COMPACTNESS AND EXISTENCE RESULTS FOR DEGENERATE CRITICAL ELLIPTIC EQUATIONS

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Abstract. This paper is devoted to the study of degenerate critical elliptic equations of Caffarelli-Kohn-Nirenberg type. By means of blow-up analysis techniques, we prove an a-priori estimate in a weighted space of continuous functions. From this compactness result, the existence of a solution to our problem is proved by exploiting the homotopy invariance of the Leray-Schauder degree.

1. Introduction

We will consider the following equation in $\mathbb{R}^N$ in dimension $N \geq 3$, which is a prototype of more general nonlinear degenerate elliptic equations describing anisotropic physical phenomena,

$$-\mathrm{div}\,(|x|^{-2\alpha}\nabla v) - \frac{\lambda}{|x|^{2(1+\alpha)}}\, v = K(x)\frac{v^{p-1}}{|x|^{\beta p}}, \quad v \geq 0, \quad v \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \setminus \{0\}, \tag{1.1}$$

where $K \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is positive and

$$\alpha < \frac{N-2}{2}, \quad \alpha \leq \beta < \alpha + 1, \tag{1.2}$$

$$\lambda < \left(\frac{N-2-2\alpha}{2}\right)^2, \quad p = p(\alpha,\beta) = \frac{2N}{N-2(1+\alpha-\beta)}. \tag{1.3}$$

We look for weak solutions in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ defined as the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)} := \left[\int_{\mathbb{R}^N} |x|^{-2\alpha}|\nabla u|^2\, dx\right]^{1/2}.$$

The range of α, β and the definition of p are related to Caffarelli-Kohn-Nirenberg inequalities, denoted by CKN-inequalities in the sequel, (see [5, 6] and the references therein), as for any α, β satisfying (1.2) there exists exactly one exponent $p = p(\alpha,\beta)$

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such that

$$\left(\int_{\mathbb{R}^N}|x|^{-\beta p}|u|^p\,dx\right)^{2/p}\leq \mathcal{C}_{\alpha,\beta}\int_{\mathbb{R}^N}|x|^{-2\alpha}|\nabla u|^2\,dx \qquad \forall u\in C_0^\infty(\mathbb{R}^N). \tag{1.4}$$

Since we are looking for nontrivial nonnegative solutions we must necessarily have that the quadratic form

$$Q(\varphi,\varphi):=\int_{\mathbb{R}^N}|x|^{-2\alpha}|\nabla\varphi|^2-\lambda|x|^{-2(1+\alpha)}|\varphi|^2$$

is positive, that is λ has to be smaller than $(N-2-2\alpha)^2/4$ the best constant in the related Hardy-type CKN-inequality for $\beta=\alpha+1$ and $p=2$. Let us define

$$a(\alpha,\lambda):=\frac{N-2}{2}-\sqrt{\left(\frac{N-2-2\alpha}{2}\right)^2-\lambda}\ \text{ and }\ b(\alpha,\beta,\lambda):=\beta+a(\alpha,\lambda)-\alpha. \tag{1.5}$$

The change of variable $u(x)=|x|^{a-\alpha}v(x)$ shows that equation (1.1) is equivalent to

$$-\text{div}\,(|x|^{-2a}\nabla u)=K(x)\frac{u^{p-1}}{|x|^{bp}},\quad u\geq 0,\quad u\in\mathcal{D}_a^{1,2}(\mathbb{R}^N)\setminus\{0\}, \tag{1.6}$$

where $a=a(\alpha,\lambda)$ and $b=b(\alpha,\beta,\lambda)$, see Lemma A.1 of the Appendix. Clearly, if we replace α by a and β by b then (1.2)-(1.3) still hold and $p(\alpha,\beta)=p(a,b)$. We will write in the sequel for short that a, b and p satisfy (1.2)-(1.3). We will mainly deal with equation (1.6) and look for weak solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$. The advantage of working with (1.6) instead of (1.1) is that we know from [10] that weak solutions of (1.6) are Hölder-continuous in $\mathbb{R}^N$ whereas solutions to (1.1), as our analysis shows, behave (possibly singular) like $|x|^{\alpha-a}$ at the origin. The main difficulty in facing problem (1.6) is the lack of compactness as p is the critical exponent in the related CKN-inequality. More precisely, if K is a positive constant equation (1.6) is invariant under the action of the non-compact group of dilations, in the sense that if u is a solution of (1.6) then for any positive μ the dilated function

$$\mu^{-\frac{N-2-2a}{2}}u(x/\mu)$$

is also a solution with the same norm in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$. The dilation invariance, as we see in (1.16) below, gives rise to a non-compact, one dimensional manifold of solutions for $K\equiv K(0)$.

Our first theorem provides sufficient conditions on K ensuring compactness of the set of solutions by means of an a-priori bound in a weighted space E defined by

$$E:=\mathcal{D}_a^{1,2}(\mathbb{R}^N)\cap C^0(\mathbb{R}^N,(1+|x|^{N-2-2a})),$$

where

$$C^0(\mathbb{R}^N,(1+|x|^{N-2-2a})):=\{u\in C^0(\mathbb{R}^N):\,u(x)(1+|x|^{N-2-2a})\in L^\infty(\mathbb{R}^N)\}$$

is equipped with the norm

$$\|u\|_{C^0(\mathbb{R}^N,(1+|x|^{N-2-2a}))}:=\sup_{x\in\mathbb{R}^N}|u(x)|(1+|x|^{N-2-2a}).$$

We endow E with the norm

$$\|u\|_E=\|u\|_{\mathcal{D}_a^{1,2}(\mathbb{R}^N)}+\|u\|_{C^0(\mathbb{R}^N,(1+|x|^{N-2-2a}))}.$$

The uniform bound in E of the set of solutions to (1.6) will provide the necessary compactness needed in the sequel. We formulate the compactness result in terms of α, β and v the parameters of equation (1.1), where we started from. Let us set

$$\tilde{K}(x) := K(x/|x|^2). \tag{1.7}$$

Theorem 1.1. (Compactness) *Let α, β, λ satisfy (1.2)-(1.3) and*

$$\lambda \geq -\alpha(N-2-\alpha), \tag{1.8}$$

$$\left(\frac{N-2-2\alpha}{2}\right)^2 - 1 < \lambda, \tag{1.9}$$

$$\beta > \alpha, \quad p > \frac{2}{\sqrt{\left(\frac{N-2-2\alpha}{2}\right)^2 - \lambda}}. \tag{1.10}$$

Suppose $K \in C^2(\mathbb{R}^N)$ satisfies

$$\tilde{K} \in C^2(\mathbb{R}^N), \text{ where } \tilde{K}(x) \text{ is defined in (1.7)}, \tag{1.11}$$

$$\nabla K(0) = 0, \quad \Delta K(0) \neq 0, \quad \text{and} \quad \nabla \tilde{K}(0) = 0, \quad \Delta \tilde{K}(0) \neq 0, \tag{1.12}$$

and for some positive constant A_1

$$1/A_1 \leq K(x), \quad \forall\, x \in \mathbb{R}^N. \tag{1.13}$$

Then there is $C_K > 0$ such that for any $t \in (0,1]$ and any solution v_t of

$$\begin{gathered} -\mathrm{div}\,(|x|^{-2\alpha}\nabla v) - \frac{\lambda}{|x|^{2(1+\alpha)}}\, v = (1 + t\,(K(x)-1))\frac{v^{p-1}}{|x|^{\beta p}}, \\ v \geq 0, \ v \in \mathcal{D}^{1,2}_{\alpha}(\mathbb{R}^N) \setminus \{0\}, \end{gathered} \tag{1.14$_t$}$$

we have $\||x|^{a-\alpha} v_t\|_E < C_K$ and

$$C_K^{-1} < |x|^{a-\alpha}(1 + |x|^{N-2-2a}) v_t(x) < C_K \ \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{1.15}$$

To prove the above compactness result we adapt the arguments of [14] to carry out a fine blow-up analysis for (1.6). Assumptions (1.8)-(1.10) imply

$$(1.8) \implies a \geq 0, \qquad (1.9) \implies \tfrac{N-4}{2} < a < \tfrac{N-2}{2}$$

$$(1.10) \implies \tfrac{4}{N-2-2a} < p < 2^* = \tfrac{2N}{N-2}.$$

A key ingredient is the exact knowledge of the solutions to the limit problem with $K \equiv$ const, which is only available for $a \geq 0$. In [8] (see also [18]) it is shown through the method of moving planes that if $a \geq 0$ then any locally bounded positive solution in $C^2(\mathbb{R}^N \setminus \{0\})$ of (1.6) with $K \equiv K(0)$ is of the form

$$z^{a,b}_{K(0),\mu} := \mu^{-\frac{N-2-2a}{2}} z^{a,b}_{K(0)}\left(\frac{x}{\mu}\right), \quad \mu > 0, \tag{1.16}$$

where $z^{a,b}_{K(0)} = z^{a,b}_1\big(x\, K(0)^{\frac{2}{(p-2)(N-2-2a)}}\big)$ and $z^{a,b}_1$ is explicitly given by

$$z^{a,b}_1(x) = \left[1 + \frac{N-2(1+a-b)}{N(N-2-2a)^2}|x|^{\frac{2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}}.$$

For $a<0$ the set of positive solutions becomes more and more complicated as $a\to-\infty$ due to the existence of non-radially symmetric solutions (see [6, 7, 9]). Up to now, our blow-up analysis is only available for $p<2^*$; the case $p=2^*$ presents additional difficulties because besides the blow-up profile $z_1^{a,b}$ a second blow-up profile described by the usual Aubin-Talenti instanton of Yamabe-type equations may occur. The further restrictions on a, p and K should be compared to the so-called flatness-assumptions in problems of prescribing scalar curvature.

Non-existence results for equation (1.6) can be obtained using a Pohozaev-type identity, i.e. any solution u to (1.6) satisfies the following identity

$$\int_{\mathbb{R}^N}(\nabla K(x)\cdot x)\frac{u^p}{|x|^{bp}}\,dx=0,$$

provided the integral is convergent and K is bounded and smooth enough (see Corollary 2.3). This implies that there are no such solutions if $\nabla K(x)\cdot x$ does not change sign in $\mathbb{R}^N$ and K is not constant.

The above compactness result allows us to exploit the homotopy invariance of the Leray-Schauder degree to pass from t small to $t=1$ in $(1.14)_t$. We compute the degree of positive solutions to $(1.14)_t$ for small t using a Melnikov-type function introduced in [2, 3] and show that it equals (see Theorem 5.3)

$$-\frac{\operatorname{sgn}\Delta K(0)+\operatorname{sgn}\Delta\tilde K(0)}{2}.$$

In particular, we prove the following existence result.

Theorem 1.2. (Existence) *Under the assumptions of Theorem 1.1, if, moreover, $p>3$ and*

$$\operatorname{sgn}\Delta K(0)+\operatorname{sgn}\Delta\tilde K(0)\neq 0$$

then equation (1.1) has a positive solution v such that $|x|^{a-\alpha}v\in B_{C_K}(0)\subset E$ and v satisfies (1.15).

The assumption $p>3$ is essentially technical and yields C^3 regularity of the functional associated to the problem which is needed in the computation of the degree.

In [9] problem (1.1) is studied in the case in which K is a small perturbation of a constant, i.e. in the case $K=1+\varepsilon k$, using a perturbative method introduced in [2, 3]. We extend some of the results in [9] to the nonperturbative case. Problem (1.1) for $\alpha=\beta=0$ (hence $p=2^*$) and $0<\lambda<(N-2)^2/4$ is treated by Smets [17] who proves that in dimension $N=4$ there exists a positive solution provided $K\in C^2$ is positive and $K(0)=\lim_{|x|\to\infty}K(x)$. Among other existence and multiplicity results, in [1] positive solutions to (1.1) for $\alpha=\beta=0$, $p=2^*$, and $0<\lambda<(N-2)^2/4$ are found via the concentration compactness argument, under assumptions ensuring that the mountain-pass level stays below the compactness threshold at which Palais-Smale condition fails. We emphasize that the solution we find in Theorem 1.2 can stay above such a threshold.

Remark 1.3. *If we drop the assumption $\alpha<\frac{N-2}{2}$ we may still change the variables $u(x)=|x|^{a-\alpha}v(x)$, where a is given in (1.5), and we still obtain weak solutions u of (1.6) in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$. But in this case the transformation $v(x)=|x|^{\alpha-a}u(x)$ gives rise* *only to classical solutions of (1.1) in* $\mathbb{R}^N \setminus \{0\}$ *but not to distributional solutions in the whole* $\mathbb{R}^N$.

The paper is organized as follows. In Section 2 we prove a Pohozaev type identity for equation (1.6). In Section 3 we introduce the notion of isolated and isolated simple blow-up point which was first introduced by Schoen [16] and provide the main local blow-up analysis. In Section 4 we prove Theorem 1.1 by combining the Pohozaev type identity with the results of our local blow-up analysis. Section 5 is devoted to the computation of the Leray-Schauder degree and to the proof of the existence theorem. Finally in the Appendix we collect some technical lemmas.

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2. A Pohozaev-type identity

Theorem 2.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ $(N \geq 3)$ *with smooth boundary,* a*,* b*, and* p *satisfy (1.2)-(1.3),* $K \in C^1(\overline{\Omega})$ *and* $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ *be a weak positive solution of*

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad x \in \Omega. \tag{2.1}$$

There holds

$$\frac{1}{p}\int_\Omega (\nabla K \cdot x)\,\frac{u^p}{|x|^{bp}}\,dx - \frac{1}{p}\int_{\partial\Omega} K(x)\frac{u^p}{|x|^{bp}}x\cdot\nu = \frac{N-2-2a}{2}\int_{\partial\Omega}|x|^{-2a}u\nabla u\cdot\nu$$
$$-\frac{1}{2}\int_{\partial\Omega}|x|^{-2a}|\nabla u|^2 x\cdot\nu + \int_{\partial\Omega}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu)$$

where ν *denotes the unit normal of the boundary.*

Proof. Note that

$$\int_0^1 ds\int_{\partial B_s(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{2a}}\right] = \int_{B_1(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{2a}}\right] < \infty$$

which implies that there exists a sequence $\varepsilon_n \to 0^+$ such that

$$\varepsilon_n\int_{\partial B_{\varepsilon_n}(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{2a}}\right] \to 0 \tag{2.2}$$

as $n \to \infty$. Let $\Omega_{\varepsilon_n} := \Omega\setminus B_{\varepsilon_n}(0)$. Multiplying equation (2.1) by $x\cdot\nabla u$ and integrating over Ω_{ε_n} we obtain

$$-\sum_{j,k=1}^N\int_{\Omega_{\varepsilon_n}}\frac{\partial}{\partial x_j}\left(|x|^{-2a}\frac{\partial u}{\partial x_j}\right)x_k\,\frac{\partial u}{\partial x_k}\,dx = \sum_{k=1}^N\int_{\Omega_{\varepsilon_n}}x_k\,\frac{\partial u}{\partial x_k}\,K(x)\frac{u^{p-1}}{|x|^{bp}}\,dx. \tag{2.3}$$

Let us first consider the right-hand side of (2.3). Integrating by parts we have

$$\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}} x_k\,\frac{\partial u}{\partial x_k}\,K(x)\frac{u^{p-1}}{|x|^{bp}}\,dx=\left(b-\frac{N}{p}\right)\int_{\Omega_{\varepsilon_n}}K(x)\frac{u^p}{|x|^{bp}}\,dx$$
$$-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega_{\varepsilon_n}}x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}. \tag{2.4}$$

Integrating by parts in the left-hand side of (2.3), we obtain

$$-\sum_{j,k=1}^{N}\int_{\Omega_{\varepsilon_n}}\frac{\partial}{\partial x_j}\left(|x|^{-2a}\frac{\partial u}{\partial x_j}\right)x_k\,\frac{\partial u}{\partial x_k}\,dx=-\frac{N-2-2a}{2}\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx$$
$$+\frac{1}{2}\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2 x\cdot\nu-\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu). \tag{2.5}$$

From (2.3), (2.4), and (2.5), we have

$$\begin{aligned}\left(b-\frac{N}{p}\right)&\int_{\Omega_{\varepsilon_n}}K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\\&+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega_{\varepsilon_n}}x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}\\&=-\frac{N-2-2a}{2}\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2 x\cdot\nu\\&\quad-\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu).\end{aligned}$$

Because of the integrability of $|x|^{-bp}u^p$ and of $|x|^{-2a}|\nabla u|^2$, it is clear that

$$\begin{aligned}\left(b-\frac{N}{p}\right)&\int_{\Omega_{\varepsilon_n}}K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\\&\underset{\varepsilon\to0^+}{\longrightarrow}\left(b-\frac{N}{p}\right)\int_{\Omega}K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega}u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\end{aligned}$$

and

$$\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx\underset{\varepsilon\to0^+}{\longrightarrow}\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx.$$

Hence, in view of (2.2), we have

$$\left(b-\frac{N}{p}\right)\int_{\Omega}K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega}u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}$$

$$+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega}x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}$$

$$=-\frac{N-2-2a}{2}\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega}|x|^{-2a}|\nabla u|^2 x\cdot\nu$$

$$-\int_{\partial\Omega}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu). \tag{2.6}$$

Multiplying equation (2.1) by u and integrating by parts, we have

$$\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx=\int_{\partial\Omega}|x|^{-2a}u\frac{\partial u}{\partial\nu}+\int_{\Omega}K(x)\frac{u^p}{|x|^{bp}}\,dx. \tag{2.7}$$

The conclusion follows from (2.6), (2.7), and from the identity $\frac{N-bp}{p}-\frac{N-2-2a}{2}=0$. □

Corollary 2.2. *If a, b, and p satisfy (1.2)-(1.3), $K\in C^1(\overline{B}_\sigma)$ and u be a weak positive solution in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ of*

$$-\mathrm{div}\,(|x|^{-2a}\nabla u)=K(x)\frac{u^{p-1}}{|x|^{bp}},\quad x\in B_\sigma:=\{x\in\mathbb{R}^N:\ |x|<\sigma\} \tag{2.8}$$

then

$$\frac{1}{p}\int_{B_\sigma}(\nabla K\cdot x)\,\frac{u^p}{|x|^{bp}}\,dx-\frac{\sigma}{p}\int_{\partial B_\sigma}K(x)\frac{u^p}{|x|^{bp}}=\int_{\partial B_\sigma}B(\sigma,x,u,\nabla u) \tag{2.9}$$

where

$$B(\sigma,x,u,\nabla u)=\frac{N-2-2a}{2}\,|x|^{-2a}u\,\frac{\partial u}{\partial\nu}-\frac{\sigma}{2}\,|x|^{-2a}|\nabla u|^2+\sigma\,|x|^{-2a}\left(\frac{\partial u}{\partial\nu}\right)^2.$$

Corollary 2.3. *Let u be a weak positive solution in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ of*

$$-\mathrm{div}\,(|x|^{-2a}\nabla u)=K(x)\frac{u^{p-1}}{|x|^{bp}},\quad x\in\mathbb{R}^N$$

where a, b, and p satisfy (1.2)-(1.3) and $K\in L^\infty\cap C^1(\mathbb{R}^N)$, $|\nabla K(x)\cdot x|\leq\mathrm{const.}$ Then

$$\int_{\mathbb{R}^N}(\nabla K(x)\cdot x)\frac{u^p}{|x|^{bp}}\,dx=0. \tag{2.10}$$

Proof. Since

$$\int_0^{+\infty}ds\int_{\partial B_s}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{2a}}\right]=\int_{\mathbb{R}^N}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{-2a}}\right]<\infty$$

there exists a sequence $R_n\to+\infty$ such that

$$R_n\int_{\partial B_{R_n}}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{2a}}\right]\underset{n\to\infty}{\longrightarrow}0. \tag{2.11}$$

From Corollary 2.2 we have that

$$\frac{1}{p}\int_{B_{R_n}} (\nabla K\cdot x)\,\frac{u^p}{|x|^{bp}}\,dx = \frac{R_n}{p}\int_{\partial B_{R_n}} K(x)\frac{u^p}{|x|^{bp}} + \frac{N-2-2a}{2}\int_{\partial B_{R_n}} |x|^{-2a}u\,\frac{\partial u}{\partial\nu} - \frac{R_n}{2}\int_{\partial B_{R_n}} |x|^{-2a}|\nabla u|^2 + R_n\int_{\partial B_{R_n}} |x|^{-2a}\left(\frac{\partial u}{\partial\nu}\right)^2. \tag{2.12}$$

In view of (2.11) and noting that from Hölder inequality

$$\begin{aligned}\int_{\partial B_{R_n}} |x|^{-2a}u\,\frac{\partial u}{\partial\nu} &= R_n^{b-a}\int_{\partial B_{R_n}} \frac{u}{|x|^b}\cdot\frac{\nabla u\cdot\nu}{|x|^a}\\ &\leq |\mathbb{S}^N|^{\frac{p-2}{2p}} R_n^{b-a+\frac{(N-1)(p-2)}{2p}-\frac{1}{p}-\frac{1}{2}}\left(R_n\int_{\partial B_{R_n}} \frac{u^p}{|x|^{bp}}\right)^{\frac{1}{p}}\left(R_n\int_{\partial B_{R_n}} \frac{|\nabla u|^2}{|x|^{2a}}\right)^{\frac{1}{2}}\\ &= |\mathbb{S}^N|^{\frac{p-2}{2p}}\left(R_n\int_{\partial B_{R_n}} \frac{u^p}{|x|^{bp}}\right)^{\frac{1}{p}}\left(R_n\int_{\partial B_{R_n}} \frac{|\nabla u|^2}{|x|^{2a}}\right)^{\frac{1}{2}}\end{aligned}$$

we can pass to the limit in (2.12) thus obtaining the claim. □

It is easy to check that the boundary term $B(\sigma,x,u,\nabla u)$ has the following properties.

Proposition 2.4.

(*i*) *For* $u(x)=|x|^{2+2a-N}$, $\sigma>0$, $B(\sigma,x,u,\nabla u)=0$ *for all* $x\in\partial B_\sigma$.

(*ii*) *For* $u(x)=|x|^{2+2a-N}+A+\zeta(x)$, *with* $A>0$ *and* $\zeta(x)$ *some function differentiable near* 0 *satisfying* $\zeta(0)=0$, *there exists* $\bar\sigma$ *such that*

$$B(\sigma,x,u,\nabla u)<0 \quad \textit{for all } x\in\partial B_\sigma \textit{ and } 0<\sigma<\bar\sigma$$

and

$$\lim_{\sigma\to 0}\int_{\partial B_\sigma} B(\sigma,x,u,\nabla u) = -\frac{(N-2-2a)^2}{2}A|\mathbb{S}^{N-1}|.$$

3. Local blow-up analysis

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, a, b, and p satisfy (1.2)-(1.3), and $\{K_i\}_i\subset C(\Omega)$ satisfy, for some constant $A_1>0$,

$$1/A_1\leq K_i(x)\leq A_1, \quad \forall\, x\in\Omega \quad\text{and}\quad K_i\to K \text{ uniformly in } \Omega. \tag{3.1}$$

Moreover, we will assume throughout this section that $a\geq 0$. We are interested in the family of problems

$$-\text{div}\,(|x|^{-2a}\nabla u) = K_i(x)\frac{u^{p-1}}{|x|^{bp}} \quad \text{weakly in } \Omega, \quad u>0 \text{ in } \Omega,\ u\in\mathcal{D}^{1,2}_a(\mathbb{R}^N). \tag{P_i}$$

Definition 3.1. *Let* $\{u_i\}_i$ *be a sequence of solutions of* (P_i). *We say that* $0\in\Omega$ *is a blow-up point of* $\{u_i\}_i$ *if there exists a sequence* $\{x_i\}_i$ *converging to* 0 *such that*

$$u_i(x_i)\to+\infty \quad \textit{and} \quad u_i(x_i)^{\frac{2}{N-2-2a}}|x_i|\to 0 \quad \textit{as } i\to+\infty. \tag{3.2}$$

Definition 3.2. *Let $\{u_i\}_i$ be a sequence of solutions of (P_i). The point 0 is said to be an isolated blow-up point of $\{u_i\}_i$ if there exist $0 < \bar r < dist(0, \partial\Omega)$, $\bar C > 0$, and a sequence $\{x_i\}_i$ converging to 0 such that $u_i(x_i) \to +\infty$, $u_i(x_i)^{\frac{2}{N-2-2a}}|x_i| \to 0$ as $i \to +\infty$, and for any $x \in B_{\bar r}(x_i)$*

$$u_i(x) \le \bar C\, |x - x_i|^{-\frac{N-2-2a}{2}}$$

where $B_{\bar r}(x_i) := \{x \in \Omega : \ |x - x_i| < \bar r\}$.

If 0 is an isolated blow-up point of $\{u_i\}_i$ we define

$$\bar{u}_i(r) = \fint_{\partial B_r(x_i)} u_i = \frac{1}{|\partial B_r(x_i)|}\int_{\partial B_r(x_i)} u_i, \quad r > 0$$

and

$$\bar{w}_i(r) = r^{\frac{N-2-2a}{2}}\,\bar{u}_i(r), \quad r > 0. \tag{3.3}$$

Definition 3.3. *The point 0 is said to be an isolated simple blow-up point of $\{u_i\}_i$ if it is an isolated blow-up point and there exist some positive $\rho \in (0, \bar r)$ independent of i and $\tilde C > 1$ such that*

$$\bar{w}_i'(r) < 0 \quad \text{for } r \text{ satisfying } \tilde C\, u_i(x_i)^{-\frac{2}{N-2-2a}} \le r \le \rho. \tag{3.4}$$

Let us now introduce the notion of blow-up at infinity. To this aim, we consider the Kelvin transform,

$$\tilde{u}_i(x) = |x|^{-(N-2-2a)} u_i\left(\frac{x}{|x|^2}\right), \tag{3.5}$$

which is an isomorphism of $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$. If u_i solves (P_i) in a neighborhood of ∞, i.e. $\Omega = \mathbb{R}^N \setminus D$ for some compact set D, then $\tilde u_i$ is a solution of (P_i) where K_i is replaced by $\tilde K_i(x) = K_i(x/|x|^2)$ and Ω by $\tilde\Omega = \mathbb{R}^N \setminus \{x/|x|^2 \mid x \in D\}$, a neighborhood of 0.

Definition 3.4. *Let $\{u_i\}_i$ be a sequence of solutions of (P_i) in a neighborhood of ∞. We say that ∞ is a blow-up point (respectively an isolated blow-up point, an isolated simple blow-up point) if 0 is a blow-up point (respectively an isolated blow-up point, an isolated simple blow-up point) of the sequence $\{\tilde u_i\}_i$ defined by the Kelvin transform (3.5).*

Remark 3.5. *It is easy to see that ∞ is a blow-up point of $\{u_i\}_i$ if and only if there exists a sequence $\{x_i\}_i$ such that $|x_i| \to \infty$ as $i \to +\infty$ and*

$$|x_i|^{N-2-2a} u_i(x_i) \underset{i\to+\infty}{\longrightarrow} \infty \quad \text{and} \quad |x_i| u_i(x_i)^{\frac{2}{N-2-2a}} \underset{i\to+\infty}{\longrightarrow} 0.$$

In the sequel we will use the notation c to denote a positive constant which may vary from line to line.

Lemma 3.6. *Let $(K_i)_{i\in\mathbb{N}}$ satisfy (3.1), $\{u_i\}_i$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then there is a positive constant $C = C(N, \bar C, A_1)$ such that for any $0 < r < \min(\bar r/3, 1)$ there holds*

$$\max_{x\in B_{2r}(x_i)\setminus B_{r/2}(x_i)} u_i(x) \le C \min_{x\in B_{2r}(x_i)\setminus B_{r/2}(x_i)} u_i(x). \tag{3.6}$$

Proof. We define $v_i(x) := r^{\frac{N-2-2a}{2}} u_i(rx + x_i)$. Then v_i satisfies in $B_3(0)$

$$0 < v_i(x) < \bar{C}|x|^{-\frac{N-2-2a}{2}}, \tag{3.7}$$

and

$$\begin{aligned}-\mathrm{div}\left(|x + r^{-1}x_i|^{-2a}\nabla v_i(x)\right) &= -r^{\frac{N-2-2a}{2}+2+2a}\mathrm{div}\left(|\cdot|^{-2a}\nabla u_i(\cdot)\right)(rx + x_i)\\ &= K_i(rx + x_i)\ |x + r^{-1}x_i|^{-bp}\ v_i^{p-1}(x),\end{aligned}$$

since

$$\frac{N-2-2a}{2} + 2 + 2a - bp - (p-1)\frac{N-2-2a}{2} = N - p\Big(\frac{N-2(1+a-b)}{2}\Big) = 0.$$

To prove the claim we use a weighted version of Harnack's inequality applied to v_i and

$$-\mathrm{div}\left(|x + r^{-1}x_i|^{-2a}\nabla v_i(x)\right) - W_i(x)v_i(x) = 0 \quad \text{ in } B_{9/4}(0) \setminus B_{1/4}(0),$$

where $W_i(x) := K_i(rx + x_i)\ |x + r^{-1}x_i|^{-bp}\ v_i^{p-2}(x)$. From (3.7) the function v_i is uniformly bounded in $B_{9/4}(0)\setminus B_{1/4}(0)$ and the claim follows from Harnack's inequality in [11]. We mention that $|\cdot + r^{-1}x_i|^{-bp}$ belongs to the class of potentials required in [11] (see Lemma A.3 of the Appendix). □

Proposition 3.7. *Let $\{K_i\}_i$ satisfy (3.1), $\{u_i\}_i$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence that:*

$$R_i u_i(x_i)^{-\frac{2}{N-2-2a}} \to 0 \ \textit{as}\ i \to \infty, \tag{3.8}$$

$$\|u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}\cdot + x_i\big) - z^{a,b}_{K(0)}(\cdot)\|_{C^{0,\gamma}(B_{2R_i}(0))} \le \varepsilon_i, \tag{3.9}$$

$$\|u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}\cdot + x_i\big) - z^{a,b}_{K(0)}(\cdot)\|_{H^1_a(B_{2R_i}(0))} \le \varepsilon_i, \tag{3.10}$$

where $H^1_a(B_{2R_i}(0))$ is the weighted Sobolev space $\{u : |x|^{-a}|\nabla u|, |x|^{-a}u \in L^2(B_{2R_i}(0))\}$.

Proof. Consider

$$\varphi_i(x) = u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big), \quad |x| < \bar{r}u_i(x_i)^{\frac{2}{N-2-2a}}.$$

We have

$$\begin{aligned}&-\mathrm{div}\left(\left|x + u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{-2a}\nabla\varphi_i(x)\right)\\ &\qquad = K_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big)\ \left|x + u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{-bp}\ \varphi_i^{p-1}(x).\end{aligned}$$

Moreover, from the definition of isolated blow-up

$$\varphi_i(0) = 1, \quad 0 < \varphi_i(x) \le \bar{C}|x|^{-\frac{N-2-2a}{2}} \qquad \text{for } |x| < \bar{r}\,u_i(x_i)^{\frac{2}{N-2-2a}}. \tag{3.11}$$

Lemma 3.6 shows that for large i and for any $0 < r < 1$ we have

$$\max_{\partial B_r}\varphi_i \le C\min_{\partial B_r}\varphi_i, \tag{3.12}$$

where $C = C(N, \bar{C}, A_1)$. Since

$$-\text{div}\left(\left|x + u_i(x_i)^{\frac{2}{N-2-2a}} x_i\right|^{-2a} \nabla\varphi_i(x)\right) \geq 0 \text{ and } \varphi_i(0) = 1$$

we may use (3.12) and the minimum principle for $|x|^{-2a}$-superharmonic functions in [12, Thm 7.12] to deduce that

$$\varphi_i(x) \leq C \quad \text{ in } B_1(0). \tag{3.13}$$

From (3.11), (3.13) and regularity results in [10] the functions φ_i are uniformly bounded in $C^{0,\gamma}_{loc}(\mathbb{R}^N)$ and $H^1_{a,loc}(\mathbb{R}^N)$ for some $\gamma \in (0,1)$. Since point-concentration is ruled out by the L^∞-bound, there is some positive function $\varphi \in C^{0,\gamma'}_{loc}(\mathbb{R}^N) \cap H^1_{a,loc}(\mathbb{R}^N)$ and some $\gamma' \in (0,1)$ such that

$$\begin{aligned}
&\varphi_i \to \varphi \text{ in } C^{0,\gamma'}_{loc}(\mathbb{R}^N) \cap H^1_{a,loc}(\mathbb{R}^N),\\
&-\text{div}\left(|x|^{-2a}\nabla\varphi\right) = \lim_{i\to\infty} K_i(x_i)\frac{\varphi^{p-1}}{|x|^{bp}}\\
&\varphi(0) = 1.
\end{aligned}$$

By uniqueness of the solutions proved in [8] we deduce that $\varphi = z^{a,b}_{K(0)}$. □

Remark 3.8. *From the proof of Proposition 3.7 one can easily check that if $x_i \to 0$ is an isolated blow-up point then there exists a positive constant C, depending on $\lim_{i\to\infty} K_i(x_i)$ and a, b, and N, such that the function $\bar{w}_i$ defined in (3.3) is strictly decreasing for $Cu_i(x_i)^{-2/(N-2-2a)} \leq r \leq r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$ (see Lemma A.2 of the Appendix).*

Lemma 3.9. *Let $x_i \to 0$ be a blow-up point. Then for any x such that $|x - x_i| \geq r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$ we have*

$$|x - x_i| = |x|(1 + o(1)).$$

In particular, $x_i \in B_{r_i}(0)$.

Proof. The assumption $\left|x_i u_i(x_i)^{\frac{2}{N-2-2a}}\right| = o(1)$ implies that $|x_i| = r_i\, o(1)$. Hence

$$|x| \geq |x - x_i| - |x_i| \geq r_i - r_i\, o(1) = r_i(1 + o(1)).$$

Therefore

$$\frac{|x_i|}{|x|} \leq \frac{r_i\, o(1)}{r_i(1 + o(1))} = o(1)$$

and hence

$$\left|\frac{x - x_i}{|x|}\right| = \left|\frac{x}{|x|} - \frac{x_i}{|x|}\right| \underset{i\to+\infty}{\longrightarrow} 1$$

thus proving the lemma. □

Proposition 3.10. *Suppose $\{K_i\}_i \subset C^1_{\text{loc}}(B_2)$ satisfy (3.1) with $\Omega = B_2$ and*

$$|\nabla K_i(x)| \leq A_2 \textit{ for all } x \in B_2. \tag{3.14}$$

Let u_i satisfy (P_i) with $\Omega = B_2$ and suppose that $x_i \to 0$ is an isolated simple blow-up point such that

$$|x - x_i|^{\frac{N-2-2a}{2}} u_i(x) \le A_3 \text{ for all } x \in B_2. \tag{3.15}$$

Then there exists $C = C(N, a, b, A_1, A_2, A_3, \bar{C}, \rho) > 0$ such that

$$u_i(x) \le C\, u_i(x_i)^{-1} |x - x_i|^{2+2a-N} \quad \text{for all } |x - x_i| \le 1. \tag{3.16}$$

Furthermore there exists a Hölder continuous function $B(x)$ (smooth outside 0) satisfying $\operatorname{div}(|x|^{-2a}\nabla B) = 0$ in B_1, such that, after passing to a subsequence,

$$u_i(x_i)u_i(x) \to h(x) = A|x|^{2+2a-N} + B(x) \quad \text{in } C^2_{\text{loc}}(B_1 \setminus \{0\})$$

where

$$A = \frac{K(0)}{(N-2-2a)|\mathbb{S}^N|} \int_{\mathbb{R}^N} \frac{\big(z^{a,b}_{K(0)}\big)^{p-1}}{|x|^{bp}}\, dx.$$

Lemma 3.11. *Under the assumption of Proposition 3.10 without (3.14) there exist a positive $\delta_i = O\Big(R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\Big)$ and $C = C(N, a, b, A_1, A_2, \bar{C}, \rho) > 0$ such that*

$$u_i(x) \le C\, u_i(x_i)^{-\lambda_i} |x - x_i|^{2+2a-N+\delta_i} \quad \text{for all } r_i \le |x - x_i| \le 1, \tag{3.17}$$

where $\lambda_i := 1 - 2\delta_i/(N-2-2a)$.

Proof. It follows from Proposition 3.7 that

$$u_i(x) \le c u_i(x_i) R_i^{2a+2-N} \text{ for } |x - x_i| = r_i. \tag{3.18}$$

From the definition of isolated simple blow-up in (3.4) there exists $\rho > 0$ such that

$$r^{\frac{N-2-2a}{2}} \bar{u}_i \text{ is strictly decreasing in } r_i < r < \rho. \tag{3.19}$$

From (3.18), (3.19) and Lemma 3.6 it follows that for all $r_i \le |x - x_i| < \rho$

$$|x - x_i|^{\frac{N-2-2a}{2}} u_i(x) \le c|x - x_i|^{\frac{N-2-2a}{2}} \bar{u}_i(|x - x_i|) \le c r_i^{\frac{N-2-2a}{2}} \bar{u}_i(r_i) \le c R_i^{\frac{2+2a-N}{2}}.$$

Therefore for $r_i < |x - x_i| < \rho$

$$u_i(x)^{\frac{4}{N-2-2a}} \le c R_i^{-2} |x - x_i|^{-2}. \tag{3.20}$$

Consider the following degenerated elliptic operator

$$\mathcal{L}_i \varphi = \operatorname{div}(|x|^{-2a}\nabla\varphi) + K_i(x)|x|^{-bp} u_i(x)^{p-2}\varphi.$$

Clearly $u_i > 0$ solves $\mathcal{L}_i u_i = 0$. Hence $-\mathcal{L}_i$ is nonnegative and the maximum principle holds for $\mathcal{L}_i$. Direct computations show for any $0 \le \mu \le N-2-2a$

$$\operatorname{div}(|x|^{-2a}\nabla(|x|^{-\mu})) = -\mu(N-2-2a-\mu)|x|^{-2-2a-\mu} \text{ for } x \ne 0. \tag{3.21}$$

From (3.20), (3.21) and Lemma 3.9 we infer

$$\mathcal{L}_i(|x|^{-\mu}) \le \Big(-\mu(N-2-2a-\mu) + c R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\Big)|x|^{-2-2a-\mu}.$$

We can choose $\delta_i = O(R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}})$ such that

$$\max(\mathcal{L}_i(|x|^{-\delta_i}), \mathcal{L}_i(|x|^{2a+2-N+\delta_i})) \le 0. \tag{3.22}$$

Set $M_i := 2\max_{\partial B_\rho(x_i)} u_i$, $\lambda_i = 1 - 2\delta_i/(N-2-2a)$, and

$$\varphi_i(x) := M_i \rho^{\delta_i} |x|^{-\delta_i} + A u_i(x_i)^{-\lambda_i} |x|^{2+2a-N+\delta_i} \text{ for } r_i \le |x - x_i| \le \rho, \tag{3.23}$$

where A will be chosen later. We will apply the maximum principle to compare φ_i and u_i. By the choice of M_i and Lemma 3.9 we infer for i sufficiently large

$$\varphi_i(x) \ge \frac{M_i}{2} \ge u_i(x) \text{ for } |x - x_i| = \rho.$$

On the inner boundary $|x - x_i| = r_i$ we have by (3.18) and for A large enough

$$\begin{aligned}\varphi_i(x) &\ge A(1+o(1)) u_i(x_i)^{-\lambda_i} r_i^{2+2a-N+\delta_i} = A(1+o(1)) R_i^{2+2a-N+\delta_i} u_i(x_i)^{2-\frac{2\delta_i}{N-2-2a}-\lambda_i} \\ &\ge A(1+o(1)) R_i^{2+2a-N} u_i(x_i) \ge u_i(x).\end{aligned}$$

Now we obtain from the maximum principle in the annulus $r_i \le |x - x_i| \le \rho$ that

$$u_i(x) \le \varphi_i(x) \text{ for all } r_i \le |x - x_i| \le \rho. \tag{3.24}$$

It follows from (3.19), (3.24) and Lemma 3.6 that for any $r_i \le \theta \le \rho$ we have

$$\begin{aligned}\rho^{\frac{N-2-2a}{2}} M_i &\le c \rho^{\frac{N-2-2a}{2}} \bar{u}_i(\rho) \le c \theta^{\frac{N-2-2a}{2}} \bar{u}_i(\theta) \\ &\le c \theta^{\frac{N-2-2a}{2}} \big(M_i \rho^{\delta_i} \theta^{-\delta_i} + A u_i(x_i)^{-\lambda_i} \theta^{2+2a-N+\delta_i} \big).\end{aligned}$$

Choose $\theta = \theta(\rho, c)$ such that

$$c \theta^{\frac{N-2-2a}{2}} \rho^{\delta_i} \theta^{-\delta_i} < \frac{1}{2} \rho^{\frac{N-2-2a}{2}}.$$

Then we have

$$M_i \le c u_i(x_i)^{-\lambda_i},$$

which, in view of (3.24) and the definition of φ_i in (3.23), proves (3.17) for $r_i \le |x - x_i| \le \rho$. The Harnack inequality in Lemma 3.6 allows to extend (3.17) for $r_i \le |x - x_i| \le 1$. □

Proof of Proposition 3.10. The inequality of Proposition 3.10 for $|x - x_i| \le r_i$ follows immediately for Proposition 3.7. Let $e \in \mathbb{R}^N$, $|e| = 1$ and consider the function

$$v_i(x) = u_i(x_i + e)^{-1} u_i(x).$$

Clearly v_i satisfies the equation

$$-\operatorname{div}(|x|^{-2a} \nabla v_i) = u_i(x_i + e)^{p-2} K_i(x) \frac{v_i^{p-1}}{|x|^{bp}} \quad \text{in } B_{4/3}. \tag{3.25}$$

Applying the Harnack inequality of Lemma 3.6 on v_i, we obtain that v_i is bounded on any compact set not containing 0. By standard elliptic theories, it follows that, up to a subsequence, $\{v_i\}_i$ converges in $C^2_{\text{loc}}(B_2 \setminus \{0\})$ to some positive function $v \in C^2(B_2 \setminus \{0\})$. Since $u_i(x_i + e) \to 0$ due to Lemma 3.11, we can pass to the limit in (3.25) thus obtaining

$$-\operatorname{div}(|x|^{-2a} \nabla v) = 0 \quad \text{in } B_2 \setminus \{0\}.$$

We claim that v has a singularity at 0. Indeed, from Lemma 3.6 and standard elliptic theories, for any $0 < r < 2$ we have that

$$\lim_{i\to\infty} u_i(x_i+e)^{-1} r^{\frac{N-2-2a}{2}} \bar{u}_i(r) = r^{\frac{N-2-2a}{2}} \bar{v}(r)$$

where $\bar{v}(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} v$. Since the blow-up is simple isolated, $r^{\frac{N-2-2a}{2}}\bar{v}(r)$ is non-increasing for $0 < r < \rho$ and this would be impossible in the case in which v is regular at 0. It follows that v is singular at 0 and hence from the Bôcher-type Theorem proved in the Appendix (see Theorem A.4)

$$v(x) = a_1|x|^{2+2a-N} + b_1(x)$$

where $a_1 > 0$ is some positive constant and $b_1(x)$ is some Hölder continuous function in B_2 such that $-\mathrm{div}\,(|x|^{-2a}\nabla b_1) = 0$.

Let us first establish the inequality in Proposition 3.10 for $|x - x_i| = 1$. Namely we prove that

$$u_i(x_i+e) \le c\, u_i(x_i)^{-1}. \tag{3.26}$$

By contradiction, suppose that (3.26) fails. Then along a subsequence, we have

$$\lim_{i\to\infty} u_i(x_i+e)u_i(x_i) = \infty. \tag{3.27}$$

Multiplying (P_i) by $u_i(x_i+e)^{-1}$ and integrating on B_1, we get

$$-\int_{\partial B_1} |x|^{-2a}\frac{\partial v_i}{\partial \nu} = \int_{B_1} K_i(x)\frac{u_i^{p-1}}{|x|^{bp}} u_i(x_i+e)^{-1}\,dx. \tag{3.28}$$

From the properties of b_1 and the convergence of v_i to v, we know that

$$\begin{aligned}\lim_{i\to\infty}\int_{\partial B_1} |x|^{-2a}\frac{\partial v_i}{\partial \nu} &= \int_{\partial B_1} |x|^{-2a}\frac{\partial}{\partial \nu}\big(a_1|x|^{2+2a-N} + b_1(x)\big)\\ &= -a_1(N-2-2a)|\mathbb{S}^N| < 0. \end{aligned}\tag{3.29}$$

From Proposition 3.7 there holds

$$\int_{|x-x_i|\le r_i} \frac{K_i(x)u_i^{p-1}}{|x|^{bp}}\,dx \le C\,u_i(x_i)^{-1} \tag{3.30}$$

while from Lemma 3.11 and Lemma 3.9 we have that

$$\begin{aligned}\int_{r_i\le|x-x_i|\le 1} \frac{K_i(x)u_i^{p-1}}{|x|^{bp}}\,dx &\le c\int_{r_i\le|x-x_i|\le 1} u_i(x_i)^{-\lambda_i(p-1)}\frac{|x-x_i|^{(2+2a-N+\delta_i)(p-1)}}{|x|^{bp}}\\ &\le c\,u_i(x_i)^{-\lambda_i(p-1)} r_i^{(2+2a-N+\delta_i)(p-1)-bp+N}\\ &= c\,u_i(x_i)^{-1}R_i^{(2+2a-N+\delta_i)(p-1)-bp+N} = o(1)u_i(x_i)^{-1}.\end{aligned}\tag{3.31}$$

Finally, (3.27), (3.29), (3.30), and (3.31) lead to a contradiction. Since we have established (3.26), the inequality in Proposition 3.10 has been established for $\rho \le |x - x_i| \le 1$ (due to Lemma 3.6). It remains to treat the case $r_i \le |x - x_i| \le \rho$. To this aim we scale the problem to reduce it to the case $|x - x_i| = 1$. By contradiction, suppose that there exists a subsequence $\tilde{x}_i$ satisfying $r_i \le |\tilde{x}_i - x_i| \le \rho$ and

$$\lim_{i\to+\infty} u_i(\tilde{x}_i)u_i(x_i)|\tilde{x}_i - x_i|^{N-2-2a} = +\infty. \tag{3.32}$$

Set $\tilde r_i = |\tilde x_i - x_i|$ and $\tilde u_i(x) = \tilde r_i^{\frac{N-2-2a}{2}} u_i(\tilde r_i x)$. We have that $\tilde u_i$ satisfies the equation

$$-\mathrm{div}\,(|x|^{-2a}\nabla \tilde u_i(x)) = K_i(\tilde r_i x)\frac{\tilde u_i(x)^{p-1}}{|x|^{bp}}.$$

Since $|x_i| = r_i\, o(1)$ and $\tilde r_i \geq r_i$ we have that $x_i/\tilde r_i \to 0$. We have that $x_i/\tilde r_i$ is an isolated simple blow-up point for $\{\tilde u_i\}_i$. From (3.26), we have that

$$\tilde u_i\left(\frac{x_i}{\tilde r_i} + \frac{\tilde x_i - x_i}{\tilde r_i}\right) \leq c\tilde u_i\left(\frac{x_i}{\tilde r_i}\right)^{-1}$$

which gives

$$\tilde r_i^{N-2-2a} u_i(\tilde x_i) u_i(x_i) \leq c.$$

The above estimate and (3.32) give rise to a contradiction. The inequality in Proposition 3.10 is thereby established.

We compute A by multiplying (P_i) by $u_i(x_i)$ and integrating over B_1. From the divergence theorem,

$$-\int_{\partial B_1} |x|^{-2a}\frac{\partial}{\partial\nu}\big(u_i(x_i)u_i\big) = u_i(x_i)\int_{B_1} K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx. \tag{3.33}$$

Let $w_i(x) = u_i(x_i)u(x)$. We have that w_i satisfies

$$-\mathrm{div}\,(|x|^{-2a}\nabla w_i) = u_i(x_i)^{2-p}K_i(x)\frac{w_i^{p-1}}{|x|^{bp}}.$$

Moreover the inequality (3.16) implies that w_i is bounded on any compact set not containing 0. Hence $w_i \to w$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ where w satisfies

$$-\mathrm{div}\,(|x|^{-2a}\nabla w) = 0 \quad \text{in } \mathbb{R}^N\setminus\{0\}.$$

From the Bôcher-type theorem proved in the Appendix (Theorem A.4), we find that $w(x) = A|x|^{2+2a-N} + B(x)$ where $B(x)$ is Hölder continuous in $\mathbb{R}^N$ and satisfies $-\mathrm{div}\,(|x|^{-2a}\nabla B) = 0$ in $\mathbb{R}^N$. Hence

$$\begin{aligned}\lim_{i\to\infty}\int_{\partial B_1}|x|^{-2a}\frac{\partial}{\partial\nu}\big(u_i(x_i)u_i\big) &= \int_{\partial B_1}|x|^{-2a}\frac{\partial}{\partial\nu}\big(A|x|^{2+2a-N}+B(x)\big)\\ &= A(2+2a-N)|\mathbb{S}^N|.\end{aligned} \tag{3.34}$$

On the other hand from (3.31) and Proposition 3.7

$$\begin{aligned}u_i(x_i)\int_{B_1}K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx &= u_i(x_i)\int_{|x-x_i|\leq r_i}K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx + o(1)\\ &= K_i(0)\int_{|y|\leq R_i}\frac{\big(z^{a,b}_{K(0)}\big)^{p-1}}{\big|y+u_i(x_i)^{\frac{2}{N-2-2a}}x_i\big|^{bp}}\,dy + o(1)\\ &= K(0)\int_{\mathbb{R}^N}\frac{\big(z^{a,b}_{K(0)}\big)^{p-1}}{|y|^{bp}}\,dy + o(1).\end{aligned} \tag{3.35}$$

By (3.33), (3.34), and (3.35) the value of A is computed and Proposition 3.10 is thereby established. $\square$

Using Proposition 3.7 and the upper bound in Proposition 3.10 it is easy to see that the following estimates hold.

Lemma 3.12. *Under the assumptions of Proposition 3.10 we have for* $s = s_1 + s_2$

$$\int_{|x-x_i|\le r_i} |x-x_i|^{s_1}|x|^{s_2}|x|^{-bp}u_i(x)^p$$
$$= \begin{cases} u_i(x_i)^{\frac{-2s}{N-2-2a}}\Big(o(1)+\int_{\mathbb{R}^N}|x|^{s-bp}z^p_{1,K_i(x_i)}\Big) & \text{if } -N+bp<s<N-bp,\\ O(u_i(x_i)^{-p}\log u_i(x_i)) & \text{if } s=N-bp,\\ o\big(u_i(x_i)^{-p}\big) & \text{if } s>N-bp.\end{cases}$$

$$\int_{r_i\le|x-x_i|\le 1} |x-x_i|^{s_1}|x|^{s_2}|x|^{-bp}u_i(x)^p$$
$$\le \begin{cases} o\big(u_i(x_i)^{\frac{-2s}{N-2-2a}}\big) & \text{if } -N+bp<s<N-bp,\\ O(u_i(x_i)^{-p}\log u_i(x_i)) & \text{if } s=N-bp,\\ O\big(u_i(x_i)^{-p}\big) & \text{if } s>N-bp.\end{cases}$$

Proposition 3.13. *Let* $a \in \big[\frac{N-4}{2}, \frac{N-2}{2}\big[$. *Suppose that* $\{K_i\}_i$ *satisfy (3.1) with* $\Omega = B_2 \subset \mathbb{R}^N$ *for some positive constant* A_1, $\nabla K_i(0) = 0$, $\{K_i\}_i$ *converge to* K *in* $C^2(B_2)$, $\{u_i\}_i$ *satisfy* (P_i) *with* $\Omega = B_2(0)$ *and* $x_i \to 0$ *is an isolated blow-up point with (3.15) for some positive constant* A_3. *Then it has to be an isolated simple blow-up point.*

Proof. From Remark 3.8 there exists a constant c such that $r^{\frac{N-2-2a}{2}}\bar u_i(r)$ is decreasing in $cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i$. Arguing by contradiction, let us suppose that the blow-up is not simple. Hence for any i there exists $\mu_i \ge r_i$, $\mu_i \to 0$, such that μ_i is the first point after r_i in which the function $r^{\frac{N-2-2a}{2}}\bar u_i(r)$ becomes increasing. In particular μ_i is a critical point of such a function. Set

$$\xi_i(x) = \mu_i^{\frac{N-2-2a}{2}} u_i(\mu_i x), \quad \text{for } |\mu_i x - x_i| \le 1.$$

Clearly ξ_i satisfies

$$-\mathrm{div}\,(|x|^{-2a}\nabla\xi_i) = K_i(\mu_i x)\frac{\xi_i^{p-1}}{|x|^{bp}}, \quad \text{for } |\mu_i x - x_i| \le 1.$$

Note that $\mu_i^{-1} \le R_i^{-1}u_i(x_i)^{\frac{2}{N-2-2a}} \le u_i(x_i)^{\frac{2}{N-2-2a}}$ and hence

$$\mu_i^{-1}|x_i| \le u_i(x_i)^{\frac{2}{N-2-2a}}|x_i| \to 0$$

in view of (3.2). Moreover (3.15) implies that

$$|x-\mu_i^{-1}x_i|^{\frac{N-2-2a}{2}}\xi_i(x) \le \text{const} \quad \text{for } |x-\mu_i^{-1}x_i| \le 1/\mu_i.$$

It is also easy to verify that

$$\lim_{i\to\infty}\xi_i(\mu_i^{-1}x_i) = \lim_{i\to\infty}\mu_i^{\frac{N-2-2a}{2}}u_i(x_i) = \infty.$$

On the other hand

$$⨍_{\partial B_r(\mu_i^{-1}x_i)}\xi_i = \mu_i^{\frac{N-2-2a}{2}} ⨍_{\partial B_{r\mu_i}(x_i)} u_i = \mu_i^{\frac{N-2-2a}{2}}\bar u_i(\mu_i r).$$

Hence

$$r^{\frac{N-2-2a}{2}}\bar{\xi}_i(r) = \bar{w}_i(\mu_i r)$$

and the function $r^{\frac{N-2-2a}{2}}\bar{\xi}_i(r)$ is decreasing in $c\xi_i(\mu_i^{-1}x_i)^{-\frac{2}{N-2-2a}} < r < 1$ so that 0 is an isolated simple blow-up point for $\{\xi_i\}$. From Proposition 3.10 we have that

$$\xi_i(\mu_i^{-1}x_i)\xi_i(x) \to h(x) = A|x|^{2+2a-N} + B(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$$

where $B(x)$ is Hölder continuous in $\mathbb{R}^N$ and satisfies $-\text{div}\,(|x|^{-2a}\nabla B) = 0 \quad \text{in } \mathbb{R}^N$. Since $h \geq 0$, the Harnack inequality implies that B is bounded and from the Liouville Theorem (see [12]) we find that B must be constant. Since

$$\frac{d}{dr}\{h(r)r^{\frac{N-2-2a}{2}}\}|_{r=1} = 0$$

we have that $A = B > 0$. From the Taylor expansion, (3.2) and the assumption $\nabla K_i(0) = 0$ we find

$$|\nabla K_i(\mu_i^{-1}x_i)| \leq \text{const}|\mu_i^{-1}x_i| = o\Big(\xi_i(\mu_i^{-1}x_i)^{-\frac{2}{N-2-2a}}\Big). \tag{3.36}$$

Using Lemma 3.12, (3.36), and the assumption on a, we have

$$\int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i \nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} = \int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\Big[\nabla K_i(\mu_i^{-1}x_i) + O(\mu_i x - \mu_i^{-1}x_i)\Big]\cdot x\frac{\xi_i^p}{|x|^{bp}}$$
$$= \int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\Big[\nabla K_i(\mu_i^{-1}x_i) + O(|x| + |x-\mu_i^{-1}x_i|)\Big]\cdot x\frac{\xi_i^p}{|x|^{bp}}$$
$$= \left\{\begin{matrix} \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-\frac{4}{N-2-2a}}\Big) & \text{if } p > \frac{4}{N-2-2a} \\ \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-p}\log u_i(x_i)\Big) & \text{if } p = \frac{4}{N-2-2a} \\ \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-p}\Big) & \text{if } p < \frac{4}{N-2-2a} \end{matrix}\right\} = o(\xi_i(\mu_i^{-1}x_i)^{-2}).$$

Hence, from Corollary 2.2 and (3.16), we have that for any $0 < \sigma < 1$

$$\begin{aligned}\int_{\partial B_\sigma(0)} & B(\sigma, x, \xi_i, \nabla\xi_i) \\ &= \frac{1}{p}\int_{B_\sigma(0)} \mu_i\nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} - \frac{\sigma}{p}\int_{\partial B_\sigma(0)} K_i(\mu_i x)\frac{\xi_i^p}{|x|^{bp}} \\ &= \frac{1}{p}\int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} + O(\xi_i(\mu_i^{-1}x_i)^{-p}) \\ &= o(\xi_i(\mu_i^{-1}x_i)^{-2}).\end{aligned}$$

Multiplying by $\xi_i(\mu_i^{-1}x_i)^2$ and letting $i \to \infty$ we find that

$$\int_{\partial B_\sigma} B(\sigma, x, h, \nabla h) = 0.$$

On the other hand Proposition 2.4 implies that for small σ the above integral is strictly negative, thus giving rise to a contradiction. The proof is now complete. □

4. A-priori estimates

To prove the a-priori estimates we first locate the possible blow-up points as in [15]. To this end we use the Kelvin transform defined in (3.5). We recall that if u solves (1.6) then $\tilde{u} = |x|^{-(N-2-2a)}u(x/|x|^2)$ solves (1.6) with K replaced by $\tilde{K}(x) = K(x/|x|^2)$. Since weak solutions to (1.6) are Hölder continuous (see [10]) we infer that

$$\lim_{|x|\to\infty} |x|^{N-2-2a}u(x) \text{ exists.} \tag{4.1}$$

Let us define $\omega_a(x) := (1 + |x|^{N-2-2a})^{-1}$.

Lemma 4.1. *Suppose $a \geq 0$, $2 < p < 2^*$, and $K \in C^2(\mathbb{R}^N)$ satisfies (1.11) and for some positive constants A_1, A_2 condition (1.13) and*

$$\|\nabla K\|_{L^\infty(B_2(0))} + \|\nabla \tilde{K}\|_{L^\infty(B_2(0))} \leq A_2. \tag{4.2}$$

Then for any $\varepsilon \in (0,1)$, $R > 1$, there exists $C_0 = C_0(\varepsilon, R, N, a, b, A_1, A_2) > 0$, such that if u is a solution of (1.6) and $\mathcal{K} = \{q_1, \dots, q_k\} \subset \mathbb{R}^N \cup \{\infty\}$ with

$$\begin{cases} \max\limits_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} \mathit{dist}(x,\mathcal{K})^{\frac{N-2-2a}{2}} > C_0, \\ u(q_i)|q_i|^{\frac{2}{N-2-2a}} < \varepsilon, \text{ and for all } 1 \leq i \leq k \\ \max\limits_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} \mathit{dist}(x,\{q_1,\dots,q_{i-1}\})^{\frac{N-2-2a}{2}} \leq \frac{u(q_i)}{\omega_a(q_i)} \mathit{dist}(q_i,\{q_1,\dots,q_{i-1}\})^{\frac{N-2-2a}{2}}, \end{cases} \tag{4.3}$$

then there exists $q^* \notin \mathcal{K}$ such that q^* is a maximum point of $(u/\omega_a)\mathit{dist}(\cdot,\mathcal{K})^{\frac{N-2-2a}{2}}$ and

(A) *if $|q^*| \leq 1$*

$$\left\| \frac{u(u(q^*)^{-\frac{2}{N-2-2a}}x + q^*)}{u(q^*)} - z^{a,b}_{K(q^*)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |q^*|u(q^*)^{\frac{2}{N-2-2a}} < \varepsilon \tag{4.4}$$

(B) *if $|q^*| > 1$*

$$\left\| \frac{\tilde{u}(\tilde{u}(\tilde{q}^*)^{-\frac{2}{N-2-2a}}x + \tilde{q}^*)}{\tilde{u}(\tilde{q}^*)} - z^{a,b}_{K(q^*)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |\tilde{q}^*|\tilde{u}(\tilde{q}^*)^{\frac{2}{N-2-2a}} < \varepsilon \tag{4.5}$$

where $\tilde{q}^* = \mathit{Inv}(q^*) := q^*/|q^*|^2$, $\tilde{u}$ is the Kelvin transform of u, $\mathit{dist}(\cdot,\cdot)$ is the distance on $\mathbb{R}^N \cup \{\infty\}$ induced by the standard metric on the sphere through the stereo-graphic projection, and $\mathit{dist}(\cdot,\emptyset) \equiv 1$.

Proof. Fix $\varepsilon > 0$ and $R > 1$. Let C_0 and C_1 be positive constants depending on $\varepsilon, R, a, b, N, A_1, A_2$ which shall be appropriately chosen in the sequel.
Let $q^* \in \mathbb{R}^N \cup \{\infty\}$ be the maximum point of $u/\omega_a \mathrm{dist}(x,\mathcal{K})^{\frac{N-2-2a}{2}}$. By (4.1) this maximum is achieved. From the first in (4.3) we have that $u(q^*)/\omega_a(q^*)\mathrm{dist}(q^*,\mathcal{K})^{\frac{N-2-2a}{2}} > C_0$. First we treat the case $|q^*| \leq 1$. We claim that there exists a constant C_1, depending only on $\varepsilon, R, a, b, N, A_1, A_2$, such that $|q^*|^{\frac{N-2-2a}{2}}u(q^*) < C_1$. If not, there exist solutions u_i of (1.6) and finite sets $\mathcal{K}_i = \{q_1^i, \dots, q_{k_i}^i\}$ satisfying (4.3) above, such that for the maximum points q_i^* of $u_i/\omega_a \mathrm{dist}(\cdot,\mathcal{K}_i)^{\frac{N-2-2a}{2}}$ there holds

$$|q_i^*| \leq 1 \text{ and } |q_i^*|^{\frac{N-2-2a}{2}}u_i(q_i^*) \to \infty.$$

Consider the functions v_i, defined by

$$v_i(x) := u_i(q_i^*)^{-1} u_i\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}} u(q_i^*)^{\frac{2-p}{2}} x + q_i^*\big),$$

which satisfy

$$\begin{aligned}&-\operatorname{div}\Big(\big||q_i^*|^{\frac{(N-2-2a)(2-p)}{4}} u_i(q_i^*)^{\frac{2-p}{2}} x + |q_i^*|^{-1} q_i^*\big|^{-2a} \nabla v_i\Big)\\ &\quad = K\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}} u_i(q_i^*)^{\frac{2-p}{2}} x + q_i^*\big) \frac{v_i^{p-1}}{\big||q_i^*|^{\frac{(N-2-2a)(2-p)}{4}} u_i(q_i^*)^{\frac{2-p}{2}} x + |q_i^*|^{-1} q_i^*\big|^{bp}}.\end{aligned}$$

Let $p_i = q_{j_i}^i \in \mathcal{K}_i$ be such that $\operatorname{dist}(q_i^*, \mathcal{K}_i) = \operatorname{dist}(q_i^*, p_i)$ and set $\hat{\mathcal{K}}_i = \{q_1^i, \dots, q_{j_i-1}^i\}$. From (4.3) we infer

$$\begin{aligned}&\operatorname{dist}(p_i, \hat{\mathcal{K}}_i) \le \operatorname{dist}(p_i, q_i^*) + \operatorname{dist}(q_i^*, \hat{\mathcal{K}}_i) \le 2\operatorname{dist}(q_i^*, \hat{\mathcal{K}}_i),\\ &u_i(p_i)|p_i|^{\frac{2}{N-2-2a}} < \varepsilon, \quad u_i(q_i^*) \le u_i(p_i) \left(\frac{\operatorname{dist}(p_i, \hat{\mathcal{K}}_i)}{\operatorname{dist}(q_i^*, \hat{\mathcal{K}}_i)}\right)^{\frac{N-2-2a}{2}} \frac{\omega_a(q_i^*)}{\omega_a(p_i)},\end{aligned}$$

and finally that if $|p_i| \le 2$

$$\begin{aligned}\varepsilon\left(\frac{|q_i^*|}{|p_i|}\right)^{\frac{2}{N-2-2a}} &> u_i(p_i)|q_i^*|^{\frac{2}{N-2-2a}} \ge u_i(q_i^*)|q_i^*|^{\frac{2}{N-2-2a}} \left(\frac{\operatorname{dist}(q_i^*, \hat{\mathcal{K}}_i)}{\operatorname{dist}(p_i, \hat{\mathcal{K}}_i)}\right)^{\frac{N-2-2a}{2}} \frac{\omega_a(p_i)}{\omega_a(q_i^*)}\\ &\ge \text{const}\, u_i(q_i^*)|q_i^*|^{\frac{2}{N-2-2a}} \to \infty.\end{aligned}$$

Consequently there exists a positive constant c such that $|q_i^*|^{-1}\operatorname{dist}(q_i^*, \mathcal{K}_i) > c$, which is trivial in the case $|p_i| > 2$ and follows from the above estimate if $|p_i| \le 2$. Thus

$$\begin{aligned}|q_i^*|^{-1-\frac{(N-2-2a)(2-p)}{4}} u_i(q_i^*)^{-\frac{2-p}{2}} \operatorname{dist}(q_i^*, \mathcal{K}_i) &\ge \big(u_i(q_i^*)|q_i^*|^{\frac{N-2-2a}{2}}\big)^{\frac{p-2}{2}} |q_i^*|^{-1}\operatorname{dist}(q_i^*, \mathcal{K}_i)\\ &\ge \mathrm{c}\big(u_i(q_i^*)|q_i^*|^{\frac{N-2-2a}{2}}\big)^{\frac{p-2}{2}} \to \infty.\end{aligned}$$

For $|x| \le \frac{c}{4}|q_i^*|^{-\frac{(N-2-2a)(2-p)}{4}} u_i(q_i^*)^{-\frac{2-p}{2}}$ we have that

$$\begin{aligned}v_i(x) &= u_i(q_i^*)^{-1} u_i\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}} u(q_i^*)^{\frac{2-p}{2}} x + q_i^*\big)\\ &\le u_i(q_i^*)^{-1} \omega_a\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}} u(q_i^*)^{\frac{2-p}{2}} x + q_i^*\big) \frac{u_i(q_i^*)}{\omega_a(q_i^*)}\\ &\le c \sup_{|x|\le \frac{c}{4}} \omega_a(x + q_i^*)\omega_a(q_i^*)^{-1} \le \text{const}.\end{aligned}$$

Up to a subsequence, we have that $q_i^* \to \bar{q}_1$ and v_i converges in $C^2_{\text{loc}}(\mathbb{R}^N)$ to a solution of

$$-\Delta w = K(\bar{q}_1) w^{p-1} \text{ in } \mathbb{R}^N, \quad w(0) = 1.$$

This is impossible since the above equation has no solution for $p < 2^*$. The claim is thereby proved. The function v_1, defined by

$$v_1(x) := u(q^*)^{-1} u\big(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*\big),$$

satisfies

$$-\operatorname{div}\big(|x+u(q^*)^{\frac{2}{N-2-2a}}q^*|^{-2a}\nabla v_1\big)=K\big(u(q^*)^{-\frac{2}{N-2-2a}}x+q^*\big)\frac{v_1^{p-1}}{|x+u(q^*)^{\frac{2}{N-2-2a}}q^*|^{bp}},$$

$$v_1(0)=1.$$

For $|x|\le C_0^{-\frac{1}{N-2-2a}}u(q^*)^{\frac{2}{N-2-2a}}\operatorname{dist}(q^*,\mathcal{K})$ we obtain

$$\begin{aligned}\operatorname{dist}(u(q^*)^{-\frac{2}{N-2-2a}}x+q^*,\mathcal{K})&\ge\operatorname{dist}(q^*,\mathcal{K})-cC_0^{-\frac{1}{N-2-2a}}\operatorname{dist}(q^*,\mathcal{K})\\&\ge\operatorname{dist}(q^*,\mathcal{K})\big(1-cC_0^{-\frac{1}{N-2-2a}}\big)\end{aligned}$$

and

$$\begin{aligned}v_1(x)&=u(q^*)^{-1}u\big(u(q^*)^{-\frac{2}{N-2-2a}}x+q^*\big)\\&\le u(q^*)^{-1}\omega_a\big(u(q^*)^{-\frac{2}{N-2-2a}}x+q^*\big)\frac{u(q^*)}{\omega_a(q^*)}\left(\frac{\operatorname{dist}(q^*,\mathcal{K})}{\operatorname{dist}(u(q^*)^{-\frac{2}{N-2-2a}}x+q^*,\mathcal{K})}\right)^{\frac{N-2-2a}{2}}\\&\le\omega_a(q^*)^{-1}\big(1-cC_0^{-\frac{1}{N-2-2a}}\big)^{-\frac{N-2-2a}{2}}.\end{aligned}$$

Notice that $|q^*|<\operatorname{const}C_1^{\frac{2}{N-2-2a}}C_0^{-\frac{2}{N-2-2a}}$ and

$$C_0^{-\frac{1}{N-2-2a}}u(q^*)^{\frac{2}{N-2-2a}}\operatorname{dist}(q^*,\mathcal{K})>\left(\frac{1}{4}C_0\right)^{\frac{1}{N-2-2a}}.$$

Hence for any $\delta>0$ we may choose C_0, depending on $a,b,N,\varepsilon,R,A_1,A_2,C_1$, such that

$$\omega_a(q^*)^{-1}\big(1-C_0^{-\frac{1}{N-2-2a}}\big)^{-\frac{N-2-2a}{2}}\le1+\delta$$

and v_1 is $\varepsilon/4$-close in $C^{0,\gamma}(B_{2R}(0))$ to a solution of

$$-\operatorname{div}\,(|x+u(q^*)^{\frac{2}{N-2-2a}}q^*|^{-2a}\nabla w)=K(q^*)\frac{w^{p-1}}{|x+u(q^*)^{\frac{2}{N-2-2a}}q^*|^{bp}}\text{ in }\mathbb{R}^N,$$

$$w(0)=1,\quad 0\le w(x)\le1+\delta.$$

If we choose δ small enough, depending on ε and R, then it is easy to see that any solution of the above equation is $\varepsilon/4$-close in $C^{0,\gamma}(B_{2R}(0))\cap H_a^1(B_{2R}(0))$ to $z_{K(q^*)}^{a,b}$ and $u(q^*)^{\frac{2}{N-2-2a}}|q^*|\le\varepsilon/2$. This gives estimate (4.4). Case (B) can be reduced to case (A) using the Kelvin transform. □

Proposition 4.2. *Under the assumptions and notations of Lemma 4.1 there exists for any* $0<\varepsilon<1$ *and* $R>1$ *a constant* $C_0=C_0(\varepsilon,R,N,a,b,A_1,A_2)>0$ *such that if* u *is a solution of (1.6) with*

$$\max_{x\in\mathbb{R}^N\cup\{\infty\}}\frac{u(x)}{\omega_a(x)}>C_0$$

then there exist $1\le k=k(u)<\infty$ *and a set* $\mathcal{S}(u)=\{q_1,q_2,\dots,q_k\}\subset\mathbb{R}^N\cup\{\infty\}$ *such that for each* $1\le j\le k$ *we have*

(A) *if* $|q_j| \leq 1$

$$\left\| \frac{u(u(q_j)^{-\frac{2}{N-2-2a}}x + q_j)}{u(q_j)} - z^{a,b}_{K(q_j)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |q_j| u(q_j)^{\frac{2}{N-2-2a}} < \varepsilon \tag{4.6}$$

(B) *if* $|q_j| > 1$

$$\left\| \frac{\tilde{u}(\tilde{u}(\tilde{q}_j)^{-\frac{2}{N-2-2a}}x + \tilde{q}_j)}{\tilde{u}(\tilde{q}_j)} - z^{a,b}_{K(q_j)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |\tilde{q}_j| \tilde{u}(\tilde{q}_j)^{\frac{2}{N-2-2a}} < \varepsilon. \tag{4.7}$$

Moreover the sets

$$U_j := \begin{cases} B_{Ru(q_j)^{-\frac{2}{N-2-2a}}}(q_j) & \text{in case (A)} \\ Inv\big(B_{R\tilde{u}(\tilde{q}_j)^{-\frac{2}{N-2-2a}}}(\tilde{q}_j)\big) & \text{in case (B)} \end{cases} \quad \textit{are disjoint.}$$

Furthermore, u satisfies

$$u(x) \leq C_0 \omega_a(x) \max_{1\leq j\leq k} dist(x, q_j)^{-\frac{N-2-2a}{2}}.$$

Proof. Fix $\varepsilon > 0$ and $R > 1$. Let C_0 be as in Lemma 4.1. First we apply Lemma 4.1 with $\mathcal{K} = \emptyset$ and find $q_1 \in \mathbb{R}^N \cup \{\infty\}$ the maximum point of u/ω_a. If $u(x) \leq C_0\omega_a(x)\text{dist}(x, q_1)^{-\frac{N-2-2a}{2}}$ holds we stop here. Otherwise we apply again Lemma 4.1 to obtain q_2. From estimates (4.6) and (4.7) it follows that U_1 and U_2 are disjoint. We continue the process. Since $u \in L^p(\mathbb{R}^N, |x|^{-bp})$ and

$$\int\limits_{U_j} \frac{K(x)}{|x|^{bp}} u(x)^p \, dx \geq \frac{1}{2A_1} \int_{B_R(0)} \frac{\big(z^{a,b}_{K(q_j)}\big)^p}{|y + \varepsilon q_j/|q_j||^{bp}} \, dy \geq c(a, b, N),$$

where $c(a, b, N)$ is independent of q_j, u, $R > 1$ and $\varepsilon < 1$, we will stop after a finite number of steps. □

Proposition 4.3. *Under the assumptions and notations of Lemma 4.1 there exist for any* $0 < \varepsilon < 1$ *and* $R > 1$ *some positive constants* $C_0 = C_0(\varepsilon, R, N, a, b, A_1, A_2)$ *and* $\delta = \delta(\varepsilon, R, N, a, b, A_1, A_2)$ *such that if u is a solution of (1.6) with*

$$\max_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} > C_0$$

then

$$dist(q_j, q_\ell) \geq \delta \textit{ for all } 1 \leq j \neq \ell \leq k,$$

where $q_j = q_j(u)$, $q_\ell = q_\ell(u)$ *and* $k = k(u)$ *are given in Proposition 4.2.*

Proof. To obtain a contradiction we assume that for some constants ε, R, A_1 and A_2 there exist sequences K_i and u_i satisfying the assumptions of Proposition 4.3 such that

$$\lim_{i\to\infty} \min_{j\neq\ell} \text{dist}(q_j(u_i), q_\ell(u_i)) = 0.$$

We may assume that

$$\sigma_i := \text{dist}(q_1(u_i), q_2(u_i)) = \min_{j\neq\ell} \text{dist}(q_j(u_i), q_\ell(u_i)) \to 0 \text{ as } i \to \infty. \tag{4.8}$$

Let us denote $q_j(u_i)$ by q^i_j. Since $U_1(u_i)$ and $U_2(u_i)$ are disjoint and (4.8) holds we have that $u_i(q^i_1) \to \infty$ and $u_i(q^i_2) \to \infty$. Therefore we can pass to a subsequence still denoted by $\{u_i\}$ and find $R_i \to \infty$, $\varepsilon_i \to 0$ such that either $q^i_1 = q_1(u_i) \to 0$ or $|q^i_1| \to \infty$, and for $j = 1, 2$

$$\left\| \frac{u_i\big(u_i(q^i_j)^{\frac{2}{N-2-2a}} x + q^i_j\big)}{u_i(q^i_j)} - z^{a,b}_{K(q^i_j)} \right\|_{C^{0,\gamma}(B_{2R_i}(0))} + |q^i_j| u_i(q^i_j)^{\frac{2}{N-2-2a}} < \varepsilon_i \text{ if } q^i_1 \to 0 \quad (4.9)$$

$$\left\| \frac{\tilde u_i\big(\tilde u_i(\tilde q^i_j)^{\frac{2}{N-2-2a}} x + \tilde q^i_j\big)}{\tilde u_i(\tilde q^i_j)} - z^{a,b}_{K(\tilde q^i_j)} \right\|_{C^{0,\gamma}(B_{2R_i}(0))} + |\tilde q^i_j| \tilde u_i(\tilde q^i_j)^{\frac{2}{N-2-2a}} < \varepsilon_i \text{ if } |q^i_1| \to \infty.$$

We first consider the case $q^i_1 \to 0$. Since $U_1(u_i)$ and $U_2(u_i)$ are disjoint we have that

$$\sigma_i > c(N) \max_{j=1,2} \{R_i u_i(q^i_j)^{-\frac{2}{N-2-2a}}\}. \qquad (4.10)$$

From (4.9) and (4.10) we get that $\sigma_i^{-1}|q^i_j| < \frac{\varepsilon_i}{c(N)R_i} \to 0$ for $j = 1, 2$ and obtain the contradiction

$$\frac{1}{2} < |\sigma_i^{-1}(q^i_2 - q^i_1)| \to 0.$$

Performing the same analysis as above for the Kelvin transform $\tilde u$ of u leads to a contradiction if $\tilde q^i_1 \to 0$. □

Remark 4.4. *Propositions 4.2 and 4.3 imply that there are only finitely many blow-up points and all are isolated.*

Proposition 4.5. *Suppose $\{K_i\}$ and $a \in](N-4)/2, (N-2)/2[$ satisfy the assumptions of Lemma 4.1 and Proposition 3.13. Let $\{u_i\}$ be solutions of (P_i) with $\Omega = \mathbb{R}^N$. Then after passing to a subsequence either $\{u_i/\omega_a\}$ stays bounded in $L^\infty(\mathbb{R}^N)$ or $\{u_i\}$ has precisely one blow-up point, which can be at 0 or at ∞.*

Proof. Suppose that $\{u_i/\omega_a\}$ is not uniformly bounded in $L^\infty(\mathbb{R}^N)$, otherwise there is nothing to prove. Consequently we may apply Proposition 4.2 and Proposition 4.3 to obtain isolated points $\{q^i_1, \dots, q^i_{k(i)}\}$ satisfying (4.6) and (4.7) with $R_i \to \infty$ and $\varepsilon_i \to 0$. To obtain a contradiction, we assume that up to a subsequence $k(i) \geq 2$. Since $u(q^i_j)/\omega_a(q^i_j) \to \infty$ for $j = 1, 2$ and $\operatorname{dist}(q^i_1, q^i_2) \geq \delta > 0$ we may assume $q^i_1 \to 0$ and $q^i_2 \to \infty$ and $k(i) = 2$ as $i \to \infty$. From Proposition 3.13 and Remark 4.4 they are isolated simple blow-up points. From Proposition 3.10 we infer that

$$\lim_{i\to\infty} u_i(q^i_1) u_i(x) = h(x) \text{ in } C^0_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}),$$

$$\operatorname{div}(|x|^{-2a}\nabla h) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using Theorem A.4 for h and its Kelvin transform and the maximum principle we obtain for some $a_1, a_2 > 0$

$$h(x) = a_1 |x|^{2+2a-N} + a_2.$$

We may now proceed as in the proof of Proposition 3.13 to see that

$$\int_{\partial B_\sigma(q^i_1)} B(\sigma, x, u_i, \nabla u_i) = o(u_i(q^i_1)^{-2}).$$

Multiplying by $u_i(q_1^i)^2$ and letting $i \to \infty$ we find that

$$\int_{\partial B_\sigma} B(\sigma, x, h, \nabla h) = 0.$$

This contradicts for small σ Proposition 2.4 and completes the proof. □

Proposition 4.6. *Suppose $K \in C^2(\mathbb{R}^N)$ satisfies (1.11)-(1.13),*

$$a \geq 0, \quad \frac{N-4}{2} < a < \frac{N-2}{2}, \quad \text{and} \quad \frac{4}{N-2-2a} < p < 2^*.$$

Then there exists $C_K > 0$ such that for any $t \in (0,1]$ and any solution u_t of

$$-\text{div}\,(|x|^{-2a}\nabla u) = (1 + t\,(K(x)-1))\frac{u^{p-1}}{|x|^{bp}}, \quad u > 0 \text{ in } \mathcal{D}_a^{1,2}(\mathbb{R}^N) \tag{P_t}$$

there holds

$$\|u_t\|_E < C_K \tag{4.11}$$

and

$$C_K^{-1} < u_t\omega_a^{-1} < C_K. \tag{4.12}$$

Proof. The bound in (4.12) follows from (4.11) and Harnack's inequality in [11]. The estimate in Lemma A.3 of the appendix shows that $(1+t\,(K(x)-1))u^{p-2}|x|^{-bp}$ belongs to the required class of potentials in [11]. To show that u_t/ω_a is bounded in $L^\infty(\mathbb{R}^N)$ we argue by contradiction and may assume in view of Proposition 4.5 that there exists a sequence $\{t_i\} \subset (0,1]$ converging to $t_0 \in [0,1]$ as $i \to \infty$ such that u_{t_i} has precisely one blow-up point (x_i), which can be supposed to be zero using the Kelvin transform. Corollary 2.3 yields

$$0 = \int_{\mathbb{R}^N} x \cdot \nabla K(x)\frac{u_{t_i}(x)^p}{|x|^{bp}}\,dx.$$

Since 0 is assumed to be the only blow-up point, the Harnack inequality and (3.16) yield, for any $\sigma \in (0,1)$,

$$\left|\int_{B_\sigma(x_i)} x \cdot \nabla K(x)\frac{u_{t_i}(x)^p}{|x|^{bp}}\,dx\right| = \left|\int_{\mathbb{R}^N \setminus B_\sigma(x_i)} x \cdot \nabla K(x)\frac{u_{t_i}(x)^p}{|x|^{bp}}\,dx\right| \leq C(\sigma)\Big(u_{t_i}(x_i)^{-p}\Big).$$

We have that from Taylor expansion, (3.2), and (1.12)

$$|\nabla K(x_i)| \leq \text{const}|x_i| = o\Big(u_{t_i}(x_i)^{-\frac{2}{N-2-2a}}\Big) \tag{4.13}$$

and

$$\begin{aligned}
&\left|\int_{B_\sigma(x_i)} \nabla K(x) \cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx\right| \\
&= \left|\int_{B_\sigma(x_i)} \nabla K(x_i) \cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx + \int_{B_\sigma(x_i)} D^2K(x_i)(x-x_i) \cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx \right.\\
&\quad \left. + \int_{B_\sigma(x_i)} o(|x-x_i|) \cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx\right|.
\end{aligned}$$

From Lemma 3.12 and (4.13) we infer

$$\left|\int_{B_\sigma(x_i)} \nabla K(x_i)\cdot x \frac{u_{t_i}^p}{|x|^{bp}}\,dx + \int_{B_\sigma(x_i)} o(|x-x_i|)\cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx\right| = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big).$$

Hence

$$\int_{B_\sigma(x_i)} D^2K(x_i)(x-x_i)\cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big).$$

Since by Lemma 3.12

$$\int_{r_i\le|x-x_i|\le\sigma} D^2K(x_i)(x-x_i)\cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big)$$

we have

$$\int_{B_{r_i}(x_i)} D^2K(x_i)(x-x_i)\cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big). \tag{4.14}$$

Making in (4.14) the change of variables $x = u_{t_i}(x_i)^{-2/(N-2-2a)}y + x_i$ and using Proposition 3.7

$$0 = \int_{\mathbb{R}^N} D^2K(0)y\cdot y|y|^{-bp}z^{a,b}_{1+t_0(K(0)-1)}(y)^p\,dy = \Delta K(0)\int_{\mathbb{R}^N}|y|^{2-bp}z^{a,b}_{1+t_0(K(0)-1)}(y)^p\,dy$$

which is not possible in view (1.12). □

Proof of Theorem 1.1. It follows from Proposition 4.6 and Lemma A.1. □

We define $f_{K,\varepsilon}:\mathcal{D}^{1,2}_a(\mathbb{R}^N)\to\mathbb{R}$ by

$$\begin{aligned} f_{K,\varepsilon}(u) &= f_0(u) - \varepsilon G_K(u)\\ f_0(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{bp}}\\ G_K(u) &= \frac{1}{p}\int_{\mathbb{R}^N}\frac{K(x)|u|^p}{|x|^{bp}}. \end{aligned}$$

We will use the notation f_ε (respectively G) instead of $f_{K,\varepsilon}$ (respectively G_K) whenever there is no possibility of confusion. Let us denote by Z the manifold

$$Z = \{z_\mu = z^{a,b}_{1,\mu}:\ \mu>0\}$$

of the solutions to (1.6) with $K\equiv 1$.

Lemma 4.7. *Suppose $p>3$. There exist constants $\rho_0,\varepsilon_0,C>0$, and smooth functions*

$$\begin{aligned} w = w(\mu,\varepsilon):&\quad (0,+\infty)\times(-\varepsilon_0,\varepsilon_0) \longrightarrow \mathcal{D}^{1,2}_a(\mathbb{R}^N)\\ \eta = \eta(\mu,\varepsilon):&\quad (0,+\infty)\times(-\varepsilon_0,\varepsilon_0) \longrightarrow \mathbb{R} \end{aligned}$$

such that for any $\mu > 0$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$w(\mu,\varepsilon) \quad \text{is orthogonal to} \quad T_{z_\mu} Z \tag{4.15}$$

$$f'_\varepsilon\big(z_\mu + w(\mu,\varepsilon)\big) = \eta(\mu,\varepsilon)\dot{\xi}_\mu \tag{4.16}$$

$$|\eta(\mu,\varepsilon)| + \|w(\mu,\varepsilon)\|_{\mathcal{D}^{1,2}_a(\mathbb{R}^N)} \le C\,|\varepsilon| \tag{4.17}$$

$$\|\dot{w}(\mu,\varepsilon)\|_{\mathcal{D}^{1,2}_a(\mathbb{R}^N)} \le C(1+\mu^{-1})\,|\varepsilon|, \tag{4.18}$$

where $\dot{\xi}_\mu$ denotes the normalized tangent vector $\frac{d}{d\mu} z_\mu$ and $\dot{w}$ stands for the derivative of w with respect to μ. Moreover, (w,η) is unique in the sense that there exists $\rho_0 > 0$ such that if $(v,\tilde{\eta})$ satisfies $\|v\|_{\mathcal{D}^{1,2}_a(\mathbb{R}^N)} + |\tilde{\eta}| < \rho_0$ and (4.15)-(4.16) for some $\mu > 0$ and $|\varepsilon| \le \varepsilon_0$, then $v = w(\mu,\varepsilon)$ and $\tilde{\eta} = \eta(\mu,\varepsilon)$.

Proof. Existence, uniqueness, and estimate (4.17) are proved in [9]. In fact w and η are implicitly defined by $H(\mu,w,\eta,\varepsilon) = (0,0)$ where

$$H : (0,\infty) \times \mathcal{D}^{1,2}_a(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to \mathcal{D}^{1,2}_a(\mathbb{R}^N) \times \mathbb{R}$$
$$H(\mu,w,\eta,\varepsilon) := (f'_\varepsilon(z_\mu + w) - \eta\dot{\xi}_\mu, (w,\dot{\xi}_\mu)).$$

Let us now show estimate (4.18). There exists a positive constant C_* such that for any $\mu > 0$ (see [9])

$$\left\| \left(\frac{\partial H}{\partial(w,\eta)}(\mu,0,0,0) \right)^{-1} \right\| \le C_*.$$

Since $\dot{w}$ satisfies

$$\begin{pmatrix} \dot{w} \\ \dot{\eta} \end{pmatrix} = -\left(\frac{\partial H}{\partial(w,\eta)} \right)^{-1} \Bigg|_{(\mu,w,\eta,\varepsilon)} \cdot \frac{\partial H}{\partial \mu} \Bigg|_{(\mu,w,\eta,\varepsilon)}$$

we have for ε small using (4.17) and the fact that $f_0 \in C^3$

$$\begin{aligned}
\|\dot{w}(\mu,\varepsilon)\| &\le C_* \frac{\partial H}{\partial \mu}\Bigg|_{(\mu,w,\eta,\varepsilon)} \\
&\le C_* \left(\left\| f''_\varepsilon(z_\mu + w(\mu,\varepsilon))\dot{z}_\mu - \eta(\mu,\varepsilon)\frac{d}{d\mu}\dot{\xi}_\mu \right\| + \left| \left(w(\mu,\varepsilon), \frac{d}{d\mu}\dot{\xi}_\mu \right) \right| \right) \\
&\le C(1+\mu^{-1})|\varepsilon| + \|f''_0(z_\mu + w(\mu,\varepsilon))\dot{z}_\mu\| \\
&\le C(1+\mu^{-1})|\varepsilon| + O(\|w(\mu,\varepsilon)\|)\|\dot{z}_\mu\| \\
&\le C(1+\mu^{-1})|\varepsilon|.
\end{aligned}$$

This ends the proof. □

Corollary 4.8. *Suppose $p > 3$ and K satisfies the assumptions of Proposition 4.6. Then there exist $t_0 > 0$ and $R_0 > 0$ such that any solution u_t of (P_t) for $t \le t_0$ is of the form $z_\mu + w(\mu,t)$, where $1/R_0 < \mu < R_0$.*

Proof. First we show that there exists $R_1 > 0$ and $t_1 > 0$ such that any solution u_t of (P_t) for $t < t_1$ satisfies

$$\operatorname{dist}(u_t, Z_{R_1}) < \rho_0,$$

where by dist we mean the distance in the $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$-norm, ρ_0 is given in Lemma 4.7, and $Z_{R_1} := \{z_\mu \,|\, 1/R_1 < \mu < R_1\}$. By contradiction, assume there exist $R_i \to \infty$, $t_i \to 0$, and solutions u_{t_i} of (P_t) such that $\operatorname{dist}(u_{t_i}, Z_{R_i}) \geq \rho_0$. From (4.11) we can pass to a subsequence converging weakly in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ to some $\bar{u}$; since in view of the regularity results of [10] $\{u_t\}$ is bounded in $C^{0,\gamma}$ and such a bound excludes any possibility of concentration, the convergence is actually strong and $\operatorname{dist}(\bar{u}, Z) \geq \rho_0$. Furthermore, $\bar{u}$ solves (P_t) with $t = 0$ and hence $\bar{u} \in Z$, which is impossible.
Fix a solution u_t of (P_t) for some $t < t_1$. A short computation shows

$$\lim_{\mu\to 0} \operatorname{dist}(z_\mu, u_t)^2 = \lim_{\mu\to\infty} \operatorname{dist}(z_\mu, u_t)^2 = \|z_1\|^2 + \|u_t\|^2 > \rho_0^2.$$

Consequently there exists $R_0 > 0$ independent of t and $z_\mu \in Z_{R_0}$ such that

$$\operatorname{dist}(u_t, Z) = \|u_t - z_\mu\| \text{ and } u_t - z_\mu \in T_{z_\mu} Z^\perp.$$

Since u_t solves (P_t) we have $f_t'(z_\mu + u_t - z_\mu) = 0$ and the uniqueness in Lemma 4.7 yields the claim. □

5. Leray-Schauder degree

We introduce the Melnikov function

$$\Gamma_K(\tau) = \frac{1}{p}\int_{\mathbb{R}^N} K(x)\frac{z_\tau^p}{|x|^{bp}}.$$

It is known (for details see [9]) that it is possible to extend the C^2- function Γ_K by continuity to $\tau = 0$ and

$$\Gamma_K'(0) = 0 \text{ and } \Gamma_K''(0) = \frac{\Delta K(0)}{Np}\int_{\mathbb{R}^N} |x|^2 \frac{z_1(x)^p}{|x|^{bp}}. \tag{5.1}$$

Furthermore, using the Kelvin transform, we find

$$\Gamma_K(\tau) = \Gamma_{\tilde{K}}(\tau^{-1}) \quad \text{where} \quad \tilde{K}(x) = K(x/|x|^2). \tag{5.2}$$

We define for small t the function $\Phi_{K,t}(\mu) := f_{K,t}(z_\mu + w(\mu, t))$ and will denote it by Φ_t whenever there is no possibility of confusion.

Lemma 5.1. *Let $p > 3$ and assume Γ_K has only non-degenerate critical points. Then there exists $t_1 > 0$ such that for any $0 < t < t_1$ any solution u_t of (P_t) is of the form $u_t = z_{\mu_t} + w(\mu_t, t)$, where $\Phi_{K,t}'(\mu_t) = 0$ and $\mu_t \in (R_0^{-1}, R_0)$ for some positive R_0. Moreover, up to a subsequence as $t \to 0$*

$$|\mu_t - \bar{\mu}| = O(t), \tag{5.3}$$

where $\bar{\mu}$ is a critical point of Γ_K. Viceversa, for any critical point $\bar{\mu}$ of Γ_K and for any $0 < t < t_1$ there exists one and only one critical point μ_t of $\Phi_{K,t}$ such that (5.3) holds.

Proof. By Corollary 4.8 any solution u_t of (P_t) is of the form $u_t = z_{\mu_t} + w(\mu_t, t)$, where $\Phi_t'(\mu_t) = 0$ and $R_0^{-1} < \mu_t < R_0$. Using the Taylor expansion and (4.17) - (4.18), we have that for $R_0^{-1} < \mu < R_0$

$$\begin{aligned}\Phi_t'(\mu) &= f_t'(z_\mu + w(\mu,t))(\dot z_\mu + \dot w(\mu,t))\\ &= f_t'(z_\mu)(\dot z_\mu + \dot w(\mu,t)) + (f_t''(z_\mu)w(\mu,t), \dot z_\mu + \dot w(\mu,t)) + O(\|w(\mu,t)\|^2))\\ &= -tG'(z_\mu)(\dot z_\mu + \dot w(\mu,t)) + (f_0''(z_\mu)w(\mu,t), \dot w(\mu,t)))\\ &\quad - t(G''(z_\mu)w(\mu,t), \dot z_\mu + \dot w(\mu,t)) + O(\|w(\mu,t)\|^2))\\ &= -t\Gamma'(\mu) + O(t^2). \end{aligned} \tag{5.4}$$

Fix a sequence (t_n) converging to 0. Since μ_t is bounded, we may assume that (μ_{t_n}) converges to $\bar\mu$. From expansion (5.4) we have that

$$0 = \Phi_{t_n}'(\mu_{t_n}) = -t_n(\Gamma'(\mu_{t_n}) + O(t_n))$$

hence $\bar\mu$ is a critical point of Γ. A further expansion yields

$$0 = \Phi_{t_n}'(\mu_{t_n}) - t_n(\Gamma''(\bar\mu)(\mu_{t_n} - \bar\mu) + o(\mu_{t_n} - \bar\mu)) + O(t_n^2)$$

which gives for $n \to \infty$

$$(\mu_{t_n} - \bar\mu)(\Gamma''(\bar\mu) + o(1)) = O(t_n)$$

proving (5.3) for $\Gamma''(\bar\mu) \neq 0$. Viceversa let $\bar\mu$ be a critical point of Γ. Arguing as above we find as $\mu \to \bar\mu$ and for any $0 < t < t_1$

$$\Phi_t'(\mu) = t(\mu - \bar\mu)(\Gamma''(\bar\mu) + o(1)) + O(t^2)$$

hence there exists μ_t such that

$$\mu_t = \bar\mu - (\Gamma''(\bar\mu) + o(1))^{-1}O(t) \quad \text{and} \quad \Phi_t'(\mu_t) = 0.$$

To prove uniqueness of such a μ_t, we follow [4] and expand Φ_t in a critical point μ_t

$$\begin{aligned}\Phi_t''(\mu_t) &= \big(f_t''(z_{\mu_t} + w(\mu_t,t))(\dot z_{\mu_t} + \dot w(\mu_t,t)), (\dot z_{\mu_t} + \dot w(\mu_t,t))\big)\\ &= \big(f_0''(z_{\mu_t} + w(\mu_t,t))(\dot z_{\mu_t} + \dot w(\mu_t,t)), (\dot z_{\mu_t} + \dot w(\mu_t,t))\big)\\ &\quad - t\big(G''(z_{\mu_t} + w(\mu_t,t))(\dot z_{\mu_t} + \dot w(\mu_t,t)), (\dot z_{\mu_t} + \dot w(\mu_t,t))\big)\\ &= (f_0''(z_{\mu_t})\dot w(\mu_t,t), \dot w(\mu_t,t)) + (f_0'''(z_{\mu_t})w(\mu_t,t)(\dot z_{\mu_t} + \dot w(\mu_t,t)), \dot z_{\mu_t} + \dot w(\mu_t,t))\\ &\quad - t\big(G''(z_{\mu_t} + w(\mu_t,t))(\dot z_{\mu_t} + \dot w(\mu_t,t)), (\dot z_{\mu_t} + \dot w(\mu_t,t))\big)\\ &= (f_0'''(z_{\mu_t})w(\mu_t,t)\dot z_{\mu_t}, \dot z_{\mu_t}) - t\big(G''(z_{\mu_t})\dot z_{\mu_t}, \dot z_{\mu_t}\big) + O(t^2). \end{aligned} \tag{5.5}$$

Since any critical point μ_t of Φ_t gives rise to a critical point $z_{\mu_t} + w(\mu_t,t)$ of f_t, we have that

$$\begin{aligned} 0 &= (f_t'(z_{\mu_t} + w(\mu_t,t)), \ddot z_{\mu_t})\\ &= (f_t'(z_{\mu_t}) + f_t''(z_{\mu_t})w(\mu_t,t) + O(\|w(\mu_t,t)\|^2), \ddot z_{\mu_t})\\ &= -t(G'(z_{\mu_t}), \ddot z_{\mu_t}) + (f_0''(z_{\mu_t})w(\mu_t,t), \ddot z_{\mu_t}) + O(t^2). \end{aligned} \tag{5.6}$$

Differentiating $f_0''(z_{\mu_t})\dot z_{\mu_t} = 0$ and testing with $w(\mu_t,t)$ we obtain

$$0 = (f_0'''(z_{\mu_t})\dot z_{\mu_t}\dot z_{\mu_t}, w(\mu_t,t)) + (f_0''(z_{\mu_t})\ddot z_{\mu_t}, w(\mu_t,t)). \tag{5.7}$$

Putting together (5.6) and (5.7) we get

$$(f_0'''(z_{\mu_t})\dot z_{\mu_t}\dot z_{\mu_t}, w(\mu_t,t)) = -t(G'(z_{\mu_t}), \ddot z_{\mu_t}) + O(t^2)$$

hence in view of (5.5)

$$\Phi_t''(\mu_t) = -t(G'(z_{\mu_t}), \ddot{z}_{\mu_t}) - t\big(G''(z_{\mu_t})\dot{z}_{\mu_t}, \dot{z}_{\mu_t}\big) + O(t^2) = -t\Gamma''(\mu_t) + O(t^2). \tag{5.8}$$

To prove uniqueness, we choose $\delta > 0$ such that $\mathrm{sgn}\Gamma''(\mu) = \mathrm{sgn}\Gamma''(\bar{\mu}) \neq 0$ for any $|\mu - \bar{\mu}| < \delta$. From (5.8), there exists $t(\delta) > 0$ such that if $t < t(\delta)$ and μ_t is a critical point of Φ_t such that $|\mu_t - \bar{\mu}| < \delta$, then

$$\mathrm{sgn}\Phi_t''(\mu_t) = -\mathrm{sgn}\Gamma''(\bar{\mu}).$$

From (5.4) we have that for $t < t(\delta)$

$$\begin{aligned}\mathrm{sgn}\Gamma''(\bar{\mu}) &= \deg(\Gamma', B_\delta(\bar{\mu}), 0) = \deg(-\Phi_t', B_\delta(\bar{\mu}), 0)\\ &= -\sum_{\substack{y\in B_\delta(\bar{\mu})\\ \Phi_t'(y)=0}} \mathrm{sgn}\Phi_t''(y) = \#\{y \in B_\delta(\bar{\mu}) :\ \Phi_t'(y) = 0\}\,\mathrm{sgn}\Gamma''(\bar{\mu}).\end{aligned}$$

Hence $\#\{y \in B_\delta(\bar{\mu}) :\ \Phi_t'(y) = 0\} = 1$, proving uniqueness. □

Lemma 5.2. *For any $K \in L^\infty(\mathbb{R}^N)$ the operator*

$$L_K :\ u \mapsto \big(-\operatorname{div}(|x|^{-2a}\nabla)\big)^{-1}\frac{K(x)}{|x|^{bp}}|u|^{p-2}u$$

is compact from E to E.

Proof. Let $\{u_n\}$ be a bounded sequence in E and set $v_n = L_K(u_n)$, i.e.

$$-\operatorname{div}(|x|^{-2a}\nabla v_n) = \frac{K(x)}{|x|^{bp}}|u_n|^{p-2}u_n.$$

By Caffarelli-Kohn-Nirenberg inequality, $\{v_n\}$ is bounded in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ and passing to a subsequence we may assume that it converges weakly in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ and pointwise almost everywhere to some limit $v \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$. Since $\{u_n\}$ is uniformly bounded in $L^\infty(B_3(0))$, from [10] the sequence $\{v_n\}$ is uniformly bounded in $C^{0,\gamma}(B_2(0))$. Using the Kelvin transform we arrive at

$$\begin{aligned}-\operatorname{div}(|x|^{-2a}\nabla\tilde{v}_n) &= |x|^{-(N+2+2a)+bp}K(x/|x|^2)\,|u_n(x/|x|^2)|^{p-2}u_n(x/|x|^2)\\ &= K(x/|x|^2)\frac{|\tilde{u}_n|^{p-2}\tilde{u}_n}{|x|^{bp}}.\end{aligned}$$

Since $\{u_n\}$ is uniformly bounded in E, $\{\tilde{u}_n\}$ is uniformly bounded in $L^\infty(B_3(0))$ and hence from [10] the sequence $\{\tilde{v}_n\}$ is uniformly bounded in $C^{0,\gamma}(B_2(0))$. Since a uniform bound in $C^{0,\gamma}(B_2(0))$ implies equicontinuity and

$$\|(v_n - v_m)\omega_a^{-1}\|_{C^0(\mathbb{R}^N\setminus B_1(0))} \leq \text{const}\,\|\tilde{v}_n - \tilde{v}_m\|_{C^0(B_1(0))}$$

from the Ascoli-Arzelà Theorem there exists a subsequence $\{v_n\}$ strongly converging in $C^0(\mathbb{R}^N, \omega_a)$ to v. Moreover, the $C^0(\mathbb{R}^N, \omega_a)$-convergence excludes any possibility of concentration at 0 or at ∞ and $\{v_n\}$ converges strongly in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$. □

From Proposition 4.6, there exists a positive constant C_K such that $\|u\|_E < C_K$ and $C_K^{-1} < u\omega_a^{-1}$ for any solution u of (P_t) uniformly with respect to $t \in (0, 1]$. By the above lemma, the Leray-Schauder degree $\deg(Id - L_K, \mathcal{B}_K, 0)$ is well-defined, where $\mathcal{B}_K := \{u \in E :\ \|u\|_E < C_K, C_K^{-1} < u\omega_a^{-1}\}$.

Theorem 5.3. *Under the assumptions of Proposition 4.6 and for $p > 3$ we have*

$$\deg(Id - L_K, \mathcal{B}_K, 0) = -\frac{\operatorname{sgn}\Delta K(0) + \operatorname{sgn}\Delta \tilde{K}(0)}{2}.$$

Proof. By transversality, we can assume that Γ_K has only non-degenerate critical points. If not, we proceed with a small perturbation of K. By Proposition 4.6 and the homotopy invariance of the Leray-Schauder degree, for $0 < t < t_1$

$$\deg(Id - L_K, \mathcal{B}_K, 0) = \deg(Id - L_{tK}, \mathcal{B}_K, 0).$$

By Lemma 5.1 we have

$$\deg(Id - L_{tK}, \mathcal{B}_K, 0) = \sum_{\mu \in (\Phi'_{t,K})^{-1}(0)} (-1)^{\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})}$$

where $\mathfrak{m}(z_\mu + w(\mu, t), f_{t,K})$ denotes the Morse index of $f_{t,K}$ in $z_\mu + w(\mu, t)$. We will only sketch the computation of $\mathfrak{m}(z_\mu + w(\mu, t), f_{t,K})$ and refer to [3, 4, 13] for details. The spectrum of $f_0''(z_\mu)$ is completely known (see [9]) and $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ is decomposed in $\langle z_\mu \rangle \oplus T_{z_\mu} Z \oplus \langle z_\mu, T_{z_\mu} Z\rangle^\perp$, where z_μ is an eigenfunction of $f_0''(z_\mu)$ with corresponding eigenvalue $-(p - 2)$, $T_{z_\mu} Z = \ker(f_0''(z_\mu))$, and $f_0''(z_\mu)$ restricted to the orthogonal complement of $\langle z_\mu, T_{z_\mu} Z\rangle$ is bounded below by a positive constant. Consequently, to compute the Morse index $\mathfrak{m}(z_\mu + w(\mu, t), f_{t,K})$ for small t it is enough to know the behavior of $f''_{t,K}(z_\mu + w(\mu, t))$ along $T_{z_\mu} Z$. From the expansion

$$f_{t,K}(z_\mu + w(\mu, t)) = f_0(z_\mu) - t\Gamma_K(\mu) + o(t^2) = \text{const } - t\Gamma_K(\mu) + o(t^2)$$

we have that for t small

$$\mathfrak{m}(z_\mu + w(\mu, t), f_{t,K}) = 1 + \begin{cases} 1 & \text{if } \Gamma''_K(\mu) > 0 \\ 0 & \text{if } \Gamma''_K(\mu) < 0. \end{cases} \tag{5.9}$$

From (5.9) and Lemma 5.1, we know that for t small

$$\sum_{\mu \in (\Phi'_{t,K})^{-1}(0)} (-1)^{\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})} = - \sum_{\mu \in (\Gamma'_K)^{-1}(0)} (-1)^{\mathfrak{m}(\mu, -\Gamma_K)}$$
$$= \deg(\Gamma'_K, ((R_0 + 1)^{-1}, R_0 + 1), 0),$$

where R_0 is given in Lemma 5.1. From (5.1) we obtain for $\mu \to 0$

$$\Gamma'_K(\mu) = \Gamma''_K(0)\mu + o(\mu) = \text{const}\Delta K(0)\mu + o(\mu).$$

Hence $\operatorname{sgn}\Gamma'_K((R_0 + 1)^{-1}) = \operatorname{sgn}\Delta K(0)$. Using (5.2) for obtain for $\mu \to \infty$

$$\Gamma'_K(\mu) = -\mu^{-2}\Gamma'_{\tilde{K}}(\mu^{-1}) = -\text{const}\Delta\tilde{K}(0)\mu^{-3} + o(\mu^{-3}).$$

Therefore $\operatorname{sgn}\Gamma'_K((R_0 + 1)) = -\operatorname{sgn}\Delta\tilde{K}(0)$ and

$$\deg(\Gamma'_K, ((R_0 + 1)^{-1}, R_0 + 1), 0) = -\frac{\operatorname{sgn}\Delta K(0) + \operatorname{sgn}\Delta\tilde{K}(0)}{2},$$

which proves the claim. □

Proof of Theorem 1.2. It follows directly from Theorem 5.3 and Lemma A.1. □

Appendix A

Lemma A.1. *v is a solution to (1.1) if and only if $u(x) = |x|^{a-\alpha}v(x)$ solves (1.6), where $a = a(\alpha, \lambda)$ and $b = b(\alpha, \beta, \lambda)$ are given in* (1.5).

Proof. By standard elliptic regularity u and v are $C^2(\mathbb{R}^N \setminus \{0\})$. Consequently we may compute for $x \in \mathbb{R}^N \setminus \{0\}$

$$\mathrm{div}\,(|x|^{-2a}\nabla u(x)) = (a-\alpha)(N-a-\alpha-2)|x|^{-a-\alpha-2}v(x) + |x|^{\alpha-a}\mathrm{div}\,(|x|^{-2\alpha}\nabla v)$$

and hence, in view of (1.1)

$$-\mathrm{div}\,(|x|^{-2a}\nabla u(x)) = \big[\lambda + (\alpha-a)(N-2-\alpha-a)\big]\frac{u(x)}{|x|^{2a+2}} + K(x)\frac{u^{p-1}}{|x|^{p(a-\alpha+\beta)}}.$$

From (1.5) we have that $\lambda + (\alpha-a)(N-2-\alpha-a) = 0$ and $a - \alpha + \beta = b$. Since $C^\infty(\mathbb{R}^N \setminus \{0\})$ is dense in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ and $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ (see [6]), the lemma is thereby proved. □

Lemma A.2. *Let $\{K_i\}_i$ satisfy (3.1), $(u_i)_{i\in\mathbb{N}}$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then for any $R_i \to \infty$, there exists a positive constant C depending on $\lim_{i\to\infty} K_i(x_i)$ and a, b, and N such that after passing to a subsequence the function $\bar{w}_i$ defined in (3.3) is strictly decreasing for $Cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i$ where $r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$.*

Proof. Making the change of variable $y = u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i$, there results

$$\begin{aligned}\bar{w}_i(r) &= \frac{r^{\frac{N-2-2a}{2}}}{|\partial B_r(x_i)|}\int_{\partial B_r(x_i)} u_i(y)\\ &= r^{\frac{N-2-2a}{2}} \fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big).\end{aligned}$$

From the proof of Proposition 3.7 we have that for some function $g_i \in C^{0,\gamma}(B_{2R_i}(0))$

$$u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big) = u_i(x_i)\big(z^{a,b}_{K(0)}(x) + g_i(x)\big)$$

where $\|g_i\|_{C^2(B_{2R_i}(0)\setminus B_C(0))} \le \varepsilon_i$. Being $z^{a,b}_{K(0)}$ a radial function, from above we find

$$\begin{aligned}\bar{w}_i(r) &= r^{\frac{N-2-2a}{2}}u_i(x_i) \fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} \big(z^{a,b}_{K(0)}(x) + g_i(x)\big)\\ &= r^{\frac{N-2-2a}{2}}u_i(x_i)\big[z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)}) + \textstyle\fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i\big].\end{aligned}$$

A direct computation shows that

$$\begin{aligned}&\frac{d}{dr}\bar{w}_i(r)\\ &= u_i(x_i)r^{\frac{N-4-2a}{2}}(z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)}))^{\frac{p}{2}}\Big[\tfrac{N-2-2a}{2}\Big(1 - K(0)u_i(x_i)^{p-2}r^{\frac{(p-2)(N-2-2a)}{2}}\Big)\\ &\quad + \tfrac{N-2-2a}{2}\big(\textstyle\fint g_i\big)z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)})^{-\frac{p}{2}} + r\big(\textstyle\fint g_i\big)' z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)})^{-\frac{p}{2}}\Big].\end{aligned}$$

Since for $Cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i$, there results $C \le ru_i(x_i)^{2/(N-2-2a)} \le R_i$, we have that

$$\fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i \le \varepsilon_i, \qquad \frac{d}{dr} \fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i \le \varepsilon_i.$$

Moreover for $C = \left(\frac{1+\delta}{K(0)}\right)^{\frac{2}{(p-2)(N-2-2a)}}$ we have $1 - K(0)u_i(x_i)^{p-2}r^{\frac{(p-2)(N-2-2a)}{2}} \le -\delta$. Choosing $\varepsilon_i = o\Big(R_i^{-\frac{p(N-2-2a)}{2}}\Big)$ the claim follows. □

Lemma A.3. *Suppose a, b, p satisfy (1.8) and (1.5). Let $(z_i)_{i\in\mathbb{N}} \subset \mathbb{R}^N$ and consider the measures $\mu_i := |x - z_i|^{-2a}\,dx$, then we have for $0 < r < 2$ as $r \to 0$*

$$\sup_{x\in B_2(0),\, i\in\mathbb{N}} \int_{B_r(x)} |y - z_i|^{-bp} \int_{|x-y|}^{8} \frac{s\,ds}{\mu_i(B_s(x))}\,dy \to 0.$$

Proof. We use as c a generic constant that may change its value from line to line. Fix $x \in B_2(0)$. From the doubling property of the measure μ_i (see [12]) we find

$$\begin{aligned} M_i(x, |x-y|) &:= \int_{|x-y|}^{8} \frac{s\,ds}{\mu_i(B_s(x))} \\ &\le c \begin{cases} |x-y|^{-N+2a+2}, & \text{if } |x-y| > \frac{1}{2}|x-z_i| \\ |x-y|^{-N+2}|x-z_i|^{2a} + |x-z_i|^{-N+2a+2}, & \text{if } |x-y| \le \frac{1}{2}|x-z_i|. \end{cases} \end{aligned}$$

An easy calculation shows that $2a - bp > -2$ and that if $a \ge 0$ then $2a - bp \le 0$. Hence, we may estimate for $0 < r \le \frac{1}{2}|x - z_i|$ and $y \in B_r(x)$

$$|y - z_i| \ge |x - z_i| - |x - y| \ge \frac{1}{2}|x - z_i|$$

and

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x-y|)\,dy \le cr^{2+2a-bp}.$$

Since $-bp > -2 - 2a > -N$ we may use the above estimate to derive

$$\int\limits_{B_{2|x-z_i|}(x)} |y - z_i|^{-bp} M_i(x, |x-y|)\,dy \le c|x - z_i|^{2+2a-bp}.$$

Consequently we obtain for $\frac{1}{2}|x - z_i| \le r \le 2|x - z_i|$

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x-y|)\,dy \le c|x - z_i|^{2+2a-bp} \le cr^{2+2a-bp}.$$

Finally we obtain for $2|x - z_i| < r \le 2$ and $|x - y| > 2|x - z_i|$

$$|y - z_i| \ge |y - x| - |x - z_i| \ge \frac{1}{2}|y - x|$$

and

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x-y|)\,dy \le cr^{2+2a-bp},$$

which ends the proof. □

A function u will be called μ-harmonic in $\Omega \subset \mathbb{R}^N$, if $u \in D^{1,2}_{a,loc}(\mathbb{R}^N)$ and for all $\varphi \in C^\infty_c(\Omega)$ there holds

$$\int_\Omega |x|^{-2a} \nabla u \nabla \varphi = 0.$$

Let us prove a Bôcher-type theorem for μ-harmonic functions.

Theorem A.4. *Let u be a nonnegative μ-harmonic function in $\mathbb{R}^N \setminus \{0\}$. Then there exist a constant $A \geq 0$ and a Hölder continuous function B, μ-harmonic in $\mathbb{R}^N$, such that*

$$u(x) = A\,|x|^{2+2a-N} + B(x).$$

Proof. We distinguish two cases.
Case 1: there exists a sequence $x_n \to 0$ and a positive constant M such that $|u(x_n)| \leq M$. In this case the Harnack Inequality (Theorem 6.2 of [12]) implies that u is bounded. Moreover from [12, Lemma 6.15] u can be continuously extended to 0 and is a weak solution of

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = 0 \quad \text{in } \mathbb{R}^N,$$

see [6, Lemma 2.1]. Therefore from the Liouville Theorem [12, Theorem 6.10] we get that u is constant and the theorem holds with $A = 0$ and $B \equiv \text{const}$.
Case 2: $u(x_n) \to +\infty$ for any sequence $x_n \to 0$. We can extend u in 0 to be $u(0) := +\infty$, thus obtaining a lower semi-continuous function in $\mathbb{R}^N$. Moreover [12, Theorem 7.35] implies that u is super-harmonic in the sense of the definition of [12, Chapter 7], i.e.

(i) u is lower semi-continuous,
(ii) $u \not\equiv \infty$ in each component of $\mathbb{R}^N$,
(iii) for each open $D \Subset \mathbb{R}^N$ and each $h \in C^0(\mathbb{R}^N)$ μ-harmonic in D the inequality $u \geq h$ on ∂D implies $u \geq h$ in D.

Let us remark that in order to apply Theorem 7.35 in [12] we need to prove that 0 has capacity 0 with respect to our weight; indeed

$$\begin{aligned}\mathrm{cap}_{|x|^{-2a}}(\{0\},\mathbb{R}^N) &:= \inf_{\substack{u \in C^\infty_0(\mathbb{R}^N),\ u\equiv 1 \\ \text{in a neighborhood of } 0}} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \leq \mathrm{cap}_{|x|^{-2a}}(B_r,\mathbb{R}^N) \\ &\leq \mathrm{cap}_{|x|^{-2a}}(B_r, B_{2r}) \leq c r^{N-2-2a}\end{aligned}$$

for any $r > 0$, where we have used [12, Lemma 2.14]. Then $\mathrm{cap}_{|x|^{-2a}}(\{0\},\mathbb{R}^N) = 0$. From [12, Corollary 7.21] there holds

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) \geq 0 \quad \text{in the sense of distributions on } \mathbb{R}^N$$

hence from the Riesz Theorem there exists a Radon measure $\mu \geq 0$ in $\mathbb{R}^N$ such that

$$\langle -\mathrm{div}\,(|x|^{-2a}\nabla u), \varphi\rangle = \int_{\mathbb{R}^N} \varphi\, d\mu \quad \forall\, \varphi \in C^\infty_0(\mathbb{R}^N).$$

Since $\langle -\mathrm{div}\,(|x|^{-2a}\nabla u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, μ must be supported in $\{0\}$ and so $\mu = A\delta_0$ for a nonnegative constant A. Since the Green's function $G_a(x) := |x|^{2+2a-N}$ satisfies

$$-\mathrm{div}\,(|x|^{-2a}\nabla G_a) = \delta_0 \quad \text{in the sense of distributions on } \mathbb{R}^N,$$

we have that

$$-\mathrm{div}\,(|x|^{-2a}\nabla(u - AG_a)) = 0$$

in the sense of distributions on $\mathbb{R}^N$. Theorem 3.70 and Lemma 6.47 in [12] imply that $B := u - AG_a$ is Hölder continuous. □

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