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FACOLTA' DI ECONOMIA

Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali

PHD in "Matematica per l'Analisi Dei Mercati Finanziari"

Portfolio allocation under general return distribution

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Abstract

Modern Portfolio theory, developed by Markowitz (1952), is based on finding the best trade-off between risk and expected return. This model assumes that returns are normally distributed. In real life, for the majority of the assets this assumption is not true, as generally the distribution of returns has negative skewness and fat tails. This is more evident in case of hedge funds, commodities or emerging markets portfolios. Therefore, in these cases, a portfolio allocation based on the first two moments does not seem to be the right procedure, because we cannot ignore the higher moments. So, we need to find a way to incorporate the higher moments in the portfolio allocation decision. This is the reason why in this dissertation we will extend the Markowitz model to the higher moments and we will analyze the impact that skewness and kurtosis have on portfolio allocation.

To introduce the higher moments in the portfolio allocation, we will approximate the expected utility by a fourth order Taylor expansion and we will compare the portfolio allocation based on four moments with the portfolio based on the first two moments. To compare two different optimal portfolios we will use a measure called, Monetary Utility Gain/Loss (MUG) .

Furthermore, in the issue of constructing the optimal portfolio allocation, we will consider different approaches for the estimation of the co-moments. We will describe in more details three different approaches:

    i.   Sample approach

   ii.   Constant Correlation approach

  iii.   Shrinkage approach

In the empirical part, we will use a fix-mixed rolling window strategy with different calibrations periods, sample periods and levels of risk aversion.

ACKONOWLEDGMENS

I would like to thank the following.

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Chapter 1

Preface

1.1 Introduction

The mean-variance portfolio model that Markowitz proposed in 1952, is widely used for asset allocation. This model assumes that returns are normally distributed, so that considering only the first moments is sufficient for portfolio selection, as skewness and excess kurtosis are zero. Also, there exists a vast literature arguing that asset returns are not always normally distributed[1]. It has been shown that the empirical distribution of asset return has tails thicker than those of the normal distribution, and negative skewness. This is more evident in case of hedge funds or commodities.

Cont (2001) indicateds some stylized facts, that are characteristic of financial data. The main stylized facts are:

- **Asymmetry**: The unconditional distribution of returns is negatively skewed, meaning that the probability that extreme negative returns will happen is higher than the probability of extreme positive returns.

- **Fat tails**: The unconditional distribution has fatter tails than the normal one. Meaning that we will underestimate the number and magnitude of crashes and booms if we use normal distribution to model financial returns.

- **Clustering of returns**: a large positive (or negative) return tends to be followed by another large (positive or negative) return.

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[1] Mandelbrot (1963) The variation of certain speculative prices. Journal of Business.

Fama. E (1965) The behavior of stock market prices. Journal of Business

Cont (2001) Empirical property of asset return: stylized facts and statistical issues. Quantitative Finance

Bekaert, Erb, Harvey and Viskanta (1998). Distributional Characteristics of Emerging Market returns and Asset Allocation.

- **Heteroskedasticity**[2]: The variance of asset's returns is not constant over time.

- **Time varying cross-correlation**: Correlation between returns of assets tends to increase during high volatility periods, this phenomena is more likely during crashes. That's why during crashes periods it is difficult to diversify the portfolio.

Looking at the stylized market facts, we can say that, the mean-variance model has two drawbacks, i.e. it doesn't consider higher moments and it ignores the dynamic movements of stocks returns. At this point one may wonder:

- What is the impact of higher moments on portfolio allocation?

- Is it better to use a portfolio allocation strategy based only on the first two moments or to construct one based on the first four moments?

- In which situation using two moments is better than using four moments?

In this work we will try to answer to all these questions, considering a port-

folio allocation, in a first step, based on the first two moments and later we will consider one based on the first four moments and we will analyze the differences between these two portfolios allocations.

Let's start our work with a brief review of the literature on this argument.

1.2   Review of literature

Given the fact that asset returns exhibit significant degrees of non-Gaussian behavior, the optimal allocation of asset returns has received a great deal of attention. Several authors have proposed improvements on the traditional model of optimal portfolio selection. Athayde and Flôres (2001 and 2004) focused on the computation of the efficient frontier when several moments are present and they concluded that higher moments are important for asset allocation. They constructed the efficient frontier fixing three moments, for e.g. fixing variance, skewness and kurtosis and they found the optimal portfolio that is the one with the higher mean.

Tibiletti (2006) focused on one side-higher moments, as measures of risk based on one side moments are coherent measures of risk according to Artzner (1999).

Bollerslev (1986) introduced the GARCH model which considers the variance to be time varying and errors normally distributed.

Bollerslev (1987) extended the conditional distribution to be a t-student. Assuming that the distribution is t-student improves the analysis as it capture

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[2]Engle R. F (1982) Auto-regressive conditional heteroskedasticity with estimates of the variance of the United Kingdom inflation. Econometrica.

the fact that conditional distribution of returns has heavy tails, but still it doesn't take into consideration the fact that returns are asymmetric.

Hansen (1994) introduced the skewed-t distribution. This distribution takes account of both skewness and kurtosis.

Premaratne and Bera (1999) introduced Pearson type IV distribution, which is a family of skewed-t distribution.

Guidolin and Timmerman (2002) discussed regime switching under higher order moments. In this model the moments change over time as regimes change.

Jondeau and Rockinger (2002) investigated how an investor will change his portfolio allocation when he cares not only about mean and variance but also about the third and fourth moments. They created a portfolio with five emerging market stocks, used CRRA utility function and the Taylor expansion of expected utility considering moments up to the fourth order. They obtained that for small values of risk-aversion parameter, non-normality does not alter significantly the optimal allocation. In contrast, when the investor is strongly risk averse they obtained significant changes in portfolio weights.

Jondeau and Rockinger (2005) use a four-term Taylor expansion but allow for time dependence distribution.

Lionel Martellini and Volker Ziemann (2009) extended the estimation methods that exist for the covariance matrix to higher order co-moments. These methods are: (i) constant correlation proposed by Elton and Gruber (1973), (ii) single factor proposed by Sharpe (1963), (iii) shrinkage estimator proposed by Ledoit and Wolf (2003).

In this work we will consider discrete models. Following Martellini and Ziemann's approach we will analyze the importance of higher moments on asset allocation and compare the impact that different estimators have on portfolio allocation.

We have chosen a portfolio composed of 13 assets (12 hedge funds indexes and 1 CTA). We will find the optimal weights considering once only the first two moments and then the first four moments and have used different approaches for the estimation of co-moments.

The approaches considered are:

- The mean-variance model using the sample estimation.

- Mean-variance-skewness-kurtosis model using the sample estimation.

- The mean-variance model estimated with constant correlation.

- Mean-variance-skewness-kurtosis model estimated with constant correlation approach.

- Mean-variance using the shrinkage estimators (where the target is covariance matrix estimated using the CC approach).

- Mean-variance-skewness-kurtosis using shrinkage estimators (the target matrices are estimated using the CC approach).

For each approach we will construct the optimal portfolios, considering the in-sample period and observing what the impact is of each estimation in the out of sample period.

Chapter 2

Basic knowledge

In this chapter we will consider some basics of Statistics regarding the definition and the calculation of the moments for a random variable and explain how we can introduce the higher moments in portfolio allocation.

2.1 Definition of the moments for a random variable

The moments of a random variable are used to present the characteristics of a probability distribution. If we denote with R a discrete random variable and let $g(R) = (R - c)^n$. Is called moment of order n from c the quantity

$$E[g(R)] = E[(R - c)^n] = \sum_{i=1}^{N} (R_i - c)^n P(R_i)$$

where , $n = 0, 1, 2, ...$ and c a constant. Particular cases of combinations between c and n are:

- if $n = 0$, for all the values of c, the moments of order zero are all equal to one.

- for $c = 0$, we obtain the moments from the origin or the un-centered moments. These moments are indicated with $_{(n)}\mu' = E[R^n]$. For $n = 1$ we obtain the mean: $_{(1)}\mu' = E[R] = \mu$.

- when $c = \mu$ we obtain the centered moments that we will indicate with $_{(n)}\mu$.

  - for $n = 1$ we obtain: $E(R - \mu) = \sum_{i=1}^{N}(R_i - \mu)^1 P(R_i) = 0$
  - for $n = 2$ we obtain $_{(2)}\mu = E(R - \mu)^2 = var(R) = \sigma^2$. The variance is a measure of dispersion of the variable from its mean.

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- for $n = 3$ we have $_{(3)}\mu = E(R - \mu)^3$, skewness is a measure of asymmetry.

- for $n = 4$ we have $_{(4)}\mu = E(R - \mu)^4$, kurtosis is a measure of the "peakedness" of the probability distribution.

If R is a continuous random variable then [1]:

$$E[g(R)] = \int_{-\infty}^{+\infty} (R_i - c)^n \ f(R) \ d(R)$$

where $f(R)$ is the probability density function.

The standardized skewness and kurtosis are defined as[2]:

$$s = E\left[\left(\frac{R - \mu}{\sigma}\right)^3\right] = \frac{_{(3)}\mu}{\sigma^3} = \frac{_{(3)}\mu}{(\sigma^2)^{3/2}}$$

$$k = E\left[\left(\frac{R - \mu}{\sigma}\right)^4\right] = \frac{_{(4)}\mu}{\sigma^4} = \frac{_{(4)}\mu}{(\sigma^2)^2}$$

For each random variable r_p (e.g it can be the portfolio return) we can write the non central moments,$_{(i)}\mu_p' = E\left[(r_p)^i\right]$ as function of central moments $_{(i)}\mu_p = E\left[(r_p - \mu_p)^i\right]$, in the following way [3]:

$$_{1}\mu_p' = \mu_p$$

$$_{(2)}\mu_p' = \sigma_p^2 + \mu_p^2$$

$$_{(3)}\mu_p' = s_p + 3\sigma_p^2 \ \mu_p + \mu_p^3$$

$$_{(4)}\mu_p' = k_p + 4s_p\mu_p + 6\sigma_p^2 \ \mu_p^2 + \mu_p^4$$

We will see in the following Section that the central moments of a portfolio with N-assets are important in the procedure of portfolio selection.

In portfolio allocation's practice the most known and used model is the one proposed by Markowitz. It is well known that this model is based on the first two moments (mean and variance). Therefore, in this model, to find the optimal weights we need to estimate the $(M_1)_{1\times n}$ (as we will see later, this is a vector containing the mean of each asset that we have in portfolio) and $(M_2)_{n\times n}$ (the co-variance matrix, that has on the diagonal the variance of each asset and in

---

[1] This can be extended as in discrete case.

[2] In portfolio allocation as we will see in the next paragraphs, we don't use the standardized moments but the central ones.

[3] This is true for any random variable. See Appendix A.1 for the proof.

off diagonal the covariances between pairs of assets). When we introduce higher moments in the portfolio allocation we need to estimate not only the first two moments but also co-skewness $(M_3)_{n \times n \times n}$ and co-kurtosis $(M_4)_{n \times n \times n \times n}$.

It looks difficult to consider M_3 and M_4 in portfolio allocation, as these matrices have three and four dimensions, respectively. To simplify our work we use the tensors matrices of co-skewness and co-kurtosis. The use of these matrices allows us to express the co-skewness and co-kurtosis as two dimensional matrices.

In the following section we will explain what tensors matrices are and how we can use them to represent co-skewness and co-kurtosis.

2.2 Tensors Matrix for skewness and kurtosis

Let us consider a portfolio with n risky assets. Following the notation of Jondeau and Rockinger (2006), the (n, n) covariance (M_2), (n, n^2) co-skewness (M_3) and (n, n^3) co-kurtosis (M_4) matrices are defined as:

$$M_2 = E\left[(R - E(R))(R - E(R))'\right] = \{\sigma_{ij}\}$$

$$M_3 = E\left[(R - E(R))(R - E(R))' \otimes (R - E(R))'\right] = \{s_{ijk}\}$$

$$M_4 = E\left[(R - E(R))(R - E(R))' \otimes (R - E(R))' \otimes (R - E(R))'\right] = \{k_{ijkl}\}$$

We have that:

$$\sigma_{ij} = E\left[(R_i - \mu_i)(R_j - \mu_j)\right] \qquad \forall \ i, \ j = 1...n$$

$$s_{ijk} = E\left[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)\right] \qquad \forall \ i, \ j, \ k = 1...n$$

$$k_{ijkl} = E\left[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)\right] \qquad \forall \ i, \ j, \ k, \ l = 1...n$$

where R_i is the individual asset return i, μ is the mean and $\otimes$ is the Kronecker product [4], σ_{ij} is the co-variance between assets i and j, s_{ijk} is what we will call

---

[4]Given two matrices $A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B_{2x3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ their Kronecker product is $A_{2x2} \otimes B_{2x3} = \begin{bmatrix} a_{11} * B & a_{12} * B \\ a_{21} * B & a_{22} * B \end{bmatrix}$

More explicitly we have:

$$K_{4x6} = \begin{bmatrix} a_{11} * b_{11} & a_{11} * b_{12} & a_{11} * b_{13} & a_{12} * b_{11} & a_{12} * b_{12} & a_{12} * b_{13} \\ a_{11} * b_{21} & a_{11} * b_{22} & a_{11} * b_{23} & a_{12} * b_{21} & a_{12} * b_{22} & a_{12} * b_{23} \\ a_{21} * b_{11} & a_{21} * b_{12} & a_{21} * b_{13} & a_{22} * b_{11} & a_{22} * b_{12} & a_{22} * b_{13} \\ a_{21} * b_{21} & a_{21} * b_{22} & a_{21} * b_{23} & a_{22} * b_{21} & a_{22} * b_{22} & a_{22} * b_{23} \end{bmatrix}$$

co-skewness between assets i, j and k and k_{ijkl} is what we will call co-kurtosis between assets i, j, k and l.

This notation allows to remain in the matrix space even when we consider a portfolio with a large number of assets. For example, for $n = 3$ assets, we have the (3×9) co-skewness matrix and (3×27) co-kurtosis matrix, that are respectively:

$$M_3 = \begin{bmatrix} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{bmatrix}$$
$$= [S_{1jk}, \ S_{2jk}, \ S_{3jk}]$$

$$M_4 = \begin{bmatrix} K_{11jk} & K_{12jk} & K_{13jk} & K_{21jk} & K_{22jk} & K_{23jk} & K_{31jk} & K_{32jk} & K_{33jk} \end{bmatrix}$$

S_{1jk} denotes the $n \times n$ matrix with elements $\{s_{1jk}\}_{j, \ k=1,2,3}$ and K_{11kl} denotes $n \times n$ matrices with elements $\{k_{11kl}\}_{k, \ l=1,2,3}$. S_{1jk} and K_{11kl} are sub-matrices and we will call super-diagonal elements for co-skewness and co-kurtosis the elements where the index is the same, s_{iii} and k_{iiii}.

Because of certain symmetries, not all the elements of this matrices have to be computed. For example the covariance matrix is $(n \times n)$ but only $n(n+1)/2$ elements need to be computed. In case of skewness[5] that is a $(n \times n^2)$ matrix and kurtosis that is $(n \times n^3)$ matricx we have to compute respectively $n(n+1)(n+2)/6$ and $n(n+1)(n+2)(n+3)/24$ different elements.

Using the tensors matrices we can calculate the unconditional mean, variance, skewness and kurtosis of a given portfolio as [6]:

$$E(R_p) = E\left[\sum_{i=1}^{N} (w_i \ R_i)\right] = w_p \ \mu'_p$$

$$\sigma^2(R_p) = E\left\{[R_p - E(R_p)]^2\right\}$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N} w_{pi} \ w_{pj} \ \sigma_{ij} = w_p \ M_2 \ w'_p$$

---

[5] Because of symmetry we have for skewness:

$$s_{ijk} = s_{ikj} = s_{jik} = s_{jki} = s_{kij} = s_{kji}$$

and for kurtosis:
$k_{ijkl} = k_{ijlk} = k_{iljk} = k_{ilkj} = k_{ikjl} = k_{iklj} = k_{jikl} = k_{jilk} = k_{jkil} = k_{jkli} = k_{jlik} = k_{jlki} =$
$k_{kijl} = k_{kjli} = k_{kilj} = k_{klij} = k_{kjil} = k_{klji} = k_{lijk} = k_{likj} = k_{ljik} = k_{ljki} = k_{lkij} = k_{lkji}$
[6] In general the n-th moment is given by $w' * M_n * w^{\otimes n-1}$.

$$s^3(R_p) = E\left\{[R_p - E(R_p)]^3\right\}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_{pi}\ w_{pj}\ w_{pk}\ s_{ijk}$$

$$= w_p M_3(w_p' \otimes w_p')$$

$$k^4(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} w_{pi}\ w_{pj}\ w_{pk}\ w_{pl}\ k_{ijkl}$$

$$= w_p M_4(w_p' \otimes w_p' \otimes w_p')$$

where N is the number of assets in portfolio, μ and w_p are vectors with dimension $1 \times N$, $M_{2,t}$ is the covariance matrix with dimension $N \times N$, $M_{3,t}$ is the co-skewness matrix with dimension $N \times N^2$ and $M_{4,t}$ is co-kurtosis matrix with dimension $N \times N^3$.

We have explained till now how we can compute the central moments and that our intention is to introduce the higher moments in the portfolio allocation. But how can we do this? When only two moments are used, to find the optimal portfolio, usually the efficient frontier is constructed and one variable is fixed, for e.g. the variance, and the optimal portfolio is the one with the higher mean, or viceversa. In case of four moments one can still construct the efficient frontier, as in Athyde and Flores (2001), but the approach that we use in our alysis is not this one. We have adopted the approach proposed by Jondeau and Rockinger (2002), that we will explain in the next Section.

2.3 The Investors problem

To introduce the higher moments in portfolio allocation we consider an investor that has a given utility function, $U(W)$. Generally the objective of an investor is to maximize the expected utility function $U(W_T)$ at the end of the period wealth W_T [7]. If we consider a portfolio with N risky assets, return vector $R_t = (R_{1,t}, ..., R_{n,t})$ and weight vector $w = (w_1, ..., w_n)$ where $\sum_{i=1}^{N} w_i = 1$, the investor problem can be written:

$$\begin{cases} \max_{\{w\}} E[U(W_T)] \\ s.t \quad \sum_{i=1}^{N} w_i = 1 \\ \quad\quad w_i \geq 0 \end{cases} \quad (2.1)$$

In this dissertation, following the approach of Jondeau and Rockinger, we approximate the expected utility by the Taylor expansion around the expected

---

[7] As we will see, in our analysis W_T is the wealth at the end of the calibration period T.

wealth .

$$U(W_T) = \sum_{k=0}^{\infty} \left[\frac{U^{(k)}(\bar{W})(W_t - \bar{W})^k}{k!} \right]$$

Focusing on terms up to the fourth (or second) order we obtain the following approximation of the expected utility [8] [9]:

$$
\begin{aligned}
E[U(W)] \quad &\approx \quad U(\bar{W}) + U^1(\bar{W}) \cdot E[W_t - \bar{W}] + \frac{U^2(\bar{W})}{2!} E[W_t - \bar{W}]^2 \\
&+ \frac{U^3(\bar{W})}{3!} E[W_t - \bar{W}]^3 + \frac{U^4(\bar{W})}{4!} E[W_t - \bar{W}]^4
\end{aligned}
$$

$$
\begin{aligned}
E[U(W_t)] \quad &\approx \quad U(\bar{W}) + \frac{U^2(\bar{W})}{2!} \sigma^2(W_t) \qquad\qquad (2.2) \\
&+ \frac{U^3(\bar{W})}{3!} s^3(W_t) + \frac{U^4(\bar{W})}{4!} k^4(W_t)
\end{aligned}
$$

The expected utility can be written as function of the derivatives of the utility function and the moments of the portfolio return distribution.

Using the notation of tensors matrices, explained in the previous Section, we can rewrite the expected utility function as:

$$
\begin{aligned}
E[U(W_t)] \quad &= \quad U(E(\mu \ w')) + \frac{U^2(\mu \ w')}{2} w \ M_2 \ w' + \qquad\qquad (2.3) \\
&+ \frac{U^3(\mu \ w')}{6} w \ M_3 \ (w' \otimes w') + \frac{U^4(\mu \ w')}{24} w \ M_4 \ (w' \otimes w' \otimes w')
\end{aligned}
$$

Let us consider two examples of utility functions, CARA and CRRA, and write the approximate Taylor expansion:

Case 1 *Consider a constant absolute risk aversion utility function (CARA):*

$$U(W_t) = -e^{-\lambda W_t}$$

where λ is the investor's risk aversion coefficient. Using the Taylor expansion the expected utility (in terms of portfolio) can be approximated by[10] [11]*:*

$$E[U(W_t)] \approx -e^{-\lambda(\mu_p)} \left[1 + \frac{1}{2}\lambda^2\sigma_p^2 - \frac{1}{6}\lambda^3 s_p^3 + \frac{1}{24}\lambda^4 k_p^4 \right]$$

---

[8] $E[W_t - \bar{W}] = E[W_t] - \bar{W} = \bar{W} - \bar{W} = 0$.

[9] In this formula skewness and kurtosis are not standardized moments, but are the third and the fourth central moments.

[10] See appendix A.2 for the proof.

[11] This approximation is in terms of moments of the portfolio return:

The optimization problem in this case is [12]:

$$
\begin{cases}
\max -e^{-\lambda(\mu \ w')}\left[1 + \frac{\lambda^2}{2}w \ M_2 \ w' - \frac{\lambda^3}{6}w \ M_3 \ (w' \otimes w') + \frac{\lambda^4}{24}w \ M_4 \ (w' \otimes w' \otimes w')\right] \\
\qquad\qquad s.t \quad \sum_{i=1}^{N} w_i = 1 \\
\qquad\qquad lb \leq w_i \leq ub
\end{cases}
$$

Case 2 *Consider a constant relative risk aversion utility function (CRRA):*

$$
U(W_T) = \begin{cases}
\frac{W_t^{1-\lambda}}{1-\lambda} & if \ \ \lambda > 1 \\
\ln(W_t) & if \ \ \lambda = 1
\end{cases}
$$

the approximated expected utility function is given by[13]:

$$
\begin{aligned}
E\left[U(W_t)\right] \ \ \approx \ \ & \frac{(\mu \ w')^{1-\lambda}}{1-\lambda} - \frac{\lambda}{2}(\mu \ w')^{-\lambda-1}\sigma_p^2 + \\
& + \frac{\lambda(\lambda+1)}{3!}(\mu \ w')^{-\lambda-2}s_p^3 - \frac{\lambda(\lambda+1)(\lambda+2)}{4!}(\mu \ w')^{-\lambda-3}k_p^4
\end{aligned}
$$

The optimization problem in this case is:

$$
\begin{cases}
\max \quad \frac{(\mu \ w')^{1-\lambda}}{1-\lambda} - \frac{\lambda}{2}(\mu \ w')^{-\lambda-1}w \ M_2 \ w' + \frac{\lambda(\lambda+1)}{3!}(\mu \ w')^{-\lambda-2}w \ M_3 \ (w' \otimes w') - \\
\qquad - \frac{\lambda(\lambda+1)(\lambda+2)}{4!}(\mu \ w')^{-\lambda-3}w \ M_4 \ (w' \otimes w' \otimes w') \\
\qquad\qquad s.t \quad \sum_{i=1}^{N} w_i = 1 \\
\qquad\qquad lb \leq w_i \leq ub
\end{cases}
$$

Notice that the first derivatives of the mean, variance, skewness and kurtosis are, respectively [14];

$$
\frac{d \ \mu_p}{d \ w} = \mu
$$

$$
\frac{d \ \sigma_p^2}{d \ w} = 2M_2 w
$$

$$
\frac{d \ s_p^3}{d \ w} = 3M_3(w \otimes w)
$$

$$
\frac{d \ k_p^4}{d \ w} = 4M_4(w \otimes w \otimes w)
$$

In general odd moments are considered as performance measures and even moments are considered as risk measures for a risk averse investor (therefore investors will prefer higher mean, higher positive skewness, lower variance and

---

[12]Where "*lb* and *ub*" are respectively the minimum and the maximum weight that can be invested in a single asset.

[13]See Appendix A.2 for the proof.

[14]In general we have $\frac{d}{dw}(w'M_j w^{\otimes j}) = jM_j w^{\otimes(j-1)}$

lower kurtosis). Since even moments capture the dispersion of the payoffs, the wider the tails of the returns distribution, the higher even moments will be. This is the reason that in finance the utility functions are strictly increasing and concave, because their expected utility increases with odd moments and decreases with even moments.

Chapter 3

Discrete time models for portfolio allocation

In the previous chapter we explained how to calculate the moments of a random variable and the approach that we will adopt, in the empirical part of this work, on how to introduce the higher moments in portfolio allocation decision. In this chapter we will explain the different approaches that we will take into consideration for the estimation of the co-moments of a random variable, when static models are considered.

The approaches that we will consider for the estimation of co-moments are:

a) The well known sample approach,

b) the Constant Correlation approach (CC),

c) the Shrinkage Constant Correlation approach (SCC).

3.1   Estimation of co-moments using different approaches

3.1.1   Estimation of co-moments with Sample approach

As we have seen in previous Chapter, the objective of an investor is to find the optimal weights that maximize the utility function. In case of CARA the optimization problem is [1]:

$$
\begin{cases}
\max -e^{-\lambda(\mu w')} \left[1 + \frac{\lambda^2}{2} w M_2 w' - \frac{\lambda^3}{6} w M_3 (w' \otimes w') + \frac{\lambda^4}{24} w M_4 (w' \otimes w' \otimes w') \right] \\
\qquad s.t \quad \sum_{i=1}^{N} w_i = 1 \\
\qquad\qquad lb \leq w_i \leq ub
\end{cases}
$$

---

[1]In the empirical analysis we will consider $lb = 0$ and $ub = 1$, meaning that we can not short sell.

Therefore to find the optimal weights we need to estimate the moments of first, second, third and fourth order.

The most natural approach for the estimation of the moments of a random variable is the Sample one, where:

The sample estimator of the mean is given by:

$$\mu_i = E[R_i] = \frac{1}{T} \sum_{t=1}^{T} R_{i,t}$$

The sample estimator of co-variance is:

$$\overset{\wedge}{\sigma}_2^{\{ij\}} = \frac{\sum_{t=1}^{T}(R_{it} - E[R_i])\ (R_{jt} - E[R_j])}{(T-1)}$$

The sample estimator of co-skewness is[2]:

$$\hat{s}_{ijk} = \frac{T}{(T-1)\ (T-2)} \sum_{t=1}^{T}(R_{it} - E[R_i])\ (R_{jt} - E[R_j])\ (R_{kt} - E[R_k])$$

The sample estimator of co-kurtosis is:

$$\hat{k}_{ijkl} = \frac{T\ (T-1)}{(T-1)\ (T-2)\ (T-3)} *$$
$$* \sum_{t=1}^{T}(R_{it} - E[R_i])\ (R_{jt} - E[R_j])\ (R_{kt} - E[R_k])\ (R_{lt} - E[R_l])$$

where T is the number of observations in the sample.

It is well known that the sample estimator of historical returns is likely to generate high sampling error [3], this is more apparent when the number of observations in the sample is close to the number of assets or when it is higher. For this reason several methods have been introduced to improve asset return co-moment matrix estimation.

The idea is to impose some structure on the co-variance matrix that reduces the number of parameters. Such "structured" estimators of the co-variance matrix include the constant correlation estimation (Elton and Gruber [1973]), the single-factor estimation (Sharpe [1963]) and the shrinkage approach proposed by Ledoit and Wolf (2003 and 2004). In these approaches sampling error is reduced at the cost of specification error.

In this work we will consider only the CC and the shrinkage toward the CC approaches.

---

[2] For more details see; Ryan J. Davies, Harry M. Kat, Sa Lu "Single strategy funds of hedge funds". January 2004.

[3] This problems, in finance, are referred as estimation risk problems. For more details see: S.J. Brown (1978) and V.s. Bawa, S.J. Brown and R.W. Klein (1979).

3.1.2 Estimation of co-moments using CC approach

Elton and Gruber [1973] proposed the CC approach for the estimation of the co-variance matrix. The idea of this approach is to estimate the co-variance based on the fact that the correlation $\hat{r}$ between two assets is constant for each pair of assets, and is given by the average of all the sample correlation coefficients, $\hat{r}_{ij}$:

$$
\begin{aligned}
\hat{r}^{(1)} &= \frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \hat{r}_{ij} = \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \frac{\hat{S}_{ij}}{\overset{\wedge}{\sigma}_i \overset{\wedge}{\sigma}_j} \\
&= \frac{1}{T\,N\,(N-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \sum_{t=1}^{T} \frac{([R_{i,t} - \mu_i]\,[R_{j,t} - \mu_j])}{\overset{\wedge}{\sigma}_i \overset{\wedge}{\sigma}_j}
\end{aligned}
\tag{3.1}
$$

Therefore, the covariance between two assets, using the CC approach, is calculated by:

$$
\hat{S}_{ij}^{CC} = \overset{\wedge}{\sigma}_i \overset{\wedge}{\sigma}_j \hat{r}^{(1)}
\tag{3.2}
$$

where $\overset{\wedge}{\sigma}_i$ is the sample standard deviation of asset i and $\hat{r}^{(1)}$ is the constant correlation coefficient. This approach decreases the dimensionality problem, as in this case we need to estimate only one correlation coefficient and N standard deviations. As a result the covariance matrix will have on the diagonal elements the sample variances and the out of diagonal elements are $\hat{S}_{ij}^{CC}$.

Martellini and Ziemann (2009) extend the concept of CC to higher order co-moments. They defined all possible combinations of higher order co-moments as:

$$
s_{iij} = E\left[(R_i - \mu_i)^2\,(R_j - \mu_j)\right]
$$

$$
s_{ijk} = E\left[(R_i - \mu_i)\,(R_j - \mu_j)\,(R_k - \mu_k)\right]
$$

$$
k_{iiij} = E\left[(R_i - \mu_i)^3\,(R_j - \mu_j)\right] \qquad \forall\, i \neq j \neq k \neq l
$$

$$
k_{iijj} = E\left[(R_i - \mu_i)^2\,(R_j - \mu_j)^2\right]
$$

$$
k_{iijk} = E\left[(R_i - \mu_i)^2\,(R_j - \mu_j)\,(R_k - \mu_k)\right]
$$

$$
k_{ijkl} = E\left[(R_i - \mu_i)\,(R_j - \mu_j)\,(R_k - \mu_k)\,(R_l - \mu_l)\right]
$$

The sample correlation coefficients are written as [4]:

$$r_{ij}^{(1)} = \frac{E\left[(R_i - \mu_i)(R_j - \mu_j)\right]}{\sqrt{\sigma_i^2 \; \sigma_j^2}}$$

$$r_{ij}^{(2)} = \frac{E\left[(R_i - \mu_i)^2 (R_j - \mu_j)\right]}{\sqrt{_{(4)}\mu_i \; \sigma_j^2}}$$

$$r_{ij}^{(3)} = \frac{E\left[(R_i - \mu_i)^3 (R_j - \mu_j)\right]}{\sqrt{_{(6)}\mu_i \; \sigma_j^2}}$$

$$r_{ijk}^{(4)} = \frac{E\left[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)\right]}{\sqrt{\sigma_k^2 \; E\left[(R_i - \mu_i)^2 (R_j - \mu_j)^2\right]}}$$

$$r_{ij}^{(5)} = \frac{E\left[(R_i - \mu_i)^2 (R_j - \mu_j)^2\right]}{\sqrt{_{(4)}\mu_i \; _{(4)}\mu_j}}$$

$$r_{ijk}^{(6)} = \frac{E\left[(R_i - \mu_i)^2 (R_j - \mu_j)(R_k - \mu_k)\right]}{\sqrt{_{(4)}\mu_i \; E\left[(R_j - \mu_j)^2 (R_k - \mu_k)^2\right]}}$$

$$r_{ijkl}^{(7)} = \frac{E\left[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)\right]}{\sqrt{E\left[(R_i - \mu_i)^2 (R_j - \mu_j)^2\right] \; E\left[(R_k - \mu_k)^2 (R_l - \mu_l)^2\right]}}$$

In the equations above $r^{(1)}$ is the standard correlation coefficient of the covariance matrix that we have seen in the previous paragraph and it indicates the correlation between the return of asset i and the return of asset j.

The interpretation of the other coefficients is:

- $\hat{r}_{ij}^{(2)}$ can be viewed as the relation between the volatility of asset i and the return of asset j. We will use this to estimate the elements, s_{iij}, of the co-skewness matrix.

- $\hat{r}_{ij}^{(3)}$ can be viewed as the relation between skewness of asset i and the return of asset j. We will use this to estimate the elements, k_{iiij}, of the co-kurtosis matrix.

---

[4] Where, $_{(n)}\mu = E((R_i - \mu_i)^n)$, is the sample centered moment of order "n" .

- $\hat{r}_{ijk}^{(4)}$ can be viewed as the relation between the return of asset k and the covariance between asset i and j. This coefficient will be used to estimate the co-skewness elements s_{ijk}.

- $\hat{r}_{ij}^{(5)}$ can be viewed as the relation between the volatility of asset i and volatility of asset j. We will use this for the estimation of co-kurtosis elements, k_{iijj}.

- $\hat{r}_{ijk}^{(6)}$ can be viewed as the relation between the volatility of asset i and the covariance between asset j and k. We will use this coefficient to estimate the elements, k_{iijk}, of the co-kurtosis matrix.

- $\hat{r}_{ijkl}^{(7)}$ can be viewed as the relation between the covariance of asset i and asset j and the covariance of asset k and l. We will use this to estimate the elements, k_{ijkl}, of the co-kurtosis matrix.

From these coefficients Martellini and Ziemann estimate all the constant correlation coefficients that are given by:

$$
\begin{aligned}
\hat{r}_{CC}^{(1)} &= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \hat{r}_{ij}^{(1)} \\
&= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \frac{E\left[(R_{it} - \mu_{it})\,(R_{jt} - \mu_j)\right]}{\sqrt{{}^{(2)}m_i\,\,{}^{(2)}m_j}} \\
&= \frac{1}{T\,N\,(N-1)} \sum_{i \neq j}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)\,(R_{jt} - \mu_j)\right]}{\sqrt{{}^{(2)}m_i\,\,{}^{(2)}m_j}}
\end{aligned}
$$

$$
\begin{aligned}
\hat{r}_{CC}^{(2)} &= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \hat{r}_{ij}^{(2)} \\
&= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \frac{E\left[(R_{it} - \mu_i)^2\,(R_{jt} - \mu_j)\right]}{\sqrt{{}^{(4)}m_i\,\,{}^{(2)}m_j}} \\
&= \frac{1}{T\,N\,(N-1)} \sum_{i \neq j}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)^2\,(R_{jt} - \mu_j)\right]}{\sqrt{{}^{(4)}m_i\,\,{}^{(2)}m_j}}
\end{aligned}
$$

From this formula we can see that $\hat{r}_{CC}^{(2)}$ is the average of all the sample correlation coefficients of skewness where only two indexes are equal: s_{iij}.

$$\hat{r}_{CC}^{(3)} = \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \hat{r}_{ij}^{(3)}$$

$$= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \frac{E\left[(R_{it} - \mu_i)^3 \,(R_{jt} - \mu_j)\right]}{\sqrt{(6)}m_i\,(2)m_j}$$

$$\frac{1}{T\,N\,(N-1)} \sum_{i \neq j}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)^3 \,(R_{jt} - \mu_j)\right]}{\sqrt{(6)}m_i\,(2)m_j}$$

$\hat{r}_{CC}^{(3)}$ is the average of all the sample correlation coefficients of kurtosis where only three indexes are equal: k_{iiij}.

$$\hat{r}_{CC}^{(4)} = \frac{1}{N\,(N-1)\,(N-2)} \sum_{i \neq j \neq k}^{N} r_{ijk}^{(4)}$$

$$= \frac{1}{N\,(N-1)\,(N-2)} \sum_{i \neq j \neq k}^{N} \frac{E\left[(R_{it} - \mu_i)\,(R_{jt} - \mu_j)\,(R_{kt} - \mu_k)\right]}{\sqrt{(2)m_k\,\hat{r}^{(5)}}\sqrt{((4)m_i\,(4)m_j)}}$$

$$= \frac{1}{T\,N\,(N-1)\,(N-2)} \sum_{i \neq j \neq k}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)\,(R_{jt} - \mu_j)\,(R_{kt} - \mu_k)\right]}{\sqrt{(2)m_k\,\hat{r}^{(5)}}\sqrt{((4)m_i\,(4)m_j)}}$$

$\hat{r}_{CC}^{(4)}$ is the average of all the sample correlation coefficients of skewness where the three indexes are different each one from the others s_{ijk}.

$$\hat{r}_{CC}^{(5)} = \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \hat{r}_{ij}^{(5)}$$

$$= \frac{1}{N\,(N-1)} \sum_{i \neq j}^{N} \frac{E\left[(R_{it} - \mu_i)^2 \,(R_{jt} - \mu_j)^2\right]}{\sqrt{(4)}m_i\,(4)m_j}$$

$$= \frac{1}{T\,N\,(N-1)} \sum_{i \neq j}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)^2 \,(R_{jt} - \mu_j)^2\right]}{\sqrt{(4)}m_i\,(4)m_j}$$

$\hat{r}_{CC}^{(5)}$ is the average of all the sample correlation coefficients of kurtosis where only two indexes are present, like the elements of k_{iijj}.

$$\hat{r}_{CC}^{(6)} = \frac{1}{N\ (N-1)\ (N-2)} \sum_{i \neq j \neq k}^{N} \hat{r}_{ijk}^{(6)}$$

$$= \frac{1}{N\ (N-1)\ (N-2)} \sum_{i \neq j \neq k}^{N} \frac{E\left[(R_{it} - \mu_i)^2\ (R_{jt} - \mu_{jt})\ (R_{kt} - \mu_k) \right]}{\sqrt{_{(4)}m_i\ \hat{r}^{(5)}} \sqrt{(_{(4)}m_j\ _{(4)}m_k)}}$$

$$= \frac{1}{T\ N\ (N-1)\ (N-2)} \sum_{i \neq j \neq k}^{N} \sum_{t=1}^{T} \frac{\left[(R_{it} - \mu_i)^2\ (R_{jt} - \mu_j)\ (R_{kt} - \mu_k) \right]}{\sqrt{_{(4)}m_i\ \hat{r}^{(5)}} \sqrt{(_{(4)}m_j\ _{(4)}m_k)}}$$

$\hat{r}_{CC}^{(6)}$ is the average of all the sample correlation coefficients of kurtosis where only three indexes are present, like the elements of k_{iijk}.

$$\hat{r}_{CC}^{(7)} = \frac{1}{N\ (N-1)\ (N-2)\ (N-3)} \sum_{i \neq j \neq k \neq l}^{N} \hat{r}_{ijkl}^{(7)}$$

$$= \frac{1}{T\ N\ (N-1)\ (N-2)\ (N-3)} *$$

$$* \sum_{i \neq j \neq k \neq l}^{N} \sum_{t=1}^{T} \frac{E\left[(R_{it} - \mu_i)\ (R_{jt} - \mu_j)\ (R_{kt} - \mu_k)\ (R_{lt} - \mu_l) \right]}{\sqrt{\hat{r}^{(5)}} \sqrt{(_{(4)}m_i\ _{(4)}m_j)} \hat{r}^{(5)} \sqrt{(_{(4)}m_k\ _{(4)}m_l)}}$$

$\hat{r}_{CC}^{(7)}$ is the average of all the sample correlation coefficients of kurtosis where all the indexes are different, like the elements k_{ijkl}.

In the formulas above, $_{(n)}m_t$ is the n-th centered sample moment of asset i given by: $_{(n)}m = \frac{1}{T} \sum_{t=1}^{T} (R_{it} - \mu_i)^n$.

Keeping the 7 correlation coefficients constant across all assets we can estimate all the elements in M_2, M_3 and M_4 by using sample estimates for the first, second, third, fourth and sixth moment for all asset results:

$$\hat{\sigma}_{ij}^{CC} = \hat{r}_{CC}^{(1)} \sqrt{_{(2)}m_i\ _{(2)}m_j}$$

$$\hat{s}_{iij}^{CC} = \hat{r}_{CC}^{(2)} \sqrt{_{(4)}m_i\ _{(2)}m_j}$$

$$\hat{s}_{ijk}^{CC} = \hat{r}_{CC}^{(4)} \sqrt{_{(2)}m_k\ \hat{r}^{(5)} \sqrt{_{(4)}m_i\ _{(4)}m_j}}$$

$$\hat{k}_{iiij}^{CC} = \hat{r}_{CC}^{(3)} \sqrt{_{(6)}m_i\ _{(2)}m_j} \qquad\qquad \forall\ i \neq j \neq k \neq l$$

$$\overset{\wedge CC}{k_{iijj}} = \overset{\wedge (5)}{r_{CC}}\sqrt{_{(4)}m_i \,_{(4)}m_j}$$

$$\overset{\wedge CC}{k_{iijk}} = \overset{\wedge (6)}{r_{CC}}\sqrt{_{(4)}m_i \overset{\wedge (5)}{r}\sqrt{_{(4)}m_j \,_{(4)}m_k}}$$

$$\overset{\wedge CC}{k_{ijkl}} = \overset{\wedge (7)}{r_{CC}}\sqrt{\overset{\wedge (5)}{r}\sqrt{_{(4)}m_i \,_{(4)}m_j} \overset{\wedge (5)}{r}\sqrt{_{(4)}m_k \,_{(4)}m_l}}$$

This approach reduces the number of parameters needed to be estimated [5]. As we have explained before, the CC co-variance matrix has on the diagonal the sample variance of each asset and on the out of diagonal the covariances estimated using the CC approach. Co-skewness matrix has on the super-diagonal [6] the sample skewness and in the out of sample diagonal the co-skewness elements estimated using the CC approach. The same applied to co-kurtosis matrix.

3.1.3 Estimation of Co-moments using the Shrinkage approach

We have seen in the previous paragraph the CC estimators for the co-matrices. This approach allows for a lower estimation risk due to the assumed structure, but on the other hand involves some misspecification in the artificial structure imposed by this estimator. As we have seen, we have imposed that the co-moments among assets are constant in each calibration period, but this is not a reasonable assumption, as in real life, for e.g. the correlation between asset 1 and asset 2 is not equal to the correlation between asset 1 and asset 3 (or asset 5 and asset 6) and so on. Therefore, the structure that we have imposed in this model reduces the sample risk at the cost of the model risk.

In the attempt to find a trade-off between the sample risk and the model risk, Ledoit and Wolf (2003) introduced in the context of the covariance matrix the asymptotically optimal linear combination of the sample estimator and the structured estimator[7] (constant correlation or single factor) with the weight given by the optimal shrinkage intensity. This approach has been extended by Martellini and Ziemann (2009) to the higher order co-moments.

Let's see how we can estimate the shrinkage intensities for each co-matrix and explain how we can estimate the co-matrices using the shrinkage approach.

In the context of the covariance matrix Ledoit and Wolf (2003) define the misspecification function L of the combined estimators as:

---

[5]For example for $n = 5$, using the sample estimation we have to estimate $\frac{5*6}{2} = 15$ covariances, $\frac{5*6*7}{6} = 35$ co-skewness and $\frac{5*6*7*8}{24} = 70$ co-kurtosis. Using the constant correlation approach the total number of parameters that we have to estimate is $4*n+7 = 4*5+7 = 27$. Using the second approach we can reduce the number of the estimated parameters from 120 to 27.

[6]The superdiagonal elements for co-skewness are all the elements where the indexes is the same, like s_{iii}.

[7]In our case the structured estimator is the Constant Correlation approach.

$$L(k) = ||kF + (1-k)S - \Omega||_F$$

where Ω is the true (unobservable) covariance matrix, S the sample estimator, F the shrinkage target, in our case is the constant correlation co-matrixe, which is a structured estimator for Ω , $||\cdot||_F$ is the Frobenius norm[8], k indicates the shrinkage intensity[9] and is a number between 0 and 1 .

We want to minimize the expected value of the misspecification function L[10], that is the expected *"distance"* between the shrinkage estimator and the true covariance matrix. Therefore the estimated co-variance using the shrinkage approach is given by:

$$\Omega_{shrinkage} = kF + (1-k)S$$

We have estimated in the previous Sections the sample co-matrix "S" and the structured co-matrix "F" that is the co-variance using the CC approach, for each calibration period. What we need to estimate now is the shrinkage intensity k.

The asymptotic estimator for k derived by Ledoit and Wolf (2004) is:[11]

$$k_{(2)} = \frac{1}{T} \frac{\pi_{(2)} - \rho_{(2)}}{\gamma_{(2)}}$$

As we can see from this formula, to estimate the shrinkage intensity we need to estimate before three parameters $\pi_{(2)}$, $\rho_{(2)}$ and $\gamma_{(2)}$.

Let explain, in the following, what each of this parameters indicate and how we can estimate these.

Ledoit and Wolf show that $\overset{\wedge\{ij\}}{\pi_2}$ represents the sum of the asymptotic variances of the sample estimator and is given by:

$$\pi_{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} AsyVar[\sqrt{T}s_{ij}] = \sum_{i=1}^{N}\sum_{j=1}^{N} \overset{\wedge\{ij\}}{\pi_2}$$

---

[8]The Frobenius norm of a matrix $A_{m \times n}$ is given by:

$$||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$$

[9]It measures the weight that is given to the structured estimator.

[10]$E(L(k)) = E(||k^*F + (1-k^*)S - \Omega||^2)$

[11]Ledoit and Wolf (2003) prove that the optimal value of the shrinkage intensity, under the assumption that the number of assets,N, is constant while T tends to infinity, asymptotically behaves like a constant over T, given by $k_{(2)} = \frac{1}{T}\frac{\pi_{(2)} - \rho_{(2)}}{\gamma_{(2)}}$.

$$\overset{\wedge\{ij\}}{\pi_2} = \frac{1}{T}\sum_{t=1}^{T}[s_{ij,t}-s_{ij}]^2$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left[(R_{it}-m_i)\ (R_{jt}-m_j)-M_2^{\{ij\}}\right]^2$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left[(R_{it}-m_i)^2\ (R_{jt}-m_j)^2-2(R_{it}-m_i)\ (R_{jt}-m_j)\ M_2^{\{ij\}}+\left(M_2^{\{ij\}}\right)^2\right]$$

$$= \frac{1}{T}\sum_{t=1}^{T}(R_{it}-m_i)^2\ (R_{jt}-m_j)^2-2\ M_2^{\{ij\}}\frac{1}{T}\sum_{t=1}^{T}(R_{it}-m_i)\ (R_{jt}-m_j)+\left(M_2^{\{ij\}}\right)^2$$

where $s_{ij}=\frac{1}{T}\sum_{t=1}^{T}s_{ij,t}$.

The $\rho_2^{\{ij\}}$ parameter represents the sum of the asymptotic covariances between the sample and the structured estimator. Ledoit and Wolf (2003, 2004) derive an explicit formula for consistent estimators of $\overset{\wedge\{ij\}}{\rho_2}$ when Ω is the covariance matrix and F is the constant correlation estimator.

$$\rho_{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N}AsyCov\left[\sqrt{T}f_{ij},\sqrt{T}s_{ij}\right]=\sum_{i=1}^{N}\sum_{j=1}^{N}\overset{\wedge\{ij\}}{\rho_2}$$

We can write this as:

$$\rho_{(2)} = \sum_{(i=j)=1}^{N}AsyVar\left[\sqrt{T}\ _{(2)}m_i\right]+\sum_{i\neq j}^{N}AsyCov\left[\sqrt{T}r^{(1)}\ \sqrt{_{(2)}m_i\ _{(2)}m_j},\sqrt{T}M_2^{\{ij\}}\right]$$

The first term indicates that the diagonal elements of $\overset{\wedge\{ij\}}{\rho_{(2)}}$ are the diagonal elements of $\overset{\wedge\{ij\}}{\pi_{(2)}}$ matrix ($\rho_{(2)}^{\{ii\}}=\pi_{(2)}^{\{ii\}}$). The second term represents the off-diagonal elements. This term is estimated by:

$$AsyCov\left[\sqrt{T}r^{(1)}\ \sqrt{_{(2)}m_i\ _{(2)}m_j},\sqrt{T}s_{ij}\right]$$

$$= \frac{r^{(1)}}{2}\left[\sqrt{\frac{_{(2)}m_j}{_{(2)}m_i}}AsyCov(\sqrt{T}_{(2)}m_i,\sqrt{T}M_2^{\{ij\}})+\right.$$

$$\left.+\sqrt{\frac{_{(2)}m_i}{_{(2)}mj}}AsyCov(\sqrt{T}_{(2)}m_j,\sqrt{T}M_2^{\{ij\}})\right]$$

Where

$$AsyCov(\sqrt{T}_{(2)}m_i,\sqrt{T}s_{ij})=\frac{1}{T}\sum_{t=1}^{T}\left\{(R_{it}-m_i)^2-_{(2)}m_i\right\}\left\{(R_{it}-m_i)\ (R_{jt}-m_j)-M_2^{\{ij\}}\right\}$$

and

$$AsyCov(\sqrt{T}_{(2)}m_j, \sqrt{T}s_{ij}) = \frac{1}{T}\sum_{t=1}^{T}\left\{(R_{jt}-m_j)^2 -_{(2)}m_j\right\}\left\{(R_{it}-m_i)\,(R_{jt}-m_j)-M_2^{\{ij\}}\right\}$$

$\overset{\wedge}{\gamma}_2$ denotes the squared error of the structured estimator and a consistent estimator is:

$$\gamma_{(2)} = \sum_{i=1}^{N}\sum_{j=12}^{N}\overset{\wedge}{\gamma}_2^{\{ij\}}$$

In Ledoit and Wolf (2003, 2004), $\overset{\wedge}{\gamma}_{(2)}$, it is given by:

$$\overset{\wedge}{\gamma}_{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\hat{F}_{ij}-M_2^{\{ij\}}\right)^2$$

Ledoit and Wolf (2003) propose an optimal estimator for the shrinkage intensity given by: $k_{(2)}^* = max\left(0,\;min\left(1,\;\frac{1}{T}\frac{\pi_{(2)}-\rho_{(2)}}{\gamma_{(2)}}\right)\right)$.

Martellini and Ziemann (2009) have developed the shrinkage estimator for higher order moment tensors.

The shrinkage estimator for co-skewness

$$k_{(3)}^* = \frac{1}{T}\frac{\pi_{(3)}-\rho_{(3)}}{\gamma_{(3)}}$$

Where, $\pi_{(3)}$ is the sum of all the asymptotic variances of the sample skewness and is defined by:

$$\begin{aligned}\pi_{(3)} &= \sum_{i,j,k=1}^{N}AsyVar[\sqrt{T}s_{ijk}] = \sum_{i=1,j=1,k=1}^{N}\overset{\wedge}{\pi}_3^{\{ijk\}}\\ &= \frac{1}{T}\sum_{i,j,k=1}^{N}\sum_{t=1}^{T}\left[(R_{it}-m_i)\,(R_{jt}-m_j)\,(R_{kt}-m_k)-M_3^{\{ijk\}}\right]^2\end{aligned}$$

The squared error of the structured estimator $(\gamma_{(3)})$ with the sample estimator is given by:

$$\gamma_{(3)} = \sum_{i,j,k=1}^{N}\overset{\wedge}{\gamma}_3^{\{ijk\}} = \sum_{i,j,k=1}^{N}\left(\hat{f}_{ijk}-M_3^{\{ijk\}}\right)^2$$

In the case of co-skewness matrixe we have to distinguish three different cases for $\rho_{(3)}^{ijk}$. This cases are:

a) $\overset{\wedge}{\rho}_{iii}$ where all the three indexes are equal,

b) $\overset{\wedge}{\rho}_{ijj}$ where only two indexes are present,

c) $\overset{\wedge}{\rho}_{ijk}$ where the three indexes are different,

Therefore, we can express $\rho_{(3)}$ as:

$$
\begin{aligned}
\rho_{(3)} &= \sum_{i,j,k=1}^{N} \overset{\wedge\{ijk\}}{\rho_3} \\
&= \sum_{\substack{i=j=k}}^{N} \overset{\wedge}{\rho}_{iii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \overset{\wedge}{\rho}_{ijj} + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{N} \overset{\wedge}{\rho}_{ijk}
\end{aligned}
$$

Note that the first sum indicates the elements of the superdiagonal, $\overset{\wedge}{\rho}_{iii}$, that are given by $\overset{\wedge}{\pi}_{iii}$.

Martellini and Ziemann (2009) found that an estimator for the other elements is given by:

$$
\overset{\wedge}{\rho}_{ijj} = \frac{\hat{r}^{(2)}}{2}\left[\sqrt{\frac{m_j^{(4)}}{m_i^{(2)}}} AsyCov\left(\sqrt{T}m_i^{(2)}, \sqrt{T}s_{ijj}\right) + \sqrt{\frac{m_i^{(2)}}{m_j^{(4)}}} AsyCov\left(\sqrt{T}m_j^{(4)}, \sqrt{T}s_{ijj}\right) \right]
$$

$$
\begin{aligned}
\overset{\wedge}{\rho}_{ijk} &= \left(\frac{\hat{r}^{(4)}}{4} \sqrt{m_k^{(2)}\hat{r}^{(5)}\sqrt{\frac{m_j^{(4)}}{\left(m_i^{(4)}\right)^3}}}\right) AsyCov\left(\sqrt{T}m_i^{(4)}, \sqrt{T}s_{ijk}\right) \\
&+ \left(\frac{\hat{r}^{(4)}}{4} \sqrt{m_k^{(2)}\hat{r}^{(5)}\sqrt{\frac{m_i^{(4)}}{\left(m_j^{(4)}\right)^3}}}\right) AsyCov\left(\sqrt{T}m_j^{(4)}, \sqrt{T}s_{ijk}\right) \\
&+ \left(\frac{\hat{r}^{(4)}}{2} \sqrt{\frac{\hat{r}^{(5)}}{m_k^{(2)}}\sqrt{m_i^{(4)}m_j^{(4)}}}\right) AsyCov\left(\sqrt{T}m_k^{(2)}, \sqrt{T}s_{ijk}\right)
\end{aligned}
$$

Where a consistent estimator for the above asymptotic covariance term is:

$$
\begin{aligned}
&AsyCov\left(\sqrt{T}m_i^{(n)}, \sqrt{T}s_{ijk}\right) = \\
&= \frac{1}{T}\sum_{t=1}^{T}\left[(R_{it}-m_i)^n - m_i^n\right] * \left[(R_{it}-m_i)(R_{jt}-m_j)(R_{kt}-m_k) - M_3^{\{ijk\}}\right]
\end{aligned}
$$

The shrinkage estimator for co-kurtosis matrix

As in the case of the co-variance and co-skewness the shrinkage intensity is given by:

$$k^*_{(4)} = \frac{1}{T} \frac{\pi_{(4)} - \rho_{(4)}}{\gamma_{(4)}}$$

Following the same logic as in the previous cases we have:

$$\pi_{(4)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \overset{\wedge}{\pi}_{ijkl}$$

where the asymptotic variance of the sample kurtosis is given by:

$$\overset{\wedge}{\pi}_{ijkl} = \frac{1}{T} \sum_{t=1}^{T} \left[(R_{it} - m_i)(R_{jt} - m_j)(R_{kt} - m_k)(R_{kt} - m_k) - M_4^{\{ijkl\}} \right]^2$$

The squared error between the sample kurtosis and CC kurtosis is given by:

$$\gamma_{(4)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \overset{\wedge}{\gamma}_{ijkl}$$

where each single element is:

$$\overset{\wedge}{\gamma}_{ijkl} = \left(\hat{f}_{ijkl} - M_4^{\{ijkl\}} \right)^2$$

In the case of the asymptotic covariances between the sample and the CC co-kurtosis we have to distinguish between six different cases. Therefore, $\rho_{(4)}$ is given by [12]:

$$
\begin{aligned}
\rho_{(4)} \quad &= \quad \sum_{i=1,j=1,k=1,l=1}^{N} \overset{\wedge}{\rho}_{ijkl} \\
&= \quad \sum_{i=j=k=l}^{N} \overset{\wedge}{\rho}_{iiii} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \overset{\wedge}{\rho}_{ijjj} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \overset{\wedge}{\rho}_{iijj} + \\
&\quad + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{N} \overset{\wedge}{\rho}_{iijk} + \sum_{\substack{i,j,k,l=1 \\ i \neq j \neq k \neq l}}^{N} \overset{\wedge}{\rho}_{ijkl}
\end{aligned}
$$

---

[12]Where $\overset{\wedge}{\rho}_{iiii}$ is a matrix with dimension (13×13^3) that has all the elements out of the super-diagonal equal to zero. $\overset{\wedge}{\rho}_{ijjj}$ is a matrix with the same dimension that has a value assigned in each element where three index are equal $(ijjj, jijj, jjij, jjji$,these are all the possible combinations) and all the other elements are zero. The same structure holds for the other coefficients $\overset{\wedge}{\rho}_{iijj}$, $\overset{\wedge}{\rho}_{iijk}$, $\overset{\wedge}{\rho}_{ijkl}$.

As in the case of co-skewness and co-variance the elements on the super diagonal are $\overset{\wedge}{\rho}_{iiii} = \overset{\wedge}{\pi}_{iiii}$, and the elements off superdiagonal are given by:

$$
\overset{\wedge}{\rho}_{ijjj} = \frac{\hat{r}^{(3)}}{2} \left[\sqrt{\frac{(6)m_j}{(2)m_i}} AsyCov\left(\sqrt{T}_{(2)}m_i, \sqrt{T}s_{ijjj}\right) \right.
$$
$$
\left. + \sqrt{\frac{(2)m_i}{(6)m_j}} AsyCov\left(\sqrt{T}_{(6)}m_j, \sqrt{T}s_{ijjj}\right) \right]
$$

$$
\overset{\wedge}{\rho}_{iijj} = \frac{\hat{r}^{(5)}}{2} \left[\sqrt{\frac{(4)m_j}{(4)m_i}} AsyCov\left(\sqrt{T}_{(4)}m_i, \sqrt{T}s_{iijj}\right) \right.
$$
$$
\left. + \sqrt{\frac{(4)m_i}{(4)m_j}} AsyCov\left(\sqrt{T}_{(4)}m_j, \sqrt{T}s_{iijj}\right) \right]
$$

$$
\overset{\wedge}{\rho}_{iijk} = \frac{\hat{r}^{(6)}}{2} \left(\sqrt{\frac{\hat{r}^{(5)}}{(4)m_i}} \sqrt{(4)m_j \, (4)m_k} * AsyCov(\sqrt{T}_{(4)}m_i, \sqrt{T}s_{iijk}) \right)
$$
$$
+ \frac{\hat{r}^{(6)}}{4} \left(\sqrt{\hat{r}^{(5)} \, (4)m_i \sqrt{\frac{(4)m_k}{\left((4)m_j\right)^3}}} * AsyCov(\sqrt{T}_{(4)}m_j, \sqrt{T}s_{iijk}) \right)
$$
$$
+ \frac{\hat{r}^{(6)}}{4} \left(\sqrt{\hat{r}^{(5)} \, (4)m_i \sqrt{\frac{(4)m_j}{\left((4)m_k\right)^3}}} * AsyCov(\sqrt{T}_{(4)}m_k, \sqrt{T}s_{iijk}) \right)
$$

$$
\overset{\wedge}{\rho}_{ijkl} = \left(\frac{\hat{r}^{(7)}}{4} \sqrt{\hat{r}^{(5)} \sqrt{\frac{(4)m_j}{\left((4)m_i\right)^3}} \hat{r}^{(5)} \sqrt{(4)m_k \, (4)m_l}} \right) AsyCov(\sqrt{T}_{(4)}m_i, \sqrt{T}s_{ijkl})
$$
$$
+ \left(\frac{\hat{r}^{(7)}}{4} \sqrt{\hat{r}^{(5)} \sqrt{\frac{(4)m_i}{\left((4)m_j\right)^3}} \hat{r}^{(5)} \sqrt{(4)m_k \, (4)m_l}} \right) AsyCov(\sqrt{T}_{(4)}m_j, \sqrt{T}s_{ijkl})
$$
$$
+ \left(\frac{\hat{r}^{(7)}}{4} \sqrt{\hat{r}^{(5)} \sqrt{(4)m_i \, (4)m_j} \hat{r}^{(5)} \sqrt{\frac{(4)m_l}{\left((4)m_k\right)^3}}} \right) AsyCov(\sqrt{T}_{(4)}m_k, \sqrt{T}s_{ijkl})
$$
$$
+ \left(\frac{\hat{r}^{(7)}}{4} \sqrt{\hat{r}^{(5)} \sqrt{(4)m_i \, (4)m_j} \hat{r}^{(5)} \sqrt{\frac{(4)m_k}{\left((4)m_l\right)^3}}} \right) AsyCov(\sqrt{T}_{(4)}m_l, \sqrt{T}s_{ijkl})
$$

Where:

$$AsyCov\left(\sqrt{T}m_i^{(n)}, \sqrt{T}s_{ijkl}\right)$$

$$= \frac{1}{T}\sum_{t=1}^{T}[(R_{it} - m_i)^n - m_i^n] * [(R_{it} - m_i)(R_{jt} - m_j)(R_{kt} - m_k)(R_{lt} - m_l) - s_{ijkl}]$$

We define three shrinkage estimators towards the constant correlation estimate, that is for each moment tensors matrix.

$$\overset{\lambda}{k}_2 = \frac{1}{T}\frac{\overset{\lambda}{\pi}_2 - \overset{\lambda}{\rho}_2}{\overset{\lambda}{\gamma}_2} \qquad \overset{\lambda}{k}_3 = \frac{1}{T}\frac{\overset{\lambda}{\pi}_3 - \overset{\lambda}{\rho}_3}{\overset{\lambda}{\gamma}_3} \qquad \overset{\lambda}{k}_4 = \frac{1}{T}\frac{\overset{\lambda}{\pi}_4 - \overset{\lambda}{\rho}_4}{\overset{\lambda}{\gamma}_4}$$

The optimal shrinkage estimators are:

$$\overset{\lambda}{k}_2^* = max\left(0, \ min\left(1, \overset{\lambda}{k}_2\right)\right)$$

$$\overset{\lambda}{k}_3^* = max\left(0, \ min\left(1, \overset{\lambda}{k}_3\right)\right)$$

$$\overset{\lambda}{k}_4^* = max\left(0, \ min\left(1, \overset{\lambda}{k}_4\right)\right)$$

After the estimation of shrinkage factors, we can estimate the co-moments using the "shrinkage approach toward the constant correlation through the following formula[13]:

$$C\_M_{shrinkage} = k^* C\_M_{CC} + (1 - k^*)C\_M_{SAMPLE}$$

---

[13] Where C_M indicate the co-moment matrix, and the subscript, CC or SAMPLE, indicate the approach used for the estimation of the co-moment.

Chapter 4

Introducing time dependency in the covariance matrix

In this chapter we will introduce briefly the dynamic model that is related with the GARCH models. To use GARCH model we need to assume a given distribution for the errors. In this chapter we will assume that the errors are normally distributed (this is the easier case) and explain the GARCH (1,1) model.

4.0.4 Introduction to GARCH models

The returns of financial instruments are generally uncorrelated but not independent. Usually this dependence can be seen in the variance of returns that, conditional on the information available, change over time. This feature gives rise to the birth of conditional and heteroskedasticity models. The time dependency of the volatility is usually modeled using a GARCH model and assuming that the errors are normally distributed.

The structure of a volatility model can be described as [1] [2],

$$r_t = \mu_t(\theta) + \varepsilon_t \tag{4.1}$$

$$\varepsilon_t = \sigma_t(\theta) \ z_t \tag{4.2}$$

where the conditional mean is:

$$\mu_t(\theta) = E[r_t | \mathcal{F}_{t-1}]$$

---

[1] The return r_t is decomposed into a conditional mean $\mu_t(\theta)$ (with parameter vector θ) and the conditional error ε_t.

[2] Innovation is given by the product of conditional variance and the residuals.

The conditional variance is:

$$
\begin{aligned}
\sigma_t^2(\theta) &= E[(r_t - \mu_t(\theta))^2 | F_{t-1}] \\
&= E[\varepsilon_t^2 | F_{t-1}]
\end{aligned}
$$

F_t is the information set available at time t and θ is a vector of unknown parameters. Using the conditional variance we obtain the standardized error $z_t = \frac{(r_t - \mu_t(\theta))}{\sigma_t(\theta)} = \frac{\varepsilon_t}{\sigma_t(\theta)}$. The variable z_t has mean zero and variance 1. In this model it is assumed that z_t are normally distributed and are independent for all "t", so that $\varepsilon_t = \sigma_t(\theta) * z_t \sim N(0, \sigma_t^2)$.

In general the conditional variance at time "t", using a GARCH (p,q) model, is given by:

$$
E[\varepsilon_t^2 | F_{t-1}] = \sigma_t^2 = \gamma + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{4.3}
$$

In words GARCH (p,q) consists of three terms:

- γ - the weighted long run variance

- $\sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2$ - the moving average term, that is the sum of the p previous lags of squared-innovations times the assigned weights α_i for each lagged squared innovation. $i = 1, ... p$

- $\sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$ - the autoregressive term, which is the sum of the q previous lagged variance times the assigned weight β_j for each lagged variance $j = 1, ..., q$.

For this model to be well defined and the variance to be positive, the parameters must satisfy the following constrains: $\gamma > 0$, $\alpha_i \geq 0$, for $i = 1, ..., p$, $\beta_j \geq 0$ for $j = 1, ..., q$ and $\left(\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j \right) < 1$[3].

The unconditional variance for this model is:

$$
\sigma^2 = \frac{\gamma}{1 - \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j} \tag{4.4}
$$

4.1 GARCH(1,1)

The simplest GARCH model is the GARCH(1,1), given by:

$$
\sigma_t^2 = \gamma + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2
$$

Iterating backward substitution of σ_t^2 to time $t - J$ yields the alternative expression of the $GARCH(1, 1)$

---

[3]This is called, the stationarity condition.

$$\sigma_t^2 = \gamma(1 + \beta + \beta^2 + ... + \beta^{J-1}) + \alpha \sum_{k=1}^{J} \beta^{k-1} \varepsilon_{t-k}^2 + \beta^J \sigma_{t-J}^2$$

$$= \gamma \frac{1 - \beta^J}{1 - \beta} + \alpha \sum_{k=1}^{J} \beta^{k-1} \varepsilon_{t-k}^2 + \beta^J \sigma_{t-J}^2$$

Estimation

To estimate the GARCH (p, q) model the Maximum log likelihood method is used. Let $\{r_1, ..., r_t\}$ be the time series of returns and the errors be normally distributed and i.i.d. The joint density function for all observations is written as:

$$f(\varepsilon_1, ..., \varepsilon_T | \theta) = f_\theta(\varepsilon_T | F_{T-1}) * f_\theta(\varepsilon_{T-1} | F_{T-2}) * ... * f_\theta(\varepsilon_1 | F_{T-T})$$

$$= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} exp\left(-\frac{\varepsilon_t^2}{2\sigma_t^2}\right)$$

where $\theta = (\gamma, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q)'$ is the vector of unknown parameters.

The ML estimator is obtained by maximizing this expression or, equivalently the log-likelihood function:

$$L_T(\theta | r_t, t = 1, ..., T) = \sum_{t=1}^{T} l_t(\theta)$$

$$= \sum_{t=1}^{T} \left(-\frac{1}{2}log(2\pi) - \frac{1}{2}log \ \sigma_t^2(\theta) - \frac{\varepsilon_t^2}{2\sigma_t^2(\theta)}\right)$$

$$= -\frac{1}{2}\left(T \ log(2\pi) + \sum_{t=1}^{T} \left(log \ \sigma_t^2(\theta) + \frac{\varepsilon_t^2}{\sigma_t^2(\theta)}\right)\right)$$

where $l_t(\theta) = -\frac{1}{2}log(2\pi) - \frac{1}{2}log \ \sigma_t^2(\theta) - \frac{\varepsilon_t^2}{2\sigma_t^2(\theta)}$, is the log-likelihood of the observation t, with $\varepsilon_t = r_t - \mu_t(\theta)$.

As the term $-\frac{1}{2}log(2\pi)$ is constant this problem can be written as;

$$\begin{cases} max \left(-\frac{1}{2}\sum_{t=1}^{T} log \ \sigma_t^2(\theta) - \frac{1}{2}\sum_{t=1}^{T} \frac{\varepsilon_t^2}{\sigma_t^2(\theta)}\right) \\ s.t \\ \sigma_t^2 = \gamma + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \\ \sum_{i=1}^{max(p,q)} \alpha_i + \beta_i \leq 1 \\ \gamma \geq 0 \\ \alpha_i \geq 0 \quad \forall i = 1, ..., p \\ \beta_i \geq 0 \quad \forall i = 1, ..., q \end{cases} \quad (4.5a)$$

We need to find θ that maximizes ML.

Analyzing a portfolio of assets we need to estimate a multivariate GARCH. There are different multivariate GARCH models, like, "VECH GARCH", "Diagonal Vech", "BEKK", "CCC", "DCC" etc.

We will not extend the dynamic models in this work as this models are left for future research.

Chapter 5

Data, Methodology and Results for Static Models

5.1 Introduction

In this chapter we perform the empirical analysis of what we have explained in the theoretical part. To analyze the impact of the different methods of estimation for the co-moments we have imposed the sample mean equal to zero. This is because the sample mean is characterized by a high sample error and in this first analysis we want to neutralize the impact of the expected return estimates on our results.

In the next Section we will use the shrinkage approach to estimate the mean, proposed by Jorion (1986, 1985), the idea of this estimator is to shrink the individual means toward the grand mean.

In the optimization part, we use the CARA utility function and in this case imposing the mean equal to zero the optimization problem is:

$$\begin{cases} max - \left[\frac{\lambda^2}{2} w_t M_{2,t} w_t' - \frac{\lambda^3}{6} w_t M_{3,t}(w_t' \otimes w_t') + \frac{\lambda^4}{24} w_t M_{4,t}(w_t' \otimes w_t' \otimes w_t') \right] \\ s.t \quad \sum_{i=1}^{N} w_{i,t} = 1 \\ lb \leq w_{i,t} \leq ub \end{cases}$$

5.2   Graphical analysis of the data

The used data consists of monthly returns of EDHEC Hedge Funds time series, on a period ranging from January 1997 to August 2009, representing 152 observations. The following figures indicate the return series, the histogram and the normal probability plot[1] for each fund.
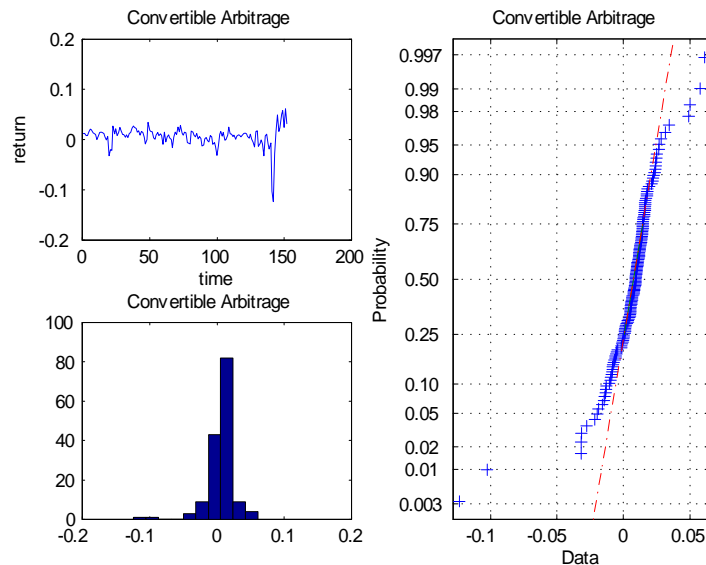
figure 1

From the graph of returns we can observe that the returns of Convertible Arbitrage fund are mean reverting and that there are no large fluctuations except the period ranging between the observations [137, 142]. Period during which the return of this fund decreases sharply from a value of 0.0107 (this is the return of observation number 137 and corresponds to the date '01/05/2008') to a value of -0,1237 (this is the return of observation number 143, '01/10/2008'). After this period the return starts increasing in value. The period from '01/05/2008' to '01/10/2008' corresponds to the last financial crisis.

There are different factors that caused the financial crisis but one of these was the Hedge Fund's crisis (Madoff fraud).

Observing the normal probability plot we can say that the returns of this fund are not normally distributed, because the empirical probability points (the bluepoints) do not fall near the normal probability line, the red one. It seems

---

[1]A normal probability plot is a useful graph for assessing whether data comes from a normal distribution.

that this distribution has fatter tails compared to the normal one. We can see this better in the histogram, it is not symmetric but has a fatter tail on the left, meaning that it is negatively skewed.
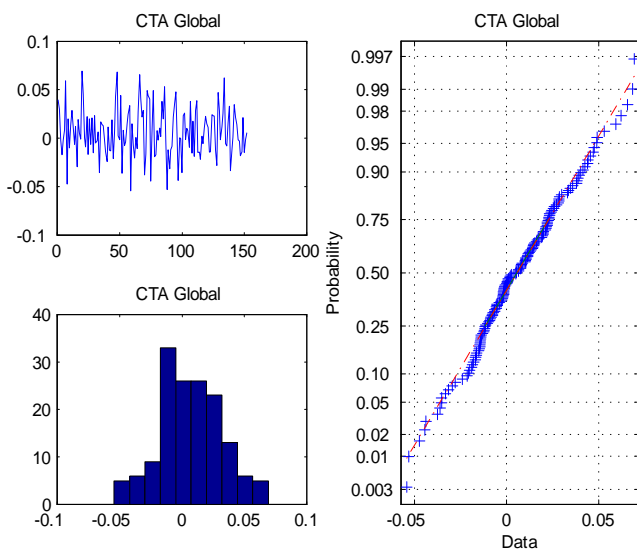
Figure 2

Observing the graphs of CTA Global fund we can say that the return distribution looks like the normal one, since the empirical probability points fall quite close to the normal probability line. Looking at the histogram, it seems that this is not symmetric and that has fatter tail on the right, meaning that it is positively skewed. The Hedge Fund crisis of 2008 did not have a big negative impact as in the previous case. By contrast, during the month of October 2008, while 'Convertible Arbitrage' was losing $12,37\%$ 'CTA Global' was scoring $3,45\%$.
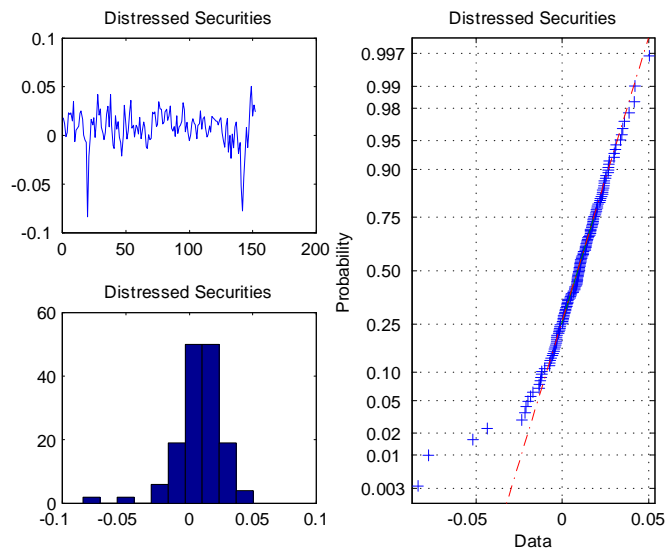
Figure 3

It is clear that Distress Securities return distribution is far from the normal one, as the empirical probability plot is not close to the normal probability plot and the histogram is not symmetric but has a fatter tail on the left, meaning that the distribution of returns is negatively skewed. From the graph of returns we can observe, as for the first fund, that during the Hedge fund crisis the returns fell down dramatically. But this is not the only interval where the returns fell down, there is a previous one from '01/02/1998' to '01/07/1998' where the returns fell down from a value of $0,0252$ to a value of $-0,0836$. This period is related with the Hedge Funds crisis of 1998, Long Term Capital Management bankruptcy.
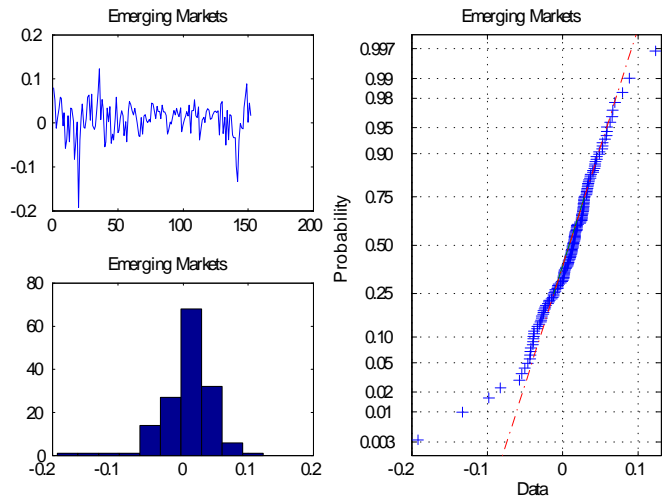
Figure 4

Emerging Markets returns are not normally distributed and are negatively skewed, having fatter tail on the left. The effect of the two crisis of Hedge Funds (1998 and 2008) is evident as in these periods the returns of this fund fell down.
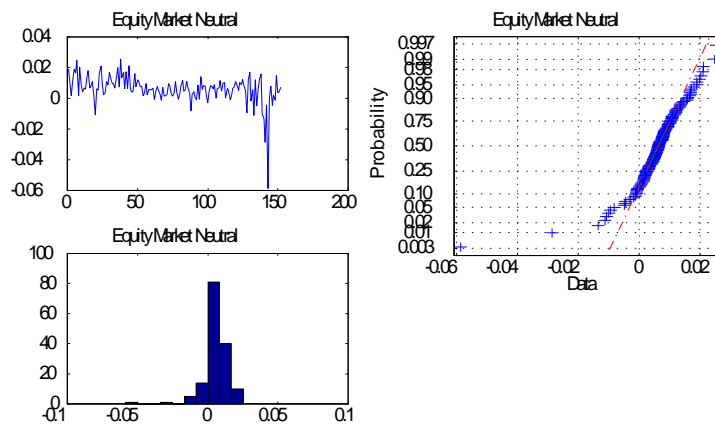
Figure 5

Equity Market Neutral fund is not normally distributed and is negatively skewed. The return of this fund fell down dramatically during the second Hedge Fund crisis, 2008.
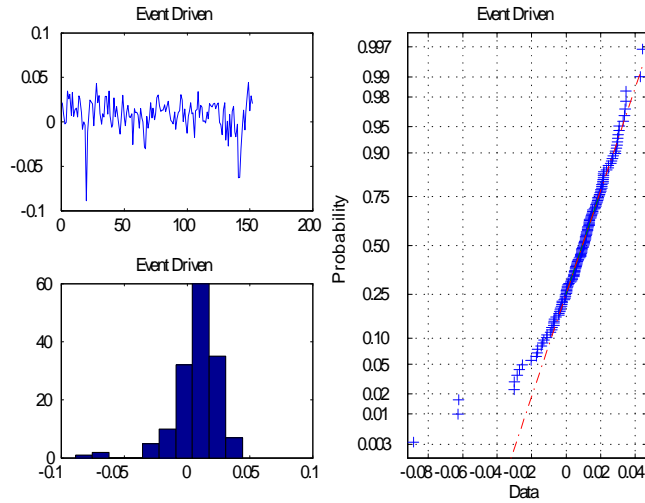
Figure 6
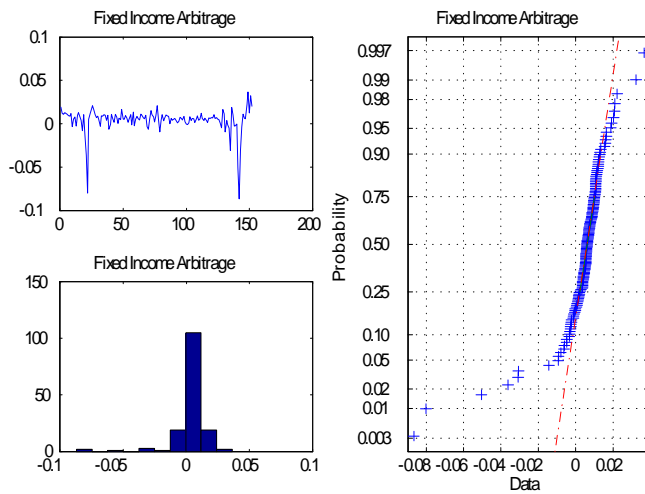
Figure 7

Event Driven and Fixed Income returns are not normally distributed and have, both, fatter tail on the left. The two Hedge Funds crisis had a big impact on the returns of these funds as during this period the return of each fund fell down and the volatility during this period was higher compared to other periods.
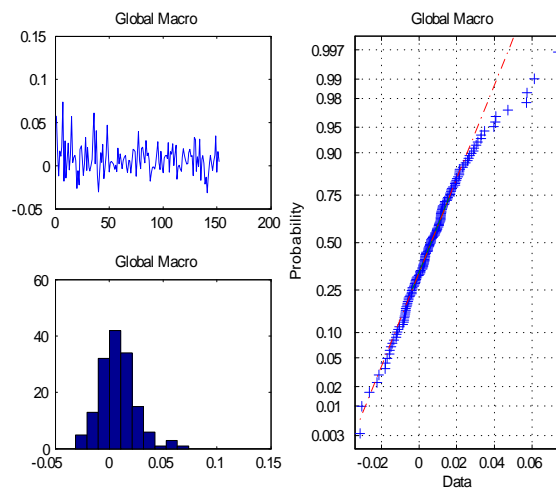
Figure 8

Global Macro returns seem to be not normally distributed and positively skewed. It seems that the Hedge Funds crisis of 2008 had not a big negative impact compared to Convertible Arbitrage or Fixed Income. As for this fund, during this period the return decreased by a maximum value of $3,13\%$ (attained on '01/09/2008').
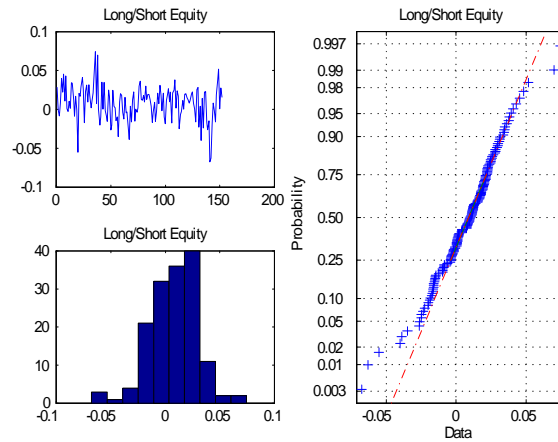
Figure 9

Long Short Equity returns seem to be not normally distributed and negatively skewed.
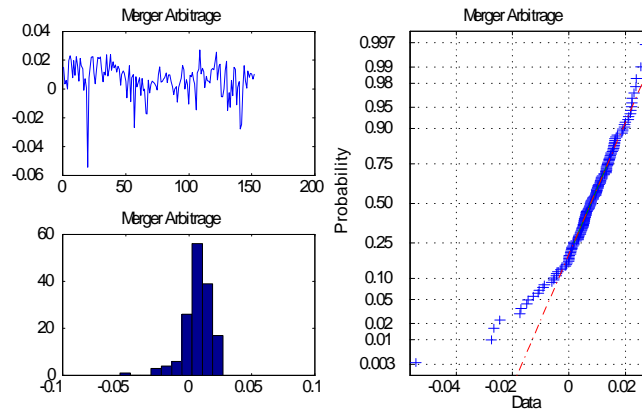
Figure 10

Merger Arbitrage returns seem to be not normally distributed and negatively skewed. The first hedge fund crisis, 1998, had a bigger impact on the returns of this fund, as during this period the return fell down to $-0,0544$, that is the minimum value of the return of this fund.
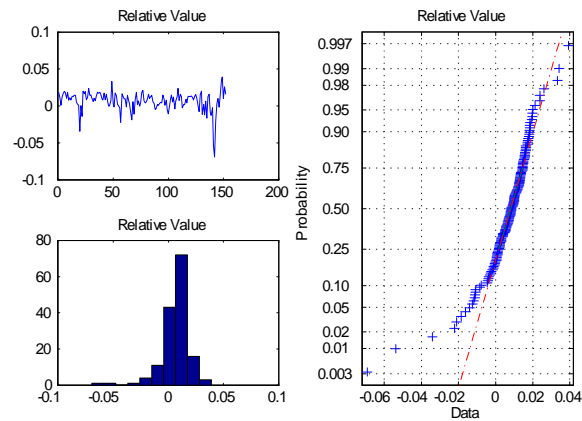
Figure 11

Relative Value is not normally distributed and is negatively skewed. The second hedge fund crisis, 2008, had a bigger impact on the returns of this fund, as in this period the return of this fund reached the minimum value of $-0,0692$.
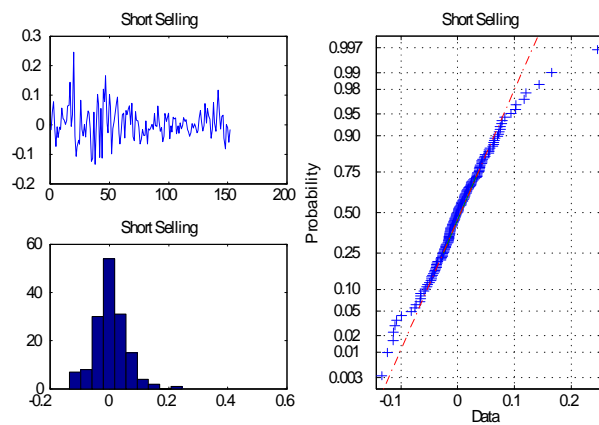
Figure 12

It seems that 'Short Selling' was not hit by the Hedge Fund Crisis[2] of 2008 as during the month where 'Convertible Arbitrage' had a loss of $12,37\%$ this fund won $11,7\%$.

---

[2] This is obvious as "Short Selling" is the best strategy during financial crisis.
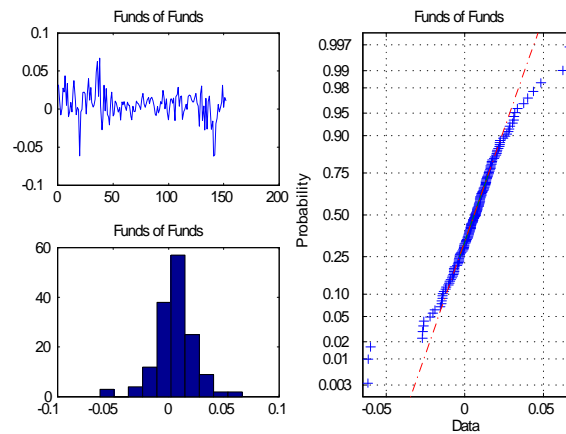
Figure 13

Observing the graphs of the last two funds "Short Selling and Fund of Funds" it seems that the returns are not normally distributed and that "short Selling is positively skewed while Funds of Funds is Negatively skewed.

In all these graphs the volatility clustering effect is easily identified because there are periods of high turbulence with many peaks clustering together and there are quiet periods where volatility stays low.

Table 1 indicates some of the descriptive statistics for each time series of returns in the whole period under consideration [3]:

---

[3] These results are given on monthly basis.

Descriptive Statistics for Assets under Consideration							
	Mean	Varaince	Skew	Kurtosis	Max	Min	JB-Test	P-value
'Convertible Arbitrage'	0,006409	0,000402	-2,6837	19,178	0,0611	-0,1237	1840,1	0
'CTA Global'	0,00649	0,000632	0,13448	2,8867	0,0691	-0,0543	0,53946	0,76359
'Distressed Securities'	0,007953	0,000337	-1,6746	9,439	0,0504	-0,0836	333,63	0
'Emerging Markets'	0,008246	0,001488	-1,2575	8,1026	0,123	-0,1922	204,96	0
'Equity Market Neutral'	0,006003	8,11E-05	-2,7476	20,407	0,0253	-0,0587	2110,3	0
'Event Driven'	0,007622	0,000337	-1,7184	9,1131	0,0442	-0,0886	311,48	0
'Fixed Income Arbitrage'	0,004231	0,000201	-3,7072	22,51	0,0365	-0,0867	2758,9	0
'Global Macro'	0,007672	0,00029	0,81531	4,7658	0,0738	-0,0313	36,586	1,14E-08
'Long/Short Equity'	0,00776	0,000492	-0,38183	4,2465	0,0745	-0,0675	13,533	0,001151
'Merger Arbitrage'	0,006785	0,000125	-1,6474	8,7932	0,0272	-0,0544	281,31	0
'Relative Value'	0,006701	0,000174	-2,1019	12,165	0,0392	-0,0692	643,86	0
'Short Selling'	0,004161	0,003036	0,57776	5,2486	0,2463	-0,134	40,479	1,62E-09
'Funds of Funds'	0,005918	0,000332	-0,45935	6,2993	0,0666	-0,0618	74,287	1,11E-16

Table 1

In the third column the skewness of each fund is reported. As we have seen, from the graphical analysis, the majority of the funds under consideration have a negative skewness[4]. The funds with positive skewness are 'CTA Global', 'Global Macro' and 'Short Selling'. The fund with the lowest value of skewness is 'Fixed Income Arbitrage' $(-3,7072)$ and the one with the highest value is 'Global Macro' $(0,81531)$.

In the fourth column the values of kurtosis are reported. This statistic range from a value of $2,8867$ for 'CTA Global' to a value of $22,51$ for 'Fixed Income Arbitrage'. Skewness and kurtosis in this table are standardized.[5]

In column seven are the values of *Jarque_Bera test,* for each fund, that measures the departure from Normality. This test is defined as [6]:

$$J - B = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right)$$

Looking at the results of JB-Test and p-value, we can say that at 76% level of significance CTA Global has a normal distribution. At a level of significance of 1% we reject the hypothesis that returns are normally distributed for all the funds, except 'CTA Global'.

The following graph report the performance of each fund in the portfolio, during the period under observation. This is constructed ex-post and I report it here just for having an idea of how each fund perform during the entire period.

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[4]This is consistent with the stylized markets facts.

[5]Max and Min are, respectively, the maximum and the minimum value of return.

[6]JB statistic is distributed as a χ^2 with two degrees of freedom. In the formula, S indicates the sample standardized skewness, K is the sample standardized kurtosis and T is the number of observations.
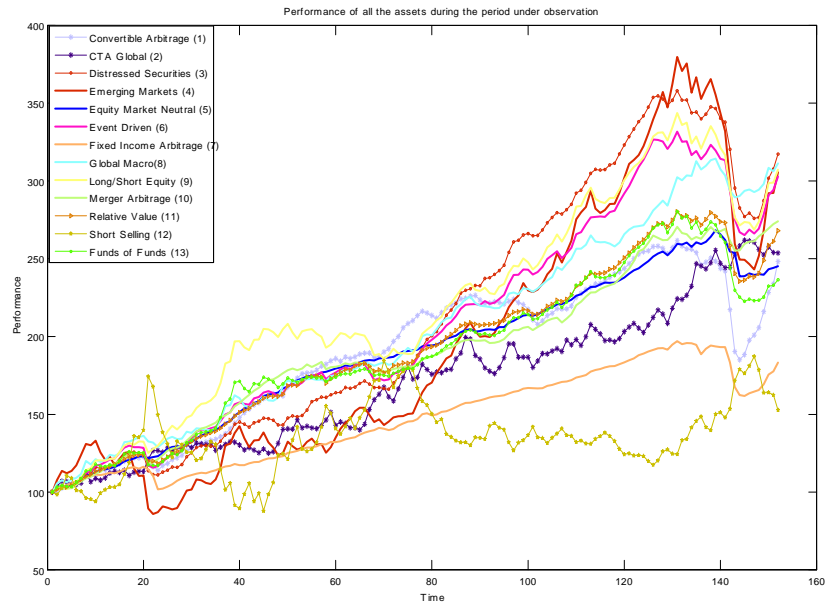
Figure 14

Observation: From this analysis we can conclude that the funds of our portfolio are not normally distributed, except 'CTA Global', meaning that they don't have skewness equal to zero and kurtosis equal to three.

How do we deal with this? Do we have to consider the higher moments in our portfolio allocation? What is the impact of higher moments on portfolio allocation?

We will try to answer to all these questions in the following sections.

Observation: It seems that all the funds under consideration have been hit dramatically by the Hedge Fund crisis of 2008, except 'Short Selling' and "CTA Global'.

5.3 Rolling window strategy

In our analysis we use the rolling window strategy with in sample period of 24 (48) and an out-of-sample period of 3 (6). We have considered four cases of rolling window strategy, that are:

- 24 months of calibration period and 3 months out-of-sample period.

- 24 months of calibration period and 6 months out-of-sample period.

- 48 months of calibration period and 3 months out-of-sample period.

- 48 months of calibration period and 6 months out-of-sample period.

Considering the CARA utility function we have found the optimal portfolio weights[7] in the in-sample period, 24 (48) months, and these weights are held constant (fixed-mixed strategy) over the next out-of sample period, 3(6) months, until a new optimization takes place based on the next in-sample window, in this way we roll over all the in-sample period.

The following figure indicates the rolling window strategy with calibration period (in black) of 24 months and out of sample computation of 3 months (in red).

Window

(k) (k+1) (k+2) (k+3) (k+4) (k+5) (k+6)   (k+8)

t   t + 3 m   t + 9 m   t + 18 m   t + 24 m

calibration period   *out of sample computation*
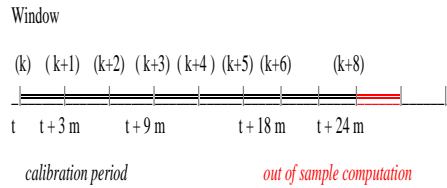
Figure 15

5.4 Estimation of co-moments with different approaches

In this section we report some of the results obtained for the estimation of the co-variance, constant correlation coefficients, shrinkage coefficients, only for the case of 24-3 rolling window strategy. We will not report the results for co-skewness and co-kurtosis because these are big matrices with dimension (13×169) and (13×2197), respectively.

Co-variance estimation using the sample approach

The first step of our analysis was to estimate the co-moments, using the sample approach, for each calibration period. In case of 24-3 rolling window strategy, given that our database has 152 observations, we have in total 42 calibration

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[7]We have found the optimal weights in the in sample period using the different approaches, proposed in this dissertation, for the estimation of the co-moments,

windows. Therefore we estimate the sample co-moments for the 42 calibration windows. In the following table the sample co-variance is reported for the first in-sample period.

| | Co-variance estimation for the first calibration period using Sample approach | | | | | | | | | | | | |
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9	Asset 10	Asset 11	Asset 12	Asset 13
Asset 1	0,00024069	-0,00006354	0,00030354	0,00051028	0,00014947	0,00032074	0,00022106	0,00027970	0,00027717	0,00021232	0,00021348	-0,00023221	0,00028231
Asset 2	-0,00006354	0,00082574	-0,00017428	-0,00036278	0,00012797	-0,00011611	-0,00003990	0,00036713	0,00008284	-0,00002958	0,00000015	0,00063661	0,00002755
Asset 3	0,00030354	-0,00017428	0,00056652	0,00109690	0,00020501	0,00059984	0,00021432	0,00041074	0,00050005	0,00038211	0,00030470	-0,00095235	0,00050959
Asset 4	0,00051028	-0,00036278	0,00109690	0,00328600	0,00030391	0,00118640	0,00035230	0,00086663	0,00094380	0,00067753	0,00049458	-0,00296980	0,00107910
Asset 5	0,00014947	0,00012797	0,00020501	0,00030391	0,00017545	0,00025565	0,00007800	0,00029528	0,00031199	0,00019888	0,00016968	-0,00021115	0,00025178
Asset 6	0,00032074	-0,00011611	0,00059984	0,00118640	0,00025565	0,00068816	0,00017492	0,00047119	0,00061218	0,00046407	0,00033946	-0,00120740	0,00057571
Asset 7	0,00022106	-0,00003990	0,00021432	0,00035230	0,00007800	0,00017492	0,00043839	0,00028773	0,00010265	0,00007158	0,00017144	0,00031392	0,00018953
Asset 8	0,00027970	0,00036713	0,00041074	0,00086663	0,00029528	0,00047119	0,00028773	0,00078703	0,00053742	0,00031631	0,00030408	-0,00030200	0,00053829
Asset 9	0,00027717	0,00008284	0,00050005	0,00094380	0,00031199	0,00061218	0,00010265	0,00053742	0,00070350	0,00044230	0,00032688	-0,00109910	0,00057977
Asset 10	0,00021232	-0,00002958	0,00038211	0,00067753	0,00019888	0,00046407	0,00007158	0,00031631	0,00044230	0,00035351	0,00024390	-0,00075136	0,00037814
Asset 11	0,00021348	0,00000015	0,00030470	0,00049458	0,00016968	0,00033946	0,00017144	0,00030408	0,00032688	0,00024390	0,00022373	-0,00034716	0,00030015
Asset 12	-0,00023221	0,00063661	-0,00095235	-0,00296980	-0,00021115	-0,00120740	0,00031392	-0,00030200	-0,00109910	-0,00075136	-0,00034716	0,00576520	-0,00091782
Asset 13	0,00028231	0,00002755	0,00050959	0,00107910	0,00025178	0,00057571	0,00018953	0,00053829	0,00057977	0,00037814	0,00030015	-0,00091782	0,00055305

Table 2

The diagonal elements of this matrix are the variances of the assets and the out of diagonal elements are the covariances. As we can see from the results, when we use sample estimation, some of the covariances are positive and some are negative depending on the sample correlation coefficients between each pair of assets.

We have estimated also the co-skewness and co-kurtosis using the sample approach. Co-skewness is a matrix with $(13 \times 13\hat{\ }2)$ dimension and the super-diagonal elements are the sample skewness and the out of the super diagonal elements are the sample co-skewness. The same is for co-kurtosis that is a matrix with $(13 \times 13\hat{\ }3)$ dimension.

As we have explained in the theoretical part, we need to estimate a lot of parameters[8] when we use the sample estimation approach. To reduce the number of parameters to be estimated and to reduce the sample error estimation, different approaches, for the estimation of co-moments were introduced. We have considered the CC approach, proposed by Elton and Gruber [1973] for the co-variance matrix and extended by Martellini and Ziemann [2009] to co-skewness and co-kurtosis matrices.

Co-variance estimation using the CC approach

It is true that using the CC approach reduces the sample error but increases the model error as we impose some structure in the model. In case of CC approach we assume that the correlation between different pairs of assets is constant over time and this is not a good assumption because correlation between two different pairs of assets varies. That is why using the CC approach decreases the sample

[8]See the theoretical part.

error, but increases the model error which depends on the structure that we have imposed in the model.

To estimate the co-moments with the CC approach we need to estimate the seven constant correlations coefficients beforehand. These values are reported in the following table for each calibration period.

	rho 1	rho 2	rho 3	rho 4	rho 5	rho 6	rho 7
t=1	0,44539	0,065264	0,28847	0,18055	0,62192	0,32067	0,13389
t=2	0,4167	0,049359	0,27628	0,15681	0,61662	0,30894	0,11709
t=3	0,43104	0,076568	0,28232	0,18027	0,61992	0,32061	0,13386
t=4	0,39415	0,000125	0,25061	0,12534	0,63444	0,29728	0,089576
t=5	0,40711	0,081541	0,25346	0,15952	0,60794	0,29525	0,094579
t=6	0,3882	0,1014	0,25169	0,1497	0,60427	0,28848	0,080894
t=7	0,37187	0,10253	0,24397	0,14711	0,60984	0,28135	0,076154
t=8	0,42881	0,47703	0,33855	0,30543	0,59204	0,37478	0,2074
t=9	0,46342	0,5148	0,38109	0,35528	0,59735	0,41783	0,26977
t=10	0,47299	0,52319	0,38961	0,36922	0,60051	0,42655	0,28229
t=11	0,42508	0,49046	0,36105	0,32255	0,58848	0,39514	0,24
t=12	0,4104	0,45915	0,34901	0,3116	0,57837	0,3813	0,23152
t=13	0,3698	0,40633	0,31916	0,2838	0,56744	0,3523	0,22585
t=14	0,33153	0,33518	0,26994	0,25498	0,52404	0,28693	0,20589
t=15	0,35321	0,31811	0,27613	0,27093	0,4956	0,30227	0,22979
t=16	0,32884	0,22884	0,26097	0,23147	0,48099	0,26766	0,20766
t=17	0,35073	0,26024	0,27985	0,23817	0,53514	0,29728	0,20526
t=18	0,34864	0,19745	0,25248	0,1891	0,51313	0,23425	0,11131
t=19	0,41641	0,32128	0,33744	0,29352	0,60001	0,36223	0,24413
t=20	0,44832	0,40083	0,36084	0,32454	0,64021	0,40567	0,2698
t=21	0,47509	0,4278	0,38219	0,34817	0,65561	0,42524	0,27821
t=22	0,49508	0,44218	0,39451	0,36923	0,65459	0,43663	0,29625
t=23	0,49825	0,45653	0,41506	0,37229	0,6662	0,4575	0,30696
t=24	0,52287	0,56326	0,44597	0,41114	0,68408	0,50223	0,34086
t=25	0,53883	0,5827	0,44998	0,42056	0,69775	0,5176	0,33924
t=26	0,52829	0,58595	0,46021	0,42059	0,72302	0,52838	0,34597
t=27	0,50788	0,48051	0,42622	0,38431	0,65346	0,482	0,33361
t=28	0,54136	0,50486	0,44587	0,41239	0,65912	0,49791	0,35064
t=29	0,51848	0,4614	0,4366	0,39408	0,64528	0,48082	0,34174
t=30	0,53256	0,50806	0,45049	0,42077	0,68659	0,52483	0,3841
t=31	0,55759	0,53817	0,4656	0,44732	0,68441	0,53891	0,40482
t=32	0,55511	0,54634	0,4546	0,4345	0,70195	0,53524	0,37902
t=33	0,57661	0,55103	0,4738	0,47328	0,69689	0,55566	0,43401
t=34	0,58117	0,5538	0,47362	0,46644	0,68846	0,54713	0,41743
t=35	0,60232	0,64391	0,49586	0,48077	0,75146	0,57175	0,39691
t=36	0,59431	0,58203	0,50645	0,44955	0,72381	0,56793	0,41467
t=37	0,62158	0,56325	0,5354	0,5068	0,72177	0,61317	0,5051
t=38	0,56267	0,27726	0,43771	0,31415	0,60158	0,48091	0,38976
t=39	0,55513	0,27361	0,42742	0,29899	0,59321	0,46642	0,38244
t=40	0,58093	-0,2399	0,46123	-0,20143	0,67661	0,54993	0,42927
t=41	0,4802	-0,34079	0,36634	-0,18195	0,63122	0,38452	0,19274
t=42	0,4792	-0,3516	0,37338	-0,18881	0,6431	0,39012	0,19579

Tabella 3

Analyzing these figures we can say that using CC approach the correlation $(r^{(1)} = rho\ 1)$ between two different assets in all the calibration periods is positive, which indicates that the returns of two different funds move in the

same direction [9] for all the calibration periods. Given the fact that $r^{(1)} > 0$ for each calibration period, it means that the co-variance matrix estimated using the CC approach will have all the elements positive[10]; this is not the case when we consider the sample co-variance (where some elements are positive and others are negative, depending on $r_{ij}^{(1)}$ that is not constant over assets).

$r^{(2)}$ and $r^{(4)}$ are the coefficients used for the calculation of the co-skewness matrix. We have used $r^{(2)}$ to estimate the elements S_{ijj}^{CC}, this means that $r^{(2)}$ measures the correlation between the return of asset i and the volatility of asset j. As we can see, this coefficient is mostly positive, except for $t =$ 40, 41, 42. This means that the elements of co-skewness matrix (S_{ijj}^{CC}) will be always positive, except in $t =$ 40, 41, 42. The calibration period 40 ranges from '01/09/2006' to '01/08/2008', the calibration period 41 ranges from '01/12/2006' to '01/11/2008' and the calibration period 42 ranges from '01/03/2007' to '01/02/2009'. As we can see all the three calibration periods include the second Hedge Fund crisis.

We have used $r^{(4)}$ to estimate S_{ijk}^{CC}, this means that $r^{(4)}$ indicates the correlation between the return of asset k and the co-variance between asset i and j. If $r^{(4)}$ is positive also S_{ijk}^{CC} will be positive. As we can see from the results reported in table 3 this coefficient is always positive, except for $t =$ 40, 41, 42.

$r^{(3)}$, $r^{(5)}, r^{(6)}$ and $r^{(7)}$ are the correlation coefficients that we have used to estimate the elements in co-kurtosis matrix. $r^{(3)}$ has been used for the estimation of K_{ijjj} [11]. $r^{(5)}$ has been used for the estimation of K_{ijjj} [12]. $r^{(6)}$ has been used for the estimation of K_{iijk} [13]. $r^{(7)}$ has been used for the estimation of K_{ijkl} [14].

Having estimated the constant correlations coefficients we can estimate the co-matrices for each calibration period. In the following table the results of the co-variance matrix for the first calibration period are reported ('01/01/1997'-'01/12/1998') .

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[9]In this analysis, this coefficient is high, ranging from $0,32884$ in $t = 16$ to $0,62158$ in $t = 37$. In $t = 16$ the calibration period ranges from '01/09/2000' to '01/08/2002' and $t = 37$ the calibration period ranges from '01/12/2005' to '01/11/2007'.

[10]This is because as we have seen in the theoretical part of this dissertation $\sigma_{ij}^{CC} = \sigma_i * \sigma_j * r^{(1)}$. If $r^{(1)}$ is positive also σ_{ij} is positive.

[11]$r^{(3)}$ measures the correlation between the return of asset i and the skewness of asset j.

[12]$r^{(5)}$ measures the correlation between the volatility of asset i and the volatility of asset j.

[13]$r^{(6)}$ measures the correlation between the volatility of asset i and the covariance between asset j and k.

[14]$r^{(7)}$ measures the correlation between the covariance of asset i and j and the covariance of asset k and l.

	Co-variance estimation for the first calibration period using CC approach												
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9	Asset 10	Asset 11	Asset 12	Asset 13
Asset 1	0,0002407	0,0001903	0,0001576	0,0003796	0,0000877	0,0001737	0,0001387	0,0001858	0,0001756	0,0001245	0,0000990	0,0005028	0,0001557
Asset 2	0,0001903	0,0008257	0,0002919	0,0007031	0,0001625	0,0003218	0,0002568	0,0003441	0,0003253	0,0002306	0,0001835	0,0009313	0,0002884
Asset 3	0,0001576	0,0002919	0,0005665	0,0005824	0,0001346	0,0002665	0,0002127	0,0002850	0,0002695	0,0001910	0,0001520	0,0007714	0,0002389
Asset 4	0,0003796	0,0007031	0,0005824	0,0032860	0,0003241	0,0006419	0,0005123	0,0006864	0,0006490	0,0004600	0,0003660	0,0018578	0,0005754
Asset 5	0,0000877	0,0001625	0,0001346	0,0003241	0,0001755	0,0001483	0,0001184	0,0001586	0,0001500	0,0001063	0,0000846	0,0004293	0,0001330
Asset 6	0,0001737	0,0003218	0,0002665	0,0006419	0,0001483	0,0006882	0,0002344	0,0003141	0,0002970	0,0002105	0,0001675	0,0008502	0,0002633
Asset 7	0,0001387	0,0002568	0,0002127	0,0005123	0,0001184	0,0002344	0,0004384	0,0002507	0,0002370	0,0001680	0,0001337	0,0006786	0,0002102
Asset 8	0,0001858	0,0003441	0,0002850	0,0006864	0,0001586	0,0003141	0,0002507	0,0007870	0,0003176	0,0002251	0,0001791	0,0009092	0,0002816
Asset 9	0,0001756	0,0003253	0,0002695	0,0006490	0,0001500	0,0002970	0,0002370	0,0003176	0,0007035	0,0002129	0,0001693	0,0008596	0,0002662
Asset 10	0,0001245	0,0002306	0,0001910	0,0004600	0,0001063	0,0002105	0,0001680	0,0002251	0,0002129	0,0003535	0,0001200	0,0006094	0,0001887
Asset 11	0,0000990	0,0001835	0,0001520	0,0003660	0,0000846	0,0001675	0,0001337	0,0001791	0,0001693	0,0001200	0,0002237	0,0004848	0,0001501
Asset 12	0,0005028	0,0009313	0,0007714	0,0018578	0,0004293	0,0008502	0,0006786	0,0009092	0,0008596	0,0006094	0,0004848	0,0057652	0,0007622
Asset 13	0,0001557	0,0002884	0,0002389	0,0005754	0,0001330	0,0002633	0,0002102	0,0002816	0,0002662	0,0001887	0,0001501	0,0007622	0,0005531

Table 4

As expected all the covariances are positive, this is true for all the other calibration periods.

Remember that the way we have constructed the CC approach, we have on the diagonal the sample variances of the assets and out of diagonal the covariance estimated using CC approach. In case of co-skewness matrix, we will have on the superdiagonal the sample skewness and on the out of superdiagonal the co-skewness estimated using CC approach. The same is for co-kurtosis.

Analyzing carefully the results of table 3 and recolling that investors dislike even moments and like odd moments, this means that we don't like a high $r_{CC}^{(1)}$, $r_{CC}^{(3)}$, $r_{CC}^{(5)}$, $r_{CC}^{(6)}$ and $r_{CC}^{(7)}$ and we don't like a low $r_{CC}^{(2)}$ and $r_{CC}^{(4)}$. In table 3 the red figures indicate the maximum value for each coefficient and the green ones indicate the minimum value for each coefficient.

Co-variance estimation using the Shrinkage approach

The third approach considered, for estimating the co-matrices, is the shrinkage towards the CC.

To estimate the co-moments using this approach we need to estimate, before, the shrinkage intensities for each co-moment and for each calibration period. We show these results in the following table.

	Shrinkage ij	Shrinkage ijk	Shrinkage ijkl
t=1	0,26777	0,76261	0,51041
t=2	0,2687	0,77844	0,51645
t=3	0,25382	0,776	0,50902
t=4	0,29839	0,79006	0,52889
t=5	0,19029	1	0,43552
t=6	0,15351	1	0,39749
t=7	0,13898	1	0,3986
t=8	0,05903	0,14803	0,12891
t=9	0,056671	0,16331	0,11578
t=10	0,053064	0,15564	0,11131
t=11	0,063667	0,19699	0,12403
t=12	0,057992	0,21322	0,11923
t=13	0,066765	0,30532	0,13894
t=14	0,063874	0,26649	0,11352
t=15	0,080822	0,31052	0,14754
t=16	0,075048	0,29703	0,16077
t=17	0,067296	0,40041	0,095215
t=18	0,086529	0,84525	0,11582
t=19	0,079099	0,40473	0,12324
t=20	0,081686	0,32947	0,13376
t=21	0,078967	0,24139	0,13104
t=22	0,079187	0,22613	0,12919
t=23	0,079686	0,23425	0,13088
t=24	0,080405	0,1386	0,13626
t=25	0,062952	0,11133	0,1098
t=26	0,058336	0,10969	0,1078
t=27	0,060049	0,17512	0,13025
t=28	0,063112	0,17517	0,12982
t=29	0,069286	0,24857	0,16415
t=30	0,065466	0,19025	0,14322
t=31	0,062467	0,16725	0,14227
t=32	0,074429	0,18941	0,1659
t=33	0,064891	0,1716	0,14637
t=34	0,06371	0,17246	0,15321
t=35	0,064605	0,11912	0,14152
t=36	0,057398	0,1247	0,11404
t=37	0,057467	0,1346	0,083972
t=38	0,074943	0,4756	0,12104
t=39	0,060116	0,38271	0,1017
t=40	0,11475	0,80119	0,47794
t=41	0,19921	0,42428	0,40084
t=42	0,19229	0,4237	0,39863

Table 5

In this table, a shrinkage coefficient equal to one means that the shrinkage co-matrix is given by the CC co-matrix for that calibration period.

Having the shrinkage coefficients we can estimate all the co-moments. The following table indicates the estimated covariance matrix for the first calibration period:

| | Co-variance estimation for the first calibration period using Shrinkage approach | | | | | | | | | | | | |
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9	Asset 10	Asset 11	Asset 12	Asset 13
Asset 1	0,0002407	0,0000044	0,0002645	0,0004753	0,0001329	0,0002814	0,0001990	0,0002546	0,0002500	0,0001888	0,0001828	-0,0000354	0,0002484
Asset 2	0,0000044	0,0008257	-0,0000494	-0,0000774	0,0001372	0,0000011	0,0000395	0,0003610	0,0001478	0,0000401	0,0000492	0,0007155	0,0000974
Asset 3	0,0002645	-0,0000494	0,0005665	0,0009592	0,0001862	0,0005106	0,0002139	0,0003771	0,0004383	0,0003309	0,0002638	-0,0004908	0,0004371
Asset 4	0,0004753	-0,0000774	0,0009592	0,0032860	0,0003093	0,0010406	0,0003951	0,0008184	0,0008649	0,0006193	0,0004601	-0,0016771	0,0009443
Asset 5	0,0001329	0,0001372	0,0001862	0,0003093	0,0001755	0,0002269	0,0000888	0,0002587	0,0002686	0,0001741	0,0001469	-0,0000397	0,0002200
Asset 6	0,0002814	0,0000011	0,0005106	0,0010406	0,0002269	0,0006882	0,0001909	0,0004291	0,0005278	0,0003962	0,0002934	-0,0006565	0,0004921
Asset 7	0,0001990	0,0000395	0,0002139	0,0003951	0,0000888	0,0001909	0,0004384	0,0002778	0,0001386	0,0000974	0,0001613	0,0004116	0,0001951
Asset 8	0,0002546	0,0003610	0,0003771	0,0008184	0,0002587	0,0004291	0,0002778	0,0007870	0,0004786	0,0002919	0,0002706	0,0000223	0,0004696
Asset 9	0,0002500	0,0001478	0,0004383	0,0008649	0,0002686	0,0005278	0,0001386	0,0004786	0,0007035	0,0003809	0,0002847	-0,0005746	0,0004958
Asset 10	0,0001888	0,0000401	0,0003309	0,0006193	0,0001741	0,0003962	0,0000974	0,0002919	0,0003809	0,0003535	0,0002107	-0,0003870	0,0003274
Asset 11	0,0001828	0,0000492	0,0002638	0,0004601	0,0001469	0,0002934	0,0001613	0,0002706	0,0002847	0,0002107	0,0002237	-0,0001244	0,0002600
Asset 12	-0,0000354	0,0007155	-0,0004908	-0,0016771	-0,0000397	-0,0006565	0,0004116	0,0000223	-0,0005746	-0,0003870	-0,0001244	0,0057652	-0,0004680
Asset 13	0,0002484	0,0000974	0,0004371	0,0009443	0,0002200	0,0004921	0,0001951	0,0004696	0,0004958	0,0003274	0,0002600	-0,0004680	0,0005531

Table 6

The Shrinkage approach was proposed to reduce the model error introduced by the CC approach.

As we can see, the diagonal elements are the sample variances for each approach, while what changes from one approach to the other are the covariances, in general the co-moments.

5.5 Utility function behavior depending on the constraints

In this section we will examine the utility function graph when we consider a portfolio of three assets. We will represent the utility function with and without constraints and analyze how the maximum value of the utility function changes depending on the constraints.

Let's see the shape of the utility function when we consider a portfolio with two assets [15], figure 16. The value of the utility function is calculated using the Taylor expansion.

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[15] We have considered the first window of the calibration period 24_6 in the case of shrinkage estimation and the risk aversion, $\lambda = 50$. We have supposed that our portfolio is composed of only two assets, asset 5 and asset 11. We have changed the weights of asset 5 from 0 to 1, using a step of 0.0001 and, as it is obvious, the weight of the second asset is $w_{(11)} = 1 - w_{(5)}$. As we can see asset 11 is used to meet the constraint of $\sum w_i = 1$. To calculate the utility function we have used the estimated co-matrixes of the first calibration period.
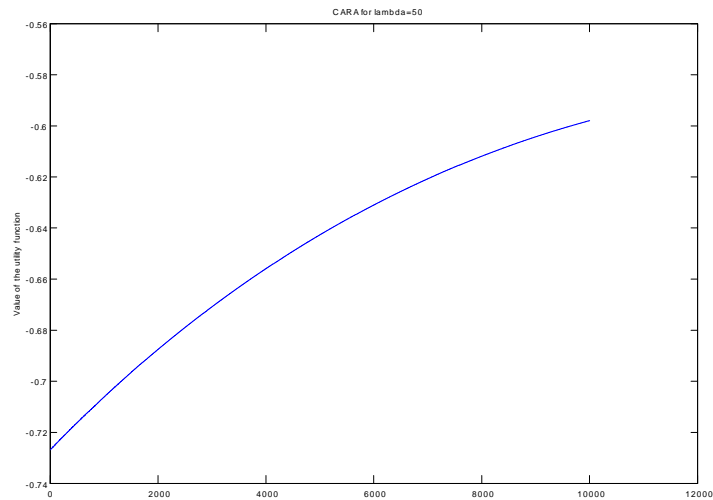
Figure 16

This graph indicates how the value of the utility function changes when we increase the weight of asset one from 0 to 1, in steps of 0.0001. The maximum value of the utility function in this case is near -0.6.

Let's consider now a portfolio composed of three assets and plot the utility function. We take account of the same portfolio as before, but now we add a third asset that is asset 12. So we have in our portfolio asset 5, asset 11 and asset 12.
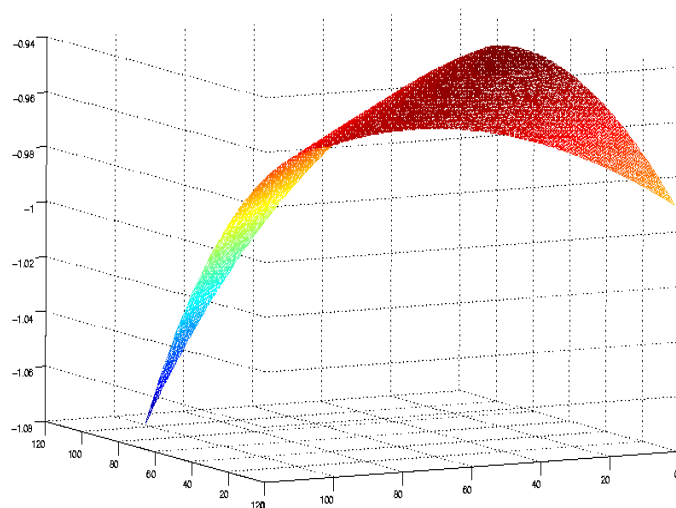
Figure 17

Figure 17 indicates the utility function of a portfolio constructed with the three selected assets, in case of the rolling window strategy 24_6, for the first calibration period and lambda=50. We have constructed this function by moving the weights of asset 5 and asset 11 from 0 to 1 with a pass of 0.1, and we have imposed the weight of asset 12 as: $w_{12} = 1 - w_5 - w_{11}$. In this case, the weight of asset 12 can be positive or negative, while each of the other two assets moves in the interval $[0,\ 1]$. Figure 17 indicates the value of the utility function as function of the weights of assets 5 and 11.

To understand better the behavior of the utility function, we have constructed in figure 18 the level curves[16] of the surfice plotted in figure 17.

_____

[16] Each isoline indicates all the admissible combinations of the variables for which the utility function has the same value.
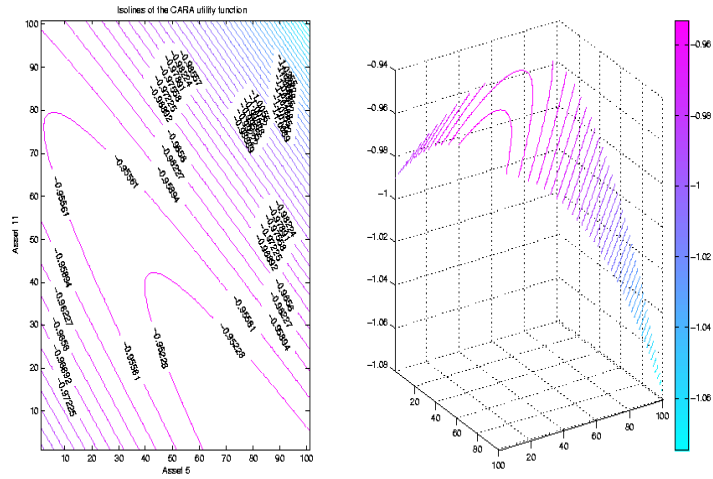
Figure 18

Let us consider the portfolio with two assets, asset 5 and asset 11. We evaluate the utility function changing the weight of asset 5 from $w_5 = 0$ to 1 with a step of 0.1 and the weight of asset 11 moves from 0 to $1 - w_5$ with the same step. The utility function in this case is:
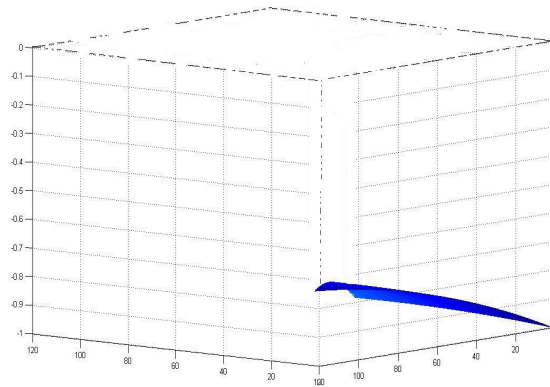
figure 19

As we can see, imposing constraints on the weights we do not consider the entire utility function, but we take into consideration only the part where our

constraints are satisfied. In this case, the utility function is defined on the set of weights $w_5 = [0, \ 1]$ and $w_{11} = [0, \ 1 - w_5]$.

In the following figure we have reported the isolines of this utility function.
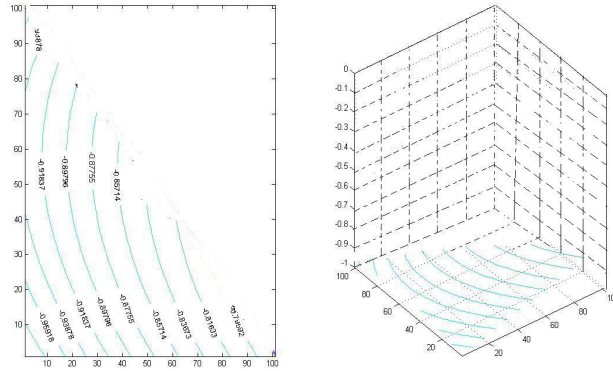
figure 20

The last case that we have considered is a portfolio with three assets, asset 5, asset 11 and 12. As before, the weight of asset 5 moves in $[0, \ 1]$ with a step of 0.1 and asset 11 moves in $[0, \ 1 - w_5]$ and we introduce asset 12 to keep the $sum = 1$, so $w_{12} = 1 - w_{11} - w_5$. The utility function and the isolines in this case are:
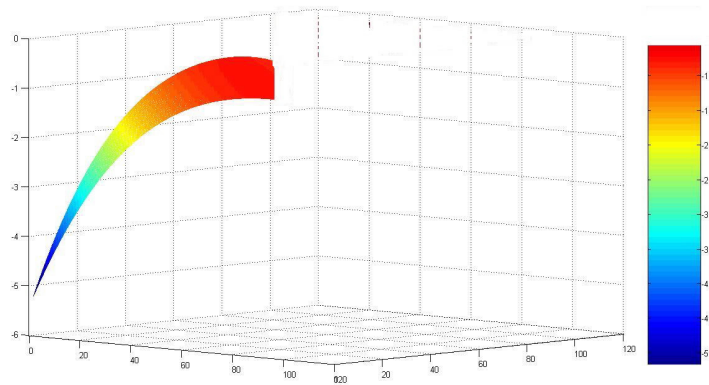
figure 21
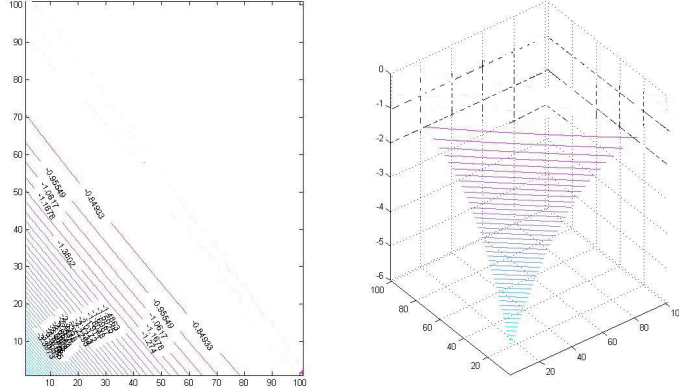
figure 22

These graphs have been reported in order to have an idea of the shape of the utility function and to understand how the maximum value of this function depends on the constraints imposed.

In this graphical analysis we have considered a portfolio of three assets. Actually our portfolio is composed of 13 assets but is impossible to plot its utility function. These graphs are important to understand how the utility function behaves and how the maximum changes when the constraints change.

5.6 Optimization

So far, we have estimated the co-moments with three different approaches (sample, CC and shrinkage) for each calibration period. In this Section, using these estimated co-moments and the CARA utility function we find the optimal weights, which are the weights where the utility function reaches the maximum value. The constraints that we have imposed are: $\sum_{i=i}^{13} w_i = 1$ and $0 \leq w_i \leq 1$, meaning that we do not allow for short selling.

We have considered six different models that are:

1.) Sample variance,

2.) CC variance,

3.) Shrinkage variance

4.) Sample variance, skewness and kurtosis,

5.) CC variance, CC skewness and CC kurtosis,

6.) Shrinkage variance, shrinkage skewness and shrinkage kurtosis.

We have found the optimal weights[17] [18] for each model using different parameters of risk aversion, $\lambda = 1$, $\lambda = 5$, $\lambda = 10$, $\lambda = 20$, $\lambda = 30$, $\lambda = 50$, $\lambda = 70$, $\lambda = 90$. The results obtained in case $\lambda = 5$ are reported in the following tables and graphs, with reference to the rolling window strategy 24_3.

	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8	asset 9	asset 10	asset 11	asset 12	asset 13
					Risk avversion=5 (sample Variance)								
t=1	0,08292	0,097128	0,070093	0,034645	0,08733	0,065911	0,083534	0,054853	0,065961	0,081031	0,082711	0,12834	0,065542
t=2	0,079486	0,10136	0,0683	0,044617	0,086006	0,06416	0,080585	0,062234	0,064426	0,077761	0,080207	0,12535	0,065501
t=3	0,080673	0,099083	0,065461	0,041156	0,088515	0,062241	0,082921	6,23E-02	0,064859	0,078118	0,081085	0,12989	0,063729
t=4	0,077422	0,1078	0,065587	0,044045	0,087841	0,062382	0,078758	0,071899	0,068862	0,075232	0,078328	0,11506	0,066786
t=5	0,078869	0,10387	0,067741	0,035892	0,08606	0,065911	0,080176	6,79E-02	0,062115	0,078491	0,079313	0,13171	0,061991
t=6	0,075147	0,098723	0,067562	0,040414	0,083139	0,067437	0,078632	0,071738	0,062688	0,076789	0,076663	0,13925	0,061822
t=7	0,074459	0,096645	0,068273	0,03894	0,082241	0,06699	0,077975	0,072059	0,062599	0,076022	0,076336	0,14542	0,062041
t=8	0,074327	0,091699	0,072276	0,046895	0,081025	0,070276	0,079156	0,070362	0,058356	0,077368	0,076342	0,14151	0,060415
t=9	0,078746	0,081951	0,075341	0,049793	0,080312	0,071865	0,086042	0,06927	0,060006	0,076344	0,078687	0,12825	0,063397
t=10	0,076681	0,081128	0,074361	0,05131	0,080156	0,071543	0,087444	0,068202	0,06312	0,077706	7,79E-02	0,12563	0,064789
t=11	0,076594	0,080047	0,076129	0,058475	0,078225	0,074919	0,084773	0,068687	0,064998	0,078283	0,077529	0,11396	0,067378
t=12	0,076686	0,079369	0,075585	0,057949	0,078759	0,074776	0,084235	0,068762	0,067087	0,078374	0,078439	0,11178	0,068196
t=13	0,074772	0,074893	0,075961	0,07036	0,078237	0,07461	0,082789	0,073263	0,074528	0,076898	0,076281	0,093128	0,074281
t=14	0,075025	0,01216	0,09115	0	0,10687	0,057223	0,12549	0,038596	0,13421	0,089574	0,065058	0,067541	0,13709
t=15	0,079327	0,06402	0,078917	0,063435	0,084211	0,077268	0,084765	0,069719	0,09318	0,083688	0,07725	0,053687	0,090531
t=16	0,080332	0,059631	0,079267	0,058529	0,087946	0,076281	0,086676	0,068442	0,096304	0,086405	0,078406	0,049091	0,092691
t=17	0,053865	0,068242	0,035122	0	0,11064	0,060762	0,088869	0,079185	0,14695	0,10953	0,074261	0,06886	0,10371
t=18	0,06349	0	0,028311	0,019526	0,11685	0,068495	0,09044	0,063085	0,15202	0,11362	0,10154	0,072105	0,11052
t=19	0,074031	0,025828	0,047485	0,027445	0,10518	0,062313	0,083046	0,055257	0,091104	0,099142	0,086721	0,15128	0,09117
t=20	0,076072	0,032004	0,043859	0,019092	0,1053	0,064284	0,083661	0,046515	0,085388	0,10265	0,085472	0,16739	0,088318
t=21	0,075156	0,03673	0,038336	0,023183	0,10379	0,063568	0,086791	0,048739	0,078305	0,10048	0,085444	0,17349	0,085996
t=22	0,081341	0,0091582	0,031856	0,017608	0,10915	0,059719	0,091326	0,04505	0,07365	0,10742	0,090218	0,19828	0,08522
t=23	0,080435	0,0074027	0,040534	0,018705	0,10552	0,059939	0,093871	0,047242	0,066853	0,10109	0,087822	0,20835	0,082242
t=24	0,084911	0,0015709	0,036929	0,012888	0,10581	0,061462	0,09341	0,04845	0,06722	0,10437	0,088755	0,21183	0,082401
t=25	0,09155	0,013758	0,04051	0,015068	0,10042	0,055139	0,095425	0,052515	0,055086	0,094477	0,084852	0,22475	0,076443
t=26	0,0994	0,037631	0,045045	0,0084223	0,096717	0,054737	0,093692	0,055797	0,047527	0,090449	0,080984	0,21685	0,072752
t=27	0,094959	0,027751	0,050544	0,016548	0,094677	0,058897	0,10099	0,060567	0,050119	0,089779	0,083963	0,20108	0,070127
t=28	0,094837	0,025975	0,052233	0,005304	0,098239	0,057312	0,10827	0,060178	0,040487	0,094453	0,083823	0,20961	0,069274
t=29	0,098555	0,029436	0,05839	0	0,099858	0,055724	0,11433	0,061521	0,035056	0,090858	0,08679	0,20629	0,06319
t=30	0,089667	0,050472	0,061457	0	0,10097	0,052084	0,11643	0,062751	0,028913	0,080464	8,11E-02	0,21589	0,059841
t=31	0,09399	0,049377	0,060659	0	0,10173	0,053734	0,11351	0,061787	0,031907	0,081855	0,082307	0,20846	0,060684
t=32	0,092449	0,050665	0,060272	0	0,10336	0,052021	0,11434	0,060057	0,030369	0,080542	0,081623	0,21474	0,059561
t=33	0,087183	0,064882	0,067918	0	0,10213	0,057255	0,11228	0,063801	0,03432	0,079593	0,079208	0,19207	0,059356
t=34	0,081412	0,068162	0,064254	0	0,10409	0,049442	0,11489	0,067961	0,032821	0,071555	0,076674	0,21128	0,057461
t=35	0,090927	0,069502	0,064346	0	0,10094	0,051547	0,11139	0,066058	0,037529	0,072925	0,07558	0,2011	0,058153
t=36	0,092293	0,049783	0,070933	0	0,099396	0,055909	0,1081	0,063649	0,042036	0,075243	0,07773	0,20928	0,055645
t=37	0,09015	0,049021	0,072173	0	0,098063	0,0587	0,10584	0,06204	0,046383	0,078064	0,077349	0,20595	0,056258
t=38	0,094712	0,045571	0,078031	0	0,095627	0,065186	0,099669	0,059511	0,050061	0,087246	0,081359	0,18815	0,054881
t=39	0,092415	0,050941	0,077782	0	0,093777	0,06247	0,098453	6,19E-02	0,046102	0,085481	0,079076	0,19875	0,052895
t=40	0,051861	0,08396	0,073935	0	0,097598	0,059237	0,091153	0,07405	0,042805	0,098936	0,075289	0,20148	0,049695
t=41	0,029966	0,11439	0,061242	0	0,087693	0,058992	0,067327	0,096839	0,049518	0,10087	0,066435	0,21197	0,05476
t=42	0,026399	0,11337	0,062627	0	0,087262	0,060799	0,067089	0,096901	0,050541	0,10321	0,066922	0,20917	0,055712

Table 7

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[17] To find the optimal weights we have used the MATLAB function "fmincon".

[18] For each model we started from a equal weighted portfolio, omega$_0$=[1/13, 1/13,....,1/13].
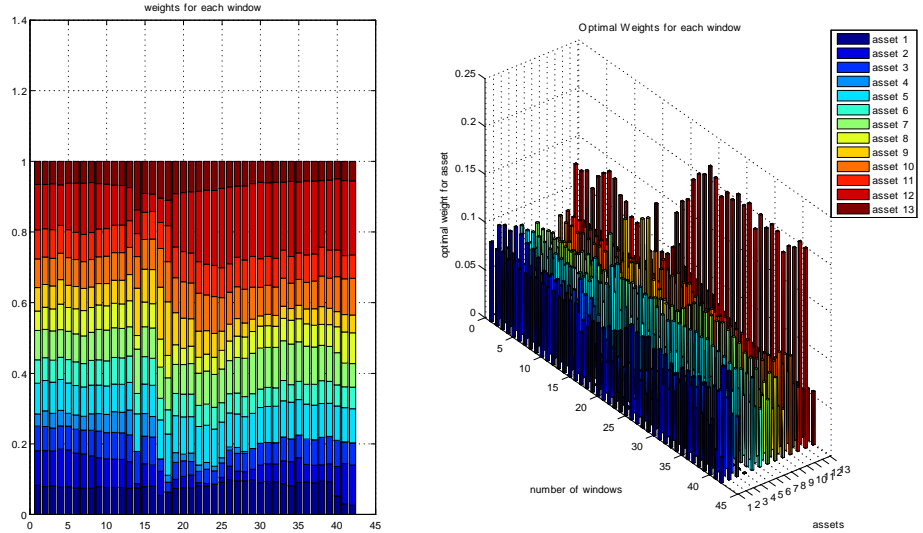
Figure 16

Considering only the first two moments, we observe that our portfolio is well diversified in almost all the calibration periods. This is coherent with the well known phrase in finance "don't put all your eggs in the same basket".

Let we focus on the last two calibration periods, number 41 and 42. The calibration period $t = 41$ covers the period from '01/12/2006' to '01/11/2008' and the calibration period $t = 42$ ranges from '01/03/2007' to '01/02/2009'. Both these calibration periods include the period of Hedge Fund crisis [19] and as we can see in these periods we invest more that 50% of our wealth in only four funds namely; 'CTA Global', 'Global Macro', 'Merger Arbitrage' and 'Short Selling'. As we have seen in the Section "Graphical analysis of the data", the 'CTA Global' and 'Short Selling' are the funds that during the Hedge Funds crisis have positive returns, which means that these founds are not affected by the crisis. This is consistent with what we have obtained from the optimization algorithm, since during these periods we invest in these two assets more than 32% of the wealth. Regarding the other two funds 'Global Macro' and 'Merger Arbitrage' during the crisis those have the smallest loss compared to that of the other funds.

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[19]It is well known that during financial crisis the correlations between assets are positive and move in the same direction rendering difficult the diversification of the portfolio.

|	Risk avversion=5 (sample Variance, sample skewness and sample kurtosis)												
	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8	asset 9	asset 10	asset 11	asset 12	asset 13
t=1	0,081942	0,096886	0,06997	0,037187	0,086594	0,066297	0,082179	0,057457	0,067115	0,080334	0,081859	0,1257	0,066478
t=2	0,078908	0,10068	0,068383	0,045939	0,085448	0,064721	0,079563	0,063961	0,065688	0,077443	0,079637	0,12324	0,066391
t=3	0,07999	0,098685	0,065826	0,042797	0,087734	0,062984	0,081677	6,40E-02	0,066063	0,077758	0,08045	0,12729	0,064781
t=4	0,077069	0,10626	0,066032	0,045569	0,087027	0,063202	0,077966	0,072577	0,069666	0,075165	0,077965	0,11393	0,06757
t=5	0,078468	0,10344	0,067777	0,03721	0,085712	0,066104	0,079416	6,86E-02	0,063122	0,078191	0,078973	0,13016	0,062795
t=6	0,075033	0,098834	0,067568	0,041057	0,083075	0,067477	0,078113	0,072068	0,063439	0,076669	0,076559	0,13769	0,062413
t=7	0,07437	0,096814	0,068267	0,039666	0,082203	0,067059	0,077492	0,072359	0,063322	0,075938	0,076246	0,14366	0,062604
t=8	0,074097	0,092188	0,072328	0,046939	0,081136	0,070332	0,078587	0,070328	0,058304	0,077466	0,076231	0,14175	0,060313
t=9	0,078657	0,082161	0,075367	0,049696	0,080278	0,071794	0,086099	0,069322	0,059802	0,076208	0,078627	0,12872	0,06327
t=10	0,076573	0,081253	0,07439	0,051316	0,080098	0,071496	0,08754	0,068228	0,063019	0,0776	7,79E-02	0,12587	0,064718
t=11	0,076513	0,080115	0,076132	0,058606	0,078175	0,0749	0,084834	0,068761	0,065022	0,078201	0,077492	0,11385	0,067401
t=12	0,076623	0,079425	0,075579	0,058086	0,078727	0,074744	0,084254	0,06885	0,067171	0,078276	0,078404	0,1116	0,068259
t=13	0,074769	0,074888	0,076005	0,070443	0,07822	0,074614	0,082804	0,073355	0,074648	0,076843	0,076258	0,092748	0,074405
t=14	0,075249	0,012093	0,091556	0	0,10694	0,056887	0,12617	0,039082	0,13564	0,088947	0,064886	0,063773	0,13877
t=15	0,079284	0,064818	0,078836	0,064339	0,083802	0,077289	0,084355	0,070256	0,092526	0,083265	0,077268	0,053955	0,090006
t=16	0,080256	0,0605	0,07918	0,059235	0,08756	0,07623	0,086392	0,068885	0,095676	0,085963	0,078367	0,049475	0,092282
t=17	0,054139	0,068613	0,035051	0	0,11063	0,060404	0,0891	0,079275	0,14667	0,10926	0,074353	0,068671	0,10383
t=18	0,063509	0	0,027392	0,01477	0,11774	0,067679	0,090955	0,062755	0,15228	0,11385	0,10156	0,07401	0,1108
t=19	0,074004	0,02695	0,047605	0,027353	0,10505	0,062147	0,083145	0,055541	0,090864	0,098817	0,086503	0,15102	0,090994
t=20	0,076038	0,032419	0,043653	0,018504	0,10545	0,064087	0,08381	0,046473	0,085348	0,1027	0,085437	0,16776	0,088314
t=21	0,075016	0,037087	0,037909	0,022626	0,10397	0,063318	0,086965	0,048683	0,078207	0,10058	0,085419	0,17422	0,086
t=22	0,081266	0,0087946	0,03105	0,016634	0,10962	0,05933	0,091648	0,044707	0,07348	0,10779	0,090351	0,20006	0,085269
t=23	0,080352	0,0070013	0,040295	0,017913	0,10574	0,059728	0,094271	0,047023	0,06652	0,10121	0,087854	0,20991	0,082171
t=24	0,085047	0,0003279	0,03639	0,011664	0,1062	0,061328	0,093837	0,048107	0,067016	0,10481	0,088943	0,21392	0,08241
t=25	0,09188	0,012397	0,039802	0,013496	0,10088	0,054769	0,095948	0,0521	0,054611	0,094848	0,085028	0,22783	0,076406
t=26	0,099813	0,037023	0,044486	0,006871	0,097039	0,054382	0,094126	0,055528	0,04694	0,090704	0,081044	0,21941	0,072639
t=27	0,094683	0,027228	0,05072	0,016245	0,094765	0,058913	0,10122	0,060556	0,049873	0,089799	0,083928	0,20203	0,07004
t=28	0,094518	0,025357	0,052359	0,0049933	0,098344	0,057276	0,10856	0,060218	0,040194	0,094464	0,083782	0,21077	0,069167
t=29	0,098165	0,028999	0,058772	0	0,09991	0,055768	0,11451	0,061605	0,034963	0,090688	0,08674	0,2068	0,063086
t=30	0,089476	0,049776	0,061698	0	0,10099	0,052151	0,11661	0,062813	0,028826	0,080456	8,11E-02	0,21634	0,059765
t=31	0,09384	0,049111	0,060805	0	0,10176	0,053724	0,11363	0,061902	0,031722	0,081813	0,082313	0,20881	0,060568
t=32	0,092259	0,05042	0,060427	0	0,10341	0,051983	0,11447	0,060153	0,030149	0,080469	0,081615	0,21522	0,059421
t=33	0,086998	0,064602	0,068072	0	0,10213	0,057243	0,11238	0,063878	0,03414	0,079537	0,079201	0,19258	0,059232
t=34	0,0812	0,067917	0,064371	0	0,10412	0,049336	0,11502	0,068073	0,032591	0,071422	0,076648	0,21201	0,057302
t=35	0,091025	0,069627	0,064426	0	0,11087	0,051548	0,11136	0,066158	0,037493	0,072862	0,075576	0,20096	0,058097
t=36	0,092311	0,049882	0,071011	0	0,099291	0,055921	0,10814	0,063831	0,042061	0,07543	0,0778	0,20887	0,055457
t=37	0,090093	0,049171	0,072102	0	0,097945	0,058653	0,10573	0,062289	0,046393	0,078229	0,077353	0,2059	0,056151
t=38	0,094	0,046991	0,077827	0	0,095582	0,064986	0,098997	0,060088	0,049805	0,087435	0,081231	0,18844	0,054616
t=39	0,091691	0,052315	0,077499	0	0,093766	0,062272	0,097716	6,24E-02	0,045941	0,085725	0,07899	0,19898	0,052661
t=40	0,047948	0,088002	0,073432	0	0,09795	0,058719	0,089779	0,07565	0,042619	0,099745	0,07474	0,20188	0,049535
t=41	0,28374	0,1167	0,061004	0	0,097137	0,059188	0,066294	0,088832	0,050381	0,10074	0,0662	0,20972	0,055428
t=42	0,025232	0,11575	0,062261	0	0,097184	0,060828	0,066095	0,088445	0,051333	0,10282	0,066651	0,20709	0,056301

Table 8
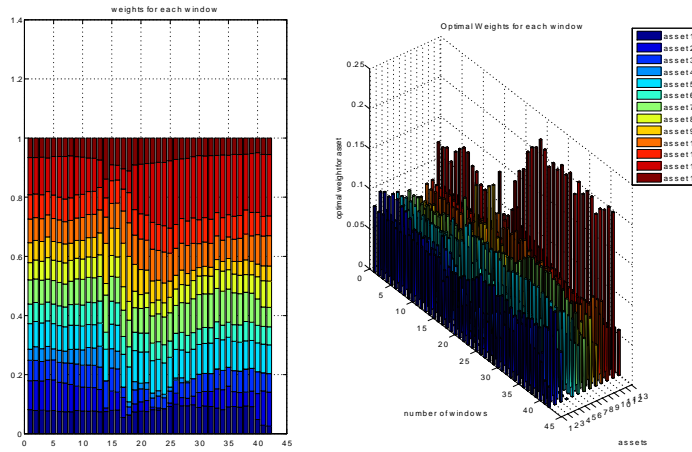
Figure 17

Table 8 and figure 17 indicate the optimal weights when we consider four moments instead of two moments and all the moments are estimated using the

sample approach. As in the case of only two moments, our portfolio is well diversified. Comparing the optimal portfolio allocation in case of two and four moments, we see that in each calibration period, the assets in the portfolio are the same, but what changes is the weight of each component.

| | Risk aversion=5 (sample Variance, sample skewness and sample kurtosis) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | asset 1 | asset 2 | asset 3 | asset 4 | asset 5 | asset 6 | asset 7 | asset 8 | asset 9 | asset 10 | asset 11 | asset 12 | asset 13 |
| t=1 | 0,081942 | 0,096886 | 0,06997 | 0,037187 | 0,086594 | 0,066297 | 0,082179 | 0,057457 | 0,067115 | 0,080334 | 0,081859 | 0,1257 | 0,066478 |
| t=2 | 0,078908 | 0,10068 | 0,068383 | 0,045939 | 0,085448 | 0,064721 | 0,079563 | 0,063961 | 0,065688 | 0,077443 | 0,079637 | 0,12324 | 0,066391 |
| t=3 | 0,07999 | 0,098685 | 0,065826 | 0,042797 | 0,087734 | 0,062984 | 0,081677 | 6,40E-02 | 0,066063 | 0,077758 | 0,08045 | 0,12729 | 0,064781 |
| t=4 | 0,077069 | 0,10626 | 0,066032 | 0,045569 | 0,087027 | 0,063202 | 0,077966 | 0,072577 | 0,069666 | 0,075165 | 0,077965 | 0,11393 | 0,06757 |
| t=5 | 0,078468 | 0,10344 | 0,067777 | 0,03721 | 0,085712 | 0,066104 | 0,079416 | 6,86E-02 | 0,063122 | 0,078191 | 0,078973 | 0,13016 | 0,062795 |
| t=6 | 0,075033 | 0,098834 | 0,067568 | 0,041057 | 0,083075 | 0,067477 | 0,078113 | 0,072068 | 0,063439 | 0,076669 | 0,076559 | 0,13769 | 0,062413 |
| t=7 | 0,07437 | 0,096814 | 0,068267 | 0,039666 | 0,082203 | 0,067059 | 0,077492 | 0,072359 | 0,063322 | 0,075938 | 0,076246 | 0,14366 | 0,062604 |
| t=8 | 0,074097 | 0,092188 | 0,072328 | 0,046939 | 0,081136 | 0,070332 | 0,078587 | 0,070328 | 0,058304 | 0,077466 | 0,076231 | 0,14175 | 0,060313 |
| t=9 | 0,078657 | 0,082161 | 0,075367 | 0,049696 | 0,080278 | 0,071794 | 0,086099 | 0,069322 | 0,059802 | 0,076208 | 0,078627 | 0,12872 | 0,06327 |
| t=10 | 0,076573 | 0,081253 | 0,07439 | 0,051316 | 0,080098 | 0,071496 | 0,08754 | 0,068228 | 0,063019 | 0,0776 | 7,79E-02 | 0,12587 | 0,064718 |
| t=11 | 0,076513 | 0,080115 | 0,076132 | 0,058606 | 0,078175 | 0,0749 | 0,084834 | 0,068761 | 0,065022 | 0,078201 | 0,077492 | 0,11385 | 0,067401 |
| t=12 | 0,076623 | 0,079425 | 0,075579 | 0,058086 | 0,078727 | 0,074744 | 0,084254 | 0,06885 | 0,067171 | 0,078276 | 0,078404 | 0,1116 | 0,068259 |
| t=13 | 0,074769 | 0,074888 | 0,076005 | 0,070443 | 0,07822 | 0,074614 | 0,082804 | 0,073355 | 0,074648 | 0,076843 | 0,076258 | 0,092748 | 0,074405 |
| t=14 | 0,075249 | 0,012093 | 0,091556 | 0 | 0,10694 | 0,056887 | 0,12617 | 0,039082 | 0,13564 | 0,088947 | 0,064886 | 0,063773 | 0,13877 |
| t=15 | 0,079284 | 0,064818 | 0,078836 | 0,064339 | 0,083802 | 0,077289 | 0,084355 | 0,070256 | 0,092526 | 0,083265 | 0,077268 | 0,053955 | 0,090006 |
| t=16 | 0,080256 | 0,0605 | 0,07918 | 0,059235 | 0,08756 | 0,07623 | 0,086392 | 0,068885 | 0,095676 | 0,085963 | 0,078367 | 0,049475 | 0,092282 |
| t=17 | 0,054139 | 0,068613 | 0,035051 | 0 | 0,11063 | 0,060404 | 0,0891 | 0,079275 | 0,14667 | 0,10926 | 0,074353 | 0,068671 | 0,10383 |
| t=18 | 0,063509 | 0 | 0,027392 | 0,017477 | 0,11774 | 0,076679 | 0,090955 | 0,062755 | 0,15228 | 0,11385 | 0,10156 | 0,07401 | 0,1108 |
| t=19 | 0,074004 | 0,02695 | 0,047605 | 0,027353 | 0,10505 | 0,062147 | 0,083145 | 0,055541 | 0,090864 | 0,098817 | 0,086503 | 0,15102 | 0,090994 |
| t=20 | 0,076038 | 0,03419 | 0,043653 | 0,018504 | 0,10545 | 0,064087 | 0,08381 | 0,046473 | 0,085348 | 0,1027 | 0,085437 | 0,16776 | 0,088314 |
| t=21 | 0,075016 | 0,037087 | 0,037909 | 0,022626 | 0,10397 | 0,063318 | 0,086965 | 0,048683 | 0,078207 | 0,10058 | 0,085419 | 0,17422 | 0,086 |
| t=22 | 0,081266 | 0,0087946 | 0,03105 | 0,016634 | 0,10962 | 0,05933 | 0,091648 | 0,044707 | 0,07348 | 0,10779 | 0,090351 | 0,20006 | 0,085269 |
| t=23 | 0,080352 | 0,0070013 | 0,040295 | 0,017913 | 0,10574 | 0,059728 | 0,094271 | 0,047023 | 0,06652 | 0,10121 | 0,087854 | 0,20991 | 0,082171 |
| t=24 | 0,085047 | 0,0003279 | 0,03639 | 0,011664 | 0,1062 | 0,061328 | 0,093837 | 0,048107 | 0,067016 | 0,10481 | 0,088943 | 0,21392 | 0,08241 |
| t=25 | 0,09188 | 0,012397 | 0,039802 | 0,013496 | 0,10088 | 0,054769 | 0,095948 | 0,0521 | 0,054611 | 0,09484 8 | 0,085028 | 0,22783 | 0,076406 |
| t=26 | 0,099813 | 0,037023 | 0,044486 | 0,006871 | 0,097039 | 0,054382 | 0,094126 | 0,055528 | 0,04694 | 0,090704 | 0,081044 | 0,21941 | 0,072639 |
| t=27 | 0,094683 | 0,027228 | 0,05072 | 0,016245 | 0,094765 | 0,058913 | 0,10122 | 0,060556 | 0,049873 | 0,089799 | 0,083928 | 0,20203 | 0,07004 |
| t=28 | 0,094518 | 0,025357 | 0,052359 | 0,0049933 | 0,098344 | 0,05727 6 | 0,10856 | 0,060218 | 0,040194 | 0,094464 | 0,083782 | 0,21077 | 0,069167 |
| t=29 | 0,098165 | 0,028999 | 0,058772 | 0 | 0,09991 | 0,055768 | 0,11451 | 0,061605 | 0,034963 | 0,090688 | 0,08674 | 0,2068 | 0,063086 |
| t=30 | 0,089476 | 0,049776 | 0,061698 | 0 | 0,10099 | 0,052151 | 0,11661 | 0,062813 | 0,028826 | 0,080456 | 8,11E-02 | 0,21634 | 0,059765 |
| t=31 | 0,09384 | 0,049111 | 0,060805 | 0 | 0,10176 | 0,053724 | 0,11363 | 0,061902 | 0,031722 | 0,081813 | 0,082313 | 0,20881 | 0,060568 |
| t=32 | 0,092259 | 0,05042 | 0,060427 | 0 | 0,10341 | 0,051983 | 0,11447 | 0,060153 | 0,030149 | 0,080469 | 0,081615 | 0,21522 | 0,059421 |
| t=33 | 0,086998 | 0,064602 | 0,068072 | 0 | 0,10213 | 0,057243 | 0,11238 | 0,063878 | 0,03414 | 0,079537 | 0,079201 | 0,19258 | 0,059232 |
| t=34 | 0,0812 | 0,067917 | 0,064371 | 0 | 0,10412 | 0,049936 | 0,11502 | 0,068073 | 0,032591 | 0,071422 | 0,076648 | 0,21201 | 0,057302 |
| t=35 | 0,091025 | 0,069627 | 0,064426 | 0 | 0,10087 | 0,051548 | 0,11136 | 0,066158 | 0,037493 | 0,072862 | 0,075576 | 0,20096 | 0,058097 |
| t=36 | 0,092311 | 0,049882 | 0,071011 | 0 | 0,099291 | 0,055921 | 0,10814 | 0,063831 | 0,042061 | 0,07543 | 0,0778 | 0,20887 | 0,055457 |
| t=37 | 0,090093 | 0,049171 | 0,072102 | 0 | 0,097945 | 0,058653 | 0,10573 | 0,062289 | 0,046393 | 0,078229 | 0,077353 | 0,2059 | 0,056151 |
| t=38 | 0,094 | 0,046991 | 0,077827 | 0 | 0,095582 | 0,064986 | 0,098997 | 0,060088 | 0,049805 | 0,087435 | 0,081231 | 0,18844 | 0,054616 |
| t=39 | 0,091691 | 0,052315 | 0,077499 | 0 | 0,093766 | 0,062272 | 0,097716 | 6,24E-02 | 0,045941 | 0,085725 | 0,07899 | 0,19898 | 0,052661 |
| t=40 | 0,047948 | 0,088002 | 0,073432 | 0 | 0,09795 | 0,058719 | 0,088779 | 0,07565 | 0,042619 | 0,099745 | 0,07474 | 0,20188 | 0,049535 |
| t=41 | 0,028374 | 0,1167 | 0,061004 | 0 | 0,088832 | 0,059188 | 0,066294 | 0,097137 | 0,050381 | 0,10074 | 0,0662 | 0,20972 | 0,055428 |
| t=42 | 0,025232 | 0,11575 | 0,062261 | 0 | 0,088445 | 0,060828 | 0,066095 | 0,097184 | 0,051333 | 0,10282 | 0,066651 | 0,20709 | 0,056301 |
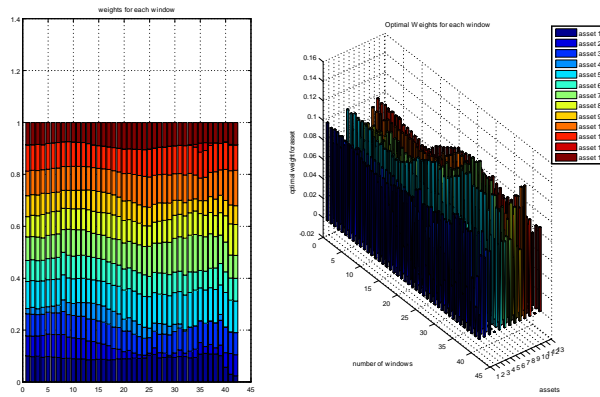
Table 9

Figure 18

Table 9 and figure 18 indicate the results obtained in case of Constant correlation approach when we use only two moments. As in the first model, the

portfolio is well diversified during all the calibration periods.

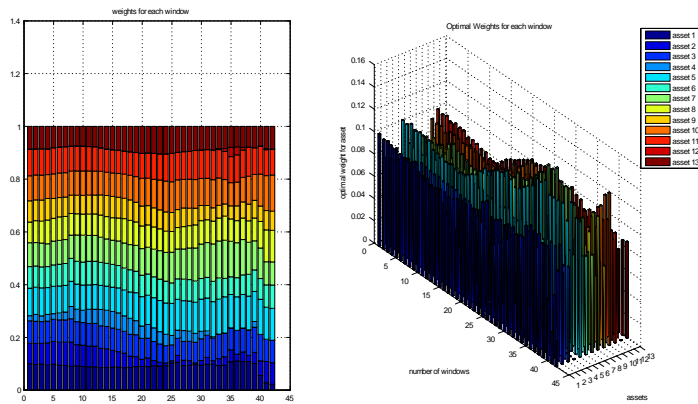|	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8	asset 9	asset 10	asset 11	asset 12	asset 13
					Risk avversion=5 (CC Variance, CC skewness and CC kurtosis)								
t=1	0,10039	0,076649	0,085811	0,019523	0,1043	0,08133	0,091027	0,077978	0,080684	0,094706	0,10136	0	0,086243
t=2	0,09782	0,078645	0,08472	0,025883	0,10326	0,080287	0,089284	0,082583	0,079209	0,093234	0,099293	0	0,085779
t=3	0,099029	0,078235	0,082801	0,02178	0,10607	0,079248	0,09096	0,082526	0,080134	0,093733	0,10066	0	0,084819
t=4	0,095319	0,083935	0,082235	0,0253	0,10245	0,078813	0,087177	0,089481	0,082196	0,090337	0,096874	0	0,085881
t=5	0,097848	0,086497	0,084082	0,021563	0,10392	0,081021	0,090326	0,08591	0,075322	0,093734	0,09912	6,94E-18	0,080655
t=6	0,09533	0,086876	0,083819	0,02654	0,10229	0,082203	0,090693	0,088652	0,073829	0,093336	0,097954	0	0,078482
t=7	0,094264	0,087959	0,083933	0,029726	0,1016	0,082139	0,091133	0,08783	0,073388	0,092787	0,097839	0	0,077407
t=8	0,090505	0,091511	0,088489	0,046327	0,095979	0,086555	0,08843	0,082119	0,069444	0,091534	0,094334	6,94E-18	0,074773
t=9	0,091369	0,081287	0,087043	0,046822	0,095359	0,086556	0,099875	0,079824	0,070996	0,091966	0,094532	0	0,07437
t=10	0,08982	0,080002	0,086615	0,046805	0,095242	0,086692	0,10221	0,079729	0,071464	0,092555	0,093816	0	0,075046
t=11	0,088606	0,078328	0,086077	0,053438	0,093017	0,088703	0,098796	0,080232	0,072313	0,091809	0,091951	0	0,076732
t=12	0,088662	0,078163	0,086343	0,053496	0,093605	0,088062	0,098503	0,080842	0,072243	0,091037	0,091637	0	0,077405
t=13	0,085803	0,072576	0,085157	0,062022	0,091557	0,086966	0,095427	0,084001	0,07037	0,089134	0,088616	0	0,081703
t=14	0,08611	0,070361	0,085369	0,061304	0,091002	0,086347	0,09303	0,084294	0,080249	0,0884	0,087453	0	0,086081
t=15	0,087282	0,063152	0,085079	0,062633	0,092361	0,085552	0,092769	0,085096	0,08046	0,089235	0,087504	0	0,088876
t=16	0,087679	0,058092	0,085191	0,062054	0,094779	0,084327	0,093744	0,08666	0,078935	0,090304	0,0875	0	0,090736
t=17	0,085483	0,05441	0,083362	0,058459	0,097922	0,083521	0,094645	0,092574	0,077098	0,092024	0,086728	0	0,093775
t=18	0,086606	0,042211	0,082623	0,060915	0,10017	0,083881	0,095549	0,091531	0,078129	0,092552	0,089305	0	0,096531
t=19	0,089166	0,03775	0,0828	0,053467	0,1056	0,081896	0,098165	0,088724	0,075211	0,09685	0,091591	0	0,098778
t=20	0,089652	0,0274	0,079162	0,046796	0,11094	0,081938	0,10079	0,085806	0,075998	0,10448	0,094963	0	0,10208
t=21	0,089591	0,032342	0,074585	0,052454	0,1109	0,080394	0,10162	0,084251	0,073522	0,10401	0,095416	0	0,10092
t=22	0,091967	0,020169	0,072157	0,051582	0,11443	0,079622	0,10485	0,083995	0,072833	0,10787	0,09835	0	0,10218
t=23	0,090988	0,01453	0,069493	0,047094	0,11726	0,081716	0,10818	0,082425	0,071893	0,11255	0,10021	0	0,10366
t=24	0,092384	0,011121	0,065045	0,043097	0,1183	0,083468	0,10925	0,080179	0,073437	0,1162	0,10334	0	0,10418
t=25	0,10248	0,0085214	0,060217	0,036086	0,12191	0,077857	0,11619	0,078625	0,071959	0,1159	0,10813	0	0,10212
t=26	0,10598	0,025864	0,063835	0,031944	0,11692	0,077771	0,11331	0,080437	0,068968	0,11178	0,10486	0	0,098333
t=27	0,093717	0,019477	0,070771	0,042442	0,11251	0,08193	0,1149	0,084007	0,07126	0,10771	0,10597	0	0,095304
t=28	0,094435	0,017227	0,07283	0,039281	0,11398	0,081587	0,1195	0,084519	0,066791	0,10972	0,10535	0	0,094782
t=29	0,096933	0,0062026	0,07829	0,033566	0,11803	0,081192	0,12452	0,087427	0,06375	0,10806	0,11057	0	0,091463
t=30	0,094687	0,017368	0,082346	0,028719	0,12171	0,078681	0,13008	0,088665	0,057944	0,10234	0,10801	0	0,089446
t=31	0,09829	0,025517	0,083986	0,016379	0,12468	0,078572	0,13228	0,086544	0,053826	0,10266	0,10933	0	0,087928
t=32	0,098154	0,013666	0,084162	0,0060012	0,13357	0,077228	0,14176	0,085973	0,051123	0,1063	0,11309	0	0,088979
t=33	0,09338	0,025886	0,092741	0	0,13303	0,080055	0,14054	0,086531	0,048469	0,10323	0,1086	0,0025996	0,084944
t=34	0,094633	0,032104	0,089547	0	0,1373	0,070404	0,14443	0,090899	0,047082	0,091739	0,1071	0,010786	0,083981
t=35	0,10761	0,044648	0,083931	0	0,12666	0,067158	0,13431	0,084617	0,050314	0,086823	0,099606	0,034847	0,07947
t=36	0,11024	0,026329	0,09025	1,39E-17	0,12707	0,07264	0,13269	0,083082	0,056908	0,089393	0,10351	0,030004	0,077884
t=37	0,1081	0,036556	0,090651	0	0,12712	0,074792	0,13345	0,081552	0,061575	0,092717	0,10363	0,01079	0,079068
t=38	0,10815	0,023742	0,096829	0	0,12753	0,082931	0,12151	0,080845	0,063968	0,1049	0,11116	0	0,078427
t=39	0,10174	0,04112	0,095526	0,022017	0,1141	0,081558	0,11007	0,086245	0,067334	0,09769	0,10211	0	0,080497
t=40	0,05498	0,056193	0,099518	0	0,13337	0,07923	0,11537	0,10085	0,059057	0,1199	0,10527	0,001801	0,074451
t=41	0,027827	0,080867	0,083443	0	0,12168	0,085666	0,087148	0,12076	0,073648	0,13209	0,098	0,0038214	0,085044
t=42	0,02203	0,08191	0,084907	0	0,12129	0,088349	0,087202	0,11975	0,07323	0,13655	0,098498	0	0,086291

Table 10

Figure 19

Table 10 and figure 19 show the optimal weight of each asset in portfolio when we consider four moments and we use the CC approach to estimate the variance, skewness and kurtosis.

In the following tables and graphs we report the results of the optimal portfolio allocation with two and four moments when we use the shrinkage approach to estimate the co-moments.

	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8	asset 9	asset 10	asset 11	asset 12	asset 13
	Risk avversion=5 (shrinkage Variance)												
t=1	0,096488	0,10215	0,073072	0	0,10368	0,065586	0,093037	0,052099	0,065493	0,090693	0,09659	0,093033	0,068072
t=2	0,093767	0,1149	0,069068	0	0,10636	0,060217	0,091042	0,060208	0,060095	0,087893	0,09555	0,095055	0,065847
t=3	0,09902	0,10912	0,064445	0	0,10804	0,058083	0,093513	0,060965	0,062204	0,087172	0,095171	0,10409	0,063292
t=4	0,090484	0,12524	0,06473	0	0,1095	0,057834	0,087657	0,079138	0,069045	0,083511	0,092641	0,071278	0,068942
t=5	0,085648	0,11155	0,068466	0,012948	0,095589	0,065317	0,085454	0,069518	0,059316	0,083619	0,086428	0,11528	0,060872
t=6	0,078962	0,10185	0,068624	0,029332	0,088342	0,068068	0,082003	0,073984	0,061542	0,079938	0,080943	0,12473	0,061679
t=7	0,077623	0,098982	0,069334	0,031026	0,086406	0,067723	0,08091	0,073859	0,061832	0,078659	0,079951	0,13166	0,06203
t=8	0,07553	0,092843	0,073167	0,044448	0,082526	0,071028	0,080223	0,07071	0,057701	0,078575	0,077789	0,13525	0,060205
t=9	0,080002	0,08255	0,07606	0,046863	0,081876	0,072539	0,088053	0,06936	0,059203	0,077646	0,080174	0,12277	0,0629
t=10	0,077748	0,08158	0,075012	0,048571	0,081622	0,072196	0,089486	0,068286	0,062442	0,078975	0,079259	0,12042	0,064402
t=11	0,077757	0,080524	0,076894	0,055427	0,079813	0,075888	0,087086	0,068673	0,064015	0,079788	0,078971	0,10827	0,066891
t=12	0,077753	0,079808	0,076287	0,055126	0,080256	0,075577	0,086308	0,068774	0,06619	0,079656	0,079718	0,10675	0,067798
t=13	0,075345	0,073988	0,076683	0,065709	0,080753	0,075083	0,087429	0,072888	0,073438	0,07863	0,077643	0,088623	0,073789
t=14	0,079385	0,060719	0,082613	0,053896	0,088063	0,075004	0,092925	0,07016	0,090707	0,083335	0,077234	0,0528	0,093158
t=15	0,080048	0,066023	0,07971	0,06491	0,084688	0,078588	0,085154	0,073339	0,08862	0,083761	0,078937	0,047522	0,0883
t=16	0,081236	0,061522	0,079981	0,060682	0,088044	0,077552	0,086935	0,072185	0,091524	0,086122	0,07966	0,04376	0,090796
t=17	0,059814	0,061836	0,042169	-1,39E-17	0,11529	0,064861	0,094604	0,085028	0,1391	0,1118	0,078132	0,04015	0,10722
t=18	0,072744	0,024817	0,052325	0,04047	0,10601	0,074389	0,090249	0,074054	0,1179	0,10174	0,094066	0,049925	0,10131
t=19	0,072253	0	0,040733	0,0049803	0,12277	0,059369	0,091825	0,052728	0,094228	0,11193	0,094084	0,14784	0,10226
t=20	0,07903	0,011929	0,037632	0	0,11926	0,062712	0,090868	0,04251	0,086564	0,11439	0,091388	0,16713	0,09659
t=21	0,077469	0,02273	0,032199	1,03E-02	0,11446	0,062137	0,093215	0,046115	0,07757	0,10919	0,090232	0,17236	0,092
t=22	0,084039	0	0,030304	0,012079	0,11617	0,059744	0,096473	0,045864	0,072557	0,11314	0,093997	0,18597	0,089672
t=23	0,082393	0	0,039439	0,014387	0,11107	0,060456	0,098208	0,048001	0,065914	0,10592	0,090905	0,19745	0,085861
t=24	0,086219	0	0,037915	0,012646	0,1089	0,06306	0,096231	0,050559	0,06717	0,10725	0,091046	0,19377	0,085227
t=25	0,093545	0,0086244	0,039165	0,01183	0,10384	0,055115	0,098433	0,052584	0,054584	0,097406	0,087177	0,2194	0,078298
t=26	0,1014	0,034203	0,044162	0,0053947	0,09945	0,054779	0,0962	0,056042	0,04695	0,092809	0,082877	0,21153	0,074205
t=27	0,096316	0,023273	0,049987	0,013756	0,09742	0,059184	0,10397	0,061039	0,049555	0,092121	0,086166	0,19574	0,07147
t=28	0,096505	0,020451	0,051673	0,0015178	0,10159	0,057519	0,11221	0,060685	0,039279	0,097386	0,086307	0,20415	0,070723
t=29	0,099349	0,02538	0,059189	0	0,10234	0,056843	0,11681	0,062953	0,035538	0,092838	0,089157	0,19471	0,064885
t=30	0,090447	0,047137	0,062429	0	0,10326	0,053126	0,11875	0,064124	0,029363	0,082117	0,083122	0,20475	0,061377
t=31	0,094571	0,047302	0,061769	0	0,10363	0,054778	0,11539	0,063006	0,032353	0,082202	0,084066	0,19791	0,062027
t=32	0,092952	0,048017	0,061569	0	0,10553	0,053296	0,11646	0,061491	0,031152	0,082163	0,083623	0,20255	0,061198
t=33	0,087466	0,062952	0,06916	0	0,10363	0,058389	0,11363	0,064937	0,035067	0,080739	0,080645	0,18275	0,060635
t=34	0,082035	0,066439	0,065479	0	0,10565	0,05046	0,11626	0,069059	0,033538	0,072511	0,078118	0,20171	0,058741
t=35	0,091742	0,068268	0,065351	1,39E-17	0,1022	0,052358	0,1125	0,067006	0,038221	0,073619	0,076787	0,1927	0,059248
t=36	0,093067	0,048771	0,071794	0	0,10061	0,056666	0,10916	0,064524	0,042725	0,075864	0,07887	0,2013	0,056656
t=37	0,090933	0,048508	0,073001	0	0,099349	0,05943	0,10705	0,062937	0,047086	0,07871	0,078525	0,19717	0,057304
t=38	0,095423	0,044452	0,079063	0	0,097406	0,066179	0,10085	0,060773	0,050854	0,088206	0,083023	0,17754	0,056234
t=39	0,09336	0,049774	0,078928	0	0,095543	0,063333	0,099698	0,063132	0,046706	0,086446	0,08062	0,18834	0,054119
t=40	0,05235	0,081785	0,076021	0	0,10052	0,06091	0,093104	0,076335	0,044224	0,10048	0,077776	0,18464	0,051857
t=41	0,029393	0,11047	0,063926	0	0,1004	0,06225	0,069796	0,092297	0,053384	0,10497	0,07049	0,18502	0,0586
t=42	0,025237	0,10996	0,065234	0	0,10049	0,064073	0,069545	0,091804	0,052999	0,10772	0,070904	0,18257	0,059466

Table 11

Figure 20

|	asset 1	asset 2	asset 3	asset 4	asset 5	asset 6	asset 7	asset 8	asset 9	asset 10	asset 11	asset 12	asset 13
				Risk avversion=5 (shrinkage Variance, shrinkage skewness and shrinkage kurtosis)									
t=1	0,095183	0,10261	0,0727	0	0,10243	0,065484	0,0923	0,052688	0,065531	0,089846	0,095308	0,098079	0,067842
t=2	0,092467	0,11493	0,068825	0,00E+00	0,10491	0,060323	0,090236	0,060714	0,060397	0,087062	0,09421	0,10017	0,065752
t=3	0,091566	0,10745	0,064998	0,0052598	0,1047	0,059184	0,091716	0,062134	0,063059	0,085689	0,092737	0,10769	0,063869
t=4	0,089625	0,12474	0,064745	0	0,10822	0,058098	0,087052	0,079019	0,069186	0,083032	0,091733	0,075596	0,068952
t=5	0,084071	0,1092	0,068706	0,018338	0,093181	0,065868	0,08449	0,069556	0,060332	0,082485	0,084771	0,11742	0,061582
t=6	0,078026	0,10077	0,068653	3,22E-02	0,086902	0,068194	0,081409	0,07362	0,062106	0,079185	0,079867	0,12696	0,06206
t=7	0,076813	0,098198	0,069303	0,033416	0,085193	0,0678	0,080415	0,07353	0,062275	0,078008	0,078995	0,13375	0,062308
t=8	0,075193	0,093096	0,073148	0,04494	0,082439	0,07102	0,079734	0,0707	0,057832	0,07851	0,07752	0,13564	0,060234
t=9	0,079716	0,08269	0,075991	0,047385	0,081592	0,072398	0,087765	0,069457	0,059259	0,077317	0,079877	0,1236	0,06295
t=10	0,077478	0,081661	0,074952	0,049157	0,081317	0,072075	0,089203	0,068357	0,062557	0,078663	0,07901	0,12109	0,064481
t=11	0,077489	0,080552	0,076779	0,056252	0,079487	0,075721	0,086698	0,068832	0,064329	0,079441	0,078694	0,10862	0,067104
t=12	0,077484	0,079821	0,076158	0,05601	0,079911	0,075397	0,085851	0,068946	0,066556	0,079292	0,07942	0,10711	0,068045
t=13	0,075229	0,074387	0,076585	0,067161	0,079992	0,075018	0,085973	0,073228	0,074016	0,078074	0,077254	0,088837	0,074246
t=14	0,079233	0,059724	0,082885	0,05263	0,088372	0,074411	0,093645	0,069537	0,092069	0,083217	0,076817	0,052924	0,094535
t=15	0,080306	0,066299	0,0796	0,065198	0,08439	0,078445	0,084848	0,073297	0,088779	0,083505	0,078766	0,0483	0,088269
t=16	0,081121	0,061812	0,079888	0,060793	0,087808	0,077392	0,086738	0,072143	0,091638	0,085903	0,079528	0,044477	0,090762
t=17	0,059205	0,063122	0,041261	0	0,11457	0,064124	0,093883	0,084297	0,14007	0,1114	0,077721	0,043532	0,10681
t=18	0,071701	0,021377	0,0494	0,037567	0,10754	0,073677	0,090375	0,072741	0,12244	0,1035	0,095112	0,051989	0,10258
t=19	0,076675	0	0,039667	0,0040953	0,12249	0,058601	0,091279	0,05198	0,094754	0,1118	0,093707	0,15317	0,10179
t=20	0,07867	0,014076	0,037763	8,02E-04	0,11812	0,062589	0,090245	0,042659	0,086601	0,11345	0,090781	0,16846	0,095781
t=21	0,077151	0,024441	0,032399	0,011025	0,11364	0,062061	0,092793	0,046437	0,077582	0,10849	0,089768	0,17276	0,091461
t=22	0,088838	0	0,029558	0,011401	0,11622	0,059361	0,09647	0,045496	0,072509	0,11321	0,093916	0,18856	0,089455
t=23	0,082193	0	0,039272	0,013974	0,11088	0,060237	0,098252	0,047795	0,065732	0,10571	0,090725	0,19969	0,085539
t=24	0,086193	0	0,037896	0,012398	0,10875	0,063081	0,096294	0,050577	0,067121	0,10718	0,090984	0,19447	0,085057
t=25	0,093802	0,007531	0,038564	0,01049	0,10416	0,054793	0,098837	0,052244	0,054182	0,097663	0,087282	0,22223	0,07822
t=26	0,10169	0,033808	0,043702	0,0040904	0,09964	0,054473	0,096515	0,055838	0,046451	0,092956	0,082872	0,21391	0,074056
t=27	0,09597	0,023233	0,050248	0,013786	0,097261	0,059219	0,1039	0,061045	0,049418	0,091939	0,08596	0,19671	0,071311
t=28	0,096104	0,020381	0,051886	0,0016022	0,10141	0,057515	0,11213	0,060739	0,039144	0,097155	0,08608	0,20532	0,070534
t=29	0,099003	0,025461	0,059484	0	0,10212	0,056809	0,11667	0,062922	0,035426	0,092497	0,088886	0,19608	0,064648
t=30	0,09025	0,04681	0,062585	0	0,10307	0,053132	0,11868	0,064093	0,029255	0,08199	0,082993	0,20594	0,061201
t=31	0,094423	0,04724	0,061849	0	0,10349	0,054728	0,11533	0,06304	0,032167	0,083073	0,083951	0,19887	0,061838
t=32	0,092781	0,048002	0,061658	0	0,10541	0,053224	0,1164	0,061506	0,030934	0,082007	0,083495	0,2036	0,060987
t=33	0,087314	0,062852	0,069217	0	0,10351	0,058333	0,11361	0,064949	0,034875	0,08063	0,080554	0,1837	0,060454
t=34	0,081845	0,066347	0,065496	0	0,10555	0,050315	0,11627	0,069105	0,0333	0,072348	0,078005	0,2029	0,058522
t=35	0,091809	0,068439	0,065389	0	0,10208	0,052345	0,11242	0,067079	0,038182	0,073548	0,076743	0,1928	0,059164
t=36	0,093049	0,048932	0,071822	0	0,10044	0,056654	0,10913	0,064669	0,042735	0,076006	0,078877	0,20125	0,056444
t=37	0,090821	0,048684	0,072872	-1,39E-17	0,099144	0,059346	0,10684	0,063124	0,047062	0,078811	0,078444	0,1977	0,057148
t=38	0,094929	0,045547	0,078762	0	0,097056	0,065905	0,10034	0,060964	0,050623	0,088136	0,082666	0,17917	0,055904
t=39	0,092763	0,050947	0,078546	0	0,095231	0,063046	0,09908	0,06338	0,046516	0,086425	0,080294	0,18998	0,053792
t=40	0,050469	0,082479	0,076294	0	0,10136	0,061005	0,093033	0,0773	0,044355	0,10125	0,078009	0,18231	0,052139
t=41	0,02747	0,11218	0,063706	0	0,10077	0,062446	0,068906	0,093443	0,053	0,10535	0,070352	0,18324	0,059139
t=42	0,02361	0,11169	0,064929	0	0,10086	0,06416	0,068679	0,093002	0,05358	0,10797	0,070751	0,18081	0,059961

Table 12
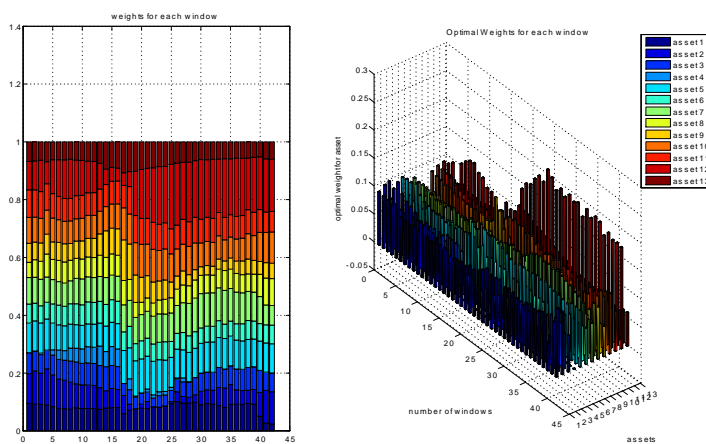
Figure 21

Observation: Analyzing these results, independently from the approach we use for the estimation of the co-moments, we can see that our portfolio

is well diversified in both cases, which are portfolio allocation based on two moments and portfolio allocation based on four moments. What changes from one portfolio to the other is the weight of each asset.

We have reported in this Section the results obtained for lambda =5 and we have seen that the optimal weights change slightly from the portfolio with two moments to that with four moments. It's not the same when we consider high values of risk aversion, because the higher is the risk aversion the higher is the impact that higher moments have on the utility function, and the difference on optimal portfolios become more evident [20].

5.6.1 Value of the CARA utility function in the optimal portfolio allocation

The following table indicates the maximum value of the CARA utility function for each calibration period and for each model, in case $\lambda = 5$ and 24-3 rolling window strategy.

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[20]Our portfolios are still well diversified but the weight of each component in the portfolio is very different when two or four moments are considered.

	Maximum Values for CARA utility function with a Risk aversion=5					
	Sample MV	Sample MVSK	CC MV	CC MVSK	Shrink MV	Shrink MVSK
t=1	-0,94951	-0,9495	-0,95204	-0,9521	-0,95033	-0,9504
t=2	-0,95412	-0,95411	-0,95535	-0,95539	-0,95467	-0,95471
t=3	-0,95144	-0,95143	-0,95287	-0,95292	-0,95203	-0,95208
t=4	-0,95279	-0,95279	-0,95621	-0,95632	-0,95387	-0,95393
t=5	-0,95224	-0,95226	-0,95048	-0,95049	-0,95207	-0,95206
t=6	-0,95063	-0,95075	-0,94925	-0,94927	-0,95024	-0,95025
t=7	-0,95035	-0,95043	-0,94882	-0,94885	-0,9501	-0,95012
t=8	-0,93074	-0,93072	-0,92975	-0,92975	-0,93068	-0,93068
t=9	-9,43E-01	-0,94298	-0,94081	-9,41E-01	-0,94294	-0,94294
t=10	-9,45E-01	-0,94474	-0,94455	-9,45E-01	-0,94479	-0,94478
t=11	-9,52E-01	-0,95205	-0,95183	-0,95182	-0,95211	-0,95209
t=12	-0,95509	-0,95509	-0,95666	-0,95665	-0,95522	-0,9552
t=13	-0,95724	-0,95725	-0,95919	-0,95919	-0,95739	-0,95738
t=14	-0,95501	-0,95503	-0,95628	-0,95632	-0,95511	-0,95515
t=15	-0,95	-0,95007	-0,95166	-0,95172	-0,95013	-0,95017
t=16	-0,94096	-0,94106	-0,93965	-0,9397	-0,94095	-0,94099
t=17	-0,96089	-0,96092	-0,9624	-0,96243	-0,96099	-0,96102
t=18	-0,96675	-0,96677	-0,96779	-0,96781	-0,96685	-0,96688
t=19	-0,9626	-0,96261	-0,96396	-0,96397	-0,96269	-0,9627
t=20	-0,95819	-0,9582	-0,95843	-0,95843	-0,95822	-0,95823
t=21	-0,95733	-0,95733	-0,95727	-0,95728	-0,95727	-0,95728
t=22	-0,95273	-0,95273	-0,95308	-0,95309	-0,95276	-0,95277
t=23	-0,95858	-0,95859	-0,95834	-0,95835	-0,95855	-0,95856
t=24	-0,95139	-0,95139	-0,95119	-0,95119	-0,95136	-0,95136
t=25	-0,94691	-0,94692	-0,94663	-0,94663	-0,94689	-0,94689
t=26	-0,94649	-0,94649	-0,94621	-0,94621	-0,94647	-0,94647
t=27	-0,95777	-0,95777	-0,95755	-0,95755	-0,95776	-0,95776
t=28	-9,57E-01	-0,95745	-0,95723	-0,95723	-0,95744	-0,95744
t=29	-9,64E-01	-0,96396	-0,96307	-9,63E-01	-0,9639	-0,9639
t=30	-9,61E-01	-0,96107	-0,95993	-9,60E-01	-0,96101	-0,96101
t=31	-0,96028	-0,96029	-0,95915	-0,95915	-0,96023	-0,96023
t=32	-0,96011	-0,96012	-0,95965	-0,95966	-0,96008	-0,96008
t=33	-0,96	-0,96001	-0,95941	-0,95941	-0,95993	-0,95993
t=34	-0,95601	-0,95602	-0,95564	-0,95565	-0,95598	-0,95598
t=35	-9,49E-01	-0,94901	-0,94864	-9,49E-01	-0,94896	-0,94897
t=36	-0,95559	-0,9556	-0,95519	-0,9552	-0,95556	-0,95557
t=37	-0,95678	-0,9568	-0,95608	-0,95609	-0,95668	-0,95669
t=38	-0,97006	-0,97006	-0,97037	-0,97039	-0,9701	-0,97011
t=39	-0,96745	-0,96746	-0,96755	-0,96757	-0,96748	-0,9675
t=40	-0,97338	-0,97331	-0,97332	-0,97323	-0,97338	-0,97328
t=41	-0,9523	-0,95213	-0,95239	-0,95232	-0,95243	-0,95236
t=42	-0,94958	-0,94944	-0,94809	-0,94797	-0,94953	-0,94948

Table 13

The analysis that we have done so far are all in-sample analysis. What we need is the out-of-sample analysis, because it's important to know which model works better in the out of sample period. Let's start the out-of-sample analysis with the calculation of the out-of-sample portfolio returns and performance.

5.6.2 Impact of co-moments on performance

An investor may ask: what is the impact of higher moments in the out-of-sample period? Is a portfolio allocation better based on two moments or on four moments?

To answer to these questions, we calculate the monthly returns and performance of the portfolio in the out-of-sample period.

Up to now, we have estimated the co-moments of the portfolio using the different approaches and we have calculated the optimal weights for each cali-

bration period. The number of calibration periods is different for each rolling window strategy that we have taken into consideration.

- The rolling window strategy 24_3 has 42 calibration periods.

- The rolling window strategy 24_6 has 21 calibration periods.

- The rolling window strategy 48_3 has 34 calibration periods.

- The rolling window strategy 48_6 has 17 calibration periods.

In this analysis we have adopted a fix-mix strategy, where we keep the weights constant for the next out-of-sample period 3 (6) months and we find the out of sample "monthly" return of the portfolio[21] that is given for each time step by:

$$R_p^i(t) = \sum_{j=1}^{J} w_{j,t} R_j(t)$$

Where $w_{j,t}$ is the weight of asset j in the out-of-sample period t and $R_j(t)$ is the return of asset j in time t. In our analysis the out of sample observations are 126 when the in-sample period is 24 and 102 when the in-sample period is 48.

Having computed the out-of-sample return of the portfolio, we can calculate the out-of-sample performance using the following definition:

$$Performance(t) = 100 \prod_{t=1}^{T} (1 + R_p(t))$$

The following graphs display the performance obtained with the different approaches of estimation for the co-moments, where the optimal weights are those that we found maximizing the CARA utility function with $\lambda = 1$, $\lambda = 5$, $\lambda = 10$, $\lambda = 20$, $\lambda = 30$, $\lambda = 50$, $\lambda = 70$ and $\lambda = 90$.

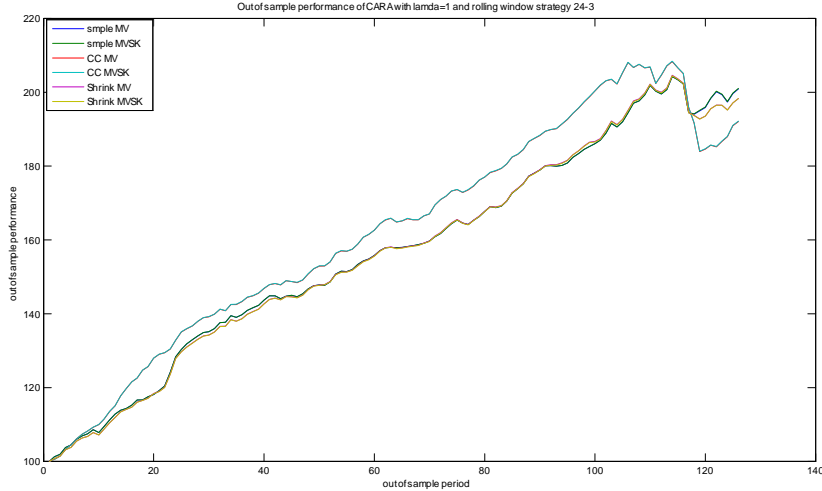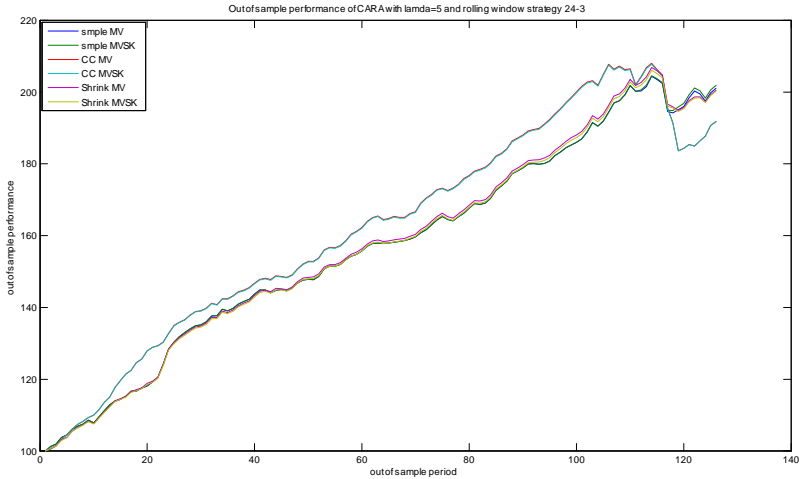[21]The return of the portfolio is equal to the weighted average of the asset returns.
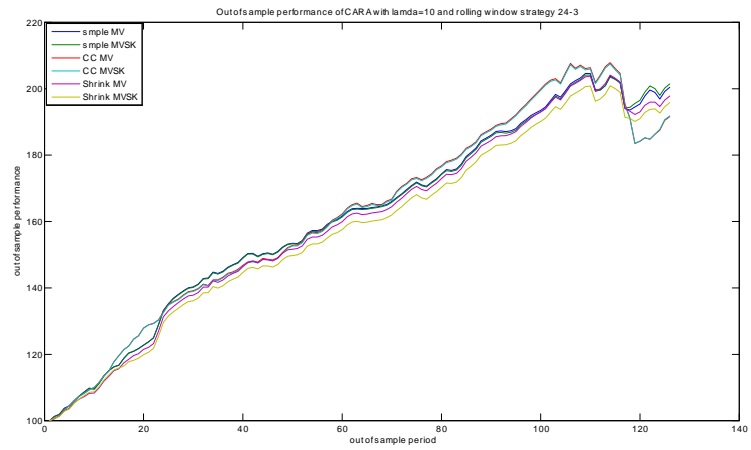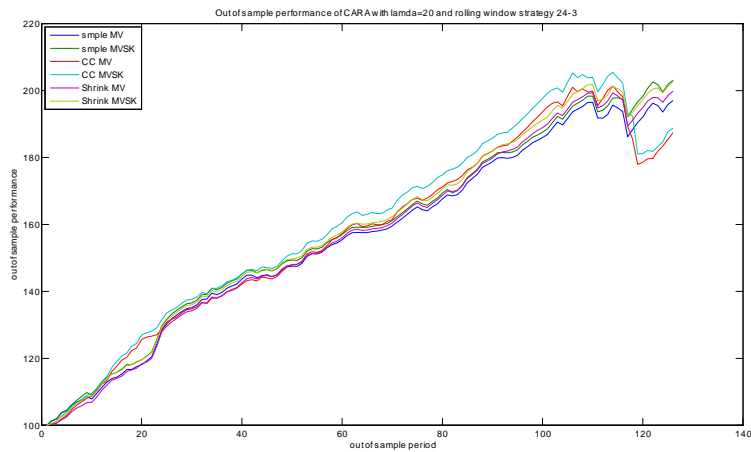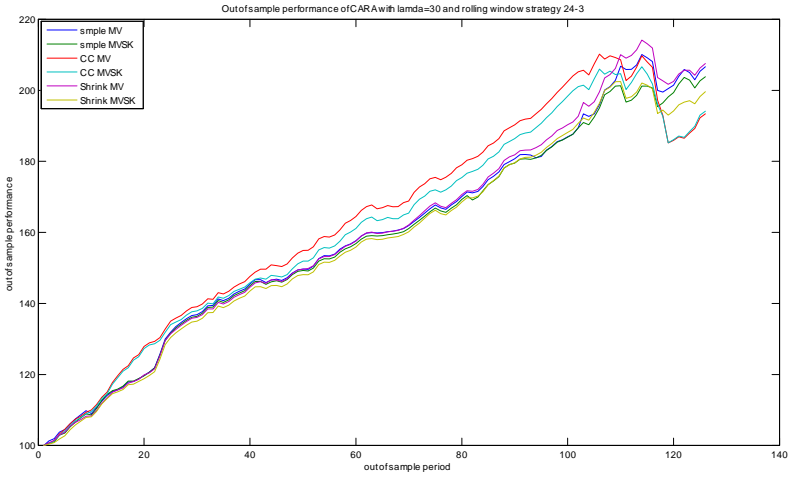
Figure 22

Figure 23

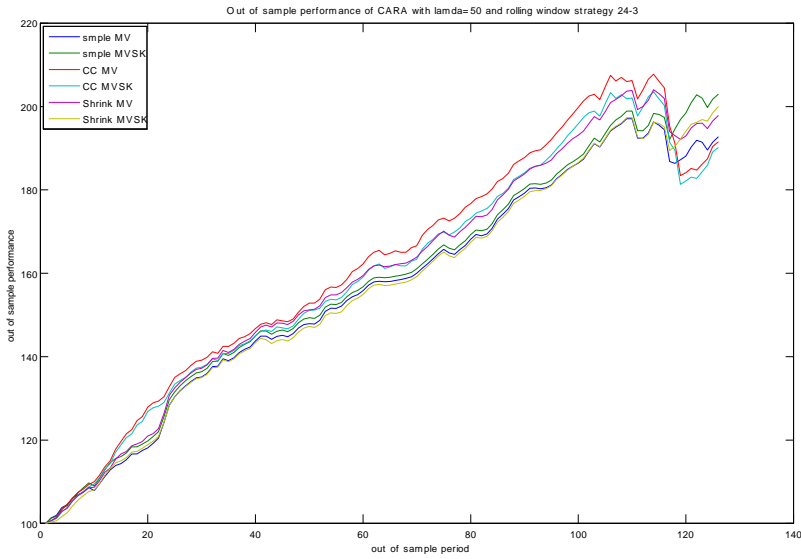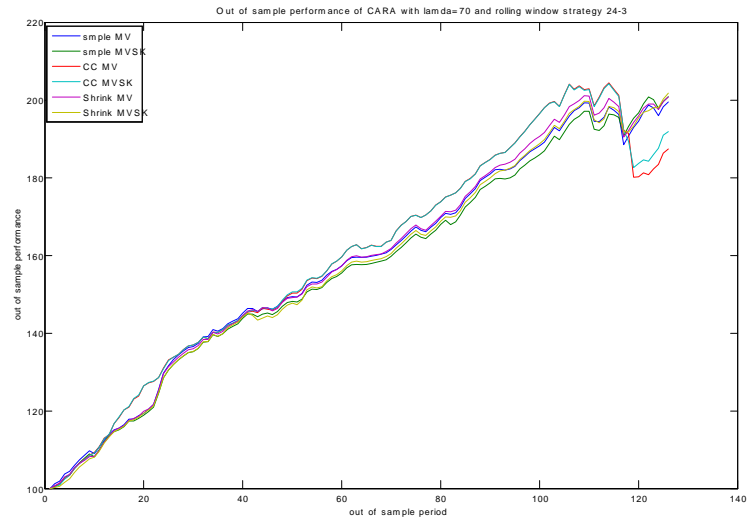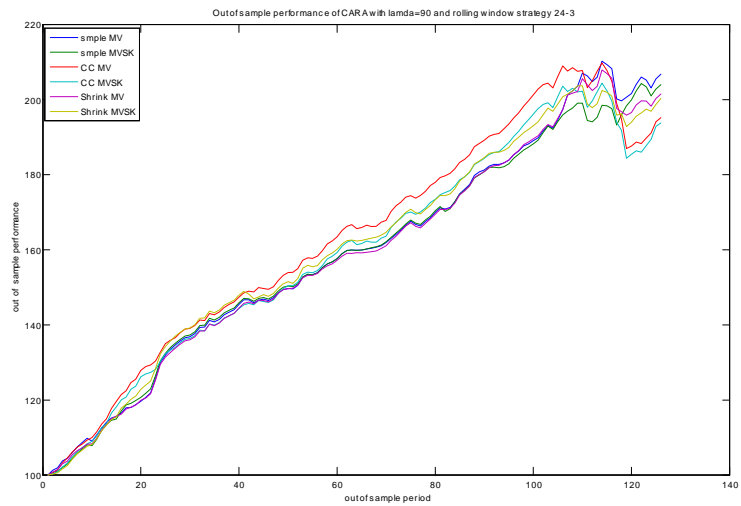Figure 24

Figure 25

Figure 26

Figure 27

Figure 28

Figure 29

We can observe that for small levels of risk aversion the performances of the portfolios, when we consider only two moments (or only four moments) estimated with different approaches, are almost overlapping. As the risk aversion coefficient increases, the difference between these performances becomes evident.

Statistics for rolling window strategy 24-3 and lamda=1			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0056254	0,0000580	-1,0319467	9,9904006
Sample MVSK 0,0056300	0,0000579	-1,0208393	9,8991150
CC MV 0,0053587	0,0000842	-2,5653975	11,4303977
CC MVSK 0,0053636	0,0000841	-2,5654296	11,4389036
Shrinkage MV 0,0055728	0,0000561	-1,1401002	11,0633880
Shrinkage MVSK 0,0055715	0,0000559	-1,1314955	11,0252280

Statistics for rolling window strategy 24-3 and lamda=30			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0058512	0,0000614	-0,8835304	9,0093649
Sample MVSK 0,0056048	0,0000539	-0,4880357	5,5596305
CC MV 0,0054160	0,0000878	-2,5876103	11,1134000
CC MVSK 0,0054320	0,0000715	-1,9768782	7,5335232
Shrinkage MV 0,0059350	0,0000576	-1,0457932	11,0080734
Shrinkage MVSK 0,0056830	0,0000542	-1,1580823	9,6410959

Statistics for rolling window strategy 24-3 and lamda=5			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0056298	0,0000570	-1,0688223	10,4211719
Sample MVSK 0,0056594	0,0000560	-1,0264459	10,1550140
CC MV 0,0053470	0,0000839	-2,5758895	11,5065419
CC MVSK 0,0053489	0,0000835	-2,5763713	11,5601806
Shrinkage MV 0,0056583	0,0000547	-1,2305546	11,8208723
Shrinkage MVSK 0,0056475	0,0000535	-1,2089105	11,7806748

Statistics for rolling window strategy 24-3 and lamda=50			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0052932	0,0000607	-1,2923184	10,4146050
Sample MVSK 0,0056845	0,0000535	-0,6498758	5,7144896
CC MV 0,0053375	0,0000838	-2,5753121	11,5121605
CC MVSK 0,0052650	0,0000809	-2,7626445	13,5016741
Shrinkage MV 0,0055492	0,0000584	-1,4009135	11,5554611
Shrinkage MVSK 0,0056154	0,0000488	-0,8691455	7,8483180

Statistics for rolling window strategy 24-3 and lamda=10			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0056099	0,0000630	-1,2161283	9,5827912
Sample MVSK 0,0056448	0,0000610	-1,1395343	9,1004184
CC MV 0,0053405	0,0000839	-2,5756768	11,5160057
CC MVSK 0,0053498	0,0000830	-2,5835105	11,6871662
Shrinkage MV 0,0055518	0,0000585	-1,3858945	11,4070755
Shrinkage MVSK 0,0054648	0,0000560	-1,3430806	11,3088205

Statistics for rolling window strategy 24-3 and lamda=70			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0055746	0,0000626	-1,3923129	10,3313499
Sample MVSK 0,0056249	0,0000522	-0,5471984	5,0194453
CC MV 0,0051726	0,0000950	-3,3857360	19,2781531
CC MVSK 0,0053305	0,0000761	-2,4185854	10,9718076
Shrinkage MV 0,0056740	0,0000580	-1,5965815	12,5659509
Shrinkage MVSK 0,0056877	0,0000492	-0,7662779	6,4544774

Statistics for rolling window strategy 24-3 and lamda=20			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0054705	0,0000620	-1,3254654	10,0527783
Sample MVSK 0,0056994	0,0000538	-0,5896971	5,7536471
CC MV 0,0051141	0,0000810	-2,6801284	12,3580773
CC MVSK 0,0052120	0,0000943	-3,4587527	19,5654293
Shrinkage MV 0,0056045	0,0000554	-1,5856362	13,2717921
Shrinkage MVSK 0,0057425	0,0000544	-1,4750799	11,9010589

Statistics for rolling window strategy 24-3 and lamda=90			
Mean	Variance	Skewness	Kurtosis
Sample MV 0,0058571	0,0000607	-0,9321474	9,2484997
Sample MVSK 0,0057457	0,0000514	-0,5049907	4,5856216
CC MV 0,0054848	0,0000751	-2,0624153	7,9398844
CC MVSK 0,0053947	0,0000695	-1,9471651	7,9347191
Shrinkage MV 0,0056992	0,0000582	-1,0164606	10,4731745
Shrinkage MVSK 0,0058834	0,0000549	-0,8009543	4,8806837

Table 14

Table 14 reports the statistics for the out of sample portfolios, obtained with the different approaches of the estimation of the co-moments for different values of lambda. Knowing our risk aversion and observing the optimal portfolios obtained for that level of risk aversion, it is not clear which portfolio is better than the other, since we prefer a portfolio with a higher mean and skewness but with a lower variance and kurtosis.

From these analysis we cannot reach a conclusion whether using two moments is better than four moments, or whether using one estimation approach is better than the other. To explore this issue, we will consider the "monetary utility gain/loss" proposed by Ang and Bekaert (2002) that we will discuss in the next paragraph.

5.7   Monetary Utility Gain/Loss

So far, when we consider the out-of-sample performance of each portfolio, we are not able to choose if one estimation approach of the co-moments is better than the other. In order to do this, we use a measure called Monetary Utility Gain/Loss (MUG) proposed by Ang and Bekaert (2002). The MUG is implicitly defined by the following equation .

$$\frac{1}{T}\sum_{t=1}^{T} -e^{-\lambda(1+r_t^{(shrink)})} = \frac{1}{T}\sum_{t=1}^{T} -e^{-\lambda W_s(1+r_t^{(samp)})} \tag{a}$$

where $MUG = (W_s - 1) * 100$ and $r_t^{(shrink)}$ $(r_t^{(samp)})$ indicates the out-of-sample annual returns [22] obtained using the shrinkage (sample) approach for the estimation of the co-moments.

We know all the parameters of the left hand side of the equation so, we can calculate the amount which indicates the wealth that we will obtain at the end of the out-of-sample period supposing that the initial wealth is equal to 1, $W_0 = 1$.

On the right hand side we do not know the parameter W_s which indicates the amount that we should invest in case we use the sample approach for the estimation of the co-moments in order to have the same wealth[23] when shrinkage approach is used. This means that, if $W_s > 1$, we prefer the shrinkage approach to the sample approach, because it permits to have the same wealth at the end of the out-of-sample period investing less in $t = 0$. Vice versa if $W_s < 1$ the sample approach is better than the shrinkage approach.

In our analysis we have solved the equation (a) using numerical algorithms. And the results that we obtained are reported in the following tables.

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[22] In the previous Section we have calculated the out-of-sample returns and performances of the portfolio on a monthly basis. In this Section we convert the monthly returns in annual returns using the compound interest:

$$i = (1 + i_{12})^{12} - 1$$

[23] In the end of the out-of-sample period.

		Ammount invested in the non improved portfolio							
		MUG of SCC versus Sample				MUG of 4-th versus 2° order			
Lambda	Cal-period	2° order		4°order		Sample		Shrinkage	
		3	6	3	6	3	6	3	6
1	24	1,0008	1,0002	1,001	1,0003	0,99993	1	1,0001	1,0002
	48	0,99992	0,99963	1	0,9997	0,99991	1	1	1,0001
5	24	1,0005	1,0002	1	0,99989	1,0002	1,0002	0,99978	0,99995
	48	0,99977	0,99955	0,99966	0,99978	1,0001	0,99998	1	1,0002
10	24	0,9985	0,9982	0,99805	0,99866	1,0011	1,0009	1,0006	1,0014
	48	0,99874	0,99851	0,99808	0,99902	1,0011	1,0006	1,0005	1,0011
20	24	0,99113	0,98969	0,99745	0,9975	1,0054	1,0056	1,0119	1,0136
	48	0,99469	0,9942	0,99731	0,99769	1,0043	1,0042	1,0069	1,0078
30	24	0,99162	0,9907	0,99448	0,99436	1,0222	1,0232	1,0252	1,0269
	48	0,9933	0,99285	0,99787	0,99831	1,0145	1,0145	1,0192	1,0201
50	24	0,99078	0,98945	0,98527	0,98511	1,0338	1,035	1,0281	1,0304
	48	0,99496	0,99438	0,99765	0,9978	1,0483	1,0492	1,0511	1,0527
70	24	0,99131	0,98844	0,97233	0,96167	1,0423	1,0463	1,0356	1,0387
	48	0,99483	0,99319	0,98583	0,98006	1,02650	1,02423	1,01030	1,01890
90	24	0,99132	0,98883	0,94156	0,91428	1,0796	1,0836	1,0254	1,0019
	48	0,99490	0,99245	0,97893	0,96876	1,03550	1,00660	1,01880	1,00300

Table 15

Table 15 indicates the results of W_s for different levels of lambda, calibration periods and in-sample periods.

- The column labeled $"2° \ order"$ indicates the initial wealth, W_s, that we invest using the sample approach with two moments, with the aim to obtain equal wealth, at the end of the period, for both sample approach and shrinkage approach with two moments. Let us note that the initial wealth of the shrinkage approach is 1. If $W_s > 1$, shrinkage approach with two moments is preferred to the sample approach with two moments.

- The column labeled $"4° \ order"$ indicates the initial wealth that we invest using the sample approach with four moments, so that the wealth at the end of the out of-sample period would be equal to that obtained using the shrinkage approach[24] with four moments. If $W_s > 1$, shrinkage approach with four moments is preferred to the sample approach with four moments.

- The column labeled $"Sample"$ indicates the initial wealth that we invest using the sample approach with two moments, so that the wealth at the

---

[24] The initial wealth of the shrinkage with four moments approach is 1.

end of the out of-sample period would be equal to that obtained using the sample approach with four moments[25]. When $W_s > 1$, it means that using the sample approach with four moments in the portfolio allocation is better than using the sample approach with two moments.

- The column labeled *"Shrinkage"* indicates the initial wealth that we invest using the shrinkage approach with two moments, so that the wealth at the end of the out of-sample period would be equal to that obtained using the shrinkage approach with four moments [26]. In case $W_s > 1$, it means that using the shrinkage approach with four moments in the portfolio allocation is better than using the shrinkage approach with two moments.

Let us consider the case of the calibration period 24, out-of-sample period 3 and $\lambda = 1$.In this case we have $W_s = 1,0008$. This result (W_s) indicates the initial wealth that we invest when we use sample approach with two moments in order to have the same wealth, W_T [27], obtained using the shrinkage approach with two moments. As $W_s = 1,0008 > 1$ the shrinkage approach yields a better result .

Yearly MUG (annualized percentage)									
		MUG of SCC versus Sample				MUG of 4-th versus 2° order			
Lambda	Cal-period	2° order		4°order		Sample		Shrinkage	
		3	6	3	6	3	6	3	6
1	24	0,08	0,02	0,1	0,03	-0,007	0	0,01	0,02
	48	-0,008	-0,037	0	-0,03	-0,009	0	0	0,01
5	24	0,05	0,02	0	-0,011	0,02	0,02	-0,022	-0,005
	48	-0,023	-0,045	-0,034	-0,022	0,01	-0,002	0	0,02
10	24	-0,15	-0,18	-0,195	-0,134	0,11	0,09	0,06	0,14
	48	-0,126	-0,149	-0,192	-0,098	0,11	0,06	0,05	0,11
20	24	-0,887	-1,031	-0,255	-0,25	0,54	0,56	1,19	1,36
	48	-0,531	-0,58	-0,269	-0,231	0,43	0,42	0,69	0,78
30	24	-0,838	-0,93	-0,552	-0,564	2,22	2,32	2,52	2,69
	48	-0,67	-0,715	-0,213	-0,169	1,45	1,45	1,92	2,01
50	24	-0,922	-1,055	-1,473	-1,489	3,38	3,5	2,81	3,04
	48	-0,504	-0,562	-0,235	-0,22	4,83	4,92	5,11	5,27
70	24	-0,869	-1,156	-2,767	-3,833	4,23	4,63	3,56	3,87
	48	-0,51700	-0,68100	-1,41700	-1,99400	2,65000	2,42300	1,03000	1,89000
90	24	-0,868	-1,117	-5,844	-8,572	7,96	8,36	2,54	0,19
	48	-0,51000	-0,75500	-2,10700	-3,12400	3,55000	0,66000	1,88000	0,30000

Table 16

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[25]The initial wealth of the sample approach with four moments is 1.

[26]The initial wealth of the of the shrinkage approach with four moments is 1.

[27]W_T is the wealth obtained at the end of the out of-sample period.

Table 16 reports the annual percentage value of the Monetary Utility Gain/Loss. From the results obtained we can conclude:

1) Sample estimation with two moments is better than the shrinkage estimation with two moments. We can see this in the column labeled *"2° order"*, where the MUG is almost always negative. This means that, in order to obtain the same wealth with the shrinkage approach, we need to invest an amount smaller than 1 when the sample approach is used.

2) Sample estimation with four moments is better that the shrinkage estimation with four moments. We can see this in the column labeled *"4° order"*. In this case the MUG is almost always negative. This proves that, in order to obtain the same wealth with the shrinkage approach with four moments, we need to invest an amount smaller than 1 when the sample approach with four moments is used.

3) If we have to make a choice between a portfolio allocation based on two moments or on four moments,independently from the approach used for the estimation of the co-moments, it is always better to choose the one based on four moments. We have the evidence due to the results of column "sample" and "shrinkage" where $MUG > 1$.

4) The results of the columns labeled *"Sample"* and *"Shrinkage"* show that MUG increases with the increase of the risk aversion,λ.This means that the higher λ the higher the gain obtained if four moments are used instead of two moments.

5.7.1 Using the shrinkage mean and robust skewness in case of CC and Shrinkage approach

In the previous Section we have presented the MUG results obtained in different cases and we have imposed the mean equal to zero as we wanted to eliminate the impact of the sample mean. As we have explained, the sample approach is characterized by a high estimation error. For this reason, we will introduce new estimators for the mean and skewness. This new estimators will be considered only in case of CC approach and Shrinkage approach.

Shrinkage estimators have been applied in order to reduce the estimation error of the sample mean [28]. We will use the James-Stein estimation approach that is called also *"shrinkage of the mean toward the grand mean"*. James-Stein shrinkage is based on averaging two different models: a high-dimensional model with low bias and high variance (this is the sample mean), and a lower dimensional model with larger bias but smaller variance (this is the target mean). The

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[28]P. Jorion (1986). Bayes-Stein Estimation for Portfolio Analysis. The Journal of Financial and Quantitative Analysis. pp (272 - 292).

D.V. Lindley, (1962), Discussion of Professor Stein's Paper, confidence sets for the mean of the multivariate normal distribution. J.R.Stat.Soc.Ser.B Stat.Methodol . 24, pp.(265 - 296).

mean of the "*Global Minimum Variance*" (GMV) portfolio will be our target in this analysis.

As in the case of shrinkage variance (skewness or kurtosis), the shrinkage mean is determined by the relative weighting of the two means estimators (sample mean and the mean of the GMV portfolio). Therefore, the Shrinkage mean is given by:

$$\mu^{shrink} = (1 - \phi) * \bar{\mu} + \phi * \mu^{t\arg et}$$

where $\bar{\mu}$ is the sample mean and $\mu^{t\arg et}$ is the target mean. The parameter ϕ is the shrinkage coefficient and is calculated as follows:

$$\phi = \min\left(1, \max\left(0, \frac{(N - 2)}{(\bar{\mu} - \mu^{t\arg et})'T\ \Sigma^{-1}(\bar{\mu} - \mu^{t\arg et})}\right)\right)$$

where N is the number of assets in portfolio, T is the number of observations and Σ is the covariance matrix estimated using the sample approach.

Given that, we have used the mean of the GMV portfolio as our target. We calculate it using the following formula:

$$\mu_j^{t\arg et} = \frac{1'\ \Sigma^{-1}}{1'\ \Sigma^{-1}1}\bar{\mu}$$

In the theoretical part we have always used the sample estimation of the skewness independently from the approach used for the estimation of the co-moments. Tae Hwan Kim and Halbert White (2003) have proposed robust estimators on skewness and kurtosis. In this Section we will use only the robust estimator for skewness, defined as:

$$Sk = \frac{Q_3 + Q_1 - 2\ Q_2}{Q_3 - Q_1}$$

where Q_i indicates the $i - th$ quartile.

In this Section, when we use sample approach, we consider all the sample co-moments and in case of CC (Shrinkage) approach we consider the James-Stein mean for the estimation of co-moments and the robust skewness for the estimation of skewness [29].

The MUG results obtained in this case are reported in the following two

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[29] We use the sample kurtosis in all the approaches.

tables.

Ammount invested in the non improved portfolio								
		MUG of SCC versus Sample				MUG of 4-th versus 2° order		
Lambda	Cal-period	2° order		4°order		Sample		Shrinkage	
		3	6	3	6	3	6	3	6
1	24	1,0001	1,0002	1,0001	1,0002	1	1	1	1
	48	0,99993	0,99938	0,99991	0,99937	1	1	0,99999	0,99999
5	24	1,004	1,0041	1,004	1,0041	1,0002	1,0003	1,0003	1,0003
	48	1,0006	1	1,0003	0,9999	1,0004	1,0001	1,0001	1
10	24	1,0177	1,0129	1,019	1,0133	1,001	1,0014	1,0023	1,0018
	48	1,0132	1,0257	1,0106	1,0223	1,0046	1,0055	1,0021	1,0022
20	24	1,0833	1,062	1,1298	1,121	0,9798	0,96258	1,0182	1,0116
	48	1,0304	1,0418	1,0212	1,0275	1,0162	1,0231	1,0074	1,0093
30	24	1,1289	1,09	1,1375	1,1435	1,0686	1,0197	1,0753	1,068
	48	1,0435	1,0496	1,0208	1,024	1,0337	1,0367	1,0114	1,0117
50	24	1,1755	1,3446	1,0206	1,1347	1,4035	1,2172	1,2193	1,0266
	48	1,0192	1,0195	1,0103	1,0109	1,042	1,0425	1,018	1,018
70	24	1,4438	1,1368	1,0129	1,0848	1,5351	1,36	1,176	1,2982
	48	1,01430	1,01430	1,00040	1,00010	1,05430	1,05450	1,02620	1,02600
90	24	1,1765	1,1379	1,0902	1,0591	1,4739	1,4505	1,3657	1,35
	48	1,02350	1,02350	1,00060	1,00030	1,02570	1,02570	1,00280	1,00240

Table 17

Yearly MUG (annualized percentage)								
		MUG of SCC versus Sample				MUG of 4-th versus 2° order		
Lambda	Cal-period	2° order		4°order		Sample		Shrinkage	
		3	6	3	6	3	6	3	6
1	24	0,01	0,02	0,01	0,02	0	0	0,00	0,00
	48	-0,007	-0,062	-0,009	-0,063	0	0	0,00	0,00
5	24	0,4	0,41	0,4	0,41	0,02	0,03	0,03	0,03
	48	0,06	0	0,03	-0,01	0,04	0,01	0,01	0,00
10	24	1,77	1,29	1,9	1,33	0,1	0,14	0,23	0,18
	48	1,32	2,57	1,06	2,23	0,46	0,55	0,21	0,22
20	24	8,33	6,2	12,98	12,1	-2,02	-3,742	1,82	1,16
	48	3,04	4,18	2,12	2,75	1,62	2,31	0,74	0,93
30	24	12,89	9	13,75	14,35	6,86	1,97	7,53	6,80
	48	4,35	4,96	2,08	2,4	3,37	3,67	1,14	1,17
50	24	17,55	34,46	2,06	13,47	40,35	21,72	21,93	2,66
	48	1,92	1,95	1,03	1,09	4,2	4,25	1,80	1,80
70	24	44,38	13,68	1,29	8,48	53,51	36	17,60	29,82
	48	1,43000	1,43000	0,04	0,01	5,43000	5,45000	2,62	2,60
90	24	17,65	13,79	9,02	5,91	47,39	45,05	36,57	35,00
	48	2,35000	2,35000	0,06000	0,03000	2,57000	2,57000	0,28	0,24

Table 18

The results in this case are different from those obtained in the previous case.

1.) The results of the column labeled *"2° order"* are always positive and increse as λ increases. This shows firstly, that the shrinkage approach with two moments is always better than the sample approach with two moments, and secondly, the gain of using shrinkage approach instead of the sample approach increases with the increase of the risk aversion.

2.) From the column labeled *"4° order"* results that the shrinkage approach with four moments is always better that the sample approach with four moments, as the result in this column are always positive.

3.) Observing the results in the last two columns we can say that taking into consideration four moments is always better than considering only two moments for the portfolio allocation. With the increase of the risk aversion the gain that we will obtain from using four moments instead of two moments increases.

5.7.2 Impact of each shrinkage estimator on the performance

In this Section we will analyze the impact that each estimation of the co-moments has on the portfolio allocation and on MUG measure. We will compare the performances and calculate the MUG of different portfolios with the optimal portfolio constructed when all the moments and co-moments are estimated using the sample approach. In order to do this we have constructed five different optimal portfolios, that are:

- the portfolio constructed using sample mean, sample co-variance, sample co-skewness and sample co-kurtosis,

- the portfolio constructed using shrinkage mean, sample co-variance, sample co-skewness and sample co-kurtosis,

- the portfolio constructed using sample mean, shrinkage co-variance, sample co-skewness and sample co-kurtosis,

- the portfolio constructed using sample mean, sample co-variance, shrinkage co-skewness and sample co-kurtosis,

- the portfolio constructed using sample mean, sample co-variance, sample co-skewness and shrinkage co-kurtosis,

The following figure shows the out-of-sample performance of all these portfolios, in case $\lambda = 20$. We have considered the calibration period 24_3.
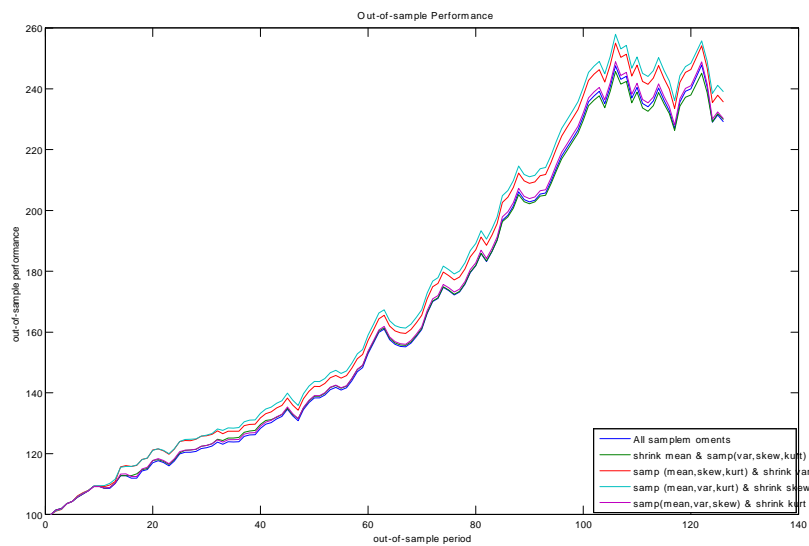
Figure 30

We report in the following graps the out-of-sample performance when all sample moments are used and the performance of the optimal portfolio when we use one shrinkage moment and three sample moments.
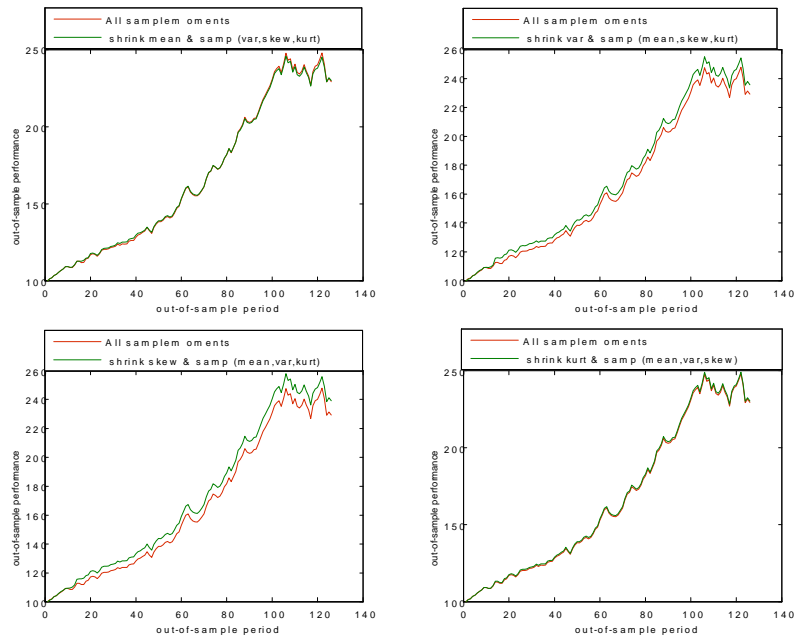
figure 31

As we can observe from this graphs, using different estimators for the co-moments changes the performance in out-of-sample portfolio.

We have calculated, also the MUG of this portfolios,for different levels of risk aversion, when we fix equal to one the initial wealth of the portfolio when only one moment is estimated using the shrinkage approach and we find the initial wealth that we should invest in the optimal portfolio obtained using the sample approach for all the moments. The results that we obtained are:

Initial wealth invested in the portfolio where all the co-moments are estimated using the sample approach				
		Shkinkage mean and sample var,skew and kurt	Shkinkage var and sample mean, skew and kurt	Shkinkage skew and sample mean, ske and kurt	Shkinkage kurt and sample mean, var and ske
lamda=1	All sample ments	1,0001	1,0001	1	1
lamda=5	All sample ments	1,004	1,004	1,0002	1,0003
lamda=10	All sample ments	1,0177	1,0189	1,001	1,0022
lamda=20	All sample ments	1,0833	1,1283	1,003	1,017
lamda=30	All sample ments	1,1287	1,1368	1,0686	1,0747
lamda=50	All sample ments	1,1753	1,0199	1,4035	1,2186
lamda=70	All sample ments	1,1969	1,0234	1,5351	1,2783
lamda=90	All sample ments	1,3194	1,0901	1,4739	1,2177

Figure 19

Annualized perentage MUG				
		Shkinkage mean and sample var,skew and kurt	Shkinkage var and sample mean, skew and kurt	Shkinkage skew and sample mean, ske and kurt	Shkinkage kurt and sample mean, var and ske
lamda=1	All sample ments	0,01	0,01	0	0
lamda=5	All sample ments	0,4	0,4	0,02	0,03
lamda=10	All sample ments	1,77	1,89	0,1	0,22
lamda=20	All sample ments	8,33	12,83	0,3	1,7
lamda=30	All sample ments	12,87	13,68	6,86	7,47
lamda=50	All sample ments	17,53	1,99	40,35	21,86
lamda=70	All sample ments	19,69	2,34	53,51	27,83
lamda=90	All sample ments	31,94	9,01	47,39	21,77

Table 20

Observations:

- As risk aversion encrese MUG increase

- The higher is the risk aversion the higher is the importance (weight) of high order co-moments.

Chapter 6

Conclusions

In this dissertation we have considered a portfolio composed of 13 assets (12 hedge funds and 1 CTA). Our data have a monthly frequency and cover the period from January 1997 to August 2009. The total number of observations in our database is 152.

We have analyzed each single time series of returns and we have reached the conclusion that the returns under consideration are not normally distributed, except 'CTA Global'. At this point we wanted to analyze the impact of higher moments on portfolio allocation. Given the fact that our funds are not normally distributed, we claimed that using the Markowitz model was not a good idea, as we needed to take into account also the higher moments in portfolio allocation. To take into consideration the higher moments we use the Taylor expansion up to the fourth order which allowed us to write the utility function as function of the co-moments.

Subsequently we wanted to take into consideration different approaches for the estimation of co-moments, since our aim was to analyze each single approach and to see which one is better than the other.

We used the monetary utility gain (MUG) as a measure to compare the different portfolios. Another measure that can be used is the difference in certainty equivalents (CE). The reason why we chose MUG instead of CE is that MUG offers a more immediate interpretation in terms of the initial investments.

We obtained that the optimal portfolio allocation is not significantly different across the models and our portfolio is well diversified in all cases. The optimal portfolios in case of two or four moments are very similar when the risk aversion is low and with the increase of the risk aversion the difference between this two portfolios becomes more evident.

We have computed the MUG in two different cases:

1. In the first case we have imposed the mean equal to zero (this because we wanted to eliminate the effect that the sample mean has on portfolio allocation). We have used the mean equal to zero not only in the optimization part but also in the calculation of all the co-moments using the

different approaches. The results obtained in this case are:

– In the out-of-sample period, two moments portfolio allocation with sample approach is better than the two moments portfolio allocation with the shrinkage approach.

– In the out-of-sample period, four moments portfolio allocation with sample approach is better than the four moments portfolio allocation with the shrinkage approach.

– In the out-of-sample period, four moments portfolio allocation with sample approach is better than the two moments portfolio allocation with the sample approach. We have seen also that in this case the MUG increases with the increase of the risk aversion.

– In the out-of-sample period, four moments portfolio allocation with shrinkage approach is better than the two moments portfolio allocation with the shrinkage approach. The MUG increases as risk aversion increases.

2. In the second case we have considered sample estimators for the mean, and the skewness in the case of sample approach and shrinkage mean, robust skewness in case of CC and Shrinkage approach. The MUG results in this case are different compared with those of the first case. We obtained that:

– In the out-of-sample period, two moments portfolio allocation with sample approach is worse than the two moments portfolio allocation with the shrinkage approach.

– In the out-of-sample period, four moments portfolio allocation with sample approach is worse than the four moments portfolio allocation with the shrinkage approach.

– In the out-of-sample period, four moments portfolio allocation with sample approach is better than the two moments portfolio allocation with the sample approach. We have seen also that in this case the MUG increases with the increase of the risk aversion (this conclusion is the same as the first case).

– In the out-of-sample period, four moments portfolio allocation with shrinkage approach is better than the two moments portfolio allocation with the shrinkage approach. The MUG increases as risk aversion increases (the same as the first case).

In the case of our portfolio we can conclude that the shrinkage approach is always better than sample approach for portfolio allocation. If we are not able to calculate all the shrinkage estimators for the moments and co-moments it is better to stick to the sample approach. Independently from the approach used, a four moments portfolio allocation is better than a two moments portfolio allocation.

For our portfolio, the shrinkage approach not only reduces dramatically the number of parameters to be estimated, but also provides a better portfolio allocation as the out-of-sample MUG is always positive.

In the last part of the analysis we have seen that the higher is the risk aversion the higher is the impact of co-skewness and co-kurtosis on the Monetary Utility Gain.

In our analysis we have considered the constraint of not allowing for short selling. In future analysis we will extend this to the case of allowing for short selling and analyze how the portfolio allocation and performance will change. Another extension of this work will be on the issue that moments are time dependent introducing GARCH models.

Chapter 7

Appendix A

7.1 Appendix A. 1

The variance (second central moment) is given by the following formula;

$$
\begin{aligned}
\sigma_{p,t}^2 &= E(r_{p,t} - \mu_{p,t})^2 \\
&= E\left[r_{p,t}^2 - 2r_{p,t}\mu_{p,t} + \mu_{p,t}^2\right] \\
&= E\left[r_{p,t}^2\right] - 2\mu_{p,t}E\left[r_{p,t}\right] + \mu_{p,t}^2 \\
&= {}_{(2)}\mu_{p,t}' - 2\mu_{p,t}^2 + \mu_{p,t}^2 \\
&= {}_{(2)}\mu_{p,t}' - \mu_{p,t}^2 \\
{}_{(2)}\mu_{p,t}' &= \sigma_{p,t}^2 + \mu_{p,t}^2
\end{aligned}
$$

The third central moment (skewness) is given by;

$$
\begin{aligned}
s_{p,t} &= E(r_{p,t} - \mu_{p,t})^3 \\
&= E\left[r_{p,t}^3 - 3r_{p,t}^2\mu_{p,t} + 3r_{p,t}\mu_{p,t}^2 - \mu_{p,t}^3\right] \\
&= {}_{(3)}\mu_{p,t}' - 3\,{}_{(2)}\mu_{p,t}'\mu_{p,t} + 3\mu_{p,t}^3 - \mu_{p,t}^3 \\
&= {}_{(3)}\mu_{p,t}' - 3\left(\sigma_{p,t}^2 + \mu_{p,t}^2\right)\mu_{p,t} + 2\mu_{p,t}^3 \\
&= {}_{(3)}\mu_{p,t}' - 3\sigma_{p,t}^2\mu_{p,t} - 3\mu_{p,t}^3 + 2\mu_{p,t}^3 \\
{}_{(3)}\mu_{p,t}' &= s_{p,t} + 3\sigma_{p,t}^2\mu_{p,t} + \mu_{p,t}^3
\end{aligned}
$$

The fourth central moment (kurtosis) is;

$$
\begin{aligned}
k_{p,t} &= E(r_{p,t} - \mu_{p,t})^4 \\
&= E\left[r_{p,t}^4 - 4r_{p,t}^3\mu_{p,t} + 6r_{p,t}^2\mu_{p,t}^2 - 4r_{p,t}\mu_{p,t}^3 + \mu_{p,t}^4\right] \\
&= {}_{(4)}\mu'_{p,t} - 4\,\mu_{p,t}\left(s_{p,t} + 3\sigma_{p,t}^2\mu_{p,t} + \mu_{p,t}^3\right) + 6\,\mu_{p,t}^2\left(\sigma_{p,t}^2 + \mu_{p,t}^2\right) - 3\mu_{p,t}^4 \\
&= {}_{(4)}\mu'_{p,t} - 4\,\mu_{p,t}s_{p,t} - 12\sigma_{p,t}^2\,\mu_{p,t}^2 - 4\mu_{p,t}^4 + 6\sigma_{p,t}^2\,\mu_{p,t}^2 + 6\mu_{p,t}^4 - 3\mu_{p,t}^4 \\
&= {}_{(4)}\mu'_{p,t} - 4\,\mu_{p,t}s_{p,t} - 6\sigma_{p,t}^2\mu_{p,t}^2 - \mu_{p,t}^4 \\
{}_{(4)}\mu'_{p,t} &= k_{p,t} + 4s_{p,t}\mu_{p,t} + 6\sigma_{p,t}^2\mu_{p,t}^2 + \mu_{p,t}^4
\end{aligned}
$$

7.2   Appendix A. 2

As we have seen using the Taylor expansion the expected utility can be approximated by:

$$
E[U(W_t)] \approx U(\bar{W}) + \frac{U^{(2)}(\bar{W})}{2!} * \sigma^2(W_t) + \frac{U^{(3)}(\bar{W})}{3!} * s^3(W_t) + \frac{U^{(4)}(\bar{W})}{4!} * k^4(W_t)
$$

Case 3 *In the case of utility(CARA) function we have :*

$$
U(W_t) = -e^{-\lambda W_t}
$$

We need to find the derivatives of the utility function of the first, second, third and fourth order, that are:

$$
\begin{aligned}
U^{(1)}(\bar{W}) &= \lambda e^{-\lambda\bar{W}} \\
U^{(2)}(\bar{W}) &= -\lambda^2 e^{-\lambda\bar{W}} \\
U^{(3)}(\bar{W}) &= \lambda^3 e^{-\lambda\bar{W}} \\
U^{(4)}(\bar{W}) &= -\lambda^4 e^{-\lambda\bar{W}}
\end{aligned}
$$

Substituting this derivatives in the Taylor expansion we obtain:

$$
\begin{aligned}
E[U(W_t)] &\approx -e^{-\lambda\bar{W}} - \frac{-\lambda^2 e^{-\lambda\bar{W}}}{2!} * \sigma^2(W_t) + \frac{\lambda^3 e^{-\lambda\bar{W}}}{3!} * s^3(W_t) + \frac{-\lambda^4 e^{-\lambda\bar{W}}}{4!} * k^4(W_t) \\
&\approx -e^{-\lambda(\mu_p)}\left[1 + \frac{1}{2}\lambda^2\sigma_p^2 - \frac{1}{6}\lambda^3 s_p^3 + \frac{1}{24}\lambda^4 k_p^4\right]
\end{aligned}
$$

This is the function that we have to maximize under the constraints considered in the theoretical part. Therefore the system that we have to solve in each calibration period is the following:

$$
\left\{
\begin{array}{c}
\max -e^{-\lambda(\mu_p)} \left[1 + \frac{1}{2}\lambda^2\sigma_p^2 - \frac{1}{6}\lambda^3 s_p^3 + \frac{1}{24}\lambda^4 k_p^4\right] \\
s.t \\
\sum_{i=1}^{N} w_i = 1 \\
0 \le w_i \le 1
\end{array}
\right.
$$

Case 4 *In the case of (CRRA) utility function :*

$$
U(W_t) = \left\{
\begin{array}{ll}
\frac{W_t^{1-\lambda}}{1-\lambda} & if \quad \lambda > 1 \\
ln(W_t) & if \quad \lambda = 1
\end{array}
\right.
$$

The derivatives of the first, second, third and fourth order are:

$$
\begin{array}{rcl}
U^{(1)}(\bar{W}) &=& \bar{W}^{-\lambda} \\
U^{(2)}(\bar{W}) &=& -\lambda\bar{W}^{-(\lambda+1)} \\
U^{(3)}(\bar{W}) &=& \lambda(\lambda+1)\bar{W}^{-(\lambda+2)} \\
U^{(4)}(\bar{W}) &=& -\lambda(\lambda+1)(\lambda+2)\bar{W}^{-(\lambda+3)}
\end{array}
$$

Substitution the expression of the derivatives in the Taylor expansion we obtain;

$$
E\left[U(W_t)\right] \approx \frac{\bar{W}^{1-\lambda}}{1-\lambda} - \frac{\lambda}{2}\bar{W}^{-\lambda-1}\sigma_t^2(W_t) + \frac{\lambda(\lambda+1)}{3!}\bar{W}^{-\lambda-2}s_t^3(W_t) + \frac{\lambda(\lambda+1)(\lambda+2)}{4!}\bar{W}^{-\lambda-3}k_t^4(W_t)
$$

In terms of portfolio moments μ_p;

$$
E\left[U(W_t)\right] \approx \frac{\mu_p^{1-\lambda}}{1-\lambda} - \frac{\lambda}{2}\mu_p^{-\lambda-1}\sigma_p^2 + \frac{\lambda(\lambda+1)}{3!}\mu_p^{-\lambda-2}s_p^3 + \frac{\lambda(\lambda+1)(\lambda+2)}{4!}\mu_p^{-\lambda-3}k_p^4
$$

Therefore, in case of CRRA the system that we have to solve is:

$$
\left\{
\begin{array}{c}
\max \frac{\mu_p^{1-\lambda}}{1-\lambda} - \frac{\lambda}{2}\mu_p^{-\lambda-1}\sigma_p^2 + \frac{\lambda(\lambda+1)}{3!}\mu_p^{-\lambda-2}s_p^3 + \frac{\lambda(\lambda+1)(\lambda+2)}{4!}\mu_p^{-\lambda-3}k_p^4 \\
s.t \\
\sum_{i=1}^{N} w_i = 1 \\
0 \le w_i \le 1
\end{array}
\right.
$$

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