

# ON SCHRÖDINGER OPERATORS WITH MULTIPOLAR INVERSE-SQUARE POTENTIALS

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ABSTRACT. Positivity, essential self-adjointness, and spectral properties of a class of Schrödinger operators with multipolar inverse-square potentials are discussed. In particular a necessary and sufficient condition on the masses of singularities for the existence of at least a configuration of poles ensuring the positivity of the associated quadratic form is established.

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

This paper deals with a class of Schrödinger operators associated with potentials possessing multiple inverse square singularities:

$$(1) \quad L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k} := -\Delta - \sum_{i=1}^k \frac{\lambda_i}{|x - a_i|^2}$$

where  $N \geq 3$ ,  $k \in \mathbb{N}$ ,  $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k$ ,  $(a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}$ ,  $a_i \neq a_j$  for  $i \neq j$ .

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From the mathematical point of view, the main reason of interest in inverse square potentials of type  $V(x) \sim \lambda|x|^{-2}$  relies in their criticality: indeed they have the same homogeneity as the laplacian and do not belong to the Kato's class, hence they cannot be regarded as a lower order perturbation term. Besides, potentials with this rate of decay are critical also in nonrelativistic quantum mechanics, as they represent an intermediate threshold between regular potentials (for which there are ordinary stationary states) and singular potentials (for which the energy is not lower-bounded and the particle falls to the center), for more details see [21, 29]. We also mention that inverse square singular potentials arise in many other physical contexts: molecular physics, see e.g. [30], quantum cosmology [4], linearization of combustion models [3, 22, 48]. Moreover we emphasize the correspondence between nonrelativistic Schrödinger operators with inverse square potentials and relativistic Schrödinger operators with Coulomb potentials, see [11].

In recent literature, several papers are concerned with Schrödinger equations with Hardy-type singular potentials, see e.g. [1, 7, 8, 16, 17, 22, 26, 41, 44, 46].

The case of multi-polar Hardy-type potentials was considered in [15, 19, 20]. More precisely, in [15] estimates on resolvent truncated at high frequencies are proved for Schrödinger operators with multiple inverse square singular potentials. In [19, 20] the existence of ground states for a class of multi-polar nonlinear elliptic equations with critical power-nonlinearity is investigated.

The purpose of the present paper is to analyze how the mutual interaction among the poles influences the spectral properties of the class of operators (1) and which configurations of singularities ensure positivity of the associated quadratic form

$$Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x - a_i|^2} dx.$$

As a natural setting to study the operators defined in (1) and the associated quadratic forms, we introduce the functional space  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  defined as the completion of  $C_c^\infty(\mathbb{R}^N)$  with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \right)^{1/2}.$$

We introduce now a notion of positivity which is closely related to the property of *strong subcriticality* introduced in [13], see also Remark 1.6 for a discussion about the relations between these notions in the framework of multi-polar inverse square potentials. We say that  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  is *positive semidefinite* whenever

$$Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(u) \geq 0, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

whereas it is said to be *positive definite* if there exists a positive constant  $\varepsilon = \varepsilon(\lambda_1, \dots, \lambda_k, a_1, \dots, a_k)$  such that

$$Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(u) \geq \varepsilon(\lambda_1, \dots, \lambda_k, a_1, \dots, a_k) \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

In the case of a single pole operator  $-\Delta - \frac{\lambda}{|x|^2}$ , a complete answer to the question of positivity is provided by the classical Hardy inequality (see for instance [22, 25]):

$$(2) \quad \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where the constant  $(\frac{N-2}{2})^2$  is optimal and not attained. The optimality of such a constant implies that the quadratic form  $Q_{\lambda,0}$  is positive definite in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  if and only if  $\lambda < (\frac{N-2}{2})^2$ . For a more detailed discussion about the properties of monopole singular Hardy type operators, we refer to [22, 46, 48].

As observed in [19], the positivity of  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  depends on the strength and the location of the singularities, and more precisely on the shape of the configuration of poles, due to scaling properties of the operator. In particular in [19, Proposition 1.2] it is proved that a sufficient condition for the quadratic form to be positive definite for any choice of  $a_1, a_2, \dots, a_k$  is that  $\sum_{i=1}^k \lambda_i^+ < \frac{(N-2)^2}{4}$ , where  $t^+ := \max\{t, 0\}$ . Conversely, if  $\sum_{i=1}^k \lambda_i^+ > \frac{(N-2)^2}{4}$ , then it is possible to find a configuration of poles such that the quadratic form is not positive definite. As a consequence, in the case  $k = 2$ , if

$$\lambda_i < \frac{(N-2)^2}{4}, \quad \text{for } i = 1, 2 \quad \text{and} \quad \lambda_1 + \lambda_2 < \frac{(N-2)^2}{4},$$

then for any choice of  $a_1, a_2$ , the quadratic form  $Q_{\lambda_1, \lambda_2, a_1, a_2}$  is positive definite.

The first result of the present paper relies in a necessary and sufficient condition on the masses  $\lambda_i$  to have positivity of the quadratic form  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  for at least a configuration of poles.

**Theorem 1.1.** *Let  $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ . Then*

$$(3) \quad \lambda_i < \frac{(N-2)^2}{4}, \quad \text{for every } i = 1, \dots, k, \quad \text{and} \quad \sum_{i=1}^k \lambda_i < \frac{(N-2)^2}{4},$$

*is a necessary and sufficient condition for the existence of a configuration of poles  $\{a_1, \dots, a_k\}$  such that the quadratic form  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  associated to the operator  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  is positive definite.*

The necessity of condition (3) follows quite directly from the optimality of the best constant in Hardy's inequality and proper interaction estimates (see the proof in Section 6). To prove the sufficiency we study the possibility of obtaining a coercive operator by summing up multisingular potentials which give rise to positive quadratic forms, after pushing them very far away from each other to weaken the interactions among poles. Since the potentials overlap at infinity, the singularity of the resulting potential is the sum of their masses, so that we need to require a control on it. To this aim, we consider the following class of potentials

$$\mathcal{V} := \left\{ \begin{array}{l} V(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, r_i)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R)}(x)}{|x|^2} + W(x) : k \in \mathbb{N}, r_i, R \in \mathbb{R}^+, \\ a_i \in \mathbb{R}^N, a_i \neq a_j \text{ for } i \neq j, \lambda_i, \lambda_\infty \in (-\infty, \frac{(N-2)^2}{4}), W \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \end{array} \right\}.$$

By Hardy's and Sobolev's inequalities, it follows that, for any  $V \in \mathcal{V}$ , the first eigenvalue  $\mu(V)$  of the operator  $-\Delta - V$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  is finite, namely

$$\mu(V) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > -\infty.$$

Hence, we shall frame in the class  $\mathcal{V}$  the analysis of coercivity conditions for Schrödinger operators. Let us notice that  $\mu(V)$  can be estimated from above as follows.

**Lemma 1.2.** *For any  $V(x) = \sum_{i=1}^k \lambda_i \chi_{B(a_i, r_i)}(x) |x - a_i|^{-2} + \lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x) |x|^{-2} + W(x) \in \mathcal{V}$ , there holds:*

- i) *if  $\lambda_i < 0$  for all  $i = 1, \dots, k, \infty$ , then  $\mu(V) \leq 1$ ;*
- ii) *if  $\max_{i=1, \dots, k, \infty} \lambda_i > 0$ , then  $\mu(V) \leq 1 - \frac{4}{(N-2)^2} \max_{i=1, \dots, k, \infty} \lambda_i$ .*

A first positivity result in the class  $\mathcal{V}$  relies in the following Shattering Lemma yielding positivity in the case of singularities localized strictly near the poles.

**Lemma 1.3. [Shattering of singularities]** *For any  $\{a_1, a_2, \dots, a_k\} \subset B(0, R_0) \subset \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ ,  $\{\lambda_1, \dots, \lambda_k, \lambda_\infty\} \subset (-\infty, (N-2)^2/4)$ , and  $0 < \alpha < 1 - \frac{4 \max_{i=1, \dots, k, \infty} \lambda_i}{(N-2)^2}$ , there exists  $\delta = \delta(\alpha) > 0$  such that*

$$\begin{aligned} \mu \left( \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2} \right) \\ \geq \begin{cases} 1 - \frac{4 \max_{i=1, \dots, k, \infty} \lambda_i}{(N-2)^2} - \alpha, & \text{if } \max_{i=1, \dots, k, \infty} \lambda_i > 0, \\ 1, & \text{if } \max_{i=1, \dots, k, \infty} \lambda_i \leq 0. \end{cases} \end{aligned}$$

In particular, the quadratic form associated to the operator

$$\mathcal{L}_{\lambda_1, \dots, \lambda_k, \lambda_\infty, a_1, \dots, a_k}^\delta := -\Delta - \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} - \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2}$$

is positive definite.

The above result can be extended to Schrödinger operators whose potentials have infinitely many singularities localized in sufficiently small neighborhoods of equidistant poles, see Lemma 3.5.

Lemma 1.3 implies that Schrödinger operators with potentials in  $\mathcal{V}$  are compact perturbations of positive operators, as stated in the following lemma.

**Lemma 1.4.** *For any  $V \in \mathcal{V}$ , there exist  $\tilde{V} \in \mathcal{V}$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that  $\mu(\tilde{V}) > 0$  and  $V(x) = \tilde{V}(x) + \tilde{W}(x)$ .*

The proof of Theorem 1.1 is based on the following result, which yields a powerful tool to obtain positive operators by choosing properly the configuration of poles.

**Theorem 1.5. [Separation Theorem]** *Let*

$$\begin{aligned} V_1(x) &= \sum_{i=1}^{k_1} \frac{\lambda_i^1 \chi_{B(a_i^1, r_i^1)}(x)}{|x - a_i^1|^2} + \frac{\lambda_\infty^1 \chi_{\mathbb{R}^N \setminus B(0, R_1)}(x)}{|x|^2} + W_1(x) \in \mathcal{V}, \\ V_2(x) &= \sum_{i=1}^{k_2} \frac{\lambda_i^2 \chi_{B(a_i^2, r_i^2)}(x)}{|x - a_i^2|^2} + \frac{\lambda_\infty^2 \chi_{\mathbb{R}^N \setminus B(0, R_2)}(x)}{|x|^2} + W_2(x) \in \mathcal{V}. \end{aligned}$$

*Assume that  $\mu(V_1), \mu(V_2) > 0$ , namely that the quadratic forms associated to the operators  $-\Delta - V_1, -\Delta - V_2$  are positive definite, and that  $\lambda_\infty^1 + \lambda_\infty^2 < (\frac{N-2}{2})^2$ . Then, there exists  $R > 0$  such that,*

for every  $y \in \mathbb{R}^N$  with  $|y| \geq R$ , the quadratic form associated to the operator  $-\Delta - (V_1 + V_2(\cdot - y))$  is positive definite.

**Remark 1.6.** We mention that the notion of positivity (respectively nullity) in the class  $\mathcal{V}$  is related to that of subcriticality (respectively criticality) of potentials arising in the classification given by Simon [43]. A recent breakthrough in the theory by Pinchover and Tintarev [37] relates subcriticality with the presence of a spectral gap. More precisely, the subcritical case occurs if and only if the corresponding Hardy type inequality can be improved, namely if there exists  $p \in C^0(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ ,  $p > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , such that

$$(4) \quad \nu_p(V) = \inf_{u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} V(x) u^2(x) dx}{\int_{\mathbb{R}^N} p(x) u^2(x) dx} > 0.$$

As observed in Remark 7.4, being any operator with potential in  $\mathcal{V}$  compact perturbation of a positive operator,  $\nu_p(V)$  has the same sign as  $\mu(V)$ . Therefore in the class  $\mathcal{V}$ , subcriticality is equivalent to positivity.

Concerning the Separation Theorem 1.5, we recall that the case of potentials with compact support was investigated in [43], while the case of potentials in the Kato class was studied in [36]. We notice that in both these papers the potential is a lower order perturbation of the laplacian and none of them include the case of potentials with singularities of inverse square type as Theorem 1.5 does. Such a case is not a trivial issue and requires some additional assumptions to control the singularity at infinity, as one can understand just summing up two mono-polar operators and observing that in this case the positivity of the resulting operator does not depend on the distance between poles, due to scaling properties.

Besides the sign, a natural question arising in the study of  $\mu(V)$  is its attainability. Indeed, while in the case of a single pole the best constant in the associated Hardy's inequality is not achieved, when dealing with multipolar Hardy-type potentials a balance between positive and negative interactions between the poles can lead, in some cases, to attainability of the best constant in the corresponding Hardy-type inequality, as proved in the proposition below. We mention that in the literature analogous phenomena are studied by [5, 6, 32] for generalized Hardy's inequalities in bounded and unbounded domains with a boundary, and by [47] for potentials satisfying Hardy type inequalities and perturbations of them in  $\mathbb{R}^N$ .

**Proposition 1.7.** Let  $V(x) = \sum_{i=1}^k \lambda_i \chi_{B(a_i, r_i)}(x) |x - a_i|^{-2} + \lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x) |x|^{-2} + W(x) \in \mathcal{V}$ . If

$$(5) \quad \mu(V) < 1 - \frac{4}{(N-2)^2} \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}$$

then  $\mu(V)$  is attained.

The properties of  $\mathcal{V}$  proved in Lemmas 1.3 and 1.4, make such a class of potentials to be a quite natural setting to study the spectral properties of multisingular Schrödinger operators in  $L^2(\mathbb{R}^N)$ . Indeed, as a direct consequence of Lemma 1.4, we obtain that if  $V = \tilde{V} + \tilde{W} \in \mathcal{V}$ , with  $\mu(\tilde{V}) > 0$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then

$$(6) \quad \nu_1(V) := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |u(x)|^2 dx} \geq -\|\widetilde{W}\|_{L^\infty(\mathbb{R}^N)} > -\infty,$$

i.e. Schrödinger operators with potentials in  $\mathcal{V}$  are semi-bounded.

In order to ensure *quantum completeness* of the quantum system associated to the Schrödinger operator, a further key aspect to be discussed is the *essential self-adjointness*, namely the existence of a unique self-adjoint extension. Semi-bounded Schrödinger operators are essentially self-adjoint whenever the potential is not too singular (see [39]). On the other hand, when dealing with inverse square potentials, the singularity is quite strong and essential self-adjointness is a nontrivial issue. In the case of one pole singularity, it was proved in [27] (see Theorem 8.1) that essential self-adjointness depends on the value of the mass of the singularity with respect to the threshold  $(N-2)^2/4-1$ . The following theorem extends such a result to potentials lying in the class  $\mathcal{V}$ .

**Theorem 1.8.** *Let  $V(x) = \sum_{i=1}^k \lambda_i |x - a_i|^{-2} \chi_{B(a_i, r_i)}(x) + \lambda_\infty |x|^{-2} \chi_{\mathbb{R}^N \setminus B(0, R)}(x) + W(x) \in \mathcal{V}$ . Then  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  if and only if  $\lambda_i \leq (N-2)^2/4-1$  for all  $i = 1, \dots, k$ .*

The proof of the above theorem is based on the regularity results proved in [18, 34, 35] for elliptic equations with singular weights (see also [14, 24]), which allow us to give the exact asymptotic behavior near the poles of solutions to Schrödinger equations with singular potentials. The characterization of essential self-adjointness for multi-singular Schrödinger operators given above can be extended to the case of infinitely many singularities distributed on reticular structures, see Theorem 8.4.

From Theorem 1.8, we have that if  $V \in \mathcal{V}$  with  $\lambda_i \leq (N-2)^2/4-1$  for all  $i = 1, \dots, k$ , then the associated Schrödinger operator  $-\Delta - V$  is essentially self-adjoint and, consequently, admits a unique self-adjoint extension, which is given by the *Friedrichs extension*  $(-\Delta - V)^F$ :

$$(7) \quad D((-\Delta - V)^F) = \{u \in H^1(\mathbb{R}^N) : -\Delta u - Vu \in L^2(\mathbb{R}^N)\}, \quad u \mapsto -\Delta u - Vu.$$

Otherwise, i.e. if  $\lambda_i > (N-2)^2/4-1$  for some  $i$ ,  $-\Delta - V$  is not essentially self-adjoint and admits many self-adjoint extensions, among which the Friedrichs extension is the only one whose domain is included in  $H^1(\mathbb{R}^N)$ , namely in the domain of the associated quadratic form (see also [15, Remark 2.5]). A complete description of the spectrum of the Friedrichs extension of operators with potentials in the class  $\mathcal{V}$  is given in the proposition below.

**Proposition 1.9.** *For any  $V \in \mathcal{V}$ , there holds:*

**1.9.1.** *the essential spectrum  $\sigma_{\text{ess}}((-\Delta - V)^F) = [0, +\infty)$ ;*

**1.9.2.** *if  $\nu_1(V) < 0$  then the discrete spectrum of  $(-\Delta - V)^F$  consists in a finite number of negative eigenvalues.*

The nature of the bottom of the essential spectrum of the Friedrichs extension of operators of type (1) is analyzed in the following theorem.

**Theorem 1.10.** *Let  $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$  satisfy (3). Then*

$$(8) \quad \sum_{i=1}^k \lambda_i^+ > \frac{(N-2)^2}{4} \quad \text{and} \quad \sum_{i=1}^k \lambda_i < \frac{(N-2)^2}{4} - 1,$$

*is a necessary and sufficient condition for the existence of at least a configuration of singularities  $(a_1, \dots, a_k) \in \mathbb{R}^{Nk}$ ,  $a_i \neq a_j$  for  $i \neq j$ , such that 0 is an eigenvalue of the Friedrichs extension of  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$ , namely there exists  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$  solving  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k} u = 0$ .*

Actually, it turns out that, under assumptions (3) and (8), the set of configurations  $(a_1, \dots, a_k)$  for which 0 is an eigenvalue of  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$ , namely for which there exists an  $L^2$ -bound state with null energy, even disconnects  $\mathbb{R}^{Nk} \setminus \Sigma$ , where  $\Sigma := \{(a_1, \dots, a_k) \in \mathbb{R}^{Nk} : a_i = a_j \text{ for some } i \neq j\}$ .

The paper is organized as follows. In Section 2 we give a condition for positivity of Schrödinger operators with potentials in  $\mathcal{V}$ . Section 3 contains the analysis of the asymptotic behavior near the poles of solutions to Schrödinger equations with Hardy type potentials, the proofs of Lemmas 1.2 and 1.3, and an extension to the case of infinitely many singularities on reticular structures. In Section 4 the possibility of perturbing positive operators at infinity is discussed, while Section 5 is devoted to the proof of the Separation Theorem 1.5. Section 6 contains the proof of Theorem 1.1. In section 7 we study the problem of attainability of  $\mu(V)$ , proving Proposition 1.7 and discussing the continuity of  $\mu(V)$  with respect to the masses and the location of singularities. In Section 8 we prove Theorem 1.8. Finally Section 9 contains a detailed description of the spectrum of Schrödinger operators with potentials in  $\mathcal{V}$  and the proofs of Proposition 1.9 and Theorem 1.10.

**Notation.** We list below some notation used throughout the paper.

- $B(a, r)$  denotes the ball  $\{x \in \mathbb{R}^N : |x - a| < r\}$  in  $\mathbb{R}^N$  with center at  $a$  and radius  $r$ .
- For any  $A \subset \mathbb{R}^N$ ,  $\chi_A$  denotes the characteristic function of  $A$ .
- $S$  is the best constant in the Sobolev inequality  $S \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2$ .
- For all  $t \in \mathbb{R}$ ,  $t^+ := \max\{t, 0\}$  (respectively  $t^- := \max\{-t, 0\}$ ) denotes the positive (respectively negative) part of  $t$ .
- For all functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $\text{supp } f$  denotes the support of  $f$ , i.e. the closure of the set of points where  $f$  is non zero.
- $\omega_N$  denotes the volume of the unit ball in  $\mathbb{R}^N$ .
- For any open set  $\Omega \subset \mathbb{R}^N$ ,  $\mathcal{D}'(\Omega)$  denotes the space of distributions in  $\Omega$ .

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## 2. A POSITIVITY CONDITION IN THE CLASS $\mathcal{V}$

Thanks to Sobolev's inequality, for a Schrödinger operator  $-\Delta - V$ ,  $V \in L^{N/2}(\mathbb{R}^N)$ , a general positivity condition is

$$\|V^+\|_{L^{N/2}(\mathbb{R}^N)} < S.$$

We mention that criteria for a potential energy operator  $V$  ( $V$  possibly changing sign or even being a complex-valued distribution) to be relative form-bounded with respect to the laplacian,

i.e. satisfying

$$(9) \quad \left| \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx \right| \leq \text{const} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

are discussed in [33]. In particular (9) implies boundedness from below of the associated quadratic form in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and positivity of Schrödinger operators with small multiple of  $V$  as a potential. However, this type of result cannot answer the question arisen in the present paper about the positivity of forms  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  for given masses  $\lambda_1, \dots, \lambda_k$ .

Furthermore, as  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  is embedded in the Lorenz space  $L^{2^*,2}(\mathbb{R}^N)$ , the positivity of the quadratic form associated to operators  $-\Delta - V$ ,  $V \in L^{N/2, \infty}$ , would be ensured by a smallness condition on  $\|V^+\|_{L^{N/2, \infty}(\mathbb{R}^N)}$ , where we recall that

$$\|f\|_{L^{N/2, \infty}(\mathbb{R}^N)} := \sup_{\substack{X \subset \mathbb{R}^N \\ \text{measurable}}} \frac{\int_X |f(x)| dx}{|X|^{1-\frac{2}{N}}}.$$

We remark that potentials in the class  $\mathcal{V}$  belong to  $L^{N/2, \infty}(\mathbb{R}^N)$ , but their norm in  $L^{N/2, \infty}(\mathbb{R}^N)$  is not small. Indeed for each pole  $a_i$ , a direct calculation yields

$$\left\| \frac{\chi_{B(a_i, r_i)}}{|x - a_i|^2} \right\|_{L^{N/2, \infty}(\mathbb{R}^N)} \geq C(N)$$

for some positive constant  $C(N)$  independent of  $r_i$ . Hence, just by increasing the number of poles, we can exhibit potentials in  $\mathcal{V}$  with as large norms as we want. In the sequel, see Remark 3.6, we will provide an example of potentials having infinite  $L^{N/2, \infty}(\mathbb{R}^N)$ -norm, but giving rise to positive quadratic forms. Therefore to provide positivity conditions in the class is a nontrivial issue.

In this section we provide a criterion for establishing positivity of Schrödinger operators with potentials in  $\mathcal{V}$  in the spirit of the well-known Allegretto-Piepenbrink theory [2, 38], which relates the existence of positive solutions to a Schrödinger equation with the positivity of the spectrum of the corresponding operator. For analogous criteria for potentials in the Kato class we refer to [10, Theorem 2.12].

**Lemma 2.1.** *Let  $V = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, r_i)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R)}(x)}{|x|^2} + W(x) \in \mathcal{V}$ . Then the two following conditions are equivalent:*

$$(i) \quad \mu(V) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0;$$

(ii) *there exist  $\varepsilon > 0$  and  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , and  $\varphi$   $C^1$ -smooth in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , such that  $-\Delta \varphi(x) - V(x)\varphi(x) > \varepsilon V^+(x)\varphi(x)$  a.e. in  $\mathbb{R}^N$ .*

PROOF. We first observe that, from Hardy's, Hölder's, and Sobolev's inequalities,

$$(10) \quad \int_{\mathbb{R}^N} V(x)u^2(x) dx \leq \left[ \frac{4}{(N-2)^2} \left( \sum_{i=1}^k \lambda_i^+ + \lambda_\infty^+ \right) + S^{-1} \|W^+\|_{L^{N/2}(\mathbb{R}^N)} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

for every  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ .



Assume that (i) holds. If  $0 < \varepsilon < \frac{\mu(V)}{2} \left[ \frac{4}{(N-2)^2} \left( \sum_{i=1}^k \lambda_i^+ + \lambda_\infty^+ \right) + S^{-1} \|W^+\|_{L^{N/2}(\mathbb{R}^N)} \right]^{-1}$ , from (10) it follows that

$$\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x) - \varepsilon V^+(x)u^2(x)) dx \geq \frac{\mu(V)}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

As a consequence, for any fixed  $p \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $p(x) > 0$  a.e. in  $\mathbb{R}^N$ , the infimum

$$\nu_p(V + \varepsilon V^+) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x) - \varepsilon V^+(x)u^2(x)) dx}{\int_{\mathbb{R}^N} p(x)u^2(x) dx}$$

is strictly positive and attained by some function  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$  satisfying

$$-\Delta \varphi(x) - V(x)\varphi(x) = \varepsilon V^+(x)\varphi(x) + \nu_p(V + \varepsilon V^+)p(x)\varphi(x).$$

By evenness we can assume  $\varphi \geq 0$ . Since  $V \in \mathcal{V}$ , the Strong Maximum Principle allows us to conclude that  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , while standard regularity theory ensures regularity of  $\varphi$  outside the poles. Hence (ii) holds.

Assume now that (ii) holds. For any  $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ , testing the inequality satisfied by  $\varphi$  with  $u^2/\varphi$  we get

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^N} V^+(x)u^2(x) dx &\leq 2 \int_{\mathbb{R}^N} \frac{u(x)}{\varphi(x)} \nabla u(x) \cdot \nabla \varphi(x) dx - \int_{\mathbb{R}^N} \frac{u^2(x)}{\varphi^2(x)} |\nabla \varphi(x)|^2 dx - \int_{\mathbb{R}^N} V(x)u^2(x) dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} V(x)u^2(x) dx. \end{aligned}$$

By density of  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , we deduce that

$$(11) \quad \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} V^+(x)u^2(x) dx} \geq \varepsilon.$$

From (11) we obtain that  $\mu(V) \geq \varepsilon/(1 + \varepsilon)$ . Indeed, for every  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ ,

$$\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx \geq \varepsilon \int_{\mathbb{R}^N} V^+(x)u^2(x) dx \geq \varepsilon \int_{\mathbb{R}^N} V(x)u^2(x) dx,$$

so that

$$\int_{\mathbb{R}^N} V(x)u^2(x) dx \leq \frac{1}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

implying

$$(12) \quad \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx \geq \frac{\varepsilon}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

□

## 3. THE SHATTERING LEMMA

A starting point for the study of positivity of Schrödinger operators with potentials in  $\mathcal{V}$ , which will be also a key ingredient for the study of their spectral structure, relies in the Shattering Lemma 1.3, which ensures the positivity of Schrödinger operators with potentials whose singularities are localized in a small neighborhood of the corresponding poles, thus avoiding mutual interactions.

The proof of the Shattering Lemma consists in constructing supersolutions for each operator  $\mathcal{L}_{\lambda_i, a_i}^\delta$  (see Lemma 2.1) and then summing up to obtain a supersolution for  $\mathcal{L}_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}^\delta$ . In order to take account of the interactions, we need to evaluate the exact behavior of such supersolutions at each pole and at  $\infty$ , as described in Lemmas 3.1 and 3.3 below, for the proof of which we refer to [18] and [34], see also [35].

**Lemma 3.1.** *Let  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\varphi \not\equiv 0$ , be a weak solution of*

$$(13) \quad -\Delta\varphi(x) = \left[ \frac{\lambda \chi_{B(0,1)}(x)}{|x|^2} + q(x) \right] \varphi(x) \quad \text{in } \mathbb{R}^N,$$

where  $\lambda < \frac{(N-2)^2}{4}$  and  $q \in L_{\text{loc}}^\infty(\mathbb{R}^N \setminus \{0\})$ . Then

- (i) if  $q(x) = O(|x|^{-(2-\varepsilon)})$  as  $|x| \rightarrow 0$  for some  $\varepsilon > 0$ , there exists a positive constant  $C$  (depending on  $q, \lambda, \varepsilon$ , and  $\varphi$ ) such that

$$\frac{1}{C}|x|^{-a_\lambda} \leq \varphi(x) \leq C|x|^{-a_\lambda} \quad \text{for all } x \in B(0,1),$$

where  $a_\lambda = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$ ;

- (ii) if  $q(x) = O(|x|^{-2-\varepsilon})$  as  $|x| \rightarrow +\infty$  for some  $\varepsilon > 0$ , there exists a positive constant  $C$  (depending on  $q, \lambda, \varepsilon$ , and  $\varphi$ ) such that

$$\frac{1}{C}|x|^{-(N-2)} \leq \varphi(x) \leq C|x|^{-(N-2)} \quad \text{for all } x \in \mathbb{R}^N \setminus B(0,1).$$

**Remark 3.2.** *Scanning through the proof of the regularity result contained in [18, Theorem 1.1], it is possible to clarify the dependence of the estimates stated in the above lemma on the data of the problem. Indeed it turns out that if  $\varphi$  solves (13) and  $u$  is given by*

$$u(x) = |x|^{a_\lambda} \varphi(x),$$

then, by [18, Theorem 1.1],  $u \in C^{0,\alpha}(B(0,1))$  for some  $\alpha > 0$  and  $\|u\|_{C^{0,\alpha}(B(0,1))} \leq c \|\varphi\|_{H^1(B(0,R))}$  for any  $R > 1$  and for some positive constant  $c$  (depending on  $R, q, \varepsilon, N$  and  $\lambda$ ) which stays bounded uniformly with respect to  $\lambda$  whenever  $\lambda$  stays bounded from below and above away from  $(N-2)^2/4$ . Hence

$$\varphi(x) \leq c(\lambda, N, q, \varepsilon, R) |x|^{-a_\lambda} \|\varphi\|_{H^1(B(0,R))}.$$

In particular, we can bound  $c(\lambda, N, q, \varepsilon, R)$  uniformly with respect to  $\lambda$  if  $\lambda$  varies in a compact subset of  $(-\infty, (N-2)^2/4)$ .

**Lemma 3.3.** *Let  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\varphi \not\equiv 0$ , be a weak solution of*

$$-\Delta\varphi(x) = \left[ \frac{\lambda \chi_{\mathbb{R}^N \setminus B(0,1)}(x)}{|x|^2} + h(x) \right] \varphi(x) \quad \text{in } \mathbb{R}^N,$$

where  $\lambda < \frac{(N-2)^2}{4}$  and  $h \in L_{\text{loc}}^\infty(\mathbb{R}^N \setminus \{0\})$ . Then

(i) if  $h(x) = O(|x|^{-2-\varepsilon})$  as  $|x| \rightarrow +\infty$  for some  $\varepsilon > 0$ , there exists a positive constant  $C$  (depending on  $h, \lambda, \varepsilon$ , and  $\varphi$ ) such that

$$\frac{1}{C} |x|^{-(N-2-a_\lambda)} \leq \varphi(x) \leq C |x|^{-(N-2-a_\lambda)} \quad \text{for all } x \in \mathbb{R}^N \setminus B(0, 1),$$

where  $a_\lambda = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$ ;

(ii) if  $h(x) = O(|x|^{-(2-\varepsilon)})$  as  $|x| \rightarrow 0$  for some  $\varepsilon > 0$ , there exists a positive constant  $C$  (depending on  $h, \lambda, \varepsilon$ , and  $\varphi$ ) such that

$$\frac{1}{C} \leq \varphi(x) \leq C \quad \text{for all } x \in B(0, 1).$$

We now prove the bound from above of the first  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -eigenvalue of Schrödinger operators with potential in  $\mathcal{V}$  stated in Lemma 1.2.

**Proof of Lemma 1.2.** Let us first consider the case in which  $\lambda_i < 0$  for all  $i = 1, \dots, k, \infty$ . Let us fix  $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  and  $P \in B(0, R_0) \setminus \{a_1, \dots, a_k\}$ . Letting  $u_\mu(x) = \mu^{-\frac{N-2}{2}} u\left(\frac{x-P}{\mu}\right)$ , for  $\mu$  small there holds

$$\mu(V) \leq 1 - \frac{\int_{\mathbb{R}^N} W(x) u_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla u_\mu(x)|^2 dx} = 1 + o(1) \quad \text{as } \mu \rightarrow 0^+.$$

Letting  $\mu \rightarrow 0^+$  we obtain that  $\mu(V) \leq 1$ .

Assume now  $\max_{i=1, \dots, k, \infty} \lambda_i > 0$ . Suppose  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$  and let  $\varepsilon > 0$ . From optimality of the best constant in the classical Hardy inequality (2) and by density of  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx < \left[ \frac{(N-2)^2}{4} + \varepsilon \right] \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx.$$

Letting  $\phi_\mu(x) = \mu^{-\frac{N-2}{2}} \phi\left(\frac{x-a_k}{\mu}\right)$ , for any  $\mu > 0$  there holds

$$\begin{aligned} \mu(V) &\leq 1 - \lambda_k \frac{\int_{B(a_k, r_k)} |x - a_k|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} - \sum_{i \neq k} \lambda_i \frac{\int_{B(a_i, r_i)} |x - a_i|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} \\ &\quad - \lambda_\infty \frac{\int_{\mathbb{R}^N \setminus B(0, R_0)} |x|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} - \frac{\int_{\mathbb{R}^N} W(x) \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} \\ &= 1 - \lambda_k \frac{\int_{\mathbb{R}^N} |x|^{-2} \phi^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx} + o(1) \quad \text{as } \mu \rightarrow 0^+. \end{aligned}$$

Letting  $\mu \rightarrow 0^+$ , by the choice of  $\phi$  we obtain

$$\mu(V) \leq 1 - \lambda_k \left[ \frac{(N-2)^2}{4} + \varepsilon \right]^{-1}$$

for any  $\varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0$  we derive that  $\mu(V) \leq 1 - \frac{4\lambda_k}{(N-2)^2}$ . Repeating the same argument with  $\phi_\mu(x) = \mu^{-\frac{N-2}{2}} \phi(x/\mu)$  and letting  $\mu \rightarrow +\infty$  we obtain also that  $\mu(V) \leq 1 - \frac{4\lambda_\infty}{(N-2)^2}$ . The required estimate is thereby proved.  $\square$

**Proof of Lemma 1.3.** Assume that  $\max_{i=1,\dots,k,\infty} \lambda_i > 0$ , otherwise there is nothing to prove. Let us fix  $0 < \varepsilon < \frac{(N-2)^2}{4 \max_{i=1,\dots,k,\infty} \lambda_i} - 1$ , so that

$$\tilde{\lambda}_\infty := \lambda_\infty + \varepsilon \lambda_\infty^+ < \frac{(N-2)^2}{4} \quad \text{and} \quad \tilde{\lambda}_i := \lambda_i + \varepsilon \lambda_i^+ < \frac{(N-2)^2}{4} \quad \text{for all } i = 1, \dots, k.$$

By scaling properties of the operator and in view of Lemma 2.1, to prove the statement it is enough to find  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  positive and smooth outside the singularities such that

$$(14) \quad -\Delta \varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) > 0 \quad \text{a.e. in } \mathbb{R}^N,$$

where

$$V_i(x) = \frac{\tilde{\lambda}_i \chi_{B(a_i/\delta, 1)}(x)}{|x - \frac{a_i}{\delta}|^2}, \quad V_\infty(x) = \frac{\tilde{\lambda}_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0/\delta)}(x)}{|x|^2},$$

and  $\delta > 0$  depends only on the location of poles and on  $\varepsilon$ . Indeed, if (14) holds for some positive  $\varphi$ , then Lemmas 2.1 and 1.2 ensure that

$$1 - \frac{4}{(N-2)^2} \max_{i=1,\dots,k,\infty} \lambda_i \geq \mu \left( \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2} \right) \geq \frac{\varepsilon}{1 + \varepsilon}$$

for all  $0 < \varepsilon < \frac{(N-2)^2}{4 \max_{i=1,\dots,k,\infty} \lambda_i} - 1$  and the result follows letting  $\varepsilon \rightarrow \frac{(N-2)^2}{4 \max_{i=1,\dots,k,\infty} \lambda_i} - 1$ .

In order to find a positive supersolution to (13), for all  $i = 1, \dots, k$  let us set, for some  $0 < \tau < 1$ ,

$$p_i(x) := p\left(x - \frac{a_i}{\delta}\right) \quad \text{where } p(x) = \frac{1}{|x|^{2-\tau}(1 + |x|^2)^\tau},$$

and

$$p_\infty(x) = \frac{\delta^\tau R_0^{-\tau}}{|x|^{2-\tau}(1 + |\delta x/R_0|^2)^\tau}.$$

Since  $p_i, p_\infty \in L^{N/2}(\mathbb{R}^N)$ , it is easy to see that the weighted first eigenvalue

$$\mu_i = \min_{\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla \varphi(x)|^2 - V_i(x) \varphi^2(x)) dx}{\int_{\mathbb{R}^N} p_i(x) \varphi^2(x) dx}$$

is positive and attained by some function  $\varphi_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi_i > 0$  and smooth in  $\mathbb{R}^N \setminus \{a_i/\delta\}$  and also

$$\begin{aligned} \mu_\infty &= \min_{\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla \varphi(x)|^2 - V_\infty(x) \varphi^2(x)) dx}{\int_{\mathbb{R}^N} p_\infty(x) \varphi^2(x) dx} \\ &= \min_{\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \varphi(x)|^2 dx - \tilde{\lambda}_\infty \int_{\mathbb{R}^N \setminus B(0,1)} \frac{\varphi^2(x)}{|x|^2} dx}{\int_{\mathbb{R}^N} p(x) \varphi^2(x) dx} \end{aligned}$$

is positive and attained by some function  $\varphi_\infty \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi_\infty > 0$  and smooth in  $\mathbb{R}^N \setminus \{0\}$ . The function  $\varphi_i$  satisfy

$$-\Delta\varphi_i(x) - V_i(x)\varphi_i(x) = \mu_i p_i(x)\varphi_i(x)$$

while  $\varphi_\infty$  satisfy

$$-\Delta\varphi_\infty(x) - V_\infty(x)\varphi_\infty(x) = \mu_\infty p_\infty(x)\varphi_\infty(x),$$

hence  $\psi(x) := \varphi_\infty\left(\frac{R_0}{\delta}x\right)$  satisfies

$$-\Delta\psi(x) - \frac{\tilde{\lambda}_\infty \chi_{\mathbb{R}^N \setminus B(0,1)}(x)}{|x|^2} \psi(x) = \mu_\infty p(x)\psi(x).$$

Lemmas 3.1 and 3.3 yield a constant  $C_0 > 0$  (independent on  $\delta$ ) such that

$$(15) \quad \frac{1}{C_0} \left| x - \frac{a_i}{\delta} \right|^{-a_{\tilde{\lambda}_i}} \leq \varphi_i(x) \leq C_0 \left| x - \frac{a_i}{\delta} \right|^{-a_{\tilde{\lambda}_i}}, \quad \text{for all } x \in B(a_i/\delta, 1)$$

$$(16) \quad \frac{1}{C_0} \left| x - \frac{a_i}{\delta} \right|^{-(N-2)} \leq \varphi_i(x) \leq C_0 \left| x - \frac{a_i}{\delta} \right|^{-(N-2)}, \quad \text{for all } x \in \mathbb{R}^N \setminus B(a_i/\delta, 1),$$

$$(17) \quad \frac{1}{C_0} \left| \frac{\delta x}{R_0} \right|^{-(N-2-a_{\tilde{\lambda}_\infty})} \leq \varphi_\infty(x) \leq C_0 \left| \frac{\delta x}{R_0} \right|^{-(N-2-a_{\tilde{\lambda}_\infty})}, \quad \text{for all } x \in \mathbb{R}^N \setminus B(0, R_0/\delta),$$

$$(18) \quad \frac{1}{C_0} \leq \varphi_\infty(x) \leq C_0, \quad \text{for all } x \in B(0, R_0/\delta).$$

Let  $\varphi = \sum_{i=1}^k \varphi_i + \eta\varphi_\infty$  for some  $0 < \eta < \inf \left\{ \frac{\mu_i}{4C_0^2 \tilde{\lambda}_i} : i = 1, \dots, k, \tilde{\lambda}_i > 0 \right\}$ . Then we have

$$\begin{aligned} -\Delta\varphi(x) - \sum_{i=1}^k V_i(x)\varphi(x) - V_\infty(x)\varphi(x) &= \sum_{i=1}^k \mu_i p_i(x)\varphi_i(x) + \mu_\infty p_\infty(x)\eta\varphi_\infty(x) \\ &\quad - \sum_{i \neq j} V_i(x)\varphi_j(x) - \eta \sum_{i=1}^k V_i(x)\varphi_\infty(x) - V_\infty(x) \sum_{i=1}^k \varphi_i(x). \end{aligned}$$

In particular a.e. in the set  $B(0, R_0/\delta) \setminus \bigcup_{i=1}^k B(a_i/\delta, 1)$ , we have

$$-\Delta\varphi(x) - \sum_{i=1}^k V_i(x)\varphi(x) - V_\infty(x)\varphi(x) = \sum_{i=1}^k \mu_i p_i(x)\varphi_i(x) + \mu_\infty p_\infty(x)\eta\varphi_\infty(x) > 0.$$

Let us consider  $B(a_i/\delta, 1)$ . If  $\tilde{\lambda}_i < 0$ , we have easily that

$$-\Delta\varphi - \sum_{i=1}^k V_i(x)\varphi(x) - V_\infty(x)\varphi(x) > 0 \quad \text{a.e. in } B(a_i/\delta, 1).$$

If  $\tilde{\lambda}_i > 0$  we can choose  $\tau < a_{\tilde{\lambda}_i}$ . Since, for  $\delta$  small,

$$B(a_i/\delta, 1) \subset B(0, R_0/\delta) \quad \text{and} \quad B(a_i/\delta, 1) \subset \mathbb{R}^N \setminus B(a_j/\delta, 1) \text{ for } j \neq i,$$

from (15), (16), and (18) it follows that, in  $B(a_i/\delta, 1)$ ,

$$\begin{aligned} -\Delta\varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) &\geq \mu_i p_i(x) \varphi_i(x) - V_i(x) \left( \sum_{j \neq i} \varphi_j(x) + \eta \varphi_\infty(x) \right) \\ &\geq \left| x - \frac{a_i}{\delta} \right|^{-2} \left[ \frac{\mu_i}{2^\tau C_0} \left| x - \frac{a_i}{\delta} \right|^{\tau - a_{\tilde{\lambda}_i}} - C_0 \tilde{\lambda}_i \left( \sum_{j \neq i} \left| x - \frac{a_j}{\delta} \right|^{-(N-2)} + \eta \right) \right]. \end{aligned}$$

It is easy to see that for  $\delta$  small

$$\left| x - \frac{a_i}{\delta} \right|^{\tau - a_{\tilde{\lambda}_i}} \geq 1 \quad \text{and} \quad \left| x - \frac{a_j}{\delta} \right|^{-(N-2)} \leq \left( \frac{2}{|a_i - a_j|} \right)^{N-2} \delta^{N-2} < \frac{\eta}{k-1},$$

and hence the choice of  $\eta$  ensures that

$$-\Delta\varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) > 0 \quad \text{a.e. in } B(a_i/\delta, 1),$$

provided  $\delta$  is sufficiently small.

Let us finally consider  $\mathbb{R}^N \setminus B(0, R_0/\delta)$ . If  $\tilde{\lambda}_\infty < 0$ , then

$$-\Delta\varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) > 0 \quad \text{a.e. in } \mathbb{R}^N \setminus B(0, R_0/\delta).$$

If  $\tilde{\lambda}_\infty > 0$  we can choose  $\tau < a_{\tilde{\lambda}_\infty}$ . From (16–17), we deduce that in  $\mathbb{R}^N \setminus B(0, R_0/\delta)$

$$\begin{aligned} -\Delta\varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) &\geq \mu_\infty p_\infty(x) \eta \varphi_\infty(x) - V_\infty(x) \sum_{i=1}^k \varphi_i(x) \\ &\geq \frac{1}{|x|^2} \left[ \frac{\mu_\infty \eta}{2^\tau C_0} \left| \frac{\delta x}{R_0} \right|^{-(N-2-a_{\tilde{\lambda}_\infty})-\tau} - C_0 \tilde{\lambda}_\infty \sum_{i=1}^k \left| x - \frac{a_i}{\delta} \right|^{-(N-2)} \right]. \end{aligned}$$

It is easy to see that, in  $\mathbb{R}^N \setminus B(0, R_0/\delta)$ ,  $|x - \frac{a_i}{\delta}| \geq (1 - \frac{\alpha}{R_0})|x|$  where  $\alpha = \max\{|a_j|\}_j$ , hence

$$\begin{aligned} -\Delta\varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) &\geq \mu_\infty p_\infty(x) \eta \varphi_\infty(x) - V_\infty(x) \sum_{i=1}^k \varphi_i(x) \\ &\geq \frac{1}{|x|^N} \left[ \frac{\mu_\infty \eta}{2^\tau C_0} \left| \frac{\delta}{R_0} \right|^{-(N-2)} - C_0 \tilde{\lambda}_\infty k \left( 1 - \frac{\alpha}{R_0} \right)^{-(N-2)} \right] > 0 \quad \text{a.e. in } \mathbb{R}^N \setminus B(0, R_0/\delta) \end{aligned}$$

provided  $\delta$  is sufficiently small. The proof is thereby complete.  $\square$

**Remark 3.4.** For  $(\lambda_1, \dots, \lambda_k, \lambda_\infty) \in (-\infty, (N-2)^2/4)^{k+1}$ , let  $\{a_1^n, a_2^n, \dots, a_k^n\} \subset B(0, R_0) \subset \mathbb{R}^N$  be a sequence of configurations approximating  $\{a_1, a_2, \dots, a_k\} \subset B(0, R_0) \subset \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ , i.e.  $a_i^n \rightarrow a_i$  as  $n \rightarrow \infty$  for all  $i = 1, \dots, k$ . From the proofs of Lemmas 2.1 and 1.3, for any  $0 < \alpha < 1 - \frac{4 \max_{i=1, \dots, k, \infty} \lambda_i}{(N-2)^2}$ , we can choose  $\delta > 0$  independently of  $n$  such that

$$\mu(\tilde{V}_n) \geq \begin{cases} 1 - \frac{4 \max_{i=1, \dots, k, \infty} \lambda_i}{(N-2)^2} - \alpha, & \text{if } \max_{i=1, \dots, k, \infty} \lambda_i > 0, \\ 1, & \text{if } \max_{i=1, \dots, k, \infty} \lambda_i \leq 0, \end{cases}$$

where

$$\tilde{V}_n(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i^n, \delta)}}{|x - a_i^n|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}}{|x|^2}.$$

Let us now deal with the case of infinitely many singularities distributed on reticular structures.

**Lemma 3.5. [Shattering of reticular singularities]** *Let  $\lambda < (N-2)^2/4$  and let  $\{a_n\}_n \subset \mathbb{R}^N$  satisfy*

$$(19) \quad \sum_{n=1}^{\infty} |a_n|^{-(N-2)} < +\infty, \quad \sum_{k=1}^{\infty} |a_{n+k} - a_n|^{-(N-2)} \quad \text{is bounded uniformly in } n,$$

and  $|a_n - a_m| \geq 1$  for all  $n \neq m$ . Then there exists  $\delta > 0$  such that

$$\inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0$$

where

$$V(x) = \lambda \sum_{n=1}^{\infty} \frac{\chi_{B(a_n, \delta)}(x)}{|x - a_n|^2}.$$

PROOF. Let  $\varepsilon > 0$  such that  $\tilde{\lambda} = \lambda + \varepsilon < (N-2)^2/4$ . Arguing as in the proof of Lemma 1.3, we can construct a function  $\psi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\psi > 0$  and smooth in  $\mathbb{R}^N \setminus \{0\}$  such that

$$-\Delta \psi(x) - \tilde{\lambda} \frac{\chi_{B(0,1)}(x)}{|x|^2} \psi(x) = \mu p(x) \psi(x),$$

where  $\mu > 0$  and  $p(x) = |x|^{\tau-2}(1 + |x|^2)^{-\tau}$ , for some  $0 < \tau < 1$ . Moreover, by Lemma 3.1,

$$\frac{|x|^{-a\tilde{\lambda}}}{C} \leq \psi(x) \leq C|x|^{-a\tilde{\lambda}} \quad \text{in } B(0, 1), \quad \text{and} \quad \frac{|x|^{-(N-2)}}{C} \leq \psi(x) \leq C|x|^{-(N-2)} \quad \text{in } \mathbb{R}^N \setminus B(0, 1).$$

Let  $\varphi(x) = \sum_{n=1}^{\infty} \psi(x - \frac{a_n}{\delta})$ . For any compact set  $K$ , we have that there exists  $\bar{n}$  such that, for all  $n \geq \bar{n}$  and  $x \in K$ ,  $|\psi(x - \frac{a_n}{\delta})| \leq C|x - \frac{a_n}{\delta}|^{-(N-2)} \leq \text{const} |\frac{a_n}{\delta}|^{-(N-2)}$ . Then

$$\varphi|_K(x) = \sum_{n=1}^{\bar{n}-1} \psi\left(x - \frac{a_n}{\delta}\right) + \sum_{n=\bar{n}}^{\infty} \psi\left(x - \frac{a_n}{\delta}\right) \in H^1(K) + L^\infty(K).$$

In particular  $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ . Moreover

$$\begin{aligned} -\Delta \varphi(x) - \tilde{\lambda} \sum_{n=1}^{\infty} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \varphi(x) \\ = \mu \sum_{n=1}^{\infty} p\left(x - \frac{a_n}{\delta}\right) \psi\left(x - \frac{a_n}{\delta}\right) - \tilde{\lambda} \sum_{m \neq n} \frac{\chi_{B(a_m/\delta, 1)}(x)}{|x - \frac{a_m}{\delta}|^2} \psi\left(x - \frac{a_n}{\delta}\right). \end{aligned}$$

Then a.e. in the set  $\mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} B(a_n/\delta, 1)$ , we have

$$-\Delta \varphi(x) - \tilde{\lambda} \sum_{n=1}^{\infty} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \varphi(x) > 0.$$

Assume  $\tilde{\lambda} > 0$ , otherwise there is nothing to prove, thus we can choose  $\tau < a_{\tilde{\lambda}}$ . Therefore in each ball  $B(a_n/\delta, 1)$

$$\begin{aligned} -\Delta\varphi(x) - \tilde{\lambda} \sum_{n=1}^{\infty} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \varphi(x) \\ \geq \mu p\left(x - \frac{a_n}{\delta}\right) \psi\left(x - \frac{a_n}{\delta}\right) - \tilde{\lambda} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \sum_{m \neq n} \psi\left(x - \frac{a_m}{\delta}\right) \\ \geq \text{const} \left|x - \frac{a_n}{\delta}\right|^{-2} \left( \text{const} \left|x - \frac{a_n}{\delta}\right|^{-a_{\tilde{\lambda}} + \varepsilon} - \sum_{m \neq n} \left|x - \frac{a_m}{\delta}\right|^{-(N-2)} \right). \end{aligned}$$

Since, for small  $\delta$ ,  $\left|x - \frac{a_m}{\delta}\right| \geq \frac{|a_m - a_n|}{\delta} - 1 \geq \frac{|a_m - a_n|}{2\delta}$  provided  $\delta$  small enough, we deduce that

$$\sum_{m \neq n} \left|x - \frac{a_m}{\delta}\right|^{-(N-2)} \leq (2\delta)^{N-2} \sum_{m \neq n} |a_m - a_n|^{-(N-2)} \leq \text{const} \delta^{N-2}.$$

Hence, we can choose  $\delta$  small enough independently of  $n$  such that

$$-\Delta\varphi(x) - \tilde{\lambda} \sum_{n=1}^{\infty} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \varphi(x) > 0$$

a.e. in  $B(a_n/\delta, 1)$ . Hence we have constructed a supersolution  $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N)$ ,  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}}$  and  $\varphi$  smooth in  $\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}}$ , such that

$$-\Delta\varphi(x) - \tilde{\lambda} \sum_{n=1}^{\infty} \frac{\chi_{B(a_n/\delta, 1)}(x)}{|x - \frac{a_n}{\delta}|^2} \varphi(x) > 0 \quad \text{a.e. in } \mathbb{R}^N,$$

where  $\tilde{\lambda} = \lambda + \varepsilon < (N-2)^2/4$ . Therefore, arguing as in Lemma 2.1 and taking into account the scaling properties of the operator, we obtain

$$\tilde{\mu}(V) := \inf_{\substack{u \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}}) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0,$$

i.e.

$$(20) \quad \int_{\mathbb{R}^N} V(x)u^2(x) dx \leq (1 - \tilde{\mu}(V)) \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx$$

for all  $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ . By density of  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and the Fatou Lemma, we can easily prove that (20) holds for all  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ .  $\square$

**Remark 3.6.** *The Lemma above can be used to construct examples of potentials having infinite  $L^{N/2, \infty}(\mathbb{R}^N)$ -norm, but giving rise to positive quadratic forms.*

**Remark 3.7.** *If the singularities  $a_n$  are located on a periodic  $M$ -dimensional reticular structure,  $M \leq N$ , i.e. if*

$$\{a_n : n \in \mathbb{N}\} = \{(x_1, x_2, \dots, x_M, 0, \dots, 0) : x_i \in \mathbb{Z} \text{ for all } i = 1, \dots, M\},$$



then  $\sum_{k=1}^{\infty} |a_{n+k} - a_n|^{-(N-2)} < +\infty$  if and only if  $\sum_{k=1}^{\infty} k^{-(N-2)+M-1} < +\infty$ , i.e. for  $M < N-2$ .

**Remark 3.8.** From Lemma 3.5, it follows that, for  $\delta$  small and any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , the series

$$\sum_{n=1}^{\infty} \int_{B(a_n, \delta)} \frac{u^2(x)}{|x - a_n|^2} dx$$

converges and

$$\sum_{n=1}^{\infty} \int_{B(a_n, \delta)} \frac{u^2(x)}{|x - a_n|^2} dx \leq \text{const} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

#### 4. PERTURBATION AT INFINITY

In this section we discuss the stability of positivity with respect to perturbations of the potentials with a small singularity sitting at infinity.

**Lemma 4.1.** *Let*

$$V(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, r_i)}(x)}{|x - a_i|^2} + \frac{\lambda_{\infty} \chi_{\mathbb{R}^N \setminus B(0, R)}(x)}{|x|^2} + W(x) \in \mathcal{V},$$

where  $W \in L^{\infty}(\mathbb{R}^N)$ ,  $W(x) = O(|x|^{-2-\delta})$ , with  $\delta > 0$ , as  $|x| \rightarrow \infty$ . Assume that  $\mu(V) > 0$ , namely that the quadratic form associated to the operator  $-\Delta - V$  is positive definite. Let  $\gamma_{\infty} \in \mathbb{R}$  such that  $\lambda_{\infty} + \gamma_{\infty} < \left(\frac{N-2}{2}\right)^2$ . Then there exist  $\tilde{R} > R$  and  $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$  and smooth in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$  such that

$$-\Delta \Phi(x) - V(x)\Phi(x) - \frac{\gamma_{\infty}}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})} \Phi(x) > 0.$$

PROOF. Let us fix

$$0 < \varepsilon < \min \left\{ \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda_{\infty}}, \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda_{\infty} - \gamma_{\infty}} \right\},$$

$C_0 > 0$  such that

$$W(x) \leq \frac{C_0}{|x|^{2+\delta}}, \quad \text{in } \mathbb{R}^N,$$

and  $R_0 > 0$  such that

$$\bigcup_{i=1}^k B(a_i, r_i) \subset B(0, R_0) \quad \text{and} \quad \left(\frac{N-2}{2}\right)^2 - \varepsilon^2 - \lambda_{\infty} \geq \frac{C_0}{R_0^{\delta}}.$$

Let  $\varphi_1 \geq 0$  be a smooth function such that

$$\varphi_1 \equiv 0 \quad \text{in } B(0, R_0), \quad \varphi_1(x) = \frac{1}{|x|^{\frac{N-2}{2} + \varepsilon}} \quad \text{in } \mathbb{R}^N \setminus B(0, 2R_0).$$

For every  $\tilde{R} > 2R_0$ , we have that

$$(21) \quad -\Delta\varphi_1(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \varphi_1(x) = \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 \right] \frac{\chi_{B(0, \tilde{R}) \setminus B(0, 2R_0)}(x)}{|x|^2} \varphi_1(x) + f_1(x) \\ + \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \gamma_\infty \right] \frac{\chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \varphi_1(x),$$

where  $f_1$  is a smooth function with compact support. Let us choose a smooth function with compact support  $f_2$  such that

$$f_1 + f_2 \geq 0 \text{ in } \mathbb{R}^N, \quad f_1 + f_2 > 0 \text{ in } B(0, 2R_0), \quad \text{and} \\ f_2 + W \chi_{B(0, 2R_0)} \varphi_1 + \frac{\lambda_\infty}{|x|^2} \chi_{B(0, 2R_0) \setminus B(0, R_0)} \varphi_1 \geq 0 \text{ in } \mathbb{R}^N.$$

Since  $f_2 + W \chi_{B(0, 2R_0)} \varphi_1 + \lambda_\infty |x|^{-2} \chi_{B(0, 2R_0) \setminus B(0, R_0)} \varphi_1 \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$  and  $\mu(V) > 0$ , in view of the Lax-Milgram Theorem there exists  $\varphi_2 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi_2 > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , satisfying

$$(22) \quad -\Delta\varphi_2(x) - V(x)\varphi_2(x) = f_2(x) + \left[ W(x)\chi_{B(0, 2R_0)}(x) + \frac{\lambda_\infty \chi_{B(0, 2R_0) \setminus B(0, R_0)}(x)}{|x|^2} \right] \varphi_1(x).$$

From Lemma 3.3, we have that, for some positive constant  $C_1$ ,

$$\frac{1}{C_1} |x|^{-(N-2-a\lambda_\infty)} \leq \varphi_2(x) \leq C_1 |x|^{-(N-2-a\lambda_\infty)} \text{ in } \mathbb{R}^N \setminus B(0, 2R_0).$$

Set  $\Phi = \varphi_1 + \varphi_2$ . We claim that for  $\tilde{R} > 0$  large enough there holds

$$-\Delta\Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \Phi(x) > 0, \quad \text{a.e. in } \mathbb{R}^N.$$

For all  $\tilde{R} > 2R_0$ , from (21–22) we deduce

$$\begin{aligned}
 (23) \quad & -\Delta\Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \Phi(x) \\
 & \geq \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \gamma_\infty \right] \frac{\chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \varphi_1(x) + \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 \right] \frac{\chi_{B(0, \tilde{R}) \setminus B(0, 2R_0)}(x)}{|x|^2} \varphi_1(x) \\
 & \quad + f_1(x) + f_2(x) + \left[ W(x) \chi_{B(0, 2R_0)}(x) + \frac{\lambda_\infty \chi_{B(0, 2R_0) \setminus B(0, R_0)}(x)}{|x|^2} \right] \varphi_1(x) \\
 & \quad - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \varphi_2(x) - V(x) \varphi_1(x) \\
 & \geq f_1(x) + f_2(x) + \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \gamma_\infty - \lambda_\infty \right] \frac{\chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \frac{1}{|x|^{\frac{N-2}{2} + \varepsilon}} \\
 & \quad + \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \lambda_\infty - \frac{C_0}{|x|^\delta} \right] \frac{\chi_{B(0, \tilde{R}) \setminus B(0, 2R_0)}(x)}{|x|^2} \varphi_1(x) \\
 & \quad - \frac{C_0}{|x|^{2+\delta}} \frac{1}{|x|^{\frac{N-2}{2} + \varepsilon}} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x) - \frac{\gamma_\infty}{|x|^2} \frac{C_1}{|x|^{N-2-a\lambda_\infty}} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x).
 \end{aligned}$$

By the choice of  $f_2$ , we have that in  $B(0, 2R_0)$

$$-\Delta\Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \Phi(x) \geq f_1(x) + f_2(x) > 0.$$

From (23) and the choice of  $\varepsilon$  and  $R_0$ , it follows that, in  $B(0, \tilde{R}) \setminus B(0, 2R_0)$ ,

$$-\Delta\Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \Phi(x) \geq \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \lambda_\infty - \frac{C_0}{|x|^\delta} \right] \frac{1}{|x|^2} \varphi_1(x) > 0.$$

From (23) and the choice of  $\varepsilon$ , we deduce that, in  $\mathbb{R}^N \setminus B(0, \tilde{R})$ ,

$$\begin{aligned}
 & -\Delta\Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}(x)}{|x|^2} \Phi(x) \geq \\
 & \geq \left[ \left( \frac{N-2}{2} \right)^2 - \varepsilon^2 - \gamma_\infty - \lambda_\infty \right] \frac{1}{|x|^{2+\frac{N-2}{2}+\varepsilon}} - \frac{C_0}{|x|^{2+\delta+\frac{N-2}{2}+\varepsilon}} - \frac{\gamma_\infty C_1}{|x|^{N-a\lambda_\infty}} > 0
 \end{aligned}$$

provided  $\tilde{R}$  is large enough. The claim is thereby proved.  $\square$

**Theorem 4.2.** *Let*

$$V(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, r_i)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R)}(x)}{|x|^2} + W(x) \in \mathcal{V},$$

where  $W \in L^\infty(\mathbb{R}^N)$ ,  $W(x) = O(|x|^{-2-\delta})$ , with  $\delta > 0$ , as  $|x| \rightarrow \infty$ . Assume that  $\mu(V) > 0$ , namely that the quadratic form associated to the operator  $-\Delta - V$  is positive definite. Let  $\gamma_\infty \in \mathbb{R}$  such

that  $\lambda_\infty + \gamma_\infty < \left(\frac{N-2}{2}\right)^2$ . Then there exists  $\tilde{R} > R$  such that  $\mu\left(V + \frac{\gamma_\infty}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}\right) > 0$ , namely the quadratic form associated to the operator  $-\Delta - V - \frac{\gamma_\infty}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}$  is positive definite.

PROOF. As already observed in the proof of Lemma 2.1, if  $V \in \mathcal{V}$  and  $\mu(V) > 0$ , then there exists  $\varepsilon > 0$  such that  $V + \varepsilon V^+ \in \mathcal{V}$ ,  $\mu(V + \varepsilon V^+) > 0$ , and  $\lambda_\infty + \gamma_\infty + \varepsilon(\lambda_\infty^+ + \gamma_\infty^+) < \left(\frac{N-2}{2}\right)^2$ . In particular  $V + \varepsilon V^+$  also satisfies the assumptions of Lemma 4.1, hence there exist  $\tilde{R} > R$  and  $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$  and smooth in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$  such that

$$-\Delta \Phi(x) - V(x)\Phi(x) - \frac{\gamma_\infty}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})} \Phi(x) > \varepsilon \left( V^+(x) + \frac{\gamma_\infty^+}{|x|^2} \right) \Phi(x) \geq \varepsilon \left( V(x) + \frac{\gamma_\infty}{|x|^2} \right)^+ \Phi(x).$$

The conclusion follows now from Lemma 2.1.  $\square$

## 5. SEPARATION THEOREM

In this section we provide a tool to construct a positive operator from two positive potentials in  $\mathcal{V}$  whose interaction at infinity is not too strong. To this aim we first show how, starting from the supersolutions corresponding to each positive given operator, it is possible to scatter the singularities and obtain a positive supersolution for the resulting operator by summation.

**Lemma 5.1.** *Let*

$$V_1(x) = \sum_{i=1}^{k_1} \frac{\lambda_i^1 \chi_{B(a_i^1, r_i^1)}(x)}{|x - a_i^1|^2} + \frac{\lambda_\infty^1 \chi_{\mathbb{R}^N \setminus B(0, R_1)}(x)}{|x|^2} + W_1(x) \in \mathcal{V},$$

$$V_2(x) = \sum_{i=1}^{k_2} \frac{\lambda_i^2 \chi_{B(a_i^2, r_i^2)}(x)}{|x - a_i^2|^2} + \frac{\lambda_\infty^2 \chi_{\mathbb{R}^N \setminus B(0, R_2)}(x)}{|x|^2} + W_2(x) \in \mathcal{V},$$

where  $W_i \in L^\infty(\mathbb{R}^N)$ ,  $W_i(x) = O(|x|^{-2-\delta})$ ,  $i = 1, 2$ , with  $\delta > 0$ , as  $|x| \rightarrow \infty$ . Assume that  $\mu(V_1), \mu(V_2) > 0$ , namely that the quadratic forms associated to the operators  $-\Delta - V_1, -\Delta - V_2$  are positive definite and that  $\lambda_\infty^1 + \lambda_\infty^2 < \left(\frac{N-2}{2}\right)^2$ . Then, there exists  $R > 0$  such that, for every  $y \in \mathbb{R}^N$  with  $|y| \geq R$ , there exists  $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi_y \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\Phi_y > 0$  and  $C^1$ -smooth in  $\mathbb{R}^N \setminus \{a_i^1, a_i^2 + y\}_{i=1, \dots, k_j, j=1, 2}$ , such that

$$-\Delta \Phi_y(x) - (V_1(x) + V_2(x - y)) \Phi_y(x) > 0 \quad \text{a.e. in } \mathbb{R}^N.$$

PROOF. Let  $0 < \varepsilon \ll 1$  be such that  $\lambda_\infty^1 + \lambda_\infty^2 < \left(\frac{N-2}{2}\right)^2 - \varepsilon$  and, for  $j = 1, 2$ , set

$$\Lambda = \left(\frac{N-2}{2}\right)^2 - \varepsilon \quad \text{and} \quad \gamma_\infty^j = \Lambda - \lambda_\infty^j.$$

Let us also choose  $0 < \eta \ll 1$  such that

$$(24) \quad \lambda_\infty^2 < \gamma_\infty^1(1 - 2\eta) \quad \text{and} \quad \lambda_\infty^1 < \gamma_\infty^2(1 - 2\eta).$$

We can choose  $\bar{R} > 0$  such that, for  $j = 1, 2$ ,  $\bigcup_{i=1}^{k_j} B(a_i^j, r_i^j) \subset B(0, \bar{R})$ , and define

$$(25) \quad p_j(x) := \begin{cases} |x - a_i^j|^{-2+\sigma} & \text{in } B(a_i^j, r_i^j), \\ 1 & \text{in } B(0, \bar{R}) \setminus \bigcup_{i=1}^{k_j} B(a_i^j, r_i^j), \\ 0 & \text{in } \mathbb{R}^N \setminus B(0, \bar{R}), \end{cases}$$

with  $\sigma > 0$ . In view of Theorem 4.2, there exist  $\tilde{R}_j$  such that the quadratic forms associated to the operators  $-\Delta - V_j - \frac{\gamma_\infty^j}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)}$  are positive definite. Therefore, since  $p_j \in L^{N/2}$ , the infima

$$\mu_j = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} [|\nabla u(x)|^2 - V_j(x)u^2(x) - \gamma_\infty^j |x|^{-2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)} u^2(x)] dx}{\int_{\mathbb{R}^N} p_j(x)u^2(x) dx}$$

are achieved by some  $\psi_j \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\psi_j \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\psi_j > 0$  in  $\mathbb{R}^N \setminus \{a_1^j, \dots, a_{k_j}^j\}$ , solving equation

$$(26) \quad -\Delta \psi_j(x) - V_j(x)\psi_j(x) = \mu_j p_j(x)\psi_j(x) + \frac{\gamma_\infty^j}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)} \psi_j(x) \quad \text{in } \mathbb{R}^N.$$

In view of Lemma 3.3, there holds

$$\lim_{|x| \rightarrow +\infty} \psi_j(x) |x|^{N-2-a_\Lambda} = \ell_j > 0,$$

hence the function  $\varphi_j := \frac{\psi_j}{\ell_j}$  solves (26) and  $\varphi_j(x) \sim |x|^{-(N-2-a_\Lambda)}$  at  $\infty$ . Then there exists  $\rho > \max\{\tilde{R}_1, \tilde{R}_2, \tilde{R}\}$  such that, in  $\mathbb{R}^N \setminus B(0, \rho)$ ,

$$(27) \quad (1 - \eta^2) |x|^{-(N-2-a_\Lambda)} \leq \varphi_j(x) \leq (1 + \eta) |x|^{-(N-2-a_\Lambda)}$$

and that

$$(28) \quad |W_1(x)| \leq \eta \gamma_\infty^2 |x|^{-2} \quad \text{and} \quad |W_2(x)| \leq \eta \gamma_\infty^1 |x|^{-2}.$$

Moreover, from Lemma 3.1 we can deduce that for some positive constant  $C$

$$(29) \quad \frac{1}{C} |x - a_i^j|^{-a_{\lambda_i^j}} \leq \varphi_j(x) \leq C |x - a_i^j|^{-a_{\lambda_i^j}} \quad \text{in } B(a_i^j, r_i^j), \quad i = 1, \dots, k_j$$

and  $\varphi_j$  are  $C^1$ -smooth outside the poles.

For any  $y \in \mathbb{R}^N$ , let us consider the function

$$\Phi_y(x) := \gamma_\infty^2 \varphi_1(x) + \gamma_\infty^1 \varphi_2(x - y) \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Then

$$\begin{aligned} & -\Delta \Phi_y(x) - (V_1(x) + V_2(x - y)) \Phi_y(x) \\ &= \mu_1 \gamma_\infty^2 p_1(x) \varphi_1(x) + \frac{\gamma_\infty^1 \gamma_\infty^2}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_1)} \varphi_1(x) \\ & \quad + \mu_2 \gamma_\infty^1 p_2(x - y) \varphi_2(x - y) + \frac{\gamma_\infty^1 \gamma_\infty^2}{|x - y|^2} \chi_{\mathbb{R}^N \setminus B(y, \tilde{R}_2)} \varphi_2(x - y) \\ & \quad - \gamma_\infty^1 V_1(x) \varphi_2(x - y) - \gamma_\infty^2 V_2(x - y) \varphi_1(x). \end{aligned}$$

From (24) and (27), it follows that in  $\mathbb{R}^N \setminus (B(0, \rho) \cup B(y, \rho))$

$$\begin{aligned}
& -\Delta\Phi_y(x) - (V_1(x) + V_2(x-y))\Phi_y(x) \\
& \geq \frac{\gamma_\infty^1 \gamma_\infty^2}{|x|^2} \varphi_1(x) + \frac{\gamma_\infty^1 \gamma_\infty^2}{|x-y|^2} \varphi_2(x-y) \\
& \quad - \gamma_\infty^1 \left( \frac{\lambda_\infty^1}{|x|^2} + W_1(x) \right) \varphi_2(x-y) - \gamma_\infty^2 \left( \frac{\lambda_\infty^2}{|x-y|^2} + W_2(x-y) \right) \varphi_1(x) \\
& > \frac{\gamma_\infty^1 \gamma_\infty^2 (1-\eta^2)}{|x|^{N-a_\Lambda}} + \frac{\gamma_\infty^1 \gamma_\infty^2 (1-\eta^2)}{|x-y|^{N-a_\Lambda}} - \frac{\gamma_\infty^1 \gamma_\infty^2 (1-\eta^2)}{|x|^2 |x-y|^{N-2-a_\Lambda}} - \frac{\gamma_\infty^1 \gamma_\infty^2 (1-\eta^2)}{|x-y|^2 |x|^{N-2-a_\Lambda}} \\
& = \gamma_\infty^1 \gamma_\infty^2 (1-\eta^2) \left( \frac{1}{|x|^{N-a_\Lambda-2}} - \frac{1}{|x-y|^{N-a_\Lambda-2}} \right) \left( \frac{1}{|x|^2} - \frac{1}{|x-y|^2} \right) \geq 0.
\end{aligned}$$

For  $|y|$  sufficiently large,  $B(0, \rho) \cap B(y, \rho) = \emptyset$ . In  $B(a_i^1, r_i^1)$ , for  $i = 1, \dots, k_1$ , from (25), (27), (28), (29) we have that

$$\begin{aligned}
& -\Delta\Phi_y(x) - (V_1(x) + V_2(x-y))\Phi_y(x) \\
& \geq \mu_1 \gamma_\infty^2 p_1(x) \varphi_1(x) + \frac{\gamma_\infty^1 \gamma_\infty^2}{|x-y|^2} \varphi_2(x-y) \\
& \quad - \gamma_\infty^1 V_1(x) \varphi_2(x-y) - \gamma_\infty^2 V_2(x-y) \varphi_1(x) \\
& \geq |x - a_i^1|^{-a_{\lambda_i^1} - 2 + \sigma} \left[ \frac{\mu_1 \gamma_\infty^2}{C} + o(1) \right], \quad \text{as } |y| \rightarrow \infty.
\end{aligned}$$

In  $B(0, \rho) \setminus \bigcup_{i=1}^{k_1} B(a_i^1, r_i^1)$ , from (25), (27), (28) and since  $\varphi_1 > c > 0$ , we obtain that

$$-\Delta\Phi_y(x) - (V_1(x) + V_2(x-y))\Phi_y(x) \geq \mu_1 \gamma_\infty^2 c + o(1), \quad \text{as } |y| \rightarrow \infty.$$

In a similar way we can prove that, if  $|y|$  is large enough,

$$-\Delta\Phi_y(x) - (V_1(x) + V_2(x-y))\Phi_y(x) \geq 0, \quad \text{a.e. in } B(y, \rho).$$

□

**Proof of Theorem 1.5.** Let us fix  $\varepsilon \in (0, 1)$  such that, for  $j = 1, 2$ ,

$$(30) \quad \varepsilon < \min \left\{ 2S \mu(V_j), \frac{\mu(V_j)}{4} \left[ \frac{4}{(N-2)^2} \left( \sum_{i=1}^k (\lambda_i^j)^+ + (\lambda_\infty^j)^+ \right) + S^{-1} \|W_j\|_{L^{N/2}(\mathbb{R}^N)} \right]^{-1} \right\}$$

and

$$\mu(V_j + \varepsilon V_j^+) > 0, \quad \lambda_\infty^1 + \lambda_\infty^2 + \varepsilon \left( (\lambda_\infty^1)^+ + (\lambda_\infty^2)^+ \right) < \left( \frac{N-2}{2} \right)^2,$$

see the proof of Lemma 2.1. Fix  $R > 0$  such that

$$(31) \quad \|W_j \chi_{\mathbb{R}^N \setminus B(0, R)}\|_{L^{N/2}(\mathbb{R}^N)} < \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{16} S \right\}.$$

Denoting  $V_{j,R} := V_j - W_j \chi_{\mathbb{R}^N \setminus B(0,R)}$ , from (30) and (31), there results

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_{j,R}(x)u^2(x)) dx \\ & \geq \left[ \mu(V_j) - \|W_j \chi_{\mathbb{R}^N \setminus B(0,R)}\|_{L^{N/2}(\mathbb{R}^N)} S^{-1} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \geq \frac{\mu(V_j)}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \end{aligned}$$

therefore, from (30), it follows

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (V_{j,R}(x) + \varepsilon V_{j,R}^+(x))u^2(x)) dx \\ & \geq \left[ \frac{\mu(V_j)}{2} - \varepsilon \left( \frac{4}{(N-2)^2} \left( \sum_{i=1}^k (\lambda_i^j)^+ + (\lambda_\infty^j)^+ \right) + S^{-1} \|W_j\|_{L^{N/2}(\mathbb{R}^N)} \right) \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \\ & \geq \frac{\mu(V_j)}{4} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx. \end{aligned}$$

Hence the potentials  $V_{j,R} + \varepsilon V_{j,R}^+$  satisfy the assumptions of Lemma 5.1, which yields, for  $|y|$  sufficiently large, the existence of  $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi_y \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\Phi_y > 0$  and  $C^1$ -smooth in  $\mathbb{R}^N \setminus \{a_i^1, a_i^2 + y\}_{i=1,\dots,k, j=1,2}$ , such that

$$-\Delta \Phi_y(x) - V_{R,y}(x)\Phi_y(x) > \varepsilon V_{R,y}^+(x)\Phi_y(x) \quad \text{a.e. in } \mathbb{R}^N,$$

where  $V_{R,y}(x) := V_{1,R}(x) + V_{2,R}(x-y)$ . As a consequence, arguing as in the proof of Lemma 2.1, we easily deduce that

$$(32) \quad \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_{R,y}(x)u^2(x)) dx}{\int_{\mathbb{R}^N} V_{R,y}^+(x)u^2(x) dx} \geq \varepsilon.$$

We claim that

$$\inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (V_1(x) + V_2(x-y))u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0.$$

We argue by contradiction and assume that there exists a sequence  $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - (V_1(x) + V_2(x-y))u_n^2(x)) dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla u_n(x)|^2 = 1.$$

From (32) and (31), we obtain, for  $n$  large enough,

$$\begin{aligned} \varepsilon &= \varepsilon \int_{\mathbb{R}^N} \left[ V_{R,y}^+(x) + W_1(x) \chi_{\mathbb{R}^N \setminus B(0,R)}(x) + W_2(x-y) \chi_{\mathbb{R}^N \setminus B(y,R)}(x) - V_{R,y}^-(x) \right] u_n^2(x) dx + o(1) \\ &\leq \int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx - \int_{\mathbb{R}^N} V_{R,y}(x)u_n^2(x) dx + \varepsilon S^{-1} \sum_{j=1}^2 \|W_j \chi_{\mathbb{R}^N \setminus B(0,R)}\|_{L^{N/2}(\mathbb{R}^N)} + o(1) \\ &\leq \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - (V_1(x) + V_2(x-y))u_n^2(x)) dx + \frac{(1+\varepsilon)}{S} \sum_{j=1}^2 \|W_j \chi_{\mathbb{R}^N \setminus B(0,R)}\|_{L^{N/2}(\mathbb{R}^N)} + o(1) \\ &\leq \frac{\varepsilon}{8}(1+\varepsilon) + o(1) \leq \frac{\varepsilon}{2}, \end{aligned}$$

which is a contradiction. The theorem is thereby proved.  $\square$

## 6. POSITIVITY OF MULTIPOLAR OPERATORS

This section is devoted to the proof of Theorem 1.1.

**Proof of Theorem 1.1.**

**Step 1 (Sufficiency.)** We prove the sufficient condition on the positivity of the quadratic form  $Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  applying an iterating process on the number of poles  $k$ . Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$ .

As observed in Section 1, if  $k = 2$  the claim is true for any choice of  $a_1, a_2$ . Suppose that the claim is true for  $k - 1$ , let us prove it for  $k$ . We may assume  $\lambda_k > 0$ , otherwise the proof is trivial. If  $\lambda_1, \dots, \lambda_k$  satisfy (3), then the same holds true for  $\lambda_1, \dots, \lambda_{k-1}$ . By the recursive assumption, there exists a configuration of poles  $\{a_1, \dots, a_{k-1}\}$  such that the quadratic form  $Q_{\lambda_1, \dots, \lambda_{k-1}, a_1, \dots, a_{k-1}}$  associated to the operator

$$L_{\lambda_1, \dots, \lambda_{k-1}, a_1, \dots, a_{k-1}} = -\Delta - \sum_{i=1}^{k-1} \frac{\lambda_i}{|x - a_i|^2}$$

is positive definite.

We claim that there exists  $a_k \in \mathbb{R}^N$  such that the quadratic form associated to the operator  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  is positive definite. Indeed, the two potentials

$$V_1(x) = \sum_{i=1}^{k-1} \frac{\lambda_i}{|x - a_i|^2}, \quad V_2(x) = \frac{\lambda_k}{|x|^2},$$

belong to the class  $\mathcal{V}$  and satisfy the assumptions of Theorem 1.5, which ensures the existence of  $a_k \in \mathbb{R}^N$  such that the quadratic form associated to the operator

$$L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k} = -\Delta - (V_1 + V_2(\cdot - a_k))$$

is positive definite.

**Step 2 (Necessity.)** Assume that for some configuration  $\{a_1, \dots, a_k\}$  and for some  $\varepsilon > 0$  there holds

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x - a_i|^2} dx \geq \varepsilon \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Arguing by contradiction, suppose that, for some  $i$ ,  $\lambda_i \geq (\frac{N-2}{2})^2$ . Let  $\delta \in (0, \varepsilon(N-2)^2/4)$ . By optimality of the best constant in the Hardy inequality (2) and by density of  $C_c^\infty(\mathbb{R}^N)$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \lambda_i \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx < \delta \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx.$$

The rescaled function  $\phi_\mu(x) = \mu^{-(N-2)/2} \phi(x/\mu)$  satisfies

$$\begin{aligned} Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(\phi_\mu(x - a_i)) &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \lambda_i \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx - \sum_{j \neq i} \lambda_j \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x - \frac{a_j - a_i}{\mu}|^2} dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \lambda_i \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx + o(1), \quad \text{as } \mu \rightarrow 0. \end{aligned}$$



Letting  $\mu \rightarrow 0$ , by Hardy's inequality, we obtain

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx &\leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \lambda_i \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx \\ &< \delta \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx \leq \frac{4\delta}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \end{aligned}$$

thus giving rise to a contradiction.

Suppose now that,  $\Lambda := \sum_{i=1}^k \lambda_i \geq \left(\frac{N-2}{2}\right)^2$ . Let  $\delta \in (0, \varepsilon(N-2)^2/4)$ . As above, there exists  $\phi \in C_c^\infty(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \Lambda \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx < \delta \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx.$$

The rescaled function  $\phi_\mu(x) = \mu^{-(N-2)/2} \phi(x/\mu)$  satisfies

$$\begin{aligned} Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(\phi_\mu(x)) &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x - a_i/\mu|^2} dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \Lambda \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx + o(1), \quad \text{as } \mu \rightarrow \infty, \end{aligned}$$

see [19, Proposition 3.1]. Letting  $\mu \rightarrow \infty$  and arguing as above, we obtain easily a contradiction.  $\square$

## 7. BEST CONSTANTS IN HARDY MULTIPOLAR INEQUALITIES

The classical Hardy's inequality states that  $\mu(|x|^{-2}) = 1 - \frac{4}{(N-2)^2}$  is not attained. On the other hand, when dealing with multipolar Hardy-type potentials, a balance between positive and negative interactions between the poles can lead to attainability of the best constant in the Hardy-type inequality associated to the multisingular potential  $V \in \mathcal{V}$ :

$$(33) \quad \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx \leq (1 - \mu(V)) \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We recall that, in view of Lemma 1.2, the best constant in inequality (33) can be estimated by terms of the best constant in the inequality associated to the potential with one singularity located at the pole carrying the largest mass, i.e.  $\mu(V) \leq 1 - \frac{4}{(N-2)^2} \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}$ . We now prove attainability of  $\mu(V)$  when it stays strictly below the bound provided in Lemma 1.2.

**Proof of Proposition 1.7.** Let us denote

$$\bar{\lambda} = \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}.$$

From assumption (5), there exists  $\alpha > 0$  such that  $\mu(V) = 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \alpha$ . From Lemma 1.3, there exists  $\delta > 0$  such that

$$(34) \quad \mu(\tilde{V}) \geq 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2},$$

where  $\tilde{V}(x) = \sum_{i=1}^k \lambda_i \chi_{B(a_i, \delta)} |x - a_i|^{-2} + \lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)} |x|^{-2}$ . We notice that we can split  $V = \tilde{V} + \tilde{W}$  for some  $\tilde{W} \in L^{N/2}(\mathbb{R}^N)$ . Let  $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  be a minimizing sequence for  $\mu(V)$ , namely

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx = \mu(V) + o(1) \quad \text{as } n \rightarrow \infty.$$

Being  $\{u_n\}_n$  bounded in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , we can assume that, up to a subsequence still denoted as  $u_n$ ,  $u_n$  converges to some  $u$  a.e. and weakly in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . Since

$$\mu(\tilde{V}) \leq \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx + \int_{\mathbb{R}^N} \tilde{W}(x)u_n^2(x) dx = \mu(V) + \int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx + o(1)$$

as  $n \rightarrow \infty$ , from (34) and the choice of  $\alpha$  it follows that

$$1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2} \leq \mu(V) + \int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx = 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \alpha + \int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx,$$

hence  $\int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx \geq \frac{\alpha}{2} > 0$ , thus implying  $u \not\equiv 0$ . From weak convergence of  $u_n$  to  $u$ , we deduce that

$$\begin{aligned} & \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} \\ &= \frac{[\int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx] - [\int_{\mathbb{R}^N} (|\nabla(u_n - u)(x)|^2 - V(x)(u_n - u)^2(x)) dx] + o(1)}{\int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)} \\ &\leq \mu(V) \frac{1 - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)}{1 - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)} = \mu(V) \left[ 1 + \frac{o(1)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx + o(1)} \right] \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain that  $u$  attains the infimum defining  $\mu(V)$ .  $\square$

As a consequence of the attainability of  $\mu(V)$ , a result of continuity follows.

**Lemma 7.1.** *Let*

$$\begin{aligned} V(x) &= \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, r_i)}(x)}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2} + W(x) \in \mathcal{V}, \\ V_n(x) &= \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i^n, r_i)}(x)}{|x - a_i^n|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2} + W_n(x) \in \mathcal{V} \end{aligned}$$

be such that  $a_i^n \rightarrow a_i$  as  $n \rightarrow \infty$ , for all  $i = 1, \dots, k$ , and  $W_n \rightarrow W$  in  $L^{N/2}(\mathbb{R}^N)$ . Then

$$\lim_{n \rightarrow \infty} \mu(V_n) = \mu(V).$$

**PROOF.** Let  $\bar{\lambda} = \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}$ . For any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , Lemma A.3 and the strong  $L^{N/2}$ -convergence of  $W_n$  imply that  $\int_{\mathbb{R}^N} V_n(x)u^2(x) dx \rightarrow \int_{\mathbb{R}^N} V(x)u^2(x) dx$ , hence

$$\mu(V_n) \leq \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_n(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} = \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx + o(1)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx}$$

for any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ . Therefore, letting  $n \rightarrow \infty$  and taking infimum over  $\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ , we obtain that

$$(35) \quad \limsup_{n \rightarrow \infty} \mu(V_n) \leq \mu(V).$$

In particular, the sequence  $\{\mu(V_n)\}_n$  is bounded. We now claim that

$$(36) \quad \mu(V) = \liminf_{n \rightarrow \infty} \mu(V_n).$$

Indeed, let  $\{\mu(V_{n_j})\}_j$  be a subsequence such that  $\lim_j \mu(V_{n_j}) = \liminf_{n \rightarrow \infty} \mu(V_n)$  and suppose, by contradiction, that  $\lim_j \mu(V_{n_j}) < \mu(V) - \alpha$ , for some  $\alpha > 0$ .

In view of Lemma 1.3, see also Remark 3.4, there exists  $\delta > 0$  independent of  $n$  such that

$$(37) \quad \mu(\tilde{V}_n) \geq 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2}, \quad \mu(\tilde{V}) \geq 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2},$$

where

$$\tilde{V}_n(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i^n, \delta)}}{|x - a_i^n|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}}{|x|^2}, \quad \tilde{V}(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, \delta)}}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}}{|x|^2}.$$

We can write  $V_n = \tilde{V}_n + \tilde{W}_n$  and  $V = \tilde{V} + \tilde{W}$ , where

$$\tilde{W}_n = W_n + \sum_{i=1}^k \frac{\lambda_i}{|x - a_i^n|^2} \chi_{B(a_i^n, r_i) \setminus B(a_i^n, \delta)} \quad \text{and} \quad \tilde{W} = W + \sum_{i=1}^k \frac{\lambda_i}{|x - a_i|^2} \chi_{B(a_i, r_i) \setminus B(a_i, \delta)}.$$

By the Dominated Convergence Theorem we deduce that  $\tilde{W}_n \rightarrow \tilde{W}$  in  $L^{N/2}(\mathbb{R}^N)$ .

From Lemma 1.2, we have that, for large  $j$ ,

$$(38) \quad \mu(V_{n_j}) < \mu(V) - \alpha < 1 - \frac{4\bar{\lambda}}{(N-2)^2},$$

hence, by Proposition 1.7,  $\mu(V_{n_j})$  is attained by some  $\varphi_j \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  satisfying

$$\int_{\mathbb{R}^N} |\nabla \varphi_j(x)|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (|\nabla \varphi_j(x)|^2 - V_{n_j}(x) \varphi_j^2(x)) dx = \mu(V_{n_j}).$$

Moreover  $\varphi_j$  satisfies the equation

$$(39) \quad -\Delta \varphi_j(x) - V_{n_j}(x) \varphi_j(x) = -\mu(V_{n_j}) \Delta \varphi_j(x).$$

Since  $\{\varphi_j\}_j$  is bounded in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists a subsequence, still denoted as  $\{\varphi_j\}_j$ , weakly converging to some  $\varphi$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . From (38) and (37) it follows that

$$\begin{aligned} 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2} &\leq \mu(\tilde{V}_{n_j}) \leq \int_{\mathbb{R}^N} (|\nabla \varphi_j(x)|^2 - V_{n_j}(x) \varphi_j^2(x)) dx + \int_{\mathbb{R}^N} \tilde{W}_{n_j}(x) \varphi_j^2(x) dx \\ &= \mu(V_{n_j}) + \int_{\mathbb{R}^N} \tilde{W}_{n_j}(x) \varphi_j^2(x) dx < \mu(V) - \alpha + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx + o(1), \end{aligned}$$

as  $j \rightarrow \infty$ . Letting  $j \rightarrow \infty$ , we obtain that

$$1 - \frac{4\bar{\lambda}}{(N-2)^2} - \frac{\alpha}{2} \leq \mu(V) - \alpha + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx \leq 1 - \frac{4\bar{\lambda}}{(N-2)^2} - \alpha + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx,$$

yielding  $\int_{\mathbb{R}^N} \widetilde{W}(x)\varphi^2(x) dx \geq \frac{\alpha}{2} > 0$  and hence  $\varphi \not\equiv 0$ . We claim that

$$(40) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \widetilde{V}_{n_j}(x)\varphi_j(x)v(x) dx = \int_{\mathbb{R}^N} \widetilde{V}(x)\varphi(x)v(x) dx \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Indeed for any  $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  and  $\varepsilon > 0$ , by density there exists  $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  such that  $\|v - \psi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} < \varepsilon$ . Since  $\psi$  lies far away from the singularities, from Hardy's inequality we have that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \widetilde{V}_{n_j}(x)\varphi_j(x)v(x) dx - \int_{\mathbb{R}^N} \widetilde{V}(x)\varphi(x)v(x) dx \right| \\ & \leq \text{const } \varepsilon + \left| \int_{\mathbb{R}^N} (\widetilde{V}_{n_j} - \widetilde{V})(x)\varphi_j(x)\psi(x) dx \right| \leq \text{const } \varepsilon + o(1) \quad \text{as } j \rightarrow \infty. \end{aligned}$$

(40) is thereby proved. From (40) and strong  $L^{N/2}$ -convergence of  $\widetilde{W}_n$  to  $\widetilde{W}$ , we can multiply (39) by  $\varphi$  and pass to limit as  $j \rightarrow \infty$  thus obtaining

$$\int_{\mathbb{R}^N} (|\nabla\varphi(x)|^2 - V(x)\varphi^2(x)) dx = \liminf_{n \rightarrow \infty} \mu(V_n) \int_{\mathbb{R}^N} |\nabla\varphi(x)|^2 dx,$$

and consequently  $\liminf_{n \rightarrow \infty} \mu(V_n) \geq \mu(V)$ , a contradiction. Claim (36) is thereby proved. The conclusion follows from (35) and (36).  $\square$

**Remark 7.2.** We emphasize that  $\mu : L^{N/2,\infty}(\mathbb{R}^N) \rightarrow \mathbb{R}$ ,  $\mu : V \mapsto \mu(V)$  is continuous with respect to the  $L^{N/2,\infty}$ -norm. In particular the first  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -eigenvalue  $\mu(V)$  is continuous not only with respect to the location of the singularities but also with respect to their masses  $\lambda_i$ 's.

**Remark 7.3.** We notice that if  $V_n \in \mathcal{V}$  converge to  $V \in \mathcal{V}$  in the sense that the poles of  $V_n$  converge to the poles of  $V$  (i.e. in the sense of Lemma 7.1), then  $\|V_n - V\|_{L^{N/2,\infty}(\mathbb{R}^N)}$  does not tend to zero. On the other hand  $\mu(V_n) \rightarrow \mu(V)$ . In other words, the first  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -eigenvalue is stable with respect to small perturbations of configurations of poles, even though such perturbations make the  $L^{N/2,\infty}$ -distance between the potentials far away from zero.

**Remark 7.4.** Since operators with potentials in  $\mathcal{V}$  are compact perturbations of positive operators, see Lemma 1.4, we can prove, for the first weighted eigenvalue  $\nu_p(V)$  defined in (4), the same type of continuity result as in Lemma 7.1. Moreover, it is not difficult to check that the infimum defining  $\nu_p$  is attained, for any  $p \in L^{N/2}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ ,  $p > 0$ . As a consequence, for any  $V \in \mathcal{V}$ , there holds

$$\text{sgn } \nu_p(V) = \text{sgn } \mu(V), \quad \text{where} \quad \text{sgn } t := \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases}$$

Indeed, it is obvious that  $\nu_p(V) < 0$  if and only if  $\mu(V) < 0$ . If  $\mu(V) > 0$ , from Hölder's and Sobolev's inequalities it immediately follows that  $\nu_p(V) > 0$ . Assume now that  $\nu_p(V) > 0$ . By continuity, we can find  $\varepsilon > 0$  such that  $\nu_p(V + \varepsilon V^+) > 0$  is attained, thus providing a positive function satisfying condition (ii) of Lemma 2.1, which yields the positivity of  $\mu(V)$ .

## 8. ESSENTIAL SELF-ADJOINTNESS

The Shattering Lemma 1.3 reveals how Schrödinger operators with potentials lying in the class  $\mathcal{V}$  are actually compact perturbations of positive operators, see Lemma 1.4. Hence they are *semi-bounded symmetric operators* and their  $L^2(\mathbb{R}^N)$ -spectrum is bounded from below. Consequently the class  $\mathcal{V}$  provides us with a good framework to study the spectral properties of multisingular Schrödinger operators in  $L^2(\mathbb{R}^N)$ .

For any  $V \in \mathcal{V}$ , let us discuss essential self-adjointness of the operator  $-\Delta - V$  on the domain  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ . In the case of just one singularity (i.e.  $k = 1$ ), a complete answer to this problem is contained in a theorem due to Kalf, Schmincke, Walter, and Wüst [27] and Simon [42] (see also [39, Theorems X.11 and X.30]):

**Theorem 8.1.** [Kalf, Schmincke, Walter, Wüst, Simon] *Let  $V(x) = \frac{\lambda}{|x|^2} + W(x)$ ,  $W \in L^\infty(\mathbb{R}^N)$ . The operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{0\})$  if and only if  $\lambda \leq (N - 2)^2/4 - 1$ .*

We are now going to extend the above result to potentials lying in the class  $\mathcal{V}$ , for which we give below a *self-adjointness criterion*. According to Lemma 1.4, we can split any  $V \in \mathcal{V}$  as  $V(x) = \tilde{V}(x) + \tilde{W}(x)$  where

$$(41) \quad \tilde{V}(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} - \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0)}(x)}{|x|^2}, \quad \delta > 0, \quad R_0 > 0, \quad \mu(\tilde{V}) > 0,$$

and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ .

**Lemma 8.2.** [Self-adjointness criterion in  $\mathcal{V}$ ] *Let  $V \in \mathcal{V}$  and  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (41) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Then the operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  if and only if  $\text{Range}(-\Delta - \tilde{V} + b)$  is dense in  $L^2(\mathbb{R}^N)$  for some  $b > 0$ .*

PROOF. For any  $b > 0$ , we can split the operator  $-\Delta - V$  as  $(-\Delta - \tilde{V} + b) - (\tilde{W} + b)$ , i.e. as a bounded perturbation of the positive operator  $-\Delta - \tilde{V} + b$ . In view of the Kato-Rellich Theorem (see e.g. [28, Theorem 4.4]), the operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  if and only if  $-\Delta - \tilde{V} + b$  is essentially self-adjoint for some  $b > 0$ . The conclusion now follows from well-known self-adjointness criteria for positive operators (see [39, Theorem X.26]).  $\square$

The above criterion provides the following *non self-adjointness condition* in  $\mathcal{V}$ .

**Corollary 8.3.** *Let  $V \in \mathcal{V}$  and  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (41) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Assume that there exist  $v \in L^2(\mathbb{R}^N)$ ,  $v(x) \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\int_{\mathbb{R}^N} v^2 > 0$ , a distribution  $h \in H^{-1}(\mathbb{R}^N)$ , and  $b > 0$  such that*

$$(42) \quad {}_{H^{-1}(\mathbb{R}^N)} \langle h, u \rangle_{H^1(\mathbb{R}^N)} \leq 0 \quad \text{for all } u \in H^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

and

$$(43) \quad -\Delta v - \tilde{V} v + b v = h \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}).$$

*Then the operator  $-\Delta - V$  is not essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ .*

PROOF. From Lemma 8.2 and the Kato-Rellich Theorem, it is enough to prove that  $\text{Range}(-\Delta - \tilde{V} + b)$  is not dense in  $L^2(\mathbb{R}^N)$ . To this aim we will show that  $v$  does not belong to the closure of  $\text{Range}(-\Delta - \tilde{V} + b)$  in  $L^2(\mathbb{R}^N)$ . Arguing by contradiction, we assume that there exist sequences  $\{u_n\}_n \subset C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  and  $\{f_n\}_n \subset L^2(\mathbb{R}^N)$  such that  $f_n \rightarrow v$  in  $L^2(\mathbb{R}^N)$  and

$$(44) \quad -\Delta u_n(x) - \tilde{V}(x)u_n(x) + bu_n(x) = f_n(x).$$

In view of the Lax-Milgram Theorem there exists  $u \in H^1(\mathbb{R}^N)$ , weakly solving

$$(45) \quad -\Delta u(x) - \tilde{V}(x)u(x) + bu(x) = v(x).$$

Testing (45) with  $-u^-$ , we easily obtain that  $u \geq 0$  a.e. in  $\mathbb{R}^N$ , hence by the Strong Maximum Principle we deduce that  $u > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ . Subtracting (45) from (44) and multiplying by  $u_n - u$ , we find that  $\|u_n - u\|_{H^1(\mathbb{R}^N)} \leq \text{const} \|f_n - v\|_{L^2(\mathbb{R}^N)}$ , hence  $u_n \rightarrow u$  in  $H^1(\mathbb{R}^N)$ . Testing (43) with  $u_n$  and using (44), we obtain

$$H^{-1}(\mathbb{R}^N) \langle h, u_n \rangle_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f_n(x)v(x) dx,$$

which, passing to the limit, yields

$$H^{-1}(\mathbb{R}^N) \langle h, u \rangle_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} v^2(x) dx > 0.$$

The above identity contradicts assumption (42).  $\square$

We now extend Theorem 8.1 to our class of multi-polar potentials, thus proving Theorem 1.8.

### Proof of Theorem 1.8.

**Step 1:** if  $\lambda_i < (N-2)^2/4 - 1$  for all  $i = 1, \dots, k$ , then  $-\Delta - V$  is essentially self-adjoint.

In view of Lemma 8.2, to prove essential self-adjointness it is enough to show that  $\text{Range}(-\Delta - \tilde{V} + b)$  is dense in  $L^2(\mathbb{R}^N)$  for some  $b > 0$ , where  $\tilde{V}$  is as in (41). Let  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  and  $b > 0$ . By the Lax-Milgram Theorem, there exists  $u \in H^1(\mathbb{R}^N)$  weakly solving

$$-\Delta u(x) - \tilde{V}(x)u(x) + bu(x) = f(x) \quad \text{in } \mathbb{R}^N.$$

From Lemma 3.1 we deduce the following asymptotic behavior of  $u$  at poles

$$(46) \quad u(x) \sim |x - a_i|^{-a\lambda_i}, \quad \text{as } x \rightarrow a_i.$$

Hence the function  $g(x) := \tilde{V}(x)u(x) - bu(x) + f(x) \sim |x - a_i|^{-a\lambda_i - 2}$  as  $x \rightarrow a_i$ . In particular, if  $\lambda_i < (N-2)^2/4 - 1$  for all  $i = 1, \dots, k$ , then  $g \in L^2(\mathbb{R}^N)$ . Green's representation formula yields

$$(47) \quad u(x) = \frac{1}{N(N-2)\omega_N} \left[ \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy + \int_{\partial B(a_i, \delta)} \frac{1}{|x-y|^{N-2}} \frac{\partial u}{\partial \nu} ds \right] \\ + \frac{1}{N\omega_N} \int_{\partial B(a_i, \delta)} \frac{u(y)}{|x-y|^{N-1}} ds, \quad x \in B(a_i, \delta),$$

where  $\omega_N$  denotes the volume of the unit ball in  $\mathbb{R}^N$ ,  $\nu$  is the unit outward normal to  $\partial B(a_i, \delta)$ , and  $ds$  indicates the  $(N-1)$ -dimensional area element in  $\partial B(a_i, \delta)$ . It is easy to verify that the functions

$$x \mapsto \int_{\partial B(a_i, \delta)} \frac{1}{|x-y|^{N-2}} \frac{\partial u}{\partial \nu} ds(y), \quad x \mapsto \int_{\partial B(a_i, \delta)} \frac{u(y)}{|x-y|^{N-1}} ds(y),$$

are of class  $C^1(B(a_i, \delta))$ . From Lemma A.1 of the appendix, we have that

$$\nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) = -\frac{1}{N\omega_N} \int_{B(a_i, \delta)} \frac{x-y}{|x-y|^N} g(y) dy.$$

Consequently

$$(48) \quad \left| \nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) \right| \leq \text{const} \int_{B(a_i, \delta)} \frac{|y-a_i|^{-a_{\lambda_i}-2}}{|x-y|^{N-1}} dy.$$

If  $\lambda_i > 1 - N$ , i.e.  $a_{\lambda_i} > -1$ , then

$$\left| \nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) \right| \leq \text{const} h_i(x - a_i),$$

where

$$h_i(x) = \int_{\mathbb{R}^N} \frac{|y|^{-a_{\lambda_i}-2}}{|x-y|^{N-1}} dy.$$

An easy scaling argument shows that  $h_i(\alpha x) = \alpha^{-a_{\lambda_i}-1} h_i(x)$  for all  $\alpha > 0$ , hence  $h_i(x) = |x|^{-a_{\lambda_i}-1} h_i(e_1)$ , where  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$ . Then, if  $\lambda_i > 1 - N$ ,

$$(49) \quad \left| \nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) \right| \leq \text{const} |x - a_i|^{-a_{\lambda_i}-1}.$$

If  $\lambda_i \leq 1 - N$ , i.e.  $a_{\lambda_i} \leq -1$ , we fix  $0 < \varepsilon < \frac{N-2}{2}$  and notice that, from (48),

$$\left| \nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) \right| \leq \delta^{-a_{\lambda_i}-1+\varepsilon} k_i(x),$$

where

$$k_i(x) = \int_{\mathbb{R}^N} \frac{1}{|y-a_i|^{1+\varepsilon} |y-x|^{N-1}} dy.$$

An easy scaling argument shows that  $k_i(\alpha x) = \alpha^{-\varepsilon} k_i(x)$  for all  $\alpha > 0$ , hence  $k_i(x) = |x|^{-\varepsilon} k_i(e_1)$ . Then, if  $\lambda_i \leq 1 - N$

$$(50) \quad \left| \nabla \left( \frac{1}{N(N-2)\omega_N} \int_{B(a_i, \delta)} \frac{g(y)}{|x-y|^{N-2}} dy \right) \right| \leq C(\varepsilon) |x - a_i|^{-\varepsilon},$$

for some positive constant  $C(\varepsilon)$  depending on  $\varepsilon$  (and also on  $N$ ,  $\lambda_i$ , and  $u$ ). Representation (47), regularity of the boundary terms, and estimates (49–50) yield

$$(51) \quad \nabla u(x) = \begin{cases} O(|x - a_i|^{-a_{\lambda_i}-1}), & \text{if } \lambda_i > 1 - N, \\ O(|x - a_i|^{-\varepsilon}), & \text{if } \lambda_i \leq 1 - N, \end{cases} \quad \text{as } x \rightarrow a_i.$$

For all  $n \in \mathbb{N}$  let  $\eta_n$  be a cut-off function such that  $\eta_n \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ ,  $0 \leq \eta_n \leq 1$ , and

$$\begin{aligned} \eta_n(x) &\equiv 0 \text{ in } \bigcup_{i=1}^k B\left(a_i, \frac{1}{2n}\right) \cup (\mathbb{R}^N \setminus B(0, 2n)), \quad \eta_n(x) \equiv 1 \text{ in } B(0, n) \setminus \bigcup_{i=1}^k B\left(a_i, \frac{1}{n}\right), \\ |\nabla \eta_n(x)| &\leq Cn \text{ in } \bigcup_{i=1}^k \left(B\left(a_i, \frac{1}{n}\right) \setminus B\left(a_i, \frac{1}{2n}\right)\right), \quad |\nabla \eta_n(x)| \leq \frac{C}{n} \text{ in } B(0, 2n) \setminus B(0, n), \\ |\Delta \eta_n(x)| &\leq Cn^2 \text{ in } \bigcup_{i=1}^k \left(B\left(a_i, \frac{1}{n}\right) \setminus B\left(a_i, \frac{1}{2n}\right)\right), \quad |\Delta \eta_n(x)| \leq \frac{C}{n^2} \text{ in } B(0, 2n) \setminus B(0, n), \end{aligned}$$

for some positive constant  $C$  independent of  $n$ . Let us set  $f_n := \eta_n f - 2\nabla \eta_n \cdot \nabla u - u \Delta \eta_n$  and  $u_n := \eta_n u$ , so that  $u_n \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  and  $-\Delta u_n(x) - \tilde{V}(x)u_n(x) + b u_n(x) = f_n(x)$ . In particular  $f_n \in \text{Range}(-\Delta - \tilde{V} + b)$ . Furthermore  $\eta_n f \rightarrow f$  in  $L^2(\mathbb{R}^N)$ , while (46) and (51) yield

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \eta_n(x)|^2 |\nabla u(x)|^2 dx \\ &\leq \text{const } n^2 \sum_{i=1}^k \int_{B(a_i, \frac{1}{n}) \setminus B(a_i, \frac{1}{2n})} |\nabla u(x)|^2 dx + \frac{\text{const}}{n^2} \int_{B(0, 2n) \setminus B(0, n)} |\nabla u(x)|^2 dx \\ &\leq \text{const } n^2 \left[ \sum_{\lambda_i > 1-N} \int_{B(0, \frac{1}{n})} |x|^{-2a_{\lambda_i} - 2} dx + \sum_{\lambda_i \leq 1-N} \int_{B(0, \frac{1}{n})} |x|^{-2\varepsilon} dx \right] + \frac{\text{const}}{n^2} \|u\|_{H^1(\mathbb{R}^N)} \\ &\leq \text{const} \left[ \sum_{\lambda_i > 1-N} n^{2a_{\lambda_i} + 4 - N} + \sum_{\lambda_i \leq 1-N} n^{2+2\varepsilon - N} + n^{-2} \right] \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} |\Delta \eta_n(x)|^2 |u(x)|^2 dx \\ &\leq \text{const } n^4 \sum_{i=1}^k \int_{B(a_i, \frac{1}{n}) \setminus B(a_i, \frac{1}{2n})} |u(x)|^2 dx + \frac{\text{const}}{n^4} \int_{B(0, 2n) \setminus B(0, n)} |u(x)|^2 dx \\ &\leq \text{const } n^4 \sum_{i=1}^k \int_{B(0, 1/n)} |x|^{-2a_{\lambda_i}} dx + \frac{\text{const}}{n^4} \|u\|_{H^1(\mathbb{R}^N)} \\ &\leq \text{const} \left[ \sum_{i=1}^k n^{2a_{\lambda_i} + 4 - N} + n^{-4} \right]. \end{aligned}$$

Since for  $\lambda_i < (N-2)^2/4 - 1$  there holds  $2a_{\lambda_i} + 4 - N < 0$ , we conclude that  $f_n \rightarrow f$  in  $L^2(\mathbb{R}^N)$ . Hence  $\text{Range}(-\Delta - \tilde{V} + b)$  is dense in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ . Since  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  is dense in  $L^2(\mathbb{R}^N)$ , we obtain that  $\text{Range}(-\Delta - \tilde{V} + b)$  is dense in  $L^2(\mathbb{R}^N)$ .

**Step 2:** if  $\lambda_i \leq (N-2)^2/4 - 1$  for all  $i \in \{1, \dots, k\}$ , then  $-\Delta - V$  is essentially self-adjoint.

Let us fix  $b > 0$ ,  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ . To prove essential self-adjointness it is enough to find some  $g \in \text{Range}(-\Delta - \tilde{V} + b)$  such that  $g$  is arbitrarily closed to  $f$  in  $L^2(\mathbb{R}^N)$ . To this aim,



we fix  $\varepsilon > 0$  and notice that there exists  $0 < \sigma < 1$  such that if  $u \in H^1(\mathbb{R}^N)$  solves

$$(52) \quad -\Delta u(x) - \tilde{V}_\sigma(x)u(x) + bu(x) = f(x),$$

where

$$\tilde{V}_\sigma(x) := \sum_{\lambda_i = \left(\frac{N-2}{2}\right)^2 - 1} \frac{(\lambda_i - \sigma) \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} + \sum_{\lambda_i < \left(\frac{N-2}{2}\right)^2 - 1} \frac{\lambda_i \chi_{B(a_i, \delta)}(x)}{|x - a_i|^2} - \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R_0/\delta)}(x)}{|x|^2},$$

then

$$(53) \quad \|(\tilde{V}_\sigma - \tilde{V})u\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

Indeed, by Remark 3.2, there exists a positive constant  $C$  independent on  $\sigma \in (0, 1)$ , such that all solutions of (52) can be estimated as

$$|u(x)| \leq C |x - a_i|^{-a(\lambda_i - \sigma)} \|u\|_{H^1(\mathbb{R}^N)} \quad \text{in } B(a_i, \delta),$$

for all  $i$  such that  $\lambda_i = \left(\frac{N-2}{2}\right)^2 - 1$ . Moreover, testing (52) by  $u$  there results that all solutions of (52) satisfy

$$\|u\|_{H^1(\mathbb{R}^N)} \leq \frac{\|f\|_{L^2(\mathbb{R}^N)}}{\min\{\mu(\tilde{V}), b\}}.$$

Then for all  $i$  such that  $\lambda_i = \left(\frac{N-2}{2}\right)^2 - 1$ , we have that

$$\begin{aligned} \left\| \frac{\sigma \chi_{B(a_i, \delta)}(x) u}{|x - a_i|^2} \right\|_{L^2(\mathbb{R}^N)}^2 &\leq C^2 \sigma^2 \|u\|_{H^1(\mathbb{R}^N)}^2 \int_0^1 r^{N-5-2a(\lambda_i - \sigma)} dr \\ &= \frac{C^2 \|f\|_{L^2(\mathbb{R}^N)}^2}{2(\min\{\mu(\tilde{V}), b\})^2} \frac{\sigma^2}{\sqrt{1 + \sigma} - 1} = o(1) \quad \text{as } \sigma \rightarrow 0. \end{aligned}$$

Therefore it is possible to choose  $\sigma$  small enough in order to ensure that all solutions of (52) satisfy (53). For such a  $\sigma$ , let  $u \in H^1(\mathbb{R}^N)$  be a solution to (52). Let  $\eta_n$  be the sequence of cut-off functions introduced in step 1. As in step 1, we have that  $f_n := \eta_n f - 2\nabla\eta_n \cdot \nabla u - u \Delta\eta_n$  converges to  $f$  in  $L^2(\mathbb{R}^N)$ . Hence, for every  $n$  large enough,  $\|f_n - f\|_{L^2(\mathbb{R}^N)} < \varepsilon$ . Moreover  $u_n := \eta_n u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  and

$$\|(\tilde{V}_\sigma - \tilde{V})u_n\|_{L^2(\mathbb{R}^N)} \leq \|(\tilde{V}_\sigma - \tilde{V})u\|_{L^2(\mathbb{R}^N)} < \varepsilon$$

for any  $n$ . Setting  $g_n(x) := f_n(x) + (\tilde{V}_\sigma(x) - \tilde{V}(x))u_n(x)$ , we obtain that  $u_n$  satisfies

$$-\Delta u_n(x) - \tilde{V}(x)u_n(x) + bu_n(x) = g_n(x),$$

i.e.  $g_n \in \text{Range}(-\Delta - \tilde{V} + b)$ , and  $\|g_n - f\|_{L^2(\mathbb{R}^N)} < 2\varepsilon$  for large  $n$ . The proof of step 2 is thereby complete.

**Step 3:** if  $\lambda_i > (N-2)^2/4 - 1$  for some  $i \in \{1, \dots, k\}$ , then  $-\Delta - V$  is not essentially self-adjoint. Let  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (41) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Let us fix  $i \in \{1, \dots, k\}$  such that  $\lambda_i > (N-2)^2/4 - 1$  and  $\alpha < 0$ , and consider the solution  $\psi \in C^1((-\infty, \ln \delta])$  of the Cauchy problem

$$\begin{cases} \psi''(s) - \omega_{\lambda_i}^2 \psi(s) = b e^{2s} \psi(s), \\ \psi(\ln \delta) = 0, \quad \psi'(\ln \delta) = \alpha, \end{cases}$$

where  $\omega_{\lambda_i} := \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda_i}$  and  $\delta$  is as in (41). In view of Lemma A.2 of the appendix, we can estimate  $\psi$  as

$$(54) \quad 0 \leq \psi(s) \leq C e^{-\omega_{\lambda_i} s} \quad \text{for all } s \leq \ln \delta,$$

for some positive constant  $C = C(\lambda_i, \delta, \alpha, b)$ . Let us set

$$v(x) := \begin{cases} |x - a_i|^{-\frac{N-2}{2}} \psi(\ln |x - a_i|), & \text{if } x \in B(a_i, \delta) \setminus \{a_i\}, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \overline{B(a_i, \delta)}. \end{cases}$$

From (54) we infer that

$$(55) \quad 0 \leq v(x) \leq C |x - a_i|^{-\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda_i}} \quad \text{in } B(a_i, \delta).$$

The assumption  $\lambda_i > (N-2)^2/4 - 1$  and estimate (55) ensure that  $v \in L^2(\mathbb{R}^N)$ . Moreover the restriction of  $v$  to  $B(a_i, \delta)$  satisfies

$$\begin{cases} -\Delta v(x) - \frac{\lambda_i}{|x - a_i|^2} v(x) + b v(x) = 0, & \text{in } B(a_i, \delta), \\ v = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \delta^{-\frac{N}{2}} \alpha, & \text{on } \partial B(a_i, \delta). \end{cases}$$

As a consequence the distribution  $-\Delta v - \tilde{V} v + b v \in \mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  acts as follows:

$$\mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}) \langle -\Delta v - \tilde{V} v + b v, \varphi \rangle_{C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})} = \delta^{-\frac{N}{2}} \alpha \int_{\partial B(a_i, \delta)} \varphi(x) ds.$$

Hence  $h = -\Delta v - \tilde{V} v + b v \in H^{-1}(\mathbb{R}^N)$  and satisfies (42) as  $\alpha < 0$ . From Corollary 8.3, we finally deduce that the operator  $-\Delta - V$  is not essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ .  $\square$

The following theorem characterizes essential self-adjointness of Schrödinger operators with potentials carrying infinitely many singularities distributed on reticular structures.

**Theorem 8.4.** *For  $\lambda < (N-2)^2/4$  and  $\{a_n\}_n \subset \mathbb{R}^N$  satisfying (19) and  $|a_n - a_m| \geq 1$  for all  $n \neq m$ , let  $0 < \delta < 1/2$  be given by Lemma 3.5 and*

$$V(x) = \lambda \sum_{n=1}^{\infty} \frac{\chi_{B(a_n, \delta)}(x)}{|x - a_n|^2}.$$

*Then  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  if and only if  $\lambda \leq (N-2)^2/4 - 1$ .*

**PROOF.** From the Kato-Rellich Theorem the operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  if and only if  $-\Delta - V + b$  is essentially self-adjoint for some  $b > 0$ . In view of Lemma 3.5, for any  $b > 0$ ,  $-\Delta - V + b$  is positive. Hence essential self-adjointness is equivalent to density of  $\text{Range}(-\Delta - V + b)$  in  $L^2(\mathbb{R}^N)$  for some  $b > 0$ .

Let us first prove that, if  $\lambda < (N-2)^2/4 - 1$ , then  $-\Delta - V$  is essentially self-adjoint. For  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and  $b > 0$ , the Lax-Milgram Theorem provides a unique  $u \in H^1(\mathbb{R}^N)$  weakly solving

$$-\Delta u(x) - V(x)u(x) + b u(x) = f(x) \quad \text{in } \mathbb{R}^N.$$

From Lemma 3.1 and arguing as in the proof of Theorem 1.8, we deduce that

$$(56) \quad u(x) \sim |x - a_n|^{-a_\lambda}, \quad \text{and} \quad \nabla u(x) = \begin{cases} O(|x - a_n|^{-a_\lambda - 1}), & \text{if } \lambda > 1 - N, \\ O(|x - a_n|^{-\varepsilon}), & \text{if } \lambda \leq 1 - N, \end{cases} \quad \text{as } x \rightarrow a_n,$$

where  $0 < \varepsilon < \frac{N-2}{2}$ . Since  $\lambda < (N-2)^2/4 - 1$ , we have that  $2a_\lambda + 4 - N < 0$ , hence, for all  $j \in \mathbb{N}$ , we can choose  $N_j \in \mathbb{N}$  such that  $N_j \rightarrow +\infty$  as  $j \rightarrow \infty$ ,  $N_j j^{2a_\lambda + 4 - N} \rightarrow 0$ , and  $N_j j^{2\varepsilon - N + 2} \rightarrow 0$ , and let  $R_j > 0$  such that  $R_j \rightarrow +\infty$  as  $j \rightarrow \infty$  and  $B(a_n, 1/j) \subset B(0, R_j)$  for all  $n = 1, \dots, N_j$ . Let  $\eta_j$  be a cut-off function such that  $\eta_j \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ ,  $0 \leq \eta_j \leq 1$ , and

$$\begin{aligned} \eta_j(x) &\equiv 0 \text{ in } \bigcup_{n=1}^{N_j} B\left(a_n, \frac{1}{2j}\right) \cup (\mathbb{R}^N \setminus B(0, 2R_j)), \quad \eta_j(x) \equiv 1 \text{ in } B(0, R_j) \setminus \bigcup_{n=1}^{N_j} B\left(a_n, \frac{1}{j}\right), \\ |\nabla \eta_j(x)| &\leq Cj \text{ in } \bigcup_{n=1}^{N_j} \left(B\left(a_n, \frac{1}{j}\right) \setminus B\left(a_n, \frac{1}{2j}\right)\right), \quad |\nabla \eta_j(x)| \leq \frac{C}{R_j} \text{ in } B(0, 2R_j) \setminus B(0, R_j), \\ |\Delta \eta_j(x)| &\leq Cj^2 \text{ in } \bigcup_{n=1}^{N_j} \left(B\left(a_n, \frac{1}{j}\right) \setminus B\left(a_n, \frac{1}{2j}\right)\right), \quad |\Delta \eta_j(x)| \leq \frac{C}{R_j^2} \text{ in } B(0, 2R_j) \setminus B(0, R_j), \end{aligned}$$

for some positive constant  $C$  independent of  $j$  and  $n$ . Let  $f_j := \eta_j f - 2\nabla \eta_j \cdot \nabla u - u \Delta \eta_j$  and  $u_j := \eta_j u$ , so that  $u_j \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and  $-\Delta u_j(x) - V(x)u_j(x) + b u_j(x) = f_j(x)$ . In particular  $f_j \in \text{Range}(-\Delta - V + b)$ . Furthermore  $\eta_j f \rightarrow f$  in  $L^2(\mathbb{R}^N)$ , while (56) yields

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \eta_j(x)|^2 |\nabla u(x)|^2 dx \\ &\leq \text{const } j^2 \sum_{n=1}^{N_j} \int_{B(a_n, \frac{1}{j}) \setminus B(a_n, \frac{1}{2j})} |\nabla u(x)|^2 dx + \frac{\text{const}}{R_j^2} \int_{B(0, 2R_j) \setminus B(0, R_j)} |\nabla u(x)|^2 dx \\ &\leq \begin{cases} \text{const } [N_j j^{2a_\lambda + 4 - N} + R_j^{-2} \|u\|_{H^1(\mathbb{R}^N)}], & \text{if } \lambda > 1 - N \\ \text{const } [N_j j^{2\varepsilon - N + 2} + R_j^{-2} \|u\|_{H^1(\mathbb{R}^N)}], & \text{if } \lambda \leq 1 - N \end{cases} \end{aligned}$$

and, in a similar way,

$$\int_{\mathbb{R}^N} |\Delta \eta_j(x)|^2 |u(x)|^2 dx \leq \text{const } [N_j j^{2a_\lambda + 4 - N} + R_j^{-4}].$$

By the choice of  $N_j$ , we deduce that  $f_j \rightarrow f$  in  $L^2(\mathbb{R}^N)$ . Hence  $\text{Range}(-\Delta - V + b)$  is dense in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and consequently in  $L^2(\mathbb{R}^N)$ .

To prove essential self-adjointness for  $\lambda = (N-2)^2/4 - 1$ , we can argue as in the proof of Theorem 1.8, Step 2, i.e. by approximation of the resonant potential  $V$  with sub-resonant potentials. To do that, we need to prove that for fixed  $b > 0$ ,  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ , and  $\varepsilon > 0$ , there exists  $0 < \sigma < 1$  such that if  $u \in H^1(\mathbb{R}^N)$  solves

$$(57) \quad -\Delta u(x) - V_\sigma(x)u(x) + b u(x) = f(x), \quad \text{where } V_\sigma(x) := (\lambda - \sigma) \sum_{n=1}^{\infty} \frac{\chi_{B(a_n, \delta)}(x)}{|x - a_n|^2},$$

then

$$(58) \quad \|(V_\sigma - V)u\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

Indeed, by Remark 3.2, there exists a positive constant  $C$  independent on  $\sigma \in (0, 1)$  and  $n \in \mathbb{N}$ , such that all solutions of (57) can be estimated as

$$|u(x)| \leq C |x - a_n|^{-a(\lambda - \sigma)} \|u\|_{H^1(B(a_n, \delta'))} \quad \text{in } B(a_n, \delta),$$

for some  $\delta < \delta' < 1/2$  and for all  $n \in \mathbb{N}$ . Consequently

$$\left\| \frac{\sigma \chi_{B(a_n, \delta)}(x) u}{|x - a_n|^2} \right\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{C^2}{2} \|u\|_{H^1(B(a_n, \delta'))}^2 \frac{\sigma^2}{\sqrt{1 + \sigma} - 1},$$

and hence

$$\begin{aligned} \|(V_\sigma - V)u\|_{L^2(\mathbb{R}^N)} &\leq \frac{C}{\sqrt{2}} \frac{\sigma}{\sqrt{\sqrt{1 + \sigma} - 1}} \|u\|_{H^1(\mathbb{R}^N)} \\ &\leq \text{const} \|f\|_{L^2(\mathbb{R}^N)} \frac{\sigma}{\sqrt{\sqrt{1 + \sigma} - 1}} \rightarrow 0 \quad \text{as } \sigma \rightarrow 0. \end{aligned}$$

Therefore it is possible to choose  $\sigma$  small enough in order to ensure that all solutions of (57) satisfy (58). In order to prove self-adjointness, it is now sufficient to repeat the argument of Theorem 1.8, Step 2.

The proof of non essential self-adjointness in the case  $\lambda > (N - 2)^2/4 - 1$  can be obtained just by mimicking the arguments of the proof of Theorem 1.8 and Corollary 8.3.  $\square$

## 9. SPECTRUM OF SCHRÖDINGER OPERATORS WITH POTENTIALS IN $\mathcal{V}$

In this section we study the spectrum of the *Friedrichs extension*  $(-\Delta - V)^F$  of Schrödinger operators with potentials in  $\mathcal{V}$ , see (7). We recall that in view of Theorem 1.8, if  $\lambda_i \leq (N - 2)^2/4 - 1$  for all  $i = 1, \dots, k$ , then  $(-\Delta - V)^F$  is the only self-adjoint extension of  $-\Delta - V$ . On the other hand, if  $\lambda_i > (N - 2)^2/4 - 1$  for some  $i$ , then  $-\Delta - V$  has many self-adjoint extensions, among which the Friedrichs extension is the only one with domain included in  $H^1(\mathbb{R}^N)$ . Due to self-adjointness, the spectrum of  $(-\Delta - V)^F$  turns out to be a subset of  $\mathbb{R}$ , which will be described below.

**9.1. Essential spectrum.** Let us start by studying the essential spectrum of the Friedrichs extension of operators  $-\Delta - V$ ,  $V \in \mathcal{V}$ , in  $L^2(\mathbb{R}^N)$ . The Friedrichs extension  $(-\Delta - V)^F : D((-\Delta - V)^F) \rightarrow L^2(\mathbb{R}^N)$  defined in (7) is self-adjoint. As a consequence, the essential spectrum  $\sigma_{\text{ess}}((-\Delta - V)^F)$  can be characterized by terms of the *Weyl sequences* as follows:  $\lambda \in \sigma_{\text{ess}}((-\Delta - V)^F)$  if and only if

$$(59) \quad \begin{cases} \text{there exists } \{f_n\}_n \subset D((-\Delta - V)^F) \text{ such that } \liminf_{n \rightarrow +\infty} \|f_n\|_{L^2(\mathbb{R}^N)} > 0, \\ f_n \rightarrow 0 \text{ weakly in } L^2(\mathbb{R}^N), \text{ and } \|-\Delta f_n - V f_n - \lambda f_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0, \end{cases}$$

see [12, p. 167].

### Proof of Proposition 1.9.1.

**Step 1:**  $[0, +\infty) \subseteq \sigma_{\text{ess}}(-\Delta - V)$ . Let  $\lambda \geq 0$ . It is well known that  $\sigma_{\text{ess}}(-\Delta) = [0, +\infty)$ , where

$$-\Delta : D(-\Delta) = H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N).$$

Hence  $\lambda \in \sigma_{\text{ess}}(-\Delta)$  and the characterization given in (59) yields a sequence  $\{f_n\}_n \subset H^2(\mathbb{R}^N)$ , such that  $\|f_n\|_{L^2(\mathbb{R}^N)} = 1$ ,  $f_n \rightharpoonup 0$  weakly in  $L^2(\mathbb{R}^N)$  and  $\|-\Delta f_n - \lambda f_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0$ . By density of  $C_c^\infty(\mathbb{R}^N)$  in  $H^2(\mathbb{R}^N)$ , for any  $n$  there exists  $g_n \in C_c^\infty(\mathbb{R}^N)$  such that  $\|g_n - f_n\|_{H^2(\mathbb{R}^N)} \leq 1/n$ . It is easy to verify that  $g_n \rightharpoonup 0$  weakly in  $L^2(\mathbb{R}^N)$ ,  $1/2 \leq \|g_n\|_{L^2(\mathbb{R}^N)} \leq 2$  for sufficiently large  $n$ , and  $\|-\Delta g_n - \lambda g_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0$ . Let us choose a sequence  $\{x_n\}_n \subset \mathbb{R}^N$  such that

$$(60) \quad \text{supp } \varphi_n \subset \mathbb{R}^N \setminus B(0, n), \quad \text{where } \varphi_n(x) := g_n(x + x_n).$$

By (60), it is easy to prove that  $\varphi_n \rightharpoonup 0$  weakly in  $L^2(\mathbb{R}^N)$  and  $1/2 \leq \|\varphi_n\|_{L^2(\mathbb{R}^N)} = \|g_n\|_{L^2(\mathbb{R}^N)} \leq 2$ . From (60), it follows also that, if  $n$  is sufficiently large, the support of  $\varphi_n$  is disjoint from all balls  $B(a_i, r_i)$  where singularities of  $V$  are located. Therefore

$$V\varphi_n = \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R)}}{|x|^2} \varphi_n + W\varphi_n \in L^2(\mathbb{R}^N)$$

and hence  $\varphi_n \in D((-\Delta - V)^F)$ .

Furthermore, letting  $h_n = -\Delta\varphi_n - \lambda\varphi_n$ , we have that  $\|h_n\|_{L^2(\mathbb{R}^N)} = \|-\Delta g_n - \lambda g_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0$ , hence

$$\int_{\mathbb{R}^N} |\nabla\varphi_n(x)|^2 dx = \lambda \int_{\mathbb{R}^N} \varphi_n^2(x) dx + \int_{\mathbb{R}^N} h_n(x)\varphi_n(x) dx \leq 4(\lambda + o(1)), \quad \text{as } n \rightarrow +\infty.$$

Since  $\varphi_n$  is bounded in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and  $W \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , from Sobolev's inequality we obtain

$$(61) \quad \|W\varphi_n\|_{L^2(\mathbb{R}^N)}^2 \leq S^{-1} \|W\|_{L^\infty(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N \setminus B(0, n)} |W(x)|^{N/2} dx \right)^{2/N} \int_{\mathbb{R}^N} |\nabla\varphi_n(x)|^2 dx \rightarrow 0$$

as  $n \rightarrow +\infty$ . Moreover

$$(62) \quad \left\| \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B(0, R)}}{|x|^2} \varphi_n \right\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{\lambda_\infty^2}{n^4} \|\varphi_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

From (61–62) we deduce that  $\lim_{n \rightarrow +\infty} \|V\varphi_n\|_{L^2(\mathbb{R}^N)} = 0$ . As a consequence

$$\|-\Delta\varphi_n - V\varphi_n - \lambda\varphi_n\|_{L^2(\mathbb{R}^N)} \leq \|h_n\|_{L^2(\mathbb{R}^N)} + \|V\varphi_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0$$

as  $n \rightarrow +\infty$ . Thus,  $\{\varphi_n\}_n$  is a Weyl's sequence and  $\lambda \in \sigma_{\text{ess}}((-\Delta - V)^F)$ .

**Step 2:**  $\sigma_{\text{ess}}((-\Delta - V)^F) \subseteq [0, +\infty)$ . Assume now that  $\lambda \in \sigma_{\text{ess}}((-\Delta - V)^F)$ . Then, from (59) there exists a sequence  $\{f_n\}_n \subset D((-\Delta - V)^F)$  such that  $\|f_n\|_{L^2(\mathbb{R}^N)} = 1$ ,  $f_n \rightharpoonup 0$  weakly in  $L^2(\mathbb{R}^N)$ , and  $h_n := -\Delta f_n - V f_n - \lambda f_n \rightarrow 0$  strongly in  $L^2(\mathbb{R}^N)$ . By Lemma 1.4, we can write  $V(x) = \tilde{V}(x) + \tilde{W}(x)$  where  $\mu(\tilde{V}) > 0$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Hence

$$\begin{aligned} \mu(\tilde{V}) \int_{\mathbb{R}^N} |\nabla f_n(x)|^2 dx &\leq \int_{\mathbb{R}^N} (|\nabla f_n(x)|^2 - \tilde{V}(x)|f_n(x)|^2) dx \\ &= \lambda \int_{\mathbb{R}^N} |f_n(x)|^2 dx + \int_{\mathbb{R}^N} \tilde{W}(x)|f_n(x)|^2 dx + \int_{\mathbb{R}^N} h_n(x)f_n(x) dx \\ &\leq \lambda + \|\tilde{W}\|_{L^\infty(\mathbb{R}^N)} + o(1) \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Being  $\{f_n\}_n$  bounded in  $H^1(\mathbb{R}^N)$ , there exists  $f \in H^1(\mathbb{R}^N)$  such that, up to a subsequence,  $f_n \rightharpoonup f$  weakly in  $H^1(\mathbb{R}^N)$ . Weak convergence of  $f_n$  to 0 in  $L^2(\mathbb{R}^N)$  implies that  $f = 0$ , hence  $f_n \rightharpoonup 0$  weakly in  $H^1(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$ . For any measurable set  $\omega$ , Sobolev's inequality implies

$$\int_{\omega} \widetilde{W}(x) f_n^2(x) dx \leq S^{-1} \left( \int_{\omega} |\widetilde{W}(x)|^{N/2} dx \right)^{2/N} \int_{\mathbb{R}^N} |\nabla f_n(x)|^2 dx \leq \text{const} \left( \int_{\omega} |\widetilde{W}(x)|^{N/2} dx \right)^{2/N},$$

hence the integral in left hand side goes to zero both for the Lebesgue measure of  $\omega$  tending to 0 and for  $\omega$  being the complement of balls with radius tending to  $+\infty$ . As a consequence, the Vitali's Convergence Theorem yields

$$(63) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \widetilde{W}(x) f_n^2(x) dx = 0.$$

From (63) and the strong convergence of  $h_n$  to 0 in  $L^2(\mathbb{R}^N)$ , we obtain

$$\begin{aligned} -\lambda &\leq \mu(\widetilde{V}) \int_{\mathbb{R}^N} |\nabla f_n(x)|^2 dx - \lambda \int_{\mathbb{R}^N} |f_n(x)|^2 dx \\ &\leq \int_{\mathbb{R}^N} (|\nabla f_n(x)|^2 - \widetilde{V}(x) |f_n(x)|^2) dx - \lambda \int_{\mathbb{R}^N} |f_n(x)|^2 dx \\ &= \int_{\mathbb{R}^N} \widetilde{W}(x) |f_n(x)|^2 dx + \int_{\mathbb{R}^N} h_n(x) f_n(x) dx = o(1) \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Letting  $n \rightarrow +\infty$ , we obtain  $\lambda \geq 0$ .  $\square$

**Remark 9.1. [Essential spectrum in the case of infinitely many reticular singularities]**  
For  $\lambda < (N-2)^2/4$ , let  $\{a_n\}_n \subset \mathbb{R}^N$  be a sequence of poles located on a periodic  $M$ -dimensional reticular structure,  $M < N-2$ . As observed in Remark 3.7, (19) is satisfied and Lemma 3.5 yields  $\delta > 0$  such that the quadratic form associated to the infinitely singular operator  $-\Delta - V$ ,  $V(x) = \lambda \sum_{n=1}^{\infty} |x - a_n|^{-2} \chi_{B(a_n, \delta)}(x)$ , is positive definite in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . Since the reticulation does not fill the whole  $\mathbb{R}^N$ , we can repeat the translation argument in Step 1 of the proof of Proposition 1.9.1 to construct Weyl's sequences. In addition, the positivity of the quadratic form allows us to mimic the procedure developed in Step 2, thus obtaining that the essential spectrum of the Friedrichs extension  $(-\Delta - V)^F$  is given by the half line  $[0, +\infty)$ .

**9.2. Discrete spectrum.** If  $\nu_1(V) < 0$ , then the spectrum of  $(-\Delta - V)^F$  below 0, namely the discrete spectrum

$$\sigma_d((-\Delta - V)^F) := \sigma((-\Delta - V)^F) \setminus \sigma_{\text{ess}}((-\Delta - V)^F) = \sigma((-\Delta - V)^F) \cap (-\infty, 0),$$

is not empty and is described as a sequence of eigenvalues

$$\nu_1(V) < \nu_2(V) < \dots < \nu_k(V) \dots$$

which admit the following variational characterization:

$$\nu_k(V) := \inf_{u \in E^k \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |u(x)|^2 dx}, \quad k = 1, 2, \dots,$$

where

$$E^k = \left\{ w \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} w(x)v_i(x) dx = 0 \quad \text{for } i = 1, \dots, k-1 \right\}$$

and  $\{v_i, i = 1, \dots, k-1\}$ , are the first  $k-1$  eigenfunctions. The following corollary of Lemma 1.4 states that whenever  $\nu_1(V) < 0$ , then it is attained. The corresponding eigenfunction thus provides a *bound state* in  $L^2(\mathbb{R}^N)$  with negative energy.

**Corollary 9.2.** *If  $V \in \mathcal{V}$  and  $\nu_1(V) < 0$ , then  $\nu_1(V)$  is attained.*

PROOF. In view of Lemma 1.4, we can write  $V$  as  $V(x) = \tilde{V}(x) + \tilde{W}(x)$  where  $\mu(\tilde{V}) > 0$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Let  $\{u_n\}_n \subset H^1(\mathbb{R}^N)$  be a minimizing sequence such that

$$\int_{\mathbb{R}^N} |u_n(x)|^2 dx = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx = \nu_1(V).$$

Since

$$\begin{aligned} \mu(\tilde{V}) \int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx &\leq \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - \tilde{V}(x)u_n^2(x)) dx \\ &= \nu_1(V) + \int_{\mathbb{R}^N} \tilde{W}(x)|u_n(x)|^2 dx + o(1) \leq \nu_1(V) + \|\tilde{W}\|_{L^\infty(\mathbb{R}^N)} + o(1) \end{aligned}$$

as  $n \rightarrow +\infty$ , we obtain that  $\{u_n\}_n$  is bounded in  $H^1(\mathbb{R}^N)$ , hence, up to a subsequence,  $u_n \rightharpoonup u$  weakly in  $H^1(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$ . Vitali's Convergence Theorem easily yields

$$\int_{\mathbb{R}^N} \tilde{W}(x)|u_n(x)|^2 dx \rightarrow \int_{\mathbb{R}^N} \tilde{W}(x)|u(x)|^2 dx.$$

Therefore, taking into account that  $\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - \tilde{V}(x)u^2(x)) dx$  is an equivalent norm, we deduce

$$\begin{aligned} (64) \quad &\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - \tilde{V}(x)u^2(x)) dx - \int_{\mathbb{R}^N} \tilde{W}(x)|u(x)|^2 dx \\ &\leq \liminf_{n \rightarrow +\infty} \left( \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - \tilde{V}(x)u_n^2(x)) dx \right) - \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \tilde{W}(x)|u_n(x)|^2 dx = \nu_1(V) < 0. \end{aligned}$$

Hence  $u \not\equiv 0$ . Then from (6) and (64) it follows

$$(65) \quad \nu_1(V) \int_{\mathbb{R}^N} |u(x)|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx \leq \nu_1(V).$$

Hence  $\int_{\mathbb{R}^N} |u(x)|^2 dx \geq 1$ . On the other hand, by weakly lower semi-continuity of the  $L^2$ -norm, we have that  $\int_{\mathbb{R}^N} |u(x)|^2 dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |u_n(x)|^2 dx = 1$ . Therefore  $\int_{\mathbb{R}^N} |u(x)|^2 dx = 1$  and, from (65),

$$\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx = \nu_1(V),$$

i.e.  $u$  attains the infimum in (6). □

Fix an integer  $k \geq 1$ . Arguing as above, Lemma 1.4 allows us to prove that whenever  $\nu_k(V) < 0$ , then it is attained, thus providing a *bound state* in  $L^2(\mathbb{R}^N)$  with negative energy.

**Corollary 9.3.** *If  $V \in \mathcal{V}$  and  $\nu_k(V) < 0$ , then  $\nu_k(V)$  is attained.*

**Proof of Proposition 1.9.2.** Since operators  $-\Delta - V$ ,  $V \in \mathcal{V}$ , are  $L^{N/2}$ -perturbations of positive operators (see Lemma 1.4), from the Cwikel-Lieb-Rosenblum inequality ([9, 31, 40]) it follows that the number of negative eigenvalues is finite. Hence the conclusion follows from Corollaries 9.2 and 9.3.  $\square$

**9.3. Eigenvalues at the bottom of the essential spectrum.** We now mean to study the nature of the bottom of essential spectrum of operators  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  defined in (1). More precisely, when the values of  $\lambda_i$ 's admit both configurations of poles corresponding to negative quadratic forms and configurations corresponding to positive quadratic forms, we will provide a necessary and sufficient condition on the masses of singularities for the existence of a configuration of  $a_i$ 's admitting a bound state with null energy.

Let  $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$  fixed. We denote as  $\Sigma$  the set of colliding configurations, namely

$$\Sigma := \{(a_1, \dots, a_k) \in \mathbb{R}^{Nk} : a_i = a_j \text{ for some } i \neq j\}.$$

For any  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^{Nk} \setminus \Sigma$ , we introduce the following notation

$$\mu_{\mathbf{a}} := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}(u)}{\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2}.$$

The following result is a direct corollary of Lemma 7.1.

**Corollary 9.4.** *For  $(\lambda_1, \dots, \lambda_k) \in (-\infty, (N-2)^2/4)^k$ , let  $\mathbf{a}_n \in \mathbb{R}^{Nk}$  be a sequence of configurations converging to  $\mathbf{a} \in \mathbb{R}^{Nk} \setminus \Sigma$ . Then  $\lim_{n \rightarrow \infty} \mu_{\mathbf{a}_n} = \mu_{\mathbf{a}}$ .*

Let us denote

$$A^+ := \{\mathbf{a} \in \mathbb{R}^{Nk} \setminus \Sigma : \mu_{\mathbf{a}} > 0\}, \quad A^- := \{\mathbf{a} \in \mathbb{R}^{Nk} \setminus \Sigma : \mu_{\mathbf{a}} < 0\}, \quad \text{and } A^0 := \{\mathbf{a} \in \mathbb{R}^{Nk} \setminus \Sigma : \mu_{\mathbf{a}} = 0\}.$$

From Corollary 9.4, it follows that  $A^+$  and  $A^-$  are open sets. Hence, whenever both  $A^+$  and  $A^-$  are nonempty, the set  $A^0$  is nonempty and disconnects  $\mathbb{R}^{Nk} \setminus \Sigma$ .

**Proof of Theorem 1.10.** Let us assume that the  $\lambda_i$ 's satisfy (3) and (8). From Theorem 1.1 and (3), there exists a configuration of poles  $\mathbf{a}^+ = (a_1^+, \dots, a_k^+)$  such that  $\mu_{\mathbf{a}^+} > 0$ . On the other hand, in [19, Proposition 1.2] it is proved that, if  $\sum_{i=1}^k \lambda_i^+ > \frac{(N-2)^2}{4}$ , then it is possible to find a configuration of poles  $\mathbf{a}^- = (a_1^-, \dots, a_k^-)$  such that  $\mu_{\mathbf{a}^-} < 0$ . It is worth noticing that, from the proofs of Theorem 1.1 and [19, Proposition 1.2], there easily results that  $\mathbf{a}^+$  and  $\mathbf{a}^-$  can be chosen to be collisionless, i.e.  $\mathbf{a}^+, \mathbf{a}^- \in \mathbb{R}^{Nk} \setminus \Sigma$ . From Corollary 9.4,  $\mathbf{a} \mapsto \mu_{\mathbf{a}} > 0$  is continuous on  $\mathbb{R}^{Nk} \setminus \Sigma$ , which is a connected open subset of  $\mathbb{R}^{Nk}$ . Therefore there exist  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^{Nk} \setminus \Sigma$  such that  $\mu_{\mathbf{a}} = 0$ . From Proposition 1.7, it follows that  $\mu_{\mathbf{a}} = 0$  is attained by some  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  weakly solving in  $\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$

$$(66) \quad -\Delta u(x) - \sum_{i=1}^k \frac{\lambda_i}{|x - a_i|^2} u(x) = 0.$$

By evenness we can assume  $u \geq 0$ , while the Strong Maximum Principle and standard regularity theory ensure that  $u$  is smooth and strictly positive in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ . Lemma 3.3 yields a precise



estimate of the decay of  $u$  at infinity, i.e.  $u(x) \sim |x|^{-(N-2-a_{\lambda_\infty})}$  as  $|x| \rightarrow \infty$ , where  $\lambda_\infty = \sum_{i=1}^k \lambda_i$ . As a consequence

$$u \in L^2(\mathbb{R}^N) \quad \text{if and only if} \quad \sum_{i=1}^k \lambda_i < \frac{(N-2)^2}{4} - 1.$$

Hence, under assumption (8), any function  $u$  attaining  $\mu_{\mathbf{a}}$  provides an eigenfunction of the Schrödinger operator  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$  associated to the null eigenvalue.

Let us now prove the necessity of condition (8). If  $\sum_{i=1}^k \lambda_i \geq \frac{(N-2)^2}{4} - 1$ , Lemma 3.3 implies that, for any  $\mathbf{a} \in \mathbb{R}^{Nk}$ , (66) cannot have any nontrivial nonnegative solution in  $H^1(\mathbb{R}^N)$ . On the other hand, if  $\sum_{i=1}^k \lambda_i^+ \leq \frac{(N-2)^2}{4}$ , we distinguish two cases:

**Case 1:**  $\sum_{i=1}^k \lambda_i^+ < \frac{(N-2)^2}{4}$ . In this case, [19, Proposition 1.2] ensures that  $\mu_{\mathbf{a}} > 0$  for any  $\mathbf{a} \in \mathbb{R}^{Nk} \setminus \Sigma$  and hence the only  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -solution to (66) is the null one.

**Case 2:**  $\sum_{i=1}^k \lambda_i^+ = \frac{(N-2)^2}{4}$ . In this case, assumption (3) implies that there exists at least one index  $i$  such that  $\lambda_i < 0$ . Arguing by contradiction, assume that (66) admits a nontrivial  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -solution  $u$  to (66) for some  $(a_1, \dots, a_k) \in \mathbb{R}^{Nk} \setminus \Sigma$ . We have that

$$\begin{aligned} 0 &= \left(1 - \frac{4}{(N-2)^2} \left(\sum_{i=1}^k \lambda_i^+\right)\right) \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2} dx = - \sum_{i=1}^k \lambda_i^- \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2} dx < 0, \end{aligned}$$

which is a contradiction.

In both cases, we have proved that, for any  $(a_1, \dots, a_k) \in \mathbb{R}^{Nk} \setminus \Sigma$ , (66) admits no nontrivial  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -solutions. In particular, for any configuration of singularities, 0 is not an eigenvalue of the Friedrichs extension of  $L_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$ .  $\square$

## APPENDIX

We collect in this appendix some technical results used in the paper. In the following lemma (which was needed in the proof of Theorem 1.8), we extend to  $L^2$  (not necessarily bounded) functions a well-known property of differentiability of *Newtonian potentials*, see [23, Lemma 4.1, p. 54].

**Lemma A.1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded smooth domain,  $p \in \Omega$ ,  $g \in L^2(\Omega)$ ,  $g$  smooth in  $\Omega \setminus \{p\}$ , and let  $u$  be the Newtonian potential of  $g$ , i.e.*

$$u(x) = \frac{1}{N(2-N)\omega_N} \int_{\Omega} \frac{g(y)}{|x-y|^{N-2}} dy, \quad x \in \mathbb{R}^N \setminus \{p\}.$$

*Then  $u \in W^{1,q}(\mathbb{R}^N)$  for all  $q \in \left(\frac{N}{N-2}, \frac{2N}{N-2}\right]$  and the weak derivatives of  $u$  are given by*

$$\frac{\partial u}{\partial x_i}(x) = \frac{1}{N\omega_N} \int_{\Omega} \frac{g(y)(x_i - y_i)}{|x-y|^N} dy, \quad x \in \mathbb{R}^N \setminus \{p\}.$$

PROOF. Let  $\tilde{g} \in L^2(\mathbb{R}^N)$  be such that  $\tilde{g}(y) = 0$  in  $\mathbb{R}^N \setminus \overline{\Omega}$  and  $\tilde{g}|_{\Omega} = g$ . Note that  $u = I_2(\tilde{g})$ , where  $I_2(\tilde{g})$  is the Riesz potential defined by

$$(I_2(\tilde{g}))(x) := \frac{1}{N(2-N)\omega_N} \int_{\mathbb{R}^N} \frac{\tilde{g}(y)}{|x-y|^{N-2}} dy.$$

For any  $1 < p < \min\{2, N/2\}$ , from [45, Theorem 1, p. 119] it is known that  $I_2$  is a linear bounded operator from  $L^p(\mathbb{R}^N)$  into  $L^{(pN)/(N-2p)}(\mathbb{R}^N)$ . It follows that  $u \in L^q(\mathbb{R}^N)$  for all  $q \in (\frac{N}{N-2}, \infty)$  if  $2 < N \leq 4$ ,  $q \in (\frac{N}{N-2}, \frac{2N}{N-4})$  if  $N > 4$ .

Let  $g_n \in C_c^\infty(\mathbb{R}^N)$  such that  $\text{supp } g_n \subset \Omega$  and  $g_n \rightarrow \tilde{g}$  in  $L^p(\mathbb{R}^N)$  for all  $p \in [1, 2]$  and set  $u_n = I_2(g_n)$ . Since for all  $1 < p < \min\{2, N/2\}$ ,  $g_n \rightarrow \tilde{g}$  in  $L^p(\mathbb{R}^N)$ , we have that  $u_n \rightarrow u$  in  $L^{(pN)/(N-2p)}(\mathbb{R}^N)$ , i.e.  $u_n \rightarrow u$  in  $L^q(\mathbb{R}^N)$  for all  $q \in (\frac{N}{N-2}, \infty)$  if  $2 < N \leq 4$ ,  $q \in (\frac{N}{N-2}, \frac{2N}{N-4})$  if  $N > 4$ . By [23, Lemma 4.1, p. 54],  $u_n \in C^1(\mathbb{R}^N)$  and

$$\frac{\partial u_n}{\partial x_i}(x) = I_1^i(g_n) := \frac{1}{N\omega_N} \int_{\mathbb{R}^N} \frac{g_n(y)(x_i - y_i)}{|x-y|^N} dy, \quad x \in \mathbb{R}^N, \quad i = 1, \dots, N.$$

From [45, Theorem 1, p. 119],  $I_1^i$  are linear bounded operators from  $L^p(\mathbb{R}^N)$  into  $L^{(pN)/(N-p)}(\mathbb{R}^N)$  for all  $p \in (1, 2]$ . Hence, for  $i = 1, \dots, N$ ,  $\frac{\partial u_n}{\partial x_i} \rightarrow I_1^i(\tilde{g})$  in  $L^{(pN)/(N-p)}(\mathbb{R}^N)$  for all  $p \in (1, 2]$ , i.e.

$$\frac{\partial u_n}{\partial x_i} \rightarrow \frac{1}{N\omega_N} \int_{\mathbb{R}^N} \frac{\tilde{g}(y)(x_i - y_i)}{|x-y|^N} dy, \quad \text{in } L^q(\mathbb{R}^N) \quad \text{for all } \frac{N}{N-1} < q \leq \frac{2N}{N-2}.$$

Therefore for all  $q \in (\frac{N}{N-2}, \frac{2N}{N-2}]$ ,  $u \in W^{1,q}(\mathbb{R}^N)$  and

$$\nabla u(x) = \frac{1}{N\omega_N} \int_{\Omega} \frac{g(y)(x-y)}{|x-y|^N} dy, \quad x \in \mathbb{R}^N \setminus \{p\}.$$

The proof is thereby complete.  $\square$

The following lemma was used in the proof of Theorem 1.8.

**Lemma A.2.** For  $\bar{s} \in \mathbb{R}$ ,  $\omega > 0$ ,  $b > 0$ , and  $\alpha < 0$ , let  $\psi \in C^1((-\infty, \bar{s}])$  be the solution of the following Cauchy problem

$$\begin{cases} \psi''(s) - \omega^2 \psi(s) = b e^{2s} \psi(s), \\ \psi(\bar{s}) = 0, \quad \psi'(\bar{s}) = \alpha. \end{cases}$$

Then

$$0 \leq \psi(s) \leq -\frac{\alpha}{2\omega} e^{\omega \bar{s}} \exp\left(\frac{b}{4\omega} e^{2\bar{s}}\right) e^{-\omega s} \quad \text{for all } s \leq \bar{s}.$$

PROOF. The initial conditions imply that  $\psi$  is positive in a left neighborhood of  $\bar{s}$ , whereas the equation forces the solution to be convex wherever it is positive. As a consequence  $\psi$  must be strictly positive in  $(-\infty, \bar{s})$ . We have that, for  $s \leq \bar{s}$ ,

$$\psi(s) = e^{-\omega s} \left[ -\frac{\alpha}{2\omega} e^{\omega \bar{s}} - \frac{b}{2\omega} \int_{\bar{s}}^s e^{\omega t} \psi(t) e^{2t} dt \right] + e^{\omega s} \left[ \frac{\alpha}{2\omega} e^{-\omega \bar{s}} + \frac{b}{2\omega} \int_{\bar{s}}^s e^{-\omega t} \psi(t) e^{2t} dt \right].$$

For any  $\tau \geq 0$ , set  $f(\tau) := e^{\omega(\bar{s}-\tau)} \psi(\bar{s}-\tau)$ , hence

$$f(\tau) = -\frac{\alpha}{2\omega} e^{\omega \bar{s}} (1 - e^{-2\omega \tau}) + \frac{b}{2\omega} \int_0^\tau (1 - e^{2\omega(t-\tau)}) f(t) e^{2(\bar{s}-t)} dt.$$

Since the function  $f \in C^1([0, +\infty))$  can be estimated as

$$f(\tau) \leq -\frac{\alpha}{2\omega} e^{\omega\bar{s}} + \frac{b}{2\omega} e^{2\bar{s}} \int_0^\tau f(t) e^{-2t} dt,$$

the Gronwall's Lemma yields

$$f(\tau) \leq -\frac{\alpha}{2\omega} e^{\omega\bar{s}} \exp\left(\frac{b}{2\omega} e^{2\bar{s}} \int_0^\tau e^{-2t} dt\right) \leq -\frac{\alpha}{2\omega} e^{\omega\bar{s}} \exp\left(\frac{b}{4\omega} e^{2\bar{s}}\right)$$

for all  $\tau \geq 0$ , thus proving the required estimate.  $\square$

We now give a result of continuity of Hardy integrals with respect to poles which was used in the proof of Lemma 7.1.

**Lemma A.3.** *For any  $a \in \mathbb{R}^N$ ,  $r > 0$ , and  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , there holds*

$$\lim_{y \rightarrow a} \int_{B(y,r)} \frac{u^2(x)}{|x-y|^2} dx = \int_{B(a,r)} \frac{u^2(x)}{|x-a|^2} dx.$$

PROOF. For any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $u \geq 0$  a.e., we consider the Schwarz symmetrization of  $u$  defined as

$$(1) \quad u^*(x) := \inf \{t > 0 : |\{y \in \mathbb{R}^N : u(y) > t\}| \leq \omega_N |x|^N\}$$

where  $|\cdot|$  denotes the Lebesgue measure of  $\mathbb{R}^N$  and  $\omega_N$  is the volume of the standard unit  $N$ -ball. For any  $\Omega \subset \mathbb{R}^N$  and for any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , let  $\Omega^* = B(0, (|\Omega|/\omega_N)^{1/N})$  and  $|u|^*$  denote the Schwarz symmetrization of  $|u|$ , see (1). From [49, Theorem 21.8] and since  $(1/|x-y|)^* = 1/|x|$ , for any  $y \in \mathbb{R}^N$ , it follows that

$$(2) \quad \int_{\Omega \cap B(y,r)} \frac{u^2}{|x-y|^2} dx \leq \int_{\Omega^* \cap B(0,r)} \frac{(|u|^*)^2}{|x|^2} dx.$$

Let  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ . It is easy to see that

$$\frac{u^2 \chi_{B(y,r)}}{|x-y|^2} \text{ converges to } \frac{u^2 \chi_{B(a,r)}}{|x-a|^2} \text{ a.e. in } \mathbb{R}^N \text{ as } y \rightarrow a.$$

Moreover, from (2), it follows that the family of functions  $\left\{ \frac{u^2 \chi_{B(y,r)}}{|x-y|^2} : y \in \mathbb{R}^N \right\}$  is equi-integrable. Hence Vitali's convergence Theorem allows to conclude.  $\square$

## REFERENCES

- [1] B. Abdellaoui, V. Felli, I. Peral, *Existence and multiplicity for perturbations of an equation involving Hardy inequality and critical Sobolev exponent in the whole  $\mathbb{R}^N$* , Adv. Differential Equations, 9 (2004), 481–508.
- [2] W. Allegretto, *On the equivalence of two types of oscillation for elliptic operators*, Pacific J. Math., 55 (1974), no. 2, 319–328.
- [3] P. Baras, J. A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc., 284 (1984), no. 1, 121–139.
- [4] H. Berestycki, M. J. Esteban, *Existence and bifurcation of solutions for an elliptic degenerate problem*, J. Differential Equations, 134 (1997), no. 1, 1–25.
- [5] H. Brezis, M. Marcus, *Hardy's inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) 25 (1997), 217–237.
- [6] H. Brezis, M. Marcus, I. Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal. 171 (2000), no. 1, 177–191.

- [7] M. Conti, S. Crotti, D. Pardo, *On the existence of positive solutions for a class of singular elliptic equations*, Adv. Differential Equations, 3 (1998), no. 1, 111–132.
- [8] M. Conti, S. Terracini, G. Verzini, *Nodal solutions to a class of nonstandard superlinear equations on  $\mathbb{R}^N$* , Adv. Differential Equations, 7 (2002), no. 3, 297–318.
- [9] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. Math. (2), 106 (1977), no. 1, 93–100.
- [10] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer Study Edition, Springer-Verlag, Berlin, 1987.
- [11] I. Daubechies, E. Lieb, *One-electron relativistic molecules with Coulomb interaction*, Comm. Math. Phys., 90 (1983), no. 4, 497–510.
- [12] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, 42, Cambridge University Press, Cambridge, 1995.
- [13] E. B. Davies, B. Simon,  *$L^p$  norms of noncritical Schrödinger semigroups*, J. Funct. Anal., 102 (1991), no. 1, 95–115.
- [14] V. De Cicco, M. A. Vivaldi, *Harnack inequalities for Fuchsian type weighted elliptic equations*, Comm. Partial Differential Equations, 21 (1996), no. 9-10, 1321–1347.
- [15] T. Duyckaerts, *Inégalités de résolvante pour l'opérateur de Schrödinger avec potentiel multipolaire critique*, Bulletin de la Société mathématique de France, 134 (2006), 201–239.
- [16] H. Egnell, *Elliptic boundary value problems with singular coefficients and critical nonlinearities*, Indiana Univ. Math., J. 38 (1989), no. 2, 235–251.
- [17] A. Ferrero, F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations*, J. Differential Equations, 177 (2001), no. 2, 494–522.
- [18] V. Felli, M. Schneider, *A note on regularity of solutions to degenerate elliptic equations of Caffarelli-Kohn-Nirenberg type*, Adv. Nonlinear Stud., 3 (2003), no. 4, 431–443.
- [19] V. Felli, S. Terracini, *Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity*, Comm. Partial Differential Equations, 31 (2006), no. 1-3, 469–495.
- [20] V. Felli, S. Terracini, *Nonlinear Schrödinger equations with symmetric multi-polar potentials*, Calc. Var. Partial Differential Equations, 27 (2006), no. 1, 25–58.
- [21] W. M. Frank, D. J. Land, R. M. Spector, *Singular potentials*, Rev. Modern Phys., 43 (1971), no. 1, 36–98.
- [22] J. García Azorero, I. Peral, *Hardy Inequalities and some critical elliptic and parabolic problems*, J. Diff. Equations, 144 (1998), no. 2, 441–476.
- [23] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order. Second edition*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin, 1983.
- [24] C. E. Gutiérrez, *Harnack's inequality for degenerate Schrödinger operators*, Trans. Amer. Math. Soc., 312 (1989), no. 1, 403–419.
- [25] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Reprint of the 1952 edition, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.
- [26] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations, 156 (1999), no. 2, 407–426.
- [27] H. Kalf, U.-W. Schmincke, J. Walter, R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), pp. 182–226. Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975.
- [28] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
- [29] L. D. Landau, E. M. Lifshitz, *Quantum mechanics*, Pergamon Press Ltd., London-Paris, 1965.
- [30] J. M. Lévy-Leblond, *Electron capture by polar molecules*, Phys. Rev., 153 (1967), no. 1, 1–4.
- [31] E. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Amer. Math. Soc., 82 (1976), no. 5, 751–753.
- [32] M. Marcus, V. Mizel, Y. Pinchover, *On the best constant for Hardy's inequality in  $\mathbb{R}^n$* , Trans. Amer. Math. Soc. 350 (1998), no. 8, 3237–3255.
- [33] V. G. Maz'ya, I. E. Verbitsky, *The Schrödinger operator on the energy space: boundedness and compactness criteria*, Acta Math., 188 (2002), no. 2, 263–302.
- [34] M. Murata, *Structure of positive solutions to  $(-\Delta + V)u = 0$  in  $\mathbb{R}^n$* , Duke Math. J. 53 (1986), no. 4, 869–943.

- [35] Y. Pinchover, *On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994), no. 3, 313–341.
- [36] Y. Pinchover, *On the localization of binding for Schrödinger operators and its extension to elliptic operators*, J. Anal. Math., 66 (1995), 57–83.
- [37] Y. Pinchover, K. Tintarev, *A ground state alternative for singular Schrödinger operators*, J. Funct. Anal., 230 (2006), no. 1, 65–77.
- [38] J. Piepenbrink, *Nonoscillatory elliptic equations*, J. Differential Equations, 15 (1974), 541–550.
- [39] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York-London, 1975.
- [40] G. V. Rosenblum, *Distribution of the discrete spectrum of singular differential operators*, Soviet Math. Izv. VUZ, 20 (1976), 63–71.
- [41] D. Ruiz, M. Willem, *Elliptic problems with critical exponents and Hardy potentials*, J. Differential Equations, 190 (2003), no. 2, 524–538.
- [42] B. Simon, *Essential self-adjointness of Schrödinger operators with singular potentials*, Arch. Rational Mech. Anal. 52 (1973), 44–48.
- [43] B. Simon, *Brownian motion,  $L^p$  properties of Schrödinger operators and the localization of binding*, J. Funct. Anal., 35 (1980), no. 2, 215–229.
- [44] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities*, Trans. AMS, 357 (2005), 2909–2938.
- [45] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.
- [46] S. Terracini, *On positive entire solutions to a class of equations with singular coefficient and critical exponent*, Adv. Diff. Equa., 1 (1996), no. 2, 241–264.
- [47] A. Tertikas, *Critical phenomena in linear elliptic problems*, J. Funct. Anal. 154 (1998), no. 1, 42–66.
- [48] J. L. Vazquez, E. Zuazua, *The Hardy inequality and the asymptotic behavior of the heat equation with an inverse-square potential*, J. Funct. Anal., 173 (2000), no. 1, 103–153.
- [49] M. Willem, *Analyse fonctionnelle élémentaire*, Cassini Éditeurs. Paris, 2003.

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