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Production and distribution in a network: the system structure predominates the individual proficiency

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Abstract

A system of producing firms is considered here. The firms own each other through fixed quotas of stocks, so they periodically share the incomes and the losses (i.e., they close the books).

A network model is introduced to describe the system. The values of the firms evolve dynamically according to the financial flows (which depend on the topology of the network) and to the distribution of the individual productivities, nevertheless, on the long run, only the system structure matters. In the limit case in which the shares matrix is irreducible (it represents a strongly connected graph), the values of the firms are determined ratios of the total production, which evolves like a Brownian motion. So the values tend to be perfectly correlated. These ratios are determined only by the shares matrix, while the individual proficiencies affect only the total production of the system. When the shares matrix is reducible (the graph is connected, but not strongly), some firms increase their value much more than the other ones.

1 Introduction

The value of a company is clearly related to the revenues coming from its productivity. However, a company can own some shares the other ones. In this case its value is affected not only by its own productivity, but also by the value of the other ones. Several works have investigated the effect of mergers and acquisitions on the stock price (e.g., [9], [6]). They actually bind each other in a network of relationships so that it is often not easy to understand who controls who. The network structure of companies is a fact, besides it seems to be scale–free ([3]). It introduces a feedback effect on the dynamics and is accountable for the systemic risks ([2] and ([4]).

In this work a basic model is introduced in order to account for the financial revenues, which clearly play a part in the total incomes. The model actually describe a distributive system since the firms, while increasing (or decreasing) their value thanks to their production (which is random, but accounts for their individual proficiency), share their values according to fixed quotas. The result is a N–dimensional, discrete time, stochastic process. However, according to the topology of the network, the values on the long run are weakly related to the individual proficiency. If the structure is strongly connected (it is a directed, weighted graph), they actually are determined only by the topology and by the total value of the system. If there are some connected components, each one evolves autonomously, while when it is connected, but not strongly, one set of firms has no shares of the remaining firms. In this case the firms of the latter group increase their value much more quickly than those of the former set.

2 The dynamics of the value

Let $\mathbf{x}_t$ be the vector of the values of the firms (their cash) at time t. $\mathbf{x}_t$ is the *value vector*. At time $t+1$, every firm closes the books, distributing its cash among the owners according to the *shares matrix* $\mathbf{P}$. Then i-th firm receives a fraction P_{ij} of j-th firm value $x_{j,t}$, besides its cash is increased by the revenue coming from the individual production $\varepsilon_{i,t+1}$ relatively to the period $[t, t+1]$. The random vector $\boldsymbol{\varepsilon}_t$ is the *output vector*. Clearly, if $x_{j,t}$ is negative, the owners must pay, according to the same ratios, to avoid the bankruptcy.[1] The following recursive identity holds

$$x_{i,t+1} = \sum_{j=1}^{N} P_{ij}\, x_{i,t} + \varepsilon_{j,t+1} \iff \mathbf{x}_{t+1} = \mathbf{P}\, \mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1} \tag{1}$$

By backward recurrence:

$$\mathbf{x}_t = \mathbf{P}^t\, \mathbf{x}_0 + \sum_{j=1}^{t} \mathbf{P}^{t-j}\, \boldsymbol{\varepsilon}_j \tag{2}$$

3 Gaussian productions

Most of the results do not depend on the probability distribution of the output vector, however if $\boldsymbol{\varepsilon}_t$ are Gaussian (and $\mathbf{x}_0$ is fixed or Gaussian), the values are Gaussian for every t. Then, to consider Gaussian output vectors improves the concision of the comments. Besides, thanks to the law of large numbers and the central limit theorem, most of the results hold independently on the probability distribution of the output vector since they concern the dynamics of the system on the long run.

[1] The inclusion of bankruptcy in the model is straightforward, but the consequences are not. So it is a subject for further research.

In this embryonic work, only simultaneous productions are supposed to be correlated, besides the probability distribution of the output vector is assumed constant:

$$\boldsymbol{\varepsilon}_t \sim \boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\Gamma}) \quad \Longleftrightarrow \quad \mathbb{E}\left[\varepsilon_{i,t}\right] = \mu_i \quad , \quad \operatorname{cov}\left[\varepsilon_{i,s}, \varepsilon_{j,t}\right] = \boldsymbol{\Gamma}_{ij}\, \delta_{st}$$

where δ_{st} is the Kronecker delta, while $\mathbf{\Gamma}$ is the covariance matrix of simultaneous productions. Finally the initial value $\mathbf{x}_0$ is assumed to be a constant (i.e., not a random vector).

The following results hold

$$\begin{aligned} \mathbb{E}\left[\mathbf{x}_t\right] &= \mathbf{P}^t\, \mathbf{x}_0 + \left(\sum_{j=0}^{t-1} \mathbf{P}^j \right) \boldsymbol{\mu} \\ \mathbf{C} &= \sum_{j=0}^{t-1} \mathbf{P}^j\, \mathbf{\Gamma}\, \left(\mathbf{P}^j\right)^T \end{aligned} \tag{3}$$

where $C_{ij} = \operatorname{cov}\left[x_{i,t}, x_{j,t}\right]$.

Equations (1) and (2) clearly show the stochastic dynamics of the value vector. The hypothesis of Gaussian output vectors simplifies the reasoning since, being the value vector Gaussian for every t, the value vector is totally identified by the vector $\mathbb{E}[\mathbf{x}_t]$ and the matrix $\mathbf{C}$.[2]

4 The dynamics of the value in the long run

The shares matrix is necessarily a stochastic matrix since

$$P_{ij} \in [0, 1] \quad \wedge \quad \mathbf{u}^T\, \mathbf{P} = \mathbf{u} \tag{4}$$

where $\mathbf{u}^T = (1, 1, 1, \ldots, 1)$. The first condition is obvious, while the second condition states that the columns sum to one. This is necessarily true since i-th column consists of the quotas of ownership of i-th firm, which must sum to one.

Consequently 1 is an eigenvalue for $\mathbf{P}$ and $\mathbf{u}$ is a left eigenvector.

Perron–Frobenius theorem and Wielandt's theorems ([7]) are useful to understand the system dynamics when the shares matrix $\mathbf{P}$ is irreducible and when it is primitive. Even though it is not used in this work, to figure out the Jordan canonical form of $\mathbf{P}$ ([1]) may help to understand the dynamics. See the Appendix for the details.

The theory states that $\mathbf{P}$ can not have (complex) eigenvalues with norm greater than 1, that is its spectral radius. Besides, when $\mathbf{u}^T\, \boldsymbol{\mu} \neq 0$, there is no need to solve the sums to know the asymptotic behavior of $\mathbf{x}_t$ for large t.

[2]This is true even when the output vectors are correlated in time, provided they are Gaussian.

4.1 Primitive shares matrix

When $\mathbf{P}$ is primitive all the (eventually complex) eigenvalues have norm less than 1. The vector $\mathbf{u}$ alone generates the left eigenspace of 1. Let $\mathbf{v}$ be a right eigenvector for $\mathbf{A}$ (the right eigenspace is clearly one–dimensional too, so all the right eigenvectors of 1 are multiples of each other).

Equation (6) states that

$$\lim_{t\to+\infty} \mathbf{P}^t = \frac{1}{\mathbf{u}^T\mathbf{v}}\,\mathbf{v}\,\mathbf{u}^T = \tilde{\mathbf{v}}\,\mathbf{u}^T$$

where

$$\tilde{\mathbf{v}} = \frac{\mathbf{v}}{\mathbf{u}^T\,\mathbf{v}}$$

is the only right eigenvector for $\mathbf{P}$ whose entries sum to one (i.e., the Perron vector of $\mathbf{P}$). The rate of convergence depends on the second eigenvalue of $\mathbf{P}$ ([5]). Then Theorem (7) in the Appendix implies

$$\sum_{j=1}^{t} \mathbf{P}^{t-j} = t\,\tilde{\mathbf{v}}\,\mathbf{u}^T + \mathbf{o}(t)$$

$$\sum_{n=1}^{t} \mathbf{P}^{t-n}\,\boldsymbol{\Gamma}\,\left(\mathbf{P}^{t-n}\right)^T = t^2\,\tilde{\mathbf{v}}\,\mathbf{u}^T\,\boldsymbol{\Gamma}\,\mathbf{u}\,\tilde{\mathbf{v}}^T + \mathbf{o}\left(t^2\right) = t^2\,\left(\mathbf{u}^T\,\boldsymbol{\Gamma}\,\mathbf{u}\right)\,\tilde{\mathbf{v}}\,\tilde{\mathbf{v}}^T + \mathbf{o}\left(t^2\right)$$

Consequently, in the long run ($t \to +\infty$), if $\mathbf{u}^T\boldsymbol{\mu} \neq 0$, the values of the firms are

$$\mathbf{x}_t \sim t\,\left(\mathbf{u}^T\,\boldsymbol{\mu}\right)\,\tilde{\mathbf{v}}$$

i.e.,

$$\mathbb{E}[\mathbf{x}_t] = t\,\left(\mathbf{u}^T\,\boldsymbol{\mu}\right)\,\tilde{\mathbf{v}} + \mathbf{o}(t)$$

$$\mathbf{C} = t^2\,\gamma\,\mathbf{v}\,\mathbf{v}^T + \mathbf{o}\left(t^2\right)$$

where

$$\gamma = \mathbf{u}^T\,\boldsymbol{\Gamma}\,\mathbf{u}$$

Let[3]

$$G_t = \mathbf{u}^T\,\sum_{j=1}^{t}\boldsymbol{\varepsilon}_j$$

the total production. Since

$$\mathbb{E}[G_t] =< G >= \left(\mathbf{u}^T\,\boldsymbol{\mu}\right)\,t \quad , \quad \mathrm{Var}[G_t] = \gamma\,t^2$$

[3]To consider $\tilde{G}_t = \mathbf{u}^T\left(\mathbf{x}_0 + \sum_{j=1}^{t}\boldsymbol{\varepsilon}_j\right)$ does not change the results, except for $\mathbb{E}\left[\tilde{G}_t\right] = \left(\mathbf{u}^T\,\boldsymbol{\mu}\right)\,t + \left(\mathbf{u}^T\,\mathbf{x}_0\right)$.

the values of the firms are asymptotically equal to

$$\mathbf{x}_t \sim G_t \, \tilde{\mathbf{v}}$$

That is: the value of i-th firm tends to be a fixed quota v_i of the total production G_t, being v_i determined only by the topology of the network $\mathbf{P}$. It implies that the correlations tend to 1:[4]

$$\mathrm{corr}[x_{i,t}, x_{j,t}] = \frac{\gamma \, t^2 \, v_i \, v_j + o\left(t^2\right)}{\sqrt{\gamma \, t^2 \, v_i^2 + o\left(t^2\right)} \, \sqrt{\gamma \, t^2 \, v_j^2 + o\left(t^2\right)}} \sim 1$$

5 Irreducible shares matrix

When $\mathbf{P}$ is irreducible, the Perron vector exists since the eigenspace of 1 is one–dimensional, but there are h unitary eigenvalues. However the eigenvalues are in the form

$$\lambda_k = e^{2 \frac{k}{h} \pi \, i}$$

with $k = 0, \ldots, h$, where h is the index of imprimitivity. This is the reason why a non–negative, irreducible but not primitive matrix is said periodic. Besides every eigenvalue is simple (i.e., its eigenspace is one–dimensional).

$\mathbf{P}^t$ does not converge in this case, however it is similar to a matrix in the following form:

$$\tilde{\mathbf{P}} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \mathbf{0} \\ 0 & e^{\phi \, i} & 0 & \ldots & 0 & \mathbf{0} \\ 0 & 0 & e^{2 \, \phi \, i} & \ldots & 0 & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & e^{(h-1) \, \phi \, i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{A} \end{pmatrix}$$

where $\phi = \frac{2\pi}{h}$ and $\mathbf{A}$ is a matrix with spectral radius less than 1. That is $\mathbf{P} = \mathbf{M} \, \tilde{\mathbf{P}} \, \mathbf{M}^{-1}$ for some non–singular (complex) matrix $\mathbf{M}$. As a consequence $\mathbf{P}^j = \mathbf{M} \, \tilde{\mathbf{P}}^j \, \mathbf{M}^{-1}$.
$\tilde{\mathbf{P}}$ is a block–diagonal matrix, then

$$\tilde{\mathbf{P}}^j = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \mathbf{0} \\ 0 & e^{j \, \phi \, i} & 0 & \ldots & 0 & \mathbf{0} \\ 0 & 0 & e^{2 \, j \, \phi \, i} & \ldots & 0 & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & e^{(h-1) \, j \, \phi \, i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{A}^j \end{pmatrix}$$

[4]By Perron–Frobenius theorem, $\mathbf{v}$ has positive entries. $\gamma > 0$ unless all the entries of the output vector have a perfect correlation (the result clearly holds in this case too).

with $\mathbf{A}^t \to \mathbf{0}$ as $t \to +\infty$. $\tilde{\mathbf{P}}^t$ can not converge (and so does $\mathbf{P}$) because the diagonal elements $\tilde{P}_{22}$, $\tilde{P}_{33}$, ..., $\tilde{P}_{hh}$ keep on cycling. However both the matrices have a Cesàro limit, which, for $\mathbf{P}$, is:

$$\lim_{t\to+\infty} \frac{1}{t} \sum_{j=1}^{t-1} \mathbf{P}^j = \tilde{\mathbf{v}}\,\mathbf{u}^T \tag{5}$$

It is easy to understand, considering the Cesàro limit of $\tilde{\mathbf{P}}$ and reminding that the first column of $\mathbf{M}$ must be a multiple of $\tilde{\mathbf{v}}$ while the first row of $\mathbf{M}^{-1}$ must be a multiple of $\mathbf{u}^T$ by a coefficient which is the reciprocal of the former.

Every stochastic matrix has a finite Cesàro limit, but it has a different form when the matrix is reducible.

Equation (5) implies

$$\sum_{j=0}^{t-1} \mathbf{P}^j = t\,\tilde{\mathbf{v}}\,\mathbf{u}^T + \mathbf{o}(t)$$

for large t. So that, again,[5]

$$\mathbb{E}(\mathbf{x}_t) = t\,\left(\mathbf{u}^T \boldsymbol{\mu}\right)\,\tilde{\mathbf{v}} + \mathbf{o}(t)$$

With some more calculations, one obtains

$$\mathbf{C} = t^2\,\gamma\,\mathbf{v}\,\mathbf{v}^T + \mathbf{o}\left(t^2\right)$$

Thus the values of the firms tend to change coherently even though the shares matrix is not primitive. The irreducibility is sufficient.

$\mathbf{P}$ is irreducible if and only if the associated directed graph is strongly connected ([7] or [8]). This means that for every ordered pair of firms f_1, f_2, a sequence s_1, ... , s_n of firms can be found such that f_1 owns some shares of s_1, which owns some shares of s_2 and so on, until s_n which owns some shares of f_2. The opposite must be true too (in general by a different sequence).

Clearly, if the shares matrix is not connected, every connected component evolves autonomously.

6 Reducible shares matrix

Only one case remains: when the shares matrix is connected but reducible (i.e., not strongly connected). In this case $\mathbf{P}$ still has a finite Cesàro limit,

[5]Since $\tilde{P^j}_{kk}$, with $k = 2, \ldots, h$, cycle, the sum on j remains bounded. $\mathbf{A}^j$ tend exponentially to zero, so their sum can not diverge and is bounded. Then the spread $\mathbf{o}(t)$ is actually $\mathbf{o}(f(t))$ for every infinite function $f(t)$, no matter how slow $f(t)$ tends to 0. It clearly holds also in the case of primitive shares matrix.

but the form is different ([7]): by a proper permutation, $\mathbf{P}$ can assume the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{0} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix}$$

with $\mathbf{T}_{11}$ lower–triangular block matrix, and

$$\mathbf{T}_{22} = \begin{pmatrix} \mathbf{A}_1 & & \\ & \ddots & \\ & & \mathbf{A}_m \end{pmatrix}$$

where $\mathbf{A}_k$ are irreducible. This means that the firms can be qualitatively grouped into "owners" (the last listed one, when the shares matrix has the above–mentioned form) and "owned firms" (even though they simply do not have shares of the second group of firms). Then, the Cesàro limit is

$$\lim_{t\to+\infty} \frac{1}{t} \sum_{j=1}^{t-1} \mathbf{P}^j = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{E}\,\mathbf{T}_{21}\,(\mathbb{I} - \mathbf{T}_{11})^{-1} & \mathbf{E} \end{pmatrix}$$

where

$$\mathbf{E} = \begin{pmatrix} \tilde{\mathbf{v}}_1\,\mathbf{u}^T & & \\ & \ddots & \\ & & \tilde{\mathbf{v}}_m\,\mathbf{u}^T \end{pmatrix}$$

$\tilde{\mathbf{v}}_k$ is the Perron vector of the matrix $\mathbf{A}_k$. Furthermore

$$\lim_{t\to+\infty} \mathbf{P}^t = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{E}\,\mathbf{T}_{21}\,(\mathbb{I} - \mathbf{T}_{11})^{-1} & \mathbf{E} \end{pmatrix}$$

if and only if $\mathbf{A}_k$ are all primitive. Otherwise the limit does not exist.

This means that the value tend to concentrate on the owners (which are "more connected"), while the owned firms tend to (relatively) decrease their value. Actually the mean value of the owners is of order t, while the one of the owned firms is just $o(t)$ (so it does not necessarily tend to zero).

7 Conclusions

In this work the evolution of the value of a set of producing firms sharing their property according to fixed quotas has been modelled. The model is still very simple and can be generalized in several manners to approach more realistic situations. The shares matrix is kept constant in time, while this does clearly not occur. Besides, the random productions are supposed to be independent on the values of the firms and correlated only at simultaneous times. However some interesting features emerged: the long run values depend weakly on the individual productivities and strongly on the topology of the network representing the shares quotas. As just as the structure becomes strongly connected, the productivities simply contribute to the total value of the system, which tends asymptotically to be shared among the firms according to quotas that depend only on the network topology. When the network (which is a weighted, directed graph) is connected, but not strongly connected, the firms can be grouped into two classes. The "less connected" ones are penalized with respect to the "more connected" ones.

The model can actually be applied to a wider range of systems, since it describes the production and the distribution on a network and shows that stronger the connection, weaker the dependence of the individual performance on the personal proficiency. Besides it shows that networks tend to get more stiffening and to trap their dynamics as they increase their connection.

Appendix

A Irreducible and primitive matrices

Here are cited some useful definitions and properties:

- A matrix $\mathbf{A}$ is reducible ([8]) if there is a permutation $\mathbf{B}$ such that

$$\mathbf{B}\,\mathbf{A}\,\mathbf{B}^{-1} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{0} & \mathbf{K}_{22} \end{pmatrix}$$

  where $\mathbf{K}_{11}$ and $\mathbf{K}_{22}$ are square matrices. Otherwise it is irreducible. With the model and the consequent formalism used in this work, a matrix is reducible if

$$\mathbf{B}\,\mathbf{A}\,\mathbf{B}^{-1} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{0} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}$$

- A non–negative matrix $\mathbf{A}$ is primitive if there is $k > 0$ such that $\mathbf{A}^k$ is positive.

- A primitive matrix must be irreducible.

- An irreducible matrix is primitive if and only if there is only one eigenvalue on its spectral circle.

- If a non–negative, irreducible matrix has a positive, diagonal entry, it is primitive.

- If $\mathbf{A}$ is primitive

$$\lim_{n\to+\infty} \left(\frac{1}{\rho(\mathbf{A})}\,\mathbf{A}\right)^n = \frac{\mathbf{r}\,\mathbf{l}^T}{\mathbf{l}^T\,\mathbf{r}} > \mathbf{0} \qquad (6)$$

where $\mathbf{r}$ and $\mathbf{l}$ are respectively a right and a left eigenvector of $\rho(\mathbf{A})$ for $\mathbf{A}$.

- A non–negative, irreducible matrix is imprimitive if there are h eigenvalues on its spectral circle. h is the index of imprimitivity.
- if $c(x) = x^n + \alpha_1 x^{n-k_1} + \alpha_2 x^{n-k_2} + \alpha_3 x^{n-k_3} + \cdots + \alpha_s x^{n-k_s}$ is the characteristic polynomial of an imprimitive matrix $\mathbf{A}$ in which only the non–zero terms are listed, then the index of imprimitivity is the greatest common divisor of k_1, k_2, ..., k_s.

B Perron-Frobenius theorem

If $\mathbf{A}$ is non–negative and irreducible then

- the spectral radius $\rho(\mathbf{A})$ is an eigenvalue for $\mathbf{A}$;
- the eigenspace of the spectral radius is one–dimensional;
- there is a positive eigenvector of the spectral radius for $\mathbf{A}$;

Besides the Collatz–Wielandt formula holds for all non–negative matrices:

$$\rho(\mathbf{A}) = \max_{\substack{\mathbf{x} \geq 0 \\ \mathbf{x} \neq \mathbf{0}}} \left(\min_{\substack{1 \leq j \leq n \\ x_j \neq 0}} \frac{[\mathbf{A}\,\mathbf{x}]_j}{x_j} \right)$$

C Wielandt's theorems

C.1

If $|\mathbf{B}| \leq \mathbf{A}$ and $\mathbf{A}$ is irreducible then $\rho(\mathbf{B}) \leq \rho(\mathbf{A})$. In the case $\rho(\mathbf{B}) = \rho(\mathbf{A})$ (i.e., $\lambda_{\mathbf{B}} = \rho(\mathbf{A})\, e^{i\,\phi}$ for some ϕ), then

$$\mathbf{B} = e^{i\,\phi}\,\mathbf{D}\,\mathbf{A}\,\mathbf{D}^{-1} \quad \text{with } D_{ij} = \begin{cases} e^{i\,\theta_i} & j = i \\ 0 & j \neq i \end{cases}$$

C.2

If $\mathbf{A}$ is non–negative and irreducible and has h eigenvalues λ_1, λ_2, ..., λ_h on its spectral circle then

- every eigenspace of λ_j is one–dimensional;
- $\lambda_k = \rho(\mathbf{A})\, e^{2\pi \frac{k}{h} i}$

D Some simple proofs

D.1 Equation 3

$$\begin{aligned}\text{cov}\left[x_{i,t}, x_{j,t}\right] &= \sum_{n=1}^{t}\sum_{m=1}^{t}\sum_{a}\sum_{b}\left[\mathbf{P}^{t-n}\right]_{ia}\left[\mathbf{P}^{t-m}\right]_{jb}\text{cov}\left[\varepsilon_{a,n}, \varepsilon_{b,m}\right] = \\ &= \sum_{n=1}^{t}\sum_{m=1}^{t}\sum_{a}\sum_{b}\left[\mathbf{P}^{t-n}\right]_{ia}\left[\mathbf{P}^{t-m}\right]_{jb}\delta_{mn}\Gamma_{ab} = \\ &= \sum_{n=1}^{t}\sum_{a}\sum_{b}\left[\mathbf{P}^{t-n}\right]_{ia}\left[\mathbf{P}^{t-n}\right]_{jb}\Gamma_{ab} = \sum_{n=1}^{t}\left[\mathbf{P}^{t-n}\,\Gamma\,\left(\mathbf{P}^{t-n}\right)^{T}\right]_{ij} = \\ &= \left[\sum_{n=1}^{t}\mathbf{P}^{t-n}\,\Gamma\,\left(\mathbf{P}^{t-n}\right)^{T}\right]_{ij}\end{aligned}$$

D.2 Theorem

If $\mathbf{M}_n$ is a converging sequence of matrices and $\mathbf{M} = \lim_{n\to\infty}\mathbf{M}_n$, then as $n \to \infty$

$$\sum_{j=1}^{n}\mathbf{M}_j = n\,\mathbf{M} + \mathbf{o}(n) \sim n\,\mathbf{M} \tag{7}$$

Proof:
Since $\mathbf{M}_n = \mathbf{M} + \mathbf{o}(1)$, for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that

$$\max_{a,b=1,\ldots,N}\left|\left[\mathbf{M}_n - \mathbf{M}\right]_{ab}\right| < \varepsilon \quad \forall n > n_\varepsilon$$

Then

$$\begin{aligned}\frac{1}{n}\left|\sum_{j=1]}^{n}\mathbf{M}_j - n\,\mathbf{M}\right| &= \frac{1}{n}\left|\sum_{j=1]}^{n}\left(\mathbf{M}_j - \mathbf{M}\right)\right| = \\ &= \frac{1}{n}\left|\sum_{j=1]}^{n_\varepsilon-1}\left(\mathbf{M}_j - \mathbf{M}\right) + \sum_{j=n_\varepsilon}^{n}\left(\mathbf{M}_j - \mathbf{M}\right)\right| \le \\ &\le \frac{1}{n}\left|\sum_{j=1}^{n_\varepsilon-1}\left(\mathbf{M}_j - \mathbf{M}\right)\right| + \frac{1}{n}\sum_{j=n_\varepsilon}^{n}\left|\mathbf{M}_j - \mathbf{M}\right| \le \\ &\le \frac{A}{n} + \frac{1}{n}\sum_{j=n_\varepsilon}^{n}N^2\,\varepsilon = \frac{A}{n} + \frac{(n - n_\varepsilon + 1)\,N^2\,\varepsilon}{n} \le \frac{A}{n} + B\varepsilon \quad \overset{n\to\infty}{\to} \quad 0\end{aligned}$$

for some positive numbers A and B.

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