How potential is the $I(p)$ inequality curve in the analysis of empirical distributions

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1. INTRODUCTION

The analysis of inequality has always aroused the interest of scholars from several subjects, and the necessary tools for this analysis have been continuously refined: from the possibility of measuring the inequality in a local way to the introduction of global inequality measures decomposable according to some criteria (Kleiber, Kotz 2003; Radaelli 2010).

A point measure represents a potential tool for analysing the inequality of a non-negative variable, and this paper is focused on this very aspect of the inequality. Many point inequality measures are obtained by comparing the cumulative distribution function with the first incomplete moment of a non-negative variable; while the $I(p)$ point measure (Zenga, 2007a) obtains information about the inequality of a non-negative variable by comparing groups of populations. By averaging these point measures, the inequality index I is defined.
Some analytical and inferential results on the $I(p)$ curve and on the I index have been developed (Zenga 2007b; Greselin, Pasquazzi 2008; Polisicchio 2008a, 2008b, Polisicchio, Porro 2008; Porro 2008; Radaelli 2008; Greselin, Pasquazzi, Zitikis 2009; Greselin, Puri, Zitikis 2009).

Here we recall some known characteristics of this point inequality measure and present other new features. Particularly, the focus is on the behaviour of this point measure and on the information provided by it, thus highlighting how this measure creates inequality contrasts. Besides, we analyse how some transformations of the observed variable influence the point measure, and special attention is paid to the translation and the equalitarian transfer. Finally, we present some real $I(p)$ curves referred to the Italian individual income distributions as to emphasize the interpretation provided by this measure.

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2. NOTATION AND DEFINITIONS

Let X be a non-negative variable and let:

$$\left\{(x_j, n_j):\ j=1,\ldots,s;\ 0 \le x_1 < x_2 < \ldots < x_s;\ \sum_{j=1}^{s} n_j = N\right\}$$

be the corresponding frequency distribution.
For this frequency distribution, we define:

$$N_j = \sum_{i=1}^{j} n_i \qquad j = 1,\ldots,s \qquad \text{with} \quad N_s = N \tag{2.1}$$

$$p_j = \frac{N_j}{N} \qquad j = 1,\ldots,s \tag{2.2}$$

$$Q_j = \sum_{i=1}^{j} x_i n_i \qquad j = 1,\ldots,s \qquad \text{with} \quad Q_s = T \tag{2.3}$$

$$M = \frac{T}{N} = \frac{1}{N}\sum_{i=1}^{s} x_i n_i \tag{2.4}$$

For every value x_j the frequency distribution of X is split into two adjacent and disjoint groups:
- the lower group $\{(x_1, n_1);\ldots;(x_j, n_j)\}$;
- the upper group $\{(x_{j+1}, n_{j+1});\ldots;(x_s, n_s);(x^*_{s+1}, 0)\}$, j=1,…, s, where x^*_{s+1} is a hypothetical value with null frequency, in order to have s non-empty pairs of groups.
For every pair of groups, we compute the mean; hence the lower mean is:

$$\overset{-}{M}(p_j) = \frac{Q_j}{N_j} = \frac{1}{N_j}\sum_{i=1}^{j} x_i n_i \qquad j=1,\ldots,s \tag{2.5}$$

and the upper mean is:

$$\overset{+}{M}(p_j) = \begin{cases} \dfrac{T - Q_j}{N - N_j} & j = 1,\ldots,s-1, \\ x^*_{s+1} & j = s, \quad \text{where } x^*_{s+1} \ge x_s \end{cases} \tag{2.6}$$

The inequality curve introduced by Zenga is the complement to 1 of the ratio between the lower mean and the upper mean:

$$I_{(p_j)} = \frac{\overset{+}{M}(p_j) - \overset{-}{M}(p_j)}{\overset{+}{M}(p_j)} = 1 - \frac{\overset{-}{M}_{(p_j)}}{\overset{+}{M}_{(p_j)}} = 1 - U_{(p_j)} \qquad j = 1,\ldots,s \tag{2.7}$$

where the ratio $U_{(p_j)}$ between the two means is the point uniformity index between the lower group and the corresponding upper group; hence the lower group, composed by p_j percent of N, has a mean of X that represents the $U_{(p_j)} \cdot 100$ of the mean of the upper adjacent group.
Considering the s points $(p_j;\ I_{(p_j)})$, j = 1,…,s , and drawing s adjacent rectangles with base in abscissa $\lfloor p_{j-1}; p_j \rfloor$, with p_o=0, and height in ordinate $\lfloor 0; I_{(p_j)} \rfloor$, we obtain the diagram of inequality $I(p_j)$.

This diagram may be obtained using the following $I(p)$ curve:

$$I(p)=\begin{cases} 1-\dfrac{\overset{-}{M}_{(p_1)}}{\overset{+}{M}_{(p_1)}} & \text{for } 0 \le p \le p_1 \\ \\ 1-\dfrac{\overset{-}{M}_{(p_j)}}{\overset{+}{M}_{(p_j)}} & \text{for } p_{j-1} < p \le p_j;\ j=2,...,s. \end{cases} \tag{2.8}$$

Finally the global index I is obtained by:

$$I=\sum_{j=1}^{s} I_{(p_j)} \cdot \frac{n_j}{N}, \tag{2.9}$$

which is the weighted mean of the $I_{(p_j)}$ point measures with weights equal to $\frac{n_j}{N}$, j=1,…,s. Alternatively I is equal to the sum of the areas of the s rectangles used to plot the $I_{(p_j)}$ inequality (Zenga, 2007a), I is also the area under the *I(p)* curve.

In order to consider some comparisons with the Lorenz curve, we define $q_j=\frac{Q_j}{T}$, j=1,…, s; hence the Lorenz curve of the frequency distribution X is obtained through the points of coordinates (p_j; q_j), j=1,…,s.

3. FEATURES AND BEHAVIOUR OF THE $I_{(p_j)}$ CURVE.

The main feature of the $I_{(p_j)}$ inequality measure is the comparison with parts of the population. By the expression (2.7) we deduce that the compared parts of population are always two adjacent and disjoint groups which compose the whole frequency distribution. The two groups differ in the value assumed by the observed variable, that is: the lower group includes values of $X \le x$; the upper group is composed by values of $X > x$.

The comparisons are made for increasing values of X, so there are s comparisons. The results of these comparisons are drawn in the graph of $I_{(p_j)}$, j=1,…,s. The comparison of the lower group and the upper group is based on the arithmetic mean, hence the $I_{(p_j)}$ measure is simple to compute and straightforward to interpret. Besides the comparison is made on the ratio between the arithmetic means of the two groups: $U_{(p_j)}=\frac{\overset{-}{M}_{(p_j)}}{\overset{+}{M}_{(p_j)}}$.

Hence the ratio $U_{(p_j)}=1-I_{(p_j)}$ is the expression of the uniformity between the compared groups and it is a uniformity point measure. In this sense the global uniformity index is the weighted mean of the uniformity point measures, that is $U=1-I=\sum_{j=1}^{s} U_{(p_j)} \cdot \frac{n_j}{N}$.

From the definition, it derives that the $I_{(p_j)}$ measure assumes values in [0; 1].

In the distribution with absence of inequality, that is {(*M*;*N*)}, the point measure $I_{(p_j)}$ is always equal to zero in the interval [0; 1], setting $x^*_{s+1} = x_s = x_1 = M$; and the inequality index *I* is equal to zero.
While in the distribution with maximum inequality, that is {(0; *N*-1),(*T*; 1)}, we have: $I_{(p_1)} = 1$ and $I_{(p_2)} = 1 - \frac{M}{x^*_{s+1}} = 1 - \frac{M}{x^*_3}$, with $x^*_3 \geq x_2 = T$, and by assuming $x^*_3 = x_2 = T = NM$, it derives $I_{(p_2)} = 1 - \frac{1}{N}$; hence under this assumption:

$$I(p) = \begin{cases} 1 & \text{for } 0 \leq p \leq \frac{N-1}{N} \\ 1 - \frac{1}{N} & \text{for } \frac{N-1}{N} < p \leq 1, \end{cases}$$

and the inequality index *I* is equal to $1 - \frac{1}{N^2}$.
In any frequentcy distribution we have $I(p) = \frac{M - x_1}{M - x_1 p_1}$ for $0 \leq p \leq p_1$ and $I(p) = 1 - \frac{M}{x^*_{s+1}}$ for $p_{s-1} < p \leq 1$, but these values depend on the frequency distribution of *X*, in relation to x_1, p_1, p_{s-1}, *M* and $x^*_{s+1} \geq x_s$.
Excluding the previous considerations, the behaviour of $I_{(p_j)}$, *j*=1,…,*s*, is free; as some empirical analyses prove there is a wide variety of behaviours, but obviously every observed $I_{(p_j)}$ curve may be decomposed in increasing, decreasing and uniform parts, in relation to increasing values of p_j , *j*=1,…,*s*.
In any way, in the plot of the observed $I_{(p_j)}$ curve, it is possible to draw the horizontal line of level equal to the value of the *I* index. Hence, we have the graph of the mean of the $I_{(p_j)}$ inequality measures and we can highlight the parts of the distribution with inequality lower, upper and equal to the inequality mean *I*. This is a very interesting representation of the inequality index *I*, and it is not shared with other global indexes derived by different inequality curves.
In order to understand the information given by the $I_{(p_j)}$ point measure and to simplify the multiplicities of its behaviour, we can consider only the following main behaviour:
i) uniform inequality;
ii) non-increasing inequality;
iii) non-decreasing inequality.

i) In the uniform inequality, the values of $I_{(p_j)}$ are equal to the same value for every *j*=1,…,*s*, hence the plot of $I_{(p_j)}$ is parallel to the abscissa. In this case, no matter how you split the values of *X* into two adjacent groups, the inequality measure based on the means of the two compared groups assumes always the same value.
The value of the *I* index is equal to the unique value assumed by the $I_{(p_j)}$ point measure. This type of behaviour is similar to those obtained for the frequency distributions with null inequality or maximum inequality, but it emphasizes the presence of inequality, because the point measure is not equal to zero or one. Thus this is a situation of uniform inequality.

The following tab.1 shows a simple distribution with N=5, M=9 and I=$I_{(p_j)}$=0.23207, j=1,…,5 (see Polisicchio, 2008b, for the generation of this kind of distribution).

x_j	$\overset{-}{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{(p_j)}$
7.24778	7.24778	9.43806	0.23207
7.98941	7.61860	9.92094	0.23207
8.85102	8.02941	10.45589	0.23207
9.85996	8.48704	11.05183	0.23207
11.0518	9.0	11.71980*	0.23207
			***I*=0.23207**

* we have set x_6^* = 11.7198 >x_5 = 11.0518 for assuring $I(p_5)$=0.23207.

Tab. 1. *A frequency distribution with uniform I(p) inequality curve.*

An analogous example is given in tab.2, where N=5, M=12.8 and I=$I_{(p_j)}$=0.63058 for j=1,…,5.

x_j	$\overset{-}{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{(p_j)}$
5.41094	5.41094	14.64727	0.63058
7.23613	6.32353	17.11765	0.63058
10.17218	7.60642	20.59038	0.63058
15.34997	9.54230	25.83078	0.63058
25.83078	12.80	34.64928*	0.63058
			***I*=0.63058**

* in this the value x_6^* = 34.64928 >x_5 = 28.83078 assures that $I(p_5)$=0.63058.

Tab. 2. *A frequency distribution with uniform I(p) inequality curve.*

Fig. 1 reports the graphs of the $I_{(p_j)}$ point measures corresponding to the two previous examples.

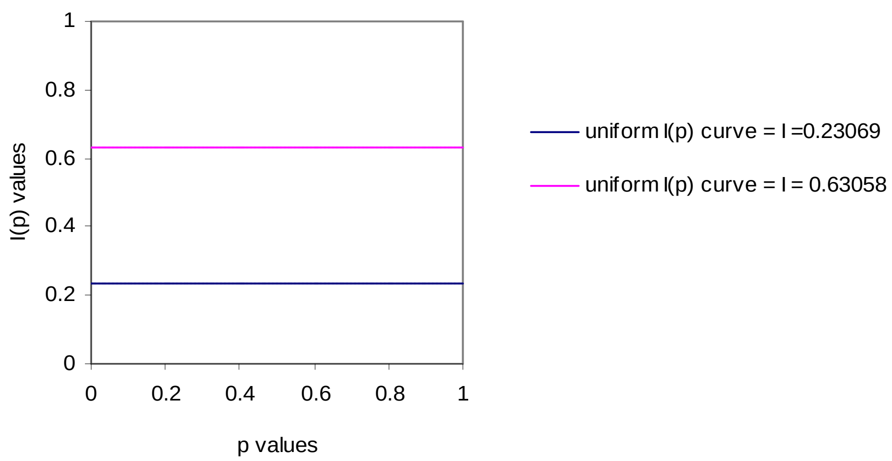

Fig. 1. *Graphs of the I(p) curves for the two frequency distributions with uniform inequality.*

ii) In the situation of non-increasing inequality, the values of $I_{(p_j)}$ are decreasing for j=1,…,s, hence the plot of $I_{(p_j)}$ is composed by rectangles with decreasing heights. For increasing values of X, the means of the two adjacent compared groups tend to be more equal and the inequality decreases.
The following tab. 3 shows a simple distribution with N=5, M=9, I=0.23207, and decreasing values of $I_{(p_j)}$, for j=1,…, 5.

x_j	$\overset{-}{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{(p_j)}$
7	7	9.5	0.26316
8	7.5	10	0.25000
9	8	10.5	0.23809
10	8.5	11	0.22727
11	9	11*	0.18182
			***I*=0.23207**

* in this case the value $x_6^* = x_5 = 11$.

Tab. 3. *A frequency distribution with non-increasing I(p) inequality curve.*

In fig. 2 we have the plot of the $I_{(p_j)}$ inequality measure and the I mean measure, where the weighted mean I of the point measures $I_{(p_j)}$ is represented with the horizontal line of level I. We can easily notice that the late behaviour is analogous to the case of uniform inequality. The plot of $I_{(p_j)}$ compared to that of I, shows the parts of the distribution that are closer to or farther from the mean I that corresponds to the case of uniform inequality. In other words with this plot we are able to compare two distributions: the observed one and the distribution with uniform inequality; obviously in the same plot we can also represent the $I(p)$ curves of the usual extreme distributions of maximum and minimum inequality.

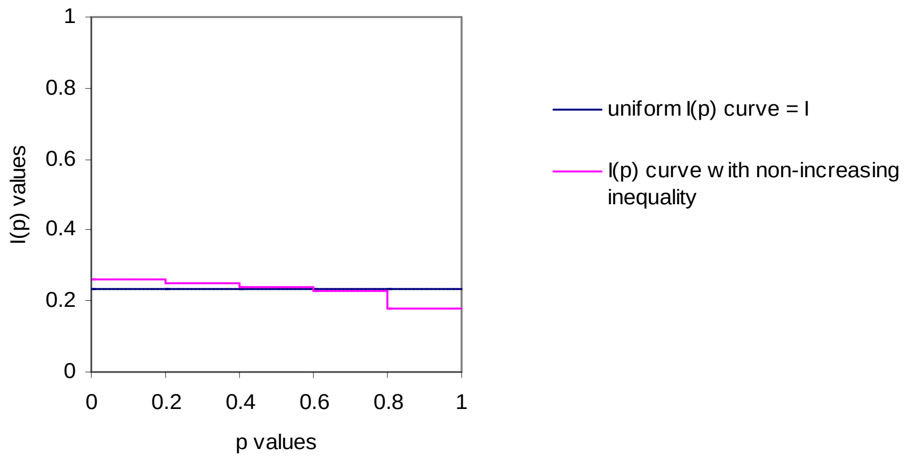

Fig. 2. *Graph of the I(p) curve of a frequency distribution with non-increasing I(p) curve and the corresponding I index.*

iii) Finally, in the situation of non-decreasing inequality the values of $I_{(p_j)}$ are increasing for j=1,…,s. The two compared groups tend to have increasing inequality for increasing values of X, because their means are less equal.
The following tab. 4 shows a distribution with N=5, M=12.8, I=0.63059, and increasing values of $I_{(p_j)}$, for j=1,…,5. Fig. 3 reports the graphs of the $I_{(p_j)}$ and I measures, and analogous considerations may be done in relation to the information given by it.

x_j	$\overset{-}{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{(p_j)}$
7	7	14.250	0.50877
8	7.5	16.33333	0.54082
9	8	20.0	0.60
10	8.5	30.0	0.71667
30	12.8	60.0(*)	0.78667
			***I*=0.63059**

* we have assumed x^*_{s+1} = 60 > x_5 = 30, for obtaining a value of $I(p_5) > I(p_4)$.

Tab. 4. *A frequency distribution with non-decreasing I(p) inequality curve.*

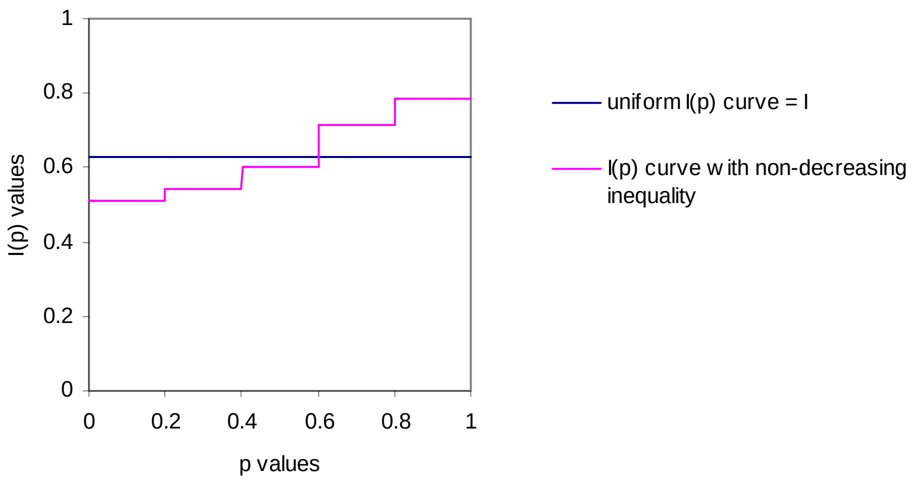

Fig. 3. *Graph of the I(p) curve of a frequency distribution with non-decreasing I(p) curve and the corresponding I index.*

The curves presented so far, may be composed resulting in mixed behaviour of the $I_{(p_j)}$ point measure; particularly, the concave, which has been often observed in the case of income distributions, can be obtained by combining a decreasing behaviour followed up by an increasing one.
These mentioned features manifest how deeply the $I(p)$ curve differs from the traditional Lorenz curve in terms of description, interpretation and behaviour; though a functional relation exists between the two curves for every p_j (Zenga, 2007a).

4. EFFECTS OF SOME TRANSFORMATIONS ON THE *I*(*p*) INEQUALITY CURVE

The *I* inequality index satisfies the usual requirements of a global index (Zenga, 2007a). As far as the point inequality measures are concerned, some properties have also been introduced (Zenga, 1990).
In relation to these requirements, it is interesting to analyse the feature of the *I*(*p*) curve following transformations which affect the frequency distribution. In particular, we analyse the effects of the translation and of the transfer from the rich to the poor on the *I*(*p*) curve.

4.1. *Translation*

Considering the frequency distribution of the non-negative variable *X*, we introduce the transformation $Y=X+h$ with $h \neq 0$.
Zenga [2007a] has proved that the *I* index is consistent with that transformation, hence if $h > 0$ it will be $I_Y < I_X$. Similarly if $h < 0$ it will be $I_Y > I_X$.
This demonstration is based on the performance of $I_{(p_j)}$, that is if $h > 0$:

$$I_{Y(p_j)} = 1 - \frac{\overset{-}{M}_{Y(p_j)}}{\overset{+}{M}_{Y(p_j)}} < I_{X(p_j)} \ , \qquad j = 1,\ldots,s. \tag{4.1}$$

where $I_{Y(p_j)}$ and $I_{X(p_j)}$ are the inequality measures of *Y* and *X*, respectively.
It is now interesting to understand how the behaviour of the $I_{(p_j)}$ point measure changes following this transformation. From the definition of the $I_{(p_j)}$ measure and the next equality:

$$\overset{-}{M}_{Y(p_j)} = \overset{-}{M}_{X(p_j)} + h \ \text{ and } \ \overset{+}{M}_{Y(p_j)} = \overset{+}{M}_{X(p_j)} + h$$

it derives:

$$I_{Y(p_j)} = 1 - \frac{\overset{-}{M}_{X(p_j)} + h}{\overset{+}{M}_{X(p_j)} + h} = \frac{\overset{+}{M}_{X(p_j)} - \overset{-}{M}_{X(p_j)}}{\overset{+}{M}_{X(p_j)} + h}$$

and by multiplying and then dividing it by $\overset{+}{M}_{X(p_j)} \neq 0$, we have:

$$I_{Y(p_j)} = I_{X(p_j)} \frac{\overset{+}{M}_{X(p_j)}}{\overset{+}{M}_{X(p_j)} + h} = I_{X(p_j)} \frac{1}{1 + \frac{h}{\overset{+}{M}_{X(p_j)}}} \qquad j=1,\ldots,s.$$

For every fixed value $h>0$, the ratio $\frac{1}{1 + \frac{h}{\overset{+}{M}_{X(p_j)}}}$ assumes values in (0;1) and it is an increasing function of p_j. Hence, $I_{Y(p_j)} < I_{X(p_j)}$ and the two curves tend to be nearer for increasing values of p_j.

An analogous consideration holds for a fixed value h, with $-x_1 \leq h < 0$, in this case the ratio $\dfrac{1}{1+\dfrac{h}{\overset{+}{M}_{X(p_j)}}}$ assumes values greater than 1 and it is a decreasing function of p_j.

Hence $I_{Y(p_j)} > I_{X(p_j)}$ and the two curves tend to be nearer for increasing values of p_j.

In conclusion, the influence of the considered transformation on the $I_{(p_j)}$ curve decreases as p_j increases no matter whether we add a positive or a negative value h to every x_j , j=1,…, N, but obviously the sign of h affects the relation between the two point measures.

Naturally, if all the non-negative and different values of a distribution are increased of the same quantity, not only the inequality of the whole distribution decreases but the reduction of the inequality involves mainly the smaller values rather than the greater ones and this feature is well highlighted by the $I_{(p_j)}$ point measure. As we will see later, the Lorenz curve is not able to point out this intuitive aspect.

The following fig. 4 reports the comparison with the uniform inequality distribution X of tab. 1 and Y=X+3. While in fig. 5 there are the I(p) curves of X and Y=X-5, being X the variable of tab. 3.

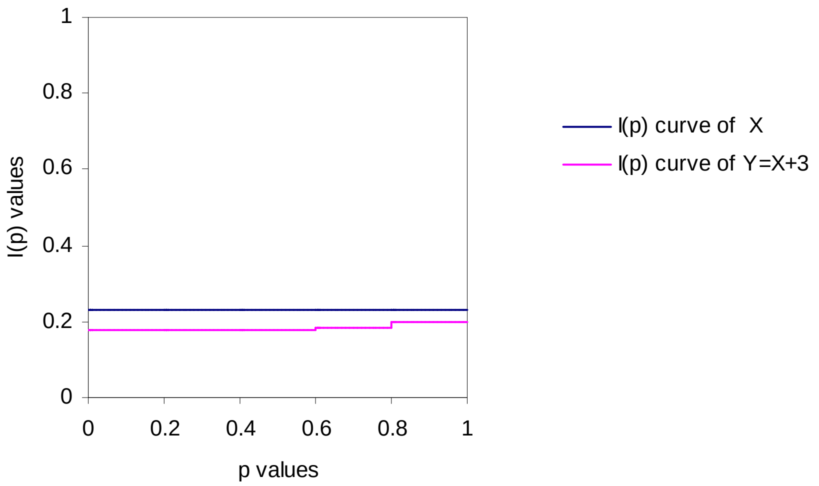

Fig. 4. *Comparison between the I(p) curves of the distribution of X reported in Tab.1 and Y=X+3.*

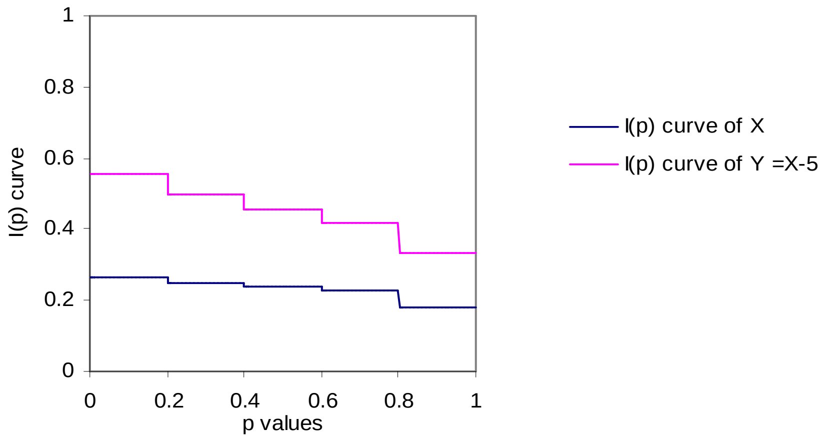

Fig. 5. *Comparison between the I(p) curves of the distribution X reported in Tab.3 and Y=X-5.*

Now it is interesting to consider the effect of this type of transformation on the Lorenz curve.
The Lorenz curve of $Y = X + h$ is obtained by means of the points with the same abscissa p_j and with ordinate $q_{Yj} = \frac{1}{NM_Y}\sum_{i=1}^{j} y_i n_i = \frac{Mq_{Xj} + p_j h}{M + h}$, $j = 1,\ldots,s$, where $M_Y = M + h$ is the mean of Y, and the points $(p_j; q_{Xj})$ for $j = 1,\ldots, s$, define the Lorenz curve of X.
The difference between the ordinates of the two Lorenz curves associated to the same abscissa p_j is:

$$q_{Y_j} - q_{X_j} = \left(p_j - q_{X_j}\right)\frac{h}{M+h} \qquad j = 1,\ldots,N,$$

hence, for $h>0$ we have a positive difference, because it depends on $(p_j - q_{xj})$ and it is an increasing function firstly and successively a decreasing function of p_j (see Nygard, Sandstrom, 1981). In other words, the variation depends directly on the prefixed behaviour of the Lorenz curve, thus the Lorenz curve has not the same reaction to the translation of the $I_{(p_j)}$ curve.

4.2 *Equalitarian transfers*

Without losing in generality, we can consider the distribution of X with unitary frequency, hence $s=N$. Considering the transfer of a quantity $h>0$ from x_{j+1} to x_j, with $h \le \frac{x_{j+1} - x_j}{2}$, $j=1,\ldots, N-1$, let Y be the distribution following the described transfer. It is easy to prove that:

$$I_{Y(p_t)} = \begin{cases} I_{X(p_t)} & t = 1,\ldots, j-1, j+1,\ldots, N, \\ 1 - \frac{N-j}{j}\cdot\frac{Q_j + h}{T - (Q_j + h)} & t = j \end{cases}$$

we notice that when $j = N-1$, in order to have the equality $I_{X(p_N)} = I_{Y(p_N)}$ it is enough to set

$x_s^* = x_N$ and $y_s^* = x_N > x_N - h = y_N$ (Zenga,2007a).
The difference between the $I_{X(p_j)}$ and $I_{Y(p_j)}$ curves is:

$$I_{X(p_t)} - I_{Y(p_t)} = \begin{cases} 0 & t = 1,\dots, j-1, j+1,\dots, N, \\ \dfrac{N-j}{j} - \dfrac{hT}{(T-Q_j)[T-(Q_j+h)]} & t = j. \end{cases}$$

Hence, the two $I(p)$ curves are unequal only for $p \in (p_{j-1}; p_j]$ and the variation depends on h, T, j and Q_j.
If the transfer of the quantity $h > 0$ is referred to two eventually non-consecutive values, such as x_{j+k} and x_j , for $j = 1,\dots, N - k$ and $k = 1,\dots, N - 1$, and this transformation does not change the order of the N values, that is $0 \le x_1 <\dots< x_{j-1} < x_j + h < x_{j+1} <\dots< x_{j+k-1} < x_{j+k} - h < x_{j+k+1} <\dots< x_N$, the two $I_{(p_j)}$ point measures are unequal for $p \in (p_{j-1}; p_{j+k-1}]$ and the difference between them has an expression analogous to that previously shown:

$$I_{X(p_t)} - I_{Y(p_t)} = \begin{cases} 0 & t = 1,\dots, j-1, j+k,\dots, N, \\ \dfrac{N-t}{t} \cdot \dfrac{hT}{(T-Q_t)[T-(Q_t+h)]} & t = j,\dots, j+k-1. \end{cases}$$

It is now interesting to analyze the effect of this type of transfer on the Lorenz curve, and to compare and contrast this effect with that on the $I(p)$ curve.
In the Lorenz curve, the effect of the described transfer between x_{j+1} and x_j is:

$$q_{Y_t} - q_{X_t} = \begin{cases} 0 & t = 1,\dots, j-1, j+1,\dots, N, \\ \dfrac{h}{T} & t = j, \end{cases}$$

hence, the two curves differ for $p \in (p_{j-1}; p_{j+1})$ and the variation depends only on h and T.
In the case of transfer referred to two non-consecutive values, under the assumptions previously described for the $I(p)$ curve, it derives:

$$q_{Yt} - q_{Xt} = \begin{cases} 0 & t = 1,\dots, j-1, j+k,\dots, N, \\ \dfrac{h}{T} & t = j,\dots, j+k-1. \end{cases}$$

Hence, the differences between the two Lorenz curves depend only on the transferred quantity h and the total T of the values of X, no matter how positions j and $j+k$ are involved in the transfer. Moreover, the plot of the two Lorenz curves differs in the part of abscissa $p \in (p_{j-1}; p_{j+k})$.
Finally, it is interesting to observe that the difference between the $I_{(p_j)}$ point measure is related to the ordinate q_j of the Lorenz curve in the following way:

$$I_{X(p_t)} - I_{Y(p_t)} = \begin{cases} 0 & t = 1,\dots, j-1, j+k,\dots, N, \\ \dfrac{h}{T}\cdot\dfrac{N-t}{t}\cdot\dfrac{1}{(1-q_{Xt})(1-q_{Yt})} & t = j,\dots, j+k-1. \end{cases}$$

Now, for pointing out these differences between the Lorenz curve and the $I(p)$ curve, some examples will be proposed.

In the following tab. 5 and 6 we consider N=5 values with M=25, but the values Y of tab. 6 are obtained by the values of tab. 5, transferring the quantity h=2 from the value x_5 to the value x_2. In the same tables we present the values of the $I_{(p_j)}$ measure and the values q_j of the Lorenz curve. While in tab. 7, for every j=1,…, s, we compute the differences between the corresponding $I_{(p_j)}$ measures and the corresponding ordinates of the Lorenz curves; these differences highlight the diverse reaction of the considered point measures to the equalitarian transfer.

x_j	$\bar{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{X(p_j)}$	Q_j	q_{Xj}
2	2	15.75	0.87302	2	0.03077
8	5	18.33333	0.72727	10	0.15385
12	7.33333	21.5	0.65891	22	0.33846
18	10	25	0.6	40	0.61538
25	13	25	0.48	65	1
			I_X = 0.66784		

Tab. 5. *Distribution of X with N*=5 *and M*=25

y_j	$\bar{M}_{(p_j)}$	$\overset{+}{M}_{(p_j)}$	$I_{Y(p_j)}$	Q_{Yj}	q_{Yj}
2	2	15.75	0.87302	2	0.03077
10	6	17.66666	0.66038	12	0.18462
12	8	20,5	0.60976	24	0.36923
18	10,5	23	0.54348	42	0.64615
23	13	25(*)	0.48	65	1
			I_Y = 0.63333		

(*) We have set $y_6^* = 25 > 23 = y_5$.

Tab. 6. *Distribution of Y with N*=5 *and M*=25 *obtained by the previous distribution X with an equalitarian transfer.*

j	$I_{X(p_j)} - I_{Y(p_j)}$	$q_{Yj} - q_{Xj}$
1	0	0
2	0.06689	0.03077
3	0.04915	0.03077
4	0.05652	0.03077
5	0	0

Tab. 7. *Differences between* $I_{(p_j)}$ *measure and Lorenz curve of X and Y.*

In tab. 8 and 9, two other examples are considered similar to the previous ones, with N = 5, M=25, the same positions of the values involved in the transfer (5^{th} and 2^{nd}) and the same transferred quantity h = 2.

x_j	$M^{-}_{(p_j)}$	$M^{+}_{(p_j)}$	$I_{X(p_j)}$	Q_j	q_{Xj}
5	5	15	0.66666	5	0.07692
7	6	17.66666	0.66038	12	0.18461
14	8.66666	19.5	0.55556	26	0.4
18	11	21	0.47619	45	0.6769
21	13	21	0.38095	65	1
			$\boldsymbol{I_X}$ **= 0.54795**		

Tab. 8. *Distribution of X with N*=5, *M*=25 *and different point inequality measure.*

y_j	$M^{-}_{(p_j)}$	$M^{+}_{(p_j)}$	$I_{Y(p_j)}$	Q_{Yj}	q_{Yj}
5	5	15	0.66666	5	0.07692
9	7	17	0.58824	14	0.21538
14	9.33333	18.5	0.49550	28	0.43077
18	11,5	19	0.39474	46	0.70769
19	13	21(*)	0.38095	65	1
			$\boldsymbol{I_Y}$ **= 0.50522**		

(*) We have set y_6^* = 21 > 19 = y_5.

Tab. 9. *Distribution of Y with N*=5 *and M*=25 *obtained by the previous distribution X with an equalitarian transfer*

These two last examples differ from the preceding ones by the point inequality measure and by the global inequality index. Tab. 10 reports for every j=1,…,5 the differences between the considered point inequality measures.
We notice that the Lorenz curve presents the same response to the previous example because of the same values h and T. On the contrary, the $I_{(p_j)}$ measure gives a diverse response to this considered transfer with respect to the preceding example, because of the dependence of the variation on the partial sum of values of X and Y.

j	$I_{X(p_j)} - I_{Y(p_j)}$	$q_{Yj} - q_{Xj}$
1	0	0
2	0.07214	0.03077
3	0.06006	0.03077
4	0.08145	0.03077
5	0	0

Tab. 10. *Differences between* $I_{(p_j)}$ *measure and Lorenz curve of X and Y.*

In other words, the different point inequality of the considered examples is well shown in the $I_{(p_j)}$ curves because it compares different and adjacent parts of distribution. This aspect is not highlighted by the Lorenz curve because the comparison of inequality is based on cumulative, ordered and relative values.

4.3 Scale transformation

The transformation $Y = aX$ ($a > 0$) has been already analysed in Zenga (2007a) and it does not determine any variation in the $I_{(p_j)}$ point inequality measure, just like in the Lorenz curve.

5. THE USE OF THE $I(p)$ CURVE IN SOME EMPIRICAL ANALYSES

The features of the $I_{(p_j)}$ inequality point measure make it especially suitable for analysing the inequality in the income distribution.

We present a study of inequality using the $I(p)$ measure, that has been carried out about some real cases referring to the individual income distribution in Italy. The considered data have been provided by the 2006 Bank of Italy sample survey on the Italian household income and wealth.

Fig. 6 reports both the $I(p)$ inequality curve and the straight line representing the I global index, referred to the non-negative individual net disposable incomes (black trait) and to the non-negative individual net disposable incomes without pensions and net transfers (grey trait). Remembering that I=1-U, where U is the global uniformity measure, for the 2006 net disposable incomes in Italy, U is equal to 0.2914, and this shows that in Italy the mean income of the lower groups is the 29.14% of the mean income of the respective upper groups, on average. To this value of U corresponds a mean value I of the $I_{(p_j)}$ inequality point measures equal to 0.7086.

The constant straight line of level 0.7086 represents the case of uniform inequality, that is to say the particular situation of equality between the means of incomes of all adjacent and disjoint groups constructed by varying the values of income. The inequality of these groups, measured by the point inequality $I_{(p_j)}$, is always equal to 0.7086, and obviously the weighted mean of these measures $I_{(p_j)}$ is equal to the same value.

Let us focus on the behaviour of the $I(p)$ inequality curve: the curve has a concave behaviour starting in the point (0; 1) and arriving at the point (1; 0.9767). In particular, for increasing values of income, the point inequality decreases with increasing decreases until the sixth decile, successively the point inequality is quite constant for a short trait and finally it increases suddenly.

This behaviour of the $I_{(p_j)}$ inequality point measure is consistent with that of the inequality in the Italian income distribution. When we compare the inequality of groups, passing from lower levels of income to intermediate levels of income, we notes that the inequality between adjacent groups of income decreases; in other words adjacent groups of income have means closer to each other, and increasing the value of income we arrive to a situation characterized by an inequality between adjacent groups that is constant. Successively, when we compare groups of intermediate incomes with groups of upper values of income, the inequality increases and this increment shows the existence of very great high values of income in the extreme groups of values.

The value of the uniform inequality line I=0.7086 is a mean value, hence it can be useful for a comparison with the graph of the $I_{(p_j)}$ point measure. From this comparison, we note that the $I_{(p_j)}$ point measure is upper than the uniform inequality line until the 25th percentile of income and then from the 90th percentile until upper extreme value of income. From the 25th percentile until the 90th percentile of income, the $I_{(p_j)}$ point inequality measure is lower than the uniform inequality line. In such a way, we are able to identify brackets of income with point inequality measure upper or lower than the mean inequality value.

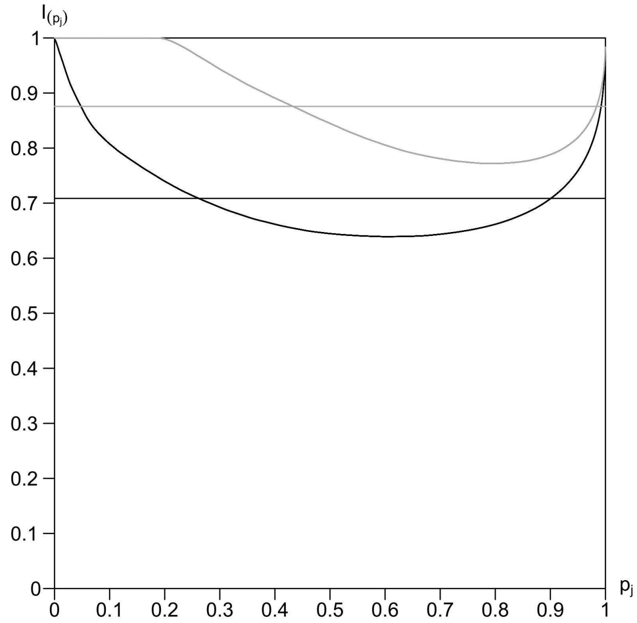

Fig. 6. *Italy. I(p) curves and I inequality indexes for non-negative net disposable income (black trait) and for non-negative net disposable income without pensions and net transfers (grey trait).*

The value of the uniform inequality line I=0.7086 is a mean value, hence it can be useful for a comparison with the graph of the $I_{(p_j)}$ point measure. From this comparison, we note that the $I_{(p_j)}$ point measure is upper than the uniform inequality line until the 25th percentile of income and then from the 90th percentile until upper extreme value of income. From the 25th percentile until the 90th percentile of income, the $I_{(p_j)}$ point inequality measure is lower than the uniform inequality line. In such a way, we are able to identify brackets of income with point inequality measure upper or lower than the mean inequality value.

Let us now highlight the potential of the $I(p)$ curve in describing the outcomes of transfers of income. In the Bank of Italy survey the net disposable income was divided into payroll income, self-employment income, pensions and net transfer, property income. Therefore it is interesting to analyze, using the $I(p)$ curve, the income distribution without pensions and net transfers. The global inequality index I is equal to 0,8758 and by its value we are given the chance to draw the line of uniform inequality (grey trait in fig.6).

By comparing the $I(p)$ inequality curve referred to this type of income (grey trait too) to the uniform inequality $I(p)$ curve, we can highlight the income levels where the point inequality measure is superior to the inequality mean, and those where the point inequality measure is lower than the inequality mean. Thus this comparison shows that these levels of income differ from those analyzed in the case of the net disposable income.

For the aim to evidence the potentiality of the $I(p)$ curve in describing the point inequality, we report in fig. 7 the corresponding Lorenz curves for the considered data.

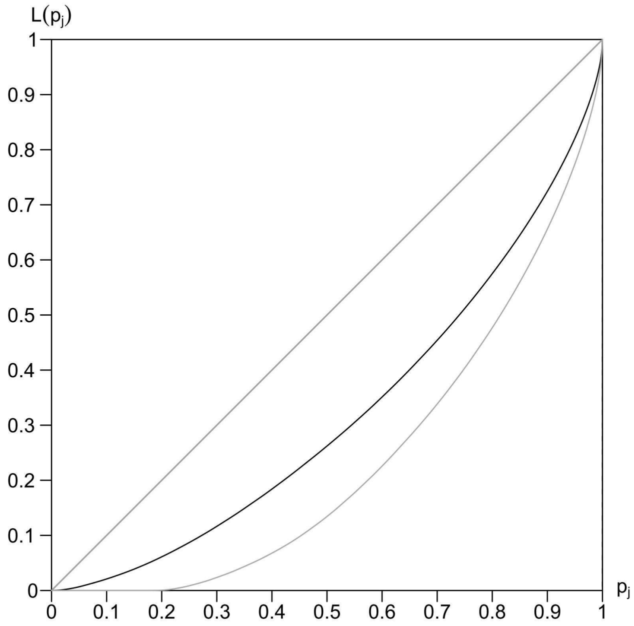

Fig.7. *Italy. Lorenz curves for non negative net disposable income (black trait) and for non negative net disposable income without pensions and net transfers (grey trait).*

In the other plots (fig. 8 - 15) the same variables are presented (net disposable individual income with or without pensions and net transfers) for particular Italian regions, for gender and for particular classes of age. Naturally every graph may be read to prove the general behaviour of the curve, the mean level of the point inequality measures, and the brackets of the distribution with lower or upper point inequality measure with respect to the mean level representing the uniform inequality distribution.

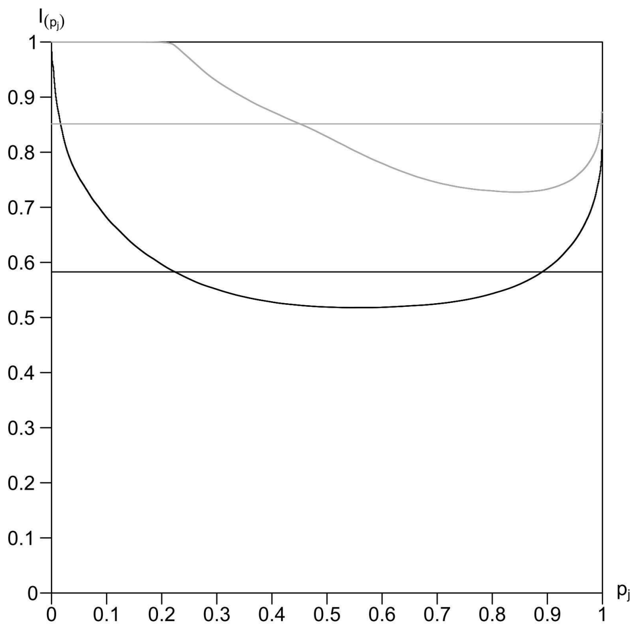

Fig. 8. *Piedmont. I(p) curves and I inequality indexes.*

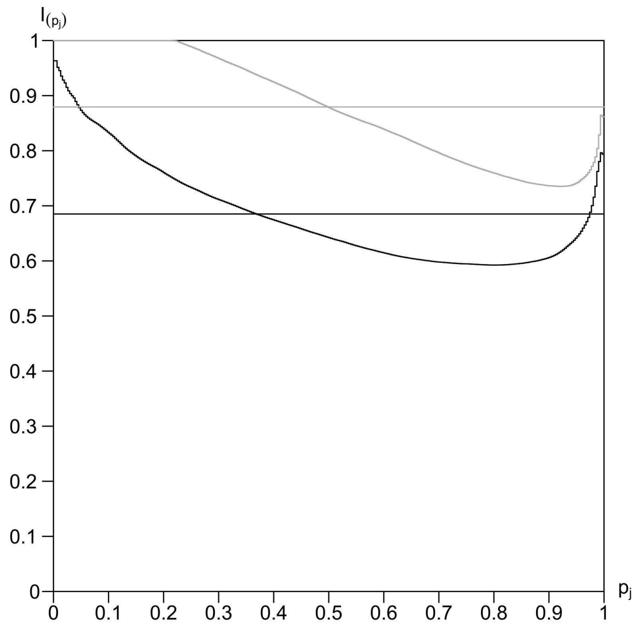

Fig. 9. *Calabria. I(p) curves and I inequality indexes.*

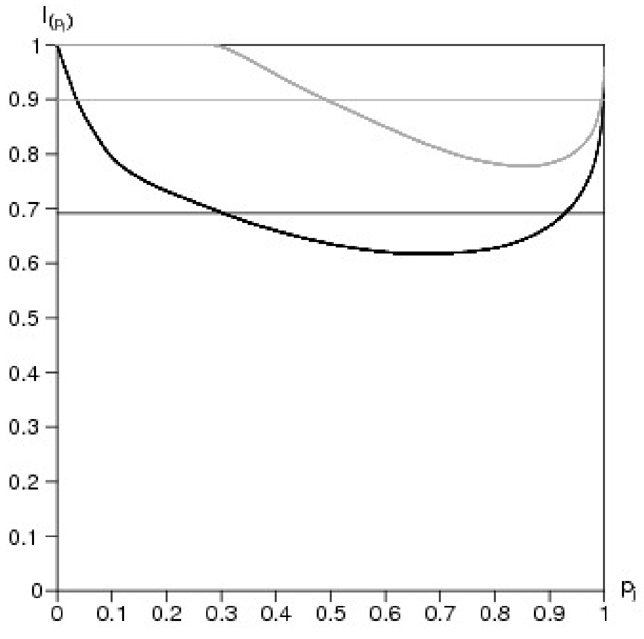

Fig. 10. *Females. I(p) curves and I inequality indexes.*

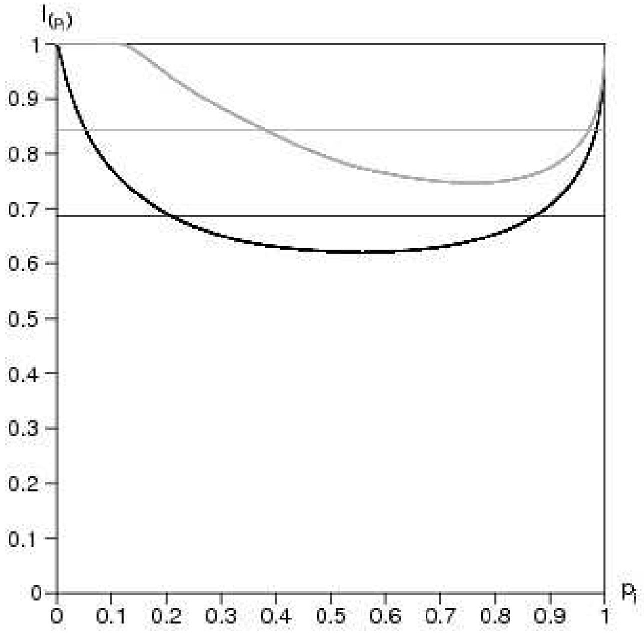

Fig. 11. *Males. I(p) curves and I inequality indexes.*

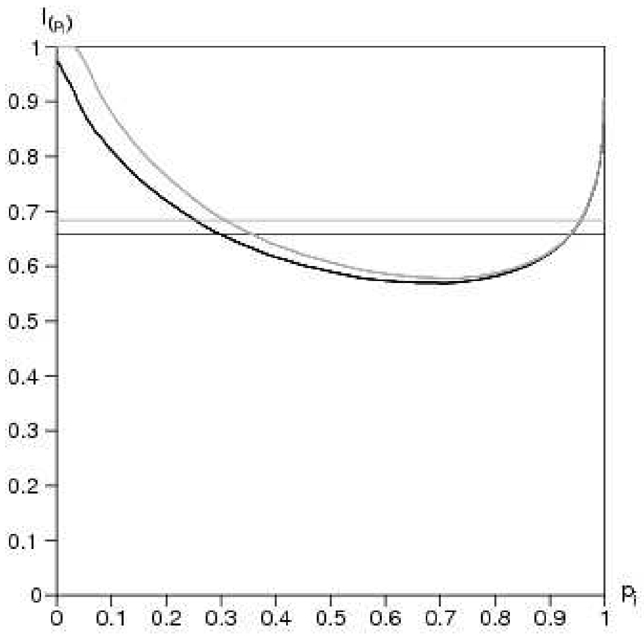

Fig. 12. *Females 31-40. I(p) curves and I inequality indexes.*

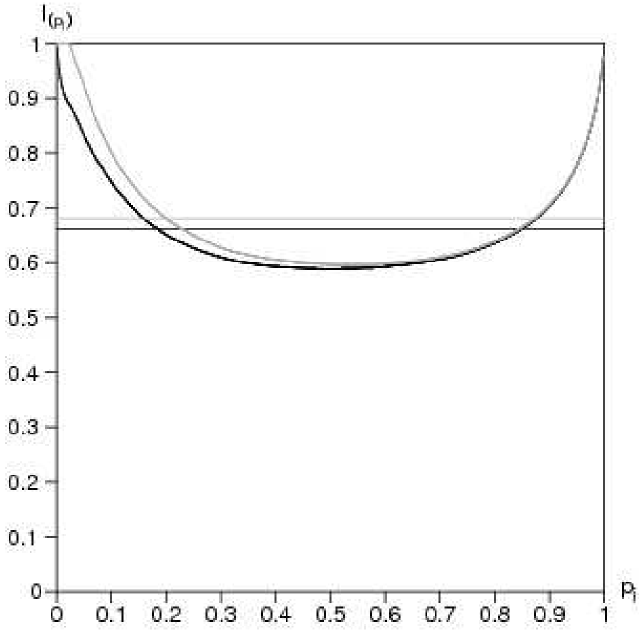

Fig. 13. *Males 31- 40. I(p) curves and I inequality indexes.*

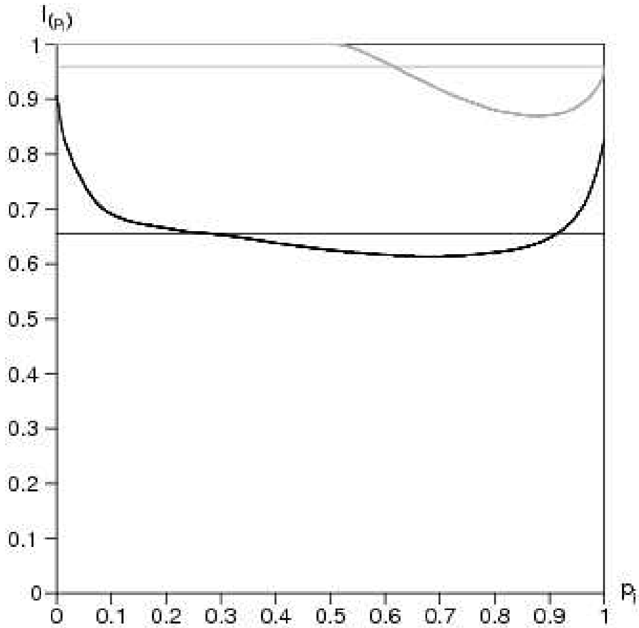

Fig. 14. *Females 66 and over. I(p) curves and I inequality indexes.*

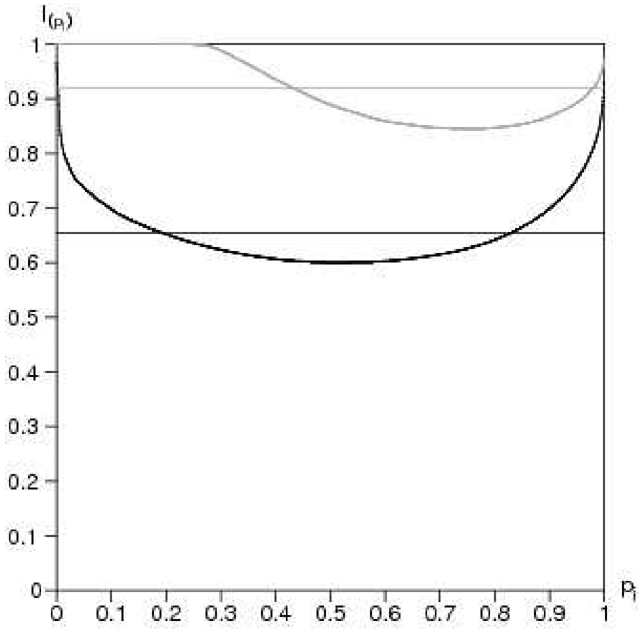

Fig. 15. *Males 66 and over. I(p) curves and I inequality indexes.*

6. CONCLUSIONS

The possibility to analyse the inequality of a frequency distribution through a point measure represents an undoubted advantage in a research. The Lorenz curve constitutes the cornerstone on which other global inequality measures have been accordingly introduced. Nonetheless other point inequality measures have been suggested so far, such as the Bonferroni curve (1930), the $\lambda(p)$ and the $Z(p)$ measure (Zenga, 1984), which have revived the debate about it. By contrasting some of them, it has also been possible to appreciate the advantage of a behaviour not bound to the definition of the measure itself.
The $I(p)$ inequality measure stands out among the others because of its straightforward interpretation, its ease of computing and its not- predetermined behaviour.
Throughout the present study, other positive aspects of the $I(p)$ measure have been pointed out. After examining the interpretative features of the point measure related to a behaviour either increasing, decreasing or constant, we have particularly highlighted its readiness in responding to translation and equalitarian transfers, and we have also suggested comparing the $I(p)$ curve with the Lorenz curve which does not show a similar response. Considering then the global I measure, obtained as weighted mean of the $I(p)$ point measures, we have showed how easily and clearly this measure can be plotted, and how it constitutes the peculiar case of uniform inequality. According to the latter, no matter how you split the population into two adjacent groups, the inequality will be constant and equal to the I level of the global measure, measuring such inequality as the complement to one of the ratio between the means of the two groups. Both those two aspects, immediate plotting of the global measure and interpretation of the uniform inequality curve respectively, are not considered in the other inequality point measures.
In order to explicit the descriptive and interpretative capacity of $I(p)$, some real cases of Italian income distribution have been analysed. The information have been provided by Bank of Italy, namely about the individual income distribution both in case of the total net disposable income, and in the case of net disposable income without pension and net transfers.

It is very interesting to notice how easily the inequality point measure can graphically show the inequality among different groups of population, and how it can express the effect of the transformations that may occur in the income distribution.

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