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Ph.D. thesis

Patching up IIA singularities

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“This possibility would probably not have direct phenomenological implications, but considering the general attractiveness of superstring theories for providing a fundamental theory of nature, any contribution to the understanding of their formal structure would be desirable.”

L. J. ROMANS

“Physics is like sex. Sure, it may give some practical results, but that’s not why we do it.”

R. P. FEYNMAN

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Chapter 1

Introduction and conclusions

It is just in the '90s, with the AdS/CFT correspondence revolution, that string theory was able to interpret the strong coupling limit in g_s, the string coupling. In this big picture, the role of type IIA is peculiar, since in the strong coupling limit it is possible to show that one dimension more is excited and the theory is eleven dimensional supergravity.

Actually this last statement is only true when the *Romans mass* is not turned on. In 1986, few years before the duality revolutions, Romans [1] proposed a generalization of type IIA supergravity in which the B-field acquires mass through Stückelberg mechanism. This extra parameter in the action has some interesting peculiarities: since it can be rearranged as a scalar Ramond Ramond field, it is a fixed parameter (it has to satisfy the Bianchi identity) and it behaves like a negative 10 dimensional cosmological constant.

Moreover, in [2] it was shown that, for classical solutions of massive IIA (i.e. with non zero Romans mass), the string coupling is bounded by the curvature in string units. In a sense, this makes the problem more rare: any solution with large g_s is already invalidated by being strongly curved. This makes the need for a non-perturbative completion less pressing.

There is still one feature of the Romans mass that should be mentioned: if one looks at the Wess Zumino coupling in string theory, the Romans mass appears in the interaction

$$F_0 \int_{D2} CS(a), \tag{1.0.1}$$

where a is the gauge field over the D2 and CS is the Chern Simons Lagrangian. This kind of coupling created a puzzle when [3] arose: without going into details, the original model proposed a duality between an AdS_4 not experiencing a Romans mass with a Chern Simons theory with levels $(k, -k)$. Where was the F_0? [4] pointed out that the F_0 was hidden in the symmetric form of the levels of the conformal theory: once a non zero Romans mass is added, the levels split to $(k, F_0 - k)$.

To summarize, the message so far is that the Romans mass avoids uplifting to M-theory and it is dual with Chern Simons theory.

However, let us consider the case of an O6 plane. If one approaches to the O6, the metric in the massless theory, i.e. with a vanishing Romans mass, reads

$$ds^2_{\mathrm{O6}} = Z^{-1/2} dx^2_{\parallel} + Z^{1/2} dx^2_{\perp} \ , \qquad Z = 1 - \frac{r_0}{r} \ , \qquad r_0 = l_s g_s \ . \tag{1.0.2}$$

Even if we excise the unphysical "hole" $r < r_0$, the metric becomes singular for $r \to r_0$. Actually, approaching to $r \sim r_0$, the dilaton starts growing, such that the supergravity approximation cannot be used and the theory should be uplifted to eleven dimension.

After quantum and instantonic corrections the metric appears to be the Atiyah-Hitchin one, studied for the first time in [5]: it is a smooth metric, without singularities, but with a minimal radius at $r = \frac{\pi}{2} r_0$.

The story of the metric near the O6 in M-theory is quite peculiar. The Atiyah-Hitchin metric studied in [5] was obtained through the analysis of hyper-Kähler manifold defined by 2 interacting monopoles with isometry $SU(2)$; the construction was pure geometric. Later, it was proposed, [6], that the moduli space for the Coulomb branch of a 3 dimensional Super Yang Mills theory with gauge group $SU(2)$ was described exactly by the Atiyah-Hitchin manifold. Moreover [6] compared the infrared limit with the long distance behaviour of the metric from [5] and they found perfect agreement. The absence of singularities in the metric was then read as the absence of singularities on the gauge theory side. The stringy explanation of this construction was given by [7]: they proposed the theory defined in [6] to be the theory living on a D2 probe in the nearby of the O6 in M-theory: in this picture, the instantons present in the theory were D0 exchanged by the D2 with its own image under the O6.

So the singularity of the massless O6 solution (1.0.2) is resolved in M-theory to a smooth hole. What about O6-planes in *massive* IIA?

Since the theory can not be uplifted, two natural questions arise:

1. What is the solution in supergravity, if there is one?

2. How should the theory on the D2 probe be deformed, in order to encode the effect of the Romans mass, if the massive system has a solution?

Solutions of massive IIA with an O6 source have been assumed to exist, especially in the context of flux compactifications. A popular trick in supergravity is to "smear" sources over the internal manifold; namely, to replace the localized source with one which is spread all over space. For an orientifold plane in string theory, this is not really physically allowed, since such sources are supposed to sit on the fixed loci of the orientifold involutions. Nevertheless, smeared solutions are often a good indicator of whether a bona fide background will exist. Using this sleight of hand, quite a few massive O6 solutions have been found. A well-known early example [8, 9] of moduli stabilization is of this type. Also, the presence of both O6's and F_0 is considered the most promising avenue for producing de Sitter vacua in string theory which are completely classical (as opposed to de Sitter vacua such as [10, 11]); examples with the smearing trick include [12, 13].

In this thesis we will find evidence for the existence of *supersymmetric* massive O6-plane solutions, [14]. We will mostly consider a spacetime of the form

$$\mathrm{AdS}_4 \times M_6 \ , \tag{1.0.3}$$

since we have already at least the example [8, 9], which is of this form. The O6 will be filling the four-dimensional spacetime, as well as three of the six directions in M_6. We will also consider the possibility $\mathrm{Mink}_4 \times M_6$; however, we do not know of any supersymmetric Minkowski compactification with O6-planes and Romans mass, and for this reason we will give more attention to (1.0.3).

Actually, although some of our considerations will be more general, we will just focus on what happens close to the O6, so taking implicitly $M_6 = \mathbb{R}^6$, meaning that corrections coming from the curvature of the internal space are not going to be considered. We cannot expect the geometry on this $\mathbb{R}^6$ to approach flat space, however, as would be the case if one factorized the metric (1.0.2) as $\mathrm{Mink}_4 \times \mathbb{R}^6$. This is because neither $\mathrm{AdS}_4 \times \mathbb{R}^6$ nor $\mathrm{Mink}_4 \times \mathbb{R}^6$ are vacua for the massive theory. We are introducing two new length scales: $\frac{1}{\sqrt{-\Lambda}}$ and $\frac{1}{g_s F_0}$ (since F_0 always appears multiplied by e^ϕ in the equations of motion). When both of these scales are large, it is possible to study the features of the geometry closer to the source (order $r_0 = g_s l_s$).

The deformation induced by the Romans mass on the D2 theory is still subject of analysis with the collaboration of my advisor Alessandro Tomasiello and with Gonzalo Torroba, from SLAC and Stanford University. Because of the coupling (1.0.1), the deformation on the theory living on the D2 due to a non vanishing F_0 is a Chern Simons interaction. In the last chapter we will present our intermediate results: we calculated the metric for the moduli space corrected by quantum effects. It is possible to see that the singularity disappears. Moreover, the deformation of the metric is proportional to the inverse of the Chern Simons level.

Let us give a synopsis of the thesis. It is split in two parts, the first dedicated to the supergravity side of the problem of the O6 singularity in IIA, the second examining the gauge theory side of the phenomenon.

Part I begins with chapter 2, introducing the main definitions in the context of supergravity for vacua configuration of type II. As an appetizer, we are going to propose some relations among the different characters in the game due to constraints from the equations of motion and we will present briefly an unlucky proposal for a dS vacuum configuration of type IIA.

In chapter 3, we will show how to relate the existence of a spinor to properties of differential forms. This chapter is going to be crucial: the first concepts in order to face supersymmetry equations on vacua of type II are presented. The main idea is to convert the supersymmetry equations (that have been shown to be solution of the equations of motions too on vacua configuration, [15, 16]), which are spinorial in two parameters, into equations for differential forms, since differential forms provide geometrical interpretation quite automatically. The case examined in this chapter is going to be the easy case in which the two spinorial parameters of supersymmetry equation are taken to be parallel. At the end of the day, the situation is like studying properties due to the existence of just one spinor.

In chapter 4 we take the most general case, i.e. the one with two independent spinorial parameter. It has been shown, [17, 18], that in that case the usual differential geometry is not enough: the geometry should be studied on the *generalized tangent bundle*,

$$T \oplus T^*, \tag{1.0.4}$$

somehow duplicating the original space. We will find that the most general case has a $SU(3) \times SU(3)-$structure and that the geometrical information is encoded in polyformic pure spinors living in $T \oplus T^*$. We will see what are the different ways to write a pure spinor, depending on the topology they define; finally we will see how to write supersymmetry equations in terms of pure spinors.

In chapter 5 we will apply the formalism defined in the previous chapter to the case at hand, i.e. the localized O6 in IIA in vacua with a non zero Romans mass. We will see that the presence of the Romans mass by itself saves the O6 from singularities. Moreover, the metric nearby the O6 can be seen as $\mathbb{R} \times S^2$ in transverse directions and the presence of the Romans mass fixes the dimension of the transverse S^2 to be non zero even in the origin, somehow substituting the O6 plane with a bubble. Furthermore, since there is no singularity, no minimal radius and everything is smooth even at the origin, the metric can be analytically continued to negative radii.

So, in part I there is the supergravity formalism and results. In the part II there is the gauge theory side.

In chapter 6 there is a brief introduction to the original Seiberg–Witten 3 dimensional model, [6]. Here we will see that, after breaking the gauge group down to $U(1)$, the low energy effective theory gets quantum and instanton corrections. As we already mentioned, the metric of the moduli space is the Atiyah-Hitchin and it can be seen as the theory on a D2 probe next to an O6 in M-theory.

In chapter 7, the last one of part II, we will present the last results on the subject of the deformed theory on the D2 probe. We deformed the $\mathcal{N} = 4$ Super Yang Mills with a Chern Simons term, breaking supersymmetry down to $\mathcal{N} = 2$. Here, we do expect the metric to be smooth even in the origin, as its supergravity counterpart is. We were able to obtain the quantum–corrected metric and in fact the metric does not exhibit singularities in the IR regime. We have an argument about the non existence of instantons in this theory, differently from the original [6].

Finally, in the appendix it is possible to find the formalism used in the different chapters and the living project E, marginally related to the main topic of this thesis.

Part I

Pure Spinors formalism for supersymmetry equations and the O6 singularity in the massive IIA

Chapter 2

String vacua basics

In this first chapter we are going to analyse implications of studying vacua in supergravity. These configurations are warped products (meaning a fibration defined through the warping factor $A(y)$) of the internal six dimensional manifold and the external one (the four dimensional spacetime), in which the external one has been chosen to be maximally symmetric, that is Minkowski, AdS or dS. As the name “vacuum” says, these configurations do not have any particles in the external spacetime, since they would break the maximal symmetry of Minkowski, AdS and dS.

We are going to work out equations of motion for vacua model and the supersymmetry ones. The former are not so trivial to solve, since they are second order differential equations; on the other hand; the latter, even if they are first order in the derivatives and, once they are satisfied, they had been shown to satisfy equations of motion, at first sight they are not so easy to work with. Anyway, in the following chapter we are going to see that it is possible to reduce supersymmetry equations on vacua in a much more friendly form, defining univocally the geometry of the internal manifold.

However, in this chapter we are going to show that even equations of motion can deliver important information: in section 2.2 we are going to show that general results for vacua, known in the literature, can be reproduced just from the equations of motion.

2.1 Vacua definition and first implications

2.1.1 Basic definitions

Before starting let us give the basic definitions of type II supergravity. In the next chapters we are going to be much more interested in type IIA, but for completeness even the IIB scenario will be presented.

The massless bosonic field contents of type II can be divided in two sectors from its very first construction: the Neveu-Schwarz-Neveu-Schwarz sector, (NSNS) and the Ramond-Ramond sector (RR). The fields contents of NSNS is the same for the two different type of supergravity theories, while RR is different due to the fact the IIA fermions have different chiralities, while IIB have not.

As it could have been guessed, the NSNS sector, since it is equal for both theories, is related to properties that are fundamental for every gravitational field theory and it contains the metric g_{MN}, the Kalb-Ramond field B_{MN} (a $2-$form) and the dilaton ϕ (a scalar). In type IIA, RR field are forms of odd degree, C_1, $C_3\,, C_5, \ldots$, while in type IIB these are forms of even degree, $C_0\,, C_2, \ldots$ In the following we are going to work much more with field strengths than with fields: for the NS sector, the only one is

$$H \equiv dB \tag{2.1.1}$$

(d is the exterior derivative[1]) for the B-field. Fields strengths for the RR sector do involve the B-field too,

$$\mathcal{F}_p \equiv dC_{p-1} - H \wedge C_{p-3}. \tag{2.1.2}$$

The RR field strenghts are constrained by the Hodge duality relation,

$$\mathcal{F}_p = (-1)^{\lfloor \frac{p}{2} \rfloor} * \mathcal{F}_{10-p}, \tag{2.1.3}$$

where $\lfloor \cdot \rfloor$ is the integer part of the argument. Because of (2.1.3), just field strength of lowest degrees are often used, i.e $\mathcal{F}_0\,, \mathcal{F}_2\,, \mathcal{F}_4$ in IIA and $\mathcal{F}_1\,, \mathcal{F}_3\,, \mathcal{F}_5$ in IIB. In the following we will use a different electric basis, the "democratic" one, using at the same time all the field strength present in the theory and later imposing the (2.1.3) as an extra condition.

Anyway, it is quite demanding to work with all these fields, so it appears much nimbler to work with polyforms: define

$$C \equiv \sum_p C_p; \tag{2.1.4}$$

this C is the sum of forms (C_p) of different degrees, all odd (IIB) or even (IIA). In terms of C defined in (2.1.4), one can write a polyformic field strength

$$\mathcal{F} \equiv dC - H \wedge C. \tag{2.1.5}$$

In terms of the previous polyforms, Bianchi Identity (BI) for IIA and IIB can be written as

$$dH = 0; \qquad d\mathcal{F} - H \wedge \mathcal{F} \equiv d_H \mathcal{F} = 0, \tag{2.1.6}$$

where in the last step we defined the operator $d_H \cdot\ = d \cdot - H \wedge \cdot$[2]. There is another subtlety to remember due to sources: as in electromagnetism,

[1] For the notation of exterior derivatives and contractions, see the appendix A.

[2] Note that, as it happens for the usual exterior derivative, $\{d_H, d_H\} = 0$, once $dH = 0$.

in the presence of a source, say an electric charge, the flux of the field strength through a closed surface is proportional to the charge inside the surface. Usual Dirac quantization arguments implies that even the field strength $\mathcal{F}_p$ experiences a quantization

$$\int_{\Sigma_p} \mathcal{F}_p \sim n, \qquad n \in \mathbb{N}, \tag{2.1.7}$$

where Σ_p is a $p-$cycle and $\sim$ means that, for the moment, we forget about factors depending on the dimension of the cycle.

The fermionic sector contains two Majorana Weyl spinors, the gravitino ψ_M and the dilatino λ. Gravitinos (dilatinos) are two and they have the same chiralities in IIB, while they are different in IIA. The chiralities of one gravitino and the corresponding dilatino is the same.

2.1.2 Vacuum definition

String theory lives in 10 dimension: among these 10, we want to take the 4 of the spacetime we experience. Let us assume that our total 10-dimensional space time M_{10} is fibred over a 4-dimensional manifold,

$$M_{10} : M_4 \times M_6. \tag{2.1.8}$$

M_4 is called the "external spacetime"; the fiber is called "internal manifold".

Vacua configurations are those with a maximally symmetric external space, so they enjoy the maximal amount of Killing vectors. In 4 dimension, this happen to be 10 and it allows just three symmetry group, determining the sign of the cosmological constant Λ:

$$\left\{ \begin{array}{cc} \text{Minkowski } (\Lambda = 0) & \text{Poincaré group} \\ \text{AdS } (\Lambda < 0) & \text{SO(3,2)} \\ \text{dS } (\Lambda > 0) & \text{SO(4,1)} \end{array} \right. \tag{2.1.9}$$

As the name suggests, in supergravity vacua the external space contains no particle: in fact, just the presence of a particle breaks the maximal symmetry since it singles out a special point or direction.

The fibration (2.1.8) and the requirement for maximal symmetry permit to write the most general metric allowed as

$$ds^2 = e^{2A(y)} g^4_{\mu\nu} dx^\mu dx^\nu + g^6_{mn} dy^m dy^n, \tag{2.1.10}$$

where $g^4_{\mu\nu}$ is one of the three metric allowed form the external space (2.1.9), g^6_{mn} is the metric in the internal manifold and $A(y)$ is called the warping factor and it depends only on the internal coordinates (again, in order not to break maximal symmetry).

Other bosonic fields in the game are the curvature for the B-field and the RR field strength. The requirement of vacuum configuration on H is trivial, since it means just that the field has to depend on y coordinates only and none of indices has to take values among the external one. For $\mathcal{F}$ the story is a little bit trickier, since a tensor with 4 indices all in the external manifold is allowed, since it treats all directions of internal space symmetrically. The way out is to write $\mathcal{F}$ in terms of forms defined over M_6:

$$\mathcal{F} = F + vol_4 \wedge \tilde{F}, \tag{2.1.11}$$

where vol_4 is the volume form for M_4. Imposing the Hodge duality condition, it is possible to gain a Hodge duality condition for F and $\tilde{F}$, restricted to M_6:

$$\tilde{F} = \lambda(*_6 F), \tag{2.1.12}$$

where the operator λ on forms of degree p is defined as

$$\lambda \cdot = (-1)^{\lfloor \frac{p}{2} \rfloor} \cdot \tag{2.1.13}$$

As we saw, bosonics fields are easy to control in this situation, but what happens to fermionic ones? The situation is a little bit more complicated: take ϵ to be a fermion in 10d. In order to satisfy the decomposition of the space (2.1.8), we should write ϵ in terms of fermions written in 4d and 6d. It turn out that the most logical thing to do is

$$\epsilon_\pm = \zeta_+ \otimes \eta_\pm + \zeta_- \otimes \eta_\mp, \tag{2.1.14}$$

where ζ is a fermion on M_4 and η one defined on M_6. Note that ϵ is automatically Weyl once $\zeta_\pm$ and $\eta_\pm$ are spinors of fixed chirality. In order to demand ϵ to be Majorana one has to impose $\epsilon = \epsilon^*$. Once chosen the right basis for the expansion of gamma matrices[3], this condition turn to

$$\eta_- = \eta_+^*; \qquad \zeta_- = \zeta_+^*. \tag{2.1.16}$$

This is a great simplification, but in not enough, since so far we did not impose the maximal symmetry of M_4.

If one ζ is specified, then it is possible to construct a vector, i.e. to find a direction among the possible four of M_4 by $v^\mu = \zeta^t \gamma^\mu \zeta$ and so to break maximal symmetry. The idea is that all fermionic equation should be solved for ϵ in such a way that they appear symmetric for a transformation of the maximal symmetry group of the external space.

If we would like to have the smallest amount possible of supersymmetry, we would like to have $\mathcal{N} = 1$ in 4d, so just one ζ with two possible

[3]Using the fact that the total manifold is a warping product between the external and the internal space, it is possible to define

$$\Gamma_\mu = e^A \gamma_\mu \otimes \mathbb{I}; \qquad \Gamma_m = \gamma_5 \otimes \gamma_m. \tag{2.1.15}$$

chiralities. But in order to have a $\mathcal{N} = 2$ (we are in type II supergravity) in 10d, then one has to have two possible η. So, at the end of the day

$$\epsilon^a_\pm = \zeta_+ \otimes \eta^a_\pm + \zeta_- \otimes \eta^a_\mp, \quad a = 1, 2. \tag{2.1.17}$$

Apart from some subtleties to take into account, working with vacua means studying what are the configuration of the fields of the theory in the *internal* space M_6.

2.2 Equations of Motions

Before going into the magic kingdom of supersymmetry, let us wait for the moment in front of the walls and look to something "easier", i.e. equations of motion. If we are interested just in the bosonic part, the equation of motion for the external graviton:

$$e^{-2\Phi}\Big(e^{-2A}\Lambda - \Box_6 A - 4(\nabla_m A)^2 + 2(\nabla_m A)(\nabla^m\Phi)\Big) + \frac{1}{4}\Big(\sum_k F_k^2 + T_p\Big) = 0 \tag{2.2.1}$$

where ϕ is the dilaton, $\Box_6 = g^{mn}\nabla_m\nabla_n$, A is the warping and Λ is the cosmological constant of the external space. T_p is the contribution from branes: it contains a density factor (which is a Dirac delta for localized sources or a finite density function for smeared ones) and a tension term, which is positive for D-branes, while negative for orientifolds.

$$F_k \equiv \frac{F_{m_1\ldots m_k} dx^{m_1} \wedge \cdots \wedge dx^{m_k}}{k!} \tag{2.2.2}$$

and the $k-$forms F_k are the RR field strengths defined earlier in (2.1.11) so

$$F_k^2 = \frac{1}{k!} F_{m_1\ldots m_k} F^{m_1\ldots m_k}. \tag{2.2.3}$$

On the other hand, the internal graviton gives:

$$\begin{aligned} e^{-2\Phi}\Big[R^{(6)}_{mn} - 4\nabla_m\nabla_n A - 4(\nabla_m A)(\nabla_n A) + 2\nabla_m\nabla_n\Phi - \frac{(H_3^2)_{mn}}{2}\Big] \\ - \frac{1}{2}\Big(\sum_k \big((F_k^2)_{mn} - \frac{g_{mn}}{2}(\mathcal{F}_k^2)\big) + T_p\big(\frac{g_{mn}}{2} - \Pi_{mn}\big)\Big) = 0 \end{aligned} \tag{2.2.4}$$

where R_{mn} is the Ricci tensor restricted to M_6 and Π_{mn} is the pullback of the metric on the source and

$$(F_k^2)_{mn} = \frac{1}{(k-1)!} F_{m\ m_1\ldots m_{k-1}} F_n^{\ m_1\ldots m_{k-1}} \tag{2.2.5}$$

Instead the dilaton equation of motion is

$$e^{-2\Phi}\Big[2\Box_6\Phi + 8(\nabla_m A)(\nabla^m\Phi) - 4(\nabla_m\Phi)^2 + H_3^2\Big] + \frac{1}{2}\Big(\sum_k [F_k^2(k-5)] - T_p\Pi^m_{\ m}\Big) = 0 \tag{2.2.6}$$

Let us calculate the trace of (2.2.4):

$$e^{-2\Phi}\Big(R-4\Box A-4(\nabla A)^2+2\Box\Phi-\frac{3}{2}H^2\Big)-\sum_k F_k^2\frac{k-3}{2}-p\frac{T_p}{2}=0. \quad (2.2.7)$$

If we focus on the bosonic sector, no other contribution can appear. As for the Bianchi identities, they can be seen as the equations of motions for the RR-fields, due to auto-duality.

2.2.1 General properties

Just combining together the equations, it is possible to obtain several properties for vacua configurations.

A = const and deSitter

Let us consider (2.2.4) and take the case A = const:

$$e^{-2(A-\Phi)}\Lambda+\frac{\sum_k F_k^2+T_p}{4}=0 \tag{2.2.8}$$

In the case of constant warping, dS (meaning $\Lambda>0$) is possible only when $T_p<0$, so in the presence of an orientifold.

A,Φ = const, with no source

Let us consider the case of A,Φ = const without sources ($T_p=0$): (2.2.7) becomes

$$R-\frac{3}{2}H^2-\sum_k F_k^2\Big(\frac{k-3}{2}\Big)=0, \tag{2.2.9}$$

while (2.2.6) is

$$H^2+\sum_k F_k^2\Big(\frac{k-5}{2}\Big)=0, \tag{2.2.10}$$

where we reabsorbed factors of $\exp(-2\Phi)$ in the definitions of the fields. Summing (2.2.9) and (2.2.10),

$$R-\frac{H^2}{2}-\sum_k F_k^2=0, \tag{2.2.11}$$

meaning that $R\geq 0$.

No deSitter without orientifolds

Let us integrate the (2.2.1), multiplied by e^{4A}:

$$\int d^6x\sqrt{-G}\,e^{4A}\Big[e^{-2\Phi}\Big(e^{-2A}\Lambda-\Box_6 A-4(\nabla_m A)^2+2(\nabla_m A)(\nabla^m\Phi)\Big)+\frac{1}{4}\Big(\sum_k F_k^2+T_p\Big)\Big]=0, \tag{2.2.12}$$

After integration by parts,

$$\int d^6x\sqrt{-G}\,e^{4A-2\Phi}\Box A = \int d^6x\sqrt{-G}\,e^{4A-2\Phi}\big(-4\nabla A\cdot\nabla A + 2\nabla A\cdot\nabla\Phi\big), \tag{2.2.13}$$

So

$$\int d^6x\sqrt{-G}\,e^{4A}\Big[e^{-2\Phi}\Big(e^{-2A}\Lambda + \frac{1}{4}\Big(\sum_k F_k^2 + T_p\Big)\Big] = 0 \tag{2.2.14}$$

means that the dS case need necessarly a source (that should be an orientifold, $T_p < 0$). The result already found in [19], but here it comes directly from the equations of motion for the external space.

2.2.2 A unlucky proposal for a dS vacua

In order to have stable dS vacua of type IIA, [20, 12, 21] showed that there are several necessary ingredients: in particular it is necessary in order to stabilize moduli to have an orientifold O6 as a source and a non zero Romans mass F_0. In my first year of the Ph.D. my supervisor and I tried to build a dS model satisfying those constraints.

The geometry of the internal space was $I \times S^2 \times S^3$, where I is the finite interval $[0, \pi]$. We supposed that the metric could be written as

$$ds^2 = d\theta^2 + a(\theta)^2 ds^2_{S^2} + b(\theta)^2 ds^2_{S^3}, \tag{2.2.15}$$

with the scaling factors a, b defined in order to vanish in the opposite boundary of I:

$$a(\theta = 0) = 0; \qquad b(\theta = \pi) = 0. \tag{2.2.16}$$

In this way it was possible to prove that the system does not experience any singularity. To avoid problems due to localization we used a smeared O6, meaning that the source, instead being localized in a point through a delta function, had a finite density. The O6 lied in the origin of the S^2, meaning that the transverse space respect to O6 is $I \times S^2$.

Field strengths were chosen in order to live completely on one of the two spheres defining the geometry, i.e., having all indices on the same subspace of $I \times S^2 \times S^3$[4]:

$$\begin{aligned} F_2 &\equiv f_2\, vol^0_{S^2} \\ F_4 &\equiv f_4\, d\theta \wedge vol^0_{S^3} \\ F_6 &\equiv \kappa_6 e^{-4A}\, vol_6 \\ H &\equiv h_1\, d\theta \wedge vol^0_{S^2}, \end{aligned} \tag{2.2.17}$$

where ^{0} indicates quantities expressed in terms of comoving coordinates.

[4] In full generality H should have a term proportional to the volume $vol^0_{S^3}$, but the Bianchi identity for F_2 with the smeared O6 forces it to be zero.

In order to simplify the problem, we asked the system to have codimension one, i.e. all fields to depend just on the variable θ. For completeness we reported the equations of motion for the system in the appendix B.

We tried to solve the equations of motion numerically, fixing the boundary condition (2.2.16). The algorithm used evolved the system from both sides of our compact space ($\theta = 0$ and $\theta = \pi$) and then tries to minimize the difference from the value of the fields at an intermediate point in θ. After several attempts, we saw that it was not possible to "close" the internal space.

Actually this project is shelved: there is the possibility of resuming the system introducing a D8, i.e. a plane that make the value of the Romans mass change from side to side. In this way we would have a free parameter to fix (the position of the D8 relative to the O6) and hence much more freedom.

2.3 Supersymmetry equations

As it can be easily seen, equations of motion are second order in the derivatives and hence not so trivial to solve. Instead the supersymmetry variation equations are first order equations in the derivatives and, in the vacua case, they have the useful property of solving the equations of motion automatically, [15, 16].

First, let us look at the supersymmetry equations in type II formalism. In order to simplify the formalism, not so easy by itself, let us convert spinor quantities into differential forms. This at first sight may appear quite odd, but it is sound. The idea is to send

$$\Gamma^M \longrightarrow dx^M, \tag{2.3.1}$$

or in other words

$$\not{\alpha} \equiv \alpha_{i_1 \dots i_n} \Gamma^{i_1 \cdots n} \longrightarrow \alpha \equiv \frac{\alpha_{i_1 \dots i_n}}{n!} dx^{i_1} \wedge \cdots \wedge dx^{i_n}, \tag{2.3.2}$$

where summation on equal indices is implicit. This mapping is called *Clifford map*.

Using this formalism, borrowed from [22], the supersymmetry variations for the gravitino are

$$\begin{aligned} \delta\psi^1_M &= \left(D_M + \frac{H_M}{4}\right)\epsilon^1 + \frac{e^{\phi}}{16}\mathcal{F}\Gamma_M\Gamma\epsilon^2 \\ \delta\psi^2_M &= \left(D_M - \frac{H_M}{4}\right)\epsilon^2 - \frac{e^{\phi}}{16}\lambda(\mathcal{F})\Gamma_M\Gamma\epsilon^1, \end{aligned} \tag{2.3.3}$$

where ϵ^a, $a = 1, 2$, is the supersymmetry parameter of the transformation and all other characters have already been met in the previous section.

The variation of the dilatino can be arranged easily in term of the supersymmetric variation of the gravitino:

$$\begin{aligned}\delta\lambda^1 - \Gamma^M\delta\psi^1_M &= -\big(D - \partial\phi + \frac{H}{4}\big)\epsilon^1\\ \delta\lambda^2 - \Gamma^M\delta\psi^2_M &= -\big(D - \partial\phi - \frac{H}{4}\big)\epsilon^2,\end{aligned} \tag{2.3.4}$$

where $D \equiv D_M\Gamma^M$ and $\partial\phi \equiv \Gamma^M\partial_M\phi$.

Let us focus on vacua configurations: if we decompose the (spinorial) parameter of the supersymmetry variation ϵ^a as we did in (2.1.17) and divide the gravitino in component with $M = \mu$ and $M = m$, our four equations (2.3.3), (2.3.4) become

$$\begin{aligned}&\big(D_m - \frac{H_m}{4}\big)\eta^1_+ \mp \frac{e^\phi}{8}F\gamma_m\eta^2_\mp = 0\\ &\big(D_m + \frac{H_m}{4}\big)\eta^2_\mp - \frac{e^\phi}{8}\lambda F\gamma_m\eta^1_+ = 0\\ &\mu\, e^{-A}\eta^1_+ + \partial A\,\eta^1_+ - \frac{e^\phi}{4}F\eta^2_\mp = 0\\ &\mu\, e^{-A}\eta^2_\pm + \partial A\,\eta^2_\mp - \frac{e^\phi}{4}\lambda(F)\eta^1_\mp = 0\\ &2\mu\, e^{-A}\eta^1_- + D\eta^1_+ + \Big(\partial(2A-\phi) + \frac{H}{4}\Big)\eta^2_+ = 0\\ &2\mu\, e^{-A}\eta^2_\pm + D\eta^2_\mp + \Big(\partial(2A-\phi) - \frac{H}{4}\Big)\eta^1_\mp = 0\end{aligned} \tag{2.3.5}$$

Even if at first sight they may not look so beatiful (and for sure they are not), they have the property that on vacua their solutions are enough to solve equations of motion. In the following chapters we are going to show how (2.3.5) can be rephrased in a more geometrical way.

Chapter 3

Spinor, G-structures and differential forms

In this chapter we are going to review some preliminar basics in order to face the formalism that is going to be used in the rest of the thesis to find solutions to the supersymmetry equations on vacua (2.3.5).

As we saw, supersymmetry equations in vacua of type II are first order differential equations on two spinors defined in 6 dimensions. For the moment, let us consider the special case in which the two supersymmetry parameters are proportional, i.e.

$$\eta^1 = a\eta^2. \tag{3.0.1}$$

(Do not worry about the most general case, it will be treated in the next section). We can say that now supersymmetry equations involve just one spinorial parameter.

In this case we can divide constraints from supersymmetry equations into two parts:

1. the existence of a well defined spinor (our supersymmetry parameter);

2. differential conditions.

Constraint **1.** fixes the topology of our problem, while **2.** constraints the possibility of "gluing" together different patches of our manifold.

In particular, constraint **1.** can be set at the same time as:

A. the existence of a well defined spinor;

B. the existence of "special" differential forms;

C. the structure group of the the tangent bundle our theory is defined on.

These three definitions are equivalent and it is possible to convert the information from one formulation to any other:

$$\begin{tikzcd} & \boxed{\text{spinor}} \arrow[dr] & \\ \boxed{\text{differential forms}} \arrow[ur] & & \boxed{\text{structure group}} \arrow[ll] \end{tikzcd}$$

Next sections will follow the previous scheme upside down, starting from the topology defined by G-structures, finishing with the one from a spinor. At the end of every section differential conditions and their meaning will be presented, together with comparisons with other topological approaches.

3.1 G-structures

First let us approach the problem from the *G-structure* door. In order to make this thesis (almost) self consistent, let us start for the very first definition of the fiber bundle.

Informally, a bundle E on a manifold M (called *base*) with a fiber F is a manifold that locally looks like the product of $M \times F$. This can be rephrased more mathematically in the following way:

Definition 3.1.1 ***Fiber Bundle***
A manifold E is called fiber bundle *with fiber F over a base manifold M if there is a projection $\pi : E \to M$ which satisfies the following condition.*

Take $x \in M$ and $U_\alpha \subset M$ a local neighbourhood of x and call Φ_α the isomorphism that sends $U_\alpha \times F$ to $\pi^{-1}(U_\alpha) \subset E$. If we denote an element of $U_\alpha \times F$ as (x, f), we ask that $\pi^{-1}(\Phi_\alpha(x, f)) = x$ as a consistency condition, called local triviality.

We call *transition functions* those which relate two isomorphism Φ_α, Φ_β, defined over two overlapping open subsets U_α, U_β in M:

$$\Phi_\beta \equiv \Phi_{\alpha\beta}\Phi_\alpha. \tag{3.1.2}$$

Let us introduce the two fiber bundles we will study most in the following: the *tangent bundle* TM is the fiber bundle whose fiber, for every $x \in M$, is T_xM; the *cotangent bundle* T^*M has fiber T^*_xM for every x on the base.

Using TM definition, the bundle over the base M with fiber an ordered basis of TM can be defined; this bundle is called *frame bundle* and it will be identified with FM. Locally it is possible to define the element of FM as (x, e_a), where $x \in M$ and e_a is a basis of T_xM; if d is the dimension of

M, e^a transforms under the action of the group $GL(d,\mathbb{R})$. The group of transition functions is the *structure group* and in this case it $GL(d,\mathbb{R})$.

Anyway, $GL(d,\mathbb{R})$ is quite a "big" group and it would be nice to reduce it to something easier to work with, say $G \subset GL(d,\mathbb{R})$. A manifold M whose tangent bundle has structure group $G \subset GL(d,\mathbb{R})$ is said to have a *G-structure*. This possibility depends on the topology of the base manifold. In the following we present several examples of G-structure manifold.

Example 3.1.3 ***Riemannian manifolds***

A Riemannian manifold *is a manifold with a symmetric positive-definite globally defined non degenerate tensor, g (the* metric*)*[1]*. Its structure group is $O(d)$. The structure group can be reduced to $SO(d)$ if it is possible to define, starting from the metric, a globally defined volume form vol_d.*

This statement is quite intuitive: the "extra" S in the group means just that the determinant of the transformation has to be 1, so no variation of the modulus is allowed, thus preserving the volume.

If the manifold is "spin", meaning that one can take $SO(d)$ to its double cover $Spin(d)$, it is possible to consider spinor bundle too. Since, at the end of the day, we are interested in working out the geometrical properties of spinorial (differential) equations, this is going to be the case.

Example 3.1.4 ***Presymplectic structures***

Let us define $J \in \Lambda^2(M)$, a globally defined $2-$form over a differential manifold M of dimension d. If it is not degenerate, i.e. there in no $x \in M$ such that $J_x = 0$, J is called presymplectic structure*. The structure group of a manifold which admit such a 2-form reduces to $Sp(d,\mathbb{R})$.*

Example 3.1.5 ***Almost Complex Structure***
Let us define a map $I : TM \to TM$ such that,

$$I^2 = -\mathbb{I}_d. \tag{3.1.6}$$

If we ask I to satisfy the structure group symmetry, $\pi(I\,v) = \pi(v)$ for every $v \in TM$. It can be proven that in this case, the structure group reduce to $GL(d/2,\mathbb{C})$ and I is called an almost complex structure*. In fact, because of (3.1.6), I has eigenvalues in $\mathbb{C}$ and those are i and $-i$, but in order to do that we should complexify our fiber: $TM \to TM \otimes \mathbb{C}$.*

[1] Even if this pletora of bombastic words may look scary, it just means that Riemannian manifolds are those manifolds which admit a customary metric.

The existence of two different eigenvalues permits to divide our bundle into two subbundles: L, the one which has eigenvalue i under I and $\bar{L}$, the one which has eigenvalue $-i$. So, every basis can be split in coordinates over L, say "holomorphic", and over $\bar{L}$, say "antiholomorphic".

However, since I is invariant under the structure group in passing from x to y (where $x, y \in M$), (3.1.6) is satisfied, but it is possible to mix coordinates from L_x to those from $\bar{L}_x$ (so holomorphic coordinates, say, are allowed to be mapped to antiholomorphic ones), even if the decomposition in L_y and $\bar{L}_y$ is present. In order to have a local *definition of holomorphicity, one has to ask for differential conditions (which are going to be discussed in the following subsection).*

Example 3.1.7 ***Hermitian metric***

Since $I : TM \to TM$, it can be seen as an element of tensor defined over $T^*M \times TM$. If over a presymplectic manifold, the almost complex structure satisfies

$$I^t J I^t = J \tag{3.1.8}$$

(hermicity condition), *it i possible to define a symmetric tensor called* hermitian metric

$$g = -JI. \tag{3.1.9}$$

It can be proven that equivalently one can ask for the satisfaction of $I^t g I = g$ over a Riemannian manifold and define a pre-symlectic structure as $J = gI$. It can be shown that the structure group is $U(d/2)$.

3.1.1 Integrability

All conditions found so far define the topology of the manifold we are interested in. Now we have to consider what are the implications coming from differential conditions. Before starting, let us give few definitions.

First, let us start with the Lie derivative.

Definition 3.1.10 ***Lie derivative***
Lie derivative $\mathcal{L}$ of a scalar function f respect to a vector field X is defined as

$$\mathcal{L}_X(f) \equiv X(f), \tag{3.1.11}$$

meaning that if $\frac{\partial}{\partial x^i}$ $(i = 1, \ldots, d)$ is a basis over TM, then $X = X^i \frac{\partial}{\partial x^i}$ and

$$\mathcal{L}_X(f) = X^i \frac{\partial f}{\partial x^i}. \tag{3.1.12}$$

Instead, Lie derivative of a vector field Y respect to a vector field X is defined as the Lie bracket of the two vector fields,

$$\mathcal{L}_X(Y) \equiv [X, Y], \quad \forall X, Y \in TM. \tag{3.1.13}$$

Now, let us call a *distribution* a subbundle locally spanned by smooth vector fields. It can be proved (Frobenius's theorem, see for example [23]) that if the distribution L is closed under the action of Lie brackets (the distribution is *involutive*), all its elements can be written locally as

$$X_a^i = \frac{\partial x^i}{\partial \sigma^a}, \tag{3.1.14}$$

where X_a, $a = 1, \dots, \mathrm{rank}(L)$ are vectors spanning L for every $x \in M$ and σ_a is a proper basis. Property (3.1.14) is called "integrability", since there is the possibility, somehow, of integrating the vector fields on a specific distribution; at first sight it would seem a trivial property (given a vector, there is always the possibility to find locally an integral curve), but the crucial aspect is that it is possible to do that without exiting from the distribution L. The importance of the integrability statement, in fact, is that it allows for coordinate transformations that collect local basis for a certain distribution. Integrability permits also to choose the most comfortable vector basis, i.e. the *adapted coordinates*, that is $\frac{\partial}{\partial x^i}$, $i = 1, \dots, \mathrm{rank}(L)$, for some x. The main implications of these properties are examined directly in the following examples.

Example 3.1.15 ***Complex structure***

Applying directly integrability to I implies that is possible to define a neighbourhood of $x \in M$ over which it is possible to define a holomorphic and antiholomorphic subbundle L and $\bar{L}$ (eigenbundle of the almost complex structure with eigenvalue respectively i and $-i$), i.e. to choose a special basis for each distribution.

We call the most convenient basis for L holomorphic coordinates $\frac{\partial}{\partial z^a}$, *while* antiholomorphic coordinates $\frac{\partial}{\partial \bar{z}^a}$ *are those for $\bar{L}$. Once integrability is imposed, it is convenient to remain on the same holomorphic (antiholomorphic) subbundle, going around in the neighbourhood of x.*

If the integrability condition is satisfied the almost complex structure I becomes a complex structure. *Using Frobenius theorem, it is possible to convert the integrability conditions in terms of closure with respect to Lie brackets:*

$$IX = iX \quad \&\& \quad IY = iY \Rightarrow I[X, Y] = i[X, Y]. \tag{3.1.16}$$

If (3.1.16) is satisfied, the Nijenhuis tensor

$$N_I(X, Y) \equiv I[IX, Y] + I[X, IY] - [IX, IY] + [X, Y] \tag{3.1.17}$$

is identically zero. The converse is also true[2]. So, a complex manifold can be defined as a manifold with an almost complex structure which satisfies $N_I(X, Y) = 0$ for every X, Y.

[2]...but it is far from being trivial to prove it.

Example 3.1.18 ***Symplectic structure and Darboux theorem***

As for the complex structure, the presympectic structure can be made a symplectic structure *once*

$$dJ = 0, \tag{3.1.19}$$

for every $x \in M$. Using Darboux theorem (that is, somehow, the integrability statement for symplectic structure, [24]), it is possible to write J as

$$J = dx^i \wedge dy^i \tag{3.1.20}$$

for a certain adapted basis $(x^1, \ldots, x^{d/2}; y^1 \ldots, y^{d/2})$.

3.2 Differential Forms vs. G-structures

At the moment we have several notions about manifold properties depending on the G-structures, i.e. how to relate one tangent bundle to another, with respect to our spacetime manifold (taken as a base for the tangent bundle). It would be great to convert these properties into differential forms properties, since they are really easy to work with.

Let us see what happens case by case, among those studied previously.

Example 3.2.1 ***Almost complex structure***

Before examining how to identify a complex structure, let us start with some definitions.

We saw that, because of the presence of an almost complex structure I, it is possible to decompose the tangent bundle into two subbundle L and $\bar{L}$, depending on the eigenvalue they take under the action of I. The same decomposition can be taken over the cotangent *bundle: if $\Lambda^p(M)$ is the set of the $p-$forms, we say that a 1-form ω_1 belongs to $\Lambda^{(1,0)}(M)$ if $\omega(\bar{X}) = 0$ for every $\bar{X} \in \bar{L}$. Analogously, one can define $\Lambda^{(0,1)}(M)$.*

So, when an almost complex structure is defined, it is possible to decompose the cotangent bundle into two subbundle $T^*M^{(1,0)}$ and $T^*M^{(0,1)}$. The same things happens for every $p-$form, that can be decompose according to previous decomposition

$$\Lambda^p(M) = \bigoplus_{0 \le q \le p} \Lambda^{p,q-p}(M). \tag{3.2.2}$$

Using this decomposition, it is possible to construct a $(d/2, 0)-$form made by the wedge product of $d/2$ $(1,0)-$forms $\omega^i_{(1,0)}$ constituting a frame:

$$\Omega \equiv \bigwedge_{i=1}^{d/2} \omega^i_{(1,0)}. \tag{3.2.3}$$

Note that it is possible to define $\bar{L}$ (and so, even L) starting from Ω:

$$\bar{L} = \{X \in TM | \iota_X \Omega = 0\} \tag{3.2.4}$$

(the equivalent version for L can be obtained by defining $\bar{\Omega}$ as wedge product of $d/2$ $(0,1)-$forms constituting a frame). So, at the end of the day, the existence of an almost complex structure implies the local existence of Ω.

In the previous section we saw that an almost complex structure has a structure group $GL(d/2, \mathbb{C})$. This means that our Ω can get a complex factor in passing from a patch to the following and so Ω is not globally defined. If we want to avoid this problem, we have to reduce the structure group to $SL(d/2, \mathbb{C})$. Moreover, Ω has to be decomposable, meaning that locally can be written as (3.2.3).

Example 3.2.5 ***Hermitian presymplectic structure and $U(d/2)$-structure***

Hermitian presymplectic structure is already a form $J \in \Lambda^2(M)$. If we ask J to satisfy hermiticity condition too, then it should be $J \in \Lambda^{(1,1)}(M)$. Since Ω is a $(3,0)-$form, the previous condition can be rephrased as

$$J \wedge \Omega = 0. \tag{3.2.6}$$

The structure group in this case, as we already saw in Example 3.1.7, is $U(\frac{d}{2})$.

Example 3.2.7 ***Hermitian symplectic structure and $SU(d/2)$-structure***

In the example 3.2.1 we saw that, when defining a $(d/2, 0)-$form Ω, thanks to the decomposition induced by an almost complex structure, there is the freedom to change Ω by a complex factor in passing from patch to patch. If we want to give a global definition of Ω we should avoid this possible factor. We already saw that it is possible, reducing the structure group from $GL(d/2, \mathbb{C})$ to $SL(d/2, \mathbb{C})$.

If we ask a non degenerate J to satisfy the hermiticity condition with a (now globally defined) Ω, as in the previous example, this condition reads

$$J \wedge \Omega = 0, \tag{3.2.8}$$

and the structure group is reduced to $SU(d/2)$.

Since Ω has a global definition, a sort of "normalization" can be fixed, just linking the adapted basis for J to the one for Ω. Using the same one for both gives

$$J^3 = \frac{3}{4} i \Omega \wedge \bar{\Omega}. \tag{3.2.9}$$

Before continuing it is important to underline the fact that all properties so far are valid point by point: again, if, for instance, we want to relate the splitting of our almost complex tangent bundle everywhere in a neighbourhood of $x \in M$, we should add some differential conditions. As we already saw in the previous sections, these are just the notion of integrability. When relating to differential forms, this statement has an easy formulation that can be applied and generalized in different situations.

3.2.1 Integrability again

Again, in order to face integrability, i.e. the possibility to upgrade the almost complex structure to a complex one, Lie derivatives are needed. When acting on differential forms, Lie derivatives can be written in terms of contractions and exterior derivatives[3]:

$$\mathcal{L}_X = \{\iota_X, d\}. \tag{3.2.10}$$

One does not need a lot of machinery in order to obtain that

$$\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y] = [\{\iota_X, d\}, \iota_Y]. \tag{3.2.11}$$

So, how do we use the integrability condition we mentioned in the previous section? If we look back at the cases examined before, the only really new argument is expressing integrability of the almost complex structure trough Ω (even in the previous section a symplectic structure is already defined in term of a globally defined differential form J). Other cases start from considering an almost complex structure and imposing integrability on it.

Example 3.2.12 ***Complex structure and*** Ω**.**
At the end of the day the integrability condition is just

$$\iota_X\Omega = \iota_Y\Omega = 0, \quad \forall X,\, Y \in \bar{L} \quad \Rightarrow \quad \iota_{[X,Y]}\Omega = 0, \tag{3.2.13}$$

but using (3.2.11), that is nothing but

$$\iota_X\iota_Y d\Omega = 0, \tag{3.2.14}$$

meaning that $d\Omega \in \Lambda^{3,1}(M)$*, condition that can be rewritten as*

$$d\Omega = \bar{W}_5 \wedge \Omega \tag{3.2.15}$$

for a certain $W_5 \in \Lambda^{1,0}(M)$*. In section 3.4 we will see why we choose such an odd name for* W_5*.*

[3]For the notation of exterior derivatives and contractions, see the appendix A.

3.3 Spinors vs. G-structures vs. Forms

So, we finally arrived to the heart of the subject of this chapter, how to relate the existence of a spinor to the G-structure of a manifold. First, let us notice that, since $GL(d, \mathbb{R})$ in general does not have a spinorial representation, we have to reduce the structure group. As we saw in the previous sections, it is possible to reduce the structure group to $SO(d, \mathbb{R})$ once we introduce a metric and an orientation. If the structure group has been reduced to $SO(d, \mathbb{R})$ its double cover Spin$(d, \mathbb{R})$ has a spinorial representation.

Anyway the story does not end here. How is this spinor related to forms? We already saw how to do that at the beginning of the section 2.3, that is using a Clifford map.

So, we saw that we can relate forms to gamma matrices (we used this fact in order to find a better way to write down supersymmetric equations). In order to simplify the problem, let us focus on 6 dimensions, the ones we are going to work with in the following sections. So, let us define two forms from a pure spinor[4,5] η_+ and its complex conjugate η_-:

$$J_{ij} \equiv \eta_+^\dagger \gamma_{ij} \eta_+; \qquad \Omega_{i_1 \ldots i_{d/2}} = \eta_-^\dagger \gamma_{i_1 \ldots i_{d/2}} \eta_+. \tag{3.3.1}$$

The definitions (3.3.1) are compatible with (3.2.8) and (3.2.9) written earlier. Since we are in 6 dimensions, that means that the structure group defined by the existence of a never vanishing spinor is $SU(3)$.

So, we were able to reproduce the scheme at the beginning of the chapter.

3.4 Differential conditions on spinors and torsion classes

Before generalizing the statements of this chapter, let us focus on how to classify possible supersymmetry solutions due to the differential condition on J and Ω.

Let us make some comments about the result found so far: when working with vacua of type II, supersymmetry equations are spinorial differential equations involving two six dimensional spinorial parameters. When considering just one of them (that is taking the two spinorial parameters to

[4]In 6 dimensions, the background we are going to work with in the following sections, every Weyl spinor is a pure spinor (a pure spinor is a spinor that is annihilated by the maximal amount of gamma matrices). That is a very lucky feature of six dimensions that really simplifies calculations so much: [25] found a very intricate way to use the same formalism of next sections to 10 dimensions and the "intricateness" relies exactly on the fact that in dimension greater that 6 not all Weyl spinors are pure spinors.

[5]The pedices indicate the chirality of the spinors.

be parallel), the geometry of the problem is constrained by the supersymmetry equations themselves: in fact it is possible to relate a spinor in six dimension to the form J and Ω and the geometry is defined by the differential condition on these two forms. But how to interpret the solutions?

Let us make a step back and introduce the concept of holonomy: in moving upon a contraible closed curve γ_i, the physical field ϕ is transformed under the action of a subgroup of $SO(d, \mathbb{R})$ (which is the structure group) and this subgroup is called *holonomy group*.
In general, a torsionful connection, compatible with metric[6] can be written as

$$\nabla_T \eta_+ = \nabla_{LC} \eta_+ + T\eta_+, \tag{3.4.1}$$

where ∇_{LC} is the Levi-Civita connection (which is torsionless) and T is the *cotorsion tensor*, somehow encoding how far from being Levi-Civita a torsionful connection is.

It is possible, in the case of $SU(3)$ structure, to show that there always exists a metric compatible connection ∇_T, in general with non zero cotorsion tensor, such that

$$\nabla_T \eta_+ = 0. \tag{3.4.2}$$

Because of relation (3.4.1) it is possible to relate every torsionful connection with its own cotorsion tensor, i.e. it is possible to classify every Riemannian manifold using the cotorsion tensor. It is possible [23], [26], to convert informations from the cotorsion tensor into five forms W_i (called *torsion classes*) such that

$$\begin{aligned} dJ &= \frac{3}{2}\mathrm{Im}(\bar{W}_1\Omega) + W_4 \wedge J + W_3; \\ d\Omega &= W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega, \end{aligned} \tag{3.4.3}$$

where W_1 is a complex scalar (or $0-$form, if you prefer), W_2 is a complex primitive $(1,1)-$form, W_3 is a real primitive $(1,2)+(2,1)$ form[7] and W_4 a real 1-form and W_5 a complex $(1,0)-$form. Nice tables on how the values of different torsion classes can define the geometry of the six dimensional manifolds can be found in [23], [26].

Here we just limit ourselves to two comments: first, W_5 found in section 3.2.1 is exactly the same of (3.4.3). Then, note just that a Calabi Yau manifold, which is defined as the manifold that allows $\nabla_{LC}\eta = 0$ for a certain well defined spinor η, has all torsion classes equal to zero.

[6] *Metric compatibility* means just $\nabla_T g = 0$.
[7] Primitivity condition for W_2 is $W_2 \wedge J^2 = 0$, while for W_3 is $W_3 \wedge J = 0$.

Chapter 4

Generalized Complex Geometry formulation of supersymmetry equations on vacua

In the previous chapter we saw that it is possible to relate the geometrical properties of a certain tangent bundle with differential conditions over its structure group, with differential conditions over “special” differential forms and with differential conditions over a never vanishing spinor.

That is not enough to solve (2.3.5): in the most general situation we have two never vanishing spinors satisfying differential conditions.

Somehow the solution can be seen as duplicating the solution found for one spinor, but instead of working on the properties of the (co)tangent bundle, we have to work on the properties of this *Generalized Tangent Bundle*,

$$T \oplus T^*. \tag{4.0.1}$$

In fact, while we saw that the structure defined by a single non vanishing spinor in 6 dimensions is $SU(3)$, we will show that the structure defined by two non vanishing spinors is $SU(3) \times SU(3)$. Complications arise even in the differential forms side of the statement: what once were J and Ω now are two polyforms $\Phi_\pm$, called *pure spinors*.

For the first time [17, 18] presented the supersymmetry equations (2.3.5) in terms of polyforms and what is its interpretation in term of generalized complex geometry.

In the following we will present first the generalized tangent bundle basics, then pure spinors and supersymmetry equations in terms of pure spinors. In the last section of this chapter we will present how a pure spinor should be written and what are its main features.

4.0.1 Before the very beginning: Clifford algebra for forms in 6d

Let us start with basic concepts of differential geometry. In what follows M is a manifold of real dimension 6, as the internal space of our theory.

If we choose a coordinates basis x on M, the derivatives $\frac{\partial}{\partial x^m}$ can be used to define contractions ι's: $\iota_m = \iota_{\frac{\partial}{\partial x^m}}$ acts on differential forms as

$$\iota_m(dx^{i_1} \wedge \cdots \wedge dx^{i_n}) \equiv p\, \delta_m^{[i_1}\, dx^{i_2} \wedge \cdots \wedge dx^{i_n]}. \qquad (4.0.2)$$

A generic vector $v \in T$ can act by contraction as $v\llcorner = v^m \iota_m$, where a sum over equal indices is implicit. So, contractions ι_m and differential forms dx^m satisfy the following algebra

$$\{dx^m \wedge, dx^n \wedge\} = \{\iota_m, \iota_n\} = 0; \qquad \{dx^m \wedge, \iota_n\} = \delta_n^m, \qquad (4.0.3)$$

(At the end of the day, the (4.0.3) is just a mathematical way of expressing the duality between tangent and cotangent space and the antisymmetry of wedge products and contractions.)

That looks quite funny: if we define an element of $T \oplus T^*$ as (v, ω) where $v \in T$ and $\omega \in T^*$, (4.0.3) is a Clifford algebra respect to the metric

$$\mathcal{G} = \begin{pmatrix} \mathbb{O}_6 & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbb{O}_6 \end{pmatrix} \qquad (4.0.4)$$

($\mathbb{O}_6$ is the six dimensional null matrix), so there is a sort of natural metric in term of which a product between elements of $T \oplus T^*$ can be defined: if $v, w \in TM$ and $\psi, \chi \in T^*M$,

$$\langle v + \psi, w + \chi \rangle \equiv \psi\llcorner w + \chi\llcorner v \qquad (4.0.5)$$

In this way the structure group reduces from $GL(12, \mathbb{R})$ to $O(6, 6)$[1].

For completeness, let us give the definition of Lie brackets acting on forms,

$$[v, w]_{Lie}\llcorner \equiv [\{d, v\llcorner\}, w\llcorner], \qquad (4.0.6)$$

where d is the exterior derivative.

4.1 Generalized Complex Structure

Let us consider the generalized tangent bundle,

$$T \oplus T^*. \qquad (4.1.1)$$

[1]Remember from (3.1.3) that in the case of "usual" differential geometry a well defined metric reduces the structure group from $GL(d, \mathbb{R})$ to $O(d)$.

We saw that the existence of a never vanishing spinor can be related to the structure group $SU(3)$, i.e. the structure group for an hermitian symplectic structure. Let us generalize it to $T \oplus T^*$.

Let us first define a *generalized almost complex structure* $\mathcal{J}$, a map

$$\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^* \tag{4.1.2}$$

such that $\mathcal{J}^2 = -\mathbb{I}_{12}$ and satisfies

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}, \tag{4.1.3}$$

where $\mathcal{I}$ is the simplectic metric on $T \oplus T^*$ that we already seen in (4.0.4) and it plays the role of the metric on $T \oplus T^*$ and the (4.1.3) is the generalized equivalent of the hermiticity condition we saw in usual differential geometry, (3.1.8). As in differential geometry the existence of a hermitian symplectic manifold reduces the structure group from $O(6)$ to $U(3)$ in 6 dimensions, here in generalized complex geometry, what happens is that the structure group is reduced from $O(6, 6)$ to $U(3, 3)$ because of the existence of $\mathcal{J}$. Hermiticity condition permits to write $\mathcal{J}$ in a matrix form:

$$\mathcal{J} = \begin{pmatrix} I & P \\ L & -I^t \end{pmatrix}, \tag{4.1.4}$$

with P, L antisymmetric matrices and $I^2 + PL = \mathbb{I}_6$[2].

4.1.1 Integrability again

In the previous chapter we saw that the solution for having a well defined holomorphic basis was asking something that could sound as

$$[L, L] \subset L, \tag{4.1.5}$$

in the sense that the tangent holomorphic bundle should be closed under Lie brackets, otherwise claimed as "holomorphic coordinates go into holomorphic coordinates". How that condition could be reinterpreted in the generalized geometry scenario?

The problems are two: first, understanding what is the generalized version of $T^{(1,0)}M$ and then understanding what is the analogous of Lie brackets. The first one can be easily solved: $T^{(1,0)}M$ is the subset of TM of holomorphic vector, i.e. those for which

$$I^m{}_n v^n = i\, v^m. \tag{4.1.6}$$

So, the analogous object to $T^{(1,0)}M$ in $T \oplus T^*$ is

$$L_{\mathcal{J}} = \{X \in T \oplus T^* | \mathcal{J}X = iX\}. \tag{4.1.7}$$

[2]The previous two condition comes directly from asking $\mathcal{J}^2 = -\mathbb{I}_{12}$.

The second problem seems worse, since the Lie brackets are just defined on T and brackets satisfying Jacobi identity over $T \oplus T^*$ in full generality do not exist. Anyway, Courant brackets, defined as a sort of generalization[3] of (4.0.6),

$$[X, Y]_{Courant} \equiv \frac{1}{2}\Big([\{X\cdot, d\}, Y\cdot] - [\{Y\cdot, d\}, X\cdot]\Big), \tag{4.1.8}$$

for $X, Y \in T \oplus T^*$, satisfy Jacobi identity once restricted to isotropic subbundles.

Having fixed all these problems, we can defined a *generalized complex structure* as a generalized almost complex structure that satisfies the *integrability condition* in $T \oplus T^*$,

$$[L_{\mathcal{J}}, L_{\mathcal{J}}]_{Courant} \subset L_{\mathcal{J}}. \tag{4.1.9}$$

At the end of this first section on generalized space, let us make two (hopefully) clarifing examples:

Example 4.1.10 ***Almost Complex Structure***
The main reason we decide to call I the element in the matrix representation of $\mathcal{J}$ in (4.1.4) is that, when $P = L = \mathbb{O}_6$, i.e.

$$\mathcal{J}_I = \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & -I^t \end{pmatrix}, \tag{4.1.11}$$

if I satisfies (3.1.6) (so it is an almost complex structure on T) it induces a natural generalized almost complex structure over $T \oplus T^*$. If (4.1.7) is satisfied too, the two almost complex structure, the usual and the generalized one, are not "almost" any more.

Example 4.1.12 ***Symplectic structure***
If $I = 0$ and $P = J$ and $L = -J^{-1}$ ($J \in \Lambda^2(M)$), i.e.

$$\mathcal{J}_J = \begin{pmatrix} \mathbb{O} & J \\ -J^{-1} & \mathbb{O} \end{pmatrix}, \tag{4.1.13}$$

it is easy to show that J is a non degenerate two form. Once (4.1.7) is satisfied, it impies $dJ = 0$, so J is a symplectic structure and the manifold is a symplectic one.

So, the two "extreme" cases, diagonal and antidiagonal form of $\mathcal{J}$, correspond respectively to complex structures and symplectic structures. The generalization of taking all $P, L, I \neq 0$ is considering all possible situations between these two.

[3] As it can be easily seen, Courant brackets reduce to Lie's once X, Y are projected to the tangent space.

4.2 Compatibility of Generalized Almost Complex Structure

So far, the structure group has been reduced from $O(6,6)$ to $U(3,3)$. Is it possible to reduce it further?

Let us suppose that, instead of only one generalized almost complex structure, we have two of them, $\mathcal{J}_1$ and $\mathcal{J}_2$, such that

$$[\mathcal{J}_1, \mathcal{J}_2] = 0. \tag{4.2.1}$$

First, note that it is possible to define

$$\mathcal{G} \equiv -\mathcal{J}_1\mathcal{J}_2 \tag{4.2.2}$$

which has the properties

$$\mathcal{G}^2 = \mathbb{I}_{12}; \qquad \mathcal{I}\mathcal{G} = \mathcal{G}^t\mathcal{I}, \tag{4.2.3}$$

where we used the definition of (4.0.4) and the defining property of the $\mathcal{J}$.

We saw in the previous chapter that a hermitian symplectic manifold, i.e. the one which admit a complex structure which is compatible with the metric, has a structure $U(3)$. In the previous section, instead, we saw that its generalized complex geometry equivalent is a $U(3,3)$ structure (since the space we are working with is bigger).

For the moment, let us forget about the freedom of varying the norm of the spinor and just consider the $U(3)$ structure (fixing the norm of the spinor is going to be the subject of the next section). Because, at the end of the day, our goal is to describe the topological locus defined by the existence of two never vanishing spinors, we could imagine that we should consider two "compatible" hermitian symplectic structures, since in "conventional" differential geometry the existence of a well defined spinor corresponds to a hermitian symplectic manifold. The condition (4.2.1) encodes exactly this idea of "compatibility". In this way, the structure group reduce from $U(3,3)$ to $U(3) \times U(3)$.

There is another reason why the generalized metric $\mathcal{G}$ is so important: the properties (4.2.3) have been shown to imply that it is possible to write

$$\mathcal{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \equiv \varepsilon \begin{pmatrix} -\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} \varepsilon^{-1}, \tag{4.2.4}$$

where g is symmetric, B is antisymmetric (both B and g are non-degenerate) and

$$\varepsilon = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ g + B & -g + B \end{pmatrix}. \tag{4.2.5}$$

So, it appears natural to identify g as the metric and B as the Kalb-Ramond field of the original theory. It can be shown, [27], that

$$\mathcal{J}_{1,2} = \varepsilon \begin{pmatrix} I_1 & \mathbb{O} \\ \mathbb{O} & \pm I_2 \end{pmatrix} \varepsilon^{-1}, \tag{4.2.6}$$

where $I_{1,2}$ are two almost complex structure.

Example 4.2.7 ***Hermitian symplectic***
(4.1.11) and (4.1.13) are compatible if and only if the hermicity condition on the metric (3.1.9) is satisfied by the g in (4.2.4). In that case, both $I_1 = I_2 = I$ in (4.1.10) and the structure group is $U(3)$, as already seen in the previous chapter.

4.2.1 Last comments before facing pure spinors definition

So, at the end of the day, we defined the main features of generalized complex tangent bundle we would like to work with: it is a generalization of the hermitian symplectic space we saw in the "conventional" differential geometry, in the sense that its metric is compatible with a almost (now) generalized complex structure.

Integrability condition has been translated into generalized complex geometry formalism by asking the "holomorphic" section of the generalized tangent bundle to close under the action of Courant brackets, i.e. the generalization of the Lie brackets. We saw, also, that it is possible to obtain back the case already seen in the previous chapter as special cases.

So, somehow the geometry is under control. How can we glue this information to forms and to spinors?

4.3 Pure spinors

The message delivered from section 4.0.1 was that in 6 dimensions forms and contractions satisfy a Clifford algebra respect to the metric (4.0.4). So, it is natural to identify the spinor bundle to the cotangent bundle of all degrees, $\Lambda(M) = \sum_p \Lambda^p(M)$ and every spinor can be mapped in a polyform. The parity of the degree of the polyforms (odd or even) appears to be the chirality (somehow it is related in the behaviour under the action of a wedge product).

Before looking to what properties may have a polyformic spinor, let us define some quantities that permits to "well" define a polyformic spinor. In order to have a good definition, a spinor should have a norm, hopefully never vanishing. It is possible to define a product between polyforms using the Mukai pairing, i.e.

$$\langle A, B\rangle \equiv (A \wedge \lambda(B))\big|_6, \tag{4.3.1}$$

where $polyform\big|_6$ indicates the coefficient of vol_6 of the part of 6-form contained in $polyform$ and λ is the operator defined in (2.1.13). Using the Mukai pairing it is possible to define the norm for a pure spinor Φ as

$$||\Phi||^2 \equiv i\langle\Phi, \bar{\Phi}\rangle. \tag{4.3.2}$$

Let us define L_Φ the annihilators set of a spinor Φ as the set of those elements $A \in T \oplus T^*$ such that

$$A \cdot \Phi = 0. \tag{4.3.3}$$

Why are we so interested in the annihilators? As always in this chapter, let us relate the ideas from "conventional" differential complex geometry to generalized complex geometry: in dimensions less or equal[4] to 6, all spinors are *pure*, meaning that the number of annihilators is maximum (in 6 dimensions is equal to 6). The notion can be translated in the language of forms adding a consistence condition, i.e. the one of never vanishing norm:

Definition 4.3.4 ***Pure spinor***
(In 6 dimension) a polyform Φ such that $\dim(L_\Phi) = 6$ and $||\Phi|| \neq 0$ in $T \oplus T^*$ is called pure spinor*. If $\dim(L_\Phi) = 6$, the set of the annihilator L_Φ is said to be* maximally isotropic*.*

As it was done in the previous chapter, this condition can be related to the geometry of the generalized tangent bundle (as the existence of a never vanishing spinor defined an $SU(3)$ structure manifold and its geometrical properties were set in terms of differential conditions for the forms defining the manifolds).

The fact is that a pure spinor can be used in order to define the eigenbundle of the generalized almost complex structure $\mathcal{J}$:

$$L_{\mathcal{J}} \leftrightarrow L_\Phi. \tag{4.3.5}$$

Note that the mapping cannot be one-to-one, since it is possible to rescale the norm of the pure spinor without changing its annihilators. So the map should be intended between the eigenbundle of $\mathcal{J}$ and a line bundle of a pure spinor.

Example 4.3.6 ***Almost Complex Structure (reprise)***
Let us reprise 4.1.10: it easy to see that the eigenbundle of (4.1.11) is just

$$L_I = T^{(1,0)} \oplus (T^*)^{(0,1)}, \tag{4.3.7}$$

[4]We are safe!

confirming our idea that it is just the complex structure eigenbundle extended to $T \oplus T^*$.

How should I write a polyformic spinor in order that the annihilators are elements of (4.3.7)? The answer is quite easy: the Ω defined in (3.2.3) has exactly the properties needed. So,

$$\Phi_I = \Omega. \tag{4.3.8}$$

Example 4.3.9 ***Symplectic structure (reprise)***
The eigenbundle of L_J is less trivial, i.e.

$$L_J = \{v^m + iv^m J_{mn} | \forall v = v^m \partial_m \in T\}. \tag{4.3.10}$$

The pure spinor that has annihilator (4.3.10) can be shown to be

$$\Phi_J = e^{-iJ}. \tag{4.3.11}$$

It has been shown, [27], (and after these two examples it appears much more believable) that the most general way to write a pure spinor is

$$\Phi = \Omega_k \wedge e^{B+ij}, \tag{4.3.12}$$

where Ω_k is a k–form and B and j are real two–forms. We will go back to this subject in 4.5.

4.3.1 Compatible pure spinors

As we saw that it was possible (and useful, from the supergravity point of view) to look for compatible generalized almost complex structure, the natural question is what are the defining properties of compatible pure spinors. As we already said, in order to reduce the structure group to $SU(3) \times SU(3)$, the norm of the two pure spinor should be under control, i.e.

$$||\Phi_1|| = ||\Phi_2||. \tag{4.3.13}$$

Moreover, it has been shown, [28], that $[\mathcal{J}_1, \mathcal{J}_2] = 0$ can be reformulated as

$$\langle \Phi_1, X\Phi_2 \rangle = 0, \qquad \forall X \in T \oplus T^*, \tag{4.3.14}$$

which implies that Φ_1 and Φ_2 must have different parity. Moreover, note that, since the generalized metric can be written in terms of the original metric and Kalb-Ramond field, it is also possible to relate a couple of compatible pure spinors to those geometrical quantities. We will come back to this issue in the following section, when we will write explicitly pure spinors for different structure group.

Example 4.3.15 $SU(3)$ ***structure***
If (4.3.8) and (4.3.11) are compatible, the structure reduces to $SU(3)$.

Compared to the case analysed in (4.2.7), the difference is that, by fixing the norm of the two pure spinors, the value of the volume is under control and the structure group is reduced from $U(3)$ to $SU(3)$.

There is just one step missing: we saw that it is possible to define the geometry of internal space in terms of polyformic spinor that can be related to "ordinary spinors", but we do not know how this map should be performed. We will not prove it (a partial, but clarifying, proof can be found in [29]) but the demonstration split into two parts: first the demonstration that $Cl(6,6) \simeq Cl(6) \times Cl(6)$ and then that it is possible write a compatible pair of pure spinors as

$$\Phi_{\pm} = e^{B\wedge}\eta^1_+ \otimes (\eta^2_{\pm})^{\dagger}, \tag{4.3.16}$$

where B is a some two form and $\eta^i_{\pm}$ are ordinary $Cl(6)$ spinors of fixed chirality.

4.4 Pure spinors supersymmetry equations

Now that we learned how to write polyformic pure spinors in term of the original spinors, it is possible to write supersymmetry equations (2.3.5) as

$$\begin{aligned} d_H\mathrm{Im}(\Phi_1) =&0\\ d_H\mathrm{Re}(\Phi_1) =&-2\mu e^{-A}\mathrm{Re}(\Phi_2)\\ d_H\big(e^{-A}\mathrm{Re}(\Phi_2)\big) =&0\\ d_H(e^{A}\mathrm{Im}(\Phi_2)) =&-3\mu\mathrm{Re}(\Phi_1) + e^{4A} *_6 \lambda(f) \end{aligned} \tag{4.4.1}$$

in the case of AdS. All characters have been already met in the previous two chapters, but $d_H = d - H\wedge$ and $\mu = \sqrt{\Lambda/3}$ (where $\Lambda < 0$ is the cosmological constant).
The Minkowski version of the previous equations is just the limit $\mu \to 0$.

Even if the simplicity of the (4.4.1) is amazing, there is one ingredient that may appear annoying: $*_6$, the 6 dimensional Hodge star, calculated respect to $g_{\Phi_{\pm}}$ is quite disturbing, since the metric informations are contained in the pure spinors, whose differential equations we are trying to solve. In order to fix this problem, there is a second version of the previous equations, [28], that is

$$\begin{aligned} d_H\big(\Phi_1\big) =&-2e^{-A}\mu\,\mathrm{Re}\Phi_2\\ F =&\mathcal{J}_1\cdot d_H(e^{-3A}\mathrm{Im}\Phi_2) + 5\mu e^{-4A}\mathrm{Re}\Phi_1, \end{aligned} \tag{4.4.2}$$

where $\mathcal{J}_1$ is the almost complex structure induced by Φ_1.

There is just the only missing point: we saw that two compatible pure spinors have to have opposite parity, but the equations look symmetric in the sense that so far we did not decide, between Φ_1 and Φ_2, which is the odd and which is the even polyform. It turns out that different choices take to different theories: if

$$\begin{aligned} \Phi_1 =&\Phi_+\\ \Phi_2 =&\Phi_- \end{aligned} \tag{4.4.3}$$

the theory is IIA. If, instead

$$\begin{aligned}\Phi_1 &= \Phi_- \\ \Phi_2 &= \Phi_+\end{aligned} \tag{4.4.4}$$

the theory is IIB.

Before going on, note that (4.4.2) is invariant under the transformation[5]

$$H \to H - d\delta b \ , \qquad F \to e^{-\delta b \wedge} F \ , \qquad \Phi_\pm \to e^{-\delta b \wedge} \Phi_\pm \ . \tag{4.4.5}$$

As it turns out, the $b_{\Phi_\pm}$ determined by $\Phi_\pm$ transforms as $b_{\Phi_\pm} \to b_{\Phi_\pm} + \delta b$ under (4.4.5). The physical NS three-form is the combination

$$H_{\rm phys} = H + db_{\Phi_\pm} \ , \tag{4.4.6}$$

which is thus invariant under (4.4.5). The physical RR field is the one which obeys physical Bianchi identities $d_{H_{\rm phys}} F_{\rm phys} = \delta$:

$$F_{\rm phys} = e^{b_{\Phi_\pm} \wedge} F \ . \tag{4.4.7}$$

4.5 How to write a pure spinor and its properties

In the following, since we will focus on type IIA compactification, we are going to consider how to write pure spinors just in the type IIA case. This situation can be generalized to IIB just switching $\Phi_+ \leftrightarrow \Phi_-$.

In (4.3.12) we saw what is the general way to write a pure spinors. Is there the possibility of restricting more the forms in the game? In order to do that, let us consider the possible way of writing a pure spinor.

Let us call the *type* of a pure spinor $\Phi = \sum_{k \geq k_0} \Phi_k$ the smallest degree k_0 that appears in the sum; in other words, Φ only contains forms of degree type(Φ) or higher. It turns out that the type of a pure spinor in dimension 6 can be at most 3. There are then three cases:

Φ_+ has type 0, and Φ_- has type 3: this is the usual SU(3) structure case that we saw;

Φ_+ has type 0, and Φ_- has type 1: this is the most generic case (so $SU(3) \times SU(3)$), and it is sometimes just called “intermediate SU(2) structure”;

Φ_+ has type 2, and Φ_+ has type 1: this is called “static SU(2) structure” case.

In this thesis, we will only need the first two cases.

[5]This property is the main reason we are using the system (4.4.2) rather than the original form of these equations, involving the Hodge star. Those equations can be made invariant under (4.4.5) only after defining a rather awkward $*_b \equiv e^b * e^b$ operator.

4.5.1 Pure spinor and SU(3)

Let us slightly modify the definitions (4.3.8) and (4.3.11), in order to have a deeper control on the pure spinors[6]:

$$\Phi_+ = \rho e^{i\theta} e^{-iJ} \;, \qquad \Phi_- = \rho\,\Omega \;, \tag{4.5.1}$$

with ρ and θ real functions and J and Ω are the ones already met, i.e. they satisfies

$$J \wedge \Omega = 0 \;, \qquad J^3 = \frac{3}{4} i \Omega \wedge \bar{\Omega} \neq 0 \;. \tag{4.5.2}$$

We will now describe how to map the metric, the dilaton and the Kalb-Ramond field for this case. The $b_{\Phi_\pm}$ obtained by it is zero:

$$b_{\Phi_\pm} = 0 \;. \tag{4.5.3}$$

The metric defined by $\Phi_\pm$ is just $g = JI$, as we already saw. Finally, the dilaton is given by

$$e^{\phi} = \frac{e^{3A}}{\rho} \;. \tag{4.5.4}$$

We also give the form of $\mathcal{J}_+ \cdot$, which enters (4.4.2):

$$\mathcal{J}_+ \cdot = J \wedge - J^{-1} \llcorner \;. \tag{4.5.5}$$

4.5.2 Pure spinors and SU(3) $\times$ SU(3) structure

In this case, one can parameterize the most general solution to (4.4.2) as [30, 31, 32]

$$\Phi_+ = \rho\, e^{i\theta} \exp[-iJ_\psi] \;, \tag{4.5.6a}$$
$$\Phi_- = \rho\, v \wedge \exp[i\omega_\psi] \;, \tag{4.5.6b}$$

where

$$J_\psi \equiv \frac{1}{\cos(\psi)} j + \frac{i}{2\tan^2(\psi)} v \wedge \bar{v} \;, \qquad \omega_\psi \equiv \frac{1}{\sin(\psi)} \left(\mathrm{Re}\omega + \frac{i}{\cos(\psi)} \mathrm{Im}\omega \right) \;, \tag{4.5.7}$$

for some (varying) angle ψ, real function ρ, one-form v and two-forms ω, j satisfying

$$j \wedge \omega = 0 \;, \qquad \omega^2 = 0 \;, \qquad \omega \wedge \bar{\omega} = 2j^2 . \tag{4.5.8}$$

(4.5.8) can be seen as the 2-dimensional version of (4.5.2), which means that ω, j define an SU(2) structure.[7] These can also be rewritten more symmetrically as

$$j \wedge \mathrm{Re}\omega = \mathrm{Re}\omega \wedge \mathrm{Im}\omega = \mathrm{Im}\omega \wedge j = 0 \;, \tag{4.5.9a}$$
$$j^2 = (\mathrm{Re}\omega)^2 = (\mathrm{Im}\omega)^2 \;; \tag{4.5.9b}$$

[6] Of course, our modifications do not change the physical contents of the theory.

[7] Actually, from the fact that the norm of pure spinors never vanishes, one would get (4.5.8) wedged with $v \wedge \bar{v}$, but one can show [32, Sec. 3.2] that these can be dropped without any loss of generality.

these equations are reminiscent of the defining relations of the quaternions i, j, k, which is ultimately because SU(2)$\cong$Sp(1). Finally, the non vanishing of the norm of the pure spinors implies that the top-form $v \wedge \bar{v} \wedge j^2$ should be non-zero everywhere.

We will now detail the map to the metric and other usual geometric quantities for this case. This can be inferred by comparing (4.5.6) to its derivation in [30, 31, 32] from spinor bilinears. For example, [32, Eq. (3.19)] can be connected to (4.5.6) by a b-transform; from this, we see that the $b_{\Phi_\pm}$ defined by the pure spinors is non-zero:

$$b_{\Phi_\pm} = \tan(\psi)\mathrm{Im}\omega \; . \tag{4.5.10}$$

The metric can be found by relating the forms j, ω and v in (4.5.6) to the spinor bilinears of an SU(3) structure. In [32] one finds $J = j + \frac{i}{2} z \wedge \bar{z}$, $\Omega = \omega \wedge z$, where $z \equiv \frac{1}{\tan(\psi)} v$. This tells us that the metric is the direct sum of a two-by-two block $z\bar{z} = \frac{1}{\tan^2(\psi)} v\bar{v}$, and of a four-by-four block determined by the SU(2) structure j, ω. In other words, we have two orthogonal distributions (namely, subbundles of T): D_2 and D_4. The explicit form of the four-by-four block in the metric is $g_4 = jI_4$, where I_4 is an almost complex structure along D_4. This means that I_4 squares to -1 along D_4:

$$I_4^2 = -\Pi_4 \; , \tag{4.5.11}$$

where $(\Pi_4)^m{}_n = \delta^m{}_n - \mathrm{Re}v^m \mathrm{Re}v_n - \mathrm{Im}v^m \mathrm{Im}v_n$ is the projector on D_4. We should now compute I_4. This can be done by writing $I_4 = (\mathrm{Re}\omega)^{-1}\mathrm{Im}\omega$ (which can be derived in holomorphic indices). Since ω only spans four directions, $\mathrm{Re}\omega$ has rank 4; so writing $(\mathrm{Re}\omega)^{-1}$ is an abuse of notation. It should be understood as an inverse along the distribution D_4. In practice, it can be computed as a matrix of minors:

$$[(\mathrm{Re}\omega)^{-1}]^{mn} = -2\frac{(dx^m \wedge dx^n \wedge \mathrm{Re}\omega \wedge v \wedge \bar{v})}{(\mathrm{Re}\omega)^2 \wedge v \wedge \bar{v}} \; . \tag{4.5.12}$$

Putting all together, we have

$$ds^2 = jI_4 + \frac{1}{\tan^2(\psi)} v\bar{v} \; , \qquad I_4 = (\mathrm{Re}\omega)^{-1}\mathrm{Im}\omega \; . \tag{4.5.13}$$

Finally, the dilaton ϕ is determined by

$$e^\phi = \frac{e^{3A}}{\rho}\cos(\psi). \tag{4.5.14}$$

We also give the form of the operator $\mathcal{J}_+ \cdot$ that appears in (4.4.2) is similar to the one in (4.5.5):

$$\mathcal{J}_+ \cdot = J_\psi \wedge - J_\psi^{-1} \llcorner \; . \tag{4.5.15}$$

Chapter 5

Localized O6 in massive IIA

5.1 Introduction

In the previous section we organized all tools needed to find solutions to supersymmetric vacua models. Converting the whole spinorial information to differential form, it is possible to analyse supersymmetry equations as differential conditions over the differential forms defined over

$$T \oplus T^*. \tag{5.1.1}$$

A part from simplifying calculations, the generalized geometry formalism for the analysis of supersymmetry equations provide a natural geometrical frame for the interpretation of the solution found.

We can apply this method to the main subject of this thesis, i.e the massive deformation of the metric in the nearby of the O6 plane. In order to understand how these deformations should appear in supergravity, let us analyse first the massless case, i.e. when $F_0 = 0$, and then the smeared massive solution proposed by [8]. These two have $SU(3)-$structure.

In the following sections we will show the explicit form of supersymmetry equation in generalized geometry formalism for the most general $SU(3) \times SU(3)$ case and we proposed a deformation due to the presence of a non zero Romans mass. In the end of this chapter we will show the numerical result to the first order in the perturbation due to Romans mass and to full order.

In order to spoil the suspense, let us introduce the final results: the presence of the Romans mass prevents the O6 to experience singularities. The metric in the transverse space is locally $\mathbb{R} \times S^2$ and the S^2 has finite dimension even in the origin, i.e. where the O6 lies; the dimension of the transverse S^2 in the origin depends on the value of the F_0. This can be interpreted as the presence of an O6 “bubble”, instead of O6 plane, in the massive case. Moreover, the absence of singularities permits to analytically continue the theory for negative radii.

5.2 O6 solution

In this brief section, we will review how the O6 solution in flat space, whose metric was given in (1.0.2), solves the system (4.4.2) in the Minkowski case.

The internal space M_6 is in this case nothing but $\mathbb{R}^6$, with coordinates x^i and y^i (to be thought of respectively as parallel and orthogonal to the O6). The O6 solution is of SU(3)-structure type (4.5.1). For cosmological constant $\Lambda = 0$, and hence $\mu = \sqrt{-\Lambda/3} = 0$, the equations in (4.4.2) read

$$\rho = e^{3A-\phi} = \text{const} \ , \qquad dJ = 0 = H \ ; \qquad d(e^{-A}\mathrm{Re}\Omega) = 0 \tag{5.2.1}$$
$$F_2 = -J^{-1} \llcorner d(e^{-\phi}\mathrm{Im}\Omega) \ , \qquad dF_2 = \delta \ . \tag{5.2.2}$$

Notice that, in this case, θ is constant, but otherwise undetermined.

In general, in (5.2.1) δ is a delta-like current supported on the sources present. For the O6 solution, it reads

$$\delta = \delta_{\mathrm{O6}} = -4\pi l_s \delta(y^1)\delta(y^2)\delta(y^3) dy^1 \wedge dy^2 \wedge dy^3 \ ; \tag{5.2.3}$$

an SU(3) structure that solves (5.2.1) can then be given as

$$\begin{aligned} J &= dx^i \wedge dy^i \ , \\ \Omega &= i(Z^{-1/4}dx^1 + iZ^{1/4}dy^1) \wedge (Z^{-1/4}dx^2 + iZ^{1/4}dy^2) \wedge (Z^{-1/4}dx^3 + iZ^{1/4}dy^3) \end{aligned} \tag{5.2.4}$$

with Z the Green function for the flat Laplacian in $\mathbb{R}^3$:

$$Z = 1 - \frac{r_0}{r} \ , \qquad r \equiv \sqrt{y^i y^i} \ , \qquad r_0 = g_s l_s \ , \tag{5.2.5}$$

as we already saw in (1.0.2).[1] We also have

$$F_2 = -\frac{l_s}{2r^3}\epsilon_{ijk} y^i dy^j \wedge dy^k \ , \qquad e^A = Z^{-1/4} \ , \qquad e^\phi = g_s Z^{-3/4} \qquad \left(\rho = \frac{1}{g_s}\right) \ ; \tag{5.2.6}$$

g_s is a constant that we can think of as the value of e^ϕ at infinity.

The SU(3) structure in (5.2.4) is one possible solution to (5.2.1), and by itself it only describes four supercharges; there are other solutions, related to the one in (5.2.4) by flipping some signs, which describe the other supercharges. In this paper, we will focus on (5.2.4): for this reason, our

[1] If we had had N D6-branes instead of an O6-plane, the function Z would have read $1 + \frac{r_0}{r}$, with $r_0 = N l_s g_s / 2$.

massive solutions will have $\mathcal{N} = 1$ supersymmetry.

Finally, notice that, since the solution stops making sense before we can get to $r \to 0$, the equation $dF_2 = \delta$ has to be understood as a Gauss' law: namely,

$$\int_{S^2} F_2 = -4\pi l_s \ , \tag{5.2.7}$$

for any S^2 that surrounds the origin, where the O6-plane is located.

5.3 Smeared O6 with Romans mass

Our aim is to find a O6 solution in the presence of Romans mass. As recalled in the introduction 1, a solution of this type can be found easily if one "smears" the O6 source; this was done in [8] in the language of effective field theory, and lifted to ten dimensions in [9].

We take a spacetime of the form (1.0.3): the four-dimensional part has non-zero cosmological constant. This means that $\mu \neq 0$, and thus we have to use the AdS version of the supersymmetry conditions (4.4.2). If we also take

$$\theta = 0 \ , \tag{5.3.1}$$

we get[2]

$$\begin{aligned} &dJ = 0 \ , \qquad d\Omega = -ig_s F_2 \wedge J \ , \qquad H = 2\mu \mathrm{Re}\Omega \ , \qquad \rho = \mathrm{const} \ , \qquad A = 0 \ ; \\ &g_s F_0 = 5\mu \ , \qquad dF_2 - HF_0 = \delta \ , \qquad g_s F_4 = \frac{3}{2}\mu J^2 \ , \qquad F_6 = 0 \ . \end{aligned} \tag{5.3.2}$$

From (4.5.4), it also follows that the dilaton is constant; $g_s \equiv e^{\phi}$.

So far, the source δ was unspecified. To find the solution in [8], take $F_2 = 0$. Then we see that the Bianchi identity for F_2 implies

$$\delta = -2\mu F_0 \mathrm{Re}\Omega \ . \tag{5.3.3}$$

This is the "smearing" proposed in [9].

To get a sense of the physics of this compactification, let us moreover assume as in [8] that F_0 is of order one, that the periods of F_4 are of order N, and that the internal space has volume $\sim R^6$. We know already that $\delta \propto \mathrm{Re}\Omega$; it makes sense to fix the proportionality constant as

$$\delta \sim -\frac{1}{R^3}\mathrm{Re}\Omega \ , \tag{5.3.4}$$

[2]The first two equations in (4.4.2), which are the ones that are equivalent to the conditions of unbroken supersymmetry, do not by themselves imply that $A = 0$. For the Romans mass they would give $g_s F_0 = 5\mu e^{4A}$; if one now also adds the Bianchi condition $dF_0 = 0$, one gets that A is constant. In (5.3.2) we set it to zero, because a non-zero value can always be reabsorbed in the definition of μ.

so that integrating δ along a 3-cycle gives an order one number. The Bianchi identity then says $F_0\mu \sim R^{-3}$; moreover, from (5.3.2) we see that $F_0 \sim \mu/g_s$ and $F_4 \sim F_0 R^4$. We thus find that the parameters scale as

$$R \sim N^{1/4} \,, \qquad g_s \sim \mu \sim R^{-3} \sim N^{-3/4} \,. \tag{5.3.5}$$

We have seen that it is easy to find a supersymmetric solution including O6 planes and Romans mass, if one is willing to smear the O6 source δ as in (5.3.4). As stressed in the introduction 1, smearing an O6 is not really meaningful in string theory, but solutions obtained with this trick are often precursors to “localized” solutions, namely ones where the source is delta-like as it should be (as in (5.2.4)). So we can take the solution reviewed in this section as an inspiration for the solution we are looking for.

The most natural course of action might seem to solve the equations (5.3.2) without assuming $F_2 = 0$, and with an unsmeared source, unlike in (5.3.4). However, we immediately face a problem: (5.3.2) imposes $A = 0$. This does not seem possible for a solution with a source: in particular, the solution with $F_0 = 0$ has a non-constant A, as we can check from (5.2.6).

So unfortunately we cannot use SU(3) structure solutions. We are left with the second and third cases in section 4.5. If we think of adding a small amount of F_0 to the massless solution, which is SU(3), it seems more natural to select second case, which is generic and can be continuously connected to the SU(3) structure case, rather than third one, which is isolated. This is the reason we did not study the third case in section 4.5. In section 4.5.2 we reviewed the solution (4.5.6) of the algebraic constraints (4.3.13, 4.3.14) for $SU(3) \times SU(3)$ structure; we will now analyze the corresponding differential equations.

5.4 SU(3) × SU(3) structure compactifications

As we just saw, a localized O6 with Romans mass cannot be an SU(3) structure solution; this motivates us to look for an SU(3) × SU(3) structure solution. For that class, the algebraic constraints have been reviewed in section 4.5.2; we will now use those results (in particular (4.5.6)) in the system (4.4.2). This section contains both a review of old results, and some new ones — most importantly, the expressions for the fluxes.

For reasons explained in the introduction 1, we will first look at the AdS case, which we divide in two sections, 5.4.1 and 5.4.2. We will then also analyze the Minkowski case, in section 5.4.3.

5.4.1 AdS: generic case

Geometry

We will start by the first equation in (4.4.2), $d_H\Phi_+ = -2\mu e^{-A}\text{Re}\Phi_-$. Using (4.5.6), the one-form part says that

$$d(\rho\,\sin(\theta)) = 0 \,, \tag{5.4.1}$$

$$\text{Re}v = \frac{e^A}{2\mu\sin(\theta)}d\theta \,. \tag{5.4.2}$$

In deriving (5.4.2), we have solved (5.4.1) by taking

$$\rho = \frac{\rho_0}{\sin(\theta)} \,, \tag{5.4.3}$$

where ρ_0 is a constant. This means that we have assumed

$$\theta \neq 0 \tag{5.4.4}$$

everywhere. In this subsection, we will continue our analysis in this assumption. The case $\theta = 0$ is quite different, and will be described in section 5.4.2.
Coming back to $d_H\Phi_+ = -2\mu e^{-A}\text{Re}\Phi_-$, its three-form part now gives

$$H = -d(\cot(\theta)J_\psi) \,, \tag{5.4.5}$$

$$d\left(\frac{1}{\sin(\theta)}J_\psi\right) = 2\mu e^{-A}\text{Im}(v\wedge\omega_\psi) \,. \tag{5.4.6}$$

Finally, the five-form part can be shown to follow from the one- and three-form parts, (5.4.2) and (5.4.6).

Flux

We will now look at the second equation in (4.4.2). We have seen that H is determined by (5.4.5). We can then use (4.4.5) with the choice $\delta b = -\cot(\theta)J_\psi$, so that we end up with $H = 0$ in (4.4.2).

However, there is a price to pay. Once we transform $\Phi_+ \to e^{-\delta b\wedge}\Phi_+$, we also have to transform the associated operator $\mathcal{J}_+\,\cdot$:

$$\mathcal{J}_+\cdot\; \to e^{-\delta b\wedge}\mathcal{J}_+\cdot e^{\delta b\wedge} \,. \tag{5.4.7}$$

For the choice $\delta b = -\cot(\theta)J_\psi$, remembering (4.5.15), we get that the new $\mathcal{J}_+$ operator is

$$\mathcal{J}_+\cdot\; = e^{\cot(\theta)J_\psi\wedge}(-J_\psi^{-1}\llcorner + J_\psi\wedge)e^{-\cot(\theta)J_\psi\wedge} \,. \tag{5.4.8}$$

This can be computed in two ways. The first is to compute the associated action on $T\oplus T^*$, where $e^{b\wedge}$ is represented by $\left(\begin{smallmatrix}1 & 0\\ -b & 1\end{smallmatrix}\right)$. The second is to just use the formula $e^{-A}Be^A = B + [B,A] + \frac{1}{2}[B,[B,A]] + \ldots$, and

$$[J_\psi^{-1}\llcorner, J_\psi\wedge] = h \,, \qquad h\omega_k \equiv (3-k)\omega_k \,, \tag{5.4.9}$$

as an example of the usual Lefschetz representation of $\mathrm{Sl}(2,\mathbb{R})$ on forms (see for example [33, Ch. 0.7]). Either way, we get

$$\mathcal{J}_+ \cdot \ = -J_\psi^{-1}\llcorner + \cot(\theta) h + \frac{1}{\sin^2(\theta)} J_\psi \wedge \ . \tag{5.4.10}$$

We can now compute the fluxes from the second equation in (4.4.2):

$$\begin{aligned}
F_0 &= -J_\psi^{-1}\llcorner d(\rho e^{-3A}\mathrm{Im}v) + 5\mu\rho e^{-4A}\cos(\theta) \ ; && (5.4.11a)\\
F_2 &= F_0 \cot(\theta) J_\psi - J_\psi^{-1}\llcorner d\,\mathrm{Re}(\rho e^{-3A} v\wedge\omega_\psi) && (5.4.11b)\\
&\quad + \mu\rho e^{-4A}\left[(5+2\tan^2(\psi))\sin(\theta)J_\psi + 2\sin(\theta)\mathrm{Re}v\wedge\mathrm{Im}v - 2\cos(\theta)\tan^2(\psi)\mathrm{Im}\omega_\psi\right] \ ; \\
F_4 &= F_0\frac{J_\psi^2}{2\sin^2(\theta)} + d\left[\rho\, e^{-3A}(J_\psi\wedge\mathrm{Im}v - \cot(\theta)\mathrm{Re}(v\wedge\omega_\psi))\right] \ ; && (5.4.11c)\\
F_6 &= -\frac{1}{\cos^2(\psi)}\mathrm{vol}_6\left(F_0\frac{\cos(\theta)}{\sin^3(\theta)} + 3\frac{\rho\mu e^{-4A}}{\sin(\theta)}\right) \ . && (5.4.11d)
\end{aligned}$$

Recall that ρ is related to the dilaton by (4.5.14). The expression for F_0 already appeared in [4]. The expressions for F_2 and F_4 are new; their expressions appear much simpler than in earlier computations, thanks in part to the δb transformation we performed earlier.

Notice that the Bianchi identities for (5.4.11) are now $dF_k = 0$, away from sources. The one for F_0 just says F_0 is constant, as usual. If we now consider dF_4, we see that the term not multiplying F_0 is exact, so it drops out. On the other hand, the form $J_\psi^2/\sin^2(\theta)$ that multiplies F_0 is easily seen to be closed as a consequence of (5.4.6). So we conclude

$$dF_0 = 0 \ \Rightarrow \ dF_4 = 0 \ . \tag{5.4.12}$$

In other words, the Bianchi identity for F_4 is redundant. This fact will be very important for the rest of this paper.

We should stress once again that the F_k given in (5.4.11) are the ones which are closed under d — and which are locally given by $F_k = dC_{k-1}$. The physical NSNS three-form is given by combining (4.4.6), (4.5.10) and (5.4.5):

$$H_{\mathrm{phys}} = dB_{\mathrm{phys}} = d(-\cot(\theta)J_\psi + \tan(\psi)\mathrm{Im}\omega) \ ; \tag{5.4.13}$$

the RR fluxes which are closed under $(d - H_{\mathrm{phys}}\wedge)$ are then given by

$$\tilde{F} = e^{B_{\mathrm{phys}}\wedge} F \ . \tag{5.4.14}$$

5.4.2 AdS: special case

We will again start by the first equation in (4.4.2), $d_H\Phi_+ = -2\mu e^{-A}\mathrm{Re}\Phi_-$. Our generic analysis in section 5.4.1 relied on the assumption that $\theta \neq 0$; in this section we will consider the case

$$\theta = 0 \ . \tag{5.4.15}$$

This obviously solves (5.4.1). The remaining one-form equation now says

$$\mathrm{Re}v = -\frac{e^{A}d\rho}{2\mu\rho} \ , \tag{5.4.16}$$

which replaces (5.4.2).

The three-form part of $d_H\Phi_+ = -2\mu e^{-A}\mathrm{Re}\Phi_-$ now gives

$$d(\rho J_\psi) = 0 \ , \qquad H = 2\mu e^{-A}\mathrm{Re}(iv\wedge\omega_\psi) \ . \tag{5.4.17}$$

Finally, the five-form part can be shown to follow from the one- and three-form parts, (5.4.16) and (5.4.17).

We now turn to the RR fluxes. Unlike in section 5.4.1, this time there is no natural b-transform to perform, because H given in (5.4.17) is not necessarily exact. So we will give the expressions of the fluxes which are closed under d_H, rather than under d:

$$F_0 = -J_\psi^{-1}\llcorner d(\rho e^{-3A}\mathrm{Im}v) + 5\mu\rho e^{-4A} \ ; \tag{5.4.18a}$$

$$F_2 = -J_\psi^{-1}\llcorner d\,\mathrm{Im}(i\rho e^{-3A}v\wedge\omega_\psi) - 2\mu\rho e^{-4A}\tan^2(\psi)\mathrm{Im}\omega_\psi \ ; \tag{5.4.18b}$$

$$F_4 = J_\psi\left[\frac{1}{2}F_0 - \mu\rho e^{-4A}\right] + J_\psi\wedge d\,\mathrm{Im}(\rho e^{-3A}v) \ ; \tag{5.4.18c}$$

$$F_6 = 0 \ . \tag{5.4.18d}$$

Unlike in section 5.4.1, this time the flux equations for F_4 are not obviously following from the ones for F_0, or from any other combination of equations.

5.4.3 Minkowski

The first equation in the Minkowski version (4.4.2), $d_H\Phi_+ = 0$, simply gives

$$\rho = \mathrm{const} \ , \qquad \theta = \mathrm{const} \ , \qquad dJ_\psi = 0 \ , \qquad H = 0 \ . \tag{5.4.19}$$

The second equation in the Minkowski (4.4.2), $d_H\mathrm{Re}\Phi_- = 0$,

$$d(e^{-A}\mathrm{Re}v) = 0 \ , \qquad d\mathrm{Re}(ie^{-A}v\wedge\omega_\psi) = 0 \ . \tag{5.4.20}$$

(The five-form part of $d_H\mathrm{Re}\Phi_- = 0$ can be shown to be redundant.)
The RR fluxes can now easily be computed from the third equation in

$$F_0 = -J_\psi^{-1}\llcorner d(\rho e^{-3A}\mathrm{Im}v) \ ; \tag{5.4.21}$$

$$F_2 = -J_\psi^{-1}\llcorner d\mathrm{Im}(i\rho e^{-3A}v\wedge\omega_\psi) \ ; \tag{5.4.22}$$

$$F_4 = \frac{1}{2}F_0 J_\psi^2 + d(\mathrm{Im}\rho e^{-3A}v\wedge J_\psi) \ ; \tag{5.4.23}$$

$$F_6 = 0 \ . \tag{5.4.24}$$

Once again, the Bianchi identity for F_4 follows from the one for F_0, as in (5.4.11c), (5.4.12).

5.5 A general massive deformation

Using the results of section 5.4, we will now point out the existence of a first-order AdS deformation of any SU(3) Minkowski solution in IIA. As we saw in the introduction, this includes any solution obtained as back-reaction of O6–D6 systems in IIA — although in section 5.6 we will specialize it to the case of a single O6 in $\mathbb{R}^6$. The expansion parameter is $\mu = \sqrt{-\Lambda/3}$. This deformation should not be taken as a modulus: as we will see below, the fluxes we will introduce contain μ, and flux quantization will in general discretize it. Rather, our expansion is to be understood as a formal device to establish the existence of a solution at finite μ.

We will start by determining how θ should be deformed. As we remarked after (5.2.2), this parameter is an undetermined constant for the O6 solution we want to deform. However, we would like our solution to have something to do with the DGKT solution we reviewed in section 5.3. More specifically, we would expect our solution to approach the DGKT solution far from the source. Remembering (5.3.1), we will take θ to be small. Since our deformation parameter is μ, we might then take θ to be of order μ.

This decision seems to run into trouble, however, as soon as we consider (5.4.2). If θ is of order μ, v seems to diverge as $\mu \to 0$, whereas we need it to go to zero.

To cure this potential disaster, we need at least two more factors of μ in the numerator of (5.4.2). One can try to postulate that these extra factors are somehow supplemented by the derivative. This leads us to

$$\theta \sim \mu + \mu^3 \tau + \dots \,. \tag{5.5.1}$$

As in [4], we also suppose that everything is either odd or even in μ, so that whatever function or form is already non-zero before the deformation will be unchanged at first order. This means, in particular, that we do not change the dilaton, internal metric and warping given in section 5.2. This gives

$$\mathrm{Re} v = \frac{\mu}{2} e^A d\tau + O(\mu^2) \,. \tag{5.5.2}$$

Also, since now v is introduced at first order, we can mimic the procedure in [4, Sec. 4.1] and use it to deform an SU(3) structure into an SU(3) × SU(3) structure. The conclusions reached in that reference can be summarized as follows. The function ψ and the one-form v start at first order:

$$\psi = \mu\, \psi_1 + O(\mu^2) \,, \qquad v = \mu\, v_1 + O(\mu^2) \,; \tag{5.5.3}$$

the pure spinors have the form

$$\Phi_+ = (1 + i\theta) e^{-iJ} + O(\mu^2) \,, \tag{5.5.4}$$

$$\Phi_- = \left(\frac{i}{\psi} v \wedge \omega \right) + v \wedge \left(1 + \frac{1}{2} j^2 \right) + O(\mu^2) \,. \tag{5.5.5}$$

Comparing the order μ^0 part of Φ_- with (4.5.1), we get

$$\Omega = \frac{i}{\psi_1} v_1 \wedge \omega \ , \tag{5.5.6}$$

which means, in particular, that v_1 is a $(1,0)$ form with respect to the almost complex structure defined by the three-form Ω of the SU(3) structure solution. This can be used to derive the imaginary part of v_1:

$$\mathrm{Im} v_1 = \frac{1}{2} e^A I \cdot d\tau \ , \qquad v = \frac{1}{2} e^A \partial \tau + O(\mu^2) \ , \tag{5.5.7}$$

where $I\cdot$ is the action of the almost complex structure determined by Ω, and ∂ is the corresponding Dolbeault operator. Finally, notice that (5.5.6) can be inverted by writing

$$\omega = -\frac{i}{2\psi_1} \bar{v}_1 \llcorner \Omega \ . \tag{5.5.8}$$

So far we have only looked at equation (5.4.2) and to the algebraic constraints on the pure spinors $\Phi_\pm$. We now turn to the other differential equations, starting with the ones that constrain the geometry.

The first equation we consider is (5.4.1), that at first order simply reads $d\rho = 0$. In view of (4.5.14), this is consistent with our postulate that A and ϕ should not be deformed at first order. Comparing with (5.4.3), we see that ρ_0 is an odd function of μ:

$$\rho_0 = \frac{1}{g_s} \mu + O(\mu^3) \ . \tag{5.5.9}$$

We have called the first coefficient in the expansion $1/g_s$, so as to conform with the value of ρ in the particular solution (5.2.6).

Equation (5.4.6) is more problematic, because of the $\sin(\theta)$ in the denominator that makes the perturbation series start at order μ^{-1} in the left-hand side. Enforcing again our policy that all our power series in μ be either even or odd function of μ, we can expand J_ψ up to second order:

$$J_\psi = J + \mu^2 J_{(2)} + O(\mu^2) \ . \tag{5.5.10}$$

Equation (5.4.6) is then, at order μ^{-1},

$$dJ = 0 \ . \tag{5.5.11}$$

This is one of the equations in the system we are deforming, as we can see from (5.2.1). At order μ, (5.4.6) then gives

$$d\left[J_{(2)} + \left(\frac{1}{6} - \tau \right) J \right] = 2e^{-A} \mathrm{Re}\Omega \ . \tag{5.5.12}$$

As we will see, this equation is the only one we will encounter in which $J_{(2)}$ appears at all, so at this order $J_{(2)}$ has nothing else to satisfy. The right hand side is automatically closed, because of (5.2.2); but saying that it should be exact is a possible obstruction to deforming a given SU(3) structure Minkowski solution.

We will now look at the fluxes. Our formula for H, (5.4.5), has a $\sin(\theta)$ in the denominator, just like (5.4.6). That would again force us to start our perturbation theory with negative powers of μ. In this case, however, we can actually use (5.4.6) to rewrite H so that it starts at first order:

$$H_{\rm phys} = \mu h + O(\mu^2) \ , \qquad h = 2{\rm Re}\Omega + d(\psi_1 {\rm Im}\omega) \ . \tag{5.5.13}$$

Notice that the first term in h is the same as the one for H in the SU(3) structure solution given in (5.3.2), and the second term vanishes wherever ψ_1 tends to a constant.

As for the RR fluxes, only F_0 and F_4 will be generated at first order; F_2 will keep the same expression it had at zeroth order, (5.2.2). F_0 is given by

$$F_0 = \mu f_0 + O(\mu^3) \ , \qquad g_s f_0 = -J^{-1} \llcorner d(e^{-\phi}{\rm Im} v_1) + 5 e^{-A-\phi} \ . \tag{5.5.14}$$

We have expanded (5.4.11a) at first order in μ, and used (4.5.14). As remarked after (5.4.12), that the Bianchi identity for F_4 follows from the one for F_0. So the only Bianchi identity we have to impose at first order is that

$$df_0 = 0 \ . \tag{5.5.15}$$

For completeness, however, we also give here the expression for F_4. Actually, the Laurent series for F_4 in (5.4.11c) starts with a term $\sim F_0 J^2/\mu^2$, which diverges like μ^{-1}. So F_4 only becomes finite once one considers a finite μ. This is not terribly worrying: as we anticipated at the beginning of this section, the expansion in μ is simply a formal device to establish the existence of a solution at finite μ. In any case, the μ^{-1} terms disappear if we go back to the $\tilde F_k$, which are closed under $(d - H_{\rm phys}\wedge)$. We get

$$\begin{aligned} \tilde F_4 =& \mu \tilde f_4 + O(\mu^3), \\ g_s \tilde f_4 =& \left(\frac{1}{2} g_s f_0 - e^{-4A}\right) J^2 + J \wedge d(e^{-3A}{\rm Im} v_1) - \psi_1 {\rm Im}\omega \wedge J^{-1}\llcorner d(e^{-3A}{\rm Im}\Omega) \ . \end{aligned} \tag{5.5.16}$$

Let us now summarize this section. We found a first-order perturbation of an SU(3) Minkowski solution which turns it into an AdS solution of SU(3) $\times$ SU(3) type. The perturbation parameter is $\mu = \sqrt{-\Lambda/3}$. The only input is the function τ in (5.5.1), which has to satisfy (5.5.15). One also has to solve (5.5.12), but this simply requires to invert d.

We are now going to apply this first-order deformation to O6 solutions.

5.6 Massive O6 solution

In section 5.5, we have found a procedure to deform any SU(3)-structure Minkowski solution at first order in $\mu = \sqrt{-\frac{\Lambda}{3}}$. In this section, we will try to promote this deformation to a fully-fledged supergravity solution.

Although the first-order deformation procedure can potentially be applied to any O6–D6 system, we will focus on the region around a single O6. This means that we will take the internal manifold to be $\mathbb{R}^6$, with a single localized source as in (5.2.3). By doing this, we gain more symmetries than would be available for a general O6–D6 system; that will help us solve the system.

However, as we anticipated in the introduction, this should not be understood too literally as a massive O6 "in flat space". Unlike for (1.0.2), in the massive case the metric will not approach flat space far away from the source, simply because flat space is not a solution in the massive case. There are two new length scales associated with the massive problem, $\frac{1}{\mu}$ and $\frac{1}{g_s F_0}$, and the deviations from flat space asymptotics will become apparent at distances of the order of the smallest of these two length scales. The solution of this section should be thought of as a "close-up" around an O6 source in an $\mathrm{AdS}_4 \times M_6$ geometry where M_6 is *compact* — so the large r-behaviour will not too important.

After some preliminaries in section 5.6.1, in section 5.6.2 we will specialize the general procedure of section 5.5 to a single O6. In section 5.6.3 we will then promote it to a finite deformation; this will culminate in the numerical study of section 5.6.3, where we will find numerical solutions and describe their physical features, some of which were described in the introduction. We will also study the system at higher order in perturbation theory, in section 5.6.4. In section 5.6.5 we will show that choosing $\theta = 0$ in the pure spinors (4.5.6) does not lead to a solution. Finally, in section 5.6.6 we will look briefly at the system for the Minkowski case; we also found numerical solutions in this case, but they do not seem to satisfy flux quantization. Moreover, we do not know of any Minkowski compactification that uses this ingredient. We will not describe these solutions in as much detail as the AdS ones.

5.6.1 Symmetries

As in section 5.2, we will denote by x^i the coordinates parallel to the O6, and by y^i the coordinates transverse to it.

The massless O6 solution is symmetric under rotations of the three y^i, rotations of the three x^i, and translations in the x^i:

$$\mathrm{ISO}(3) \times \mathrm{SO}(3) \,. \tag{5.6.1}$$

It is already clear that the massive solution will not be symmetric under the whole group (5.6.1). As we have argued in section 5.3, we need to consider an SU(3) × SU(3) solution. One of the data in its definition is a complex one-form v; as we saw in section 4.5.2, the algebraic constraints in (4.3.13, 4.3.14) demand in particular that $v \wedge \bar{v} \wedge j^2 \neq 0$ everywhere. So the real and imaginary part of v are two linearly independent one-forms. However, the only linearly indepedent one-form which does not break any of the symmetries in (5.6.1) is

$$dr = \frac{1}{r} y^i dy^i \ . \tag{5.6.2}$$

Thus, in the massive solution the symmetry group (5.6.1) will be broken. In section 5.6.2, we will see that a natural subgroup emerges when one applies the general first-order procedure of section 5.5 to the O6 solution of section 5.2.

5.6.2 First order deformation

We will still demand that translation along the three internal coordinates x^i parallel to the O6 should remain a symmetry. This will not be valid for a solution where there are several O6 sources, such as the one reviewed in section 5.3. However, this invariance will be restored when we get closer to an individual O6, which is the focus of the present paper.

Since everything can only depend on the transverse coordinates y^i, from now on we will use the notation

$$\partial_i \equiv \partial_{y^i} \ . \tag{5.6.3}$$

Using (5.5.2) and (5.5.7), we then have

$$v = -\frac{i}{2} \mu Z^{-1/2} \partial_i \tau (Z^{-1/4} dx^i + i Z^{1/4} dy^i) \ . \tag{5.6.4}$$

Since τ depends on r only, we have $\partial_i = \frac{y^i}{r} \partial_r$, and Im$v$ is proportional to

$$y^i dx^i \ , \tag{5.6.5}$$

which breaks the symmetries (5.6.1) of the massless O6 solution, as anticipated in section 5.6.1. Indeed, the one-form (5.6.5) is neither invariant under either the SO(3) that rotates the transverse y^i, nor under the SO(3) that rotates the parallel x^i. It is still invariant, however, under the diagonal SO(3) that rotates both the x^i and the y^i simultaneously. Also, it is still invariant under translations along the x^i, as we stipulated at the beginning of this section. So (5.6.4) breaks (5.6.1) to

$$\mathrm{ISO}(3) \ . \tag{5.6.6}$$

It is not hard to list all the possible forms invariant under (5.6.6); we have done so in appendix C. We will see that the rest of the solution respects this smaller symmetry group.

Let us now go back to applying the first order procedure of section 5.5 to the O6 solution.[3] The next step is to impose (5.5.15), namely that F_0, calculated at first order, is constant:

$$df_0 = 0 \ , \qquad g_s f_0 = -\frac{1}{2}\Delta\tau + 5Z = \text{const.} \ , \tag{5.6.7}$$

where $\Delta \equiv \partial_i\partial_i$, and g_s is the value of e^{ϕ} at infinity in the unperturbed solution (5.2.6). Explicitly, using (5.2.5), we get

$$\tau = \frac{1}{3}\left(5 - g_s f_0\right) r^2 - 5 r_0 r \ , \tag{5.6.8}$$

setting to zero an inconsequential integration constant.

The other equation to be solved is (5.5.12). This can be inverted to give

$$J_{(2)} = -2\left(\frac{1}{3} - \frac{r_0}{2r} + \frac{p}{r^3}\right)\omega_{2,1} + 2\omega_{2,2} - \frac{\alpha'}{r}\omega_{2,3} + \left(\tau - \frac{1}{6} + \alpha\right)\omega_{2,4} \tag{5.6.9}$$

where a prime denotes ∂_r. We have used the two-forms defined in (C.0.2); those forms are invariant under (5.6.6), as promised. The constant p and the function $\alpha = \alpha(r)$ are as yet undetermined.

At this point, we have already demonstrated the existence of a solution at first order. For completeness, however, let us also give the physical fluxes explicitly. First of all, we can determine ψ_1 from imposing that $J_\psi \to J$. Looking at the expression of J_ψ in (4.5.7), this can be done by checking that $J^{-1} \llcorner \left(\frac{i}{2\psi_1^2} v_1 \wedge \bar{v}_1\right) = 1$; we get

$$\psi_1 = \frac{\tau'}{2\sqrt{Z}} \ . \tag{5.6.10}$$

Now we can compute the first-order fluxes $\tilde{f}_4$ and h from (5.5.13), (5.5.16):

$$\begin{aligned} g_s\tilde{f}_4 &= \frac{1}{r^3}\left(-\frac{5}{2}r_0 + Z^{-1}\right)\omega_{4,1} + \left(\frac{r_0}{2r} - \frac{1}{3}(4 + g_s f_0)\right)\omega_{4,4} \ ; \\ h &= d\left[\left(-\frac{\tau' + 2r_0}{2r} + \frac{2}{3}\right)\omega_{2,1} + \left(\frac{\tau'}{2rZ} - 2\right)\omega_{2,2} + \frac{1}{2}\omega_{2,4}\right] \ . \end{aligned} \tag{5.6.11}$$

As already stressed, the flux F_2 will not get deformed at first order in μ.

[3] As remarked in section 5.2, we will deform one particular SU(3) structure which solves (5.2.1); for this reason, our massive solution will have only four supercharges, or $\mathcal{N} = 1$ in four dimensions, just like the solutions in [8, 9]. Incidentally, it is easy to show that *any* supersymmetric SU(3) structure solution with Romans mass has only four supercharges.

Let us now pause to consider the properties of the first-order solution we have just obtained. First of all, we note that we have a certain freedom: we have left undetermined a function $\alpha(r)$ in (5.6.9), which does not enter in the fluxes, and a constant f_0, defined in (5.5.14) as the ratio between the deformation of F_0 and μ. Let us see what happens if we set

$$f_0 = \frac{5}{g_s} \, , \tag{5.6.12}$$

inspired by (5.3.2), which is valid for SU(3) structures. We see that we cancel the r^2 term in (5.6.8), which now goes linearly. One can then check that

$$r \to \infty \quad \Rightarrow \quad f_4 \to \frac{3}{2} J^2 \, , \qquad h \to 2\mathrm{Re}\Omega \, ; \tag{5.6.13}$$

in other words, far from the O6 source the solution approaches the SU(3) solution in (5.3.2).

The perturbative procedure, however, can only work in an appropriate regime. We have already determined $J_{(2)}$ in (5.6.9). Since τ actually grows with r, $J_{(2)}$ seems to grow large at large r, thus invalidating the first-order procedure. If $f_0 = 5/g_s$, for example, we see from (5.6.8) that τ grows linearly; if $\alpha = 0$, since $J_\psi = J + \mu^2 J_{(2)} + \ldots$, and recalling that $r_0 = g_s l_s$, we have that the perturbation procedure is valid only if

$$r \ll \frac{1}{g_s l_s \mu^2} \, . \tag{5.6.14}$$

We are not necessarily interested, however, in what happens outside this region, because eventually we want to compactify the six "internal" directions, and in particular the three directions y^i. In the smeared solution we reviewed in section 5.3, we see from (5.3.5) that the compactification radius in string units goes like $R \sim \mu^{-1/3}$, whereas $1/(g_s\mu^2) \sim \mu^{-3}$. In other words, the perturbative procedure breaks down for distances of order μ^{-3}, which are much larger than the compactification radius $\mu^{-1/3}$.

In any case, we are now going to set up the study of the system of differential equations at all orders, guided by the results of this section. We will come back to perturbation theory in μ in section 5.6.4.

5.6.3 Full solution

We now want to check whether the solution we just found at first order in μ survives beyond first order. We are not going to use perturbation theory in this section; we will go back to using it in section 5.6.4.

Variables

At first order, the whole solution was determined by a single piece of data, the function τ in (5.5.1), which then has to solve (5.5.15).

Beyond first order, however, the input data are many more: the functions ψ, θ and the forms v, j, ω, in (4.5.6), as well as the warping function A in (2.1.10). At first order, the continuous symmetry (5.6.6) emerged, and we are going to assume that it is not broken in the full solution. This means that we should expand v in terms of the one-forms (C.0.1), and j, ω in terms of the two-forms (C.0.2).

There is also a discrete symmetry that we can use to our advantage. The solution we are looking for contains an O6, which is defined by quotienting the theory by the symmetry $\Omega(-)^{F_L} I_y$, where Ω is the world-sheet parity, F_L is the fermionic number for left-movers, and

$$I_y : \begin{cases} x^i \to x^i \\ y^i \to -y^i \end{cases} \tag{5.6.15}$$

is the inversion in the three y^i directions. The pure spinors $\phi_\pm$ should then transform as [34]

$$I_y^* \phi_+ = \lambda(\phi_+) \ , \qquad I_y^* \phi_- = \lambda(\bar{\phi}_-) \ , \tag{5.6.16}$$

where λ is the sign operator defined in (2.1.13). This implies

$$I_y^* v = \bar{v} \ , \qquad I_y^* j = -j \ , \qquad I_y^* \omega = -\bar{\omega} \ . \tag{5.6.17}$$

All the invariant forms in appendix C transform by simply picking up a sign, as detailed in table C.1. Using that table, (5.6.17) implies

$$v = v_r\, \omega_{1,0} + i\, v_i\, \omega_{1,1} \ , \qquad j = \sum_{i=1}^{4} j_i\, \omega_{2,i} \ , \qquad \omega = a_0\, \omega_{2,0} + i \sum_{i=1}^{4} a_i\, \omega_{2,i} \ ; \tag{5.6.18}$$

the coefficients v_r, v_i, j_i, a_i are now all real.

Algebraic equations

With this parameterization in hand, we can now proceed to imposing the algebraic equations (4.5.8). These give:

$$j_4 = j_3 r^2 \ , \qquad a_4 = a_3 r^2 \ , \qquad a_2 j_1 + a_1 j_2 = 2 a_3 j_3 r^2 \ , \tag{5.6.19a}$$

$$a_3^2 r^2 - a_1 a_2 = a_0^2 = j_3^2 r^2 - j_1 j_2 \ . \tag{5.6.19b}$$

Specifically, (5.6.19a) comes from (4.5.9a), whereas (5.6.19b) comes from (4.5.9b). Moreover, the requirement in (4.3.13) that $(\Phi_-, \bar{\Phi}_-) \neq 0$ demands[4]

$$a_0 \neq 0 \ . \tag{5.6.20}$$

Given a solution to the algebraic constraints (5.6.19), one can also compute the internal, six-dimensional metric associated to the pure spinors. This

[4] In fact, the first two equations in (5.6.19a) are linear precisely because we divided by a common factor a_0, since it cannot vanish.

is not really needed in finding a solution, except for one important check: that its signature should be Euclidean. Applying (4.5.13), we find

$$ds^2 = (\alpha_1\delta^{ij} + \alpha_2 y^i y^j)dx^i dx^j + (\alpha_3\delta^{ij} + \alpha_4 y^i y^j)dy^i dy^j + \alpha_5\epsilon_{ijk}y^i dx^j dy^k \ , \tag{5.6.21}$$

where the $\alpha_i = \alpha_i(r)$ are given by

$$\begin{aligned} \alpha_1 &= \frac{-a_2 j_3 + a_3 j_2}{a_0} r^2 \ , & \alpha_2 &= \frac{a_2 j_3 - a_3 j_2}{a_0} + \frac{v_i^2}{\tan^2(\psi)} \ , & & \\ \alpha_3 &= \frac{a_1 j_3 - a_3 j_1}{a_0} r^2 \ , & \alpha_4 &= \frac{-a_1 j_3 + a_3 j_1}{a_0} + \frac{v_r^2}{\tan^2(\psi)} & \alpha_5 &= \frac{a_2 j_1 - a_1 j_2}{a_0} \ . \end{aligned} \tag{5.6.22}$$

The metric (5.6.21) is symmetric under ISO(3), as we argued above (5.6.6). If we go to polar coordinates for the y^i, by defining $r = \sqrt{y^i y^i}$ as in (5.2.5), and

$$\hat{y}^i \equiv \frac{y^i}{r} \ , \tag{5.6.23}$$

we can write (5.6.21) as

$$\begin{aligned} ds^2 &= (\alpha_1\delta^{ij} + r^2\alpha_2\hat{y}^i\hat{y}^j)Dx^i Dx^j + (\alpha_3 + r^2\alpha_4)dr^2 + r^2\left(\alpha_3 - \frac{r^2\alpha_5^2}{4\alpha_1}\right)ds^2_{S^2} \ , \\ Dx^i &= dx^i - \frac{r^2\alpha_5}{2\alpha_1}\epsilon^{ijk}\hat{y}^j d\hat{y}^k \ , \end{aligned} \tag{5.6.24}$$

where $ds^2_{S^2}$ is the round metric of unit radius on the S^2 in the y^i directions (which is the one that surrounds the O6). This exhibits the metric as a fibration of the $\mathbb{R}^3$ spanned by the x^i (along which the O6-plane is wrapped) over the $\mathbb{R}^3$ spanned by the y^i, or by r and the $\hat{y}^i$. Since the connection is a globally defined one-form, this fibration is topologically trivial. Notice that the function multiplying dr^2 simplifies to

$$\alpha_3 + r^2\alpha_4 = \left(\frac{r\, v_r}{\tan(\psi)}\right)^2 \ , \tag{5.6.25}$$

using (5.6.22).

Differential equations

The differential equations we have to impose are (5.4.2), (5.4.6), $dF_0 = 0$, and $dF_2 = \delta_{\text{O6}}$, where F_0 and F_2 are given by (5.4.11a) and (5.4.11b), and δ_{O6} is given by (5.2.3). Recall that $dF_4 = 0$ follows from $dF_0 = 0$, as pointed out before (5.4.12).

First of all, (5.4.2) gives

$$v_r = -\frac{e^A}{2\mu r}\frac{\theta'}{\sin(\theta)} \ . \tag{5.6.26}$$

(5.4.6) is clearly odd under I_y. From table C.1, we see that there are four odd three-forms; so (5.4.6) has four non-trivial components. One of these turns out to be algebraic:

$$v_i = \frac{e^A}{2\mu r^2}\frac{j_2}{a_0}\frac{\tan(\psi)}{\sin(\theta)} . \tag{5.6.27}$$

So v is completely determined algebraically, at all orders. The other three components in (5.4.6) are

$$\begin{aligned}
\partial_r \log\left(\frac{j_1 r^3}{\sin(\theta)\cos(\psi)}\right) &= \frac{a_1}{j_1}\frac{\theta'}{\sin(\psi)} ,\\
\partial_r \log\left(\frac{j_2 r}{\sin(\theta)\cos(\psi)}\right) &= \frac{a_2}{j_2}\frac{\theta'}{\sin(\psi)} ,\\
\partial_r \log\left(\frac{j_3 r^3}{\sin(\theta)\cos(\psi)}\right) &= \left(a_3 - \frac{j_2 e^{2A}}{4a_0 r^4\mu^2}\frac{\cos^2(\psi)}{\sin^2(\theta)}\right)\frac{\theta'}{j_3\sin(\psi)} .
\end{aligned} \tag{5.6.28}$$

We now turn to the Bianchi identities. We have one first-order equation that reads $F_0 = \text{const}$. After some manipulation we write it as an equation linear in the derivatives of the variables:

$$\partial_r \log\left(\frac{v_i r e^{-3A}}{\sin(\theta)}\right) = \theta' \cot^2(\psi)\left(\frac{5}{2}\cot(\theta) - \frac{F_0 e^{-4A}}{2\mu\rho_0} + \frac{j_3 v_i\cos(\psi)e^A}{a_0^2\mu\sin(\theta)}\right) . \tag{5.6.29}$$

We also have $dF_2 = \delta_{O6}$. A priori, this would seem to have four components, since F_2 is odd under I_y. However, closer inspection reveals that only three components are non-trivial:

$$F_2 = \sum_{i=1}^{4} f_{2,i}\omega_{2,i} , \qquad dF_2 = (3f_{2,1}+rf'_{2,1})\omega_{3,1}+f_{2,2}\omega_{3,3}-\left(f_{2,3}+\frac{1}{r}f'_{2,4}\right)\omega_{3,5}+\frac{1}{r}f'_{2,2}\omega_{3,7} . \tag{5.6.30}$$

The component of dF_2 along $\omega_{3,1}$ can be set to zero by taking $f_{2,1}$ proportional to r^{-3}; the proportionality constant can be fixed by requiring that it reproduces the correct factor in δ_{O6}. This can be read off (5.2.6). Thus the non-trivial equations are three:

$$f_{2,1} = -\frac{l_s}{r^3} , \qquad f_{2,2} = 0 , \qquad f'_{2,4} = -rf_{2,3} . \tag{5.6.31}$$

These $f_{2,i}$ are determined by (5.4.11b) in terms of the data j_i, a_i, ψ, θ, A and their first derivatives. The equations for $f_{2,1}$ and $f_{2,2}$ give two equations which are again linear in the derivatives of the variables:

$$\partial_r \log\left(\frac{a_1 v_i r^4 e^{-3A}}{\sin^2(\theta)\sin(2\psi)}\right) = \frac{\theta'}{2a_1}\left[j_1\left(-\frac{5}{\sin(\psi)}+3\sin(\psi)\right)\right.$$
$$\left.+\frac{\cos^3(\psi)}{\sin(\psi)}\left(-\frac{l_s e^{4A}}{\rho_0\mu r^3}\cos(\psi)-\frac{4a_0^2}{j_2}-\frac{F_0 e^{4A} j_1\cot(\theta)}{\rho_0\mu}+\frac{a_3 j_1 j_2 e^{2A}}{a_0^2\mu^2 r^2\sin^2(\theta)}\right)\right] , \tag{5.6.32a}$$

$$\partial_r \log\left(\frac{a_2 v_i r^2 e^{-3A}}{\sin^2(\theta)\sin(2\psi)}\right) =$$
$$\frac{j_2\theta'}{2a_2}\left[-2\sin(\psi)+\frac{\cos^3(\psi)}{\sin(\psi)}\left(-5-\frac{F_0 e^{4A}\cot(\theta)}{\rho_0\mu}+\frac{a_3 j_2^2 e^{2A}}{a_0^2 r^2\mu^2\sin^2(\theta)}\right)\right] . \tag{5.6.32b}$$

Remarkably, by using these two equations and (5.6.28), one can show that the last equation in (5.6.31) is actually automatically satisfied.

All in all, we have three differential equations from (5.6.28) (coming from (5.4.6)), one from (5.6.29) (coming from F_0 =const), and two from (5.6.32) (coming from $dF_2 = \delta_{O6}$), for a total of six. All of these are first-order, and linear in the first derivatives.

Having counted our equations, let us now count our variables. We can use (5.6.26) and (5.6.27) to eliminate v_r and v_i from the system; moreover, we can use the first two in (5.6.19a) to eliminate j_4 and a_4. It is less clear how to use the remaining three equations in (5.6.19); one possibility is to derive a_1, j_1 and j_3. This leaves us with the variables

$$a_0\ , \qquad a_2\ , \qquad a_3\ ; \qquad j_2\ ; \qquad A\ , \qquad \theta\ , \qquad \psi\ , \tag{5.6.33}$$

for a total of seven variables. We should also notice, however, that we have not yet fixed the gauge invariance coming from reparameterizations of the radial direction:

$$r \to \tilde{r}(r)\ . \tag{5.6.34}$$

Under these reparameterizations, the coefficients of j and ω a priori could mix. It turns out, however, that only the coefficients of ω_3 and ω_4 mix; if we impose the algebraic equations in (5.6.19), even the coefficients along those two are proportional. So, in particular we have

$$a_0 \to \left(\frac{r}{\tilde{r}}\right)^2 a_0\ , \qquad (a_2, j_2) \to \left(\frac{r}{\tilde{r}}\right)(a_2, j_2)\ , \qquad a_3 \to \left(\frac{r}{\tilde{r}}\right)^3 a_3\ , \tag{5.6.35}$$

whereas of course A, θ, ψ transform as functions.

Thus, out of the seven variables in (5.6.33), one is redundant because of the gauge invariance (5.6.34). This effectively leaves us with six variables, which is as many as the differential equations (5.6.28), (5.6.29), (5.6.32). So we have as many equations as variables, and we expect a solution to exist. We will now study the system numerically.

Numerics

The system we found in section 5.6.3 is first-order, and linear in the derivatives of our variables. We found it useful to fix the gauge invariance (5.6.34) by demanding θ to be exact at order μ^3; namely,

$$\theta_{\rm gf} = \mu + \mu^3 \tau \ , \tag{5.6.36}$$

with τ given (5.6.8). In other words, the ... terms in (5.5.1) are absent. This gauge makes it easier to compare the massless limit of our numerical solutions with the solution in section 5.2.

Also, we imposed boundary conditions at an r much larger than $r_0 = g_s l_s$, but much smaller than the scales $(g_s F_0)^{-1}$ and μ^{-1}, where deviations from the massless asymptotics become apparent. Using the first-order solution in section 5.6.2 as a clue, we identified a family of boundary conditions (depending on F_0 and μ) such that, when one takes the limit $F_0 \to 0$ and $\mu \to 0$ (thus forgetting for a moment about flux quantization), one recovers the massless solution[5]. This works quite well, especially if one takes the limit by keeping $\frac{g_s F_0}{\mu} = 5$, as in the special choice (5.6.12) for the first-order solution. We take all this as a check that our numerical analysis is sound.

We then increased F_0 until it satisfied the flux quantization condition $F_0 = \frac{n_0}{2\pi}$, $n_0 \in \mathbb{Z}$. The behavior of the solutions for $n_0 \neq 0$ is qualitatively different from the massless solution: notably, it does not display the divergence at $r_0 = g_s l_s$ that plagues the massless solution (1.0.2) — see figure 5.1. We checked that the eigenvalues of the metric (5.6.21) remain positive in our numerical solutions.

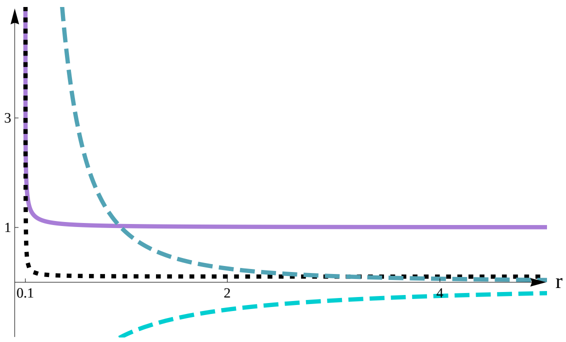
Let us now focus on the asymptotic behavior of our solutions at $r \to 0$. In our gauge, θ tends to a constant at $r \to 0$; numerically, one can see ψ and A also tend to constants ψ_0 and A_0. We can then use the differential equations (5.6.28), (5.6.29), (5.6.32) to find the asymptotic behavior of the coefficients a_i, j_i:

$$\begin{gathered} a_0 \sim a_{00} r^{-2} \ , \qquad a_1 \sim a_{10} r^{-3} \ , \qquad a_2 \sim a_{20} r^{-1} \ , \qquad a_3 \sim a_{30} r^{-3} \ ; \\ j_1 \sim j_{10} r^{-3} \ , \qquad j_2 \sim j_{20} r^{-1} \ , \qquad j_3 \sim j_{30} r^{-3} \ , \end{gathered} \tag{5.6.37}$$

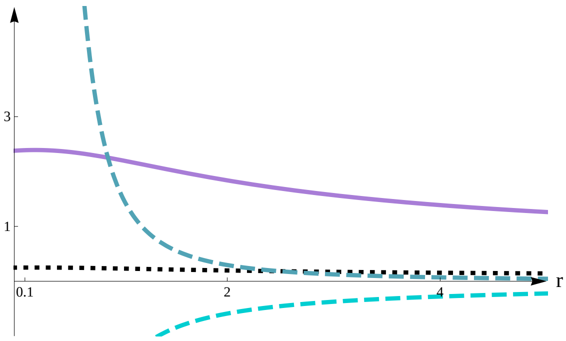
where the a_{i0} and j_{i0} are constants. These are also in agreement with the algebraic constraints (5.6.19).
From (5.6.37) it follows that the α_i in (5.6.21) behave as

$$\alpha_1 \to \alpha_{10} \ , \qquad \alpha_2 \sim \alpha_{20} r^{-2} \ , \qquad \alpha_3 \sim \alpha_{30} r^{-2} \ , \qquad \alpha_4 \sim \alpha_{40} r^{-4} \ , \qquad \alpha_5 \sim \alpha_{50} r^{-2} \ , \tag{5.6.38}$$

[5]The family is obtained with the help of the perturbative expansion we will consider in section 5.6.4; actually, besides F_0 and μ, the family also depends on an integration constant in a_3. This constant has no influence on the massless limit.



(a) The massless O6 solution.



(b) A O6 solution with Romans mass.

Figure 5.1: Comparison between the massless O6 solution and a solution with Romans mass. The solid line is e^A; the dotted line is e^ϕ; the dashed lines are j_3 (positive) and a_0 (negative). On the left we plot these coefficients (in string units, for $g_s = 0.1$) for the solution with $F_0 = 0$: from (1.0.2) and (5.2.4) we get $e^A = (1 - r_0/r)^{-1/4}$, $j_3 = 1/r^2$, $a_0 = -1/r$. In particular, the solution diverges at $r = r_0 = 0.1\, l_s$. On the right, we plot the same coefficients for a supersymmetric solution with localized O6 source, for $\mu \sim .055$, $F_0 = \frac{4}{2\pi l_s}$. j_3 and a_0 retain a power-law behavior, while e^A no longer diverges at $r_0 = 0.1$. At larger distances, one can see deviations from the flat-space behavior, due to the fact that flat space is not a solution for $F_0 \neq 0$, as observed earlier.

where α_{i0} are non-zero constants. For the crucial combination $\alpha_3 + r^2\alpha_4$, however, which multiplies dr^2 in (5.6.24), from (5.6.25) and (5.6.26) we see that

$$\alpha_3 + r^2\alpha_4 \to \left(\frac{5}{2} g_s \mu\right)^2 ; \tag{5.6.39}$$

thus, the r^{-2} divergencies cancel out, and this coefficient goes to a constant.

As $r \to 0$, the metric (5.6.24) then tends to

$$ds^2 = (\alpha_{10}\delta^{ij} + \alpha_{20}\hat{y}^i\hat{y}^j) D_0x^i D_0x^j + \left(\frac{5}{2} g_s\mu\right)^2 dr^2 + \left(\alpha_{30} - \frac{\alpha_{50}^2}{4\alpha_{10}}\right) ds^2_{S^2} \, ,$$
$$D_0x^i = dx^i - \frac{\alpha_{50}}{2\alpha_{10}} \epsilon^{ijk}\hat{y}^j d\hat{y}^k \, . \tag{5.6.40}$$

This metric factorizes in a factor dr^2, and a five-dimensional $\mathbb{R}^3$ fibration over S^2. Thus, asymptotically we have $\mathbb{R} \times M_5$.

For most values of μ, the curvature of M_5 is small, and we can trust the supergravity approximation. However, the size of the S^2 remains finite, and the metric is no longer geodesically complete. Fortunately, it is possible to perform an analytic continuation by going to polar coordinates for the y^i. One can then see that, in the system described in sections 5.6.3 and 5.6.3, all explicit dependence on r drops out; the only dependence is introduced by the way we fix the gauge freedom (5.6.34). One can then continue r to negative values. With our gauge choice (5.6.36), one can see that for $r < 0$ the metric gets continued essentially to a mirror copy of itself.

One might feel unsatisfied by the fact that the S^2 that surrounds the orientifold never shrinks to a zero size; so the O6-plane locus does not really exist in these metrics, even though all fields transform as they should under the antipodal map $\hat{y}^i \to -\hat{y}^i$ of an O6 projection. Even in the massless case, however, the transverse S^2 does not shrink in the smooth Atiyah–Hitchin metric (see for example the discussion in [35, Sec. 3]).

For special choices of μ, the curvature of M_5 gets large; in that case, the supergravity approximation breaks down. It is possible that α' corrections make the size of the S^2 shrink, but this is of course speculation.

5.6.4 Back to perturbation theory in μ

In section 5.6.2 we considered our equations to order μ, and found an explicit solution. In section 5.6.3 we analyzed the conditions for unbroken supersymmetry in the setup suggested by the first-order solution, culminating in the numerical analysis in 5.6.3. In this section we will go back to perturbation theory in $\mu = \sqrt{-\frac{\Lambda}{3}}$, to see how explicit can the solution be made.

First, a bit of notation: we are going to expand the various coefficients and functions as a power series in μ, keeping the same assumptions in section 5.5 about which expansions contain even or odd powers:

$$\begin{aligned} j_i &= j_{i,0} + \mu^2 j_{i,2} + \mu^4 j_{i,4} + O(\mu^4) \,, \qquad & \psi &= \mu\psi_1 + \mu^3\psi_3 + O(\mu^5) \,, \\ a_i &= a_{i,0} + \mu^2 a_{i,2} + \mu^4 a_{i,4} + O(\mu^4) \,, \qquad & A &= A_0 + \mu^2 A_2 + O(\mu^4) \,, \\ & & \theta &= \mu + \mu^3\tau + \mu^5\theta_5 + O(\mu^7) \,. \end{aligned} \tag{5.6.41}$$

As it turns out, the equations at order μ^2 and μ^3 mix quite a bit. Using the algebraic equations, we found it convenient to use the variables

$$A_2 \,, \qquad \theta_5 \,, \qquad \psi_3 \,; \qquad j_{1,4} \,, \qquad j_{2,4} \,, \qquad j_{3,2} \,, \qquad a_{2,2} \,. \tag{5.6.42}$$

For example, even if we have already solved $J_{(2)}$ at second order in (5.6.9), we did so only up to an unknown function $\alpha(r)$. This means that one component was actually undetermined; in terms of the expansion (5.6.41), this remaining equation can be written in terms of the variables (5.6.42). At the same time, of the equations in (5.4.6) only two contain the variables in (5.6.42); the third involves variables at higher order, and we can ignore it at this level. We then have one equation $F_0 =$ const. and three equations from $dF_2 = \delta_{O6}$, just like in our discussion at all orders in section 5.6.3.

In section 5.6.3, the system of differential equations was first-order and linear in the derivatives of the variables. The perturbative system we are considering in this subsection, once we use the solution found at first order in section 5.6.2, is also linear (inhomogenous) in the variables themselves. This means that we can write it as

$$v' = Mv + b \ , \qquad v = (\theta_5, \psi_3, j_{1,4}, j_{2,4}, j_{3,2}, a_{2,2})^t \ . \tag{5.6.43}$$

The matrix M is particularly simple in the gauge $A = A_0 = \log(Z^{-3/4})$, and with the simplifying assumption $f_0 = 5$:

$$M = \begin{pmatrix} 0 & -2\sqrt{Z} & 0 & 0 & 10r_0r^2 & \tau Z \\ 0 & \frac{1}{2}\left(\frac{3}{r} - \frac{Z}{r_0}\right) & 0 & 0 & -\frac{\tau}{Z} & -\frac{5}{2}r_0 \\ 0 & 0 & \frac{3}{r} & 0 & 8rZ & -4Z^{3/2} \\ 0 & 0 & 0 & \frac{1}{r} & -4r & 0 \\ 0 & 0 & 0 & -\frac{1}{2r^3} & \frac{2}{r} & \frac{\sqrt{Z}}{r^2} \\ 0 & 0 & 0 & -\frac{1}{r^2\sqrt{Z}} & \frac{2}{\sqrt{Z}} & \frac{1}{2}\left(\frac{3}{r} - \frac{Z}{r_0}\right) \end{pmatrix} . \tag{5.6.44}$$

The expression for the vector b is more complicated, and we see no reason to inflict it on the reader. The first three columns of (5.6.44) show three obvious eigenvalues; the variables θ_5, ψ_3, $j_{1,4}$ are determined once the other three are. So the crucial part of M is the lower-right 3×3 block, concerning the variables $j_{2,4}$, $j_{3,2}$, $a_{2,2}$. The eigenvalues of this block can be found by the Cardano–Tartaglia formula, and so in principle the system at this order can be solved analytically.

5.6.5 The special case $\theta = 0$

In section 5.4 we have divided the analysis of SU(3) × SU(3) structure solutions in three cases: AdS for $\theta \neq 0$ (the "generic" case of section 5.4.1), AdS for $\theta = 0$ (the "special" case of section 5.4.2), and the Minkowski case (in section 5.4.3). So far, in this section we have analyzed the system in detail in the generic AdS case $\theta \neq 0$. We now want to go back to the other two cases. We will begin in this subsection by the special AdS case, $\theta = 0$.

We will again work with the symmetry group (5.6.6), for the same reasons explained in section 5.6.1 and 5.6.2. The parameterization of the forms v, j, ω is still the same as in section 5.6.3. The algebraic equations satisfied by them can still be written as in (5.6.19).

Since in this case H in (5.4.17) is not already exact (as for (5.5.13)), we have to impose by hand that $dH = 0$. Since H is odd, the only non-zero component of this equation is the one along $\omega_{4,0}$:

$$a_2\, v_r\, \mu = 0\ . \tag{5.6.45}$$

v_r cannot be zero because of the requirement $(\Phi_-, \bar{\Phi}_-) = 0$ in (4.3.13). Also, $\mu \neq 0$ by assumption; so we get $a_2 = 0$.

We then look at $d(\rho J_\psi) = 0$, again from (5.4.17). This has four non-zero components, but in particular the one along $\omega_{3,3}$ tells us that

$$j_2 = 0\ . \tag{5.6.46}$$

We can now go back to the algebraic system (5.6.19), and use that $a_2 = j_2 = 0$. The last equation of (5.6.19a) tells us that $a_3 j_3 = 0$. But, both if $a_3 = 0$ and if $j_3 = 0$, (5.6.19b) now tells us $a_0 = 0$. This means that $\mathrm{Re}\omega = 0$, which is not possible, again because of the requirement $(\Phi_-, \bar{\Phi}_-) = 0$ in (4.3.13).

Thus, in this section we have quickly disposed of the case $\theta = 0$. This case cannot lead to massive O6 solutions with the symmetry (5.6.6).

5.6.6 Minkowski

Finally, in this section we will look at the Minkowski case.

Once again, we can use the parameterization of the forms v, j, ω in section 5.6.3, whose coefficients have to satisfy the algebraic equations in (5.6.19).

The relevant differential equations were given in 5.4.3. We start with (5.4.19). This says

$$j_2 = 0\ , \qquad \frac{r^3 j_1}{\cos(\psi)} = \text{const.}\ , \qquad \partial_r \log\left(\frac{r^3 j_3}{\cos(\psi)}\right) = -\frac{v_r v_i}{r j_3}\frac{\cos^3(\psi)}{\sin^2(\psi)}\ . \tag{5.6.47}$$

We then turn to (5.4.20). The first is trivially satisfied, using the symmetries of our setup. The second gives

$$\partial_r \log\left(\frac{a_0 v_i r^3 e^{-A}}{\sin(\psi)}\right) = \frac{a_2}{a_0}\frac{v_r}{v_i \cos(\psi)}\ . \tag{5.6.48}$$

We now turn to the Bianchi identities. They can be discussed along the lines of the AdS case in section 5.6.3. One consists in imposing that F_0 is constant, and can be written as

$$\partial_r \log(v_i r e^{-3A}) = \frac{v_r r}{\tan^2(\psi)}\left(\frac{2 j_3 v_i \cos(\psi)}{a_0^2} - F_0 e^{3A}\right)\ . \tag{5.6.49}$$

As in (5.6.31), F_2 would seem to give three equations. The ones for $f_{2,1}$ and $f_{2,2}$ read:

$$\partial_r \log\left(\frac{a_1 v_i r^4 e^{-3A}}{\sin(2\psi)}\right) = \frac{v_r}{a_1}\cos(\psi)\left(-\frac{2a_0}{v_i r} + \frac{2a_3 j_1 v_i r}{a_0^2\tan^2(\psi)} - \frac{l_s\cos^2(\psi)e^{3A}}{\sin(\psi)r^2}\right) \tag{5.6.50a}$$

$$\partial_r \log\left(\frac{a_2 v_i r^2 e^{-3A}}{\sin(2\psi)}\right) = 2\frac{\cos^3(\psi)}{\sin^2(\psi)}\frac{a_3 j_2 v_r v_i r}{a_2 a_0^2} \tag{5.6.50b}$$

Once again, the third equation in (5.6.31) can be shown to be automatically implied by (5.6.50) and by (5.6.47), (5.6.48).

So we have one differential equation from (5.6.47), one from (5.6.48), one from (5.6.49), and two from (5.6.50). This gives a total of five differential equations, which are all first order, and linear in the derivatives.

Let us now count our variables. Unlike in the AdS case, v_r and v_i are now independent variables. On the other hand, (5.6.47) allows us to eliminate j_2 (which vanishes) and j_1 (which is a function of other variables). All in all, we can take as independent variables

$$a_3\,,\qquad j_3\,,\qquad v_r\,,\qquad v_i\,,\qquad A\,,\qquad \psi\,. \tag{5.6.51}$$

Just as in section 5.6.3, we still have the gauge freedom (5.6.34). This means that one of these six variables is actually redundant, and we effectively have five variables.

So we again have as many variables as equations. We have studied the system numerically. The solutions share some qualitative features with the ones for the AdS case (see figure 5.1(b)); for example, the warping A stays flat rather than diverging. However, they only survive for small values of F_0, which do not satisfy the flux quantization condition $F_0 = \frac{n_0}{2\pi l_s}$. For values of F_0 that do satisfy flux quantization, the system seems to crash in a singularity before it gets to $r = 0$.

It is also possible to set up a perturbative study. Since $\Lambda = 0$ in this case, we cannot perturb in μ. We introduce a new perturbation parameter ν, such that $v \to 0$ as $\nu \to 0$. This can be achieved by taking the coefficients v_r and v_i to be odd functions of ν, while the other coefficients a_i, j_i will be even functions of ν. We solved the resulting system at first order in ν, similarly to section 5.6.2.

Finally, it would presumably also be possible to deform the Minkowski solutions discussed in this section into an AdS solution, by generalizing the procedure in section 5.5.

Part II

Topological resolution of Coulomb-branch singularities

Chapter 6

$\mathcal{N} = 4$ Super Yang Mills in 3d

6.1 Introduction and motivations

In the previous chapter we saw that Romans mass can patch up the singularity of the O6 plane. Summarizing, in the presence of an O6 plane in the IIA supergravity with zero Romans mass, there is a singularity in the O6. Anyway, the behaviour of the dilaton next to the singularity shows that supergravity is no longer reliable, since it starts growing. The theory has to be uplifted to 11 dimensions and the metric gets quantum and instantons corrections. After this treatment, the metric transverse to the O6 is smooth and there is no more a singularity, but just a minimal radius that can be accessible.

When a non zero Romans mass is added, it is no longer possible to uplift to M-theory, so the same procedure cannot be used. The fact is that, without uplifting, the O6 plane is able to protect itself from singularity, due to the presence of a non zero F_0. Moreover, there is no longer a minimal radius and the metric can be continued even for negative radii.

Let us go back to the massless case and give more details. Atiyah and Hitchin calculated the metric of 2 BPS monopoles in the center of mass system. It is a family of 4 dimensional hyper-Kähler manifold with a $SO(3)$ action that rotates the three inequivalent complex structures.

Seiberg and Witten in [6] considered the theory dimensional reduced to three dimensions of the model proposed in [36]. It is a $\mathcal{N} = 4$ theory with gauge group $SU(2)$, in which the Coulomb branch is studied. After considering quantum and instantonic corrections, the moduli space of the effective theory has been proposed to be the Atiyah-Hitchin one. To check their proposal, Seiberg and Witten compared the weak coupling behaviour from supersymmetry with the limit of large spatial separation between the monopoles for the Atiyah-Hitchin metric and there is perfect agreement.

The relation between the three dimensional supersymmetric gauge theory and the geometry defined by two BPS monopoles was explained by [7, 37]: the theory is the one defined on a D2 probe next to the O6 plane in M-theory. The 4 dimensional effective theory describes the metric in the direction that are orthogonal both to the O6 and the D2. After quantum correction, instantons corrections are the exchange of D0 between the D2 and its own image from the O6.

That was the massless case, i.e. when there is a vanishing Romans mass. What happens when F_0 is non zero?

My supervisor Alessandro, our collaborator Gonzalo Torroba from SLAC and Stanford University, and I are still working on this subject and we can present some results.

The idea is that the theory on the D2 probe should be modified by a Chern Simons interaction, since it is the natural coupling (from the Wess-Zumino interaction) with the Romans mass. Moreover [4] showed that Chern Simons and Romans mass are deeply related even in the context of ABJM theory [3]: in the usual formulation of this AdS_4/CFT_3 correspondence, the supergravity side did not enjoy a non zero Romans mass, while the conformal field theory side has Chern Simons terms with levels $(k, -k)$. Once F_0 is given a non zero value, the Chern Simons levels are no longer symmetric, but they become $(k, F_0 - k)$.

So it appears natural to expect the Chern Simons term should be the modification needed in order to describe the gauge theory side of the resolution of the O6 singularity via Romans mass. In fact we were able to show that this is the case: in a $\mathcal{N} = 2$ simplified model[1], the one-loop corrections does not show the singularity any more (for the moment our result is restricted to the IR regime, the calculations in full generality are still under study).

Work in progress is about the presence of instantons and the explicit calculation of the metric.

In this chapter we are going to introduce the original $\mathcal{N} = 4$ theory, then we will focus to its CS deformation in the next chapter. In this chapter we will present first the gauge theory of [6], introducing first symmetries and the Lagrangian, then considering the low energy effective theory; when the action in the low energy regime is computed, we will compare

[1] As it can be seen just from counting the number of supercharges, this model cannot be really the field theory on a D2 probing the solution found in the previous chapter. However, it is similar enough that it should capture the relevant physics. The $\mathcal{N} = 1$ Chern Simons deformation is going to be the subject of future research. Details of the model studied can be found in future chapter.

the metric of the moduli space with the celebrated Atiyah-Hitchin metric and study some of its properties.

6.1.1 Symmetries

The original model by Seiberg and Witten [6] is a $\mathcal{N} = 4$ supersymmetric theory in 3 dimensions, obtained as a dimensional reduction of the $\mathcal{N} = 2$ theory in 4 dimensions [36] defined over $\mathbb{R}^3 \times S^1$. It is an example of extended supersymmetry, where all fields are in the adjoint representations of the gauge group.

Let us focus on the structure of this system. The Lagrangian, as we are going to see in the next section, exhibits three scalars. From the string theory point of view, they represent directions that are transverse both to the D2 and the O6. The Lagrangian is symmetric under rotations in the three scalar fields: it can be seen as the invariance under rotation in the space transverse to the O6 and because of this intuition we will call it $SU(2)_N$[2], where N means "normal".

There is also the usual $SU(2)_R$ symmetry, acting just on the fermions. Our theory can be obtained even as dimensional reduction of an $\mathcal{N} = 1$ Super Yang Mills in 6 dimensions; in the original theory there is this $SU(2)_R$ that goes through the dimensional reduction: fermions, then transform as a doublet under $SU(2)_R$

Moreover there is the Lorentz group, which in the Euclidean is $SO(3)_E$; we will take the double cover of this group, $SU(2)_E$. So, the total structure is $SU(2)_N \times SU(2)_R \times SU(2)_E \times G$, where G is the gauge group.

We are in the context of extended supersymmetry, so scalar fields are in the same supermultiplet as gauge bosons and thus transform under the adjoint representation of the gauge group. If the scalars are given a non zero vacuum expectation value, we fall in the part of the moduli space called Coulomb branch. This name has been given because the vacuum expectation value of the scalar leaves one of the gauge boson massless, so generating a long distance electromagnetic-like forces. From the gauge group point of view, this happens because the gauge group is reduced by spontaneous symmetry breaking to $U(1)$ factors.

6.1.2 Three-dimensional supersymmetric theories

Let us start by reviewing the structure of 3 dimensional theories with $\mathcal{N} = 2$ supersymmetry (4 supercharges).

The vector superfield contains the gauge field A^a_μ and gaugino λ^a, a real scalar σ^a (the extra component of the gauge field in reducing from 4 to

[2]Instead of the original $SO(3)$ we took its double cover $SU(2)$.

3 dimensions) and an auxiliary D-term D^a. Their Lagrangian is[3]

$$L_{gauge} = \frac{1}{g^2}\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu\sigma^a D^\mu\sigma^a + i\bar{\lambda}^a \not{D}\lambda^a - \bar{\lambda}\sigma\lambda + \frac{1}{2}D^a D^a\right]. \tag{6.1.1}$$

The gaugino is a 3 dimensional Dirac fermion.

Moving to the matter sector, a chiral superfield contains a complex scalar ϕ, a Dirac fermion ψ and an auxiliary field F. The Lagrangian reads

$$\begin{aligned} L_{matter} =& (D_\mu\phi_i)^\dagger D^\mu\phi_i + i\bar{\psi}_i \not{D}\psi_i - \phi^\dagger\sigma^2\phi + \phi^\dagger D\phi - \bar{\psi}\sigma\psi + i\sqrt{2}\phi^\dagger\bar{\lambda}\psi - i\sqrt{2}\bar{\psi}\lambda\phi \\ &+ F_i^\dagger F_i + F_i\frac{\partial W}{\partial\phi_i} + F_i^\dagger\left(\frac{\partial W}{\partial\phi_i}\right)^\dagger - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)^\dagger\bar{\psi}_i\bar{\psi}_j. \end{aligned} \tag{6.1.2}$$

The fields from the vector superfield act on the matter ones as matrices. For instance, $\phi^\dagger D\phi \equiv \phi_i^\dagger(T^a D^a)_{ij}\phi_j$. Similarly,

$$\phi^\dagger\sigma^2\phi \equiv \phi_i^\dagger(\sigma^a T^a_{ij})(\sigma^b T^b_{jk})\phi_k\,.$$

This term can be understood as coming from $g^{AB}D_A\phi^\dagger D_B\phi$ and the extra 4 dimensional component of the gauge field, in the four dimensional theory.

Note that integrating out the D-term sets $D^a = -g^2\phi^\dagger T^a\phi$, and the relevant part of the Lagrangian becomes

$$\frac{1}{2g^2}D^a D^a + \phi^\dagger D\phi \to -\frac{g^2}{2}(\phi^\dagger T^a\phi)(\phi^\dagger T^a\phi)\,. \tag{6.1.3}$$

6.1.3 The $\mathcal{N} = 4$ theory without flavors

Now we specialize to the simplest case in which singularities in the Coulomb branch can be studied, namely a 3 dimensional $\mathcal{N} = 4$ theory with gauge group $SU(N)$ and no flavors. In the notation of 6.1.2, this arises for the special case of an $\mathcal{N} = 2$ theory with a single matter superfield in the adjoint representation, and vanishing superpotential, $W = 0$.

In the $\mathcal{N} = 4$ theory, the normalization of the kinetic terms of the vector and matter superfields are related by supersymmetry. It is simplest to choose the normalization to be $1/g^2$. The Lagrangian of the 3d $\mathcal{N} = 4$

[3] See the appendix D.1 for the notation.

theory is then given by[4]

$$
\begin{aligned}
L &= \frac{1}{g^2}\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu\sigma^a D^\mu\sigma^a + (D_\mu\phi^a)^\dagger D^\mu\phi^a + i\bar\lambda^a \not{D}\lambda^a + i\bar\psi^a \not{D}\psi^a\right. \\
&- \left.\bar\lambda\sigma\lambda - \bar\psi\sigma\psi - \phi^\dagger\sigma^2\phi + i\sqrt{2}\,\phi^\dagger\bar\lambda\psi - i\sqrt{2}\,\bar\psi\lambda\phi - \frac{1}{2}(\phi^\dagger T^a\phi)(\phi^\dagger T^a\phi)\right].
\end{aligned}
\tag{6.1.4}
$$

For our purposes, it will be enough to consider an $SU(2)$ gauge group. In the string theory side, this arises from a D2 probe near an O6 plane.

For $SU(2)$, $f^{abc} = \epsilon^{abc}$, and the adjoint representation is three dimensional. Explicitly, the different terms in the action are as follows. Using the expression for the covariant derivative

$$
D_\mu\phi_a = \partial_\mu\phi_a + \epsilon_{abc}A_{\mu\, b}\phi_c\,,
$$

one obtains a scalar kinetic term

$$
\begin{aligned}
(D_\mu\phi)^\dagger D^\mu\phi =& \partial_\mu\phi^\dagger_a\partial^\mu\phi_a - \epsilon_{abc}A^\mu_a(\partial_\mu\phi^\dagger_b\,\phi_c + \partial_\mu\phi_b\,\phi^\dagger_c) \\
&+ (A_{\mu\, a}A^\mu_a)(\phi^\dagger_b\phi_b) - (A_{\mu\, a}\phi_a)(A^\mu_b\phi^\dagger_b)\,.
\end{aligned}
\tag{6.1.5}
$$

For the potential term $\phi^\dagger\sigma^2\phi$ we find

$$
\phi^\dagger\sigma^2\phi = (\phi^\dagger_a\phi_a)(\sigma_b\sigma_b) - (\phi^\dagger_a\sigma_a)(\phi_b\sigma_b)\,. \tag{6.1.6}
$$

Similarly,

$$
(\phi^\dagger T^a\phi)^2 = (\phi^\dagger_a\phi_a)(\phi^\dagger_b\phi_b) - (\phi^\dagger_a\phi^\dagger_a)(\phi_b\phi_b)\,. \tag{6.1.7}
$$

Let us make a few comments before approaching the Coulomb branch: in the theory the three scalars representing directions transverse to the O6 are the complex field ϕ (which contains two fields, its real and imaginary part) and σ. Giving non zero vacuum expectation value to one of them can be seen, from the string theory point of view, as moving the D2 probe out of the O6 in that direction: the non zero vacuum expectation value can be related to the distance respect to the O6.

Anyway, remember that we do expect that the effective action moduli space should be 4 dimensional, i.e. that the massless boson field of the effective Lagrangian should be four. As we saw, we considered just three scalars, so it seems that we are missing something. The answer is that also one of the $SU(2)$ gauge components of the gauge vector boson remains massless. In three dimensions the photon is dual to a scalar and so, the net result is of 4 scalars, 3 from the original scalars of the theory and one from the dual to the photon (the massless gauge vector boson). In section 6.2 we will see some details about the arising of this scalar.

[4]Notice that compared to [38], the action (6.1.4) has extra factors of $\sqrt{2}$. These are a consequence of the normalization of the gaugino kinetic term, which is chosen to be the same as that of ψ.

6.1.4 Gauge fixing Lagrangian in R_ξ-gauge

Now, let us give an expectation constant value just for the third component of the real part of ϕ, so breaking gauge symmetry,

$$\phi_3 \equiv v + \chi, \tag{6.1.8}$$

where $v \in \mathbb{R}$ is a constant and χ is the complex Higgs field. As it usually happens, the non zero vacuum expectation value generates an interaction between the gauge vector boson and the scalar in the kinetic term:

$$\begin{aligned} |D\phi|^2 =& \frac{(\partial\varphi)^2 + (\partial\tilde{\varphi})^2}{2} - \sqrt{2}v\epsilon_{\alpha\beta}A^\mu_\beta\partial_\mu\varphi_\beta \\ &- \epsilon_{\alpha\beta}A^\mu_\beta\Big[(\partial_\mu\phi^\dagger_\beta)\chi + (\partial_\mu\phi_\beta)\chi^\dagger - (\partial_\mu\chi)\phi^\dagger_\beta - (\partial_\mu\chi^\dagger)\phi_\beta\Big] \\ &+ (A^\mu_\alpha)^2(v^2 + \sqrt{2}v\varphi_3 + \frac{\varphi_3^2 + \tilde{\varphi}_3^2}{2}) + \ldots \end{aligned} \tag{6.1.9}$$

where we defined

$$\chi \equiv \frac{\varphi_3 + i\,\tilde{\varphi}_3}{\sqrt{2}} \qquad \text{and} \qquad \phi_\alpha \equiv \frac{\varphi_\alpha + i\,\tilde{\varphi}_\alpha}{\sqrt{2}}, \tag{6.1.10}$$

where φ, $\tilde{\varphi}$ are real fields, $\alpha = 1, 2$ and "$\ldots$" are terms that are not going to be interesting in the one-loop calculations.

The annoying term $\sqrt{2}v\epsilon_{\alpha\beta}A^\mu_\beta\partial_\mu\varphi_\beta$ can be canceled, as usual, using the gauge fixing Lagrangian,

$$\mathcal{L}_{gf} = -\frac{(G_a)^2}{2g^2\xi}, \tag{6.1.11}$$

where

$$G_a \equiv \partial_\mu A^\mu_a + \delta_{a\alpha}\xi\epsilon_{\alpha\beta}\big(\phi^\dagger_\beta\phi_3 + \phi_\beta\phi^\dagger_3\big). \tag{6.1.12}$$

Respect to the usual gauge fixing, i.e. the one present in the most common quantum field theory books like [39], (6.1.11) has been chosen to be quadratic in the scalar fields.

The previous Lagrangian (6.1.11) has been proven to be well-defined, i.e. BRST exact, and the quadratic contribution has been use to simplify the $p-$dependent interaction between the scalar and the vector boson gauge due to the kinetic term of the scalar.

Once the spontaneous symmetry breaking is performed, every field with a gauge index $a = \alpha = 1, 2$ has been given a mass

$$m = \sqrt{2}v, \tag{6.1.13}$$

while all 3-components remain massless.

Plugging the information obtained so far, the interacting Lagrangian involved in the one-loop corrections for the scalar φ_3 is

$$\begin{aligned}\mathcal{L}_{\text{int}} =& - m\Big[\tilde{\varphi}_\alpha^2 + \sigma_\alpha^2 - \left(A_\alpha^\mu\right)^2\Big]\varphi_3 + 2\epsilon_{\alpha\beta}A_\alpha^\mu\left(\varphi_\beta\partial_\mu\varphi_3 + \tilde{\varphi}_\beta\partial_\mu\tilde{\varphi}_3\right)\\ &+ \epsilon_{\alpha\beta}\left[\varphi_3\left(\bar{\psi}_\alpha\lambda_\beta - \bar{\lambda}_\alpha\psi_\beta\right) + i\,\tilde{\varphi}_3\left(\bar{\psi}_\alpha\lambda_\beta + \bar{\lambda}_\alpha\psi_\beta\right)\right]\\ &- \Big[\tilde{\varphi}_\alpha^2 + \sigma_\alpha^2 - \left(A_\alpha^\mu\right)^2\Big]\frac{\varphi_3^2}{2} - \Big[\varphi_\alpha^2 + \sigma_\alpha^2 - \left(A_\alpha^\mu\right)^2\Big]\frac{\tilde{\varphi}_3^2}{2}\\ &- \xi\, m\Big[\varphi_3\Big(\varphi_\alpha^2 + 2\bar{c}_\alpha c_\alpha\Big)\Big] + (1-\xi)m\,\varphi_\alpha\tilde{\varphi}_\alpha\tilde{\varphi}_3\\ &- \xi\frac{\varphi_3^2}{2}\Big(\varphi_\alpha^2 + 2\bar{c}_\alpha c_\alpha\Big) - \xi\frac{\tilde{\varphi}_3^2}{2}\Big(\tilde{\varphi}_\alpha^2 + 2\bar{c}_\alpha c_\alpha\Big) + \ldots\end{aligned} \tag{6.1.14}$$

In the following we will work in the Landau-Lorentz gauge ($\xi = 0$). This choice is useful in order to the decouple the ghost for the physical fields and it give a natural geometrical interpretation of the propagator for the vector boson field in terms of projectors.

6.2 The low energy theory

A useful reference for the material covered in this section can be found in [40]; we will borrow some of their arguments. At the classical level, after breaking $SU(2)_N$, the bosonic sector of the effective action is simply the action of all massless fields:

$$S_{eff}^{B,class} = \frac{1}{g^2}\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{2}\partial_\mu\varphi_3\partial^\mu\varphi_3 + \frac{1}{2}\partial_\mu\tilde{\varphi}_3\partial^\mu\tilde{\varphi}_3 + \frac{1}{2}\partial_\mu\sigma_3\partial^\mu\sigma_3 + \frac{1}{4}F_{\mu\nu}^3F_3^{\mu\nu}\right) \tag{6.2.1}$$

Before focusing on the quantum corrections, let us remember that the theory we are studying comes from the celebrated [36], one of whose most peculiar features is the presence of monopoles. Here we have something analogous to a $\theta-$ term, that can be written, in the low energy theory as

$$S_{surface} = \frac{i}{8\pi}\tilde{\sigma}^3\int\frac{d^3q}{(2\pi)^3}d^3x\,\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}^3. \tag{6.2.2}$$

This surface term can be seen as a constraint on the Bianchi identity for $F_{\mu\nu}^3$. One can obtain an alternative description of our system just promoting $\tilde{\sigma}^3$ to a dynamical field, so considering $\tilde{\sigma}^3$ as a Lagrangian multiplier for the Bianchi identity: Naively[5] using the equations of motion for $F_{\mu\nu}^3$,

$$\begin{aligned}F_{\mu\nu}^3 =& \frac{i}{4\pi}g^2\epsilon^{\mu\nu\rho}\partial_\rho\tilde{\sigma}^3;\\ \partial_\rho\tilde{\sigma}^3 =& -2\pi i\frac{\epsilon_{\mu\nu\rho}F^{\mu\nu}}{g^2},\end{aligned} \tag{6.2.3}$$

[5]We should use the equations of motion for A_μ^3, since $F_{\mu\nu}^3$ is a field strength, not a field. Anyway, the result is the same. We do prefer this naive formulation here since it appears much more intuitive.

we can substitute in the low energy effective lagrangian for $F^3_{\mu\nu}$:

$$\begin{aligned}\mathcal{S}^{A^3_\mu} =& \frac{1}{g^2}\int\frac{d^3q}{(2\pi)^3}\Big(\frac{1}{4}F^3_{\mu\nu}F^{\mu\nu}_3 + \frac{i}{8\pi}\tilde{\sigma}^3\epsilon^{\mu\nu\rho}\partial_\mu F^3_{\nu\rho}\Big)\\ =& \frac{g^2}{(4\pi)^2}\int\frac{d^3q}{(2\pi)^3}\partial_\mu\tilde{\sigma}_3\partial^\mu\tilde{\sigma}_3.\end{aligned} \tag{6.2.4}$$

We said that we can think of $\tilde{\sigma}^3$ as a Lagrangian multiplier for the Bianchi identity, but we did not use so far the Dirac quantization of the charge: that gives

$$k = \frac{i}{8\pi}\int\frac{d^3q}{(2\pi)^3}d^3x\,\epsilon^{\mu\nu\rho}\partial_\mu F^3_{\nu\rho} \in \mathbb{Z}. \tag{6.2.5}$$

Since the surface term enters in the path integral as $e^{i\,k\tilde{\sigma}_3}$ and the kinetic term for $\tilde{\sigma}^3$ involves (as always) just derivatives, we see that the system is invariant under

$$\tilde{\sigma}^3 \to \tilde{\sigma}^3 + 2\pi, \tag{6.2.6}$$

so, locally the system looks like $\mathbb{R}^3 \times S^1$. It will be useful, in the following, to define

$$Y \equiv \begin{pmatrix}\varphi_3\\ \tilde{\varphi}_3\\ \sigma_3\end{pmatrix}, \tag{6.2.7}$$

which transforms as $\mathbf{3}$ of $SO(3)_N$. Using Y the classical bosonic effective theory can be written as

$$\mathcal{S}^{B,class}_{eff} = \frac{1}{g^2}\int\frac{d^3q}{(2\pi)^3}\big(\frac{1}{2}\partial_\mu Y^i\partial^\mu Y^i + \frac{g^4}{16\pi^2}\partial_\mu\tilde{\sigma}_3\partial^\mu\tilde{\sigma}_3\big). \tag{6.2.8}$$

In order to simplify the notation, we can define

$$X \equiv \begin{pmatrix}\varphi_3\\ \tilde{\varphi}_3\\ \sigma_3\\ \dfrac{g^2}{2\sqrt{2}\pi}\tilde{\sigma}_3\end{pmatrix}, \tag{6.2.9}$$

such that

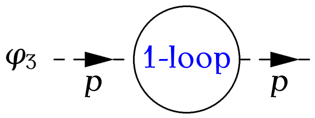
$$\mathcal{S}^{B,class}_{eff} = \frac{1}{g^2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2}\partial_\mu X^i\partial^\mu X^i. \tag{6.2.10}$$

On the other hand, the fermion sector of the low energy Lagrangian is much less exotic,

$$\mathcal{S}^{F,class}_{eff} = \frac{1}{g^2}\int\frac{d^3q}{(2\pi)^3}\Big(i\bar{\lambda}_3\partial\!\!\!/\lambda_3 + i\bar{\psi}_3\partial\!\!\!/\psi_3\Big). \tag{6.2.11}$$

6.2.1 Quantum corrections to the effective action

Consider the one loop correction to $\langle\varphi_3\varphi_3\rangle$: after (quite boring) calculations, it can been shown that, using dimensional regularization,



$$= \frac{i\,p^2}{2m\pi} + O(p^4). \tag{6.2.12}$$

Since the corrections are finite, it is possible to reinterpret them as a renormalization of the gauge coupling,

$$\frac{1}{g^2} \to \frac{1}{g^2} - \frac{1}{2m\pi}, \tag{6.2.13}$$

such that the effective action, quantum corrected, looks like

$$S^{B,\,one-loop}_{eff} = \Big(\frac{1}{g^2} - \frac{1}{2m\pi}\Big) \int \frac{d^3q}{(2\pi)^3} \Big(\frac{1}{2}\partial_\mu X^i \partial^\mu X^i\Big). \tag{6.2.14}$$

Before going on, let us comment briefly the situation so far. As we said previously, the scalars in the theory are directions that are orthogonal to both the O6 and the D2: giving a non-zero vacuum expectations value to one of them can be seen as taking the D2 away from the O6 by a distance that is proportional to v. In terms of the metric in the moduli space v is going to be our radial variable, up to some constant.

Moreover, notice that one-loop corrections introduced a singularity. The classical theory moduli space was just flat, as obvious, but quantum corrections go like $\sim v^{-1}$, meaning that the metric should go as $\sim r^{-1}$, so it has a singularity in the origin. We will see that instantons corrections save the theory from singularities.

We avoid the discussion about the fermion sector: at the end of the day the one-loop corrections are the same.

6.2.2 Instantons corrections

Let us briefly discuss the role and the shape of the instanton corrections in [6].

Using the usual Bogomol'ny approach for the euclidean bosonic action:

$$
\begin{aligned}
S_B^{Euc} &= \frac{1}{g^2}\int \frac{d^3q}{(2\pi)^3}\Big[\frac{F^a_{\mu\nu}F^{\mu\nu}_a}{4} + \frac{D_\mu\sigma_a D^\mu\sigma_a}{2} + |D\phi_a|^2 + \phi^\dagger\sigma^2\phi + \frac{(\phi^\dagger T^a\phi)^2}{2}\Big] \\
&= \frac{1}{g^2}\int \frac{d^3q}{(2\pi)^3}\Big[\frac{B^a_\mu B^\mu_a}{2} + \frac{D_\mu\sigma_a D^\mu\sigma_a}{2} + |D\phi_a|^2 + \phi^\dagger\sigma^2\phi + \frac{(\phi^\dagger T^a\phi)^2}{2}\Big] \\
&\geq \frac{1}{2g^2}\int \frac{d^3q}{(2\pi)^3}\Big[B^a_\mu B^\mu_a + D_\mu\varphi_a D^\mu\varphi_a\Big] \\
&= \frac{1}{2g^2}\int \frac{d^3q}{(2\pi)^3}\Big[(B^a_\mu + D_\mu\varphi_a)^2 - 2B^a_\mu D^\mu\varphi_a\Big] \\
&\geq -\frac{1}{g^2}\int \frac{d^3q}{(2\pi)^3} B^a_\mu D^\mu\varphi_a,
\end{aligned}
\tag{6.2.15}
$$

where $B^a_\mu = \frac{\epsilon_{\mu\nu\rho}F^a_{\nu\rho}}{2}$. The chain of inequalities is saturated once

$$
B^a_\mu = D_\mu\varphi_a; \qquad \sigma_a = \tilde{\varphi}_a = 0. \tag{6.2.16}
$$

Using the same ansatz of Polyakov, [41],

$$
\begin{cases}
\varphi_a(x) \equiv f(x)\dfrac{x^a}{x^2} \\[2ex]
A^\mu_a(x) \equiv a(x)\epsilon_{a\mu\nu}\dfrac{x^\nu}{x^2},
\end{cases}
\tag{6.2.17}
$$

it is possible to check that

$$
\begin{cases}
f(x) = m|x|\coth(m|x|) - 1 \\[2ex]
a(x) = 1 - \dfrac{m|x|}{\sinh(m|x|)}
\end{cases}
\tag{6.2.18}
$$

(m is the mass defined in (6.1.13)) is a classical solution with boundary conditions at $|x| \to \infty$ for the non vanishing fields

$$
\varphi_a \xrightarrow{|x|\to\infty} m\frac{x^a}{x}; \qquad B^a_\mu \xrightarrow{|x|\to\infty} -\frac{x^\mu x^a}{x^4}. \tag{6.2.19}
$$

Integrating, we can see that the action of the instanton is, [40]

$$
S_0 = \frac{4\pi}{g^2}m. \tag{6.2.20}
$$

As always, the contribution from the instanton is going to appear in the Lagrangian as e^{-S_0}. This is not the only contribution: as we already saw, in the spectrum of the bosonic low energy theory we also have the scalar dual to the photon $\tilde{\sigma}$, which should contribute to the action, [6, 41], incorporating the long distance interaction of the photon. So, the net contribution from the instanton should be

$$
\mathcal{L}_{inst} \sim e^{-S_0 - i\tilde{\sigma}}. \tag{6.2.21}
$$

What are modifications induced by this term? We should know if there are zero modes in the theory. It is possible to show [40] that there are 4 bosonic and 4 fermionic zero modes for every instanton. The net modification from the boson zero modes affects directly the metric, modifying the measure of the path integral; the fermionic one gives contribution to the metric too, but it introduces also a new term in the low energy effective action. Since, as shown by [40], just the $k = 0, \pm 1$ charged instantons contribute to the low energy effective Lagrangian, we will focus on this case.

As we said at the beginning of this section, instantons in the 3 dimensional theory are solutions of the broken phase. So, if the supercharges transform as $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ under $SU(2)_R \times SU(2)_N \times SU(2)_N$, once a real non zero $\langle \phi_3 \rangle$ breaks $SU(2)_N$ down to $U(1)_N$, in the broken phase the supercharge should transform as $(\mathbf{2}, \mathbf{2})^{1/2} \oplus (\mathbf{2}, \mathbf{2})^{-1/2}$, where the exponent are the $U(1)_N$ charge (they are half integers since the supercharges transforms as $\frac{1}{2}$ spin of $SU(2)_N$). Considered that at the end of the day there are just 4 fermion zero modes for one instanton, the interaction we expect should be in the form

$$\sim \psi^4 e^{-S_0 - i\tilde{\sigma}} \tag{6.2.22}$$

which at first sight, seems to generate an anomaly, since the total charge carried by the four fermions is $2 (= 4 \cdot \frac{1}{2})$. However, we did not fix the charge of $\tilde{\sigma}$ under $U(1)_N$: if the transformation induced by $U(1)_N$ acts as

$$\tilde{\sigma} \longrightarrow \tilde{\sigma} + 2\alpha \tag{6.2.23}$$

the symmetry of the system is restored, [6].

At the end of the day, the instanton corrections enter into the metric of the low energy effective theory, making it smooth and introduce a term involving 4 fermions. The effective theory can be shown to be a non linear supersymmetric $\sigma-$model and the potential for the 4 fermion modes due to the instanton can be read as the Riemann tensor of the moduli space:

$$Seff = k \int \frac{d^3q}{(2\pi)^3} g_{ij}(X) \frac{\partial_m X^i \partial_m X^i - \Omega^i \not{D} \Omega^j}{2} - \frac{R_{ijkl}(X)(\Omega^i \cdot \Omega^j)(\Omega^k \cdot \Omega^l)}{12}, \tag{6.2.24}$$

where X has already been introduced, while Ω is a linear composition ψ_3 and λ_3.

6.3 The metric of the moduli space

After a plethora of different corrections, in the previous section we were able to find the low energy effective action (6.2.24): it is a supersymmetric non linear $\sigma-$model where the target space is 4 dimensional. The aim of this section is to work out the metric of the moduli space.

What kind of metric do we expect? First, let us consider the holonomy of our system: for every $d-$dimensional Riemann manifold, the holonomy group $H \subset SO(d)$. In $\mathcal{N} = 4$ the commutativity relation of the supersymmetry generators with H generators implies that d should be divisible by four and $H \subset Sp(d/4)$. By definition, a manifold with symplectic holonomy is a hyper-Kähler.

Atiyah and Hitchin calculated the metric of the moduli space of two BPS monopoles, just using the fact that it is a hyper-Kähler manifolds with a $SU(2)$ isometry . It is exactly what we have on both side: a 4 dimensional hyper-Kähler as the moduli space of a 3 dimensional $\mathcal{N} = 4$ Super Yang Mills theory with gauge group $SU(2)$[6]. Moreover, Atiyah and Hitchin showed that the only one without singularities was the one that then got their names. This metric is known to be complete, i.e. any curve of finite length has a limiting point.

Seiberg and Witten, [6], proposed that the metric of the effective theory could be the Atiyah-Hitchin one, based on the geometrical properties and the fact that the absence of singularities could be related to the strong coupling behaviour of the supersymmetric gauge theory.

Following [35], we parametrize orbits of $SO(3)_N$ (whose double cover is $SU(2)_N$) by Euler angle (θ, ϕ, ψ) ($\theta \in [0, \pi]$, $\phi, \psi \in [0, 2\pi]$). In terms of them we can define a new coordinate system $(\sigma_1, \sigma_2, \sigma_3)$[7]:

$$\begin{cases} \sigma_1 = -\sin(\psi)\, d\theta + \cos(\psi)\sin(\theta)\, d\phi \\ \sigma_2 = \cos(\psi)\, d\theta + \sin(\psi)\sin(\theta)\, d\phi \\ \sigma_3 = d\psi + \cos(\theta)\, d\phi \end{cases} \tag{6.3.1}$$

Let us label the direction transverse to $SO(3)_N$ with r: with $SO(3)$ isometry the most general metric can be written in the form

$$ds^2 = f^2(r)d^2r + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2. \tag{6.3.2}$$

Before analysing the properties of (6.3.2), let us focus on the correspondence with the boson fields in our Lagrangian (6.2.24): let us define a new set of coordinates, Cartesian ones:

$$\begin{cases} x = r\cos(\phi)\sin(\theta) \\ y = r\sin(\phi)\sin(\theta) \\ z = r\cos(\theta) \end{cases} \tag{6.3.3}$$

[6]The isometry $SU(2)$ is the $SU(2)_N$ in the moduli space.

[7]The choice of the name of the coordinate may cause confusion respect to the field σ_a; we chose to agree with the notation of [35].

By definition they do transform as **3** of $SO(3)$, as Y defined in (6.2.7), so, up to some constant,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{const.} \begin{pmatrix} \varphi_3 \\ \tilde{\varphi}_3 \\ \sigma_3 \end{pmatrix} \tag{6.3.4}$$

The constant can be found looking at the classical bosonic effective action: in order to have just $g_{ij} = \delta_{ij}$ as the metric for the target space, the field should be renormalized taking the constant in (6.3.4) to be equal to S_0/m, meaning that $r = S_0$.

The angle ψ remained out of the previous definitions; under rotation of an angle α around the axis (x, y, z), ψ change as

$$\psi \longrightarrow \psi + \alpha. \tag{6.3.5}$$

The only field out of the game is $\tilde{\sigma}_3$, which transforms under the unbroken $U(1)_N$ as

$$\tilde{\sigma}_3 \longrightarrow \tilde{\sigma}_3 + 2\alpha, \tag{6.3.6}$$

so it appears sufficiently natural to set

$$\psi = \frac{\tilde{\sigma}_3}{2}. \tag{6.3.7}$$

Let us focus on the metric (6.2.24): it can be shown, [40], that introducing one-loop effects

$$\begin{aligned} a^2 =& b^2 = S_0(S_0 - 2) + O(S_0^{-1}); \\ c^2 =& 4 + \frac{8}{S_0} + O(S_0^{-2}); \\ f^2 =& 1 - \frac{2}{S_0} + O(S_0^{-2}). \end{aligned} \tag{6.3.8}$$

Two loops or higher give contributions that are suppressed by powers of S_0^{-1}.

At this point the hyper-Kähler condition can be rephrased as a system of differential equations,

$$\begin{cases} \dfrac{2bc}{f}\dfrac{da}{dr} = (b - c)^2 - a^2 \\ \\ \text{(cyclic permutation of } a,\, b,\, c) \end{cases} \tag{6.3.9}$$

In order to find a solution to the previous equations we should make an ansatz on f. In any case, consider that, because of the role that it has in the metric, the function f just controls the definition of the radial coordinate r. In order to check our correspondence, we should use (6.3.8), but, pay attention since (6.3.8) is not defined to every order.

Anyway, in order to have a look to the whole solution and simplify calculations, let us use $f = -b/r$, the same used by [35]: in this case

$$r = 2K\Big(\sin\big(\frac{\beta}{2}\big)\Big), \tag{6.3.10}$$

where

$$K(k) \equiv \int_0^{\frac{\pi}{2}} \frac{d\tau}{\sqrt{1 - k^2\sin^2(\tau)}} \tag{6.3.11}$$

is the elliptic integral. With this choice $r \in [\pi, \infty)$ as $\beta \in [0, \pi]$ and integrating numerically (6.3.9) one can find the behaviour of the functions a, b, c shown in Figure 6.1.

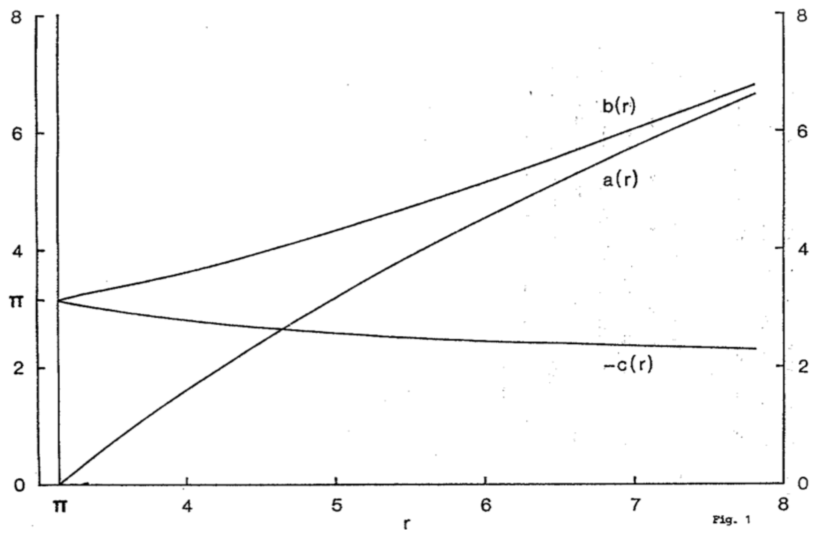


Figure 6.1: The Atiyah-Hitchin metric defining functions with the choice $f = -b/r$, from [35]. Note that in this choice $r \in [\pi, \infty)$.

With the choice $f = -\dfrac{b}{r}$ the asymptotic expansions of functions a, b, c is

$$\begin{cases} a = r\sqrt{1 - \dfrac{2}{r}} - 4r^2\big(1 - \dfrac{1}{2r^2}\big)e^{-r} + \dots \\[2ex] b = r\sqrt{1 - \dfrac{2}{r}} + 4r^2\big(1 - \dfrac{2}{r} - \dfrac{1}{2r^2}\big)e^{-r} + \dots \\[2ex] c = -\dfrac{2}{\sqrt{1 - \dfrac{2}{r}}} \end{cases} \tag{6.3.12}$$

where all term decaying faster than e^{-r} has been neglected. Note that terms e^{-r} has to be related to instantons, since, even with the choice from [35], $r \sim S_0$.

Chapter 7

Chern Simons deformation

As we already said in the introduction to the previous chapter 6.1, the aim of my present research, in collaboration with my supervisor Alessandro and Gonzalo Torroba from SLAC and Stanford University, is to find the properties of the system described in [6], deformed by the presence of a non zero Romans mass. The idea is that, if the massless system correspond to the 3 dimensional theory on a D2 probe next to an O6, once you add a non vanishing F_0, the theory on the D2 should be deformed by a Chern Simons term.

We recall why we expect the deformation to be Chern Simons. First, the natural coupling due to Wess Zumino interaction for the Romans mass with the D2 is

$$F_0 \int_{D2} CS(a), \tag{7.0.1}$$

where a is the gauge field over the brane, and $CS(a)$ is the Chern Simons Lagrangian for a. There are other hints that Chern Simons should be the right deformation: in [3], a correspondence from AdS_4/CFT_3, the conformal 3 dimensional theory is a Chern Simons one with levels $(k, -k)$, while the AdS_4 side has a vanishing F_0. Following [4], once the supergravity side enjoys a non zero Romans mass, what happens is that the two Chern Simons levels are shifted to $(k, F_0 - k)$.

We are going to add to the system studied in the previous section a Chern Simons terms, breaking supersymmetry down to $\mathcal{N} = 2$[1], so having 4 supercharges. In this way we do expect to find something similar, but not equal to the system studied from the supergravity side. Why?

Let us consider the number of supercharges. At first sight, one may expect that a non zero Romans mass should not break any supersym-

[1]The Lagrangian in (6.1.4) was obtained somehow “gluing” together Lagrangians (6.1.1) and (6.1.2), using supersymmetry. Since the Chern Simons term involves just the gauge fields, it is natural to expect to break one half of the supersymmetry.

metries, so the total amount of supercharges should be exactly the same of the massless case. But in order to have a non vanishing F_0, there should be a source for it, i.e. a D8, and so we do expect to break some supersymmetries. When consider a couple of D-branes, say Dp and Dp′, with $p > p'$, the condition in order to have a supersymmetric solution, if Dp′ lies completely on Dp, is

$$p - p' = 0 \quad (mod\, 4); \tag{7.0.2}$$

then the total amount of supercharges is reduced by a factor 1/4, [42]. This description fits with the previous system: we have an O6 and a D2, so $p - p' = 0\,(mod\, 4)$ and the original 32 supercharges of type IIA are reduced to 8, as in $\mathcal{N} = 4$ in 3 dimensions.

For the couple O6-D8, $p - p' = 2$ and one should have broken all supersymmetries. How to build a supersymmetric solution, in this case?

The (7.0.2) condition is valid just when the Dp′ brane lies completely on the Dp: the difference between the dimensions of the two sources is not really important, but transverse directions to them is. In fact, the condition comes from directions not in common between the two branes. If one chooses the system to have, for example, the configuration 7.1, there is an extra 1/4 for the total amount of supercharges, such that the final number of supercharges is 2, which means $\mathcal{N} = 1$ in 3 dimensions. There are still conditions about the compatibility of the D2 with the D8: they imply constraints on the B–fields, [43]. According to the configuration described in the table 7.1, compatibility constraints read as conditions on B_{45}, B_{67} and B_{89}.

Table 7.1: Configuration of branes in the massive theory (B–fields have to be switched on in direction 45, 67, 89).

	0	1	2	3	4	5	6	7	8	9
D2	•	•	•							
O6	•	•	•	•	•		•		•	
D8	•	•	•		•	•	•	•	•	•

In this chapter we are going to study a system with more supersymmetry than the one studied in 5, say $\mathcal{N} = 2$ in 3 dimensions, in order to simplify the problem of the effect of a Chern Simons term. The study of deformed $\mathcal{N} = 1$ is going to be the subject of the future research.

But what result do we expect? We think that the Chern Simons term in the Lagrangian will act like a patch on the singularity, as we saw the Romans mass does in the chapter 5. Moreover, in 5 we saw that the “size” of the patch depends on the value of the F_0: this result has been obtained numerically and we saw just a dependence, whose analytical form we did not calculate. So, somehow the patch even in the field theory side should depend on the Chern Simons level; in section 7.1.1 we will give more details to strengthen this expectations.

Moreover, is the metric of the moduli space for the deformed case expected to be hyper-Kähler? No, it is not: since we are going to break supersymmetry down to $\mathcal{N} = 2$, there is no way to expect the hyper-Kähler condition to be valid. Moreover, in the previous case the expected moduli space was made by the direction transverse to both the O6 and the D2 probe in M-theory, so 4 dimensional. What is the dimensionality we expect for $\mathcal{N} = 2$?

We are already in the position of saying which are the fields that are not going to participate to the new low energy effective Lagrangian. One is σ_3, since it is the one in both the Chern Simons and Yang Mills Lagrangian, so it has a mass coming from the topological term.

In the previous chapter we saw that $\tilde{\sigma}$ was the dual to the photon and in the moduli space it appears as an angular coordinate on a S^1; in fact $\tilde{\sigma}$ is the coordinate which has opened up because of the strong coupling limit. Since we are not going to experience M-theory, $\tilde{\sigma}$ is not going to appear.

So, the moduli space is two dimensional. How can it be? As we explained, the fact is that there is a stringy interpretation for the $\mathcal{N} = 1$ system, but not for the $\mathcal{N} = 2$.

We have some intermediate result that confirm this idea for the IR regime that we are enlarging to the entire system.

7.1 Yang-Mills-Chern-Simons theory

In order to have a protected moduli space, we consider an $\mathcal{N} = 2$ Chern Simons deformation. This is given by[2]

$$L_{CS} = \frac{k}{8\pi}\left(\epsilon^{\mu\nu\rho}(A^a_\mu \partial_\nu A^a_\rho - \frac{1}{6} f^{abc} A^a_\mu A^b_\nu A^c_\rho) - 2\bar{\lambda}^a\lambda^a + 2D^a\sigma^a\right) . \qquad (7.1.1)$$

The full theory is now Yang-Mills-Chern-Simons with matter,

$$L = L_{gauge} + L_{matter} + L_{CS} \qquad (7.1.2)$$

where L_{gauge} and L_{matter} were defined in (6.1.1) and (6.1.2), respectively. As in 6 the matter content is taken to be a single chiral superfield in the

[2]Our gaugino kinetic term has a Dirac normalization, and this is the reason for the extra factor of 2 in the gaugino mass in (7.1.1), as compared to [38]. This is verified by showing that all the vector supermultiplet gets the same supersymmetric mass (7.1.5) .

adjoint, with no superpotential. This theory has $\mathcal{N} = 2$ supersymmetry.

The final form for the Lagrangian is obtained by integrating out the auxiliary D-term,

$$D^a = -\frac{g^2 k}{4\pi}\sigma^a - g^2\phi_i^\dagger T^a_{ij}\phi_j \,.$$

We then arrive at

$$\begin{aligned} L &= \frac{1}{g^2}\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\sigma^a)^2 + i\bar{\lambda}^a \not{D}\lambda^a - \bar{\lambda}\sigma\lambda\right] \\ &+ \frac{k}{8\pi}\left[\epsilon^{\mu\nu\rho}(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho) - 2\bar{\lambda}^a\lambda^a\right] - \frac{1}{2}g^2\left(\frac{k}{4\pi}\sigma^a + \phi_i^\dagger T^a_{ij}\phi_j\right)^2 \\ &+ (D_\mu\phi_i)^\dagger(D^\mu\phi_i) + i\bar{\psi}_i \not{D}\psi_i - \phi^\dagger\sigma^2\phi - \bar{\psi}\sigma\psi + \sqrt{2}i(\phi^\dagger\bar{\lambda}\psi - \bar{\psi}\lambda\phi)\,. \end{aligned} \tag{7.1.3}$$

We will also be interested in the IR limit $g^2 \to \infty$, where the kinetic terms for the vector supermultiplet are set to zero, and $L = L_{CS} + L_{matter}$. Now σ, D and λ are all auxiliary, and integrating them out sets

$$\lambda^a = \frac{4\pi}{k}\sqrt{2}i(\phi^\dagger T^a\psi)\ ,\ \sigma^a = -\frac{4\pi}{k}(\phi^\dagger T^a\phi)$$

where $(\phi^\dagger T^a\phi) = \phi_i^\dagger T^a_{ij}\phi_j$. Therefore, the Lagrangian becomes

$$\begin{aligned} L &= (D_\mu\phi_i)^\dagger D^\mu\phi_i + i\bar{\psi}_i \not{D}\psi_i + \frac{k}{8\pi}\epsilon^{\mu\nu\rho}(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho) \qquad (7.1.4) \\ &- \frac{16\pi^2}{k^2}(\phi^\dagger T^a T^b\phi)(\phi^\dagger T^a\phi)(\phi^\dagger T^b\phi) + \frac{4\pi}{k}(\bar{\psi}T^a\psi)(\phi^\dagger T^a\phi) + \frac{8\pi}{k}(\bar{\psi}T^a\phi)(\phi^\dagger T^a\psi)\,. \end{aligned}$$

7.1.1 Coulomb branch of the Chern-Simons- deformed theory in IR regime

Let us begin by analyzing the IR limit $g^2 \to \infty$, described by (7.1.4), for an $SU(2)$ gauge group. As in 6, the theory is studied around the Coulomb branch point $\langle\phi_a\rangle = \delta_{a3}v$. Note that now the dimension of ϕ, and hence of v, is 1/2, because the kinetic term is no longer multiplied by $1/g^2$.

Before proceeding, it is useful to understand what kind of corrections can appear along the Coulomb branch. The crucial difference between the Chern Simons matter theory (7.1.4) and the previous case with nonzero gauge coupling is that, at least in perturbation theory, there is no dimensionful parameter. The Chern Simons level k, which is dimensionless and quantized, determines all the interactions. Next, turning on a Coulomb branch expectation value v, this will be the only dimensionful parameter, and hence it cannot appear in the loop corrected kinetic term for ϕ_3. In the Yang-Mills case there was a dimensionless parameter v/g^2 that was appearing in loop corrections, but this is no longer possible in the pure Chern Simons-matter case. We thus conclude that, if there are quantum corrections to the Coulomb branch metric, they have to be independent of v. As a consequence, this guarantees the absence of singularities along the Coulomb branch.

Let us calculate the Coulomb branch of the IR limit, (7.1.4): first, consider the massless boson fields. Using the definition $w = \sqrt{2}v = \sqrt{2}\langle\phi\rangle$ and $\mu \equiv \frac{k}{8\pi}$, it is possible to define

$$m_{CS} \equiv \frac{w^2}{2\mu}. \tag{7.1.5}$$

As happened in the previous chapter, even in this case just the components 3 of the fields remain massless, but the great difference with respect to chapter 6 is that in (7.1.4) the field σ_α has been cancelled, since it acts like an auxiliary field.

We use the gauge fixing lagrangian,

$$\mathcal{L}_{gf} = A^\mu_\alpha \frac{\partial_\mu \partial_\nu}{2\xi} A^\nu_\alpha + \epsilon_{\alpha\beta} A^\mu_\alpha \, \partial_\mu \left(\phi^\dagger_\beta \phi_3 + \phi^\dagger_3 \phi_\beta\right) - \frac{\xi}{2}(\phi^\dagger_\alpha \phi_3 + \phi^\dagger_3 \phi_\alpha)^2, \tag{7.1.6}$$

that is similar to the one used in the previous chapter (6.1.11) but for the fact that, since the dimension of the field is $[\phi] = 1/2$ and we no longer have the constant g^2 ($[g^2] = 1$), the gauge fixing parameter has dimension $[\xi] = 1$.

Even in this case we will use Landau gauge, since it automatically decouples ghosts from other fields, as already seen in the previous chapter.

At the end of the day, the interaction terms involved in the one-loop calculations in Landau gauge are

$$\begin{aligned}\mathcal{L}^{int}_{one-loop} =& \bar{\psi}_\alpha \frac{2w\varphi_3 + \varphi_3^2 + \tilde{\varphi}_3^2}{2\mu} \psi_\alpha + 2\epsilon_{\alpha\beta} A^\mu_\alpha \left(\varphi_\beta \partial_\mu \varphi_3 + \tilde{\varphi}_\beta \partial_\mu \tilde{\varphi}_3\right) \\ &+ \varphi_3 \left(w \left(A^\alpha_\mu\right)^2 - \frac{w^3}{2\mu^2} \tilde{\varphi}^2_\alpha \right) + \frac{w^3}{4\mu^2} \tilde{\varphi}_3 \varphi_\alpha \tilde{\varphi}_\alpha \\ &+ \frac{\varphi_3^2}{2} \left(- \frac{3w^2}{2\mu^2} \tilde{\varphi}^2_\alpha + \left(A^\alpha_\mu\right)^2 \right) + \frac{\tilde{\varphi}_3^2}{2} \left(- \frac{w^2}{4\mu^2} (\varphi^2_\alpha + \tilde{\varphi}^2_\alpha) + \left(A^\alpha_\mu\right)^2 \right)\end{aligned} \tag{7.1.7}$$

Let us calculate one-loop corrections to $\langle \varphi_3 \varphi_3 \rangle$; as predicted, they depend just on μ,

$$\varphi_3 \; \underset{p}{\longrightarrow} \; \text{1-loop} \; \underset{p}{\longrightarrow} \; \varphi_3 \; = -\frac{i\,p^2}{4\pi\mu} + O(p^4) \tag{7.1.8}$$

So, modifying the bosonic effective action through

$$S^{B, one-loop}_{eff} = \int d^3x \left(1 + \frac{1}{4\pi\mu}\right) \partial_\mu X^i_{CS} \partial^\mu X^i_{CS}, \tag{7.1.9}$$

where

$$X_{CS} = \begin{pmatrix} \varphi_3 \\ \tilde{\varphi}_3 \end{pmatrix}, \tag{7.1.10}$$

we have cancelled the singularity.

In the previous chapter we saw that there are corrections coming from instantons too. Are there instantons in our theory? The subject is discussed at the moment in the team, but we may say that they are not.

We said that the instantons in the original model described in 6, can be seen from the stringy point of view as D0 exchanged by the D2 probe with its own image from the O6. In fact, similarly to (7.0.1), there are even interactions of the type

$$\int_{D0} F_0\, a, \tag{7.1.11}$$

where again a is the gauge field over the $D0$. This contribution gives a tadpole in the massive case, so there is no classical solution and so no instanton solution can be added. This proposal is being checked at the moment.

7.1.2 Coulomb branch of the Chern-Simons deformed theory for finite g

Let us take the full lagrangian, $L_{CS} + L_{matter} + L_{gauge}$, with finite g:

$$\begin{aligned} L &= \frac{1}{g^2}\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\sigma^a)^2 + i\bar{\lambda}^a \not{D}\lambda^a - \bar{\lambda}\sigma\lambda\right] \\ &+ \frac{k}{8\pi}\left[\epsilon^{\mu\nu\rho}(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho) - 2\bar{\lambda}^a\lambda^a\right] - \frac{1}{2}g^2\left(\frac{k}{4\pi}\sigma^a + \phi_i^\dagger T^a_{ij}\phi_j\right)^2 \\ &+ (D_\mu\phi_i)^\dagger(D^\mu\phi_i) + i\bar{\psi}_i \not{D}\psi_i - \phi^\dagger\sigma^2\phi - \bar{\psi}\sigma\psi + \sqrt{2}i(\phi^\dagger\bar{\lambda}\psi - \bar{\psi}\lambda\phi)\,. \end{aligned} \tag{7.1.12}$$

At first sight one can think that σ_3 could remain massless, so taking the moduli space to be 3 dimensional. Because of the non zero vacuum expectation value for ϕ_3, this is not the case; in fact, due to the interaction term

$$-\frac{1}{2}g^2\left(\frac{k}{4\pi}\sigma^a + \phi_i^\dagger T^a_{ij}\phi_j\right)^2, \tag{7.1.13}$$

σ_a is given a mass $M = 2\mu g^2$. This term is not exactly beautiful: if we take $\langle\phi_3\rangle = \frac{w}{\sqrt{2}}$, among other term, we have

$$2g^2\mu\, w\, \epsilon_{\alpha\beta}\, \sigma_\alpha \tilde{\varphi}_\beta, \tag{7.1.14}$$

that implies the presence of a two particles vertex, or mixed propagator.

Except for the term (7.1.13), the interaction terms are exactly those from the original theory (6.1.4), so vertices should be similar (once you pay attention to g^2 factors).

At the end of the day, there are two masses, M and m. The former is the mass of every gauge fields and depends just on μ and g, so it's not related to the non zero vacuum expectation value of ϕ_3; the latter is generated by the spontaneous symmetry breaking (it depends on w). In the original non deformed theory, we had a 4 dimensional moduli space, given by the fact that there are 3 scalars in the theory and the dual to the photon, related to the opening of the new direction of M-theory. One of the "original" scalars in the game there was σ_3.

The deformed theory has masses for all gauge fields given by the interaction terms in Chern Simons. Moreover, the spontaneous symmetry breaking gives an extra mass to all directions but the Higgs one of $SU(2)_N$. So, σ_3 is not given an extra mass by symmetry breaking, but it has already a mass due to Chern Simons. Moreover, we are not going to be uplifted to M-theory, so even the dual to the photon is not going to appear in the effective Lagrangian, i.e appear as a coordinate of the moduli space. So, the moduli space should be still 2 dimensional (as in the IR regime, where $g^2 \to \infty$).

Part III

Appendix

Appendix A

General definition: signature, indices and so on...

In the present thesis the signature of the metric has been chosen to be $(-, +, +, \ldots, +)$. M, N indices are ten dimensional, μ, ν are used for the external spacetime, while m, n indicates quantities defined over the internal space. In dimension d the Hodge star $*$ action over a $k-$form is defined as

$$*_d\, e^{a_1} \wedge \cdots \wedge e^{a_k} \equiv \frac{1}{(d-k)!} \epsilon_{a_{k+1} \cdots a_d}{}^{a_1 \cdots a_{k+1}} e^{a_{k+1}} \wedge \cdots \wedge e^{a_d}. \tag{A.0.1}$$

If we choose a coordinates basis x on M, the derivatives $\frac{\partial}{\partial x^m}$ can be used to define contractions ι's: $\iota_m = \iota_{\frac{\partial}{\partial x^m}}$ acts on differential forms as

$$\iota_m (dx^{i_1} \wedge \cdots \wedge dx^{i_n}) \equiv p\, \delta_m^{[i_1}\, dx^{i_2} \wedge \cdots \wedge dx^{i_n]}. \tag{A.0.2}$$

Let us suppose that the $k-$form ω can be written as

$$\omega = \omega_m\, dx^m = \omega_{m_1 \cdots m_k} dx^{m_1} \wedge \cdots \wedge dx^{m_k}, \tag{A.0.3}$$

where dx^m is the basis dual to $\frac{\partial}{\partial x^m}$ and ω_m are components. The exterior derivative in terms of local coordinates is $d \equiv dx^m \partial_m$, such that

$$d\omega = dx^{[m} \wedge \partial_m \omega_{m_1 \cdots m_k} dx^{m_1} \wedge \cdots \wedge dx^{m_k}. \tag{A.0.4}$$

By definition, $d^2 = 0$.

Appendix B

Equations of motion for the dS proposal of 2.2.2

In the next formulas the $'$ is going to be used to indicate derivation respect to θ. The following equations of motion are valid outside the sources. For the source corrections, look at the modifications in 2.2.

Because of the symmetry of the system, the non equivalent equations of motion from the internal graviton are three, one from I,

$$2\frac{a''}{a} + 3\frac{b''}{b} + 4A'' + 4A'^2 - 2A'\phi' = \frac{e^{2\phi}}{4}\left(f_0^2 + \frac{f_2^2}{a^2} + \frac{f_4^2}{b^6} + \kappa_6^2 e^{-8A}\right), \tag{B.0.1}$$

one coming from the S^2 components,

$$\frac{1}{a^2} - \left(\frac{a'}{a}\right)^2 - 3\frac{a'}{a}\frac{b'}{b} - \frac{a''}{a} + \frac{a'}{a}(-4A' + 2\phi') = \frac{e^{2\phi}}{4}\left(-f_0^2 + \frac{f_2^2}{a^2} - \frac{f_4^2}{b^6} + \kappa_6^2 e^{-8A}\right), \tag{B.0.2}$$

and one from S^3's,

$$\frac{2}{b^2} - 2\left(\frac{b'}{b}\right)^2 - 2\frac{a'}{a}\frac{b'}{b} - \frac{b''}{b} + \frac{b'}{b}(-4A' + 2\phi') = \frac{e^{2\phi}}{4}\left(-f_0^2 - \frac{f_2^2}{a^2} + \frac{f_4^2}{b^6} + \kappa_6^2 e^{-8A}\right). \tag{B.0.3}$$

The external graviton equation of motion,

$$\Lambda e^{-2A} = A'' + \left(2\frac{a'}{a} + 3\frac{b'}{b}\right)A' + 4A'^2 - 2A'\phi' - \frac{e^{2\phi}}{4}\left(f_0^2 + \frac{f_2^2}{a^2} + \frac{f_4^2}{b^6} + \kappa_6^2 e^{-8A}\right), \tag{B.0.4}$$

is the one involving the internal cosmological constant Λ.

The dilaton evolution is given by

$$2\phi'' + 2\left(2\frac{a'}{a} + 3\frac{b'}{b}\right)\phi' - 4\phi'^2 + 8A'\phi' + \frac{h_1^2}{a^4} + \frac{e^{2\phi}}{2}\left(-5f_0^2 - 3\frac{f_2^2}{a^4} - \frac{f_4^2}{b^6} + \kappa_6^2 e^{-8A}\right) = 0. \tag{B.0.5}$$

Appendix C

Forms used in chapter 5

In this appendix, we will give a basis of forms symmetric under the symmetry ISO(3) we identified in section 5.6.2. This consists of translations in the directions x^i parallel to the O6-plane, and of simultaneous rotations of both the x^i and of the y^i, transverse to the O6-plane. In the main text, we have used this basis to expand both our pure spinors and fluxes.
The one-forms are:

$$\omega_{1,0} = y^i\, dy^i \equiv r\, dr \ , \tag{C.0.1a}$$
$$\omega_{1,1} = y^i\, dx^i \ . \tag{C.0.1b}$$

A 2-form basis compatible with the symmetry is:

$$\omega_{2,0} = \epsilon_{ijk}\, y^i\, dy^j \wedge dx^k \ , \tag{C.0.2a}$$
$$\omega_{2,1} = \epsilon_{ijk}\, y^i\, dy^j \wedge dy^k \ , \tag{C.0.2b}$$
$$\omega_{2,2} = \epsilon_{ijk}\, y^i\, dx^j \wedge dx^k \ , \tag{C.0.2c}$$
$$\omega_{2,3} = y^i\, dy^i \wedge y^j\, dx^j = \omega_{1,0} \wedge \omega_{1,1} \ , \tag{C.0.2d}$$
$$\omega_{2,4} = dx^i \wedge dy^i = J \ ; \tag{C.0.2e}$$

we recalled here that the last form is nothing but the two-form J of the massless O6 solution, (5.2.4).
The 3-forms can be written in terms of:

$$\omega_{3,0} = \frac{1}{6}\epsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k \equiv \mathrm{vol}_{\parallel} \ , \tag{C.0.3a}$$
$$\omega_{3,1} = \frac{1}{6}\epsilon_{ijk}\, dy^i \wedge dy^j \wedge dy^k \equiv \mathrm{vol}_{\perp} \ , \tag{C.0.3b}$$
$$\omega_{3,2} = \epsilon_{ijk}\, dx^i \wedge dy^j \wedge dy^k \ , \tag{C.0.3c}$$
$$\omega_{3,3} = \epsilon_{ijk}\, dx^i \wedge dx^j \wedge dy^k \ , \tag{C.0.3d}$$
$$\omega_{3,4} = \epsilon_{ijk} y^i\, y^m\, dx^m \wedge dy^j \wedge dy^k = \omega_{1,1} \wedge \omega_{2,2} \ , \tag{C.0.3e}$$
$$\omega_{3,5} = y^i\, dx^j \wedge dy^i \wedge dy^j = \omega_{1,1} \wedge \omega_{2,4} \ , \tag{C.0.3f}$$
$$\omega_{3,6} = y^i\, dx^j \wedge dx^i \wedge dy^j = \omega_{1,0} \wedge \omega_{2,4} \ , \tag{C.0.3g}$$
$$\omega_{3,7} = \epsilon_{ijk}\, y^i\, r\, dr \wedge dx^j \wedge dx^k = \epsilon_{ijk}\, y^i\, y^m\, dy^m \wedge dx^j \wedge dx^k \ , \tag{C.0.3h}$$

4-forms and 5-forms can then be obtained as wedge products from the previous definitions:

$$\omega_{4,0} = \epsilon_{ijk} y^i\, dx^m \wedge dx^j \wedge dy^m \wedge dy^k = \omega_{2,0} \wedge \omega_{2,4} \ , \tag{C.0.4a}$$

$$\omega_{4,1} = \epsilon_{ijk}\epsilon_{lmn}\, y^i\, y^l\, dx^j \wedge dx^m \wedge dy^k \wedge dy^n = \omega_{2,1} \wedge \omega_{2,2} \ , \tag{C.0.4b}$$

$$\omega_{4,2} = y^i\, dx^i \wedge \mathrm{vol}_\perp = \omega_{1,1} \wedge \mathrm{vol}_\perp \ , \tag{C.0.4c}$$

$$\omega_{4,3} = \mathrm{vol}_\parallel \wedge y^i\, dy^i = \mathrm{vol}_\parallel \wedge \omega_{1,0} \ , \tag{C.0.4d}$$

$$\omega_{4,4} = dx^i \wedge dx^j \wedge dy^i \wedge dy^j = -J^2 \ ; \tag{C.0.4e}$$

$$\omega_{5,0} = \omega_{2,2} \wedge \mathrm{vol}_\perp \ , \tag{C.0.5a}$$

$$\omega_{5,1} = \omega_{2,1} \wedge \mathrm{vol}_\parallel \ . \tag{C.0.5b}$$

Crucially, this basis is closed under exterior derivative d wedge product. One can then express both in terms of appropriate tensors: for example, the wedge product between the 2-form $\Psi = \Psi_i \omega_{2,i}\,, (i = 0, \ldots, 4)$ and the 3-form $\Omega = \Omega_I \omega_{3,I}\,, (I = 0, \ldots, 7)$ can be written in terms of a tensor W_{23}:

$$\Psi \wedge \Omega = \Psi_i\, \Omega_I\, \omega_{2,i} \wedge \omega_{3,I} \equiv \Psi_i\, \Omega_I\, (W_{23})_{i,I,\alpha} \omega_{5,\alpha} = (\Psi \wedge \Omega)_\alpha \omega_{5,\alpha} \ , \tag{C.0.6}$$

where $\alpha = 0, 1$. The same idea can be applied to the exterior derivative. For example:

$$d\,\Psi = d(\Psi_i\, \omega_{2,i}) = \frac{\Psi_i'}{r} \omega_{1,0} \wedge \omega_{2,i} + \Psi_i d(\omega_{2,i}) \equiv \left[\frac{\Psi_i'}{r} (W_{12})_{0,i,I} + \Psi_i D_{i,I} \right] \omega_{3,I} \ , \tag{C.0.7}$$

with $D_{i,I}$ an appropriate tensor. Working out all these tensors speeds up computations significantly.
Under the a parity transformation

$$\sigma : y_i \longrightarrow -y_i \tag{C.0.8}$$

in the directions perpendicular to the O6-plane, the forms defined above transform by picking up signs. These signs are summarized in table C.1.

Table C.1: Parity properties of our form basis under I_y in (5.6.15).

	Even	Odd
1-forms	$\omega_{1,0}$	$\omega_{1,1}$
2-forms	$\omega_{2,0}$	$\omega_{2,1}, \omega_{2,2}, \omega_{2,3}, \omega_{2,4}$
3-forms	$\omega_{3,0}, \omega_{3,2}, \omega_{3,4}, \omega_{3,6}$	$\omega_{3,1}, \omega_{3,3}, \omega_{3,5}, \omega_{3,7}$
4-forms	$\omega_{4,1}, \omega_{4,2}, \omega_{4,3}, \omega_{4,4}$	$\omega_{4,0}$
5-forms	$\omega_{5,0}$	$\omega_{5,1}$

Appendix D

Formalism used in part II

D.1 Notations

Even if it can generate confusion, we decide to use the metric with signature $(+--)$ in the last chapter, since it is the one traditionally used in [6, 40] and it makes the comparison with other results in the literature easier. The covariant derivative is given by

$$D_\mu \phi_i = \partial_\mu \phi_i + i A^a_\mu T^a_{ij} \phi_j \,. \tag{D.1.1}$$

where T^a_{ij} are the generators of the gauge group. Consider the gauge group $SU(N)$ and recall that for a field in the adjoint representation, the generators are given in terms of the structure constants by

$$T^a_{ij} = -i f^a_{ij} \,.$$

These generators are normalized according to

$$\mathrm{tr}(T^a T^b) = N\, \delta^{ab} \,.$$

Also, in this case the covariant derivative is given by

$$D_\mu \phi_c = \partial_\mu \phi_c + f^a_{cb} A^a_\mu \phi_b \,.$$

When specializing to $SU(2)$ gauge group ($f^{abc} = \epsilon^{abc}$), our convention for the generators becomes

$$T^a_{bc} = -i \epsilon_{abc} \,, \tag{D.1.2}$$

so, such that

$$\mathrm{tr}[T^a, T^b] = 2\delta_{ab} \tag{D.1.3}$$

and

$$D_\mu \phi_a = \partial_\mu \phi_a + \epsilon_{abc} A_{\mu\, b} \phi_c \,. \tag{D.1.4}$$

3d fermions are Dirac, with $\bar{\psi} = \psi^\dagger \gamma^0$. We choose the representation of gamma matrices with $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, $\gamma^2 = i\sigma_1$.

Appendix E

Towards a family of (non-)supersymmetric solutions

We saw how is possible to convert supersymmetry equations in terms of differential conditions on forms. In the case of $SU(3)$ structure, the whole information is encoded into the simplectic form J and the holomorphic volume Ω, while in the $SU(3) \times SU(3)$ case we have a sort of generalization of it, involving some J_ψ and Ω_ψ.

We saw J_ψ and Ω_ψ can be expressed in term of j and ω, defining an $SU(2)$ structure. It is possible to apply this decomposition to an $SU(3)$ structure too, since $SU(3) \subset SU(3) \times SU(3)$.

Once a supersymmetric solution with structure $SU(3)$ is found, is it possible to deform its $SU(2)$ structure in order to find a new supersymmetric solution, again with $SU(3)$ structure? In this way it would be possible to build families of supersymmetric solutions from an original forefather. Moreover, note that in the previous proposal there is not the necessity of starting from, or finishing to, a certain curvature in the internal space, so it seems to be possible to pass from an AdS to a Minkowski solution, or viceversa.

There is a second consideration to analyse: as we said in the second chapter, supersymmetry equations are differential equations of the first order, while the equations of motion are of the second. Suppose we know how to write equations of motion in terms of pure spinors: then, the same algorithm of the $SU(2)$ deformation could be applied in order to find non supersymmetric solution. In fact, it could be possible to use a supersymmetric solution, i.e. a solution of a first degree differential system of equations, in order to reduce the degree of the equations of motion.

At the end of the day, the main target of this construction is to check if there is the possibility to build non-supersymmetric solutions starting from supersymmetric ones in terms of pure spinors. In order to do that, one has to write the equations of motion in terms of pure spinors. Some attempts to go in that direction are known in the literature, [44, 45, 46], but at the moment of the writing of this thesis none had done completely in a satisfying way. This project is (slowly...) evolving with the collaboration of Stefano Massai, a colleague from Saclay.

Anyway, at the moment, the $SU(2)$ deformation has been studied only for the first simple step from AdS to AdS (the result is not so exciting, but it is the first step).

In the whole chapter we will use the following notation: "unhatted" quantities are the known one (the original supersymmetric solution), "hatted" quantities are the one "deformed" (the output). If $\Phi_\pm$ and $\hat{\Phi}_\pm$ are pure spinors satisfying the previous notations, the main idea can be rephrased as:

$$\hat{\Phi}_\pm = \hat{\Phi}_\pm(\Phi_\pm, \text{parameters}). \tag{E.0.1}$$

E.1 $SU(3)$ structure for every pure spinor

Let us ask all pure spinors to have a $SU(3)$-structure:

$$\begin{aligned} \hat{\Phi}_+ &= \hat{\rho}\, e^{i\hat{\theta}}\, e^{-i\hat{J}}; & \hat{\Phi}_- &= \hat{\rho}\hat{\Omega}; \\ \Phi_+ &= \rho\, e^{i\theta}\, e^{-iJ}; & \Phi_- &= \rho\Omega. \end{aligned}$$

and express the forms in the game in terms of $SU(2)$ structure:

$$J = j + \mathrm{Re}(v) \wedge \mathrm{Im}(v); \qquad \Omega = iv \wedge \omega, \tag{E.1.1}$$

where

$$\begin{aligned} j \wedge \omega &= 0; & \mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega) &= 0; \\ j^2 &= \mathrm{Re}(\omega)^2 = \mathrm{Im}(\omega)^2; & \omega\, &\text{pure} \end{aligned} \tag{E.1.2}$$

(the meaning of "pure" will be explained later).

In principle there is no way to know what the shape of $\hat{J}, \hat{\Omega}$ should be. In the game there are just 3 real 2-forms:

$$j,\ \mathrm{Re}(\omega),\ \mathrm{Im}(\omega) \tag{E.1.3}$$

and 2 real 1-forms:

$$\mathrm{Re}(v),\ \mathrm{Im}(v). \tag{E.1.4}$$

Is it possible to decompose $\hat{J}$ and $\hat{\Omega}$ in terms of these forms? At first sight one would say that there is no so much freedom to built a new (1,1) and a new (3,0) from the forms in the game. Anyway, remember that modifying Ω, somehow the concept of $(1,1)$ is changed . In other words, what decide what is a (1,1) form, for instance, is the new Ω.

So, if $\hat{\Omega}$ is pure, it will be enough.

Before going on, note that in the most general case, the warping too should be modified:

$$\hat{A} \equiv A_{\zeta} + A \tag{E.1.5}$$

The shape of the deformation is due to Bianchi identities for the Romans mass in AdS[1].

E.2 From AdS

In the following we will focus on supersymmetric solutions obtained starting from AdS and going to AdS. The result is not so exciting, just a specific class of known solutions can be built from an AdS supersymmetric solution. Anyway, the same idea could be used in order to build Minkowski solutions (quite rare) from AdS one.

The presentation of this living project is the occasion of presenting in details supersymmetry equations in terms of pure spinors in the special case of $SU(3)$ structure.

$$d_H(\mathrm{Re}(\Phi)_+) = -2\mu e^{-A}\mathrm{Re}(\Phi)_-; \tag{E.2.1a}$$
$$d_H(\mathrm{Im}(\Phi)_+) = 0; \tag{E.2.1b}$$
$$\frac{F}{\rho} = J \wedge d\left(e^{-3A}\mathrm{Im}(\Omega)\right) - J^{-1}\llcorner\left[d\left(e^{-3A}\mathrm{Im}(\Omega)\right) - H \wedge \left(e^{-3A}\mathrm{Im}(\Omega)\right)\right]$$
$$+ 5\mu e^{-4A}\,\mathrm{Re}(e^{i\theta}e^{-iJ}) \tag{E.2.1c}$$

E.2.1 Supersymmetry equations non involving fluxes

First, let us focus on (E.2.1a) and (E.2.1b). The 1-form contributions say that

$$d\rho = d\theta = 0. \tag{E.2.2}$$

Instead, the next degree is much more interesting: from (E.2.1a) and (E.2.1b)

$$\begin{aligned} dJ &= -\tan\theta\, H = -2\mu\, e^{-A}\sin\theta\,\mathrm{Re}(\Omega); \\ H &= 2\mu\, e^{-A}\cos\theta\,\mathrm{Re}(\Omega). \end{aligned} \tag{E.2.3}$$

[1]In Minkowski with $SU(3)$ structure the Romans mass is identically zero.

Implications on torsion classes

Note that the non zero value for dJ turns on the W_1 torsion class: in fact,

$$\begin{aligned} 0 = d(J\wedge\Omega) =& dJ\wedge\Omega + J\wedge d\Omega \\ =& -\frac{4}{3} i\,\mu e^{-A}\sin\theta\, J^3 + W_1 J^3 \end{aligned} \tag{E.2.4}$$

so $W_1 = \frac{4}{3} i\,\mu\, e^{-A}\sin(\theta)$. Let us look at the Ω side: the most general case is

$$d\Omega = W_1 J^2 + W_2\wedge J + \bar{W}_5\wedge\Omega, \tag{E.2.5}$$

with $W_2\wedge J^2 = 0$. We are asking W_5 to be (1,0) and W_2 to be a (1,1)-form. Since we have only a 1-form that is (1,0) (and it is v),

$$W_5 = w_5\, v, \tag{E.2.6}$$

and analogously

$$W_2 = w_{2,j}\, j + w_{2,v}\,\mathrm{Re}(v)\wedge\mathrm{Im}(v). \tag{E.2.7}$$

On (E.2.7) let us impose the primitivity condition:

$$W_2\wedge J^2 = (2w_{2,j} + w_{2,v}) j^2\wedge\mathrm{Re}(v)\wedge\mathrm{Im}(v) = 0, \tag{E.2.8}$$

implying $w_{2,v} = -2w_{2,j} \equiv -2\,w_2$. So,

$$W_2 = w_2\Big(j - 2\,\mathrm{Re}(v)\wedge\mathrm{Im}(v)\Big). \tag{E.2.9}$$

Bianchi identities on H implies

$$d(e^{-A}\mathrm{Re}(\Omega)) = 0, \tag{E.2.10}$$

which means

$$\begin{cases} \mathrm{Re}(w_2) = 0 \\ \mathrm{Im}(w_5) = -\frac{\dot{A}}{2} \\ \mathrm{Re}(w_5) = -\tau_0\frac{\dot{A}}{2} = \tau_0\,\mathrm{Im}(w_5) \end{cases} \tag{E.2.11}$$

(since W_1 is purely imaginary, it has no contribution from the previous calculations).

E.2.2 Supersymmetry equations non involving fluxes

In order to avoid useless constant factors, let us renormalize fluxes: $\frac{F_k}{\rho} \to F_k$[2].
From (E.2.1c), one can calculate the Romans mass:

$$F_0 = 5\mu e^{-4A}\cos\theta \tag{E.2.12}$$

[2]Note that it is just a renormalization respect to a constant, see (E.2.2)

Now you can see why we changed the warping factor as in (E.1.5): in AdS, the Bianchi identity for the Romans mass,

$$dF_0 = 0 \tag{E.2.13}$$

is equivalent to

$$dA = 0, \tag{E.2.14}$$

which implies that $W_5 = 0$. This is not true for Minkowski, so going to a space with different external curvature would imply a variation in the warping as (E.1.5).

Let us calculate F_2:

$$\begin{aligned} F_2 =& -J^{-1}\llcorner d\left(e^{-3A}\mathrm{Im}(\Omega)\right) + 5\mu e^{-4A}\sin\theta J \\ =& \frac{e^{-3A}}{3}\Big[-(3w+\mu e^{-A}\sin\theta)j + (6w-\mu e^{-A}\sin\theta)\mathrm{Re}(v)\wedge\mathrm{Im}(v)\Big] \end{aligned} \tag{E.2.15}$$

Before going on, let us notice that since $j\wedge\Omega = \mathrm{Re}(v)\wedge\mathrm{Im}(v)\wedge\Omega = 0$,

$$H\wedge F_2 = 0. \tag{E.2.16}$$

For the calculation of F_4, it has been used $J^{-1}\llcorner J^3 = 3J^2$:

$$F_4 = \frac{3}{2}\mu e^{-4A}\cos\theta J^2 = \frac{3}{10}F_0 J^2. \tag{E.2.17}$$

Last, for completeness, F_6:

$$F_6 = \frac{\mu e^{-4A}\sin\theta}{2}J^3 = -3\mu e^{-4A}\sin\theta\,\mathrm{vol}_6. \tag{E.2.18}$$

Non trivial Bianchi identities

Let us start with F_4: these are quite straighforward since $dF_0 = 0$ and $J\wedge dJ = 0$ imply $dF_4 = 0$. Moreover, thanks to (E.2.16), we see that

$$d_H F_4 = 0 \tag{E.2.19}$$

and no source for F_4 ($D4$ or $O4$) is allowed.

Let us look to the remaining F_2: with lots of patience, the result is

$$\begin{aligned} d_H F_2 =& e^{-3A}\Big\{ -\dot{w}\,j\wedge dx - 3w\,dj + \mu e^{-A}\Big[-4w\sin\theta + \mu e^{-A}\Big(\frac{2}{3}\sin^2\theta + 10\cos^2\theta\Big)\Big] \\ & \cdot\Big(\mathrm{Re}(v)\wedge\mathrm{Im}(\omega) + \mathrm{Im}(v)\wedge\mathrm{Re}(\omega)\Big)\Big\} \\ =& (source)_{AdS}. \end{aligned} \tag{E.2.20}$$

As it is easy to imagine, the problem is going to be fixing sources: this can be easily done once the beginning and the ending space have the same curvature in the external manifold, but it is not so clear what to impose when the external spacetimes enjoy different curvature. The subject is under study, at the moment.

E.2.3 Exterior derivatives

In what follows, just the forms already presented are going to be used, i.e. all forms defining the $SU(3)$-structure are the only forms in the game. In order to have a complete control over them, we should know the exterior derivative over them in full generality. Because the differential forms in the game satisfy different conditions according to the curvature of the external space, we proposed the most general expansion of the exterior derivative of our forms; the expansion depends on the curvature of the external spacetime.

We will not present here how the exterior derivative expansion looks like in its whole ugliness. Anyway, we just say that in passing from AdS to AdS lots of simplifications occur and the system give a readable answer (even if not really interesting...).

E.3 Deformations

Now, let us go to the most (hopefully) interesting part of this chapter, that is the way the system is deformed.

E.3.1 Conditions for $SU(3)$ structure for $\hat{\Phi}_\pm$

In full generality, it is possible to write

$$\hat{J} = a\, j + b\,\mathrm{Re}(\omega) + c\,\mathrm{Im}(\omega) + \frac{\mathrm{Re}(v)\wedge\mathrm{Im}(v)}{e^2}; \qquad \hat{\Omega} = i\, v \wedge (\alpha\, j + \beta\,\mathrm{Re}(\omega) + \gamma\,\mathrm{Im}(\omega)), \tag{E.3.1}$$

where latin letters are real coefficient, while greek ones are complex (the coefficient to $\mathrm{Re}(v)\wedge\mathrm{Im}(v)$ may look odd, but the shape chosen will clarify its geometrical interpretation in the following). j and ω satisfy (E.1.2), as already said.

Let us apply all conditions that define an $SU(3)$ structure: asking that

$$\hat{J}^3 = \frac{3}{2}\mathrm{Re}(\Omega)\wedge\mathrm{Im}(\Omega) \tag{E.3.2}$$

implies

$$a^2 + b^2 + c^2 = e^2; \tag{E.3.3a}$$
$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 2. \tag{E.3.3b}$$

Instead, $\hat{J}\wedge\hat{\Omega} = 0$ gives

$$(\alpha a + \beta b + \gamma c) = 0, \tag{E.3.4}$$

that can be separated into the real and the imaginary part:

$$\begin{aligned} \mathrm{Re}(\alpha)a + \mathrm{Re}(\beta)b + \mathrm{Re}(\gamma)c &= 0;\\ \mathrm{Im}(\alpha)a + \mathrm{Im}(\beta)b + \mathrm{Im}(\gamma)c &= 0. \end{aligned} \tag{E.3.5}$$

Now, purity on $\hat{\Omega}$ means just that, if $\hat{\Omega} \equiv i\, v \wedge \hat{\omega}$, then $\hat{\omega}^2 = 0$, so

$$\hat{\omega}^2 = 0 = (\alpha^2 + \beta^2 + \gamma^2) j^2 = 0, \tag{E.3.6}$$

which implies

$$\mathrm{Re}(\alpha)^2 + \mathrm{Re}(\beta)^2 + \mathrm{Re}(\gamma)^2 = 1 = \mathrm{Im}(\alpha)^2 + \mathrm{Im}(\beta)^2 + \mathrm{Im}(\gamma)^2 \tag{E.3.7}$$

and, together (E.3.3b),

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 0 \\ |\alpha|^2 + |\beta|^2 + |\gamma|^2 &= 2. \end{aligned} \tag{E.3.8}$$

Note that the case $\hat{J} = J$ and $\hat{\Omega} = \Omega$ automatically satisfies all our conditions.

E.3.2 Geometrical interpretation

Note that, just considering the parameters for $\hat{\Omega}$ defines a special surface through (E.3.8), i.e. a locus in $\mathbb{R}^3$ in which the vector of the real part is orthogonal to the vector of the imaginary part and both have unitary modulus. It can be easily seen in formulae. Let us define:

$$\underline{\alpha} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{E.3.9}$$

Eqts (E.3.8) can be rewritten as

$$\mathrm{Re}(\underline{\alpha}) \cdot \mathrm{Re}(\underline{\alpha}) = \mathrm{Im}(\underline{\alpha}) \cdot \mathrm{Im}(\underline{\alpha}) = 1; \qquad \mathrm{Re}(\underline{\alpha}) \cdot \mathrm{Im}(\underline{\alpha}) = 0. \tag{E.3.10}$$

First condition implies that $\mathrm{Re}(\underline{\alpha})$ define an S^2. Anyway, for every $\mathrm{Re}(\underline{\alpha})$ there is an orthogonal $\mathrm{Im}(\underline{\alpha})$, which fix a S^1. So the solutions live over $S^2 \times S^1 = S^3$, that is the famous Hopf fibrations.

Anyway, since equations (E.3.10) are quadratic, we have an invariance for the change of sign, so at the end of the day, the parameters for the definition of $\hat{\Omega}$ is

$$S^3/\mathbb{Z}_2. \tag{E.3.11}$$

If we now add the condition from (E.3.4), the result is quite interesting: define a vector

$$\underline{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{E.3.12}$$

which, because of (E.3.3a), has modulus e (in principle e is allowed to vary as it prefers). The condition (E.3.4) (and following) implies, in terms of 3 dimensional vectors,

$$\underline{a} \cdot \mathrm{Re}(\underline{\alpha}) = \underline{a} \cdot \mathrm{Im}(\underline{\alpha}) = 0. \tag{E.3.13}$$

So, the main difference of the vector $\underline{a}$ respect to $\mathrm{Re}(\underline{\alpha})$ and $\mathrm{Im}(\underline{\alpha})$ is the fact that the modulus is not fixed to 1, but it is controlled by the value of the coefficient e. The conditions for the deformed $SU(2)$ just define a system of orthonormal three vectors, so it is like choosing a coordinate basis in $\mathbb{R}^3$ to define a point.

E.4 To AdS

Obviously, AdS supersymmetry equations are the same that we already saw in E.2, but with "hatted" quantities:

$$d_{\hat{H}}(\mathrm{Re}(\hat{\Phi})_+) = -2\hat{\mu}e^{-\hat{A}}\mathrm{Re}(\hat{\Phi})_-; \tag{E.4.1a}$$

$$d_{\hat{H}}(\mathrm{Im}(\hat{\Phi})_+) = 0; \tag{E.4.1b}$$

$$\begin{aligned}\hat{F} &= \hat{J}\wedge d\Big(e^{-3\hat{A}}\mathrm{Im}(\hat{\Omega})\Big) - \hat{J}^{-1}\llcorner\Big[d\Big(e^{-3\hat{A}}\mathrm{Im}(\hat{\Omega})\Big)\hat{H}\wedge\Big(e^{-3\hat{A}}\mathrm{Im}(\hat{\Omega})\Big)\Big]\\ &\quad + 5\hat{\mu}e^{-4\hat{A}}\,\mathrm{Re}(e^{i\hat{\theta}}e^{-i\hat{J}})\end{aligned} \tag{E.4.1c}$$

As we did in E.2, we normalized the fluxes as $\dfrac{\hat{F}}{\hat{\rho}} \longrightarrow \hat{F}$.

Of course, there is no need to calculate again everything, we can just "hat" quantities already calculated:

$$d\hat{\rho} = d\hat{\theta} = 0 \tag{E.4.2a}$$

$$d\hat{J} = -2\hat{\mu}e^{-\hat{A}}\sin\hat{\theta}\,\mathrm{Re}(\hat{\Omega}) \tag{E.4.2b}$$

$$d\mathrm{Re}(\hat{\Omega}) = 0 \tag{E.4.2c}$$

$$d\mathrm{Im}(\hat{\Omega}) = \big(\mathrm{Im}(\hat{W}_1) + \hat{w}\big)\hat{j}^2 + \big(2\mathrm{Im}(\hat{W}_1) - \hat{w}\big)\hat{j}\wedge\mathrm{Re}(v)\wedge\mathrm{Im}(v) \tag{E.4.2d}$$

$$\begin{aligned}d_{\hat{H}}\hat{F}_2 = e^{-3\hat{A}}\Big\{ &-\dot{\hat{w}}\,\hat{j}\wedge dx - 3\hat{w}\,d\hat{j}\\ &+\hat{\mu}e^{-\hat{A}}\Big[-4\hat{w}\sin\hat{\theta} + \hat{\mu}e^{-\hat{A}}\Big(\frac{2}{3}\sin^2\hat{\theta} + 10\cos^2\hat{\theta}\Big)\Big]\\ &\cdot\Big(\mathrm{Re}(\hat{\omega})\wedge\mathrm{Im}(v) + \mathrm{Im}(\hat{\omega})\wedge\mathrm{Re}(v)\Big)\Big\}\end{aligned} \tag{E.4.2e}$$

where $\hat{j} \equiv \hat{J} - \mathrm{Re}(v)\wedge\mathrm{Im}(v)$.

E.4.1 From AdS to AdS: final comments

Plugging all ingredients together is hard, but not so complicated (just a huge amount of algebra) and some interesting (maybe) results can be obtained. First, consistency of the algorithm fixes

$$\theta = \hat{\theta} = 0. \tag{E.4.3}$$

Moreover, there are some differential constraints on a, b, c and α, β, γ currently under study. Anyway, the result is that if there are solutions (and the system does not have any big evidences of fighting constraints), there should send a nearly Kähler to another nearly Kähler. This should not come as a surprise: the building blocks are defined over a nearly Kähler manifold and it seems strange that the system could go to something completely different than a from a nearly Kähler.

Much more interesting results should come from changing the curvature in the external manifold, i.e. going from AdS to Minkowski, since in this case the system is different and there should be enough freedom to find a solution.

So, at the end of the day, the idea is to check if there is the possibility for going from and to spacetime with different external curvature. The situation, anyway, has a difficulty: how should I work out the shape of sources? In the system analysed the only possible sources are $D6$ or $O6$, but how to relate the ones in input to the output ones? The easiest idea should be to ask them to be equal in form[3], but the subject is still under study.

[3]With "equal in form" we mean that if in the original theory, for instance, the source is $f(\Omega)$, the source contribution of the theory in output should be $f(\hat{\Omega})$.

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