NECESSARY CONDITIONS FOR SOLUTIONS TO VARIATIONAL PROBLEMS

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Abstract. We prove necessary conditions for a solution u to the problem of minimizing

$$\int_\Omega [f(\|\nabla v(x)\|) + g(x, v(x))]\, dx$$

in the form of a Pontryagin Maximum Principle, for f convex and satisfying a growth assumption, but without assuming differentiability.

Key words. Euler-Lagrange equation, Pontryagin Maximum Principle

AMS subject classification. 49K10

1. Introduction. This paper deals with the necessary conditions satisfied by a solution u to the problem of minimizing

$$\int_\Omega [f(\|\nabla v(x)\|) + g(x, v(x))]\, dx \qquad \text{on } v_0 + W_0^{1,1}(\Omega)$$

where f is a convex function defined on $\Re^+$ and g is a Carathéodory function, differentiable with respect to v, and whose derivative g_v is also a Carathéodory function. The main point of the paper is that we do not assume further assumptions on f, with the exception of a growth estimate. For functionals of the form

$$\int_\Omega [F(\nabla v(x)) + g(x, v(x))]\, dx$$

with F a convex function defined on $\Re^N$, it has been conjectured in [2] that the suitable form of the Euler-Lagrange equations satisfied by a solution u should be

$$\exists p(\cdot) \in L^1(\Omega), \text{ a selection from } \partial F(\nabla u(\cdot)), \text{ such that } \operatorname{div} p(\cdot) = g_v(\cdot, u(\cdot))$$

in the sense of distributions.

Equivalently, the condition can be expressed as

$$\exists p(\cdot) \in L^1(\Omega) : \operatorname{div} p(\cdot) = g_v(\cdot, u(\cdot))$$

and, for a.e. x and every $\xi \in \Re^N$, we have

$$\langle p, \nabla u(x)\rangle - [F(\nabla u(x)) + g(x, u(x))] \geq \langle p, \xi\rangle - [F(\xi) + g(x, u(x))].$$

In this form, this condition is the equivalent of the Pontryagin Maximum Principle [4]. The purpose of the present paper is to prove this condition for the class of mappings under consideration.

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2. Necessary conditions. For the properties of convex functions we refer to [5] and, for those of Sobolev functions, to [1]. In what follows, $B[0,1]$ denotes the closed unit ball of $\Re^N$. We set $F(\xi) = f(\|\xi\|)$ and $\partial f^+(t) = \sup\{\lambda : \lambda \in \partial f(t)\}$. We consider mappings satisfying the following growth assumption.

Assumption A. The convex function f is such that there exist K and t_0 such that, for $t \geq t_0$,

$$\partial f^+(t) \leq K f(t).$$

THEOREM 2.1.

Let Ω be a bounded open subset of $\Re^n$. Let $f : \Re \to \Re$ be symmetric, convex, and satisfying the growth assumption A. Let $g(\cdot,\cdot)$ and $g_v(\cdot,\cdot)$ be Carathéodory functions and assume that for every U there exists $\xi_U \in L^1_{loc}$ such that $|v| \leq U$ implies $|g_v(x,v)| \leq \xi_U(x)$. Let u be a locally bounded solution to the problem of minimizing

$$\int_\Omega [f(\|\nabla v(x)\|) + g(x, v(x))]\, dx \qquad \text{on } v_0 + W_0^{1,1}(\Omega).$$

Then there exists $p \in L^1(\Omega)$, a selection from the map $x \to \partial F(\nabla u(x))$, such that

$$div\ \ p(\cdot) = g_v(\cdot, u(\cdot))$$

in the sense of distributions.

Notice that, although p has a weak divergence, there is no claim that it belongs to $W^{1,1}(\Omega)$.

Proof.

By the assumption of convexity, f is not differentiable at most on a countable set, possibly containing 0. Set $k_0 = 0$ and call $0 < k_1 < k_2 < \ldots$ the other points of non differentiability for f. Set

$$A_i = \{x : \|\nabla u(x)\| = k_i\}$$

$$B_i = \{x : k_i < \|\nabla u(x)\| < k_{i+1}\}$$

for $i = 0, \ldots, \infty$. We shall also set $A_i^+ = \{x \in A_i : \langle \nabla u, \nabla \eta\rangle \geq 0\}$ and $A_i^- = \{x \in A_i : \langle \nabla u, \nabla \eta\rangle < 0\}$. For $\eta \in C_0^1(\Omega)$ and $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon}\left\{\int_\Omega [f(\|\nabla u + \varepsilon\nabla\eta\|) + g(x, u+\varepsilon\eta) - f(\|\nabla u\|) - g(x,u)]\right\} \geq 0. \tag{2.1}$$

Fix one such η; consider a compact set O containing supp(η); let $D = \sup(\|\nabla\eta\|)$, U be $\sup(|u|)$ and H be $\sup(|\eta|)$ on O. Then

$$\left|\frac{g(x, u(x)+\varepsilon\eta(x)) - g(x, u(x))}{\varepsilon}\right| \leq H\xi_{U+\varepsilon H}(x)$$

so that, by dominated convergence,

$$\int_\Omega \frac{g(x, u(x)+\varepsilon\eta(x)) - g(x,u(x))}{\varepsilon}\, dx \longrightarrow \int_\Omega g_v(x, u(x))\,\eta(x)\, dx.$$

Consider A_0. On A_0, as $\varepsilon \to 0^+$, pointwise with respect to x, we have that

$$\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon} = \frac{f(\|\varepsilon\nabla\eta\|) - f(0)}{\varepsilon} \longrightarrow \partial f^+(0)\|\nabla\eta\|;$$

For $i = 0, \ldots, \infty$, on B_i we obtain

$$\frac{f\left(\|\nabla u + \varepsilon \nabla \eta\|\right) - f\left(\|\nabla u\|\right)}{\varepsilon} \longrightarrow f'\left(\|\nabla u\|\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle;$$

and, on A_i, $i = 1, \ldots, \infty$, we have

$$\frac{f\left(\|\nabla u + \varepsilon \nabla \eta\|\right) - f\left(\|\nabla u\|\right)}{\varepsilon} \longrightarrow \partial f^-(k_i)\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \rangle,$$

when $\langle \nabla u, \nabla \eta \rangle < 0$, and

$$\frac{f\left(\|\nabla u + \varepsilon \nabla \eta\|\right) - f\left(\|\nabla u\|\right)}{\varepsilon} \longrightarrow \partial f^+\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle$$

otherwise.

Moreover, we have

$$\left|\frac{f\left(\|\nabla u + \varepsilon \nabla \eta\|\right) - f(\|\nabla u\|)}{\varepsilon}\right| = \left|\frac{f(\|\nabla u\| + \theta(\varepsilon, x)) - f(\|\nabla u\|)}{\varepsilon}\right| = s\frac{|\theta(\varepsilon, x))|}{\varepsilon}$$

where $|\theta(\varepsilon, x)| \leq \varepsilon D$, and for some $s(x) \in \partial f(\xi(x))$, with $\xi(x)$ in the interval of extremes $\|\nabla u(x)\|$ and $\|\nabla u(x)\| + \theta(\varepsilon, x)$. Consider assumption A. Then, either $\max\{\|\nabla u(x)\|, \|\nabla u(x)\| + \theta(\varepsilon, x)\} \leq t_0 + D$, and in this case $s(x) \leq \partial f^+(t_0 + D)$; or, $\max\{\|\nabla u(x)\|, \|\nabla u(x)\| + \theta(\varepsilon, x)\} > t_0 + D$, i.e. both $\|\nabla u(x)\|$ and $\xi(x)$ are $> t_0$, so that $f(\xi(x)) \leq f(\|\nabla u(x)\|)e^{K\varepsilon D} \leq f(\|\nabla u(x)\|)e^{KD}$ and $\partial f^+(\xi(x)) \leq K f(\|\nabla u(x)\|)e^{KD}$. Hence

$$\left|\frac{f\left(\|\nabla u + \varepsilon \nabla \eta\|\right) - f(\|\nabla u\|)}{\varepsilon}\right| \leq \max\{D\partial f^+(t_0 + D), DKf(\|\nabla u\|)e^{KD}\},$$

an integrable function independent of ε. By dominated convergence, from (2.1), we obtain

$$\int_{A_0} \partial f^+(0)\|\nabla \eta\| + \sum_{i=1}^{\infty} \int_{A_i^+} \partial f^+\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{A_i^-} \partial f^-\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle$$

$$+ \sum_{i=0}^{\infty} \int_{B_i} f'\left(\|\nabla u\|\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{\Omega} g_v\left(x, u\right)\eta \geq 0.$$

The same considerations, when applied to the variation $-\eta$, yield

$$\int_{A_0} \partial f^+\left(0\right)\|\nabla \eta\| - \sum_{i=1}^{\infty}\left[\int_{A_i^-} \partial f^+\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{A_i^+} \partial f^-\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle\right]$$

$$- \sum_{i=0}^{\infty} \int_{B_i} f'\left(\|\nabla u\|\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle - \int_{\Omega} g_v\left(x, u\right)\eta \geq 0.$$

From these two inequalities we obtain

$$- \int_{A_0} \partial f^+\left(0\right)\|\nabla \eta\| - \sum_{i=1}^{\infty}\left[\int_{A_i^+} \partial f^+\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{A_i^-} \partial f^-\left(k_i\right)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle\right]$$

$$\leq \sum_{i=0}^{\infty} \int_{B_i} f'(\|\nabla u\|) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{\Omega} g_v(x,u)\,\eta \leq \int_{A_0} \partial f^+(0)\,\|\nabla \eta\| -$$

$$- \sum_{i=1}^{\infty} \left[\int_{A_i^-} \partial f^+(k_i) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \int_{A_i^+} \partial f^-(k_i) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle \right].$$

Adding the term $\sum_{i=1}^{\infty} \int_{A_i} \frac{1}{2} \left[\partial f^+(k_i) + \partial f^-(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle$ to all sides, we have the estimate

$$-\int_{A_0} \partial f^+(0)\,\|\nabla \eta\| - \sum_{i=1}^{\infty} \int_{A_i} \frac{1}{2} \left[\partial f^+(k_i) - \partial f^-(k_i)\right] \left| \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle \right| \tag{2.2}$$

$$\leq \sum_{i=0}^{\infty} \int_{B_i} f'(\|\nabla u\|) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle + \sum_{i=1}^{\infty} \int_{A_i} \frac{1}{2} \left[\partial f^+(k_i) + \partial f^-(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle$$

$$+ \int_{\Omega} g_v(x,u)\,\eta \leq$$

$$\int_{A_0} \partial f^+(0)\,\|\nabla \eta\| + \sum_{i=1}^{\infty} \int_{A_i} \frac{1}{2} \left[\partial f^+(k_i) - \partial f^-(k_i)\right] \left| \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle \right|.$$

Set

$$X = \Big\{ (v,w) \in L^1(A_0, \Re^n) \times L^1(\cup_{i=1}^{\infty} A_i, \Re) : \exists \eta \in C_0^1(\Omega) : v = \partial f^+(0)\,\nabla \eta|_{A_0},$$

$$w|_{A_i} = \left[\partial f^+(k_i) - \partial f^-(k_i)\right] \langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \rangle|_{A_i}, i = 1 \ldots, \infty \Big\}.$$

Define the map $T : X \to \Re$ as follows:

$$T(v,w) = -\sum_{i=0}^{\infty} \int_{B_i} f'(\|\nabla u\|) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \right\rangle \tag{2.3}$$

$$-\sum_{i=1}^{\infty} \frac{1}{2} \int_{A_i} [\partial f^+(k_i) + \partial f^-(k_i)] \langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta \rangle - \int_{\Omega} g_v(x,u)\,\eta.$$

We claim that T is well defined and that it is a continuous linear functional on X.

In fact, consider (v,w) in X, and assume that there exist η_1 and η_2 such that $v = \partial f^+(0)\,\nabla \eta_1|_{A_0} = \partial f^+(0)\,\nabla \eta_2|_{A_0}$ and

$$w|_{A_i} = [\partial f^+(k_i) - \partial f^-(k_i)] \langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta_1 \rangle|_{A_i} = [\partial f^+(k_i) - \partial f^-(k_i)] \langle \frac{\nabla u}{\|\nabla u\|}, \nabla \eta_2 \rangle|_{A_i}.$$

From (2.2) we have

$$|-\sum_{i=0}^{\infty}\int_{B_i} f'\left(\|\nabla u\|\right)\left\langle\frac{\nabla u}{\|\nabla u\|},\nabla\eta_1-\nabla\eta_2\right\rangle$$

$$-\sum_{i=1}^{\infty}\frac{1}{2}\int_{A_i}\left[\partial f^+\left(k_i\right)+\partial f^-\left(k_i\right)\right]\langle\frac{\nabla u}{\|\nabla u\|},\nabla\eta_1-\nabla\eta_2\rangle-\int_\Omega g_v\left(x,u\right)\left[\eta_1-\eta_2\right]|\leq$$

$$\int_{A_0}\partial f^+\left(0\right)\|\nabla\eta_1-\nabla\eta_2\|+\sum_{i=1}^{\infty}\int_{A_i}\frac{1}{2}\left[\partial f^+\left(k_i\right)-\partial f^-\left(k_i\right)\right]|\left\langle\frac{\nabla u}{\|\nabla u\|},\nabla\eta_1-\nabla\eta_2\right\rangle|=0,$$

so that T is well defined. It is clearly linear and, from

$$|T\left(v,w\right)|\leq\int_{A_0}\|v\|+\frac{1}{2}\int_{\cup A_i}|w|\quad\forall\left(v,w\right)\in X,$$

it is bounded. Hence, by the Hahn-Banach theorem, there exists L, a continuous linear functional on $L^1\left(A_0,\Re^n\right)\times L^1\left(\cup_{i=1}^{\infty}A_i,\Re\right)$, such that $L|_X\equiv T$ and

$$|L\left(v,w\right)|\leq\int_{A_0}\|v\|+\frac{1}{2}\int_{\cup A_i}|w|\quad\forall\left(v,w\right)\in L^1\left(A_0,\Re^n\right)\times L^1\left(\cup_{i=1}^{\infty}A_i,\Re\right).$$

Let us define $L^*:L^1\left(A_0,\Re^n\right)\to\Re$, setting

$$L^*\left(v\right)=L\left(v,0\right)$$

and $L^{**}:L^1\left(\bigcup_{i=1}^{\infty}A_i,\Re\right)\to\Re$, setting

$$L^{**}\left(w\right)=L\left(0,w\right).$$

We have that

$$|L^*\left(v\right)|\leq\int_{A_0}\|v\|\quad\forall v\in L^1\left(A_0,\Re^n\right)$$

and

$$|L^{**}\left(w\right)|\leq\frac{1}{2}\int_{\bigcup_{i=1}^{\infty}A_i}|w|\quad\forall w\in L^1\left(\bigcup_{i=1}^{\infty}A_i,\Re\right),$$

so that $\|L^*\|\leq 1$ and $\|L^{**}\|\leq\frac{1}{2}$.

By Riesz's Theorem, there exists $\alpha\in L^\infty\left(A_0,\Re^n\right)$, supess$\|\alpha\|\leq 1$, such that, for every $v\in L^1\left(A_0,\Re^n\right)$,

$$L^*\left(v\right)=\int_{A_0}\langle\alpha,v\rangle$$

and there exists $\beta\in L^\infty\left(\cup_{i=1}^{\infty}A_i,\Re\right)$, with $|\beta|\leq\frac{1}{2}$ a.e., such that, for every $w\in L^1\left(\bigcup_{i=1}^{\infty}A_i,\Re\right)$,

$$L^{**}(w) = \int_{\bigcup A_i} \beta w.$$

Hence, we can conclude that, for $\eta \in C_0^1(\Omega)$, we have

$$T\left(\partial f^+(0)\nabla\eta|_{A_0}, [\partial f^+(k_i) - \partial f^-(k_i)]\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle |_{\bigcup A_i}\right)$$

$$= L\left(\partial f^+(0)\nabla\eta|_{A_0}, [\partial f^+(k_i) - \partial f^-(k_i)]\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle |_{\bigcup A_i}\right)$$

$$= L\left(\partial f^+(0)\nabla\eta|_{A_0}, 0\right) + L\left(0, [\partial f^+(k_i) - \partial f^-(k_i)]\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle |_{\bigcup A_i}\right)$$

$$= L^*\left(\partial f^+(0)\nabla\eta|_{A_0}\right) + L^{**}\left([\partial f^+(k_i) - \partial f^-(k_i)]\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\rangle|_{\cup A_i}\right)$$

$$= \int_{A_0} \partial f^+(0)\langle\alpha, \nabla\eta\rangle + \int_{\cup A_i} \beta[\partial f^+(k_i) - \partial f^-(k_i)]\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\rangle.$$

Equating the definition (2.3) to the equality above, we obtain

$$\int_{A_0} \partial f^+(0)\langle\alpha, \nabla\eta\rangle$$

$$+\sum_{i=1}^{\infty}\int_{A_i}\left[\frac{1}{2}[\partial f^+(k_i) + \partial f^-(k_i)] + \beta[\partial f^+(k_i) - \partial f^-(k_i)]\right]\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle$$

$$+\sum_{i=0}^{\infty}\int_{B_i} f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_\Omega g_v(x,u)\eta = 0.$$

Since

$$\partial F(\xi) = \begin{cases} \partial f^+(0)B[0,1] & \text{for } \xi = 0 \\ \{b\frac{\xi}{\|\xi\|} : \partial f^-(k_i) \le b \le \partial f^+(k_i)\} & \text{for } \|\xi\| = k_i \\ f'(\|\xi\|)\frac{\xi}{\|\xi\|} & \text{otherwise} \end{cases},$$

from the properties of α and β we have that the map

$$p(x) = \partial f^+(0)\alpha(x)\chi_{A_0}(x)+$$

$$+\sum_{i=1}^{\infty}\left[\frac{1}{2}[\partial f^+(k_i) + \partial f^-(k_i)] + \beta(x)[\partial f^+(k_i) - \partial f^-(k_i)]\right]\frac{\nabla u(x)}{\|\nabla u(x)\|}\chi_{A_i}(x)+$$

$$+\sum_{i=0}^{\infty} f'(\|\nabla u(x)\|)\frac{\nabla u(x)}{\|\nabla u(x)\|}\chi_{B_i}(x)$$

is a selection from $\partial F(\nabla u(x))$ and

$$\int_{\Omega} [\langle p(x), \nabla\eta(x)\rangle + g_v(x,u)\eta(x)]\, dx = 0$$

for every $\eta \in C_c^1(\Omega)$. Moreover, from our assumptions on g and the local boundedness of u, we have that $f(\|\nabla u(\cdot)\|) \in L^1(\Omega)$; then, from assumption A, we obtain that every selection from $\partial f(\|\nabla u(\cdot)\|)$ is integrable, thus proving the Theorem. □

EXAMPLE 1. *In* $\Re^2$, *let* $g(x,u) = u$ *and*

$$F(\xi) = f(\|\xi\|) = \begin{cases} \sqrt{2}\|\xi\| & \text{for } \|\xi\| \le \sqrt{2} \\ 1 + \frac{1}{2}\|\xi\|^2 & \text{for } \|\xi\| \ge \sqrt{2} \end{cases} \tag{2.4}$$

We have that $\partial F(0) = \sqrt{2}B[0,1]$. *Then, as described in [3],*

$$u(x) = \begin{cases} 0 & \text{for } \frac{\|x\|}{2} \le \sqrt{2} \\ (\frac{\|x\|}{2})^2 - 2 & \text{for } \frac{\|x\|}{2} \ge \sqrt{2} \end{cases} \tag{2.5}$$

is a solution to the minimization problem, among those functions satisfying the same values as u on $\partial\Omega$. *We have*

$$\nabla u(x) = \begin{cases} 0 & \text{for } \frac{\|x\|}{2} < \sqrt{2} \\ \frac{1}{2}x & \text{for } \frac{\|x\|}{2} > \sqrt{2} \end{cases} \tag{2.6}$$

Hence,

$$\partial F(\nabla u(x)) = \begin{cases} \sqrt{2}B[0,1] & \text{for } \frac{\|x\|}{2} < \sqrt{2} \\ \nabla F(\nabla u(x)) = \nabla u(x) = \frac{1}{2}x & \text{for } \frac{\|x\|}{2} > \sqrt{2} \end{cases}. \tag{2.7}$$

Then, although the function $\nabla u(x)$ *is discontinuous, the vector function*

$$p(x) = \frac{1}{2}x$$

is an everywhere smooth selection from the map $x \to \partial F(\nabla u(x))$ *and has everywhere divergence equal* 1.

REFERENCES

[1] R. A. ADAMS, Sobolev spaces, Academic Press, New York, 1975

[2] A. CELLINA *The Euler Lagrange equation and the Pontriagin maximum principle.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 8 (2005), no. 2, 323–347

[3] A. CELLINA *Uniqueness and comparison results for functionals depending on* ∇u *and on* u SIAM J. Optim. 18 (2007), 711–716

[4] L. PONTRIAGIN, V. BOLTYIANSKII, R. GAMKRELIDZE, E. MISHCHENKO *The Mathematical Theory of Control Processes* Interscience, New York, 1962

[5] R. T. ROCKAFELLAR, Convex Analysis, Princeton University Press, Princeton, NJ, 1972.