Università degli Studi di Milano–Bicocca
Dottorato di Ricerca in Matematica Pura ed Applicata
XXII Ciclo

PhD Thesis

Hyperbolic Conservation Laws: $\mathbf{L}^{\infty}$ Data and Granular Matter

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Contents

Introduction

Description of the Problems

This thesis is composed of three parts. The first and the second concern the existence of solutions to a system of conservation laws without assuming the usual hypothesis on the smallness of the total variation of the initial datum. The third presents a new model for the movement of granular matter.

These results require entirely different techniques and have different aims. The first ones are analytical theorems whose relevance consists in relaxing the hypotheses found in the literature. In the latter one, the original element is the model itself, whose relevance consists in improving the descriptions provided by previous models.

In the first Chapter (see also [25]) we consider the possibly non conservative strictly hyperbolic $n \times n$ system

$$\partial_t u + A(u)\,\partial_x u = 0$$

where each characteristic field is genuinely nonlinear. Assume moreover that it is a *straight line* system i.e., with standard notation, $Dr_i \cdot r_i = 0$ for $i = 1, \ldots, n$.

When the total variation of the initial data is sufficiently small, systems of this kind generate an $\mathbf{L}^1$ Lipschitz continuous semigroup, see [10, Theorem 2]. When the total variation of the initial data is large, though finite, well posedness was proved in [4, Theorem 1] in the case of conservative systems. Stability for $\mathbf{L}^\infty$ data was proved in [17, Theorem 1].

We consider a perturbation of the system above, say

$$\partial_t u + B(u)\,\partial_x u = 0$$

where $\|B - A\|_{\mathbf{C}^{1,1}}$ is sufficiently small. For this perturbed system, the characteristic families are still genuinely nonlinear, but the straight line condition may well fail. Therefore, for data with small total variation, the perturbed system above falls in the class for which well posedness was proved in [11, Theorem 1] by means of the vanishing viscosity approach. The wave front tracking technique also proved to be effective, see [2].

In the first part of this thesis, we prove the existence of a global solution to the perturbed system under the assumption that the initial datum has finite, but not necessarily small, total variation.

As it is standard in the framework of wave front tracking, the first step is the solution of Riemann problems. We achieve it through the construction of *generalized rarefaction* and *generalized shock curves*, extending the results in [8, 38, 39] to the case of large total variation. When B is a Jacobian matrix, these generalized Lax curves coincide with the classical Lax curves.

Then, we pass to the Cauchy problem. Inserting careful estimates in the standard wave front tracking procedure, we are able to prove the convergence of piecewise constant approximate solutions, thus obtaining the existence of *generalized solutions*. This construction requires neither the total variation of the initial data to be small, nor the conservative form, nor that shock and rarefaction curves in the perturbed system coincide. In the conservative case, these solutions are the standard weak entropy solutions, see [14, 27].

Remark that as soon as the initial datum is allowed to have large total variation and the system is not of Temple type, blow up in the $\mathbf{L}^\infty$ or $\mathbf{BV}$ norms may well take place, also in the conservative case, see [35] or [27, Section 9.10].

In the case of conservation laws with $\mathbf{L}^\infty$ data, the reference result is the classical Glimm–Lax paper [31]. It refers to a 2×2 strictly hyperbolic system with each characteristic field genuinely nonlinear. Under a further assumption on the geometry of the Lax curves, [31, Theorem 5.1] proves the existence of solutions with data in $\mathbf{L}^\infty$.

In the second Chapter (see also [13]), we prove the Glimm–Lax result without the assumption taken therein on the geometry of the shock–rarefaction curves.

More precisely, the assumption [31, (c)] ensures that the interaction of two shocks of the same family yields a shock of that family and a *rarefaction* of the other family. In our construction, no assumption whatsoever of this kind is assumed.

Other attempts towards an extension of Glimm–Lax result are found in the literature. In the case of systems with coinciding shock and rarefaction curves, the well posedness in $\mathbf{L}^\infty$ is proved in [7, Theorem 1.1.], extending the previous results [5, 17].

Our proof relies on the construction of a solution as limit of ε–front tracking approximations v^ε, defined through a suitable modification of the algorithm in [15]. First, as in [31], careful decay estimates on a trapezoid allow to bound the positive variation and the $\mathbf{L}^\infty$ norm of v^ε. A further *a priori* assumption on the $\mathbf{L}^\infty$ norm of v^ε allows to inductively extend the definition of the approximate solutions globally in time. A key point is now to provide estimates so that the *a priori* assumption is abandoned. This is achieved through $\mathbf{L}^\infty$ estimates essentially based on the conservative form of the system and on the previous results on the trapezoids. It is thanks to these estimates that the result in [31] can be extended. We remark that also a significant simplification was achieved, for the original Glimm–Lax paper counts more than 100 pages, while our construction less than 30.

As a byproduct, we also obtain an existence result valid for all initial data having small $\mathbf{L}^\infty$ norm and bounded total variation, under the standard Lax condition, i.e. that each characteristic field is either genuinely non linear or linearly degenerate.

The third Chapter (see also [24]) concerns the dynamics of granular matter. In this context, two models widely considered in the literature are the Savage–Hutter [42] and the Hadeler–Kuttler [33] ones. The first reminds of the shallow water equation. It is based on the conservation of mass and on the balance of linear momentum. However, this model does not allow any evolution in the *a priori* fixed bed. The Hadeler–Kuttler model does consider erosion–deposition effects, but it lacks any energy inequality, possibly giving rise to somewhat unphysical solutions, as we show through numerical integrations.

We propose the following synthesis of the two models above, which includes both the physics of the sliding material, as in the Savage–Hutter model, and the exchange of mass between the standing material and the sliding one, as in the Hudler–Kuttler model:

$$\begin{cases} \partial_t h + \nabla \cdot (h\, v) = -\gamma h \left(\alpha - \|\nabla u\|\right) + H \\ \partial_t v + \nabla \cdot \left(\frac{1}{2} v \otimes v + gh\,\mathrm{Id}\right) = -g\nabla u - \nu(v, u_x) - \gamma [\![\alpha - \|\nabla u\|]\!]_- v + V \\ \partial_t u = \gamma h \left(\alpha - \|\nabla u\|\right) \end{cases}$$

where $[\![\cdot]\!]_-$ denotes the negative part, i.e. $[\![\xi]\!]_- = \left(\xi - |\xi|\right)/2$. Here, v defines the velocity vector field at which matter slides along the slope with profile u, g is gravity and the function ν reflects the friction between the sliding material and the surface. The constant α is the angle above which matter erodes the slope whereas below this angle matter tends to deposit, γ is the speed of this process. The terms H and V allow to describe the effects of matter being poured, or falling, on all or part of the considered slope.

Among other analytical properties of this system, we prove that the smooth solutions to the equations above dissipate the physically reasonable energy

$$E = \int_{\mathbb{R}^2} \left(\frac{1}{2}\, h\, \|v\|^2 + \frac{1}{2}\, g\, (h+u)^2\right) \mathrm{d}x$$

under realistic assumptions on the friction term $\nu(v, \nabla u)$, as it is explained in this Chapter.

Then, several numerical integrations show the main qualitative differences among the three systems. We remark that, in many instances, the asymptotic behavior of the three models is very similar. On the other hand, the transient qualitative features of the Savage–Hutter and Hadeler–Kuttler model are somewhat surprising, as the figures in Chapter 3 show. On the other hand, the behavior of the solutions to our model appears reasonable for all times.

Articles and Preprints

This Phd thesis collects the results presented in the following papers:

(1) R.M. Colombo, F. Monti: *Solutions with Large Total Variation to Nonconservative Hyperbolic Systems.* To appear on *Communications in Pure and Applied Analysis*, 2009.

(2) S. Bianchini, R.M. Colombo, F. Monti: 2×2 *Systems of Conservation Laws with* $\mathbf{L}^\infty$ *Data.* Preprint, 2009. Submitted.

(3) R.M. Colombo, G. Guerra, F. Monti: *Modeling the Dynamics of Granular Matter.* Preprint, 2009. Submitted.

Communications and Advanced Courses

I already communicated some of the results presented in this PhD thesis in the following meetings:

- "Solutions with Large Total Variation to Nonconservative Hyperbolic Systems"
  Conservation Laws and Applications
  Brescia 28.05.2008
- "Solutions with Large Total Variation to Nonconservative Hyperbolic Systems"
  Sixth meeting on Hyperbolic Conservation Laws
  L'Aquila 17-19.07.2008
- **As invited speaker:** "2×2 Systems of Conservation Laws with $\mathbf{L}^\infty$ Data"
  Seventh meeting on Hyperbolic Conservation Laws and Fluid Dynamics: Recent Results and Research Perspectives
  Trieste 31.08-4.09.2009

During the preparation of this Phd thesis I attained the following courses, schools and conferences:

- Conference: "Nonlinear Hyperbolic problems"
  Roma 28.05-01.06.2007
- Conference:"Fifth Meeting on Hyperbolic Conservation Laws"
  Trieste 21-22.06.2007
- School: "Partial Differential Equations"
  Courses of C.Dafermos and P.D'Ancona
  Cortona 15-28.07.2007
  (SMI)
- Conference: "Conservation Laws and Applications"
  Brescia 28.05.2008
- School: Nonlinear Partial Differential Equations and Applications
  Courses of S.bianchini, A.Carlen, A.Mielke, F.Otto, C.Villani
  Cetraro, 22-28.06.2008
  (CIME)
- Conference: "Sixth meeting on Hyperbolic Conservation Laws"
  L'Aquila 17-19.07.2008
- School: "Optimal Transportation, Geometry and Functional Inequalities"
  Courses of F.Barthe, W.Gangbo, F.Maggi, R.McCann
  Pisa, 28-31.10.2008
- School: "Regional PDE winter school"
  Courses of P.Raphael, S.Serfaty, S.Terracini
  Oxford, 12-13.12.2008
- School: "Modelling and Optimisation of Flows on Networks"
  Courses of:
  L.Ambrosio, A.Bressan, D.Helbing, A.Klar, C.Ringhofer, E.Zuazua
  Cetraro, 15-19.06.2009
  (CIME)

- From February 2009 to May 2009 I have been at S.I.S.S.A–I.S.A.S (Trieste) and I have partecipate to the following courses:

  "Optimal Transportation"
  Professor Stefano Bianchini.
  "Introduction to Geometric Measure Theory and BV functions"
  Dr. Massimiliano Morini and Dr. Maria Giovanna Mora

- Conference: "Nonlinear conservation laws and Applications"
  Minneapolis, 13-31.07.2009

- School: "Seventh meeting on Hyperbolic Conservation Laws and Fluid Dynamics: Recent Results and Research Perspectives"
  Courses of L.Berselli, C.Mascia, N.Gigli, L.Spinolo Trieste 31.08-4.09.2009

Chapter 1

Solutions with large total variation to nonconservative hyperbolic systems

1.1 Introduction

Consider the $n \times n$ quasilinear hyperbolic system

$$\partial_t u + A(u)\, \partial_x u = 0\,. \tag{1.1.1}$$

Call $\lambda_1^A(u) < \cdots < \lambda_n^A(u)$ the n eigenvalues of $A(u)$ and $r_1^A(u), \ldots, r_n^A(u)$ the corresponding right eigenvectors. As in [9, formula (1.12)], we assume throughout that

$$\nabla \lambda_i^A(u) \cdot r_i^A(u) > 0 \quad \text{and} \quad D r_i^A(u) \cdot r_i^A(u) = 0\,. \tag{1.1.2}$$

Remark that any genuinely nonlinear Temple system of conservation laws, when written in the Riemann coordinates, is in the form (1.1.1) and satisfies (1.1.2). With initial data having sufficiently small TV, it is well known that (1.1.1)–(1.1.2) generates an $\mathbf{L}^1$ Lipschitz continuous semigroup, see [10, Theorem 2]. In the conservative case, the well posedness of (1.1.1) was proved in [4] for initial data having large total variation, in [17] for $\mathbf{L}^\infty$ data and, in [6] for not necessarily genuinely nonlinear characteristic fields.

We consider below a perturbation of (1.1.1), i.e.

$$\partial_t u + B(u)\, \partial_x u = 0 \tag{1.1.3}$$

with B sufficiently near to A. Note that the characteristic families of system (1.1.3) are genuinely nonlinear, but the latter condition in (1.1.2) may well fail.

When the initial datum has sufficiently small total variation, solutions to (1.1.3) were defined in [11] by means of the vanishing viscosity method and in [2] using the wave front tracking technique. In the conservative case, the solutions constructed below coincide with the standard weak entropy solutions, see [14, 27].

In Section 1.2, we solve the Riemann problem for (1.1.3) through the construction of the *generalized rarefaction* and *shocks curves*, thus extending the results in [8, 38, 39] to the case of large total variation. When B is a Jacobian matrix, the generalized Lax curves defined below reduce to the classical Lax curves. Furthermore, Proposition 1.2.4 furnishes estimates on the dependence of these generalized solutions from B.

Section 1.3 is devoted to the Cauchy problem for (1.1.3). We exhibit its *generalized solutions* using an algorithm that generates approximate solutions and we show their convergence. This construction requires neither the total variation of the initial data to be small, nor the conservative form, nor that shock and rarefaction curves coincide. In fact, the results obtained in [4] in the conservative case hold also for system (1.1.1) under assumption (1.1.2). When B is sufficiently near to A, (1.1.3) inherits similar properties, without requiring that (1.1.3) is a straight line system.

It is well known that, in general, blow up in the $\mathbf{L}^\infty$ or $\mathbf{BV}$ norms may take place, also in the conservative case, as soon as the initial datum is allowed to have large total variation and the system is not of Temple type, see [35] or [27, Section 9.10].

Below we provide estimates on the distance between solutions to (1.1.1) and to (1.1.3), see 4. in Theorem 1.3.1. Under the stronger assumption that both systems be in conservation form, slightly stronger stability estimates on the dependence of solutions from the flow are proved in [12, Theorem 2.1].

When (1.1.1) is in conservation form, this algorithm yields the existence of generalized solutions with large total variation to possibly non-conservative perturbations of Temple systems. Note that, in particular, if also B is in conservation form, it is not required that (1.1.3) be a Temple system.

In the case of data with small total variation, this algorithm yields the existence of vanishing viscosity solutions to perturbations of general quasilinear hyperbolic systems. Here, *"solution"* is meant in the sense of [9, 11].

Finally, Section 1.4 is devoted to the technical details.

1.2 The Riemann problem

Throughout, $\Omega \subseteq \mathbb{R}^n$ is the closure of a nonempty open set. On the $n \times n$ matrix A, we assume the following condition:

(H) Let A belongs to $\mathbf{C}^{1,1}(\Omega; \mathbb{R}^{n\times n})$. For all $u \in \Omega$, the matrix $A(u)$ admits n real eigenvalues $\lambda_1^A(u), \ldots, \lambda_n^A(u)$, satisfying $\sup_\Omega \lambda_{i-1}^A < \inf_\Omega \lambda_i^A$, with n linearly independent right eigenvectors $r_i^A(u)$, for $i = 1, \ldots, n$. Moreover, the conditions (1.1.2) hold for all $i = 1, \ldots, n$ and all $u \in \Omega$.

Temple systems furnish a well known example of systems satisfying **(H)**, when written in the Riemann coordinates.

The left eigenvectors of $A(u)$ are $l_i^A(u)$, for $i = 1, \ldots, n$. Here and in what follows, we choose the following normalization, see also [14, formula (5.4)]:

$$\|r_i^A(u)\| = 1\,, \; l_i^A(u) \cdot r_i^A(u) = 1 \text{ and } l_j^A(u) \cdot r_i^A(u) = 0 \text{ for } i \neq j\,. \tag{1.2.1}$$

More precisely, we may also assume that $(r_1^A, \ldots, r_n^A)$ is the canonical base in $\mathbb{R}^n$. By means of a further linear transformation, we assume that $\Omega := I^n$, with $I := [-a/2, a/2]$ for a suitable $a > 0$. Introduce for a positive and sufficiently small η the interval $I_\eta := \big[-(a-\eta)/2, (a-\eta)/2\big]$ and the set $\Omega_\eta := {I_\eta}^n$.

As it is usual in Temple systems, see [6], we use the following norm and total variation in Ω:

$$\|u\| := \sum_{i=1}^{n} |u_i| \quad \text{and} \quad \mathrm{TV}(u) := \sum_{i=1}^{n} \mathrm{TV}(u_i)\,.$$

Concerning the perturbed system (1.1.3), Proposition 1.4.2 below ensures that $B(u)$ admits eigenvalues λ_i^B, right and left eigenvectors r_i^B and l_i^B, normalized as in (1.2.1), for $i = 1, \ldots, n$. For any $\eta > 0$ sufficiently small, on the set $\mathbf{C^{1,1}}(\Omega_\eta; \mathbb{R}^{n\times n})$ we use the $\mathbf{C^{1,1}}$ norm

$$\begin{aligned} \|B\|_{\mathbf{C^{1,1}}} &:= \max\left\{ \|B\|_{\mathbf{C^0}}, \|DB\|_{\mathbf{C^0}}, \sup_{u,w\in\Omega_\eta,\, u\neq w} \frac{\left\|DB(u) - DB(w)\right\|}{\|u-w\|} \right\} \\ \|B\|_{\mathbf{C^0}} &:= \sup_{u\in\Omega_\eta} \sup_{\|v\|_{\mathbb{R}^n}\leq 1} \left\|B(u)v\right\|. \end{aligned}$$

We denote by $\mathcal{B}_\eta(A, \delta)$ the open sphere centered at A with radius δ with respect to the $\mathbf{C^{1,1}}$ norm on Ω_η. $\mathcal{O}(1)$ denotes a real number dependent on Ω and $\mathcal{B}_0(A, \delta_o)$, for a fixed $\delta_o > 0$.

In [9], a Riemann solver for (1.1.1) was defined, its meaning being justified, *a posteriori*, by the vanishing viscosity limit, provided the jump in the initial data is sufficiently small. Here, we extend the construction therein to the perturbation (1.1.3) of (1.1.1). As soon as (1.1.3) is in conservation form, the construction below yields the standard Lax solutions to Riemann problems.

We first construct the *"rarefaction curves"* for (1.1.3) as integral curves of the right eigenvectors.

Proposition 1.2.1. *For all $\eta > 0$, there exists a $\delta > 0$ such that for all $B \in \mathcal{B}_\eta(A, \delta)$, for all $i = 1, \ldots, n$, all $u_i \in I$ and all $u^l \in \Omega_\eta$, there exists a unique curve $u_i \mapsto R_i^B(u^l, u_i)$ with $R_i^B \in \mathbf{C^{1,1}}(\Omega_\eta \times I; \Omega)$ and*

$$\begin{aligned} R_i^B(u^l, u_i^l) &= u^l \qquad \frac{\partial}{\partial u_i} R_i^B(u^l, u_i) = r_i^B(R_i^B(u^l, u_i)) \\ \frac{\partial}{\partial u_i} \lambda_i^B(R_i^B(u^l, u_i)) &> 0\,. \end{aligned}$$

Moreover, for all $B_1, B_2 \in \mathcal{B}_\eta(A, \delta)$,

$$\|R_i^{B_1} - R_i^{B_2}\|_{\mathbf{C^{1,1}}(\Omega_\eta\times I;\Omega)} \leq \mathcal{O}(1)\, \|B_1 - B_2\|_{\mathbf{C^{1,1}}}\,.$$

The proof relies on the basic theory of ordinary differential equations and is deferred to Section 1.4.

Passing to the shock curves we extend the procedure based on [14, formula (5.24)]. For $u^l, u \in \Omega$, introduce the eigenvalue $\lambda_i(u^l, u)$ and the left eigenvector $l_i^B(u^l, u)$ of the averaged matrix

$$B(u^l, u) := \int_0^1 B(\vartheta u + (1-\vartheta)u^l)\, d\vartheta\,. \tag{1.2.2}$$

For B in a neighborhood of A, $S\colon \Omega_\eta \times I \to \Omega$ and any i, define the map G^i by

$$\Big[(G^i(B, S))(u^l, u_i)\Big]_j := \begin{cases} l_j^B(u^l, S(u^l, u_i))t(S(u^l, u_i) - u^l) & j \neq i \\ (S(u^l, u_i))_i & j = i\,. \end{cases}$$

Proposition 1.2.2. *For all $\eta > 0$, there exists a $\delta > 0$ such that for all $B \in \mathcal{B}_\eta(A,\delta)$, for all $i = 1,\ldots,n$, all $u_i \in I$ and all $u^l \in \Omega_\eta$, there exists a unique curve $u_i \mapsto S_i^B(u^l,u_i)$ with $S_i^B \in \mathbf{C^{1,1}}(\Omega_\eta \times I;\Omega)$ and*

$$\begin{array}{rclcrcl} S_i^B(u^l,u_i^l) & = & u^l & \quad & \dfrac{\partial}{\partial u_i} S_i^B(u^l,u_i^l) & = & r_i^B(u^l) \\ \dfrac{\partial}{\partial u_i}\lambda_i^B(S_i^B(u^l,u_i)) & > & 0 & & \Big[(G^i(B,S_i^B))(u^l,u_i)\Big]_j & = & \left\{\begin{array}{ccc} 0 & j \neq i \\ u_i & j = i. \end{array}\right. \end{array}$$

Moreover, for all $B_1, B_2 \in \mathcal{B}_\eta(A,\delta)$,

$$\left\| S_i^{B_1} - S_i^{B_2} \right\|_{\mathbf{C^{1,1}}(\Omega_\eta \times I;\Omega)} \leq \mathcal{O}(1) \left\| B_1 - B_2 \right\|_{\mathbf{C^{1,1}}}.$$

The proof relies on a careful application of the Implicit Function Theorem in $\mathbf{C^{1,1}}$ and is deferred to Section 1.4.

For $i = 1,\ldots,n$ and $u \in \Omega$, introduce the generalized i-th Lax curve

$$\begin{array}{rcl} L_i^B(u^l,u_i) & := & \left\{\begin{array}{ll} R_i^B(u^l,u_i) & \text{if } u_i \geq u_i^l \\ S_i^B(u^l,u_i) & \text{if } u_i < u_i^l \end{array}\right. \\ L^B(u^l,u) & := & L_n^B(\ldots L_2^B(L_1^B(u^l,u_1),u_2),\ldots,u_n) \end{array}$$

where $u = (u_1,\ldots,u_n)$. Note that by propositions 1.2.1 and 1.2.2,

$$L^B \in \mathbf{C^{1,1}}(\Omega_\eta \times \Omega;\Omega) \tag{1.2.3}$$

$$\left\| L^{B_1} - L^{B_2} \right\|_{\mathbf{C^{1,1}}(\Omega_\eta \times \Omega;\Omega)} \leq \mathcal{O}(1) \left\| B_1 - B_2 \right\|_{\mathbf{C^{1,1}}} \tag{1.2.4}$$

for all B_1 and $B_2 \in \mathcal{B}_\eta(A,\delta)$. For later use, we write the Lax curves also as

$$w = \mathcal{L}_i^B(u^l,\sigma_i^B) \quad \text{with parameter} \quad \sigma_i^B = L_i^B(u^l,w_i) - u_i^l.$$

Denote $\sigma^B = (\sigma_1^B,\ldots,\sigma_n^B)$ and note that, in particular,

$$\sigma^A(u^l,u^r) = \left(u_1^r - u_1^l,\ldots,u_n^r - u_n^l\right).$$

We are now ready to globally solve the Riemann Problem for (1.1.3), i.e.

$$\left\{\begin{array}{l} \partial_t u + B(u)\,\partial_x u = 0 \\ u(0,x) = \left\{\begin{array}{lll} u^l & \text{if } x & < 0 \\ u^r & \text{if } x & \geq 0 \end{array}\right. \end{array}\right. \tag{1.2.5}$$

Proposition 1.2.3. *For all $\eta > 0$, there exists a positive δ such that for all $B \in \mathcal{B}_\eta(A,\delta)$, there exists a map $U^B \in \mathbf{C^{1,1}}(\Omega_\eta \times \Omega_\eta;\Omega)$ such that*

$$L^B\left(u^l, U^B(u^l,u^r)\right) = u^r$$

and, in particular, $U^A(u^l,u^r) = u^r$.

The proof is deferred to Section 1.4. We may now define the solution to the Riemann problem (1.2.5) as in [14, formula (5.45)]: letting $w^0 = u^l$ and $w^n = u^r$, this solution is a sequence of constant states $w^i = \mathcal{L}_i^B(w^{i-1}, \sigma_i^B)$ separated by a shock traveling with speed $\lambda_i(w^{i-1}, w^i)$ if $\sigma_i^B < 0$, or by a rarefaction, if $\sigma_i^B > 0$.

Proposition 1.2.4. *For all $\eta > 0$, there exists $\delta > 0$ such that for all $B_1, B_2 \in \mathbf{C^1}(\Omega_\eta; \mathbb{R}^{n\times n})$, with $\|B_1 - A\|_{\mathbf{C^1}} < \delta$ and $\|B_2 - A\|_{\mathbf{C^1}} < \delta$, the corresponding solutions u^{B_1}, u^{B_2} to* (1.2.5) *satisfy*

$$\left\|u^{B_1}(t) - u^{B_2}(t)\right\|_{\mathbf{L}^1} \leq \mathcal{O}(1)\ \|B_1 - B_2\|_{\mathbf{C^1}} \left\|u^l - u^r\right\| t\,.$$

The proof is an immediate consequence of Lemma 1.4.3 in Section 1.4.

Whenever $B = Dg$ for a suitable smooth flow g, the above construction yields the standard Lax solution to Riemann problems.

When the jump $\left\|u^l - u^r\right\|$ is sufficiently small, we recover the construction in [8, Section 4.1], see also [39, Section 5] and [38].

1.3 The Cauchy problem

In this section we prove the existence of solutions to the Cauchy problem

$$\begin{cases} \partial_t u + B(u)\,\partial_x u = 0 \\ u(0,x) = \overline{u}(x) \end{cases} \tag{1.3.1}$$

In the conservative case, the results in [4, 6, 17] ensure that the unperturbed system (1.1.1) generates a *Standard Riemann Semigroup*, as defined in [14, Chapter 8], defined on all functions with uniformly bounded total variation.

Theorem 1.3.1. *Let A satisfy assumption **(H)**. For all $\eta \in \,]0, a[$, there exists a positive δ such that for all $B \in \mathcal{B}_\eta(A, \delta)$ and for all*

$$\overline{u} \in \mathbf{L}^1(\mathbb{R}; \Omega) \qquad \textit{with} \qquad \mathrm{TV}(\overline{u}) < a - 2\eta \tag{1.3.2}$$

the Cauchy problem (1.3.1) *admits a generalized solution $u = u(t,x)$ defined for every $t \geq 0$. Moreover,*

1. *$u(t, \mathbb{R}) \subseteq \Omega_\eta$ for all $t \geq 0$.*
2. *$\mathrm{TV}\left(u(t)\right) \leq a - \eta$ for all $t \geq 0$.*
3. *There exists a positive H such that for all $t, s \geq 0$*
$$\left\|u(t) - u(s)\right\|_{\mathbf{L}^1(\mathbb{R};\Omega_\eta)} \leq H\,|t - s|\,.$$
4. *If $A = Df$ for a smooth f, then there exists a constant $\mathcal{H}$ such that*
$$\left\|u(t) - S_t\overline{u}\right\|_{\mathbf{L}^1} \leq \mathcal{H} \cdot \|B - A\|_{\mathbf{C^1}} \cdot \mathrm{TV}(\overline{u}) \cdot t$$
where S is the Standard Riemann Semigroup generated by f. If $\overline{w}$ satisfies (1.3.2) *and w is the corresponding solution to* (1.3.1)*, then*
$$\left\|u(t) - w(t)\right\|_{\mathbf{L}^1} \leq \mathcal{H} \cdot \Big(\|\overline{u} - \overline{w}\|_{\mathbf{L}^1} + \left(\mathrm{TV}(\overline{u}) + \mathrm{TV}(\overline{w})\right) \|B - A\|_{\mathbf{C^1}} t\Big)\,.$$

5. *If* $\mathrm{TV}(\overline{u})$ *is sufficiently small, then the solution* u *coincides with the vanishing viscosity solution constructed in [2, 9, 11].*

6. *If* $B = Dg$ *for a suitable flow* g*, then the solution* u *is a weak entropy solution to* (1.3.1).

We now describe the wave front tracking algorithm that generates approximate solutions to (1.3.1) and on which the proof of Theorem 1.3.1 is based.

Fix a positive η. For all $\nu \in \mathbb{N}$, at $t = 0$ we consider a piecewise constant approximation $\overline{u}^\nu$ of $\overline{u}$, i.e. a function $\overline{u}^\nu(x) = \sum_{\alpha=1}^{N_\nu} \overline{u}_\alpha \chi_{[x_\alpha, x_{\alpha+1}[}(x)$ such that: $\overline{u}_\alpha \in \Omega_\eta$, $\forall \alpha = 1, \ldots, N_\nu$; $\mathrm{TV}(\overline{u}_\nu) \leq \mathrm{TV}(\overline{u})$ and $\|\overline{u}^\nu - \overline{u}\|_{\mathbf{L}^1} < 1/\nu$. By Proposition 1.2.3, every Riemann problem (1.2.5) with $u^l = \overline{u}_{\alpha-1}$ and $u^r = \overline{u}_\alpha$ admits a solution, provided B is sufficiently near to A. We do not approximate the shocks in $u^\nu(x,t)$. On the contrary, rarefaction waves are substituted by rarefaction fans as in [14, formula (7.25)], each wavelet having size at most $1/\nu$.

Then, $u^\nu(x,t)$, defined as gluing of these approximate solutions to the Riemann problems above, is a piecewise function well defined up to the first time t when the first set of interactions occurs.

By slightly perturbing the speed of waves, we can assume that every collision involves only two incoming fronts, as in [14, Chapter 7].

Beyond an interaction time, the solution u^ν is extended by means of the *Accurate Riemann Solver* or the *Simplified Riemann Solver*, see [14, Paragraph 7.2], possibly obtaining *non-physical* waves. More precisely, following [5], we fix a parameter $\rho_\nu > 0$ and use the latter solver whenever the product of the sizes of the interacting waves is smaller than ρ_ν, or when one of the interacting waves is non-physical. In the other cases we use the Accurate solver.

This algorithm allows to extend u beyond time t provided

- $u^\nu(t-,\cdot) \in \Omega_\eta$, so that the Riemann problems generated by wave-front interactions are solvable, and
- the number of wave fronts in $u^\nu(t-,\cdot)$ is finite.

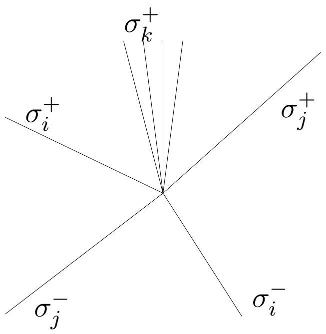
Following the classical strategy by Glimm, we seek a functional equivalent to the total variation and non increasing along approximate solutions. The starting point is a set of estimates for the change in the wave sizes at interactions.

At time $t > 0$, for $B \in \mathcal{B}_\eta(A,\delta)$, consider the piecewise constant approximate solution $u = \sum_\alpha u_\alpha \chi_{[x_\alpha, x_{\alpha+1}[}$. Define $\sigma_\alpha^B = (\sigma_{1,\alpha}^B, \ldots, \sigma_{n,\alpha}^B)$ as the size of the wave at x_α as $\sigma_\alpha^B = \sigma^B(u_{\alpha-1}, u_\alpha)$. Non-physical waves are assigned to a fictitious $(n+1)$-th family and we set $\sigma_{n+1,\alpha} := \left\| u(t, x_\alpha+) - u(t, x_\alpha-) \right\|$.

The next proposition provides the key estimates used in the sequel.

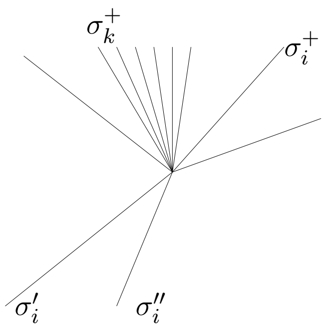
Proposition 1.3.2. *Fix a positive* η*. Then, there exist positive* δ *and* C *such that for all* $B \in \mathcal{B}_\eta(A,\delta)$*, the following estimates hold.*

1. *Let* σ_i^-*, respectively* σ_j^-*, be the size of the incoming* i*-wave, respectively* j*-wave, with* $j > i$*. Let* $\sigma_1^+, \ldots, \sigma_n^+$ *the sizes of the outgoing waves. Then,*
$$\left|\sigma_i^+ - \sigma_i^-\right| + \left|\sigma_j^+ - \sigma_j^-\right| + \sum_{k \neq i,j} \left|\sigma_k^+\right| \leq C \left\|B - A\right\|_{\mathbf{C}^{1,1}} \left|\sigma_i^- \sigma_j^-\right|$$



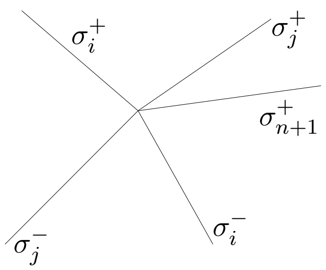
2. *Let σ_i' and σ_i'' be the sizes of the two incoming waves, both belonging to the i-family. Let $\sigma_1^+, \ldots, \sigma_n^+$ the sizes of the outgoing waves. Then,*

$$\left|\sigma_i^+ - \sigma_i' - \sigma_i''\right| + \sum_{k \neq i} \left|\sigma_k^+\right| \leq C \left\|B - A\right\|_{\mathbf{C^{1,1}}} \left|\sigma_i'\sigma_i''\right|$$



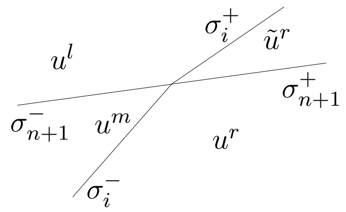
3. *Let σ_i^-, σ_j^- be the sizes of the two incoming waves. Their interaction produces one or two outgoing physical waves and a non-physical wave of size σ_{n+1}^+. Then,*

$$\sigma_{n+1}^+ \leq C \left\|B - A\right\|_{\mathbf{C^{1,1}}} \left|\sigma_i^- \sigma_j^-\right|$$



4. *Let a non-physical wave of size σ_{n+1}^-, connecting the states u^l and u^m, interact with a physical wave of size σ_i^-. Their interaction produces a non-physical wave of size σ_{n+1}^+ that connects the states $\tilde{u}^r$ and u^r. Then,*

$$\left|\sigma_{n+1}^+ - \sigma_{n+1}^-\right| \leq C \left\|B - A\right\|_{\mathbf{C^{1,1}}} \left|\sigma_i^- \sigma_{n+1}^-\right|$$



The proof is deferred to Section 1.4.

Define the *total strength of waves*, the *wave interaction potential* and the *Glimm functional* respectively by

$$\begin{aligned} V^B(u) &:= \sum_\alpha \left\|\sigma_\alpha^B\right\|, \qquad Q^B(u) := \sum_{(\sigma^B_{i,\alpha},\sigma^B_{j,\beta})\in\mathcal{A}^B} \left|\sigma^B_{i,\alpha}\sigma^B_{j,\beta}\right| \\ \Upsilon^B(u) &:= V^B(u) + K\,\|B-A\|_{\mathbf{C^{1,1}}}\, Q^B(u) \end{aligned} \tag{1.3.3}$$

where $\mathcal{A}^B$ is the natural extension to the present case of the set of all couples of approaching wave-fronts, see [14, Paragraph 3, Section 7.3]. Indeed, $(\sigma^B_{i,\alpha},\sigma^B_{j,\beta}) \in \mathcal{A}^B$ if and only if either $i > j$ and $x_\alpha < x_\beta$, or $i = j \leq n$ and $\min\{\sigma^B_{i,\alpha},\sigma^B_{j,\beta}\} < 0$. The constant K is defined in (1.3.5) below.

Consider a time t when two fronts σ' and σ'' interact. Using Proposition 1.3.2, in any of the possible interactions, we have

$$\begin{aligned} \Delta V^B(t) &\leq C\,\|B-A\|_{\mathbf{C^{1,1}}}\,\left|\sigma'\sigma''\right| \\ \Delta Q^B(t) &\leq \Big(C\,\|B-A\|_{\mathbf{C^{1,1}}}\,V^B(t-) - 1\Big)\,\left|\sigma'\sigma''\right| \\ \Delta\Upsilon^B(t) &\leq \Big(C + CKV^B(t-)\|B-A\|_{\mathbf{C^{1,1}}} - K\Big)\,\|B-A\|_{\mathbf{C^{1,1}}}\,\left|\sigma'\sigma''\right| \end{aligned} \tag{1.3.4}$$

where $\Delta V^B(t) := V^B(u(t+)) - V^B(u(t-))$. Then, it is easy to see that $\Delta\Upsilon^B(t) \leq 0$ as soon as

$$\delta < 1/(2\,C\,\Upsilon^B(0)) \quad \text{and} \quad K \geq 2C\,. \tag{1.3.5}$$

The above construction ensures that Υ^B is a non increasing function of time along any approximate solution.

The following proposition provides a bound on the total number of waves, in order to avoid the generation of cluster points of interactions.

Proposition 1.3.3. *Let $u^\nu(t,x)$ be an approximate solution. Then, the number of wave fronts is finite.*

The proof is deferred to Section 1.4.

Now, we have a sequence of well defined approximate solutions u^ν such that $u^\nu(t,\cdot) \in \Omega_\eta$ for every t. In fact, using (1.4.2), the properties of the Glimm functional and possibly reducing the choice of δ,

$$\begin{aligned} \mathrm{TV}\left(u^\nu(t)\right) &\leq \left(1+\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^{1,1}}}\right)\Upsilon^B(u^\nu(t)) && \text{by (1.3.3)} \\ &\leq \left(1+\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^{1,1}}}\right)\Upsilon^B(u^\nu(0)) && \text{by (1.3.4) and (1.3.5)} \\ &\leq \left(1+\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^{1,1}}}\right)^2\,V^B(u^\nu(0)) && \text{by (1.3.3)} \\ &\leq \left(1+\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^{1,1}}}\right)\,\mathrm{TV}(u^\nu(0)) && \text{by (1.3.3)} \\ &\leq \left(1+\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^{1,1}}}\right)\,(a-2\eta) && \text{by (1.3.2)} \\ &\leq a-\eta \end{aligned}$$

provided $\|B - A\|_{\mathbf{C}^{1,1}} < \delta$ and δ is sufficiently small.

Proceeding as in [14, Paragraph 7.4], we prove the uniform Lipschitz continuity of the approximate solution with respect to time. Then, by the refinement [14, Theorem 2.4] of Helly Compactness Theorem, we can extract a subsequence of approximate solutions, say u^ν, that converges to a limit function u in $\mathbf{L}^1_{\mathbf{loc}}$ with $u(t,\mathbb{R}) \subset \Omega_\eta$ and $\mathrm{TV}\,\big(u(t)\big) \le a - \eta$ for all $t \ge 0$, proving 1., 2. and 3. in Theorem 1.3.1. We define this function u as the generalized solution to the Cauchy problem (1.3.1).

To prove the first estimate in 4., we use [14, Theorem 2.9]. Indeed,

$$
\begin{aligned}
\big\|u(t) - S_t\overline{u}\big\|_{\mathbf{L}^1} &= \lim_{\nu\to+\infty} \big\|u^\nu(t) - S_t\overline{u}^\nu\big\|_{\mathbf{L}^1} \\
&\le \lim_{\nu\to+\infty} H \int_0^t \liminf_{h\to 0} \frac{1}{h}\big\|u^\nu(\tau+h) - S_h u^\nu(\tau)\big\|_{\mathbf{L}^1}\, d\tau\,.
\end{aligned}
\tag{1.3.6}
$$

We localize the computation of the $\mathbf{L}^1$-norm above in a suitable neighborhood $\mathcal{I}_\alpha =]x_\alpha - \varepsilon, x_\alpha + \varepsilon[$ of a point of jump x_α in $u^\nu(\tau)$. Let u^B_α be the solution to the Riemann problem (1.2.5) with $u^l = u^\nu(\tau, x_\alpha-)$ and $u^r = u^\nu(\tau, x_\alpha+)$. Then, summing up over all points of jump in $u^\nu(\tau)$, by Proposition 1.2.4 we have

$$
\begin{aligned}
& \big\|u^\nu(\tau+h) - S_h u^\nu(\tau)\big\|_{\mathbf{L}^1} \\
=\ & \sum_\alpha \big\|u^\nu(\tau+h) - S_h u^\nu(\tau)\big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} \\
\le\ & \sum_\alpha \Big\|u^\nu(\tau+h) - u^B_\alpha(\tau+h)\Big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} + \Big\|u^B_\alpha(\tau+h) - S_h u^\nu(\tau)\Big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} \\
\le\ & \sum_{x_\alpha \text{ rarefaction or non-physical}} \Big\|u^\nu(\tau+h) - u^B_\alpha(\tau+h)\Big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} \\
& + \mathcal{O}(1)\,\|B - A\|_{\mathbf{C}^1} \left(\sum_\alpha \big\|u^\nu(\tau, x_\alpha+) - u^\nu(\tau, x_\alpha-)\big\|\right) h \\
\le\ & \mathcal{O}(1)\,\big(1 + \|B - A\|_{\mathbf{C}^{1,1}}\big)\,\frac{1}{\nu} h + \mathcal{O}(1)\,\|B - A\|_{\mathbf{C}^1}\,\mathrm{TV}\,\big(u^\nu(\tau)\big)\, h
\end{aligned}
\tag{1.3.7}
$$

Here, we used the estimates

$$
\begin{aligned}
\sum_{x_\alpha \text{ rarefaction}} \Big\|u^\nu(\tau+h) - u^B_\alpha(\tau+h)\Big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} &\le \sum_{x_\alpha \text{ rarefaction}} \mathcal{O}(1)\,\frac{1}{\nu^2}\, h \\
&\le \mathcal{O}(1)\,\frac{1}{\nu}\, h \\
\sum_{x_\alpha \text{ non-physical}} \Big\|u^\nu(\tau+h) - u^B_\alpha(\tau+h)\Big\|_{\mathbf{L}^1(\mathcal{I}_\alpha)} &\le \mathcal{O}(1)\,\|B - A\|_{\mathbf{C}^{1,1}}\,\frac{1}{\nu}\, h
\end{aligned}
$$

that are proved exactly as in [14, steps 5 and 6 in Paragraph 7.3]. Inserting (1.3.7) in (1.3.6), using the fact that Υ is non increasing and passing to the limit $\nu \to +\infty$, the former estimate in 4. is proved.

The latter estimate in 4. then follows by the triangle inequality.

To prove 5., assume that the total variation of the initial data $\overline{u}$ is sufficiently small. Then, the vanishing-viscosity solution coincides with the unique limit of front tracking approximations [11, Theorem 1]. Hence, the solution u defined in the previous section can be regarded as the limit of solutions to the parabolic problems

$$\begin{cases} \partial_t u + B(u)\,\partial_x u = \varepsilon\,\partial_{xx} u \\ u(0,x) = \overline{u}(x) \end{cases}$$

as $\varepsilon \to 0$. Recall that if $\mathrm{TV}(\overline{u})$ is sufficiently small, the results in [2, 11] ensure the convergence of the above limit and the existence of a solution to (1.3.1) without requiring condition (1.1.2).

Finally, to prove 6., assume that $B = Dg$, for a suitable smooth flow g. The standard limiting procedure, shows that the solution constructed above satisfies the integral equality in the definition of weak solution and the integral inequality in the definition of entropy admissible solution.

1.4 Technical proofs

For the sake of completeness, we state without proof, the Implicit Function Theorem in the form that is used below.

Theorem 1.4.1. *Let X, Y be Banach spaces and $F\colon \mathcal{X} \times \mathcal{Y} \longrightarrow Y$ be a continuous function, with $\mathcal{X} \subset X$ and $\mathcal{Y} \subset Y$. Let $\overline{x} \in \mathring{\mathcal{X}}$ and $\overline{y} \in \mathring{\mathcal{Y}}$ be such that $F(\overline{x}, \overline{y}) = 0$, $D_y F(x,y)$ exists and is continuous in $(\overline{x}, \overline{y})$, $D_y F(\overline{x}, \overline{y})$ is invertible. Then, there exist neighborhoods $\tilde{\mathcal{X}}$ of $\overline{x}$, $\tilde{\mathcal{Y}}$ of $\overline{y}$ and a continuous function $\varphi\colon \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{Y}}$ such that:*

$$y = \varphi(x) \iff x \in \tilde{\mathcal{X}}\,,\ y \in \tilde{\mathcal{Y}} \text{ and } F(x,y) = 0$$

Moreover, if F is Lipschitz continuous in x, then φ is Lipschitz continuous.

The next proposition is referred to the averaged matrix (1.2.2). To simplify the notation, we let $\lambda_i^B(u) = \lambda_i^B(u,u)$, $r_i^B(u) = r_i^B(u,u)$ and $l_i^B(u) = l_i^B(u,u)$.

Proposition 1.4.2. *Fix $\eta > 0$. Let A satisfy **(H)**. Then, there exists a positive δ such that for all $B \in \mathcal{B}_\eta(A,\delta)$, the averaged matrix (1.2.2) satisfies:*

1. *for all $u^l, u \in \Omega$, $B(u^l,u)$ admits the n uniformly strictly separated real eigenvalues $\lambda_1^B(u^l,u), \ldots, \lambda_n^B(u^l,u)$ with the corresponding right, respectively left, eigenvectors $r_i^B(u^l,u)$, respectively $l_i^B(u^l,u)$, for $i = 1\ldots,n$ and normalized as in (1.2.1);*

2. *for $i = 1,\ldots,n$, the i-th characteristic field is genuinely non linear, i.e. for all $u \in \Omega$, $\nabla\lambda_i^B(u) \cdot r_i^B(u) > 0$;*

3. *for $i = 1,\ldots,n$ and for all $B_1, B_2 \in \mathcal{B}_\eta(A,\delta)$,*

$$\begin{aligned} \left\| \lambda_i^{B_1} - \lambda_i^{B_2} \right\|_{\mathbf{C}^{1,1}} &\le \mathcal{O}(1)\, \|B_1 - B_2\|_{\mathbf{C}^{1,1}} \\ \left\| r_i^{B_1} - r_i^{B_2} \right\|_{\mathbf{C}^{1,1}} &\le \mathcal{O}(1)\, \|B_1 - B_2\|_{\mathbf{C}^{1,1}}\,. \end{aligned}$$

Proof. Denote by Δ the determinant function. Let $X := \mathbf{C^{1,1}}(\Omega\times\Omega;\mathbb{R}^{n\times n})$, $Y := \mathbf{C^{1,1}}(\Omega\times\Omega;\mathbb{R})$ and define $F\colon X\times Y \mapsto Y$ by $F(B,\lambda) := \Delta(B-\lambda\,\mathrm{Id})$. Clearly, $F \in \mathbf{C^0}(X\times Y;Y)$. Moreover, F is Fréchet differentiable with respect to λ, $D_\lambda F(B,\lambda)\colon v \mapsto \Delta'(B-\lambda\,\mathrm{Id})v$ is linear and, by the compactness of Ω, bounded. The map $(B,\lambda)\mapsto D_\lambda F(B,\lambda)$ is continuous with respect to the operator norm by the compactness of Ω and the regularity of Δ.

For any index i in $\{1,\ldots,n\}$, $F(A,\lambda_i^A)=0$. Moreover, since λ_i^A is a simple eigenvalue, $D_\lambda F(A,\lambda_i^A) = \Delta'\left(A-\lambda_i^A\,\mathrm{Id}\right)\neq 0$. Furthermore, by the compactness of Ω, there exists $c>0$ such that for $u^l,u\in\Omega$, $1/c < \left|\Delta'(A(u^l,u)-\lambda_i^A(u^l,u)\right| < c$. Hence, the map $D_\lambda F(A,\lambda_i^A)$ is invertible and its inverse is continuous.

An application of Theorem 1.4.1 yields the existence of a positive δ_i and, for any $B\in\mathcal{B}_\eta(A,\delta_i)$, of a map λ_i such that $\lambda_i^B(u^l,u)$ is an eigenvalue of $B(u^l,u)$, for all $u^l,u\in\Omega$.

Define $\delta := \min_i \delta_i$. Possibly reducing δ, we also have the inequalities: $\sup\lambda_{i-1}^B < \inf\lambda_i^B$.

For $B\in\mathcal{B}_\eta(A,\delta)$ and $i=1,\ldots,n$, let $H_i(B) = B-\lambda_i^B\,\mathrm{Id}$. The matrix $H_i(A)$ has rank $n-1$. Hence, there exist $n-1$ row vectors linearly independent, say $h_1^i(A),\cdots,h_{n-1}^i(A)$. Possibly reducing the choice of δ, also $h_1^i(B),\cdots,h_{n-1}^i(B)$ are linearly independent. In a neighborhood of (A,r_i^A), apply the Implicit Function Theorem to the map

$$(B,r)\to\left[(r\cdot r-1),(h_1^i(B)\cdot r),\cdots,(h_{n-1}^i(B)\cdot r)\right]$$

and obtain the existence of a positive δ_i such that for all $B\in\mathcal{B}_\eta(A,\delta_i)$, $r_i^B(u^l,u)$ is the normalized right eigenvector of $B(u^l,u)$ corresponding to $\lambda_i^B(u^l,u)$, for all $u^l,u\in\Omega$.

The proof of 1. is thus completed. By continuity, 2. immediately follows. The estimates in 3. follow from the Lipschitz continuity of the implicit function, see Theorem 1.4.1. $\square$

Proof of Proposition 1.2.1. By Proposition 1.4.2, if $B\in\mathcal{B}_\eta(A,\delta)$, the Cauchy problem

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial u_i}R_i^B(u^l,u_i) = r_i^B(R_i^B(u^l,u_i))\\ R_i^B(u^l,u_i^l)=u^l\end{array}\right. \tag{1.4.1}$$

where $u^l\in\Omega_\eta$ and $u_i\in I$, satisfies the classical well posedness theorems on o.d.e.s. Hence, it admits a unique global solution $R_i^B\in\mathbf{C^{1,1}}$. Straightforward computations show that its first and second derivative satisfy the equalities in the statement. Apply 3. in Proposition 1.4.2 and use (1.2.1) to obtain:

$$\begin{aligned}
&\left\|R_i^A(u^l,u_i)-R_i^B(u^l,u_i)\right\|\\
\leq\;&\int_{u_i^l}^{u_i}\left\|r_i^A(R_i^A(u^l,w_i))-r_i^A(R_i^B(u^l,w_i))\right\|dw_i\\
&\quad+\int_{u_i^l}^{u_i}\left\|r_i^A(R_i^B(u^l,w_i))-r_i^B(R_i^B(u^l,w_i))\right\|dw_i\\
\leq\;&\left\|r_i^A\right\|_{\mathbf{C^1}}\int_{u_i^l}^{u_i}\left\|R_i^A(u^l,w_i))-R_i^B(u^l,w_i))\right\|dw_i+2a\left\|r_i^A-r_i^B\right\|_{\mathbf{C^0}}\\
\leq\;&\mathcal{O}(1)\,\|B-A\|_{\mathbf{C^0}}+\int_{u_i^l}^{u_i}\left\|R_i^A(u^l,w_i))-R_i^B(u^l,w_i))\right\|dw_i
\end{aligned}$$

By Grönwall lemma

$$\left\| R_i^A - R_i^B \right\|_{\mathbf{C^0}} \leq \mathcal{O}(1) \, \|B - A\|_{\mathbf{C^0}} e^{2a\|r_i^A\|_{\mathbf{C^1}}} \leq \mathcal{O}(1) \, \|B - A\|_{\mathbf{C^0}} .$$

If $\|B - A\|_{\mathbf{C^0}}$ is sufficiently small, then the latter estimate ensures that R_i^B is defined on all $\Omega_\eta \times I$, attains values in Ω and that genuinely non-linearity holds. Similar computations allow to prove also that

$$\left\| R_i^{B_1} - R_i^{B_2} \right\|_{\mathbf{C^0}} \leq \mathcal{O}(1) \, \|B_1 - B_2\|_{\mathbf{C^0}} .$$

whenever $B_1, B_2 \in \mathcal{B}_\eta(A, \delta)$.

Moreover since the first derivatives of rarefaction curves with respect to u^l and u_i depend on the right eigenvectors and on their derivatives, using 3. in Proposition 1.4.2, we have

$$\left\| R_i^{B_1} - R_i^{B_2} \right\|_{\mathbf{C^{1,1}}(\Omega_\eta \times I;\Omega)} \leq \mathcal{O}(1) \, \|B_1 - B_2\|_{\mathbf{C^{1,1}}} .$$

By possibly reducing δ, we have that the map $\mathcal{R} \colon \mathcal{B}_\eta(A, \delta) \mapsto \mathbf{C^{0,1}}(\Omega_\eta \times I, \mathbb{R})$ defined by

$$(\mathcal{R}(B)(u^l, u_i)) := \frac{\partial}{\partial u_i} \lambda_i^B \left(R_i^B(u^l, u_i) \right)$$

is in $\mathbf{C^0} \left(\mathcal{B}_\eta(A, \delta); \mathbf{C^{0,1}}(\Omega_\eta \times I, \mathbb{R}) \right)$.
Hence, the set defined by $\mathcal{A} := \{h \in \mathbf{C^{0,1}}(\Omega_\eta \times I, \mathbb{R}) \colon \|h\|_{\mathbf{C^0}} > 0\}$ is open. Then, $\mathcal{R}^{-1}(\mathcal{A})$ is open and using (1.1.2) we can assume $\mathcal{B}_\eta(A, \delta) \subset \mathcal{R}^{-1}(\mathcal{A})$. □

Proof of Proposition 1.2.2. Let $X := \mathbf{C^{1,1}}(\Omega; \mathbb{R}^{n \times n})$, $\mathcal{X} := \mathcal{B}_\eta(A; \delta)$, $Y := \mathbf{C^{1,1}}(\Omega_\eta \times I; \mathbb{R}^n)$ and $\mathcal{Y} := \mathbf{C^{1,1}}(\Omega_\eta \times I; \Omega)$. Since $G^i(B, \cdot) \in \mathbf{C^{1,1}}(\Omega; \mathbb{R}^n)$ and Ω is a compact set, there exists a linear bounded operator $D_S G^i(B, S)$ such that $G^i(B, S + hV) = G^i(B, S) + h \, D_S G^i(B, S) \, V + o(h)$ uniformly in Ω, for $h \to 0$. Moreover

$$\left\| D_S G^i(B, S) - D_S G^i(A, S_i^A) \right\|_{\mathbf{C^0}} \leq \mathcal{O}(1) \, \left(\|B - A\|_{\mathbf{C^0}} + \left\| S - S_i^A \right\|_{\mathbf{C^0}} \right)$$

It is easy to prove that $D_S G^i(A, S_i^A)$ is invertible in fact $D_S G^i(A, S_i^A)$ is the identity $n \times n$ matrix. Then, by an application of the Implicit Function Theorem, for every $\eta > 0$ there exists δ and a function

$$S_i \colon \mathcal{B}_\eta(A, \delta) \mapsto \mathbf{C^{1,1}}(\Omega_\eta \times I; \Omega)$$

(with a possibly smaller δ) such that

$$\begin{aligned} \left[(G^i(B, S_i^B))(u^l, u_i) \right]_j &= \begin{cases} 0 & \text{if } j \neq i \\ u_i & \text{if } j = i \end{cases} \\ \left\| S_i^{B_1} - S_i^{B_2} \right\|_{\mathbf{C^{1,1}}(\Omega_\eta \times I;\Omega)} &\leq \mathcal{O}(1) \, \|B_1 - B_2\|_{\mathbf{C^{1,1}}} . \end{aligned}$$

for B_1 and $B_2 \in \mathcal{B}_\eta(A, \delta)$.

The evaluation of the first derivatives of the shock curve is exactly as in [14, (5.17) in Theorem 5.1]. The latter inequality is proved as the analogous estimate in Proposition 1.2.1. □

Proof of Proposition 1.2.3. Fix $\eta > 0$. Then, there exists a positive δ such that the map $F(B, u, u^l, u^r) := L^B(u^l, u) - u^r$ is well defined on $\mathcal{B}_\eta(A, \delta) \times \Omega \times \Omega_\eta \times \Omega_\eta$.

Clearly, $F(A, u^r, u^l, u^r) = 0$ and $D_u F(B, u, u^l, u^r)$ exists and is continuous in (A, u^r) for every $(u^l, u^r) \in \Omega_\eta \times \Omega_\eta$. Moreover $D_u F(A, u^r, u^l, u^r)$ is invertible, in fact the matrix

$$D_u F\left(A, u^r, u^l, u^r\right) = \begin{bmatrix} r_1^A & r_2^A & \dots & r_n^A \end{bmatrix}$$

is constant and has non zero determinant by **(H)**. The Implicit Function Theorem, applied with $X := \mathbf{C^{1,1}}(\Omega; \mathbb{R}^{n\times n}) \times \mathbb{R}^n \times \mathbb{R}^n$, $\mathcal{X} := \mathcal{B}_\eta(A, \delta) \times \Omega_\eta \times \Omega_\eta$, $Y := \mathbf{C^{1,1}}(\mathcal{X}; \mathbb{R}^n)$, $\mathcal{Y} := \mathbf{C^{1,1}}(\mathcal{X}; \Omega)$, yields the existence of a map $U^B(u^l, u^r)$ such that $L^B\left(u^l, U^B(u^l, u^r)\right) = u^r$.
□

Moreover, the following estimates hold.

Lemma 1.4.3. *For all $\eta > 0$, there exist $\delta > 0$ and C such that for all $B_1, B_2 \in \mathbf{C^1}(\Omega_\eta; \mathbb{R}^{n\times n})$ with $\|B_1 - B_2\|_{\mathbf{C^1}} < \delta$,*

$$\begin{aligned}
\left\|U^{B_1} - U^{B_2}\right\|_{\mathbf{C^1}(\Omega_\eta\times\Omega_\eta;\Omega)} &\le C\,\|B_1 - B_2\|_{\mathbf{C^1}} \\
\left\|\sigma^{B_2} - \sigma^{B_1}\right\|_{\mathbf{C^1}} &\le C\,\|B_1 - B_2\|_{\mathbf{C^1}} \\
\left\|U^{B_1}(u^l, u^r) - U^{B_2}(u^l, u^r)\right\| &\le C\,\|B_1 - B_2\|_{\mathbf{C^1}} \left\|u^l - u^r\right\| \\
\left\|\sigma^{B_2}(u^l, u^r) - \sigma^{B_1}(u^l, u^r)\right\| &\le C\,\|B_1 - B_2\|_{\mathbf{C^1}} \left\|u^l - u^r\right\|.
\end{aligned} \tag{1.4.2}$$

If moreover $B_1, B_2 \in \mathcal{B}_\eta(A, \delta)$, then

$$\begin{aligned}
\left\|U^{B_1} - U^{B_2}\right\|_{\mathbf{C^{1,1}}(\Omega_\eta\times\Omega_\eta;\Omega)} &\le C\,\|B_1 - B_2\|_{\mathbf{C^{1,1}}} \\
\left\|\sigma^{B_2} - \sigma^{B_1}\right\|_{\mathbf{C^{1,1}}} &\le C\,\|B_1 - B_2\|_{\mathbf{C^{1,1}}}.
\end{aligned} \tag{1.4.3}$$

Proof. The first two estimates follow from Theorem 1.4.1 applied in $\mathbf{C^1}$. To obtain the second and third bounds, apply again Theorem 1.4.1 with $X := \mathbf{C^1}(\Omega; \mathbb{R}^{n\times n}) \times \mathbb{R}^n \times \mathbb{R}^n$, $\mathcal{X} := \mathcal{B}_\eta(A, \delta) \times \Omega_\eta \times \Omega_\eta$, $Y := \mathbf{C^1}(\mathcal{X}; \mathbb{R}^n)$, $\mathcal{Y} := \mathbf{C^1}(\mathcal{X}; \Omega)$. Letting $v := u^r - u^l$ and $\mathcal{U}(v) := U^{B_1}(u^l, v + u^l) - U^{B_2}(u^l, v + u^l)$, we have

$$\left\|\mathcal{U}(v)\right\| \le \left\|\frac{\partial U^{B_1}}{\partial u^r} - \frac{\partial U^{B_1}}{\partial u^r}\right\|_{\mathbf{C^0}} \|v\| \le C\,\|B_1 - B_2\|_{\mathbf{C^1}} \left\|u^l - u^r\right\|.$$

The latter two estimates are proved applying Theorem 1.4.1 in $\mathbf{C^{1,1}}$, as used in the proof of Proposition 1.2.3. □

The next lemma is a slight extension of [14, Lemma 2.5] to the case of $\mathbf{C^{1,1}}$ functions.

Lemma 1.4.4. *Let $f \in \mathbf{C^{1,1}}(\mathbb{R}^2; \mathbb{R})$ be such that $f(x, 0) = f(0, y) = 0$ for all x, y. Then,*

$$\left|f(x, y)\right| \le \mathbf{C^{0,1}}(Df) \cdot |xy|.$$

Proof. Simply compute

$$\begin{aligned}\left|f(x,y)\right| &= \left|f(x,y)-f(x,0)\right| &\leq \left|\int_0^y \left|\frac{\partial f}{\partial y}(x,s)\right| ds\right| \\ &= \left|\int_0^y \left|\frac{\partial f}{\partial y}(x,s)-\frac{\partial f}{\partial y}(0,s)\right| ds\right| &\leq \mathbf{C^{0,1}}(Df)\,|xy|\end{aligned}$$

completing the proof. □

Proof of Proposition 1.3.2. Consider the different interactions separately.
1. We observe that $\sigma_k^+ = \sigma^B(u^l,u^r) = \sigma^B(u^l,\mathcal{L}_i^B(\mathcal{L}_j^B(u^l,\sigma_j^-,\sigma_i^-))$.
For $k=i,j$ define $\Sigma_k^B(\sigma_i^-,\sigma_j^-) := \sigma_k^+ - \sigma_k^-$. Otherwise, let $\Sigma_k^B(\sigma_i^-,\sigma_j^-) := \sigma_k^+$. Then, for all $k=1,\ldots,n$ we have that $\Sigma_k^B(\sigma_i^-,0)=0$ and $\Sigma_k^B(0,\sigma_j^-)=0$. Moreover, Σ_k^B is in $\mathbf{C^{1,1}}$, so that we may apply Lemma 1.4.4 using the bound

$$\mathbf{C^{0,1}}(D\Sigma_k^B) \leq \left\|\Sigma_k^B\right\|_{\mathbf{C^{1,1}}} = \left\|\Sigma_k^B - \Sigma_k^A\right\|_{\mathbf{C^{1,1}}} \leq C\left\|B-A\right\|$$

where the latter estimate follows from (1.2.4) and (1.4.3).
2. Set $\sigma_k^+ := \sigma_k^B(u^l,u^r) = \sigma_k^B(u^l,\mathcal{L}_i^B(\mathcal{L}_i^B(u^l,\sigma_i''),\sigma_i'))$ and $\Sigma_i^B(\sigma_i',\sigma_i'') := \sigma_i^+ - \sigma_i' - \sigma_i''$ otherwise $\Sigma_k^B(\sigma_i',\sigma_i'') := \sigma_k^+$, for $k=1,\ldots,n$. As above, we have that $\Sigma_k^B(\sigma_i',0)=0$ and $\Sigma_k^B(0,\sigma_i'')=0$. Moreover

$$\mathbf{C^{0,1}}(D\Sigma_k^B) \leq C\|B-A\|_{\mathbf{C^{1,1}}}\,.$$

Then, by Lemma 1.4.4

$$\left|\Sigma_k^B(\sigma_i',\sigma_i'')\right| \leq C\|B-A\|_{\mathbf{C^{1,1}}}\left|\sigma_i'\sigma_i''\right|.$$

3. In the case $j>i$ we have:

$$\Sigma_{n+1}^+(\sigma_i^-,\sigma_j^-) := \sigma_{n+1}^+(u^l,u^r) = \sigma_{n+1}^+(u^l,\mathcal{L}_i^B(\mathcal{L}_j^B(u^l,\sigma_j^-),\sigma_i^-))$$

It holds: $\Sigma_{n+1}^+(\sigma_i^-,0)=0$ and $\Sigma_{n+1}^+(0,\sigma_j^-)=0$. Using (1.2.4), (1.4.3) and the compactness of Ω_η

$$\mathbf{C^{0,1}}(D\Sigma_{n+1}^+) \leq C\|B-A\|_{\mathbf{C^{1,1}}}$$

and an application of Lemma 1.4.4 completes the proof of this bound. In the case $i=j$ the proof is the same.
4. Define $T^B(v,\sigma_i^-) := \mathcal{L}_i^B(u^l,\sigma_i^-) - \mathcal{L}_i^B(u^l+v,\sigma_i^-) - v$ with $v := (u^m-u^l)\in\mathbb{R}^n$. Clearly: $T^B(0,\sigma)=0$ and $T^B(v,0)=0$. Being $T^A=0$, we have also that:

$$\mathbf{C^{0,1}}(T^B) \leq C\|B-A\|_{\mathbf{C^{1,1}}}\,.$$

Then, using the previous arguments,

$$\|u^r-\tilde{u}^r\| - \left\|u^m-u^l\right\| \leq \left\|T^B(v,\sigma_i^-)\right\| \leq C\|B-A\|_{\mathbf{C^{1,1}}}\left\|u^m-u^l\right\|\left|\sigma_i^-\right|$$

completing the proof. □

Proof of Proposition 1.3.3. In fact, formula (1.3.5) implies that also Q is non-increasing along approximate solutions. Then, following [14, Paragraph 7.3, Section 4], the Accurate Riemann Solver can be used only finitely many times (more precisely not more than $2Q(0)/\rho_\nu$), hence the total number of physical fronts is finite. A non-physical wave is generated only by the interaction between two physical fronts, so that also the number of non-physical waves is bounded. □

Chapter 2

2×2 Systems of Conservation Laws with $\mathbf{L}^\infty$ Data

2.1 Introduction

Consider the following non-linear 2×2 system of conservation laws

$$\partial_t u + \partial_x \left[f(u) \right] = 0 \tag{2.1.1}$$

and the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \left[f(u) \right] = 0 \\ u(0,x) = \bar{u}(x) \,. \end{cases} \tag{2.1.2}$$

Our aim is to extend the classical result [31, Theorem 5.1] relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves. More precisely, as is well known, the assumptions in [31] ensure that the interaction of two shocks of the same family yields a shock of that family and a *rarefaction* of the other family. Here, no assumption whatsoever of this kind is assumed. Nevertheless, the result of Theorem 2.1.1 is the same of that in [31, Theorem 5.1], namely the existence of a weak entropy solution to (2.1.2) for all initial data with sufficiently small $\mathbf{L}^\infty$ norm.

On the flow f in (2.1.1) we assume the following Glimm-Lax condition, analogously to [31, formula (1.4)]:

(GL) $f \colon B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and with both characteristic fields genuinely non linear

where $B(0,r)$ is the ball of $\mathbb{R}^2$ with center 0 and radius r. The main result of this chapter is the following:

Theorem 2.1.1. *Under the assumption* ***(GL)****, there exists a sufficiently small* $\eta > 0$ *such that for every initial condition* $\bar{v} \in \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^2)$ *with:*

$$\|\bar{v}\|_\infty \leq \eta \tag{2.1.3}$$

the Cauchy problem (2.1.2) *admits a weak entropy solution for all* $t \geq 0$.

The solution is constructed as limit of the ε–approximations v^ε constructed through the front tracking algorithm used in [15], suitably adapted to the present situation. First, as in [31], careful decay estimates on a trapezoid (see Figure 2.2) allow to bound the positive variation and the $\mathbf{L}^\infty$ norm of v^ε on the upper side of the trapezoid. Under the further assumption that a suitable $\mathbf{L}^\infty$ estimate on v^ε holds, see condition **(A)** , a technique based on the hyperbolic rescaling allows to extend the previous bound to any positive time. The approximate solutions can hence be defined globally in time.

A key point is now to provide estimates that allow to abandon condition **(A)**. This is achieved through $\mathbf{L}^\infty$ estimates essentially based on the conservation form of (2.1.1) and on the previous results on the trapezoids. It is here that the integral estimates in Section 2.6 allow us to extend the result in [31].

As a byproduct, we also obtain Theorem 2.3.12, under the standard Lax condition

(L) $f\colon B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and each characteristic field is either genuinely non linear or linearly degenerate.

Indeed, Theorem 2.3.12 is an existence result valid for all initial data having small $\mathbf{L}^\infty$ norm and bounded, not necessarily small, total variation.

In this connection, we recall that in the case of systems with coinciding shock and rarefaction waves, the well posedness of (2.1.2) in $\mathbf{L}^\infty$ was proved in [7] under condition **(GL)**, extending the previous results [5, 17]. Another attempt towards an extension of Glimm–Lax result is in [21].

This chapter is organized as follows. Section 2.2 is devoted to introduce the notation. Then, ε–approximate solutions are defined in Section 2.3 and suitable bounds are proved, in the case of bounded total variation. Section 2.4 uses the previous results to construct the ε–approximate solutions globally in time under the further assumption **(A)**. This latter assumption is abandoned in Section 2.5, which relies on the integral estimates in Section 2.6. The more technical details are collected in the final Section 2.7.

2.2 Notations

As a general reference on the theory of conservation laws, we refer to [14, 27]. Throughout, we let $B(u,r)$ be the open sphere in $\mathbb{R}^2$ centered at u with radius r.

Denote by $A(u)$ the 2×2 hyperbolic matrix $Df(u)$, by λ_1, λ_2 its eigenvalues and by l_1, l_2 (resp. r_1, r_2) its left (resp. right) eigenvectors, normalized so that

$$\left\|r_i(u)\right\| = 1, \qquad \langle l_j(u), r_i(u)\rangle = \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right. \qquad i,j = 1,2\,.$$

If the i–th characteristic field is genuinely nonlinear, we choose r_i oriented so that

$$D\lambda_i(u)\, r_i(u) \geq c > 0 \quad \text{for} \quad i = 1,2 \quad \text{and} \quad u \in B(0,r) \tag{2.2.1}$$

for a suitable c. In the linearly degenerate case, we do not need to specify this orientation. By **(L)**, $\sup_{B(0,r)} \lambda_1 < \inf_{B(0,r)} \lambda_2$.

By a linear change of coordinates, we can assume that $f(0) = 0$, $A(0) = \mathrm{diag}\left(\lambda_1(0), \lambda_2(0)\right)$ and that $\lambda_1(0) = -1$, $\lambda_2(0) = 1$. We are thus led to assume that f can be written as follows:

$$\begin{array}{rcl} f_1(u) & = & -u_1 + \frac{1}{2}\alpha_{11}\, u_1^2 + \alpha_{12}\, u_1\, u_2 + \frac{1}{2}\alpha_{22}\, u_2^2 + \mathcal{O}(1)\, \|u\|^3 \\ f_2(u) & = & u_2 + \frac{1}{2}\beta_{11}\, u_1^2 + \beta_{12}\, u_1\, u_2 + \frac{1}{2}\beta_{22}\, u_2^2 + \mathcal{O}(1)\, \|u\|^3 \end{array} \tag{2.2.2}$$

with $\alpha_{ij} := \dfrac{\partial^2 f_1}{\partial u_i\, \partial u_j}(0)$ and $\beta_{ij} := \dfrac{\partial^2 f_2}{\partial u_i\, \partial u_j}(0)$.

Following [14, formula (5.38)], introduce the Lax curves as the gluing of the shock and rarefaction curves:

$$L_i(u,\sigma) := \left\{ \begin{array}{ll} S_i(u,\sigma) & \sigma < 0\,, \\ R_i(u,\sigma) & \sigma \geq 0\,. \end{array} \right. \tag{2.2.3}$$

As in [14, formula (7.36)], call $E = E(u^-, u^+)$ the map giving the sizes of the waves in the solution to the Riemann problem for (2.1.1) with data u^- and u^+:

$$(\sigma_1, \sigma_2) = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = L_2\Big(L_1(u^-, \sigma_1), \sigma_2\Big)\,.$$

Recall now the continuous version of the Glimm potentials, see [16, (1.14) and (1.15)] or [23, (4.2)–(4.4)]. Throughout, we assume that any $u \in \mathbf{BV}\left(\mathbb{R}; B(0,r)\right)$ is right continuous. For a Borel $\Omega \subseteq \mathbb{R}$, define the wave measures μ_i for $i = 1, 2$, as

$$\mu_i(\Omega) := \int_\Omega \langle l_i(u), d\mu_c\rangle + \sum_{x\in\Omega} E_i\left(u(x-), u(x+)\right)$$

where by μ_c we mean the continuous part of the weak derivative of u and $\langle l_i(u), d\mu_c\rangle := \sum_{j=1}^n l_i^j(u) d\mu_c^j$. Below, we consider also the positive part of the signed measure μ_i, denoted by μ_i^+, and the positive total variation of the i–th component of u, denoted by $\mathrm{TV}^+(u_i)$. Then, let

$$\rho := |\mu_2| \otimes |\mu_1| + \sum_{i=1}^{2} \left(\mu_i^- \otimes \mu_i^- + \mu_i^+ \otimes \mu_i^- + \mu_i^- \otimes \mu_i^+\right) \tag{2.2.4}$$

and, as in [4, 14, 16, 23], set

$$\begin{array}{rcll} Q(u) & := & \rho\left(\left\{(x,y) \in \mathbb{R}^2 \colon x < y\right\}\right) & \\ V(u,I) & := & |\mu_1|(I) + |\mu_2|(I) & \quad I \subseteq \mathbb{R} \text{ interval} \\ \Upsilon(u) & := & V(u,\mathbb{R}) + Q(u) & \end{array}$$

where $|\mu_i|$ is the total variation of measure μ, $V(u,\mathbb{R})$ is the *total strength of waves* in u and $Q(u)$ is the *interaction potential* of u. For a $u \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^2)$, define its total variation by:

$$\mathrm{TV}(u) := \sup\left\{ \sum_{i=1}^{2}\sum_{l=1}^{N} \left|u_i(x_l) - u_i(x_{l-1})\right| \colon \begin{array}{l} x_1, \ldots, x_N \in \mathbb{R} \text{ with} \\ x_1 < \cdots < x_N \end{array} \right\}. \tag{2.2.5}$$

Obviously, the total variation and the functional $V(\cdot, \mathbb{R})$ are equivalent. In the following, for $L > 0$, it will be useful also the notation:

$$\mathrm{TV}(u; L) := \sup_{x \in \mathbb{R}} \mathrm{TV}\left(u_{|[x,x+L]}\right)$$

where $u_{|[x,x+L]}$ is the restriction of u to the interval $[x, x+L]$.

For a function $u \colon \mathbb{R} \to B(0,r)$, we use below the $\mathbf{L}^\infty$ norm

$$\|u\|_\infty := \operatorname*{supess}_{x \in \mathbb{R}} \left|u_1(x)\right| + \operatorname*{supess}_{x \in \mathbb{R}} \left|u_2(x)\right| .$$

Below, $\hat{\lambda}$ denotes an upper bound for the moduli of the characteristic speeds in $B(0,r)$, i.e.

$$\hat{\lambda} > \sup_{i=1,2;\, \|u\| \le r} \left|\lambda_i(u)\right| . \tag{2.2.6}$$

2.3 Bounded Total Variation and Small $\mathbf{L}^\infty$ Norm

In this section, we modify the wave front tracking algorithm in [15, Section 2] to construct a solution to (2.1.2) under the assumption that the initial datum has bounded total variation and small $\mathbf{L}^\infty$ norm. More precisely, let $\bar{u}$ belong to

$$\mathcal{D}(\eta, \bar{K}) := \left\{u \in \mathbf{L}^1_{\mathbf{loc}}\left(\mathbb{R}; B(0,\eta)\right) : \mathrm{TV}(u) \le \bar{K}\right\} , \tag{2.3.1}$$

where $\bar{K}, \eta$ are positive constants.

Moreover, in the first two paragraphs below, it is not necessary to assume that both characteristic fields be genuinely nonlinear. The standard Lax [37, Section 9] condition **(L)** is sufficient.

2.3.1 The Algorithm

Fix $\varepsilon > 0$. Denote by v the Riemann coordinates of (2.1.1), see [27, Definition 7.3.2], and call $\mathcal{L}_i$, $\mathcal{R}_i$ and $\mathcal{S}_i$ the Lax, the rarefaction and the shock curves in the Riemann coordinates:

$$\mathcal{L}_i(v,\sigma) := \begin{cases} \mathcal{S}_i(v,\sigma) & \sigma < 0\,, \\ \mathcal{R}_i(v,\sigma) & \sigma \ge 0\,. \end{cases} \tag{2.3.2}$$

In these variables, as in [15], we parametrize the rarefaction and the shock curves as follows:

$$\begin{aligned} \mathcal{R}_1(v,\sigma) &= (v_1+\sigma, v_2), \quad \mathcal{S}_1(v,\sigma) = \left(v_1+\sigma, v_2+\hat{\psi}_2(v,\sigma)\,\sigma^3\right) \\ \mathcal{R}_2(v,\sigma) &= (v_1, v_2+\sigma), \quad \mathcal{S}_2(v,\sigma) = \left(v_1+\hat{\psi}_1(v,\sigma)\,\sigma^3, v_2+\sigma\right) \end{aligned} \tag{2.3.3}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are suitable smooth functions of their arguments. First, the initial datum $\bar{v}$ is substituted by a piecewise constant $\bar{v}^\varepsilon$ such that:

$$\lim_{\varepsilon \to 0+} \|\bar{v}^\varepsilon - \bar{v}\|_{\mathbf{L}^1} = 0\,, \quad \mathrm{TV}(\bar{v}^\varepsilon) \le \mathrm{TV}(\bar{v}) \le \bar{K}\,, \quad \|\bar{v}^\varepsilon\|_\infty \le \eta\,.$$

At each point of jump in $\bar{v}^\varepsilon$, the resulting Riemann problem is solved as in [15, Section 2]. Let $\varphi \in \mathbf{C}^\infty(\mathbb{R};\mathbb{R})$ be such that

$$\begin{array}{rclcrcl} \varphi(\sigma) & = & 1 & \text{for} & \sigma & \leq & -2 \\ \varphi(\sigma) & = & 0 & \text{for} & \sigma & \geq & -1 \\ \varphi'(\sigma) & \in & [-2,0] & \text{for} & \sigma & \in & [-2,-1] \end{array}$$

and introduce the ε-approximate Lax curves

$$\mathcal{L}_i^\varepsilon(v,\sigma) = \varphi(\sigma/\sqrt{\varepsilon})\,\mathcal{S}_i(v,\sigma) + \Big(1-\varphi(\sigma/\sqrt{\varepsilon})\Big)\,\mathcal{R}_i(v,\sigma) \quad \text{for } i=1,2\,.$$

An ε–solution to the Riemann problem for (2.1.1) with data v^-, v^+ is obtained gluing ε–rarefactions and ε–shocks. ε–rarefactions of the first, respectively second, family are substituted by rarefaction fans attaining values in $\varepsilon\mathbb{Z}\times\mathbb{R}$, respectively $\mathbb{R}\times\varepsilon\mathbb{Z}$, traveling with the characteristic speed of the state on the right of each wave. More precisely, similarly to [15, formulæ (2.13)–(2.16)], in the case $i=1$ of the first family, define $h,k\in\mathbb{Z}$ such that

$$h\varepsilon \leq v_1^- < (h+1)\varepsilon \quad \text{and} \quad k\varepsilon \leq \mathcal{R}_1(v^-,\sigma_1) < (k+1)\varepsilon\,.$$

Introducing $\omega_1^j = (j\varepsilon, v_2^-)$ for $j=h,\ldots,k$, define

$$v(t,x) := \begin{cases} v^- & x < \lambda_1(\omega_i^{h+1})\cdot t \\ \omega_1^j & \lambda_1(\omega_1^j)t \leq x < \lambda_1(\omega_1^{j+1})t \qquad \text{for } h+1\leq j\leq k-1 \\ \omega_1^k & \lambda_1(\omega_1^k)t \leq x < \lambda_1\left(\mathcal{R}_1(v^-,\sigma_1)\right)t \\ \mathcal{R}_1(v^-,\sigma_1) & \lambda_1\left(\mathcal{R}_1(v^-,\sigma_1)\right)t \leq x\,. \end{cases} \tag{2.3.4}$$

The case of rarefaction waves of the second family is entirely similar.

A 1–shock with left state v^- and size σ_1, such that $\sigma_1 < -2\sqrt{\varepsilon}$, travels with the exact Rankine–Hugoniot speed $\lambda_1^s(v^-,\sigma_1)$. When $\sigma_1 > -2\sqrt{\varepsilon}$, we assign to this jump an interpolated speed λ_1^φ defined as an average between the exact Rankine-Hugoniot speed $\lambda_1^\varphi(v,\sigma)$ and an approximate characteristic speed, see [15, formulæ (2.17), (2.18) and (2.19)]

$$\begin{array}{rcl} \lambda_1^\varphi(v^-,\sigma_1) & := & \varphi(\sigma_1/\sqrt{\varepsilon})\,\lambda_1^s(v^-,\sigma_1) + \left(1-\varphi(\sigma_1/\sqrt{\varepsilon})\right)\lambda_1^r(v^-,\sigma_1) \\ \lambda_1^r(v^-,\sigma_1) & := & \sum_j \dfrac{\mathrm{meas}\Big([j\varepsilon,(j+1)\varepsilon]\cap\big[\left(\mathcal{S}_1(v^-,\sigma_1)\right)_1, v_1^-\big]\Big)}{|\sigma_1|}\,\lambda_1(\omega_1^{j+1})\,. \end{array} \tag{2.3.5}$$

For every $\sigma_i < 0$, it holds

$$\lambda_i\Big(\mathcal{S}_i(v^-,\sigma_i)\Big) < \lambda_i^\varphi(v^-,\sigma_i) < \lambda_i(v^-)\,. \tag{2.3.6}$$

2-shocks are treated similarly, we refer to [15, Section 2] for further details.

If the i–th characteristic family is linearly degenerate, the shock, the rarefaction and the ε–approximate Lax curves coincide. Moreover, the characteristic speed is constant along these curves, so that the interpolation (2.3.5) is trivial. Gluing the solutions to the Riemann problems at the points of jump in $\bar{v}^\varepsilon$ we obtain an ε–solution defined on a non trivial time interval $[0,t_1]$, t_1 being the first time at which two or more waves interact. Any interaction yields a new Riemann problem, so that a piecewise constant ε–solution of the form

$$v^\varepsilon = \sum_\alpha v^\alpha \chi_{[x_\alpha, x_{\alpha+1}[} \quad \text{with} \quad v^{\alpha+1} = \mathcal{L}_2^\varepsilon\left(\mathcal{L}_1^\varepsilon(v^\alpha,\sigma_{1,\alpha}),\sigma_{2,\alpha}\right) \tag{2.3.7}$$

is recursively extended in time. Hence, we obtain a sequence of ε–approximate solutions. Here, the meaning of by ε–approximate solutions is slightly different from that in [15, Definition 1], namely:

Definition 2.3.1. *A piecewise constant function $v^\varepsilon = v^\varepsilon(t,x)$ is an ε–approximate solution if all its lines of discontinuities are ε–admissible wave fronts.*

By an ε–admissible wavefront of the first family we mean a line $x = x(t)$ across which a function v^ε has a jump, say with $v^- = (v_1^-, v_2^-)$, $v^+ = (v_1^+, v_2^+)$, satisfying the following conditions:

- *If $v_1^+ \geq v_1^-$, then $v_2^+ = v_2^-$ and*

$$v_1^+ \leq v_1^- + \varepsilon, \quad \dot{x} = \lambda_1(v^+)\,. \tag{2.3.8}$$

- *If $v_1^+ \leq v_1^-$, then $v^+ = \mathcal{L}_1^\varepsilon(v^-, \sigma_1)$ for some $\sigma_1 < 0$, $\dot{x}$ coincides with the speed λ_1^φ defined in [15, formula (2.19)] and satisfies*

$$\lambda_1(v^+) < \dot{x} < \lambda_1(v^-). \tag{2.3.9}$$

The ε–admissible wave fronts of the second family are defined in an entirely similar way.

It may happen that three or more fronts interact at the same point. Due to the above algorithm, at least one of the interacting waves needs to be a shock. Then, similarly to [14, Remark 7.1] it is sufficient to slightly modify the speed of this incoming shock to avoid the multiple interaction. If this perturbation is small enough, the bound (2.3.9) is still true.

Above, we modified the wave propagation speed adopted in [15, Section 2]. The speeds defined therein have an essential role in the proof of the uniform Lipschitz dependence of the approximate solution from the initial datum. The present choice (2.3.4)–(2.3.5) is sufficient for [14, propositions 2 and 3] to hold and allows for simpler proofs in the sequel.

2.3.2 Existence and Properties of the Approximate Solutions

In this paragraph we show that the ε–approximate solutions constructed by the previous algorithm are well defined, see Theorem 2.3.10.

Throughout, by C we denote a positive constant dependent only on f and r as in **(L)**.

The following Lemma provides the standard interaction estimates.

Lemma 2.3.2. *There exists a positive C such that for any interaction resulting in the waves σ_1^+ and σ_2^+, the following estimates hold.*

1. *If the interacting waves are σ_1^- of the first family and σ_2^- of the second family,*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| = C\left|\sigma_1^-\sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right).$$

2. *If the interacting waves σ' and σ'' both belong to the first family, we have*

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| = C\left|\sigma'\sigma''\right| \left(\left|\sigma'\right| + \left|\sigma''\right|\right).$$

3. If the interacting waves σ' and σ'' both belong to the second family, we have

$$\left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| = C\left|\sigma'\sigma''\right| \left(\left|\sigma'\right| + \left|\sigma''\right|\right) .$$

The proof is in [15, Lemma 2. and Lemma 3.].

Assume now that the ε–approximate solution v^ε is defined up to time $T > 0$. For $i = 1, 2$, $t \in [0, T]$ and $x \in \mathbb{R}$, introduce the quantities

$$\begin{aligned} \check\lambda_i(t,x) &:= \min\left\{\lambda_i\left(v^\varepsilon(t,x-)\right), \lambda_i\left(v^\varepsilon(t,x+)\right)\right\} \\ \hat\lambda_i(t,x) &:= \max\left\{\lambda_i\left(v^\varepsilon(t,x-)\right), \lambda_i\left(v^\varepsilon(t,x+)\right)\right\} . \end{aligned}$$

For any $X \in \mathbb{R}$, the generalized i-th characteristic through (T, X) is an absolutely continuous solution $x(t)$ to the differential inclusion

$$\begin{cases} \dot x \in \left[\check\lambda_i(t,x)\,,\ \hat\lambda_i(t,x)\right] \\ x(T) = X\,. \end{cases}$$

The *minimal* backward i-th characteristic through (T, X) is the generalized i-th characteristic such that, for $t \in [0, T]$,

$$y_i(t) := \min\left\{x(t)\colon x \text{ is a generalized } i\text{-th characteristic through } (T,X)\right\} ,$$

where we omit the dependence of $y_i(t)$ from (T, X). It is clear that $y_i(t)$ is well defined, for v^ε piecewise constant, see [3, Theorem 2, Chapter 2, § 1].

As a reference about minimal backward characteristics on exact solutions, see [27, Paragraph 10.3]. Backward characteristics on wave front tracking solutions were used, for instance, in [16, Section 4].

To estimate the norm $\left\|v^\varepsilon(T)\right\|_\infty$, for $T > 0$, we follow backward the i–coordinate v_i^ε along the minimal characteristic $y_i(t)$ through (T, X), for all $X \in \mathbb{R}$. Using the Lax inequality (2.3.6) and the choice adopted for the speed of rarefaction waves, we can conclude that y_i does not interact with any i–shock with size $\sigma < -\sqrt{\varepsilon}$, it can coincide on a non-trivial time interval with an i–wave with size $\sigma \geq -\sqrt{\varepsilon}$, it can cross a wave of the other family or pass through an interaction point where a rarefaction of its family arises, see Figure 2.1.

In the lemma below, we denote $v\left(t^\pm, y_i(t^\pm)\right) := \lim_{\tau\to t^\pm} v\left(\tau, y_i(\tau)\right)$.

Lemma 2.3.3. *Let $t > 0$ be such that $v_1\left(t^+, y_1(t^+)\right) \neq v_1\left(t^-, y_1(t^-)\right)$. Then, either y_1 crosses a 2–wave σ_2, and*

$$\left|v_1^\varepsilon\left(t^+, y_1(t^+)\right)\right| - \left|v_1^\varepsilon\left(t^-, y_1(t^-)\right)\right| \leq C\left|\sigma_2\right|^3, \tag{2.3.10}$$

or y_1 passes through an interaction point between two waves σ', σ'' of the second family and

$$\left|v_1^\varepsilon\left(t^+, y_1(t^+)\right)\right| - \left|v_1^\varepsilon\left(t^-, y_1(t^-)\right)\right| \leq C\left(\left|\sigma'\right| + \left|\sigma''\right|\right)^3 . \tag{2.3.11}$$

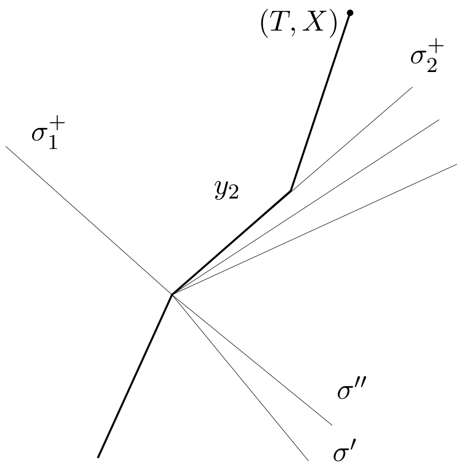


Figure 2.1: Two 1-shock σ' and σ'' interact resulting in a 1-shock σ_1^+ and a 2-rarefaction σ_2^+. A 2-characteristic y_2 (thick line) is superimposed to the 2-rarefaction and passes through the interaction point.

The proof directly follows from (2.3.3) and 3. in Lemma 2.3.2. An entirely analogous result holds along 2-characteristics.

The total size of the j-waves, with $j \neq i$, which may potentially interact with $y_i(t)$ after time t is given by the functionals

$$\tilde{Q}_1(t) := \sum_{\alpha:\, x_\alpha < y_1(t)} \left|\sigma_{2,\alpha}\right| \quad \text{and} \quad \tilde{Q}_2(t) := \sum_{\alpha:\, x_\alpha > y_2(t)} \left|\sigma_{1,\alpha}\right| \tag{2.3.12}$$

where we referred to the form (2.3.7) of v^ε. To estimate $\Delta\tilde{Q}_i(t)$, we analyze all the cases:

Lemma 2.3.4. *Let $i, j = 1, 2$ and $i \neq j$. Fix $t > 0$. If at time t there is*

1. *no interaction and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) = 0$;*
2. *no interaction and $y_i(t)$ crosses a j–wave σ_j, then $\Delta\tilde{Q}_i(t) = -\left|\sigma_j\right|$;*
3. *an interaction between σ' and σ'', and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) \leq C\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$;*
4. *an interaction between the waves σ' and σ'', and $y_i(t)$ crosses a j–wave σ_j, then $\Delta\tilde{Q}_i(t) \leq C\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right) - \left|\sigma_j\right|$;*
5. *an interaction between the j-waves σ' and σ'', and $y_i(t)$ crosses the interaction point, then $\Delta\tilde{Q}_i(t) \leq -\left|\sigma'\right| - \left|\sigma''\right|$.*

Proof. Points 1., 2. and 5. directly follow from the definition (2.3.12).
Points 3. and 4. follow from Lemma 2.3.2 and (2.3.12). □

Now we also define, as usual, the *total strength of waves* and the *interaction potential*:

$$V(v^\varepsilon) := \sum_{i,\alpha} \left|\sigma_{i,\alpha}\right|, \qquad Q(v^\varepsilon) := \sum_{(\sigma_{i,\alpha},\sigma_{j,\beta})\in\mathcal{A}} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|, \tag{2.3.13}$$

where $\mathcal{A}$ is the set of all couples of approaching wave-fronts, see [14, Paragraph 3, Section 7.3].

Proposition 2.3.5. *Fix a positive M'. Let the ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ be defined up to time $t > 0$. At time t an interaction between two waves σ' and σ'' takes place. If* $\mathrm{TV}\left(v^\varepsilon(t^-)\right) < M'$ *and* $\left\|v^\varepsilon(t^-)\right\|_\infty$ *is sufficiently small, then v^ε can be defined beyond time t and*

$$\Delta Q(v^\varepsilon(t)) \leq -\frac{\left|\sigma'\sigma''\right|}{2}.$$

Proof. Using Lemma 2.3.2 and (2.3.13), we have

$$\begin{aligned}\Delta Q(v^\varepsilon(t)) &\leq -\left|\sigma'\sigma''\right| + C\,\mathrm{TV}\left(v^\varepsilon(t^-)\right)\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\\ &\leq \left|\sigma'\sigma''\right|\left(-1 + CM'\left\|v^\varepsilon(t^-)\right\|_\infty\right)\end{aligned}$$

Choosing $\left\|v^\varepsilon(t^-)\right\|_\infty < 1/(2CM')$, we obtain

$$\Delta Q(v^\varepsilon(t)) \leq -\frac{\left|\sigma'\sigma''\right|}{2}.$$

□

We introduce now the following two functionals:

$$\Upsilon^\varepsilon(t) := V(v^\varepsilon(t)) + K\,Q(v^\varepsilon(t)) \tag{2.3.14}$$

$$\Theta_i^\varepsilon(t) := \left(\left|v_i^\varepsilon\left(t, y_i(t)\right)\right| + \|\bar{v}^\varepsilon\|_\infty\right) e^{\tilde{H}\,\tilde{Q}_i(t) + H\,Q(v^\varepsilon(t))} \tag{2.3.15}$$

where $i = 1,2$, $\tilde{H}, H$ and K are positive constants to be precisely defined below.

Proposition 2.3.6. *Fix positive M, M'. Choose an initial datum $\bar{v}^\varepsilon$ such that $\|\bar{v}^\varepsilon\|_\infty < \eta$. Assume that the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ is defined up to time $t > 0$. If η is sufficiently small,* $\mathrm{TV}\left(v^\varepsilon(t^-)\right) < M'$ *and* $\left\|v^\varepsilon(t-)\right\|_\infty < M\|\bar{v}^\varepsilon\|$*, then, there exist positive $\tilde{H}, H$ and K such that*

$$\Delta\Upsilon^\varepsilon(t) \leq 0 \tag{2.3.16}$$

$$\Delta\Theta_i^\varepsilon(t) \leq 0 \quad \textit{for } i = 1,2. \tag{2.3.17}$$

Proof. First, we suppose that at time t there is no interaction and that y_i crosses the wave

σ_j. Obviously, $\Delta\Upsilon^\varepsilon = 0$ and $\left\|v^\varepsilon(t^+)\right\|_\infty = \left\|v^\varepsilon(t^-)\right\|_\infty$. Moreover:

$$
\begin{aligned}
& \Delta\Theta_i^\varepsilon(t) \\
= & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} \\
& - \left(\left|v_i^\varepsilon\left(t^-, y_i(t)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) e^{\tilde H \tilde Q_i(t^-) + H\, Q(v^\varepsilon(t^-))} \\
= & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} \\
& + \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) \left(e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} - e^{\tilde H \tilde Q_i(t^-) + H\, Q(v^\varepsilon(t^-))}\right) \\
\leq & \ C \left|\sigma_j\right|^3 e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} - \tilde H \, \|\bar{v^\varepsilon}\|_\infty \left|\sigma_j\right| e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} \\
\leq & \ 0 \,,
\end{aligned}
$$

provided $\tilde H \geq C M^2 \|\bar{v^\varepsilon}\|_\infty$.

Suppose now that at time t the waves σ' and σ'' interact and y_i does not pass through the interaction point. Hence, using Lemma 2.3.2 and the estimate of Proposition 2.3.5,

$$
\Delta\Upsilon^\varepsilon(t) \leq C \left(\left|\sigma'\right| + \left|\sigma''\right|\right) \left|\sigma'\sigma''\right| - \frac{K}{2}\left|\sigma'\sigma''\right| \leq 0 \tag{2.3.18}
$$

if $K \geq 2C \left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. For the functional Θ_i^ε, we consider separately two cases. If $y_i(t)$ does not cross any wave at time t, we get:

$$
\begin{aligned}
& \Delta\Theta_i^\varepsilon(t) \\
\leq & \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) \\
& \left(e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} - e^{\tilde H \tilde Q_i(t^-) + H\, Q(v^\varepsilon(t^-))}\right) \\
\leq & \ \|\bar{v^\varepsilon}\|_\infty \left(\tilde H \left(\left|\sigma'\right| + \left|\sigma''\right|\right) - \frac{H}{2}\right) \left|\sigma'\sigma''\right| e^{\tilde H \tilde Q_i(t^+) + H\, Q(v^\varepsilon(t^+))} \\
\leq & \ 0 \,,
\end{aligned}
$$

provided $H \geq 2\tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. If $y_i(t)$ crosses a j–wave:

$$\begin{aligned}
&\Delta\Theta_i^\varepsilon(t)\\
\leq\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
&+\left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right)\\
&\left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right)\\
\leq\ & C\left|\sigma_j\right|^3 e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
&+\|\bar{v^\varepsilon}\|_\infty\left(-\tilde{H}\left|\sigma_j\right| + \tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\left|\sigma'\sigma''\right| - \frac{H}{2}\left|\sigma'\sigma''\right|\right)\\
&e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
\leq\ & 0
\end{aligned}$$

provided $\tilde{H} > CM^2\,\|\bar{v^\varepsilon}\|_\infty$ and $H \geq 2\tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$.

Finally, we consider the case in which $y_i(t)$ is an interaction point where an i-rarefaction arises. Then, $\Delta\Upsilon(t) \leq 0$, as in (2.3.18), provided $K \geq 2C\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. Concerning $\Delta\Theta_i^\varepsilon(t)$, call σ', σ'' the sizes of the interacting j-waves.

$$\begin{aligned}
&\Delta\Theta_i^\varepsilon(t)\\
\leq\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
&+\left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right)\\
&\left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right)\\
\leq\ & C\left(\left|\sigma'\right| + \left|\sigma''\right|\right)^3 e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
&+\|\bar{v^\varepsilon}\|_\infty\left(-\tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right) + \tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\left|\sigma'\sigma''\right| - \frac{H}{2}\left|\sigma'\sigma''\right|\right)\\
&e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\
\leq\ & 0
\end{aligned}$$

provided $\tilde{H} > 4CM^2\,\|\bar{v^\varepsilon}\|_\infty$ and $H \geq 2\tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. □

Proposition 2.3.7. *There exist positive M and C_2 such that, for all η, ε sufficiently small, if the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ corresponding to the initial datum $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ is defined up to time T, then, for all $t \in [0,T]$,*

$$\mathrm{TV}\left(v^\varepsilon(t)\right) \leq C_2\bar{K} \qquad \textit{and} \qquad \left\|v^\varepsilon(t)\right\|_\infty \leq M\|\bar{v}^\varepsilon\|_\infty\,.$$

Proof. Let $t \in [0,T]$. To bound the $\mathbf{L}^\infty$ norm, for any $x \in \mathbb{R}$, first choose $\tilde{H} = CM^2\eta$ and $H = 4CM^3\eta^2$, as in Proposition 2.3.6. Then, recursively,

$$\begin{aligned}
\left\|v_i^\varepsilon(t)\right\| &\leq \Theta_i^\varepsilon(t) && \text{by (2.3.15)}\\
&\leq \Theta_i^\varepsilon(0) && \text{by Proposition 2.3.6}\\
&\leq 2\eta e^{CM^2\eta\left(\tilde{Q}(0)+4M\eta Q(0)\right)} && \text{by (2.3.15).}\\
&\leq 2\eta e^{CM^2\eta\,(1+M\eta)} && \text{for a suitably large } C\\
&\leq M\eta && \text{for } M=2e^2 \text{ and } \eta<1/(CM^2)
\end{aligned}$$

for $i = 1,2$. Taking the supremum with respect to x, we obtain the desired bound.

Similarly, to bound the total variation, apply recursively the previous results:

$$\begin{aligned}
\mathrm{TV}(v^\varepsilon(t)) &\leq C_1\,\Upsilon^\varepsilon(t) && \text{by (2.3.14)}\\
&\leq C_1\Upsilon^\varepsilon(0) && \text{by Proposition 2.3.6}\\
&\leq C_2\,\mathrm{TV}(\bar{v}^\varepsilon) && \text{by (2.3.14)}\\
&\leq C_2\bar{K} && \text{by (2.3.1)}
\end{aligned}$$

completing the proof. □

Hence, by the Proposition 2.3.7, if $\bar{v}^\varepsilon \in \mathcal{D}(\eta,\bar{K})$ and if the approximate solution v^ε can be constructed on some initial interval $[0,T]$, then $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, C_2\bar{K})$ for all $t \in [0,T]$. In order to prove that v^ε can actually be defined for all $t > 0$, it remains to show that the total number of wave fronts and of points of interaction remains finite. For this aim, we use the next two propositions.

Proposition 2.3.8. *[15, Proposition 2] Let $v^\varepsilon = v^\varepsilon(t,x)$ be an ε–approximate solution constructed by the previous algorithm, with $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, C_2\bar{K})$ for all $t > 0$. Then, all of the shocks with size $\sigma < -\sqrt{\varepsilon}$ are located along a finite number of polygonal lines.*

Proposition 2.3.9. *[15, Proposition 3] Let $v^\varepsilon = v^\varepsilon(t,x)$ be an ε–approximate solution constructed by the previous algorithm, with $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, \bar{K})$ for all $t > 0$. Then, the set of all points where two fronts interact has no limit point in the (t,x)–plane.*

These two propositions are proved exactly as in [15]. The above results complete the proof of the following Theorem.

Theorem 2.3.10. *Let **(L)** hold. Fix a positive $\bar{K}$. Then, there exist positive η and M such that for every initial condition $\bar{v} \in \mathcal{D}(\eta,\bar{K})$ and for every sufficiently small $\varepsilon > 0$, the Cauchy problem (2.1.2) admits an ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ such that*

$$\left\|v^\varepsilon(t)\right\|_\infty \leq M\left\|\bar{v}\right\|_\infty. \tag{2.3.19}$$

Under condition **(GL)**, we also have the following decay estimate.

Theorem 2.3.11. *Let **(GL)** hold. Fix a positive $\bar{K}$. Then, there exist positive η and $\mathcal{M}$ such that for every initial condition $\bar{v} \in \mathcal{D}(\eta,\bar{K})$ and for every sufficiently small $\varepsilon > 0$, the ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ to the Cauchy problem (2.1.2) constructed in Theorem 2.3.10 satisfies for all $t > 0$, for all $a,b \in \mathbb{R}$ and for $i = 1,2$:*

$$\mathrm{TV}^+\left(v_i^\varepsilon(t);[a,b]\right) \leq \frac{b-a}{c\,t} + \mathcal{M}\left(\left\|\bar{v}\right\|_\infty \mathrm{TV}\left(\bar{v};[a-\hat{\lambda}t, b+\hat{\lambda}t]\right) + \varepsilon\right) \tag{2.3.20}$$

with c as in (2.2.1) and $\hat{\lambda}$ as in (2.2.6).

Proof. Under the present hypotheses, we use the usual decay estimate, see [14, Theorem 10.3] or [16, Theorem 1]:

$$\begin{aligned}
\mathrm{TV}^+\left(v_i^\varepsilon(t);[a,b]\right) &\le \frac{b-a}{ct} + C\left[Q\left(\bar v_{|[a-\hat\lambda t, b+\hat\lambda t]}\right) - Q\left(v^\varepsilon(t)_{|[a,b]}\right) + \varepsilon\right] \\
&\le \frac{b-a}{ct} + C\,Q\left(\bar v_{|[a-\hat\lambda t, b+\hat\lambda t]}\right) + C\varepsilon \\
&\le \frac{b-a}{ct} + \mathcal{M}\left(\|\bar v\|_\infty \mathrm{TV}\left(\bar v; [a-\hat\lambda t, b+\hat\lambda t]\right) + \varepsilon\right)
\end{aligned}$$

completing the proof. □

2.3.3 Existence of Solutions

For the sake of completeness, we pass the ε–approximate solutions to the limit $\varepsilon \to 0$. This standard application of Helly compactness Theorem yields a slight extension of the wave front tracking construction exhibited in [15]. Indeed, the mere existence of solutions to (2.1.2) is here obtained under the assumptions that the total variation of the initial datum be bounded.

Theorem 2.3.12. *Let **(L)** hold. Fix a positive $\bar K$. Then, there exist positive η, M such that for all $\bar u \in \mathcal{D}(\eta, \bar K)$, the Cauchy problem (2.1.2) admits a weak entropy solution, which is the limit of the wave front tracking approximate solutions constructed above and satisfying*

$$\left\|v(t)\right\|_\infty \le M \left\|\bar v\right\|_\infty .$$

*Moreover, if also **(GL)** holds, then there exists a positive $\mathcal{M}$ such that for all $t > 0$, for all $a, b \in \mathbb{R}$ and for $i = 1, 2$,*

$$\mathrm{TV}^+\left(v_i(t);[a,b]\right) \le \frac{b-a}{c\,t} + \mathcal{M}\left\|\bar v\right\|_\infty \mathrm{TV}\left(\bar v; [a-\hat\lambda t, b+\hat\lambda t]\right)$$

with c as in (2.2.1) and $\hat\lambda$ as in (2.2.6).

Thanks to the estimates proved above, the proof is standard and, hence, omitted.

2.4 Construction of a Solution with small $\mathbf{L}^\infty$ norm

We now prove Theorem 2.1.1 in the case of initial data satisfying the stronger conditions

$$\bar v \in \mathbf{C^1}\left(\mathbb{R}; B(0,\eta)\right) \quad \text{with} \quad \left\|\frac{d\bar v}{dx}\right\|_\infty \le \mathcal{L}\,, \tag{2.4.1}$$

see [31, i), ii) and iii) in Section 5].

We are going to use an inductive method. Define, for $m = 0, 1, 2, \ldots$ and for every $L > 0$, the m-trapezoid by

$$\triangle_m := \left\{ (t,x) \in [0,+\infty[\, \times \mathbb{R} \colon \begin{array}{rcl} t & \in & [t_m, t_m + \Delta t_m] \text{ and} \\ x & \in & [-2^m L + \hat\lambda t, 2^m L - \hat\lambda t] \end{array} \right\} \tag{2.4.2}$$

see Figure 2.2, where:

$$t_m = (2^m - 1)L/2\hat{\lambda} \quad \text{and} \quad \Delta t_m = 2^{m-1}L/\hat{\lambda}\,. \tag{2.4.3}$$

The upper side of $\triangle_m$ measures $2^m L$ and the lower one $2^{m+1}L$. The upper bases of 4

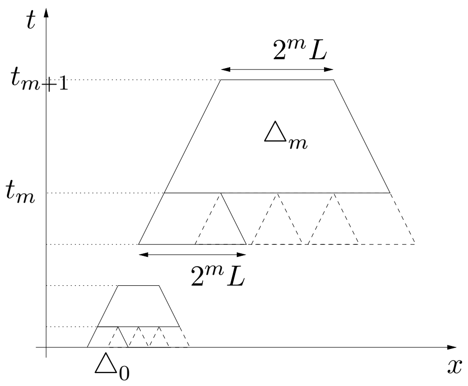


Figure 2.2: Construction of the trapezoids.

trapezoids $\triangle_{m-1}$ cover the lower basis of $\triangle_m$. We denote by $\triangle_m(x)$ the translation of the m–trapezoid: $\triangle_m(x) := (0,x) + \triangle_m$. Correspondingly, we introduce the domains

$$\mathcal{D}_m(\delta, 20\frac{\hat{\lambda}}{c}) := \left\{ v \in \mathbf{L}^1_{\mathbf{loc}}\left(\mathbb{R}; B(0,\delta)\right) \,:\, \mathrm{TV}(v; 2^{m+1}L) \leq 20\frac{\hat{\lambda}}{c} \right\}. \tag{2.4.4}$$

2.4.1 Construction in the 0–Trapezoid

In this paragraph we show that we are able to construct a solution in $\triangle_0(x)$, for all $x \in \mathbb{R}$. In fact, since the initial datum satisfies (2.4.1), we can always choose $L > 0$ such that

$$\mathrm{TV}(\bar{v}, 2L) \leq 20\hat{\lambda}/c\,. \tag{2.4.5}$$

Then, with reference to (2.4.4), we prove the following result.

Proposition 2.4.1. *Let **(GL)** and (2.4.1) hold. Then, there exist a sufficiently small $\eta > 0$ and $M, \mathcal{M} > 0$ such that for every initial condition $\bar{v} \in \mathcal{D}_0(\eta, 20\hat{\lambda}/c)$, the Cauchy problem* (2.1.2) *admits a weak entropy solution $v = v(t,x)$ defined for all $t \in [0, L/2\hat{\lambda}]$ and*

$$\begin{aligned} &\left\|v(t)\right\|_\infty \leq M\,\|\bar{v}\|_\infty \\ &\mathrm{TV}^+\left(v_i(t); 2(L - \hat{\lambda}t)\right) \leq \frac{2}{c}\,\frac{L - \hat{\lambda}t}{t} + \mathcal{M}\,\|\bar{v}\|_\infty\,\mathrm{TV}(\bar{v}; 2L)\,. \end{aligned}$$

The proof follows directly from Theorem 2.3.12.

2.4.2 Construction in the m–Trapezoid

Now we prove that, if a solution v to (2.1.2) satisfies suitable conditions at time $t = t_m$, then this solution can be extended on all the interval $[t_m, t_{m+1}]$. We also provide suitable estimates for later use.

Proposition 2.4.2. *Let **(GL)** hold. Then, there exists a sufficiently small $\eta > 0$ and positive $M, \mathcal{M}$ such that if $v(t_m) \in \mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$, then the problem (2.1.1) with datum $v(t_m)$ admits a weak entropy solution $v = v(t,x)$ defined for $t \in [t_m, t_{m+1}]$ satisfying*

$$\left\|v(t)\right\|_\infty \le M \left\|v(t_m)\right\|_\infty \tag{2.4.6}$$

$$\mathrm{TV}^+\left(v_i(t); 2(2^m L - \hat{\lambda}t)\right) \le \frac{2}{c}\,\frac{2^m L - \hat{\lambda}t}{t - t_m} + \mathcal{M}\left\|v(t_m)\right\|_\infty \mathrm{TV}(\bar{v}; 2^{m+1}L). \tag{2.4.7}$$

Above, $\mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$ is defined in (2.4.4). The proof is entirely similar to that of Proposition 2.4.1.

2.4.3 Existence of a Global Solution

In this paragraph we assume the following a priory bound:

(A) Whenever it is possible to define up to time t_m a solution v to (2.1.2) with an initial datum satisfying (2.4.1), then there exists $K > 0$ such that, for all $m \in \mathbb{N}$, $\left\|v(t_m)\right\|_\infty \le K\sqrt{\eta}$, where η is an upper bound for $\|\bar{v}\|_\infty$.

It is motivated by the recursive proof of Theorem 2.1.1 and by the following Proposition.

Proposition 2.4.3. *Suppose there exists up to time t_m a weak entropy solution $v = v(t,x)$ to* (2.1.2) *with an initial datum satisfying* (2.4.1)*. Let **(GL)**, (2.4.5) and **(A)** hold. Then, for all sufficiently small $\eta > 0$, if $\|\bar{v}\|_\infty \le \eta$, for all $m \in \mathbb{N}$ we have the estimate*

$$\mathrm{TV}\left(v(t_m); 2^{m+1}L\right) \le 20\frac{\hat{\lambda}}{c}\,.$$

Proof. Condition (2.4.5) immediately implies the desired bound for $m = 0$.

Let $m \ge 1$ and proceed by induction. Using the definition (2.4.2) of $\triangle_m(x)$ and the estimate (2.4.7), we get:

$$\begin{aligned}
& \mathrm{TV}^+\left(v_i(t_m); 2^{m+1}L\right) \\
\le\ & 4\,\mathrm{TV}^+\left(v_i(t_m); 2^{m-1}L\right) \\
\le\ & \frac{2^{m+1}L}{c(t_m - t_{m-1})} + 4\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right) \\
\le\ & 8\frac{\hat{\lambda}}{c} + 4\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right)\,.
\end{aligned}$$

Since $\mathrm{TV}(v) \le \left(\mathrm{TV}^+(v_1) + \mathrm{TV}^+(v_2)\right) + 2\|v\|_\infty$, we obtain:

$$\mathrm{TV}\left(v(t_m); 2^{m+1}L\right) \le 16\frac{\hat{\lambda}}{c} + 8\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right) + 2\left\|v(t_m)\right\|_\infty.$$

By **(A)** and choosing η small enough we get the thesis. □

Proof of Theorem 2.1.1 under condition (A).

Assume first that the initial data satisfies (2.4.1). By an application of Proposition 2.4.1, we are able to construct a solution for all $t \in [0, L/2\hat{\lambda}]$. Now, assume that a solution exists up to time t_m, with $m \geq 1$. Then, by **(A)**, we may apply Proposition 2.4.3 to obtain the TV bound at time t_m. Therefore, again thanks to **(A)**, we apply Proposition 2.4.2 to extend the solution up to time t_{m+1}. The proof is thus obtained inductively.

Consider now a general initial datum satisfying only (2.1.3). As in [31, Section 5], we approximate the initial datum $\bar{v}$ by a sequence of mollified data $\bar{v}_n$ such that each $\bar{v}_n$ satisfies (2.4.1). So, we are able to construct a sequence of solutions v_n to (2.1.1) related to the initial data $\bar{v}_n$. Then by [27, Theorem 1.7.3] we can select a subsequence that converges to a limit v, which is a weak entropy solution to (2.1.2). □

2.5 The $\mathbf{L}^\infty$ Estimate

The next step consists in proving that the a priori bound **(A)** is in fact a consequence of the other assumptions in Theorem 2.1.1 when the initial datum satisfies (2.4.1).

Proposition 2.5.1. *There exists a positive K such that for all initial datum $\bar{v}$ in (2.1.2), satisfying (2.1.3) and for all $m \in \mathbb{N}$, on the solution $v = v(t,x)$ to (2.1.2) the following estimate holds:*

$$\left\|v(t_m)\right\|_\infty \leq K\sqrt{\eta}\,,$$

where t_m is defined in (2.4.3).

Proof. For $m = 0$ the thesis holds, provided $K > \sqrt{\eta}$. Now, by induction, suppose that the theorem holds true up to $m-1$.

The lower basis of $\triangle_m$ is covered exactly by the upper basis of 4 $(m-1)$–trapezoids. Denote by T_{m-1} the union of these trapezoids. Then, divide T_{m-1} by horizontal segments $b^0_{m-1}, \ldots, b^N_{m-1}$ into N sub-trapezoids, say $T^1_{m-1}, \ldots, T^N_{m-1}$. Each sub-trapezoid T^j_{m-1} has height $h_N = 2^{m-2}L/(N\hat{\lambda})$, upper basis b^j_{m-1} and lower basis b^{j-1}_{m-1}, for $j = 1, \ldots, N$. Obviously, b^0_{m-1} and b^N_{m-1} are the lower and upper basis of T_{m-1}.

At least one of these trapezoids, call it T^n_{m-1}, is such that

$$\begin{aligned}
& Q\left(v\left(t_{m-1}+(n-1)h_N\right)_{|b^{n-1}_{m-1}}\right) - Q\left(v\left(t_{m-1}+nh_N\right)_{|b^n_{m-1}}\right) \\
\leq\ & \frac{1}{N}\left[Q\left(v(t_{m-1})_{|b^0_{m-1}}\right) - Q\left(v(t_m)_{|b^N_{m-1}}\right)\right] \\
\leq\ & \frac{1}{N}Q\left(v(t_{m-1})_{|b^0_{m-1}}\right) \\
\leq\ & \frac{1}{N}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}(v(t_{m-1})) \\
\leq\ & \frac{1}{N}\left\|v(t_{m-1})\right\|_\infty \frac{20\hat{\lambda}}{c}
\end{aligned} \tag{2.5.1}$$

by Proposition 2.4.3. Now, fix (t,x) and (t,y) on b^n_{m-1} with $x<y$. Then, using together the usual decay estimate [14, Theorem 10.3] or [16, Theorem 1] on the region T^n_{m-1}, together with (2.5.1), we have:

$$v_i(t,y) \quad \leq \quad v_i(t,x) + \frac{N}{L}\frac{y-x}{2^{m-2}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\left\|v(t_{m-1})\right\|_\infty .$$

Integrate in y to obtain

$$\frac{1}{l}\int_x^{x+l} v_i(t,y)\,dy \leq v_i(t,x) + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\left\|v(t_{m-1})\right\|_\infty. \tag{2.5.2}$$

Similarly, integrating in x, we get

$$v_i(t,y) \leq \frac{1}{l}\int_{y-l}^{y} v_i(t,x)\,dx + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\left\|v(t_{m-1})\right\|_\infty. \tag{2.5.3}$$

Using together (2.5.2) and (2.5.3), we obtain

$$\left|v_i(t,x)\right| \leq \frac{1}{l}\left|\int_{y-l}^{y} v_i(t,x)\,dx\right| + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\left\|v(t_{m-1})\right\|_\infty . \tag{2.5.4}$$

At this point we consider three different cases, depending on which coefficients in (2.2.2) vanish. We defer the proofs of the corresponding integral estimates to Section 2.6.

1. $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$. Hence by Proposition 2.6.2,

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l+C''t) \qquad \text{for } i=1,2\,. \tag{2.5.5}$$

(Note that it is this case that covers the situation considered in [31]).

2. $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$. Then, using Proposition 2.6.3

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l+C''t) + C\left\|v(t)\right\|^3_\infty t \qquad \text{for } i=1,2\,.$$

3. $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$). Hence, by an application of Proposition 2.6.4:

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l+C''t) + C\left\|v(t)\right\|^3_\infty t \qquad \text{for } i=1,2\,.$$

Using the (worst) estimate of cases 2. and 3., we have

$$\left|v_i(t,x)\right| \leq C'\eta\left(1+C''\frac{t}{l}\right) + C\left\|v(t)\right\|^3_\infty \frac{t}{l} + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat\lambda}{c} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\left\|v(t_{m-1})\right\|_\infty .$$

Setting $l/t = \sqrt{\eta + \|v(t)\|_\infty^3}$, using the fact that $t \le t_m$ and the inductive assumption $\|v(t)\|_\infty \le MK\sqrt{\eta}$, we have

$$\begin{aligned}\|v(t)\|_\infty &\le C\left(\eta + \sqrt{\eta}\right) + C\|v(t)\|_\infty^{3/2} + \frac{N}{c}\frac{\sqrt{\eta + \|v(t)\|_\infty^3}}{2^{m-1}}\frac{\hat\lambda}{L}t \\ &\quad + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\|v(t_{m-1})\|_\infty \\ &\le C\sqrt{\eta} + \frac{CN}{c}\sqrt{\eta} + \frac{\mathcal{M}}{N}\frac{20\hat\lambda}{c}\|v(t_{m-1})\|_\infty \\ &\le CN\sqrt{\eta} + \frac{C}{N}\|v(t_{m-1})\|_\infty .\end{aligned}$$

Choosing $N = 4CM$ and $K = 4MNC$, by the inductive hypothesis, we get $\|v(t)\|_\infty \le \frac{K}{2M}\sqrt{\eta}$. So, we can conclude:

$$\|v(t_m)\|_\infty \le 2M\|v(t)\|_\infty \le K\sqrt{\eta}$$

completing the proof. Obviously, the proof is exactly the same if, instead of $\triangle_m$, we consider a generic trapezoid $\triangle_m(x)$ for some $x \in \mathbb{R}$. □

Remark that in the previous proof, case 1 covers the situation treated in [31]. Indeed, in (2.5.5) the optimal choice for l/t is $l/t = \sqrt{\eta}$, exactly as in [31].

2.6 The Integral Estimate

Lemma 2.6.1. *Let $u = u(t,x)$ be the solution to* (2.1.2) *constructed in the previous sections, such that $\|u(t)\|_\infty \le C\sqrt{\eta}$, with an initial data satisfying* (2.1.3) *and* (2.4.1). *If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \ne 0$ (respectively $\frac{\partial^2 f_2}{\partial u_1^2}(0) \ne 0$), then there exists an invariant region for the variable u_1 (respectively u_2). More precisely, there exists a positive constant $\mathcal{K}$ such that, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, it holds:*

$$u_1(t,x) \ge -\mathcal{K}\eta\,, \qquad \textit{respectively} \quad u_2(t,x) \ge -\mathcal{K}\eta\,.$$

Proof. At first we consider the ε–approximate solutions constructed above. Let v_1 and v_2 be the corresponding Riemann coordinates. The map $\mathcal{T}\colon v = (v_1, v_2) \mapsto u = (u_1, u_2)$ is smooth and maps the origin into the origin. So, using the hypothesis $\frac{\partial^2 f_1}{\partial u_2^2}(0) \ne 0$, Lemma 2.7.2 implies that

$$\left[\dddot{\mathcal{S}}_2(v,\sigma) - \dddot{\mathcal{R}}_2(v,\sigma)\right]_1 = \left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \ne 0 \tag{2.6.1}$$

for v sufficiently small.

Let u^- and u^+ denote the left and the right states in a Riemann initial value problem, and let u^* denote the intermediate state, connected to u^- by a 1-wave and to u^+ by a 2-wave.

If $\left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \geq 0$ then we have that the Riemann invariant v_1^ε doesn't change along a right rarefaction and increases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \leq v_1^\varepsilon(u^+). \tag{2.6.2}$$

Obviously, this inequality holds also whenever the right shock has strength less then $2\sqrt{\varepsilon}$, in fact in this case we interpolate a rarefaction and an entropic shock. Using (2.6.2) and the fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \leq \eta$, we obtain $v_1^\varepsilon(t,x) \leq \eta$ for any $t > 0$. By a liner change of coordinates, we can assume that $\mathcal{T}_1(0,0) = 0$, $\frac{\partial \mathcal{T}_1}{\partial v_2}(0,0) = 0$ $\frac{\partial \mathcal{T}_1}{\partial v_1}(0,0) = -\mathcal{K}_1$, with $\mathcal{K}_1 > 0$. By this choice, it holds that $u_1^\varepsilon(t,x) = \mathcal{T}_1(v_1^\varepsilon(t,x), v_2^\varepsilon(t,x)) = -\mathcal{K}_1\, v_1^\varepsilon(t,x) + \mathcal{K}_2 \left(v_1^\varepsilon(t,x)\right)^2 + \mathcal{K}_3\, v_1^\varepsilon(t,x)\, v_2^\varepsilon(t,x) + \mathcal{K}_4 \left(v_2^\varepsilon(t,x)\right)^2$, where $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are the second derivatives of $\mathcal{T}_1$ computed in an intermediate point. Since $v_1^\varepsilon(t,x) < \eta$ and $\left\|v(t)\right\|_\infty \leq C\sqrt{\eta}$, we have $u_1^\varepsilon(t,x) \geq -\tilde{C}\mathcal{K}_1\,\eta - |\mathcal{K}_4|\eta$, for a suitable $\tilde{C} > 0$. Now, choosing $\mathcal{K} = \tilde{C}\mathcal{K}_1 + |\mathcal{K}_4|$, we obtain

$$u_1^\varepsilon(t,x) \geq -\mathcal{K}\eta.$$

Similarly, if $\left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \leq 0$, v_1^ε doesn't change along a right rarefaction and decreases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \geq v_1^\varepsilon(u^+). \tag{2.6.3}$$

Now, using the fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \geq -\eta$ and (2.6.3), we get: $v_1^\varepsilon(t,x) \geq -\eta$ for any $t > 0$. As above, we can suppose that the map $\mathcal{T}_1$ is such that:

$$u_1^\varepsilon(t,x) \geq -\mathcal{K}\eta.$$

Clearly, the result still holds when we pass to the limit.

Similarly, if $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, it holds $u_2(t,x) \geq -\mathcal{K}\eta$. □

Proposition 2.6.2. *Let $v = v(t,x)$ be the solution to* (2.1.2) *constructed in the previous sections, with an initial data satisfying* (2.1.3) *and* (2.4.1). *If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, then, for all segment l and for all $\bar{t} \geq 0$:*

$$\left|\int_l v_i(\bar{t},x)dx\right| \leq C'\eta\Big(l + C''\bar{t}\Big). \tag{2.6.4}$$

Proof. By an application of Lemma 2.6.1, we get:

$$|u_1| \leq u_1 + 2\mathcal{K}\eta \;\;,\;\; |u_2| \leq u_2 + 2\mathcal{K}\eta. \tag{2.6.5}$$

Then, let us consider in the t,x plain the trapezoid with the lower basis l_0 equals to $[(0,x^l),(0,x^r)]$ and the upper basis l equals to $[(\bar{t}, x^l + \vartheta\bar{t}), (\bar{t}, x^r - \vartheta\bar{t})]$, where ϑ is positive.

Then, using the Divergence Theorem

$$\int_l [u_1(\bar t,x)+u_2(\bar t,x)]dx = \int_{l_0}[u_1(0,x)+u_2(0,x)]dx$$

$$-\int_{x^l}^{x^l+\vartheta\bar t}\left\{[u_1(\frac{x-x^l}{\vartheta},x)+u_2(\frac{x-x^l}{\vartheta},x)]-\frac{1}{\vartheta}\,[f_1(u(\frac{x-x^l}{\vartheta},x))+f_2(u(\frac{x-x^l}{\vartheta},x))]\right\}dx$$

$$-\int_{x^r-\vartheta\bar t}^{x^r}\left\{[u_1(\frac{x^r-x}{\vartheta},x)+u_2(\frac{x^r-x}{\vartheta},x)]+\frac{1}{\vartheta}\,[f_1((\frac{x^r-x}{\vartheta},x))+f_2((\frac{x^r-x}{\vartheta},x))]\right\}dx\,. \tag{2.6.6}$$

Since f_1 and f_2 depend smoothly on u_1 and u_2 it holds that $|f_1|+|f_2|\le C\,(|u_1|+|u_2|)$. Then, using this last estimate and (2.6.5) we get

$$\begin{aligned}&[u_1(\frac{x-x^l}{\vartheta},x)+u_2(\frac{x-x^l}{\vartheta},x)]-\frac{1}{\vartheta}\,[f_1(u(\frac{x-x^l}{\vartheta},x))+f_2(u((\frac{x-x^l}{\vartheta},x)]\\ &\ge\left(\left|u_1(\frac{x-x^l}{\vartheta},x)\right|+\left|u_2(\frac{x-x^l}{\vartheta},x)\right|\right)(1-\frac{C}{\vartheta})-2\mathcal{K}\eta\end{aligned} \tag{2.6.7}$$

and

$$\begin{aligned}&[u_1(\frac{x^r-x}{\vartheta},x)+u_2(\frac{x^r-x}{\vartheta},x)]+\frac{1}{\vartheta}\,[f_1(u(\frac{x^r-x}{\vartheta},x))+f_2(u(\frac{x^r-x}{\vartheta},x))]\\ &\ge[u_1(\frac{x^r-x}{\vartheta},x)+u_2(\frac{x^r-x}{\vartheta},x)]-\frac{1}{\vartheta}\,\left[\left|f_1(u(\frac{x^r-x}{\vartheta},x))\right|+\left|f_2(u((\frac{x^r-x}{\vartheta},x))\right|\right]\\ &\ge\left(\left|u_1(\frac{x^r-x}{\vartheta},x)\right|+\left|u_2(\frac{x^r-x}{\vartheta},x)\right|\right)(1-\frac{C}{\vartheta})-2\mathcal{K}\eta\end{aligned} \tag{2.6.8}$$

We can choose $\vartheta=C$; now using (2.6.7) and (2.6.8) in the two last integrals on the right in (2.6.6) and (2.6.5) on the left, we get

$$\int_l\Big[\big|u_1(\bar t,x)\big|+\big|u_2(\bar t,x)\big|-2\mathcal{K}\eta\Big]\,dx=\int_{l_0}\Big[\big|u_1(0,x)\big|+\big|u_2(0,x)\big|\Big]\,dx+4\mathcal{K}C\bar t\eta$$

then

$$\int_l\Big[\big|u_1(\bar t,x)\big|+\big|u_2(\bar t,x)\big|\Big]\,dx\le C'\eta(l+C''\bar t)$$

Since v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (2.6.4) is proved. □

Proposition 2.6.3. *Let $v=v(t,x)$ be the solution to* (2.1.2) *constructed in the previous sections, with an initial data satisfying* (2.1.3) *and* (2.4.1). *If $\dfrac{\partial^2 f_1}{\partial u_2^2}(0)=0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0)=0$, then, for all segment l and for all $\bar t\ge 0$:*

$$\left|\int_l v_i(\bar t,x)dx\right|\le C'\eta\Big(l+C''\bar t\Big)+C\big\|v(\bar t)\big\|_\infty^3\bar t. \tag{2.6.9}$$

Proof. Let us call l^- and l^+ the initial and the terminal point of l. For any curves $x^-(t)$ and $x^+(t)$ such that $x^-(\bar{t}) = l^-$ and $x^+(\bar{t}) = l^+$, by the Divergence Theorem, we get:

$$\begin{aligned}\int_l u_i(\bar{t},x)dx &= \int_{x^-(0)}^{x^+(0)} u_i(0,x)dx + \int_0^{\bar{t}} [f_i(u(t,x^-(t))) - \dot{x}^-(t)\, u_i(t,x^-(t))]dt \\ &\quad + \int_0^{\bar{t}} [-f_i(u(t,x^+(t))) + \dot{x}^+(t)\, u_i(t,x^+(t))]dt\end{aligned}$$

for $i = 1, 2$. Hence, to obtain

$$\left|\int_l u_i(\bar{t},x)dx\right| \le C'\eta(l + C''\bar{t}) + C\big\|u(\bar{t})\big\|_\infty^3 \bar{t} \tag{2.6.10}$$

it is sufficiently to solve on $[0,\bar{t}]$ and out of shocks, up to terms of the order of $\big\|u(t)\big\|_\infty^2$, the ordinary differential equations:

$$\dot{x}^-(t) = \frac{f_i(u(t,x^-(t)))}{u_i(t,x^-(t))}, \qquad \dot{x}^+(t) = \frac{f_i(u(t,x^+(t)))}{u_i(t,x^+(t))}, \tag{2.6.11}$$

with the initial conditions $x^\pm(\bar{t}) = l^\pm$. By the hypothesis $\dfrac{\partial^2 f_i}{\partial u_j^2}(0) = 0$, (2.6.11) admit generalized solutions $x_i^-(t)$ and $x_i^+(t)$ in the sense of Filippov (see [30, Chapter 2, Section 4]). It may happen that their graph coincides with the support of shocks of the function u on sets of positive $\mathcal{H}^1$–measure. By Proposition 2.7.3, there exist two Lipschitz functions $\tilde{x}_i^\pm$ with $\tilde{x}_i^\pm(\bar{t}) = l^\pm$ and

$$\left\|\dot{x}_i^- - \dot{\tilde{x}}_i^-\right\|_\infty \le \|u\|_\infty^2, \qquad \left\|\dot{x}_i^+ - \dot{\tilde{x}}_i^+\right\|_\infty \le \|u\|_\infty^2$$

such that their graphs coincide with the shock of u on sets of zero $\mathcal{H}^1$–measure. Then, we have that (2.6.10) holds and, by the smoothness of v_1 and v_2, also the inequality (2.6.9) is proved. □

Proposition 2.6.4. *Let $v = v(t,x)$ be the solution to* (2.1.2) *constructed in the previous sections, with an initial data satisfying* (2.1.3) *and* (2.4.1)*. If $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), then, for all segment l and for all $\bar{t} \ge 0$:*

$$\left|\int_l v_i(\bar{t},x)dx\right| \le C'\eta\Big(l + C''\bar{t}\Big) + C\big\|v(\bar{t})\big\|_\infty^3 \bar{t}. \tag{2.6.12}$$

Proof. Let us consider $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, in fact in the opposite case the proof is exactly the same. By an application of Lemma 2.6.1, we get:

$$|u_1| \le u_1 + 2\mathcal{K}\eta. \tag{2.6.13}$$

Proceeding as in Proposition 2.6.2, we get:

$$\int_l \Big[\big|u_1(\bar t, x)\big| - 2\mathcal{K}\eta\Big]\, dx = \int_{l_0} \big|u_1(0,x)\big| dx + 4\mathcal{K}C\bar t\eta$$

then:

$$\left|\int_l u_1(\bar t, x) dx\right| \le \int_l \big|u_1(\bar t, x)\big| dx \le C'\eta(l + C''\bar t). \tag{2.6.14}$$

For the variable u_2 we follow exactly the same strategy used in the Proposition 2.6.3, so that we obtain:

$$\int_l \big|u_2(\bar t, x)\big| dx \le C'\eta(l + C''\bar t) + C\big\|u(\bar t)\big\|_\infty^3 \bar t. \tag{2.6.15}$$

Now, using together (2.6.14) and (2.6.15) and the fact that v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (2.6.12) is proved. □

2.7 Technical Details

Lemma 2.7.1. *If f is as in* (2.2.2), *then*

$$(Dr_2\, r_2)(0) = [-\alpha_{22}, 0]^T \qquad \textit{and} \qquad (Dr_1\, r_1)(0) = [-\beta_{11}, 0]^T \tag{2.7.1}$$

Proof. Recall the definition of the resolvent: $R(\xi, u) := (A(u) - \xi I)^{-1}$ (see [36]). We have:

$$\begin{aligned} R(\xi,u) &= \Big(A(0) + \big(A(u) - A(0)\big) - \xi I\Big)^{-1} \\ &= \big(A(0) - \xi I\big)^{-1}\Big(I + \big(A(u) - A(0)\big)\big(A(0) - \xi I\big)^{-1}\Big)^{-1} \\ &= (A(0) - \xi I)^{-1} - (A(0) - \xi I)^{-1}\big(A(u) - A(0)\big)\,(A(0) - \xi I)^{-1} + \\ &\quad +\mathcal{O}\left(u^2\right). \end{aligned}$$

Choose a closed curve Γ such that $\lambda_2(u)$ is the unique eigenvalue inside it. The projection P_2 can then be computed as:

$$P_2(u) = -\frac{1}{2\pi i}\oint_\Gamma R(\xi,u) d\xi = -\frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi}\end{bmatrix} d\xi$$

$$+\frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi}\end{bmatrix}\begin{bmatrix} \frac{\partial f_1}{\partial u_1}(u) + 1 & \frac{\partial f_1}{\partial u_2}(u) \\ \frac{\partial f_2}{\partial u_1}(u) & \frac{\partial f_2}{\partial u_2}(u) - 1\end{bmatrix}\begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi}\end{bmatrix} d\xi + \mathcal{O}\left(u^2\right)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} + \frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} 0 & -\frac{\frac{\partial f_1}{\partial u_2}(u)}{(\xi+1)(1-\xi)} \\ -\frac{\frac{\partial f_2}{\partial u_1}(u)}{(\xi+1)(1-\xi)} & 0\end{bmatrix}$$

$$+\mathcal{O}\left(\tfrac{1}{(1-\xi)^2}\right) + \mathcal{O}\left(\tfrac{1}{(\xi+1)^2}\right) + \mathcal{O}(u^2)$$

$$= \begin{bmatrix} 0 & -\alpha_{12}u_1 - \alpha_{22}u_2 \\ -\beta_{11}u_1 - \beta_{12}u_2 & 1 \end{bmatrix} + \mathcal{O}\left(u^2\right)$$

Since $P_2(u) = r_2(u) \otimes l_2(u)$,

$$r_2(u) = [-\alpha_{12}u_1 - \alpha_{22}u_2, 1]^T + \mathcal{O}(1)\,\|u\|^2. \tag{2.7.2}$$

and

$$l_2(u) = [-\beta_{11}u_1 - \beta_{12}u_2, 1]^T + \mathcal{O}(1)\,\|u\|^2. \tag{2.7.3}$$

Finally $Dr_2(0) = \begin{bmatrix} -\alpha_{12} & -\alpha_{22} \\ 0 & 0 \end{bmatrix}$ and $(Dr_2\, r_2)(0) = [-\alpha_{22}, 0]^T$.

To prove the second equation it is sufficient to repeat the previous arguments. □

Lemma 2.7.2. *If* $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = \alpha_{22} \neq 0$, $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = \beta_{11} \neq 0$ *and condition* ***(GL)*** *holds, then*

$$\begin{aligned} \left[\dddot{S}_2(0,0) - \dddot{R}_2(0,0)\right]_1 &= \frac{1}{2}\frac{\langle (D\lambda_2\, r_2)(Dr_2\, r_2), r_1\rangle}{\lambda_2 - \lambda_1} \neq 0\,, \\ \left[\dddot{S}_1(0,0) - \dddot{R}_1(0,0)\right]_2 &= \frac{1}{2}\frac{\langle (D\lambda_1\, r_1)(Dr_1\, r_1), r_2\rangle}{\lambda_1 - \lambda_2} \neq 0\,. \end{aligned}$$

Proof. Let us denote by $S_2(\sigma)$ and $R_2(\sigma)$ the shock and the rarefaction curve of the second family with starting point 0, by $A(\sigma)$ the Jacobian matrix $Df(S_2(\sigma))$, by $r_i(\sigma)$ ($l_i(\sigma)$) the right (left) eigenvector $r_i(S_2(\sigma))$ ($l_i(S_2(\sigma))$) and by Λ the Rankine–Hugoniot speed.

Differentiating three times the Rankine-Hugoniot conditions w.r.t. σ we obtain:

$$\ddot{A}\dot{S}_2 + 2\dot{A}\ddot{S}_2 + A\dddot{S}_2 = \ddot{\Lambda}S_2 + 3\dot{\Lambda}\ddot{S}_2 + 3\ddot{\Lambda}\dot{S}_2 + \Lambda\dddot{S}_2.$$

At $\sigma = 0$ it becomes

$$\ddot{A}r_2 + 2\dot{A}(Dr_2\, r_2) = \frac{3}{2}(D\lambda_2\, r_2)(Dr_2\, r_2) - A\dddot{S}_2 + 3\ddot{\Lambda}r_2 + \lambda_2\dddot{S}_2\,. \tag{2.7.4}$$

Differentiating twice w.r.t. σ the identity $Ar_2 = \lambda_2 r_2$ at $\sigma = 0$ we find

$$\begin{aligned} &\ddot{A}r_2 + 2\dot{A}(Dr_2\, r_2) + A(D^2r_2\, r_2)r_2 + ADr_2(Dr_2\, r_2) \\ &= \langle D^2\lambda_2\, r_2, r_2\rangle r_2 + \langle D\lambda_2\, Dr_2, r_2\rangle r_2 + 2(D\lambda_2\, r_2)(Dr_2\, r_2) \\ &+\lambda_2(D^2r_2\, r_2)r_2 + \lambda_2 Dr_2(Dr_2\, r_2). \end{aligned}$$

Using (2.7.4) in the last equation:

$$\begin{aligned} &(A - \lambda_2\,\mathrm{Id})(D^2r_2\, r_2)r_2 + (A - \lambda_2\,\mathrm{Id})Dr_2(Dr_2\, r_2) - (A - \lambda_2\,\mathrm{Id})\dddot{S}_2 + 3\ddot{\Lambda}r_2 \\ &= \langle D^2\lambda_2\, r_2, r_2\rangle r_2 + D\lambda_2\,(Dr_2\, r_2)r_2 + \frac{1}{2}(D\lambda_2\, r_2)(Dr_2\, r_2). \end{aligned} \tag{2.7.5}$$

Then, multiplying on the left by $l_2(0)$, it holds:

$$\ddot{\Lambda} = \frac{1}{3}D(D\lambda_2\, r_2)\, r_2. \tag{2.7.6}$$

We can now substitute (2.7.6) in (2.7.5) and obtain

$$\begin{aligned} &(\lambda_2\,\mathrm{Id} - A)\dddot{S}_2 = \tfrac{1}{2}(D\lambda_2\, r_2)(Dr_2\, r_2) + (\lambda_2\,\mathrm{Id} - A)(D^2r_2\, r_2)r_2 \\ &+(\lambda_2\,\mathrm{Id} - A)Dr_2(Dr_2\, r_2). \end{aligned}$$

Hence, multiplying on the left by $l_1(0) = [1,0] = r_1^T(0)$, we have that

$$\langle \dddot{S}_2, r_1\rangle = \frac{1}{2}\frac{\langle (D\lambda_2\, r_2)(Dr_2\, r_2), r_1\rangle}{\lambda_2-\lambda_1} + \langle (D^2 r_2\, r_2) r_2, r_1\rangle + \langle Dr_2(Dr_2\, r_2), r_1\rangle\,.$$

Now, since $\langle \ddot{R}_2, r^1\rangle = \langle (D^2 r_2\, r_2) r_2, r_1\rangle + \langle Dr_2(Dr_2\, r_2), r_1\rangle$, using (2.7.1) and the genuine non linearity, we can conclude that:

$$\langle \dddot{S}_2, r_1\rangle - \langle \dddot{R}_2, r^1\rangle = \frac{1}{2}\frac{\langle (D\lambda_2\, r_2)(Dr_2\, r_2), r_1\rangle}{\lambda_2-\lambda_1} \neq 0.$$

The second part of the statement is proved repeating the same arguments. □

Proposition 2.7.3. *Let $u = u(t,x)$ be a weak entropy solution to* (2.1.2) *and denote by $\{y_m(t)\}_{m\in\mathbb{N}}$ the countable family of its shocks (see [14, Section 10.3]). Setting $L(T,X) := \{\varphi \in W^{1,\infty}[0,T] : \varphi(T) = X\}$ and $J := \bigcup_m \mathrm{graph}(y_m)$, we have that the set*

$$\mathcal{F} := \{\varphi \in L : \mathcal{H}^1(\mathrm{graph}(\varphi)\cap J) = 0\}$$

is dense in $L(T,X)$ endowed with the usual norm of $W^{1,\infty}$ (i.e. $\|\varphi\|_{W^{1,\infty}} := \|\varphi\|_\infty + \|\varphi'\|_\infty$).

Proof. L is complete, being a closed subset of a complete metric space. Observe that $\mathcal{F} = \bigcap_{m,n}\mathcal{F}_{n,m}$, where:

$$\mathcal{F}_{n,m} := \left\{\varphi \in L(T;X) \colon \mathcal{H}^1(\mathrm{graph}(\varphi)\cap \mathrm{graph}(y_m)) < 1/n\right\}\,.$$

By Baire Theorem, see [46, Proposition 3.5.4], it is sufficient to prove that each $\mathcal{F}_{n,m}$ is an open and dense subset of $L(T,X)$.

$\mathcal{F}_{n,m}$ **is open:** Fix $\varphi \in \mathcal{F}_{n,m}$ and define

$$\begin{aligned} D_\varphi &:= \left\{(t, y_m(t)) \in [0,T]\times\mathbb{R} \colon \varphi(t) = y_m(t)\right\}\\ D_\varphi^d &:= \left\{(t, y_m(t)) \in [0,T]\times\mathbb{R} \colon \left|\varphi(t) - y_m(t)\right| \le d\right\}\,. \end{aligned}$$

For every $\varepsilon \in \left]0, 1/n - \mathcal{H}^1(D_\varphi)\right[$, there exists a positive δ such that $\mathcal{H}^1(D_\varphi^\delta) = 1/n - \varepsilon$. Now, consider the open ball $\mathcal{B}(\varphi,\delta)$ in the space $(L(T,X), \|\cdot\|_{W^{1,\infty}})$. For every $\psi \in \mathcal{B}(\varphi,\delta)$, we have that $\psi(t) \neq y_m(t)$ whenever $(t, y_m(t)) \in \mathbb{R}^2 \setminus D_\varphi^\delta$. In fact, if $\psi(t) = y_m(t)$ with $(t, y_m(t)) \in \mathbb{R}^2\setminus D_\varphi^\delta$, then $\left|\varphi(t) - \psi(t)\right| > \delta$ which is impossible since $\psi \in \mathcal{B}(\varphi,\delta)$. Hence, we obtain that $D_\psi \subseteq D_\varphi^\delta$, for all $\psi \in \mathcal{B}(\varphi,\delta)$, i.e. $\mathcal{B}(\varphi,\delta) \subset \mathcal{F}_{n,m}$. By the arbitrariness of φ, we conclude that $\mathcal{F}_{n,m}$ is open.

$\mathcal{F}_{n,m}$ **is dense:** Choose a $\varphi \in L$. We show that φ can be arbitrarily approximated by functions in $\mathcal{F}_{n,m}$, hence we can assume that $\mathcal{H}^1(\mathrm{graph}(\varphi)\cap\mathrm{graph}(y_m)) \ge 1/n$. By [14, Theorem 10.4]), $\varphi - y_m$ is Lipschitz on $[0,T]$. Then, call $\mathcal{C} = \left\{t \in [0,T] \colon \varphi(t) = y_m(t)\right\}$. $\mathcal{C}$ is closed and can be represented as $\mathcal{C} \subseteq \bigcup_{k=1}^N [a_k, b_k]$, for a suitable $N \ge 1$. Define, for instance, ψ as

$$\psi(t) := \varphi(t) + \delta^2 \sum_{k=1}^N e^{-1/\left((t-a_k)^2(b_k-t)^2\right)}\chi_{[a_k,b_k]}(t) \tag{2.7.7}$$

Clearly, $\psi \in \mathcal{F}_{n,m}$. Moreover $\|\varphi - \psi\|_{W^{1,\infty}} \le \delta$, for δ small. Hence, $\psi \in \mathcal{B}(\varphi,\delta)$, proving the density of $\mathcal{F}_{n,m}$ in $L(T,X)$. □

Chapter 3

Modeling the Dynamics of Granular Matter

3.1 Introduction

Consider a slope with profile $u = u(t, x)$, where $x \in \mathbb{R}^2$ and $t \geq 0$. On this bed, some kind of material with thickness $h = h(t, x)$ is free to slide, subject to gravity. The sliding matter may well erode or deposit, thus modifying the slope as well as its distribution over it. This explains why both functions u and h are also time dependent. We describe the complex dynamics that arises through the following equations:

$$\begin{cases} \partial_t h + \nabla \cdot (h\, v) = -\gamma \left(\alpha - \|\nabla u\|\right) h + H \\ \partial_t v + \nabla \cdot \left(\frac{1}{2} v \otimes v + gh\, \mathrm{Id}\right) = -g\nabla u + \nu(v, \nabla u) - \gamma [\![\, \alpha - \|\nabla u\| \,]\!]_- v + V \\ \partial_t u = \gamma \left(\alpha - \|\nabla u\|\right) h \end{cases} \tag{3.1.1}$$

Here, v defines the velocity vector field at which matter slides over the bed. g is gravity and α is the critical angle: at slopes higher than α, the sliding matter erodes the bed while falling, whereas at lower slopes it deposits over the bed. The constant γ is the speed at which erosion–deposition takes place. More precisely, in the first equation, the term $-\gamma h(\alpha - \|\nabla u\|)$ corresponds to the quantity of matter that deposits, when $\|\nabla u\| < \alpha$, or that is eroded, when $\|\nabla u\| > \alpha$. Conservation of mass requires that the same term appears, with the opposite sign, in the third equation. The right hand side of the second equation contains a first term $-g\,\nabla u$, describing the component of gravity parallel to the slope. The vector $\nu(v, \nabla u)$ describes the friction between the sliding matter and the slope. Therefore, we require the following physically obvious conditions:

$$v \cdot \nu(v, \nabla u) \leq 0 \quad \text{and} \quad \nu(0, \nabla u) = 0 \tag{3.1.2}$$

for all v and u. The term $-[\![\, \gamma(\alpha - \|\nabla u\|) \,]\!]_- v$ in the second equation is due to the eroded material that starts moving and affects the speed of the sliding matter. Finally, H and V are functions of (t, x), presumably known, modeling sources such as material falling or being poured over the bed.

A priori, the first two equations in (3.1.1) can be justified through the balance of mass and of linear momentum, see [28], the third equation by the erosion–deposition dynamics. Part of these terms, in fact, are found also in the Hadeler–Kuttler model introduced in [33, formula (5)]:

$$\textbf{(HK)} \qquad \begin{cases} \partial_t h - \nabla \cdot (\beta\, h \nabla u) = -\gamma\left(\alpha - \|\nabla u\|\right) h + H \\ \partial_t u = \gamma\left(\alpha - \|\nabla u\|\right) h \end{cases}$$

where we used the same notations as above. β is the rate that connects the velocity of the rolling layer to the gradient of u, see [34]. This model has been widely considered in the literature, both from the point of view of stationary asymptotic solutions and from that of evolution problems, see for instance [1, 18, 19, 20, 26, 29, 45]. Some of the above studies are purely analytical, others are of a more numerical nature.

On the other hand, the convective part in the first two equations of (3.1.1) reminds of that of the Savage–Hutter model, see [42, 47], which, for smooth solutions, can be rewritten as

$$\textbf{(SH)} \qquad \begin{cases} \partial_t h + \nabla \cdot (h\, v) = 0 \\ \partial_t v + \nabla \cdot \left(\frac{1}{2} v \otimes v + h\, G\right) = s\,. \end{cases}$$

Here, the vector s and the 2×2 diagonal matrix G are functions of the unknowns and of the space variables, see [47, formulæ (1)–(10)], whose role is to accurately describe the given fixed geometry of the slope and the effect of gravity. Remark that **(SH)** is essentially equivalent to the shallow water equations with a bed having a given fixed geometry and a drift term in the moment equation, see [28]. Several papers consider the model **(SH)** from various points of view, see for instance [32, 41, 43, 44].

Below, we study (3.1.1) and compare it with **(HK)** and **(SH)**. To this aim, we first scale out the various constant parameters in (3.1.1) and **(HK)**, obtaining

$$\begin{cases} \partial_t h + \nabla \cdot (hv) = -\left(1 - \|\nabla u\|\right) h + H \\ \partial_t v + \nabla \cdot \left(\frac{1}{2} v \otimes v + h\, \mathrm{Id}\right) = -\nabla u + \nu(v, \nabla u) - [\![\, 1 - \|\nabla u\| \,]\!]_{-}\, v + V \\ \partial_t u = \left(1 - \|\nabla u\|\right) h \end{cases} \tag{3.1.3}$$

in the case of model (3.1.1), see Lemma 3.5.2. In the case of **(HK)**, Lemma 3.5.1 yields the rescaling

$$\begin{cases} \partial_t h - \nabla \cdot (h \nabla u) = -\left(1 - \|\nabla u\|\right) h + H \\ \partial_t u = \left(1 - \|\nabla u\|\right) h\,. \end{cases} \tag{3.1.4}$$

A first key difference between (3.1.1) and **(HK)** is the energy balance. Indeed, smooth solutions to (3.1.1) dissipate the energy

$$E = \int_{\mathbb{R}^2} \left(\frac{1}{2}\, h \, \|v\|^2 + \frac{1}{2} \, (h+u)^2\right) \mathrm{d}x \,, \tag{3.1.5}$$

see Proposition 3.2.1. On the other hand, the oscillations arising in the solutions to **(HK)**, see Paragraph 3.3.1, show that **(HK)** can hardly be energy dissipating.

An obvious difference between (3.1.1) and **(SH)** is that the latter model does not take into account the erosion–deposition phenomena. Therefore, below, we compare the qualitative behaviour of the uppermost moving profile $u + h$ in (3.1.1) with its analog h in **(SH)**.

Then, we pass to the 1D case. After the standard preliminary study, we consider some numerical integrations of the different models and compare the corresponding solutions. It is immediate to see that as soon as a change in the slope of the bed is present, the two models **(HK)** and **(SH)** may display a somewhat surprising behavior. In particular, in the case of the former model, unphysical oscillations may arise in the short term and then disappear for large times, see Section 3.3.1. In the case of the latter model, when the slope changes sign, the sliding matter may accumulate creating unexpected peaks, see Section 3.3.2. This somewhat unphysical behaviour has to be expected, for the **(SH)** system is suited to bed whose slope has small variations.

The present model (3.1.1) and **(HK)** may differ also in the asymptotic behaviour, as shown in Section 3.3.3. There, the final profiles given by (3.1.1) and **(HK)** left from the fall of some granular material over a flat bed have in fact different concavities.

The paper is organized as follows. First, in Section 3.2, we consider the main analytical properties of (3.1.1). Secondly, in Section 3.3, several numerical integrations show the main differences between the three models. The technical details are collected in Section 3.5.

3.2 Analytical Preliminaries

This section is devoted to the analytical properties of the models. First, it is immediate to note that all systems are invariant with respect to the symmetry $x \to -x$, $t \to t$, $h \to h$, $v \to -v$ and $u \to u$, as it is physically necessary.

In the case of (3.1.3), we have the following energy dissipation property.

Proposition 3.2.1. *Let $H = 0$ and $V = 0$. Choose smooth (h_o, v_o, u_o) such that (3.1.3) with initial datum (h_o, v_o, u_o) admits a smooth solution with compact support up to time $T > 0$. Consider the energy (3.1.5). If (3.1.2) holds, then $E\big((h, v, u)(t)\big) \leq E(h_o, v_o, u_o)$ for all $t \in [0, T[$, more precisely,*

$$\begin{aligned} \frac{d}{dt}E &= \int_{\mathbb{R}^2} hv \cdot \nu(v,p)\,\mathrm{d}x - \int_{\mathbb{R}^2} h\|v\|^2 \left(\big|1 - \|\nabla u\|\big| - \frac{1}{2}\left(1 - \|\nabla u\|\right)\right) \mathrm{d}x \\ &\leq 0\,. \end{aligned}$$

The proof is immediate and, hence, omitted. Note that the energy decay has to terms: the former one is due to friction and the latter one to erosion–deposition.

Above, by *smooth* solution we mean that $(h, v, u) \in \mathbf{C^1}(I \times \mathbb{R}^2; \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R})$. However, as is well known, the smoothness of solutions does not persist, for singularities may arise. Therefore, we define a measurable map $(h, v, u) \colon \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}$ to be a *distributional* solution to (3.1.3) if (h, v, u) satisfies (3.1.3) in the sense of distributions.

We now pass to the 1D case, so that (3.1.3) simplifies to

$$\begin{cases} \partial_t h + \partial_x (hv) = -\big(1 - |\partial_x u|\big)\, h + H \\ \partial_t v + \partial_x (\frac{1}{2}v^2 + h) = -\partial_x u + \nu(v, \partial_x u) - [\![1 - |\partial_x u|]\!]_{-}\, v + V \\ \partial_t u = \big(1 - |\partial_x u|\big)\, h \end{cases} \tag{3.2.1}$$

and (3.1.4) to

$$\begin{cases} \partial_t h - \partial_x (h\, \partial_x u) = -\big(1 - |\partial_x u|\big)\, h + H \\ \partial_t u = \big(1 - |\partial_x u|\big)\, h\,. \end{cases} \tag{3.2.2}$$

To study (3.2.1) and (3.2.2) as 1D systems of balance laws, it is useful to introduce the variable $p = \partial_x u$, obtaining

$$\begin{cases} \partial_t h + \partial_x (hv) = -\big(1 - |p|\big)\, h + H \\ \partial_t v + \partial_x \left(\frac{1}{2} v^2 + h\right) = -p + \nu(v,p) - [\![\, 1 - |p|\,]\!]_{-} v + V \\ \partial_t p - \partial_x \Big(\big(1 - |p|\big)\, h\Big) = 0 \end{cases} \tag{3.2.3}$$

and, in the case of the Hadeler-Kuttler model,

$$\begin{cases} \partial_t h + \partial_x (h\, p) = -\big(1 - |p|\big)\, h + H \\ \partial_t p - \partial_x \big((1 - |p|)h\big) = 0 \end{cases} \tag{3.2.4}$$

Both these systems fall within the class of 3×3 systems of balance laws, see [27, Chapter VII] as a general reference on this subject.

In the case of distributional solutions, the equivalence between the two systems (3.2.1) and (3.2.3) is proved by the following lemma.

Lemma 3.2.2. *Let $I = [0,T]$ for a $T > 0$. If (h,v,u) is a distributional solution to (3.2.1) satisfying*

$$\begin{gathered} (h,v,u) \in \mathbf{L}^\infty(I \times \mathbb{R}; \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}), \\ \textit{with } h(t) \textit{ and } \partial_x u(t) \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}) \quad \textit{for a.e. } t \in I \\ \textit{and } \partial_x u \in \mathbf{L}^\infty(I \times \mathbb{R}; \mathbb{R}) \end{gathered}$$

then, setting $p = \partial_x u$,

$$\begin{gathered} (h,v,p) \in \mathbf{L}^\infty(I \times \mathbb{R}; \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}), \\ \textit{with } h(t) \textit{ and } p(t) \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};\mathbb{R}) \quad \textit{for a.e. } t \in I \end{gathered}$$

is a distributional solution to (3.2.3). And viceversa.

The proof is deferred to Section 3.5. The equivalence of (3.1.4) and (3.2.4) is stated and proved similarly.

The first step in the analytical study of (3.2.3) is the computation of eigenvalues and eigenvectors of the Jacobian of the flow, which is the content of the next lemma.

Lemma 3.2.3. *The Jacobian of the flow of system (3.2.3) is the matrix*

$$\begin{bmatrix} v & h & 0 \\ 1 & v & 0 \\ |p| - 1 & 0 & h\, \mathrm{sgn}\, p \end{bmatrix}$$

its eigenvalues and eigenvectors are

$$\lambda_1 = v - \sqrt{h} \qquad \lambda_2 = v + \sqrt{h} \qquad \lambda_3 = h \,\mathrm{sgn}\, p$$

$$r_1 = \begin{bmatrix} -1 \\ +\frac{1}{\sqrt{h}} \\ \frac{1-|p|}{v - h\,\mathrm{sgn}\,p - \sqrt{h}} \end{bmatrix} \quad r_2 = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{h}} \\ \frac{|p|-1}{v - h\,\mathrm{sgn}\,p + \sqrt{h}} \end{bmatrix} \quad r_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If $p \neq 0$, $h > 0$ and $v \neq h \pm \sqrt{h}$ then system (3.2.3) *is strictly hyperbolic. Moreover,*

$$\nabla\lambda_1 \cdot r_1 = \frac{3}{2\sqrt{h}}, \quad \nabla\lambda_2 \cdot r_2 = \frac{3}{2\sqrt{h}}, \quad \nabla\lambda_3 \cdot r_3 = 0\,.$$

so that the first two fields are genuinely nonlinear while the third one is linearly degenerate. For $p > 0$, the ordering of the eigenvalues is as follows:

$$\begin{array}{lcl} \lambda_3 < \lambda_1 < \lambda_2 & \Longleftrightarrow & v > h + \sqrt{h} \\ \lambda_1 < \lambda_3 < \lambda_2 & \Longleftrightarrow & h - \sqrt{h} < v < h + \sqrt{h} \\ \lambda_1 < \lambda_2 < \lambda_3 & \Longleftrightarrow & v < h - \sqrt{h} \end{array}$$

and symmetric relations hold for $p < 0$.

The proof is straightforward and hence omitted. Remark that when $v = h + \sqrt{h}$ then, not only $\lambda_1 = \lambda_3$, but also the corresponding eigenspaces coincide, therefore hyperbolicity is lost. The same happens when $v = h - \sqrt{h}$. It is remarkable that, due to the loss of hyperbolicity at $h = 0$ and to the form of the source term in (3.2.3), the well posedness of this system does not follow from the standard results on systems of balance laws. Indeed, fix any state (h_o, v_o, u_o) where (3.2.3) is strictly hyperbolic. Then, $(h_o, v_o, u_o) + \mathbf{L}^1(\mathbb{R};\mathbb{R})$ is not invariant with respect to the ordinary differential equation defined by the right hand side in (3.2.3). Nevertheless, given a positive L, there exists a $T > 0$ such that the construction in [22] can be localized to any trapezoid of the type $\left\{(t,x) \in [0,T] \times \mathbb{R} \colon |x| < L + \hat\lambda t\right\}$. This procedure ensures the local well posedness of (3.2.3).

For analytical results about (3.2.4) we refer to [1, 45]. Recall that it is a 2×2 system of balance laws with a Lipschitz flow, hyperbolic for $p \neq 0$. In 1D, the Savage–Hutter model has the simpler form, see [42, formulæ (2.25)–(2.26)]:

$$\begin{cases} \partial_t h + \partial_x (hv) = 0 \\ \partial_t v + \partial_x \left(\frac{1}{2} v^2 + \delta \cos\zeta \; h\right) = \sin\zeta - \delta\cos\zeta\;\partial_x b + \nu \;\mathrm{sgn}\, v \;\cos\zeta \end{cases} \tag{3.2.5}$$

where ζ is a constant slope angle, $\nu\,\mathrm{sgn}(v)\cos\zeta$ describes the friction of the sliding material with respect to the bed and b describes the deviation of the bed from the constant angle ζ, see Figure 3.2. In other words, the relation between the slope u in (3.2.1) and the functions b and ζ in (3.2.5) is

$$u(x)\cos\zeta + x\sin\zeta = b\left(x\cos\zeta - u(x)\sin\zeta\right)\,.$$

But, as already remarked, due to the absence of the erosion–deposition phenomena, in (3.2.5) b and ζ are time *independent*, whereas u is time *dependent* in (3.2.1).

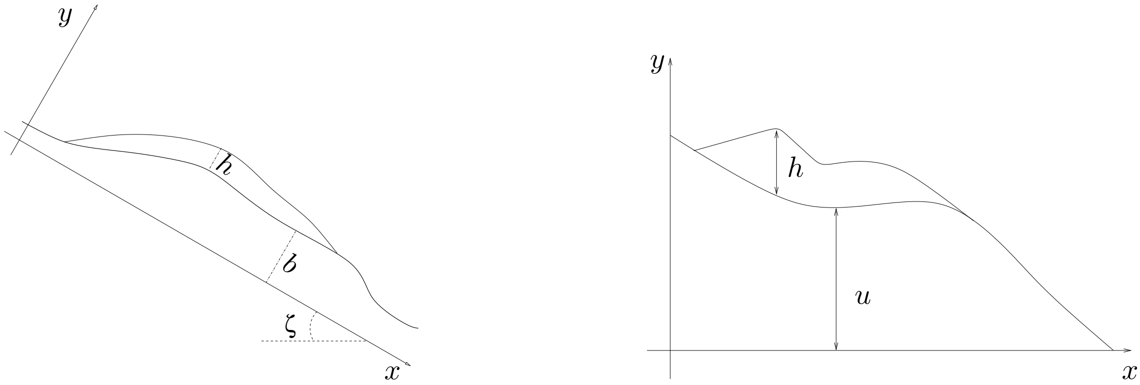


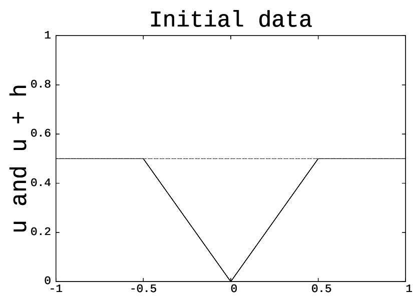
Figure 3.1: Left: notation for the 1D Savage–Hutter model (3.2.5). Right, notation for the 1D model (3.2.1).

3.3 Qualitative Behavior of the Solutions

This section is devoted to various comparisons among the solutions to the systems (3.2.3), (3.2.4) and (3.2.5), setting $\nu(v, \partial_x u) = -\nu v$. In all the numerical integrations, we use the standard Lax–Friedrichs method, see [40, § 12.5] coupled with Euler polygonals to deal with the source term, through the operator splitting method, see [40, § 17.1] or [22].

3.3.1 Evolution of a Horizontal Profile

As a first example, we consider the initial datum



$$\begin{aligned} h_o(x) &= \begin{cases} 0 & x > 1/2 \\ \frac{1}{2} - |x| & x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & x < -1/2 \end{cases} \\ v_o(x) &= 0 \\ p_o(x) &= \begin{cases} 0 & x < -1/2 \\ -1 & x \in [-1/2, 0] \\ 1 & x \in \left]0, 1/2\right] \\ 0 & x > 1/2 \end{cases} \end{aligned} \tag{3.3.1}$$

which represents a hole filled with snow at rest. Choose $H = 0$ and $V = 0$.

Independently from the choice of ν, the solution to (3.2.3)–(3.3.1) is stationary, which is physically reasonable. Indeed, where $h = 0$ nothing may move. Where $h \neq 0$, the initial slope satisfies $|p| = 1$ so that neither erosion nor deposition may take place. Besides, the effects of gravity disappear due to the fact that the profile is horizontal.

More formally, we prove the following lemma.

Lemma 3.3.1. *Let $H = 0$, $V = 0$ and choose ν satisfying (3.1.2). Then, $(h, v, p) = (h_o, v_o, p_o)$ is a distributional stationary solution to (3.2.3)–(3.3.1).*

The proof is in Section 3.5.

On the contrary, the solution to (3.2.4)–(3.3.1) is not stationary and displays a somewhat unexpected behavior, depicted in Figure 3.2. Indeed, the change in the slope at $x = 0$

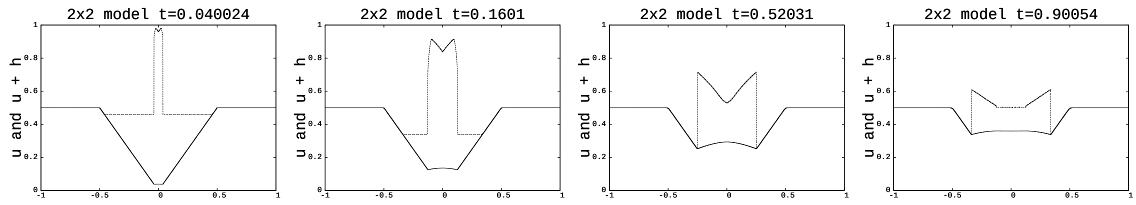


Figure 3.2: Integration of (3.2.4)–(3.3.1). The change in the slope at $x = 0$ leads to the immediate formation of two large shocks which are eventually smeared out by the right hand side in (3.2.4).

leads to the creation of two large shocks. These discontinuities are due to the convective part of (3.2.4), which dominates the source term at the small time scale. Consider the

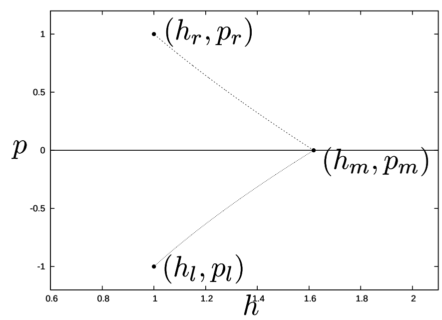


Figure 3.3: Shock curves, or Hugoniot loci, for to the solution of the Riemann problem (3.3.2), corresponding to the homogeneous part of (3.2.4) with the initial data (3.3.1). The solution to (3.3.2) governs that of (3.2.4)– (3.3.1) over the short time scale.

following Riemann problem for the convective part of (3.2.4):

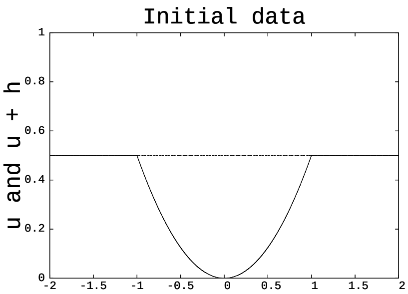
$$\left\{ \begin{array}{l} \partial_t h + \partial_x (h\, p) = 0 \\ \partial_t p - \partial_x \big((1 - |p|) h \big) = 0 \\ (h, p)(0, x) = \left\{ \begin{array}{lcl} (1, -1) & x < & 0\,, \\ (1, 1) & x > & 0\,. \end{array} \right. \end{array} \right. \tag{3.3.2}$$

Its solution consists of two (relatively) large shocks: see Figure 3.3 for the location of the Hugoniot loci displaying the solution to (3.3.2) and Figure 3.2, left, for the corresponding oscillations in the solution to (3.2.4)–(3.3.1). These shocks are eventually smeared out by the source terms and the solution to (3.2.4)–(3.3.1) approaches asymptotically the constant solution $h = 0$, $p = 0$.

We remark that the unphysical oscillations displayed in Figure 3.2 by the solutions to (3.2.4) are thus analytically justified consequences of the equations and are not due to numerical problems.

In this example, the asymptotic state reached by the solution to (3.2.3) differs from that of (3.2.4). However, in the case of the initial datum (3.3.1), this appears to be a *non generic* situation. Indeed, generically, small perturbations of the initial datum (3.3.1) lead to solutions of (3.2.3) that eventually tend to the asymptotic solution $h(x) = 0$, $v = 0$, $p = 0$.

The rise of large shocks due to changes in the slope of the bed does not depend on the smoothness of this change. Indeed, consider the initial datum



$$\begin{aligned} h_o(x) &= \begin{cases} 0 & x > 1 \\ \frac{1}{2}(1 - x^2) & x \in [-1, 1] \\ 0 & x < -1 \end{cases} \\ v_o(x) &= 0 \\ p_o(x) &= \begin{cases} 0 & x > 1 \\ x & x \in [-1, 1] \\ 0 & x < -1 \end{cases} \end{aligned} \tag{3.3.3}$$

representing a (smooth) hole filled with snow. We set $\nu = 1$, $H = 0$ and $V = 0$. Then, the uppermost profile $u + h$ in the solution to (3.2.3)–(3.3.3) is again stationary, although deposition now takes place since p attains values in $]-1, 1[$: h diminishes to 0 and the sliding matter becomes part of the bed, see Figure 3.4.

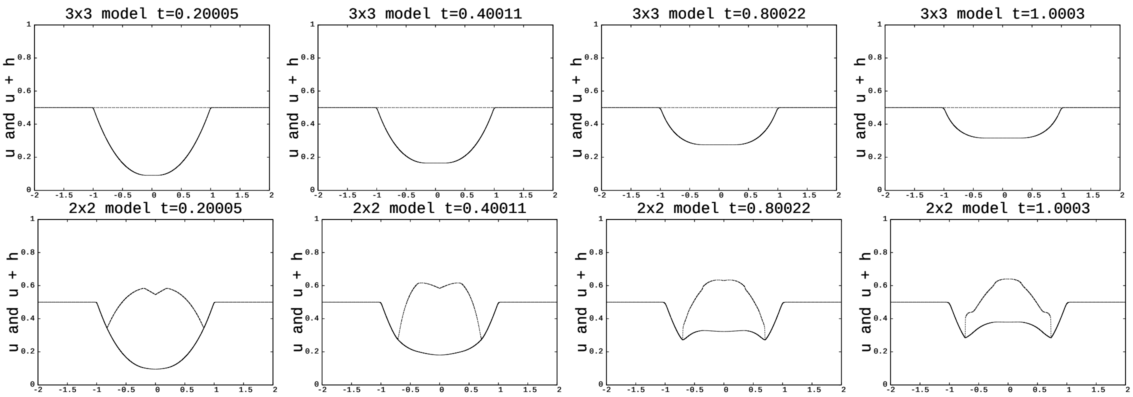


Figure 3.4: Above, the solution to (3.2.3)–(3.3.3): note that deposit takes place faster where the slope of the bed is lower. Below, the solution to (3.2.4)–(3.3.3): note the formation of unexpected peaks near to $x = 0$ where the slope smoothly changes sign.

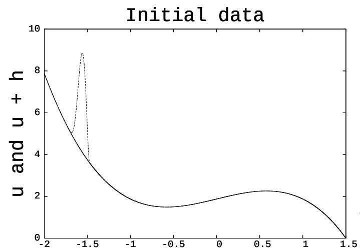
On the other hand, in the case of (3.2.4), once more we have the shocks due to the convective part are present and lead to the formation of a sort of hill. The sliding matter accumulates at the center of the hole and its level gets higher than the initial one, see Figure 3.4.

Remark that asymptotically, the solutions to both models tend to $h = 0$, $p = 0$.

The solution to (3.2.5)–(3.3.3) is stationary and hence it is not displayed in Figure 3.4. Note that this behavior is physically acceptable, in spite of the fact that the Savage–Hutter model is adapted to describe only *small* variations in the average slope of the bed.

3.3.2 Falling Matter

Consider now an avalanche or a landslide falling along a bed with varying slope. In other words we consider the initial datum:



$$
\begin{aligned}
h_o(x) &= (x-\alpha)^4(x-\beta)^2\,\chi_{[\alpha,\beta]}(x)\cdot 10^6 \\
& \qquad \alpha = -1.720 \text{ and } \beta = -1.475 \\
v_o(x) &= 0 \\
p_o(x) &= 1-3x^2
\end{aligned}
\tag{3.3.4}
$$

with $H = 0$, $V = 0$ and $\nu = 0.1$. In (3.2.5) we also set $\delta = 0.1$, $\zeta = 0$ and $\partial_x b = p_o$.

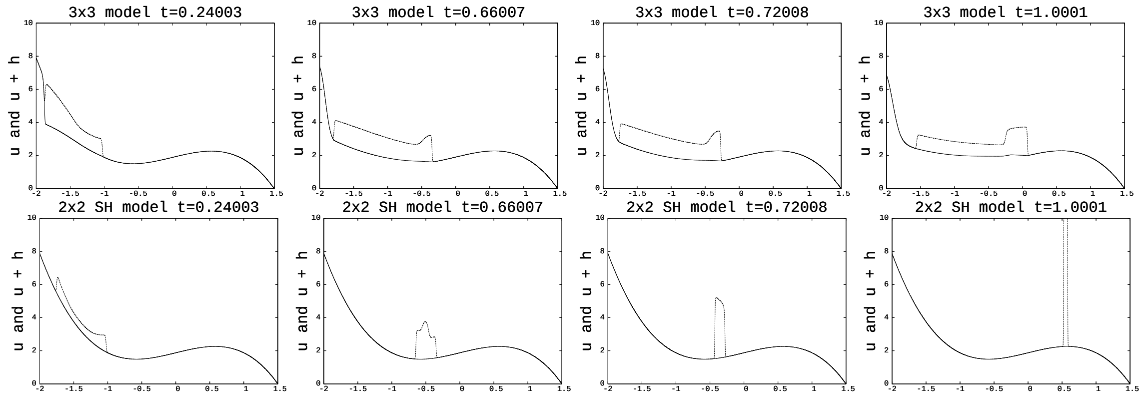


Figure 3.5: Above, integration of (3.2.3)–(3.3.4): first, the sliding matter erodes the bed while falling, then it deposits while slowing down. Below, integration of (3.2.5)–(3.3.4): neither erosion nor deposit may take place. Besides, the slower deceleration in the middle part causes the formation of a peak.

As long as the avalanche does not reach the change in the slope, the solution to (3.2.3) displays a bunch of matter moving downwards along the slope and, at the same time, eroding the steepest part of the bed. In the case of the Savage–Hutter model, the profile of the bed does not change, while that of the solution is quickly deeply modified, see Figure 3.5.

Where the bed's slope is small, the sliding matter in the solution to (3.2.3) slows down and starts depositing. In the solution to (3.2.5), by the absence of erosion–deposition term the sliding matter goes down faster than in the previous case, hence it concentrates and creates a peak, see Figure 3.5.

3.3.3 On the Role of H

We now compare the three models (3.2.3), (3.2.4) and (3.2.5) in the case of a flat horizontal bed on which a material is being poured. Thus, assume that

$$
\begin{aligned}
h_o(x) &= 0 & & & H(t,x) &= 1.5\,\chi_{[-0.1,0.1]}(x)\cdot\chi_{[0,0.5]}(t) \\
v_o(x) &= 0 & &\text{with} & V(t,x) &= 0 \\
p_o(x) &= 0 & & & \nu &= 0.1\,.
\end{aligned}
\tag{3.3.5}
$$

The results of the corresponding numerical integrations are collected in Figure 3.6. We remark that, initially, the uppermost profiles $y = h(x) + u(x)$, in the case of (3.2.3) and (3.2.4), $y = h(x)$, in the case of (3.2.5), are similar, see the first column in Figure 3.6.

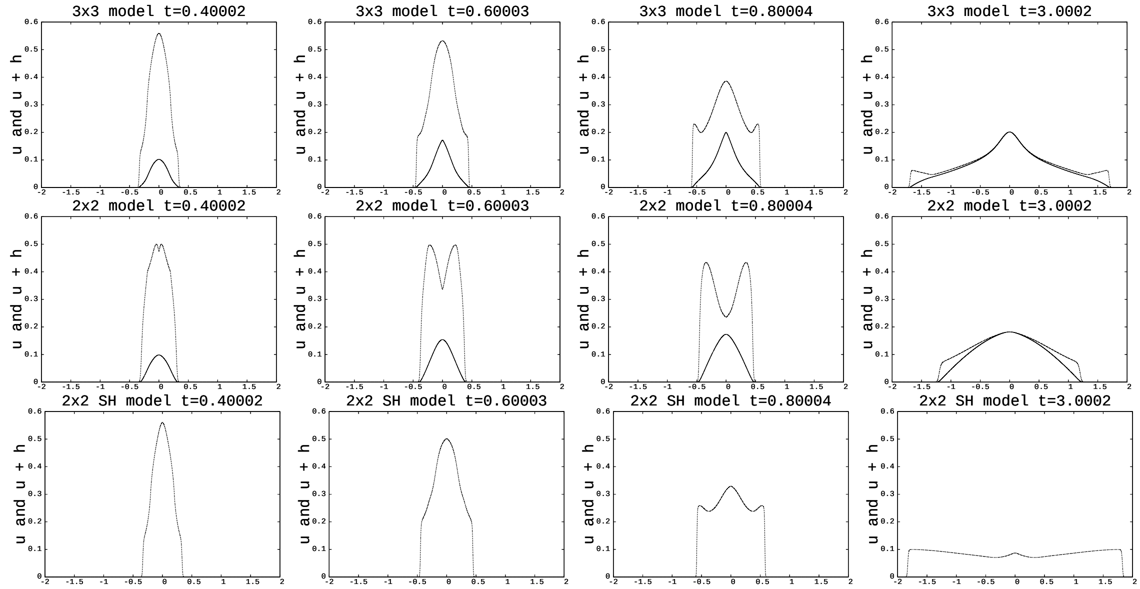


Figure 3.6: First row, solution to (3.2.3)–(3.3.5); second row, solution to (3.2.4)–(3.3.5); third row, solution to (3.2.5)–(3.3.5). Note

As soon as the lower deposited part takes a shape with a significant change in its slope, the solution to (3.2.4) displays the behaviour already noted above. Two symmetric shocks start moving off from the vertex of the deposited part, see the second and third columns in Figure 3.6.

Eventually, the solution to (3.2.5) spread all over the real line, for the **(SH)** model does not account for any deposit. However, as long as h in (3.2.3) is positive, the qualitative aspects of the uppermost profiles in (3.2.3) and in (3.2.5) are analogous. Note that the asymptotic shape of the profile in the solutions to (3.2.3) and (3.2.4), rightmost column in Figure 3.6, are rather different.

3.4 Conclusions

We presented a new model for the movement of granular matter. It is a synthesis of the Hadeler–Kuttler and of the Savage–Hutter models. The result is the 3×3 system (3.1.1), which we proved to be compliant with energy dissipation, similarly to the Savage-Hutter model, but able to describe the erosion–deposition dynamics, which is not considered in the **(SH)** model.

Moreover, (3.1.1) seems to describe better than the Hadeler-Kuttler model the evolution of the falling matter, in particular in case of changes in the bed slope. Indeed, the solutions to (3.1.1) do not display the sudden oscillations due to the convective part of **(HK)**. Furthermore, we believe it is relevant that an initial horizontal profile evolves remaining horizontal, as in the cases examined in Paragraph 3.3.1.

3.5 Technical Details

We omit the proofs of the next two lemmas, since they are straightforward.

Lemma 3.5.1. *[33, Appendix A] With the rescaling $x \to \frac{\beta}{\gamma}x$, $t \to \frac{1}{\alpha\gamma}t$, $u \to \frac{\alpha\beta}{\gamma}u$, $h \to \frac{\alpha\beta}{\gamma}h$ and $f \to \frac{1}{\alpha^2\beta}f$, system **(HK)** reduces to (3.1.4).*

Similarly, in the case of (3.1.1), we have the following lemma.

Lemma 3.5.2. *With the rescaling $x \to \frac{g}{\alpha\gamma^2}x$, $t \to \frac{1}{\alpha\gamma}t$, $u \to \frac{g}{\gamma^2}u$, $h \to \frac{g}{\gamma^2}h$, $v \to \frac{g}{\gamma}v$, $H \to \frac{\gamma}{\alpha g}H$, $V \to \frac{\gamma^2}{\alpha g}V$ and $\nu \to \frac{\gamma^2}{\alpha g}\nu$ system (3.1.1) reduces to (3.1.3).*

Proof of Lemma 3.2.2. Let (h, v, u) be a distributional solution to (3.2.1) and set $p := \partial_x u$. It is straightforward to show that the first two equations of (3.2.3) are satisfied in the sense of distributions. Moreover, for every test function $\varphi \in \mathbf{C_c^\infty}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$,

$$\begin{aligned}
& \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\partial_t\varphi\, p + \partial_x\varphi\, \big(1 - |p|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t \\
= & \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\partial_t\varphi\, \partial_x u + \partial_x\varphi\, \big(1 - |\partial_x u|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t \\
= & \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[-\partial_t\partial_x\varphi\, u + \partial_x\varphi\, \big(1 - |\partial_x u|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t\ .
\end{aligned}$$

Now, using the fact that $\partial_x\varphi$ belongs to $\mathbf{C_c^\infty}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ and the hypothesis that (h, v, u) satisfies the latter equation of (3.2.1), we get

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\partial_t\varphi\, p + \partial_x\varphi\, \big(1 - |p|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t = 0\,,$$

i.e. also the third equation of (3.2.3) holds in the distributional sense.

Let (h, v, p) be a distributional solution to (3.2.3) and define the function: $u(t, x) := \int_{-\infty}^{x} p(t, \xi)\, \mathrm{d}\xi$. As above, the proof for the first two equations is trivial. Moreover, using (3.2.3) and [27, Theorem 4.3.1]:

$$\begin{aligned}
& \int_a^x p(t, \xi)\, \mathrm{d}\xi - \int_a^x p(0, \xi)\, \mathrm{d}\xi \\
= & \int_0^t \Big(1 - \big|p(\vartheta, x-)\big|\Big)\, h(\vartheta, x-)\, \mathrm{d}\vartheta - \int_0^t \Big(1 - \big|p(\vartheta, a+)\big|\Big)\, h(\vartheta, a+)\, \mathrm{d}\vartheta\ .
\end{aligned}$$

Since $h(t)$ and $p(t)$ belong to $(\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; \mathbb{R})$ for a.e. $t \in I$, when $a \to -\infty$ we get:

$$u(t, x) = \int_{-\infty}^{x} p(t, \xi)\, \mathrm{d}\xi = \int_{-\infty}^{x} p(0, \xi)\, \mathrm{d}\xi + \int_0^t \Big(1 - \big|p(\vartheta, x)\big|\Big)\, h(\vartheta, x)\, \mathrm{d}\vartheta\ .$$

Introduce, for simplicity, the quantities $g(x) := \int_{-\infty}^{x} p(0, \xi)\, \mathrm{d}\xi$ and $l(t, x) := \int_0^t \Big(1 - \big|p(\vartheta, x)\big|\Big)\, h(\vartheta, x)\, \mathrm{d}\vartheta$. Then, for every test function $\varphi \in \mathbf{C_c^\infty}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$,

$$\begin{aligned}
& \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\partial_t\varphi\, u + \varphi\, \big(1 - |\partial_x u|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t \\
= & \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\partial_t\varphi\, g + \partial_t\varphi\, l + \varphi\, \big(1 - |p|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t\ .
\end{aligned}$$

Now, integrating by parts and using the fact that $\partial_t g = 0$ and $\partial_t l = \big(1 - |p|\big)\, h$, we obtain:

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big[\partial_t \varphi\, u + \varphi\, \big(1 - |\partial_x u|\big)\, h\Big]\, \mathrm{d}x\ \mathrm{d}t = 0\,.$$

Hence, also the third equation in (3.2.1) is satisfied in distributional sense and the proof is completed. □

Proof of Lemma 3.3.1. Let h, v, p be the stationary functions $h(t,x) = h_o(x)$, $v(t,x) = 0$, $p(t,x) = p_o(x)$. Separately, in each of the regions $\mathbb{R}^+ \times \big]-\infty, -1/2\big[$, $\mathbb{R}^+ \times \big]-1/2, 0\big[$, $\mathbb{R}^+ \times \big]0, 1/2\big[$ and $\mathbb{R}^+ \times \big]1/2, +\infty\big[$, (h, v, p) is a smooth solution to (3.2.3). On the other hand, the traces of (h, v, p) along the three boundaries $\mathbb{R}^+ \times \{-1/2\}$, $\mathbb{R}^+ \times \{0\}$ and $\mathbb{R}^+ \times \{1/2\}$, satisfy the Rankine–Hugoniot conditions with zero speed. Hence (h, v, p) is a stationary solution. □

Bibliography

[1] D. Amadori and W. Shen. Global existence of large BV solutions in a model of granular flow. *Preprint 2008*, 63(1):259–278 (electronic), 2002.

[2] F. Ancona and A. Marson. Existence theory by front tracking for general nonlinear hyperbolic systems. *Arch. Ration. Mech. Anal.*, 185(2):287–340, 2007.

[3] J.-P. Aubin and A. Cellina. *Differential inclusions.* Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[4] P. Baiti and A. Bressan. The semigroup generated by a Temple class system with large data. *Differ. Integral Equ.*, 10(3):401–418, 1997.

[5] P. Baiti and H. K. Jenssen. On the front-tracking algorithm. *J. Math. Anal. Appl.*, 217(2):395–404, 1998.

[6] S. Bianchini. The semigroup generated by a Temple class system with non-convex flux function. *Differential Integral Equations*, 13(10-12):1529–1550, 2000.

[7] S. Bianchini. Stability of L^∞ solutions for hyperbolic systems with coinciding shocks and rarefactions. *SIAM J. Math. Anal.*, 33(4):959–981 (electronic), 2001.

[8] S. Bianchini. On the Riemann problem for non-conservative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 166(1):1–26, 2003.

[9] S. Bianchini and A. Bressan. BV solutions for a class of viscous hyperbolic systems. *Indiana Univ. Math. J.*, 49(4):1673–1713, 2000.

[10] S. Bianchini and A. Bressan. Viscosity solutions for hyperbolic systems where shock curves are straight lines. In *Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000)*, volume 141 of *Internat. Ser. Numer. Math., 140*, pages 159–167. Birkhäuser, Basel, 2001.

[11] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.

[12] S. Bianchini and R. M. Colombo. On the stability of the standard Riemann semigroup. *Proc. Amer. Math. Soc.*, 130(7):1961–1973 (electronic), 2002.

[13] S. Bianchini, R. M. Colombo, and F. Monti. 2×2 Systems of Conservation Laws with L^∞ Data. *Submitted, preprint*, 2009.

[14] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[15] A. Bressan and R. M. Colombo. The semigroup generated by 2×2 conservation laws. *Arch. Rational Mech. Anal.*, 133(1):1–75, 1995.

[16] A. Bressan and R. M. Colombo. Decay of positive waves in nonlinear systems of conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):133–160, 1998.

[17] A. Bressan and P. Goatin. Stability of L^∞ solutions of Temple class systems. *Differential Integral Equations*, 13(10-12):1503–1528, 2000.

[18] P. Cannarsa and P. Cardaliaguet. Representation of equilibrium solutions to the table problem for growing sandpiles. *J. Eur. Math. Soc. (JEMS)*, 6(4):435–464, 2004.

[19] P. Cannarsa, P. Cardaliaguet, G. Crasta, and E. Giorgieri. A boundary value problem for a PDE model in mass transfer theory: representation of solutions and applications. *Calc. Var. Partial Differential Equations*, 24(4):431–457, 2005.

[20] P. Cannarsa, P. Cardaliaguet, and E. Giorgieri. Hölder regularity of the normal distance with an application to a PDE model for growing sandpiles. *Trans. Amer. Math. Soc.*, 359(6):2741–2775 (electronic), 2007.

[21] C. Cheverry. Systèmes de lois de conservation et stabilité BV. *Mém. Soc. Math. Fr. (N.S.)*, (75):vi+103, 1998.

[22] R. M. Colombo and G. Guerra. Hyperbolic balance laws with a non local source. *Comm. Partial Differential Equations*, 32(10-12):1917–1939, 2007.

[23] R. M. Colombo and G. Guerra. On the stability functional for conservation laws. *Nonlinear Anal.*, 69(5-6):1581–1598, 2008.

[24] R. M. Colombo, G. Guerra, and F. Monti. Modeling the Dynamics of Granular Matter. *Submitted, preprint*, 2009.

[25] R. M. Colombo and F. Monti. Solutions with Large Total Variation to Nonconservative Hyperbolic Systems. *Communications in Pure and Applied Analysis, to appear.*, 2009.

[26] G. Crasta and S. Finzi Vita. An existence result for the sandpile problem on flat tables with walls. *Netw. Heterog. Media*, 3(4):815–830, 2008.

[27] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2005.

[28] A. de Saint-Venant. *Thorie du mouvement non-permanent des eaux, avec application aux crues des rivires et l'introduction des mares dans leur lit.* C.R.Acad.Sci.Paris 73, 1871.

[29] M. Falcone and S. Finzi Vita. A finite-difference approximation of a two-layer system for growing sandpiles. *SIAM J. Sci. Comput.*, 28(3):1120–1132 (electronic), 2006.

[30] A. Filippov. *Differential Equations with Discontinuous Righthand Sides.* Kluwer Academic Publisher, Dordrecht, The Netherlands, first edition, 1988.

[31] J. Glimm and P. D. Lax. *Decay of solutions of systems of nonlinear hyperbolic conservation laws.* Memoirs of the American Mathematical Society, No. 101. American Mathematical Society, Providence, R.I., 1970.

[32] P. Gwiazda and A. Świerczewska. Multivalued equations for granular avalanches. *Nonlinear Anal.*, 62(5):895–912, 2005.

[33] K. Hadeler and C. Kuttler. Dynamical models for granular matter. *Granular Matter*, 2(9–18), 1999.

[34] K. Hadeler and C. Kuttler. Granular matter in a silo. *Granular Matter*, 3(193–197), 2001.

[35] H. K. Jenssen. Blowup for systems of conservation laws. *SIAM J. Math. Anal.*, 31(4):894–908 (electronic), 2000.

[36] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[37] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[38] P. G. LeFloch and A. E. Tzavaras. Existence theory for the Riemann problem for non-conservative hyperbolic systems. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(4):347–352, 1996.

[39] P. G. Lefloch and A. E. Tzavaras. Representation of weak limits and definition of nonconservative products. *SIAM J. Math. Anal.*, 30(6):1309–1342 (electronic), 1999.

[40] R. J. LeVeque. *Finite volume methods for hyperbolic problems.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[41] M. Pelanti, F. Bouchut, and A. Mangeney. A Roe-type scheme for two-phase shallow granular flows over variable topography. *M2AN Math. Model. Numer. Anal.*, 42(5):851–885, 2008.

[42] S. B. Savage and K. Hutter. The motion of a finite mass of granular material down a rough incline. *J. Fluid Mech.*, 199:177–215, 1989.

[43] S. B. Savage and K. Hutter. The dynamics of avalanches of granular materials from initiation to runout. I. Analysis. *Acta Mech.*, 86(1-4):201–223, 1991.

[44] M. Shearer, J. M. N. T. Gray, and A. R. Thornton. Stable solutions of a scalar conservation law for particle-size segregation in dense granular avalanches. *European J. Appl. Math.*, 19(1):61–86, 2008.

[45] W. Shen. On the shape of avalanches. *J. Math. Anal. Appl.*, 339(2):828–838, 2008.

[46] S. M. Srivastava. *A course on Borel sets*, volume 180 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[47] Y. Wang, K. Hutter, and S. P. Pudasaini. The Savage-Hutter theory: a system of partial differential equations for avalanche flows of snow, debris, and mud. *ZAMM Z. Angew. Math. Mech.*, 84(8):507–527, 2004.