lambda_{cr} = 0$ . Unfortunately, due to the monotonicity of  $c$ , in the case  $\phi \cdot c(\infty) = 1$ , Theorem 3.2(2) is useless (since the series is always divergent). We do not know of any example when  $\lambda_{cr}(\phi, c, N) > 0$ .

Again by monotonicity, if  $N$ ,  $c$  and  $\lambda$  are fixed then there exists a critical  $\phi_{cr} = \phi_{cr}(\lambda, c, N)$  such that given  $\phi > \phi_{cr}$  the PRP survives with positive probability, while  $\phi < \phi_{cr}$  implies almost sure extinction. In general it could happen that  $\phi_{cr} = \infty$  (almost sure extinction for all  $\phi$  - for instance when  $N = 1$ ,  $c = \delta_0$  and  $\lambda < \lambda_{CP}(\mathbb{Z}^d)$ ) or  $\phi_{cr} = 0$  (survival for all positive  $\phi$  - for instance when  $\lambda > \lambda_{CP}(\mathcal{E}_N^d)$ ). The following theorem gives sufficient conditions for  $\phi_{cr} \in (0, \infty)$ .

**Theorem 3.4.** (1) If  $\lambda > \lambda_{CP}(\mathcal{E}_N^d)$  then  $\phi_{cr}(\lambda, c, N) = 0$  for all  $c, N$  and the PRP survives with positive probability when  $\phi = 0$ .

(2) For any  $\lambda \in (0, 1/2d)$  and  $c$  such that  $c(1) > 0$  then  $\phi_{cr}(\lambda, c, N) \in (0, +\infty)$ .

Actually to prove  $\phi_{cr} < \infty$  we only need the hypothesis  $c(1) > 0$  while  $\lambda \in (0, 1/2d)$  is needed to prove  $\phi_{cr} > 0$ . One guesses that  $\phi_{cr} > 0$  could be proved under the milder assumption  $\lambda \in (0, \lambda_{CP}(\mathcal{E}_N^d))$ ; in order to do so one could adapt the proofs of [3, Theorem 3] and of [11, Theorem 1(c)]. The main difficulty is to prove the analog of [1, Theorem 1.7] for the CP on  $\mathcal{E}_N^d$ . Since  $\lambda_{CP}(\mathcal{E}_N^d) \downarrow 1/2d$ , extending the interval for  $\lambda$  to  $(0, \lambda_{CP}(\mathcal{E}_N^d))$  seems to be a minor improvement at least for large values of  $N$ .

Now we want to study the dependence of the behavior of the PRP on the underlying space: note that “space” here is described both by  $N$  (the horizontal space) and  $c$  (the vertical space). Hence it is natural to define a sequence of PRPs by means of nondecreasing sequences of functions  $\{c_i\}_{i \in \mathbb{N}}$  and of patch dimensions  $\{N_i\}_{i \in \mathbb{N}}$ . More precisely the  $i$ th process is  $\mathbb{N}^{\mathcal{E}_{N_i}^d}$ -valued and its transition rates are defined as in equation (2.1) with  $c = c_i$  (being  $d, \lambda, \phi$  fixed for all  $i$ ).

**Theorem 3.5.** Consider a sequence of PRPs on  $\mathcal{E}_{N_i}^d$  with parameters  $\phi$  and  $\lambda$  and control functions  $c_i$ . If  $\lambda > 0$  and one of the following conditions holds

(1)  $\phi + 2d\lambda > 1$  and  $N_i \rightarrow \infty$ ;

(2)  $\phi > 1$  and  $c_\infty \equiv 1$ ;

then there exists  $i_{cr} = i_{cr}(\lambda, \phi, c)$  such that for all  $i \geq i_{cr}$  there is survival with positive probability.

Roughly speaking, Theorem 3.5 states that if the space is sufficiently large then the PRP survives with positive probability provided that the breeding parameters are not too small. In particular Theorem 3.5(1) is a partial converse of Theorem 3.1. In the proof we need to mimic a technique introduced in [6] in order to show the “convergence” of the projections on  $\mathbb{Z}^d$  of the PRPs to a branching random walk.

**Remark 3.6.** Theorems 3.1 and 3.5 imply, roughly speaking, that when  $N_i \rightarrow \infty$ , the projection of the  $i$ -th PRP on  $\mathbb{Z}^d$  behaves, in the limit  $i \rightarrow \infty$ , as the branching random walk on  $\mathbb{Z}^d$  with intra-site breeding rate  $\phi$  and inter-site rate  $\lambda$ , in the sense that it survives eventually if and only if  $\phi + 2d\lambda > 1$ .

The following proposition deals with the asymptotic behavior of the critical parameters of various processes on  $\mathcal{E}_{N_i}^d$  as  $N_i \rightarrow \infty$ . The results follow from Theorems 3.1 and 3.5.

**Proposition 3.7.** If  $N_i \uparrow \infty$  the following hold.

(1)  $\lambda_{CP}(\mathcal{E}_{N_i}^d) \downarrow 1/2d$ .

(2) Consider the biparametric contact process, for all fixed  $\phi \geq 0$  we have that

$$\lambda_{BCP}(\mathcal{E}_{N_i}^d, \phi) \downarrow \max\left(\frac{1-\phi}{2d}, 0\right),$$

while for all fixed  $\lambda > 0$  we have that

$$\phi_{BCP}(\mathcal{E}_{N_i}^d, \lambda) \downarrow \max(1 - 2d\lambda, 0).$$

In the particular case  $\phi = \lambda$  (the intra-patch reproduction has the same rate as the inter-patch one), the critical parameter converges to  $1/(2d + 1)$ .

(3) For a sequence of PRPs,

$$\begin{aligned} \lim_{i \rightarrow \infty} \lambda_{cr}(\phi, c_i, N_i) &= 0 && \text{if } \phi > 1, \\ \lim_{i \rightarrow \infty} \phi_{cr}(\lambda, c_i, N_i) &= 0 && \text{if } \lambda > 1/2d. \end{aligned}$$

We note that Proposition 3.7(3) gives the limit of  $\lambda_{cr}$  as the size of the patch goes to infinity. The following proposition deals with the limit of  $\lambda_{cr}$  in the case  $c_\infty \equiv 1$  (being  $N$  and  $\phi$  fixed). It is a consequence of Corollary 3.3 and Theorem 3.5(2) and establishes the continuity of  $\lambda_{cr}$  with respect to  $c$ .

**Proposition 3.8.** *Given  $N \in \mathbb{N}$ ,  $\phi > 1$  and a sequence of functions  $\{c_i\}_{i \geq 0}$  such that  $c_\infty \equiv 1$ , then*

$$\lim_{i \rightarrow \infty} \lambda_{cr}(\phi, c_i, N) = \lambda_{cr}(\phi, c_\infty, N).$$

We recall that the IRP with cluster size  $\kappa$  is a particular case of the PRP (taking  $N = 1$  and  $c = \mathbb{1}_{\{0, \dots, \kappa-1\}}$ ). Thus if  $\bar{\lambda}^\kappa(\phi)$  is the critical parameter of this IRP, from Proposition 3.8 we get the following result on the asymptotic behavior of  $\bar{\lambda}^\kappa(\phi)$  as  $\kappa$  goes to infinity.

**Corollary 3.9.** *If  $\phi > 1$  then  $\lim_{\kappa \rightarrow \infty} \bar{\lambda}^\kappa(\phi) = 0$ .*

We note that, if  $\bar{\lambda}^\infty(\phi)$  is the critical parameter of the IRP with cluster size equal to infinity (see [2] for some details on this process), then as a consequence of Corollary 3.3, if  $\phi > 1$  then  $\bar{\lambda}^\infty(\phi) = 0$ . Thus the previous corollary is a result of continuity at infinity for  $\bar{\lambda}^\kappa(\phi)$  with respect to the cluster size.

#### 4. PROOFS

*Proof of Theorem 3.1.* It is enough to note that the total reproduction rate of each particle is bounded from above by  $\phi + 2d\lambda$ . Indeed the projection  $\sum_{k \in \mathbb{K}_n} \tilde{\eta}_t(x, k)$  is dominated by a branching random walk on  $\mathbb{Z}^d$  with intra-patch infection rate  $\phi$  and inter-patch rate  $\lambda$  on each edge. In this branching random walk each particle breeds at rate  $\phi + 2d\lambda$  and dies at rate 1.  $\square$

In order to prove Theorem 3.2 we need the following lemma on discrete time random walks.

**Lemma 4.1.** Let  $\{Z_n\}_{n \geq 0}$  be a random walk on  $\mathbb{N}^N$  with the following transition probabilities:

$$\begin{cases} (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_j + 1, \dots, i_N) & \text{with probability } \frac{\phi c(i_j)}{(1+\phi)N} \\ (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_j - 1, \dots, i_N) & \text{with probability } \frac{1}{(1+\phi)N} \\ (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_N) & \text{with probability } \frac{\phi}{(1+\phi)N} \sum_{j=1}^N (1 - c(i_j)). \end{cases} \quad (4.4)$$

- (1) If  $\sum_{n=0}^{\infty} (\phi^n \prod_{i=0}^n c(i))^{-1} < +\infty$  then the random walk is transient.  
(2) The random walk is positive recurrent if and only if  $\sum_{n=0}^{\infty} \phi^n \prod_{i=0}^n c(i) < +\infty$ .

*Proof.* (1) We note that the projection of the  $N$ -dimensional process on the  $j$ th coordinate is a birth and death process with forward probabilities  $\phi c(i)/(N(1+\phi))$  and backward probabilities  $1/(N(1+\phi))$  which by [13, Theorem 5.9] is transient if and only if  $\sum_{n=0}^{\infty} (\phi^n \prod_{i=0}^n c(i))^{-1} < +\infty$ . But transience of at least one projection implies transience of the  $N$ -dimensional process.

- (2) Positive recurrence is equivalent to the existence of a finite invariant measure. It is not difficult to prove that

$$\nu(i_1, \dots, i_N) = \prod_{j: i_j > 0} \left( \phi^{i_j} \prod_{i=0}^{i_j-1} c(i) \right), \quad \nu(0, \dots, 0) = 1,$$

is a reversible measure. The claim follows noting that

$$\nu(\mathbb{N}^N) = \sum_{i_1, \dots, i_N} \nu(i_1, \dots, i_N) = \left( 1 + \phi \sum_{n=0}^{+\infty} \phi^n \prod_{i=0}^n c(i) \right)^N.$$

□

*Proof of Theorem 3.2.* (1) Take the PRP with  $\lambda = 0$ . Since it cannot leave the original patch, this is a (continuous time) random walk on  $\mathbb{N}^N$  and it is stochastically dominated by the original PRP; in this case survival means “avoiding the origin”. The survival of the (continuous time) random walk is equivalent to the survival of its discrete counterpart (see equation (4.4)) which is guaranteed by Lemma 4.1(1) noting that transience implies survival.

- (2) By (1),  $\lambda_{cr} \leq \lambda_{CP}(\mathcal{E}_N^d)$ . We now prove that  $\lambda_{cr} > 0$ . Let us start by considering the case  $N = 1$  (remember that  $\mathcal{E}_1^d$  can be identified with  $\mathbb{Z}^d$ ). For  $\lambda$  small, we want to find a subcritical branching process which dominates the total number of successful inter-patch reproductions. Indeed in this case the PRP cannot survive without leaving the first patch, hence survival is equivalent to the positive probability of having an infinite number of successful inter-patch infections. Once a particle appears at  $x$ , the total number of particles living at  $x$  up to the next time when the colony at  $x$  has no individuals is, according to Lemma 4.1, a positive recurrent random walk (let us call  $\tau_0$  the number of steps before reaching 0) with the following rates:

$$\begin{cases} i \longrightarrow i + 1 & \text{with rate } i\phi c(i) \\ i \longrightarrow i - 1 & \text{with rate } 1 \\ i \longrightarrow i & \text{with rate } i\phi(1 - c(i)). \end{cases}$$

Note that the exponential clock of the transition (including the loop) has parameter  $i(1 + \phi)$ .

The number of inter-patch reproduction trials made from the site  $x$  depends on the number of particles at  $x$ : indeed, when the number of particles at  $x$  is  $i$ , the total reproduction rate towards the neighboring patches is  $i\lambda$ . Hence the number of inter-patch reproduction trials between two transitions does not depend on  $i$ , since it is the number of times a Poisson clock with rate  $i\lambda$  rings before the transition clock with rate  $i(1 + \phi)$  does. It is clear that the total number of reproduction trials (before the progeny of the original particle at  $x$  dies out) from  $x$  to the neighbors is  $\sum_{k=1}^{\tau_0} \sum_{j=1}^{2d} Y_k(j)$  where  $\{Y_k\}_{k \geq 1}$  is an i.i.d. sequence of random  $2d$ -dimensional vectors; to be precise, if  $n_i$  is the number of reproduction trials on the neighbor  $i$  then

$$\mathbb{P}(Y_k = (n_1, \dots, n_{2d})) = \frac{(\sum_{i=1}^{2d} n_i)! (\lambda / (1 + \phi))^{\sum_{i=1}^{2d} n_i}}{(1 + 2d\lambda / (1 + \phi))^{1 + \sum_{i=1}^{2d} n_i} \prod_{i=1}^{2d} n_i!}$$

(see [7] for details). Thus

$$\mathbb{E} \left( \sum_{k=1}^{\tau_0} \sum_{j=1}^{2d} Y_k(j) \right) = 2d \frac{\lambda}{1 + \phi} \mathbb{E}(\tau_0) \rightarrow 0$$

as  $\lambda \rightarrow 0$ . By choosing  $\lambda < (1 + \phi) / (2d\mathbb{E}(\tau_0))$  the total number of successful trials (which cannot exceed the total number of trials) is dominated by a (subcritical) branching process with expected number of offsprings  $2d \frac{\lambda}{1 + \phi} \mathbb{E}(\tau_0) < 1$ .

Let us take  $N > 1$ : we cannot repeat exactly the same argument since generations inside a patch are not independent, due to the presence of the controlling function  $c$ . Hence we couple with a suitable process. The coupled process has the following rules:

- a. every inter-patch reproduction is successful (i.e. we do not use  $c$ );
- b. we define the particle born from an inter-patch reproduction as the ancestor of its descendance inside its patch;
- c. inside a patch particles compete for resources if and only if they have the same ancestor (i.e. the function  $c$  applies inside the same progeny).

Clearly this process dominates the original PRP and progenies of different ancestors are independent.

The number of generations of the progeny of a particle in the dominating process is equal to the absorption time  $\tau_0$  of a random walk with rates

$$\begin{cases} (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_j + 1, \dots, i_N) & \text{with rate } \phi c(i_j) \\ (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_j - 1, \dots, i_N) & \text{with rate } 1 \\ (i_1, \dots, i_N) \longrightarrow (i_1, \dots, i_N) & \text{with rate } \phi \sum_{j=1}^N (1 - c(i_j)). \end{cases}$$

As before, the behavior of this random walk is equivalent to the behavior of its discrete-time counterpart which, according to Lemma 4.1, is positive recurrent (thus  $\mathbb{E}(\tau_0) < +\infty$ ). Hence survival is equivalent to the positive probability of having an infinite number of progenies. But

the expected number of successful inter-patch reproductions is equal to  $2d\frac{\lambda}{1+\phi}\mathbb{E}(\tau_0)$ . Choosing  $\lambda$  small enough yields the conclusion.  $\square$

*Proof of Theorem 3.4.* (1) It follows from the fact that for all  $\phi \geq 0$  the PRP stochastically dominates the CP on  $\mathcal{E}_N^d$ .

(2) The hypothesis  $c(1) > 0$  yields  $\phi_{cr} < \infty$ . Indeed let  $\tilde{\phi}_{cr}(\lambda, 2) > 0$  be the critical threshold in [3, Theorem 3] and choose  $\phi > \tilde{\phi}_{cr}(\lambda, 2)/c(1)$ . Let  $\tilde{\eta}_t : \mathbb{Z}^d \rightarrow \{0, 1, 2\}$  be the IRP with parameters  $\tilde{\lambda} = \lambda$ ,  $\tilde{\phi} = \phi c(1)$  and  $\tilde{\kappa} = 2$ . By these choices of the parameters,  $\tilde{\eta}_t$  is supercritical. On the other hand, the projection  $\xi_t(x) = \sum_{i \in \mathbb{K}_n} \tilde{\eta}_t(x, i)$  dominates  $\tilde{\eta}_t$  and this proves that the PRP survives when  $\phi > \tilde{\phi}_{cr}(\lambda, 2)/c(1)$ .

To show that  $\phi_{cr} > 0$  in the case where  $\lambda < 1/2d$ , note that by Theorem 3.1 if we choose a positive  $\phi$  smaller than  $1 - 2d\lambda$  then the population dies out.  $\square$

Before proving Theorem 3.5 we prove an equivalence between conditions on the sequence of parameters.

**Lemma 4.2.** *The condition*

$$N_i \rightarrow \infty \vee c_\infty \equiv 1 \tag{4.5}$$

is equivalent to

$$\min_{f \in \mathbb{N}^{\mathbb{K}_N} : \sum_j f(j) = M} \frac{1}{N_i} \sum_{j=1}^{N_i} c_i(f(j)) \rightarrow 1 \text{ as } i \rightarrow \infty, \quad \forall M \in \mathbb{N}. \tag{4.6}$$

*Proof.* We first prove that (4.5) implies (4.6). Since  $c_i = \delta_0$ , if  $N_i \rightarrow \infty$  then

$$\min_{f \in \mathbb{N}^{\mathbb{K}_N} : \sum_j f(j) = M} \frac{1}{N_i} \sum_{j=1}^{N_i} c_i(f(j)) = 1 - M/N_i,$$

thus condition (4.6) holds. For general  $c_i$  the same follows from  $c_i \geq \delta_0$ . If  $N_i$  is bounded then  $N_i = N$  eventually for some  $N \in \mathbb{N}$  (since the sequence is nondecreasing). Hence for any  $f$  such that  $\sum_j f(j) = M$  we have  $\frac{1}{N} \sum_{j=1}^N c_i(f(j)) \rightarrow \frac{1}{N} \sum_{j=1}^N c_\infty(f(j))$  which is equal to 1 if  $c_\infty = 1$ .

Suppose now that (4.5) does not hold, that is  $N_i$  is bounded and  $c_\infty(M) < 1$  for some  $M \in \mathbb{N}$ . Choose  $\tilde{f}$  such that  $\tilde{f}(1) = M$  and 0 elsewhere. Thus, eventually

$$\min_{f \in \mathbb{N}^{\mathbb{K}_N} : \sum_j f(j) = M} \frac{1}{N_i} \sum_{j=1}^{N_i} c_i(f(j)) \leq \frac{1}{N} \sum_{j=1}^N c_i(\tilde{f}(j)) = 1 - \frac{1 - c_\infty(M)}{N} < 1,$$

which proves that (4.6) does not hold.  $\square$

*Proof of Theorem 3.5.* (1) The PRPs of the sequence dominate the ones where the control functions are equal to  $\delta_0$  for all  $i$ . Hence it is enough to prove the statement for the latter case.

The projection of this process on  $\mathbb{Z}^d$ , namely  $\xi_t^i(x) := \sum_{r \in \mathbb{K}_N} \eta_t^i(x, r)$ , has transition rates at  $x \in \mathbb{Z}^d$

$$\begin{aligned} j \rightarrow j+1 & \text{ at rate } \left( j\phi + \lambda \sum_{y \sim x} \xi_t^i(y) \right) \left( 1 - \frac{j}{N_i} \right); \\ j \rightarrow j-1 & \text{ at rate } j. \end{aligned} \quad (4.7)$$

This process dominates eventually (as  $N_i \rightarrow \infty$ ) the process  $\{\tilde{\xi}_t\}_{t \geq 0}$  with rates

$$\begin{aligned} j \rightarrow j+1 & \text{ at rate } \left( j\phi + \lambda \sum_{y \sim x} \tilde{\xi}_t(y) \right) (1 - \varepsilon) \mathbb{1}_{[0, \bar{N}]}(j); \\ j \rightarrow j-1 & \text{ at rate } j; \end{aligned} \quad (4.8)$$

for all  $\bar{N} > 0$  and  $\varepsilon > 0$  (it suffices that  $N_i > \bar{N}/\varepsilon$ ). Choose  $\varepsilon > 0$  such that  $(\phi + 2d\lambda)(1 - \varepsilon) > 1$ . We will prove now, following [6], that for all sufficiently large  $\bar{N}$ , the process  $\{\tilde{\xi}_t\}_{t \geq 0}$  survives with positive probability. The strategy is to study the branching random walk  $\{\bar{\xi}_t\}_{t \geq 0}$  with rates

$$\begin{aligned} j \rightarrow j+1 & \text{ at rate } \left( j\bar{\phi} + \bar{\lambda} \sum_{y \sim x} \bar{\xi}_t(y) \right), \\ j \rightarrow j-1 & \text{ at rate } j, \end{aligned} \quad (4.9)$$

starting with one particle at the origin  $0 \in \mathbb{Z}^d$ , where  $\bar{\phi} = (1 - \varepsilon)\phi$  and  $\bar{\lambda} = (1 - \varepsilon)\lambda$ . Note that the process  $\{\bar{\xi}_t\}_{t \geq 0}$  can be seen as a truncation (at height  $\bar{N}$ ) of  $\{\xi_t\}_{t \geq 0}$ .

It is not difficult to verify that  $\mathbb{E}^{\delta_0}(\bar{\xi}_t)$  satisfies the following differential equation (see [6, Section 5], where we did the same with the classical BRW):

$$\frac{d}{dt} \mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) = -\mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) + \phi \mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) + \lambda \sum_{\substack{y \sim x \\ y \neq x}} \mathbb{E}^{\delta_0}(\bar{\xi}_t(y)), \quad (4.10)$$

whose solution is

$$\mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-|x|_{\mathbb{Z}^d}} \mu^{(n,k)}(0, x) \frac{\bar{\phi}^k \bar{\lambda}^{n-k} t^n}{n!} e^{-t}, \quad (4.11)$$

where  $\mu^{(n,k)}(0, x)$  is the number of paths from 0 to  $x$  of length  $n$  and  $k$  loops. Moreover, if  $|x|_{\mathbb{Z}^d} = 1$  then, taking  $t = n$  large enough,

$$\begin{aligned} \mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) & \geq \sum_{k=0}^{n-1} \mu^{(n,k)}(0, x) \frac{\bar{\phi}^k \bar{\lambda}^{n-k} t^n}{n!} e^{-t} = \sum_{k=0}^{n-1} \mu^{(n,k)}(0, x) \frac{\bar{\phi}^k \bar{\lambda}^{n-k}}{(\bar{\phi} + 2d\bar{\lambda})^n} \frac{n^n (\bar{\phi} + 2d\bar{\lambda})^n}{n!} e^{-n} \\ & \stackrel{n \rightarrow \infty}{\sim} \frac{(\bar{\phi} + 2d\bar{\lambda})^n}{\sqrt{2\pi n}} \sum_{k=0}^{n-1} \mu^{(n,k)}(0, x) \frac{\bar{\phi}^k \bar{\lambda}^{n-k}}{(\bar{\phi} + 2d\bar{\lambda})^n} \geq \frac{(\bar{\phi} + 2d\bar{\lambda})^n}{\sqrt{2\pi n}} C n^{-d/2} \end{aligned}$$

for some  $C = C(\bar{\lambda}, \bar{\phi}) > 0$ , since  $\sum_{k=0}^{n-1} \mu^{(n,k)}(0, x) \bar{\phi}^k \bar{\lambda}^{n-k} / (\bar{\phi} + 2d\bar{\lambda})^n$  is the probability of being in  $x$  ( $|x|_{\mathbb{Z}^d} = 1$ ) after  $n$  steps for a discrete-time random walk with transition probabilities

$$p(x, y) = \begin{cases} \frac{\bar{\lambda}}{\bar{\phi} + 2d\bar{\lambda}} & \text{if } x \sim y, \\ \frac{\bar{\phi}}{\bar{\phi} + 2d\bar{\lambda}} & \text{if } x = y, \end{cases}$$

(see [12, Corollary 13.11] for the asymptotic estimates of the  $n$ -step probabilities). Hence there exists  $t = n$  sufficiently large such that  $\mathbb{E}^{\delta_0}(\bar{\xi}_t(x)) > 1$  (this result is analogous to Lemmas 5.2 and 5.3 of [6]). Using the same arguments as in Lemma 5.4, Remark 5.5 and Theorem 5.6 of [6], it is possible to prove that the process  $\{\tilde{\xi}_t\}_{t \geq 0}$  survives when  $\bar{N}$  is sufficiently large and this implies the survival of  $\{\xi_t^i\}_{t \geq 0}$  when  $i$  is sufficiently large.

- (2) Let  $\kappa_c(\lambda, (1 + \phi)/2)$  as in [3, Theorem 5]. By Lemma 4.2 we may pick  $i_{cr}$  such that for all  $i \geq i_{cr}$

$$\min_{f \in \mathbb{N}^{\mathbb{K}^N} : \sum_j f(j) = M} \frac{1}{N_i} \sum_{j=1}^{N_i} c_i(f(j)) > \frac{1 + \phi}{2\phi}, \quad \forall M \leq \kappa_c + 1.$$

In this case, if in a patch there are at most  $\kappa_c + 1$  particles then the effective reproduction rate for each particle within the patch is at least  $(1 + \phi)/2$ , which implies that the PRP dominates the IRP with  $\kappa = \kappa_c + 1$  and reproduction rates  $\lambda$  and  $(1 + \phi)/2$  respectively. □

## 5. OPEN QUESTIONS

In the phase diagram the answer to some questions remains unknown. For instance, we do not know if  $\lambda_{cr}(\phi, c, N) = 0$  when  $\phi \cdot c(\infty) = 1$  (we know there are examples where this happens). Moreover we do not know if  $\phi_{cr}(\lambda, c, N) > 0$  when  $\lambda \in (1/2d, \lambda_{CP}(\mathcal{E}_N^d))$ . One interesting question is whether a nontrivial invariant measure exists when there is survival with positive probability. The answer is positive if  $c(i) = 0$  eventually (in this case the configuration space is compact and the invariant measure is obtained as the limit of the process starting from the maximal configuration). In the noncompact case one could try to mimic the technique employed in [5].

On the other hand, one may consider other models which, although similar to those considered here, cannot be obtained as particular cases of the PRP. One of particular interest is a self-regulating IRP where the inter-cluster infection is always active. In each site of  $\mathbb{Z}^d$  there is a (possibly infinite) cluster and, given a regulating function  $c$ , the transition rates at site  $x$  are:

$$\begin{aligned} i &\rightarrow i + 1 \text{ at rate } \left( i\phi + \lambda \sum_{z \sim x} \eta(z) \right) c(i), \text{ for } 0 \leq i; \\ i &\rightarrow i - 1 \text{ at rate } i, \text{ for } i \geq 1. \end{aligned}$$



In the present work, we dealt with the case  $c(i) = \max(0, 1 - i/N)$  (see Subsection 2.3). Further investigations may be made on the processes where we add a rate  $\delta$  for the transition  $i \rightarrow 0$  - the so-called *catastrophe* or *mass extinction*.

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#### REFERENCES

- [1] C. Bezuidenhout, G. Grimmett, Exponential decay for subcritical contact and percolation processes, *Ann. Probab.* **19** (1991), no.3, 984–1009.
- [2] L. Belhadji, Ergodicity and hydrodynamic limits for an epidemic models, arXiv:0710.5185.
- [3] L. Belhadji, N. Lanchier, Individual versus cluster recoveries within a spatially structured population, *Ann. Appl. Probab.* **16** (2006), no.1, 403–422.
- [4] L. Belhadji, N. Lanchier, Two scale contact process and effects of habitat fragmentation on metapopulations. *Markov Process. Related Fields.* **14** (2008), 487–514.
- [5] D. Bertacchi, G. Posta, F. Zucca, Ecological equilibrium for restrained random walks, *Ann. Appl. Probab.* **17** n. 4 (2007), 1117–1137.
- [6] D. Bertacchi, F. Zucca, Approximating critical parameters of branching random walks, arXiv:0710.3792
- [7] D. Bertacchi, F. Zucca, Characterization of the critical values of branching random walks on weighted graphs through infinite-type branching processes, to appear on *J. Stat. Phys.*, arXiv:0804.0224.
- [8] M. Bramson, R. Durrett, G. Swindle, Statistical mechanics of crabgrass, *Ann. Probab.* **17** (1989), no. 2, 444–481.
- [9] R. Durrett, S. Levin, The importance of being discrete (and spatial), *Theor. Popul. Biol.* **46** (1994), 363–394.
- [10] T. E. Harris, Contact interactions on a lattice, *Ann. Prob.* **2** (1974), 969–988.
- [11] R. B. Schinazi, On the role of social clusters in the transmission of infectious diseases, *Theor. Popul. Biol.* **61** (2002), 163–169.
- [12] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, **138**, Cambridge Univ. Press, 2000.
- [13] W. Woess, *Denumerable Markov Chains*, European Mathematical Society, in press.

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