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PhD Thesis

Conservation Laws in Gas Dynamics and Traffic Flow

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Contents

Introduction

Description of the problems

This PhD thesis is concerned with applications of nonlinear systems of conservation laws to gas dynamics and traffic flow modeling.

The first result is contained in **(1)** and is here presented in Chapter 1. It is devoted to the analytical description of a fluid flowing in a tube with varying cross section. When the section $a(x)$ varies smoothly, a classical model is based on the *p–system*

$$\textbf{(p)} \quad \begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a}\, \partial_x a \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = -\dfrac{q^2}{a\rho}\, \partial_x a \end{cases}$$

$t \in \mathbb{R}^+$	time
$x \in \mathbb{R}$	space
$\rho = \rho(t,x)$	fluid density
$q = q(t,x)$	linear momentum density
$a = a(x)$	pipe section
$p = p(\rho)$	pressure.

Here, the source term takes into account the inhomogeneities in the tube geometry, see for instance [40, Remark 2.7]. In this case, the regularity of the pipe automatically selects the appropriate definition of weak solution.

The mathematical problem related to a junction has been widely considered in the recent literature, see [11, 21, 26] and the references therein. Analytically, it consists of a *sharp discontinuity* in the pipe's geometry, say sited at $x = 0$. More precisely, it corresponds to the section $a(x) = a^-$ for $x < 0$ and $a(x) = a^+$ for $x > 0$. Thus, in each of the two pipes, the model reads

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = 0 \end{cases}$$

where the *coupling condition*

$$\Psi \left(a^-, (\rho, q)(t, 0-); a^+, (\rho, q)(t, 0+) \right) = 0$$

imposes suitable physical requirements, such as the conservation of mass and the equality of the hydrostatic pressure, see [11], or the partial conservation of momentum, see [21]. In **(1)** we introduce a choice of Ψ motivated as limit of the smooth case, see Figure 1.

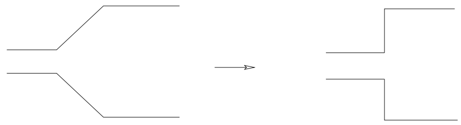


Figure 1: The unique concept of solution in the case of a smooth section induces a definition of solution in the case of the junction.

With this definition, we first prove the well posedness of the resulting model, also in the case of a piecewise constant pipe's section. The bounds on the total variation obtained in this construction allow to pass, through a suitable limit, to the case of a pipe's section a of class $\mathbf{W^{1,1}}$, see Figure 2. In particular, by means of this latter limit, we also prove the well posedness of the smooth case.

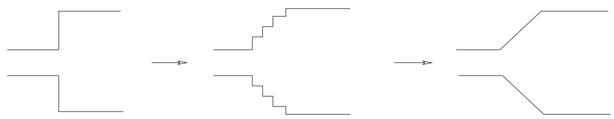


Figure 2: The construction for a single jump is first extended to general piecewise constant sections and then, through an approximating procedure, to a section function a of class $\mathbf{W^{1,1}}$.

Above, as usual in the theory of conservation laws, by *well posedness* we mean that we construct an $\mathbf{L^1}$ Lipschitz semigroup whose orbits are solutions to the Cauchy problem. Moreover, the formal convergence of the problem with piecewise constant section to that one with $\mathbf{W^{1,1}}$ section is completed by the rigorous proof of the convergence of the corresponding semigroups.

In **(1)**, a careful estimate on the total variation of the solution shows that, at lower fluid speeds, higher total variations of the pipe's section are acceptable for the solution to exist. On the contrary, an explicit example computed in the case of the isothermal pressure law shows that, if the fluid speed is sufficiently close to the sound speed, a shock entering a pipe may have its strength arbitrarily magnified due to its interaction with the pipe's walls. In other words, near to the sonic speed and with a having large total variation, the total variation of the solution may grow arbitrarily in finite time.

A key role in the result above is played by wave front tracking solutions to conservation laws and by the *operator splitting method*. The former technique allows the construction of very accurate piecewise constant approximations. In general, properties of solutions are first proved on these approximations, then passing to the limit we show that they hold on the exact solutions too. In the present case, the standard wave front tracking procedure [16, Chapter 7] needs to be adapted to the presence of the junction.

In **(2)**, presented in Chapter 2, as a first result we study the Cauchy problem for an $n \times n$ strictly hyperbolic system of balance laws. More precisely, we consider

$$\begin{cases} \partial_t u + \partial_x f(u) = g(x,u) \\ u(0,x) = u_o(x) \end{cases} \qquad \begin{array}{r} t \in \mathbb{R}^+ \\ x \in \mathbb{R} \\ u \in \mathbb{R}^n \\ u_o \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n) \end{array}$$

with each characteristic field being genuinely nonlinear or linearly degenerate (see [16, Definition 5.2, Chapter 7]. Under the *nonresonance* assumption

$$\left|\lambda_i(u)\right| \geq c > 0 \text{ for all } i \in \{1, \ldots, n\} \text{ and for all } u\,,$$

and the boundedness condition

$$\left\|g(x,\cdot)\right\|_{\mathbf{C}^2} \leq M(x) \text{ with } M \in \mathbf{L^1}(\mathbb{R};\mathbb{R}),$$

we prove the global existence, uniqueness and regularity of entropy solutions with bounded total variation provided, as usual, that the $\mathbf{L^1}$ norm of $\|g(x,\cdot)\|_{\mathbf{C}^1}$ and $\mathrm{TV}(u_o)$ are small enough. In [1] an analogous result was proved, but under the stronger condition $M \in \big(\mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \cap \mathbf{L^1}(\mathbb{R};\mathbb{R})\big)$.

This general result allows to compute the limit considered in **(1)** and illustrated in Figure 1, also in the case of the full 3×3 Euler system. Indeed, in **(2)**, we derive existence and uniqueness of solutions in the case of a discontinuous pipe's section as limit of solutions corresponding to smooth pipe's section.

Among the results in **(2)** there is also a characterization of solutions in the case of a general balance law. When applied in the 2×2 case of the p–system, it ensures that the solutions constructed in [26, Theorem 3.2] coincide with those in **(1)**, whenever the coupling condition induced by the smooth junction is considered.

Here, the technique is again based on the wave front tracking algorithm but, differently from **(1)**, we do not use the operator splitting procedure. On the contrary, as in [1] here the source is approximated by a sequence of Dirac deltas; careful estimates allow us to use the $\mathbf{L^1}$ norm on the bound on M instead of its $\mathbf{L}^\infty$ norm, so that we can go to the limit as in the scheme of Figure 1.

In **(3)**, presented in Chapter 3, the basic analytical properties of the equations governing a fluid flowing in a pipe with varying section, proved in **(1)** for the case of the p-system, are extended to the full 3×3 Euler equations.

In particular, we consider two tubes separated by a junction sited at, say, $x = 0$. In each pipe, the fluid dynamics is described by the full 3×3 Euler system:

$$
\textbf{(e)} \quad \begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho, e) \right) = 0 \\ \partial_t E + \partial_x \left(\dfrac{q}{\rho} \left(E + p(\rho, e) \right) \right) = 0 \end{cases}
\qquad
\begin{array}{rl}
\rho{=}\rho(t,x) & \text{fluid density} \\
q{=}q(t,x) & \text{linear momentum} \\
e{=}e(t,x) & \text{internal energy density} \\
p{=}p(\rho,e) & \text{pressure} \\
E{=}\frac{1}{2}\frac{q^2}{\rho}{+}\rho e & \text{total energy density.}
\end{array}
$$

We extend the results proved for the p-system in [26, Theorem 3.2] to the 3×3 case of a general coupling condition at the junction, that is

$$
\Psi \left(a^-, (\rho, q, E)(t, 0-); a^+, (\rho, q, E)(t, 0+) \right) = 0 \, .
$$

This framework comprises various choices of the coupling condition found in the literature, such as for instance in [11, 21], once they are extended to the 3×3 case and [31] for the full 3×3 system. We also extend the condition inherited from the smooth case introduced in **(1)**.

Within this setting, we prove the well posedness of the Cauchy problem for **(e)** and, then, the extension to pipes with several junctions and to pipes with a $\mathbf{W^{1,1}}$ section.

As in the 2×2 case of the p-system, here a key assumption is the boundedness of the total variation of the pipe section. We provide explicit examples to show that this bound is necessary for each of the different coupling conditions considered.

The analytical techniques used here are the same to that ones in **(1)**. To show the necessity of the bound on the total variation, in a part of the proof, we used a software for symbolic computations.

Concerning traffic flow, in **(4)**, presented in Chapter 4, we introduce a new macroscopic traffic model, based on a non-smooth 2×2 system of conservation laws. Consider the classical LWR model

$$
\begin{cases} \partial_t \rho + \partial_x (\rho V) = 0 \\ V = w \, \psi(\rho) \end{cases}
\qquad
\begin{array}{rl}
\rho{=}\rho(t,x) & \text{traffic density} \\
V{=}V(w,\rho) & \text{traffic speed} \\
w{>}0 & \text{maximal traffic speed.}
\end{array}
$$

Here, ψ describes the attitude of drivers to choose their speed depending on the traffic density at their location. First, we assume that each driver has his proper maximal speed, so that the constant w above becomes a variable quantity transported by traffic. Then, we assume that there exists an overall maximal speed $V_{\max}$. Therefore, we obtain the following model, which can be seen as a development of those presented in [6, 18]:

$$
\textbf{(t)} \quad \begin{cases} \partial_t \rho + \partial_x \left(\rho \, v(\rho, w) \right) = 0 \\ \partial_t w + v(\rho, w) \, \partial_x w = 0 \end{cases} \qquad \text{with} \qquad v(\rho, w) = \min \left\{ V_{\max}, w \, \psi(\rho) \right\}
$$

and can be rewritten as a 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ flow:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \left(\rho\, v(\rho,w)\right) = 0 \\ \partial_t (\rho\, w) + \partial_x \left(\rho\, w\, v(\rho,w)\right) = 0 \end{array}\right. \qquad \text{with} \qquad v(\rho,w) = \min\left\{V_{\max}, w\,\psi(\rho)\right\}\,.$$

A different approach leads to a similar model in [14].

In **(4)**, we study the Riemann problem for **(t)** and the qualitative properties of its solutions that are relevant from the point of view of traffic.

It is remarkable that the introduction of the speed bound $V_{\max}$ induces the formation of two distinct *phases*, similarly to the model in [18] and coherently with common traffic observations. The former one corresponds to the *free phase*, i.e. high speed and low density, while the latter describes the *congested phase*, i.e. low speed and high density. The first one is 1D in the density–flow plane $(\rho, \rho v)$, but 2D in the plane of the conserved variables $(\rho, \rho w)$. The second one is 2D in both planes, as it has to be expected from the traffic point of view.

Moreover, we establish a connection between this model and other approaches found in the literature, considering also kinetic and microscopic descriptions. In the case of other macroscopic models, we compare the various fundamental diagrams among each other and with the experimentally observed ones. In particular, we develop a rigorous connection between **(t)** and the microscopic *Follow-The-Leader* class of models, based on ordinary differential equations. As a result, we show *directly* that **(t)** can be viewed as the limit of the microscopic model when the number of vehicles increases to infinity. Our approach is different from others found in the literature, see for instance [5]; in our approach neither the Lagrangian description, nor Godunov scheme are considered.

Articles and Preprints

This PhD thesis collects the results presented in the following papers:

(1) R.M. Colombo, F. Marcellini. Smooth and discontinuous Junctions in the p-system. *Journal of Mathematical Analysis and Applications*, 361: 440-456, 2010.

(2) G. Guerra, F. Marcellini, V. Schleper. Balance Laws with Integrable Unbounded Sources. *SIAM Journal of Mathematical Analysis*, 41(3): 1164–1189, 2009.

(3) R.M. Colombo, F. Marcellini. Coupling Conditions for the 3×3 Euler System. *Quaderno di Matematica n. 15/2009, Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca. Submitted*, 2009.

(4) R.M. Colombo, F. Marcellini, M. Rascle. A 2–Phase Traffic Model Based on a Speed Bound. *Quaderno di Matematica n. 13/2009, Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca. Submitted*, 2009.

Communications and Advanced Courses

I communicated some of the results presented in this PhD thesis at the following meetings:

- Communication: ”Junctions in gas pipelines”
  during the meeting ”Conservation Laws and Applications”
  Brescia 28.05.2008

- Communication: ”Smooth and discontinuous junctions in the p-system”
  during the meeting: ”Sixth meeting on Hyperbolic Conservation Laws”
  L’Aquila 17-19.07.2008.

During the preparation of this PhD thesis I attained the following courses, schools and conferences:

- “Nonlinear Hyperbolic problems”
  Roma 28.05-01.06.2007

- “Partial Differential Equations”
  Cortona 15-28.07.07
  (SMI)
  Courses by Prof. Constantine Dafermos (Brown University) and Prof. Piero D’Ancona (”La Sapienza” University)

- “Evolution Equations in Pure and Applied Sciences”
  Firenze 18-19.04.2008

- “Conservation Laws and Applications”
  Brescia 28.05.2008

- Nonlinear Partial Differential Equations and Applications
  Cetraro, 22-28.06.2008
  (CIME)
  Courses by Prof. Stefano Bianchini (SISSA/ISAS), Prof. Eric A. Carlen (School Math. Georgia Inst. Technology), Prof. Alexander Mielke (WIAS Berlin), Prof. Felix Otto (Inst. Appl. Math., Univ. Bonn), Prof. Cedric Villani (ENS Lyon)

- “Sixth meeting on Hyperbolic Conservation Laws”
  L’Aquila 17-19.07.2008

- “Optimal Transpportation, Geometry and Functional Inequalities”
  Pisa, 28.10-31.10.2008

- “First Winter School at IMEDEA on PDEs and Inequalities”
  Madrid, 25.01-30.01.2009

- From 08 to 15 February I visited the University of Nice, France, for a collaboration with Professor Michel Rascle

- “Modelling and Optimisation of Flows on Networks”
  Cetraro, 15-19.06.2009
  (CIME)
  Courses by Prof. Luigi Ambrosio (SNS, Pisa), Prof. Alberto Bressan (Penn State University), Prof. Dirk Helbing (ETH), Prof. Axel Klar (Kaiserlautern University), Prof. Christian Ringhofer (Arizona University), Prof. Enrique Zuazua (Basque Center for Applied Mathematics).

- “Nonlinear Conservation Laws and Applications”
  Minneapolis, USA, 13-31.07.2009
  (SUMMER PROGRAM at the University of Minnesota)

- “Seventh meeting on Hyperbolic Conservation Laws”
  S.I.S.S.A., Trieste 31.08-04.09.2009.

Chapter 1

Smooth and Discontinuous Junctions in the p-System

1.1 Introduction

Consider a gas pipe with smoothly varying section. In the isentropic or isothermal approximation, the dynamics of the fluid in the pipe is described by the following system of Euler equations:

$$\begin{cases} \partial_t(a\rho) + \partial_x(aq) = 0 \\ \partial_t(aq) + \partial_x \left[a \left(\dfrac{q^2}{\rho} + p(\rho) \right) \right] = p\,(\rho)\ \partial_x a, \end{cases} \tag{1.1.1}$$

where, as usual, ρ is the fluid density, q is the linear momentum density, $p = p(\rho)$ is the pressure and $a = a(x)$ is *cross-sectional area* of the tube. We provide a basic well posedness result for (1.1.1), under the assumptions that the initial data is subsonic, has sufficiently small total variation and the oscillation in the pipe section $a = a(x)$ is also small. We provide an explicit bound on the total variation of a. As it is physically reasonable, as the fluid speed increases this bound decreases and vanishes at sonic speed, see (1.2.14).

As a tool in the study of (1.1.1) we use the system recently proposed for the case of a sharp discontinuous change in the pipe's section between the values a^- and a^+, see [10, 21, 26]. This description is based on the p-system

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = 0 \end{cases} \tag{1.1.2}$$

equipped with a *coupling condition* at the junction of the form

$$\Psi\left(a^-, (\rho, q)(t, 0-); a^+, (\rho, q)(t, 0+)\right) = 0 \tag{1.1.3}$$

whose role is essentially that of selecting stationary solutions.

Remark that the introduction of condition (1.1.3) is necessary as soon as the section of the pipe is not smooth. The literature offers different choices for this condition, see [10, 21, 26]. The construction below does not require any specific choice of Ψ in (1.1.3), but applies to all conditions satisfying minimal physically reasonable requirements, see **(Σ0)**–**(Σ2)**.

On the contrary, if $a \in \mathbf{W^{1,1}}$ the product in the right hand side of the second equation in (1.1.1) is well defined and system (1.1.1) is equivalent to the 2×2 system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x q = -\frac{q}{a}\, \partial_x a \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho)\right) = -\frac{q^2}{a\rho}\, \partial_x a\,. \end{cases} \tag{1.1.4}$$

Systems of this type were considered, for instance, in [17, 35, 40, 52, 56, 65, 74]. In this case the stationary solutions to (1.1.1) are characterized as solutions to

$$\begin{cases} \partial_x (a(x)\, q) = 0 \\ \partial_x \left(a(x) \, \left(\frac{q^2}{\rho} + p(\rho)\right)\right) = p(\rho)\, \partial_x a \end{cases} \quad \text{or} \quad \begin{cases} \partial_x q = -\frac{q}{a}\, \partial_x a \\ \partial_x \left(\frac{q^2}{\rho} + p(\rho)\right) = -\frac{q^2}{a\rho}\, \partial_x a\,, \end{cases} \tag{1.1.5}$$

see Lemma 1.2.6 for a proof of the equivalence between (1.1.4) and (1.1.1).

Thus, the case of a smooth a induces a unique choice for condition (1.1.3), see (1.2.3) and (1.2.19). Even with this choice, in the case of the isothermal pressure law $p(\rho) = c^2 \rho$, we show below that a shock entering a pipe can have its strength arbitrarily magnified, provided the total variation of the pipe's section is sufficiently high and the fluid speed is sufficiently near to the sound speed, see Section 1.2.2. Recall, from the physical point of view, that the present situation neglects friction, viscosity and the conservation of energy. Moreover, this example shows the necessity of a bound on the total variation of the pipe section in any well posedness theorem for (1.1.1).

The next section is divided into three parts, the former one deals with a pipe with a single junction, the second with a pipe with a piecewise constant section and the latter with a pipe having a $\mathbf{W^{1,1}}$ section. All proofs are gathered in Section 1.3.

1.2 Notation and Main Results

Throughout this chapter, u denotes the pair (ρ, q) so that, for instance, $u^\pm = (\rho^\pm, q^\pm)$, $\bar u = (\bar\rho, \bar q)$, ... Correspondingly, we denote by $f(u) = \big(q, P(\rho, q)\big)$ the flow in (1.1.2). Introduce also the notation $\mathbb{R}^+ = [0, +\infty[$, whereas $\mathring{\mathbb{R}}^+ = \,]0, +\infty[$. Besides, we let $a(x\pm) = \lim_{\xi \to x\pm} a(\xi)$. Below, $B(u; \delta)$ denotes the open ball centered in u with radius δ.

The pressure law p is assumed to satisfy the following requirement:

(P) $p \in \mathbf{C^2}(\mathbb{R}^+;\mathbb{R}^+)$ is such that for all $\rho > 0$, $p'(\rho) > 0$ and $p''(\rho) \geq 0$.

The classical example is the γ-law, where $p(\rho) = k\,\rho^\gamma$, for a suitable $\gamma \geq 1$.

Recall the expressions of the eigenvalues $\lambda_{1,2}$ and eigenvectors $r_{1,2}$ of the p-system, with c denoting the sound speed,

$$\begin{array}{ccc} \lambda_1(u) = \frac{q}{\rho} - c(\rho)\,, & c(\rho) = \sqrt{p'(\rho)}\,, & \lambda_2(u) = \frac{q}{\rho} + c(\rho)\,, \\ r_1(u) = \begin{bmatrix} -1 \\ -\lambda_1(u) \end{bmatrix}, & & r_2(u) = \begin{bmatrix} 1 \\ \lambda_2(u) \end{bmatrix}. \end{array} \tag{1.2.1}$$

The *subsonic* region is given by

$$A_0 = \left\{ u \in \mathring{\mathbb{R}}^+ \times \mathbb{R} \colon \lambda_1(u) < 0 < \lambda_2(u) \right\}. \tag{1.2.2}$$

For later use, we recall the quantities

$$\begin{array}{rrl} \text{flow of the linear momentum:} & P(u) & = \dfrac{q^2}{\rho} + p(\rho) \\ \text{total energy density:} & E(u) & = \dfrac{q^2}{2\rho} + \rho \displaystyle\int_{\rho_*}^{\rho} \frac{p(r)}{r^2}\,dr \\ \text{flow of the total energy density:} & F(u) & = \dfrac{q}{\rho} \cdot \big(E(u) + p(\rho)\big) \end{array}$$

where $\rho_* > 0$ is a suitable fixed constant. As it is well known, see [33, formula (3.3.21)], the pair (E, F) plays the role of the (mathematical) entropy - entropy flux pair.

1.2.1 A Pipe with a Single Junction

This paragraph is devoted to (1.1.2)–(1.1.3). Fix the section $\bar{a} > \Delta$, with $\Delta > 0$ and the state $\bar{u} \in A_0$.

First, introduce a function $\Sigma = \Sigma(a^-, a^+; u^-)$ that describes the effects of the junction when the section changes from a^- to a^+ and the state to the left of the junction is u^-. We specify the choice of (1.1.3) writing

$$\Psi(a^-, u^-; a^+, u^+) = \begin{bmatrix} a^+q^+ - a^-q^- \\ a^+P(u^+) - a^-P(u^-) \end{bmatrix} - \Sigma(a^-, a^+; u^-)\,. \tag{1.2.3}$$

We pose the following assumptions on Σ:

($\boldsymbol{\Sigma}$0) $\Sigma \in \mathbf{C^1}\left([\bar{a} - \Delta, \bar{a} + \Delta] \times B(\bar{u}; \delta); \mathbb{R}^2\right)$.

($\boldsymbol{\Sigma}$1) $\Sigma(a, a; u^-) = 0$ for all $a \in [\bar{a} - \Delta, \bar{a} + \Delta]$ and all $u^- \in B(\bar{u}; \delta)$.

Condition **($\boldsymbol{\Sigma}$0)** is a natural regularity condition. Condition **($\boldsymbol{\Sigma}$1)** is aimed to comprehend the standard *"no junction"* situation: if $a^- = a^+$, then the junction has no effects and Σ vanishes.

Conditions **($\boldsymbol{\Sigma}$0)**–**($\boldsymbol{\Sigma}$1)** ensure the existence of stationary solutions to problem (1.1.2)–(1.1.3).

Lemma 1.2.1 *Let* **(Σ0)**–**(Σ1)** *hold. Then, for any* $\bar a \in \mathring{\mathbb{R}}^+$, $\bar u \in A_0$, *there exists a positive* $\bar\delta$ *and a Lipschitz map*

$$T \colon \left]\bar a - \bar\delta, \bar a + \bar\delta\right[\times \left]\bar a - \bar\delta, \bar a + \bar\delta\right[\times B\left(\bar u; \bar\delta\right) \to A_0 \tag{1.2.4}$$

such that

$$\left.\begin{array}{l} \Psi(a^-, u^-; a^+, u^+) = 0 \\ a^- \in \left]\bar a - \bar\delta, \bar a + \bar\delta\right[\\ a^+ \in \left]\bar a - \bar\delta, \bar a + \bar\delta\right[\\ u^-, u^+ \in B\left(\bar u; \bar\delta\right) \end{array}\right\} \iff u^+ = T(a^-, a^+; u^-)\,.$$

In particular, $T(\bar a, \bar a, \bar u) = \bar u$. We may now state a final requirement on Σ:

(Σ2) $\Sigma(a^-, a^0; u^-) + \Sigma\left(a^0, a^+; T(a^-, a^0; u^-)\right) = \Sigma(a^-, a^+; u^-)\,.$

With T as in Lemma 1.2.1. Alternatively, by (1.2.3), the above condition **(Σ2)** can be restated as

$$\left.\begin{array}{rcl} \Psi(a^-, u^-; a^0, u^0) & = & 0 \\ \Psi(a^0, u^0; a^+, u^+) & = & 0 \end{array}\right\} \Rightarrow \Psi(a^-, u^-; a^+, u^+) = 0\,.$$

Condition **(Σ2)** says that if the two Riemann problems with initial states $(a^-, u^-), (a^0, u^0)$ and $(a^0, u^0), (a^+, u^+)$ both yield the stationary solution, then also the Riemann problem with initial state (a^-, u^-) and (a^+, u^+) is solved by the stationary solution.

Remark that the *"natural"* choice (1.2.19) implied by a smooth section satisfies **(Σ0)**, **(Σ1)** and **(Σ2)**.

Denote now by $\hat u$ a map satisfying

$$\hat u(x) = \begin{cases} \hat u^- & \text{if } x < 0 \\ \hat u^+ & \text{if } x > 0 \end{cases} \quad \text{with} \quad \begin{array}{l} \Psi\left(a^-, \hat u^-; a^+, \hat u^+\right) = 0, \\ \hat u^-, \hat u^+ \in A_0. \end{array} \tag{1.2.5}$$

The existence of such a map follows from Lemma 1.2.1. Recall first the definition of weak Ψ-solution, see [21, Definition 2.1] and [26, Definition 2.1].

Definition 1.2.2 *Let* Σ *satisfy* **(Σ0)**–**(Σ2)**. *A weak* Ψ*-solution to (1.1.2)–(1.1.3) is a map*

$$\begin{array}{rcl} u & \in & \mathbf{C^0}\left(\mathbb{R}^+; \hat u + \mathbf{L^1}(\mathbb{R}^+; \mathring{\mathbb{R}}^+ \times \mathbb{R})\right) \\ u(t) & \in & \mathbf{BV}(\mathbb{R}; \mathring{\mathbb{R}}^+ \times \mathbb{R}) \quad \textit{for a.e. } t \in \mathbb{R}^+ \end{array} \tag{1.2.6}$$

such that

(W) *for all* $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$ *whose support does not intersect* $x = 0$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(u\, \partial_t \varphi + f(u)\, \partial_x \varphi\right) dx\, dt = 0\,;$$

($\mathbf{\Psi}$) *for a.e. $t \in \mathbb{R}^+$ and with Ψ as in (1.2.3), the junction condition is met:*

$$\Psi\left(a^-, u(t,0-); a^+, u(t,0+)\right) = 0\,.$$

It is also an entropy solution if

(E) *for all $\varphi \in \mathbf{C}^1_{\mathbf{c}}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R}^+)$ whose support does not intersect $x = 0$*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(E(u)\,\partial_t \varphi + F(u)\,\partial_x \varphi\right)\,dx\,dt \geq 0\,.$$

In the particular case of a Riemann Problem, i.e. of (1.1.1) with initial datum

$$u(0,x) = \left\{\begin{array}{lccc} u^- & \text{if} & x > & 0 \\ u^+ & \text{if} & x < & 0\,, \end{array}\right.$$

Definition 1.2.2 reduces to [26, Definition 2.1].

To state the uniqueness property in the theorems below, we need to introduce the following integral conditions, following [16, Theorem 9.2], see also [45, Theorem 8] and [1]. Given a function $u = u(t,x)$ and a point (τ, ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the homogeneous Riemann Problem consisting of (1.1.2)–(1.1.3)–(1.2.3) with initial datum at time τ

$$w(\tau,x) = \left\{\begin{array}{lccc} \lim_{x\to\xi-} u(\tau,x) & \text{if} & x < & \xi \\ \lim_{x\to\xi+} u(\tau,x) & \text{if} & x > & \xi\,. \end{array}\right. \tag{1.2.7}$$

and with Σ satisfying **($\mathbf{\Sigma 0}$)**, **($\mathbf{\Sigma 1}$)** and **($\mathbf{\Sigma 2}$)**. Moreover, define $U^\flat_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$\left\{\begin{array}{ll} \partial_t \omega + \partial_x \widetilde{A}\omega = 0 & t \geq \tau \\ w(\tau,x) = u(\tau,x)\,, & \end{array}\right. \tag{1.2.8}$$

with $\widetilde{A} = Df\left(u(\tau,\xi)\right)$.

The next theorem applies [26, Theorem 3.2] to (1.1.2) with the choice (1.2.3) to construct the semigroup generated by (1.1.2)–(1.1.3)–(1.2.3). The uniqueness part follows from [45, Theorem 2].

Theorem 1.2.3 *Let p satisfy **(P)** and Σ satisfy **($\mathbf{\Sigma 0}$)–($\mathbf{\Sigma 2}$)**. Choose any $\bar a > 0$, $\bar u \in A_0$. Then, there exist a positive Δ such that for all a^-, a^+ with $\left|a^- - \bar a\right| < \Delta$ and $\left|a^+ - \bar a\right| < \Delta$, there exist a map $\hat u$ as in (1.2.5), positive δ, L and a semigroup $S\colon \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$ such that*

1. *$\mathcal{D} \supseteq \left\{u \in \hat u + \mathbf{L}^1(\mathbb{R}; A_0)\colon \mathrm{TV}(u - \hat u) < \delta\right\}$.*
2. *For all $u \in \mathcal{D}$, $S_0 u = u$ and for all $t, s \geq 0$, $S_t S_s u = S_{s+t} u$.*

3. *For all* $u, u' \in \mathcal{D}$ *and for all* $t, t' \geq 0$,

$$\left\|S_t u - S_{t'} u'\right\|_{\mathbf{L}^1} \leq L \cdot \left(\left\|u - u'\right\|_{\mathbf{L}^1} + \left|t - t'\right|\right).$$

4. *If* $u \in \mathcal{D}$ *is piecewise constant, then for* t *small,* $S_t u$ *is the gluing of solutions to Riemann problems at the points of jump in* u *and at the junction at* $x = 0$.

5. *For all* $u \in \mathcal{D}$*, the orbit* $t \to S_t u$ *is a weak* Ψ*-solution to (1.1.2).*

6. *Let* $\hat\lambda$ *be an upper bound for the moduli of the characteristic speeds in* $\bar B\left(\hat u(\mathbb{R}), \delta\right)$. *For all* $u \in \mathcal{D}$*, the orbit* $u(t) = S_t u$ *satisfies the integral conditions*

   (i) For all $\tau > 0$ *and* $\xi \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{1}{h} \int_{\xi - h\hat\lambda}^{\xi + h\hat\lambda} \left\|u(\tau + h, x) - U^\sharp_{(u;\tau,\xi)}(\tau + h, x)\right\| dx = 0\,. \quad (1.2.9)$$

   (ii) There exists a $C > 0$ *such that for all* $\tau > 0$, $a, b \in \mathbb{R}$ *and* $\xi \in\,]a, b[$,

$$\begin{aligned} &\frac{1}{h} \int_{a + h\hat\lambda}^{b - h\hat\lambda} \left\|u(\tau + h, x) - U^\flat_{(u;\tau,\xi)}(\tau + h, x)\right\| dx \\ &\qquad \leq C \left[\mathrm{TV}\left\{u(\tau);]a, b[\right\}\right]^2. \end{aligned} \quad (1.2.10)$$

7. *If a Lipschitz map* $w\colon \mathbb{R} \to \mathcal{D}$ *satisfies (2.1.8)–(2.1.9), then it coincides with the semigroup orbit:* $w(t) = S_t\left(w(0)\right)$.

The proof is deferred to Paragraph 1.3.1. Note that, similarly to what happens in the standard case of [16, Theorem 9.2], condition (2.1.9) is always satisfied at a junction.

1.2.2 A Pipe with Piecewise Constant Section

We consider now a tube with piecewise constant section

$$a = a_0\, \chi_{]-\infty, x_1]} + \sum_{j=1}^{n-1} a_j\, \chi_{[x_j, x_{j+1}[} + a_n\, \chi_{[x_n, +\infty[}$$

for a suitable $n \in \mathbb{N}$. The fluid in each pipe is modeled by (1.1.2). At each junction x_j, we require condition (1.1.3), namely

$$\Psi(a_{j-1}, u_j^-; a_j, u_j^+) = 0 \qquad \begin{array}{l} \text{for all } j = 1, \ldots, n, \text{ where} \\ u_j^\pm = \lim_{x \to x_j \pm} u_j(x)\,. \end{array} \quad (1.2.11)$$

We omit the formal definition of Ψ-solution to (1.1.2)–(1.1.3) in the present case, since it is an obvious iteration of Definition 1.2.2.

Theorem 1.2.4 *Let p satisfy **(P)** and Σ satisfy **(Σ0)**–**(Σ2)**. For any $\bar{a} > 0$ and any $\bar{u} \in A_0$ there exist positive $M, \Delta, \delta, L, \mathcal{M}$ such that for any profile satisfying*

(A0) $a \in \mathbf{PC}\left(\mathbb{R};]\bar{a}-\Delta, \bar{a}+\Delta[\right)$ *with* $\mathrm{TV}(a) < M$,

there exists a piecewise constant stationary solution

$$\hat{u} = \hat{u}_0 \chi_{]-\infty, x_1[} + \sum_{j=1}^{n-1} \hat{u}_j \chi_{]x_j, x_{j+1}[} + \hat{u}_n \chi_{]x_n, +\infty[}$$

to (1.1.2)–(1.2.11) satisfying

$$\begin{array}{l} \hat{u}_j \in A_0 \text{ with } \left|\hat{u}_j - \bar{u}\right| < \delta \text{ for } j = 0, \ldots n \\ \Psi\left(a_{j-1}, \hat{u}_{j-1}; a_j, \hat{u}_j\right) = 0 \text{ for } j = 1, \ldots, n \\ \mathrm{TV}(\hat{u}) \leq \mathcal{M}\, \mathrm{TV}(a) \end{array} \tag{1.2.12}$$

and a semigroup $S^a : \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

1. $\mathcal{D}^a \supseteq \left\{u \in \hat{u} + \mathbf{L^1}(\mathbb{R}; A_0) : \mathrm{TV}(u - \hat{u}) < \delta\right\}$.
2. *S_0^a is the identity and for all $t, s \geq 0$, $S_t^a S_s^a = S_{s+t}^a$.*
3. *For all $u, u' \in \mathcal{D}^a$ and for all $t, t' \geq 0$,*

$$\left\|S_t^a u - S_{t'}^a u'\right\|_{\mathbf{L^1}} \leq L \cdot \left(\left\|(u) - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right|\right).$$

4. *If $u \in \mathcal{D}^a$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at each junction x_j.*
5. *For all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is a weak Ψ-solution to (1.1.2)–(1.2.11).*
6. *The semigroup satisfies the integral conditions (2.1.8)–(2.1.9) in 6. of Theorem 1.2.3.*
7. *If a Lipschitz map $w : \mathbb{R} \to \mathcal{D}$ satisfies (2.1.8)–(2.1.9), then it coincides with the semigroup orbit: $w(t) = S_t\left(w(0)\right)$.*

Remark that δ and L depend on a only through $\bar{a}$ and $\mathrm{TV}(a)$. In particular, all the construction above is independent from the number of points of jump in a. For every $\bar{u}$, we provide below an estimate of M at the leading order in δ and Δ, see (1.3.11) and (1.3.8). In the case of Σ as in (1.2.19) and with the isothermal pressure law, which obviously satisfies **(P)**,

$$p(\rho) = c^2 \rho\,, \tag{1.2.13}$$

the bounds (1.3.11) and (1.3.8) reduce to the simpler estimate

$$M = \begin{cases} \dfrac{\bar a}{4e} & \text{if } \bar v/c \in \left]0, 1/\sqrt{2}\right], \\ \dfrac{\bar a}{4e}\dfrac{1-(\bar v/c)^2}{(\bar v/c)^2} & \text{if } \bar v/c \in \left]1/\sqrt{2}, 1\right[, \end{cases} \tag{1.2.14}$$

where $\bar v = \bar q/\bar\rho$. Note that, as it is physically reasonable, M is a weakly decreasing function of $\bar v$, so that at lower fluid speeds, higher values for the total variation of the pipe's section can be accepted.

Furthermore, the estimates proved in Section 1.3.2 show that the total variation of the solution to (1.1.2)–(1.2.11) may grow unboundedly if $\mathrm{TV}(a)$ is large. Consider the case in Figure 1.1. A wave σ_2^- hits a junction where

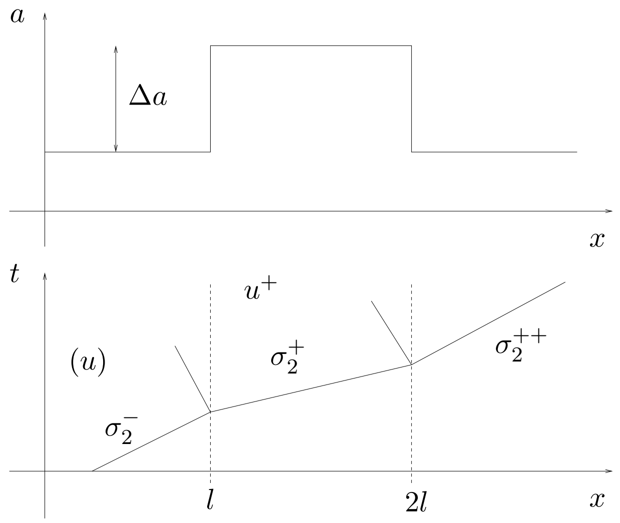


Figure 1.1: A wave σ_2^- hits a junction, giving rise to σ_2^+ which hits a second junction.

the pipe's section increases by $\Delta a > 0$. From this interaction, the wave σ_2^+ of the second family arises, which hits the second junction where the section diminishes by Δa. At the leading term in Δa, we have the estimate

$$\left|\sigma_2^{++}\right| \le \left(1 + \mathcal{K}(\bar v/c)\left(\frac{\Delta a}{a}\right)^2\right)\left|\sigma_2^-\right|, \qquad \text{where} \tag{1.2.15}$$

$$\mathcal{K}(\xi) = \frac{-1 + 8\,\xi^2 - 7\,\xi^4 + 2\,\xi^6}{2(1-\xi)^3\,(1+\xi)^3}, \tag{1.2.16}$$

see Section 1.3.2 for the proof. Note that $\mathcal{K}(0) = -1$ whereas $\lim_{\xi\to 1-}\mathcal{K}(\xi) = +\infty$. Therefore, for any fixed Δa, if $\bar v$ is sufficiently near to c, repeating the interactions in Figure 1.1 a sufficient number of times makes the 2 shock waves arbitrarily large.

1.2.3 A Pipe with a $\mathbf{W^{1,1}}$ Section

In this paragraph, the pipe's section a is assumed to satisfy

$$\textbf{(A1)} \qquad \left\{\begin{array}{l} a \in \mathbf{W^{1,1}}\left(\mathbb{R};]\bar a - \Delta, \bar a + \Delta[\right) \text{ for suitable } \Delta > 0,\ \bar a > \Delta \\ \mathrm{TV}(a) < M \text{ for a suitable } M > 0 \\ a'(x) = 0 \text{ for a.e. } x \in \mathbb{R} \setminus [-X, X] \text{ for a suitable } X > 0\,. \end{array}\right.$$

For smooth solutions, the equivalence of (1.1.1) and (1.1.4) is immediate. Note that the latter is in the standard form of a 1D conservation law and the usual definition of weak entropy solution applies, see for instance [68, Definition 3.5.1] or [32, Section 6]. The definition below of weak entropy solution to (1.1.1) makes the two systems fully equivalent also for non smooth solutions.

Definition 1.2.5 *A weak solution to* (1.1.1) *is a map*

$$u \in \mathbf{C^0}\left(\mathbb{R}^+; \hat u + \mathbf{L^1}(\mathbb{R}; \mathring{\mathbb{R}}^+ \times \mathbb{R})\right)$$

such that for all $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a\rho \\ aq \end{bmatrix} \partial_t \varphi + \begin{bmatrix} aq \\ aP(u) \end{bmatrix} \partial_x \varphi + \begin{bmatrix} 0 \\ p(\rho)\partial_x a \end{bmatrix} \varphi \right) dx\, dt = 0\,. \qquad (1.2.17)$$

u is an entropy weak solution if, for any $\varphi \in \mathbf{C^1_c}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R})$, $\varphi \geq 0$,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(a\, E(u)\, \partial_t \varphi + a\, F(u)\, \partial_x \varphi\right)\, dx\, dt \geq 0\,. \qquad (1.2.18)$$

Lemma 1.2.6 *Let a satisfy **(A1)**. Then, u is a weak entropy solution to (1.1.1) in the sense of Definition 1.2.5, if and only if it is a weak entropy solution of (1.1.4).*

The proof is deferred to Section 1.3.3.

Now, the section a of the pipe is sufficiently regular to select stationary solutions as solutions to either of the systems (1.1.5), which are equivalent by Lemma 1.2.6. Hence, the smoothness of a also singles out a specific choice of Σ, see [45, formula (14)].

Proposition 1.2.7 *Fix* $a^-, a^+ \in\ ]\bar a - \Delta, \bar a + \Delta[$ *and* $u^- \in A_0$. *Choose a function a strictly monotone, in* $\mathbf{C^1}$, *that satisfies **(A1)** with* $a(-X-) = a^-$ *and* $a(X+) = a^+$. *Call* $\rho = R^a(x; u^-)$ *the ρ-component of the corresponding solution to either of the Cauchy problems (1.1.5) with initial condition* $u(-X) = u^-$. *Then,*

1. *the function*

$$\Sigma(a^-, a^+, u^-) = \left[\begin{array}{c} 0 \\ \int_{-X}^{X} p\left(R^a(x; u^-)\right) a'(x)\, dx \end{array} \right] \tag{1.2.19}$$

satisfies **(Σ0)**–**(Σ2)**;

2. *if* $\tilde{a}$ *is a strictly monotone function satisfying the same requirements above for* a*, the corresponding map* $\tilde{\Sigma}$ *coincides with* Σ.

The basic well posedness theorem in the present $\mathbf{W^{1,1}}$ case is stated similarly to Theorem 1.2.4.

Theorem 1.2.8 *Let* p *satisfy* **(P)**. *For any* $\bar{a} > 0$ *and any* $\bar{u} \in A_0$ *there exist positive* M, Δ, δ, L *such that for any profile* a *satisfying* **(A1)** *there exists a stationary solution* $\hat{u}$ *to (1.1.1) satisfying*

$$\hat{u} \in A_0 \text{ with } \left\|\hat{u}(x) - \bar{u}\right\| < \delta \text{ for all } x \in \mathbb{R}$$

and a semigroup $S^a \colon \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ *such that*

1. $\mathcal{D}^a \supseteq \left\{u \in \hat{u} + \mathbf{L^1}(\mathbb{R}; A_0) \colon \mathrm{TV}(u - \hat{u}) < \delta\right\}$.

2. S_0^a *is the identity and for all* $t, s \geq 0$, $S_t^a S_s^a = S_{s+t}^a$.

3. *for all* $u, u' \in \mathcal{D}^a$ *and for all* $t, t' \geq 0$,

$$\left\|S_t^a u - S_{t'}^a u'\right\|_{\mathbf{L^1}} \leq L \cdot \left(\left\|u - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right|\right).$$

4. *for all* $u \in \mathcal{D}^a$*, the orbit* $t \to S_t^a u$ *is solution to (1.1.2) in the sense of Definition 1.2.5.*

5. *Let* $\hat{\lambda}$ *be an upper bound for the moduli of the characteristic speeds in* $\bar{B}\left(\hat{u}(\mathbb{R}), \delta\right)$. *For all* $u \in \mathcal{D}$*, the orbit* $u(t) = S_t u$ *satisfies the integral conditions*

   (i) For all $\tau < 0$ *and* $\xi \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{1}{h} \int_{\xi - h\hat{\lambda}}^{\xi + h\hat{\lambda}} \left\|u(\tau + h, x) - U^{\sharp}_{(u;\tau,\xi)}(\tau + h, x)\right\| dx = 0\,. \tag{1.2.20}$$

   (ii) There exists a $C > 0$ *such that, for all* $\tau > 0$, $a, b \in \mathbb{R}$ *and* $\xi \in \left]a, b\right[$,

$$\begin{aligned} &\frac{1}{h} \int_{a + h\hat{\lambda}}^{b - h\hat{\lambda}} \left\|u(\tau + h, x) - U^{\flat}_{(u;\tau,\xi)}(\tau + h, x)\right\| dx \\ &\leq C \left[\mathrm{TV}\left\{u(\tau); \left]a, b\right[\right\} + \mathrm{TV}\left\{a; \left]a, b\right[\right\}\right]^2. \end{aligned} \tag{1.2.21}$$

6. *If a Lipschitz map $w: \mathbb{R} \to \mathcal{D}$ solves (1.1.1), then it coincides with the semigroup orbit:* $w(t) = S_t\left(w(0)\right)$.

Thanks to Theorem 1.2.4, the proof is obtained approximating a with a piecewise constant function a_n. The corresponding problems (1.1.2)–(1.2.11) generate semigroups defined on domains characterized by uniform bounds on the total variation and with a uniformly bounded Lipschitz constants for their time dependence. Then, we pass to the limit (see Section 4 for the proof) and we follow the same procedure as in [16, Theorem 9.2] and [45, theorems 2 and 8] to characterize the solution.

As a byproduct of the proof of Theorem 1.2.8, we also obtain the following convergence result, relating the construction in Theorem 1.2.4 to that of Theorem 1.2.8.

Proposition 1.2.9 *Under the same assumptions of Theorem 1.2.8, for every $n \in \mathbb{N}$, choose a function β_n such that:*

(i) β_n *is piecewise constant with points of jump* $y_n^1, \ldots, y_n^{m_n}$, *with* $y_n^1 = -X$, $y_n^{m_n} = X$, *and* $\max_j (y_n^{j+1} - y_n^j) \leq 1/n$.

(ii) $\beta_n(x) = 0$ *for all* $x \in \mathbb{R} \setminus [-X, X]$.

(iii) $\beta_n \to a'$ *in* $\mathbf{L^1}(\mathbb{R};\mathbb{R})$ *with* $\|\beta_n\|_{\mathbf{L^1}} \leq M$, *with M as in Theorem 1.2.8.*

Define $\alpha_n(x) = a(-X-) + \int_{-X}^{x} \beta_n(\xi)\, d\xi$ *and points* $x_n^j \in \left]y_n^j, y_n^{j+1}\right[$ *for* $j = 1, \ldots, m_n - 1$ *and let*

$$a_n = a(-X-)\, \chi_{]-\infty, x_j^1[} + \sum_{j=1}^{m_n-1} \alpha_n(y_n^{j+1})\, \chi_{[x_n^j, x_n^{j+1}[} + a(X+)\, \chi_{[x_n^{m_n}, +\infty[}$$

*(see Figure 1.2) . Then, a_n satisfies **(A0)** and the corresponding semigroup S^n constructed in Theorem 1.2.4 converges pointwise to the semigroup S constructed in Theorem 1.2.8.*

1.3 Technical Proofs

1.3.1 Proofs Related to Section 1.2.1

The following equalities will be of use below:

$$\partial_\rho P = -\lambda_1\, \lambda_2 \qquad \text{and} \qquad \partial_q P = \lambda_1 + \lambda_2\,. \tag{1.3.1}$$

Proof of Lemma 1.2.1. Apply the Implicit Function Theorem to the equality $\Psi = 0$ in a neighborhood of $(\bar{a}, \bar{u}, \bar{a}, \bar{u})$, which satisfies $\Psi = 0$

by **(Σ1)**. Observe that $\partial_u \Sigma(a, a; u^-) = 0$ by **(Σ1)**. Using (1.3.1), compute

$$\begin{aligned}
\det \partial_{u^+} \Psi(\bar a, \bar u, \bar a, \bar u) &= \det \begin{bmatrix} -\partial_{\rho^+}\Sigma_1 & \bar a \\ \bar a\, \partial_{\rho^+} P & \bar a\, \partial_{q^+} P \end{bmatrix} \\
&= \det \begin{bmatrix} 0 & \bar a \\ \bar a\, \partial_{\rho^+} P & \bar a\, \partial_{q^+} P \end{bmatrix} \\
&= \bar a^2\, \lambda_1(\bar u)\, \lambda_2(\bar u) \\
&\neq 0\,,
\end{aligned}$$

completing the proof. □

Proof of Theorem 1.2.3. Let Δ be defined as in Lemma 1.2.1. Assumption **(F)** in [26, Theorem 3.2] follows from **(P)**, thanks to (1.2.1) and to the choices (1.2.2)–(1.2.5). We now verify condition [26, formula (2.2)]. Recall that $D_{u^-}\Sigma(\bar a, \bar a; \bar u) = 0$ by **(Σ1)**. Hence, using (1.3.1),

$$\begin{aligned}
& \det \left[D_{u^-}\Psi(\bar a, \bar u; \bar a, \bar u) \cdot r_1(\bar u) \quad D_{u^+}\Psi(\bar a, \bar u; \bar a, \bar u) \cdot r_2(\bar u) \right] \\
= \;& \det \begin{bmatrix} \bar a \lambda_1(\bar u) + \partial_{\rho^-}\Sigma_1(\bar a, \bar a; \bar u) + \lambda_1 \partial_{q^-}\Sigma_1(\bar a, \bar a; \bar u) & \bar a \lambda_2(\bar u) \\ \bar a\left(\lambda_1(\bar u)\right)^2 + \partial_{\rho^-}\Sigma_2(\bar a, \bar a; \bar u) + \lambda_1^- \partial_{q^-}\Sigma_2(\bar a, \bar a; \bar u) & \bar a \left(\lambda_2(\bar u)\right)^2 \end{bmatrix} \\
= \;& \det \begin{bmatrix} \bar a\, \lambda_1(\bar u) & \bar a\, \lambda_2(\bar u) \\ \bar a\, \left(\lambda_1(\bar u)\right)^2 & \bar a\, \left(\lambda_2(\bar u)\right)^2 \end{bmatrix} \\
= \;& \bar a^2\, \lambda_1(\bar u)\, \lambda_2(\bar u) \left(\lambda_2(\bar u) - \lambda_1(\bar u)\right) \\
\neq \;& 0\,.
\end{aligned}$$

The proof of 1.–5. is completed applying [26, Theorem 3.2]. The obtained semigroup coincides with that constructed in [45, Theorem 2], where the uniqueness conditions 6. and 7. are proved. □

1.3.2 Proofs Related to Section 1.2.2

We now work towards the proof of Theorem 1.2.4. We first use the wave front tracking technique to construct approximate solutions to the Cauchy problem (1.1.2)–(1.2.11) adapting the wave front tracking technique introduced in [16, Chapter 7].

Fix an initial datum $u_o \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0)$ and an $\varepsilon > 0$. Approximate u_o with a piecewise constant initial datum u_o^ε having a finite number of discontinuities and so that $\lim_{\varepsilon \to 0} \left\| u_o^\varepsilon - u_o \right\|_{\mathbf{L^1}} = 0$. Then, at each junction and at each point of jump in u_o^ε along the pipe, we solve the corresponding Riemann Problem according to Definition 1.2.2. If the total variation of the initial datum is sufficiently small, then Theorem 1.2.3 ensures the existence and uniqueness of solutions to each Riemann Problem. We approximate each rarefaction wave with a rarefaction fan, i.e. by means of (non entropic) shock waves traveling at the characteristic speed of the state to the right of the shock and with size at most ε.

This construction can be extended up to the first time $\bar{t}_1$ at which two waves interact in a pipe or a wave hits the junction. At time $\bar{t}_1$ the functions so constructed are piecewise constant with a finite number of discontinuities. At any subsequent interaction or collision with the junction, we repeat the previous construction with the following provisions:

1. no more than 2 waves interact at the same point or at the junction;

2. a rarefaction fan of the i-th family produced by the interaction between an i-th rarefaction and any other wave, is *not* split any further;

3. when the product of the strengths of two interacting waves falls below a threshold $\check{\varepsilon}$, then we let the waves cross each other, their size being unaltered, and introduce a *non physical* wave with speed $\hat{\lambda}$, with $\hat{\lambda} > \sup_{(u)} \lambda_2(u)$; see [16, Chapter 7] and the refinement [9].

We complete the above algorithm stating how Riemann Problems at the junctions are solved. We use the same rules as in [21, § 4.2] and [26, § 5]. In particular, at time $t = 0$ and whenever a physical wave with size greater than $\check{\varepsilon}$ hits the junction, the accurate solver is used, i.e. the exact solution is approximated replacing rarefaction waves with rarefaction fans. When a non physical wave hits the junction, then we let it be refracted into a non physical wave with the same speed $\hat{\lambda}$ and no other wave is produced.

Repeating recursively this procedure, we construct a wave front tracking sequence of approximate solutions u_ε in the sense of [16, Definition 7.1].

At interactions of waves in a pipe, we have the following classical result.

Lemma 1.3.1 *Consider interactions in a pipe. Then, there exists a positive K with the properties:*

1. *An interaction between the wave σ_1^- of the first family and σ_2^- of the second family produces the waves σ_1^+ and σ_2^+ with*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| \leq K \cdot \left|\sigma_1^- \sigma_2^-\right| . \tag{1.3.2}$$

2. *An interaction between σ_i' and σ_i'' both of the same i-th family produces waves of total size σ_1^+ and σ_2^+ with*

$$\begin{aligned} \left|\sigma_1^+ - (\sigma_1'' + \sigma_1')\right| + \left|\sigma_2^+\right| &\leq K \cdot \left|\sigma_1' \sigma_1''\right| \qquad \textit{if } i = 1\,, \\ \left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma_2'' + \sigma_2')\right| &\leq K \cdot \left|\sigma_2' \sigma_2''\right| \qquad \textit{if } i = 2\,. \end{aligned}$$

3. *An interaction between the physical waves σ_1^- and σ_2^- produces a non physical wave σ_3^+, then*

$$\left|\sigma_3^+\right| \leq K \cdot \left|\sigma_1^- \sigma_2^-\right|.$$

4. *An interaction between a physical wave σ and a non physical wave σ_3^- produces a physical wave σ and a non physical wave σ_3^+, then*

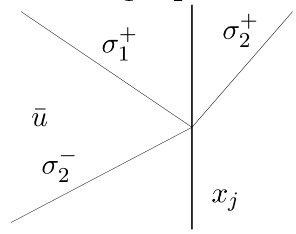
$$\left|\sigma_3^+\right| - \left|\sigma_3^-\right| \leq K \cdot \left|\sigma\sigma_3^-\right|.$$

For a proof of this result see [16, Chapter 7]. Differently from the constructions in [21, 26], we now can not avoid the interaction of non physical waves with junctions. Moreover, the estimates found therein do not allow to pass to the limit $n \to +\infty$, n being the number of junctions.

Lemma 1.3.2 *Consider interactions at the junction sited at x_j. There exist positive K_1, K_2, K_3 with the following properties.*

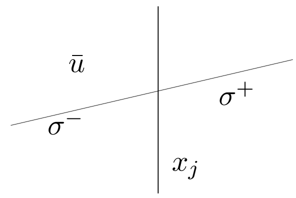
1. *The wave σ_2^- hits the junction. The resulting waves σ_1^+, σ_2^+ satisfy*

$$\begin{aligned}
\left|\sigma_1^+\right| &\leq K_1 \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right|, \\
\left|\sigma_2^+\right| &\leq \left(1 + K_2 \left|a_j - a_{j-1}\right|\right) \left|\sigma_2^-\right| \\
&\leq e^{K_2 \left|a_j - a_{j-1}\right|} \left|\sigma_2^-\right|.
\end{aligned}$$



2. *The non-physical wave σ^- hits the junction. The resulting wave σ^+ satisfies*

$$\begin{aligned}
\left|\sigma^+\right| &\leq \left(1 + K_3 \left|a_j - a_{j-1}\right|\right) \left|\sigma^-\right| \\
&\leq e^{K_3 \left|a_j - a_{j-1}\right|} \left|\sigma^-\right|.
\end{aligned}$$



Proof. Use the notation in the figure above. Recall that σ_1^+ and σ_2^+ are computed through the Implicit Function Theorem applied to a suitable combination of the Lax curves of (1.1.2), see [21, Proposition 2.4] and [26, Proposition 2.2]. Repeating the proof of Theorem 1.2.3 one shows that the Implicit Function Theorem can be applied. Therefore, the regularity of the Lax curves and **(P)** ensure that $\sigma_1^+ = \sigma_1^+\left(\sigma_2^-, a_j - a_{j-1}; \bar{u}\right)$ and $\sigma_2^+ = \sigma_2^+\left(\sigma_2^-, a_j - a_{j-1}; \bar{u}\right)$. An application of [16, Lemma 2.5], yields

$$\left.\begin{aligned}
\sigma_1^+\left(0, a_j - a_{j-1}; \bar{u}\right) &= 0 \\
\sigma_1^+\left(\sigma_2^-, 0; \bar{u}\right) &= 0
\end{aligned}\right\} \Rightarrow \left|\sigma_1^+\right| \leq K_1 \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right|,$$

$$\left.\begin{array}{rcl} \sigma_2^+\left(0, a_j - a_{j-1}; \bar u\right) & = & 0 \\ \sigma_2^+\left(\sigma_2^-, 0; \bar u\right) & = & \sigma_2^- \end{array}\right\} \Rightarrow \left|\sigma_2^+ - \sigma_2^-\right| \le K_2 \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| \quad (1.3.3)$$

$$\Rightarrow \left|\sigma_2^+\right| \le \left[1 + K_2\left|a_j - a_{j-1}\right|\right]\left|\sigma_2^-\right|,$$

completing the proof of 1. The estimate at 2. is proved similarly. □

We now aim at an improvement of (1.3.3). Solving the Riemann problem at the interaction in case 1. amounts to solve the system

$$\mathcal{L}_2\left(T\left(\mathcal{L}_1(\bar u; \sigma_1^+)\right); \sigma_2^+\right) = T\left(\mathcal{L}_2(\bar u; \sigma_2^-)\right). \quad (1.3.4)$$

By (1.2.1), the first order expansions in the wave's sizes of the Lax curves exiting u are

$$\mathcal{L}_1(u;\sigma) = \left[\begin{array}{c} \rho - \sigma + o(\sigma) \\ q - \lambda_1(u)\sigma + o(\sigma) \end{array}\right] \text{ and } \mathcal{L}_2(u;\sigma) = \left[\begin{array}{c} \rho + \sigma + o(\sigma) \\ q + \lambda_2(u)\sigma + o(\sigma) \end{array}\right],$$

while the first order expansion in the size's difference $\Delta a = a^+ - a^-$ of the map T defined at (1.2.4), with $v = q/\rho$, is

$$\begin{array}{rcl} T(a, a+\Delta a; u) & = & \left[\begin{array}{c} \left(1 + H\frac{\Delta a}{a}\right)\rho + o(\Delta a) \\ \left(1 - \frac{\Delta a}{a}\right) q + o(\Delta a) \end{array}\right], \text{ where} \quad (1.3.5) \\ H & = & \dfrac{v^2 + \frac{\partial_{a^+}\Sigma - p(\rho)}{\rho}}{c^2 - v^2}. \end{array}$$

Inserting these expansions in (1.3.4), we get the following linear system for σ_1^+, σ_2^+:

$$\left\{\begin{array}{rcrcl} -\left(1 + \bar H\dfrac{\Delta a}{\bar a}\right)\sigma_1^+ & + & \sigma_2^+ & = & \left(1 + \bar H\frac{\Delta a}{\bar a}\right)\sigma_2^- \\ -\left(1 - \dfrac{\Delta a}{\bar a}\right)\bar\lambda_1\,\sigma_1^+ & + & \left(1 + \bar G\frac{\Delta a}{\bar a}\right)\bar\lambda_2\,\sigma_2^+ & = & \left(1 - \frac{\Delta a}{\bar a}\right)\bar\lambda_2\sigma_2^- \end{array}\right.$$

where

$$\bar H = \frac{\bar v^2 + \left(\partial_{a^+}\Sigma(\bar a, \bar a, \bar u) - p(\bar\rho)\right)/\bar\rho}{c^2 - \bar v^2} \quad \text{and} \quad \bar G = \frac{(c'(\bar\rho)\bar\rho - \bar v)\bar H - \bar v}{\bar v + c}$$

and all functions are computed in $\bar u$. The solution is

$$\sigma_1^+ = -\frac{\bar\lambda_2}{2c}\left(1 + \bar G + \bar H\right)\frac{\Delta a}{a}\,\sigma_2^- \quad (1.3.6)$$

$$\sigma_2^+ = \left(1 - \frac{\bar\lambda_1 \bar H + \bar\lambda_2(1 + \bar G)}{2c}\,\frac{\Delta a}{a}\right)\sigma_2^- \quad (1.3.7)$$

which implies the following first order estimate for the coefficients in the interaction estimates of Lemma 1.3.2:

$$
\begin{aligned}
K_1 &= \frac{1}{2a}\left|\frac{1+\frac{c'\rho}{c}\left(\frac{v}{c}\right)^2+\frac{1}{c^2}\left(\frac{c'\rho}{c}+1\right)\frac{\partial_{a^+}\Sigma-p(\rho)}{\rho}}{1-\left(\frac{v}{c}\right)^2}\right|, \\
K_2 &= \frac{1}{2a}\left|\frac{1-2\left(\frac{v}{c}\right)^2+\frac{c'\rho}{c}\left(\frac{v}{c}\right)^2+\frac{1}{c^2}\left(\frac{c'\rho}{c}-1\right)\frac{\partial_{a^+}\Sigma-p(\rho)}{\rho}}{1-\left(\frac{v}{c}\right)^2}\right|.
\end{aligned}
\tag{1.3.8}
$$

The estimate (1.3.7) directly implies the following corollary.

Corollary 1.3.3 *If $\left|a_j - a_{j-1}\right|$ is sufficiently small, then σ_2^+ and σ_2^- are either both rarefactions or both shocks.*

Denote by $\sigma_{i,\alpha}^j$ the wave belonging to the i-th family and sited at the point of jump x^α, with x^α in the j-th pipe I_j, where we set $I_0 = \left]-\infty, x_1\right[$, $I_j = \left]x_j, x_{j+1}\right[$ for $j = 1, \ldots, n-1$ and $I_n = \left]x_n, +\infty\right[$. Aiming at a bound on the Total Variation of the approximate solution, we define the Glimm-like functionals, see [16, formulæ (7.53) and (7.54)] or also [33, 38, 58, 70],

$$
\begin{aligned}
V &= \sum_{j=0}^{n}\sum_{x^\alpha\in I_j}\left(\left|\sigma_{1,\alpha}^j\right| e^{C\sum_{h=1}^{j}|a_h-a_{h-1}|}+\left|\sigma_{2,\alpha}^j\right| e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\right) \\
&\quad+\sum_{j=0}^{n} e^{C\sum_{h=j}^{n-1}|a_{h+1}-a_h|}\sum_{\sigma \text{ non physical in } I_j}|\sigma|\,, \\
Q &= \sum_{(\sigma_{i,\alpha}^j,\sigma_{i',\alpha'}^{j'})\in\mathcal{A}}\left|\sigma_{i,\alpha}^j\sigma_{i',\alpha'}^{j'}\right|, \\
\Upsilon &= V+Q\,,
\end{aligned}
\tag{1.3.9}
$$

where C is a positive constant to be specified below. $\mathcal{A}$ is the set of pairs $(\sigma_{i,\alpha}^j, \sigma_{i',\alpha'}^{j'})$ of approaching waves, see [16, Paragraph 3, Section 7.3]. The i-wave $\sigma_{i,\alpha}^j$ sited at x_α and the i'-wave $\sigma_{i',\alpha'}^{j'}$ sited at $x_{\alpha'}$ are approaching if either $i < i'$ and $x_\alpha > x_{\alpha'}$, or if $i = i' < 3$ and $\min\{\sigma_{i,\alpha}^j, \sigma_{i',\alpha'}^{j'}\} < 0$, independently from j and j'. As usual, non physical waves are considered as belonging to a fictitious linearly degenerate 3rd family, hence they are approaching to all physical waves to their right.

It is immediate to note that the weights $\exp\left(C\sum_{h=1}^{j}|a_h - a_{h-1}|\right)$ and $\exp\left(C\sum_{h=j}^{n-1}|a_{h+1} - a_h|\right)$ in the definition of V are uniformly bounded:

$$
\forall j \qquad \left\{
\begin{array}{lllll}
1 & \le & \exp\left(C\sum_{h=1}^{j}|a_h-a_{h-1}|\right) & \le & \exp\left(C\,\mathrm{TV}(a)\right), \\
1 & \le & \exp\left(C\sum_{h=j}^{n-1}|a_{h+1}-a_h|\right) & \le & \exp\left(C\,\mathrm{TV}(a)\right).
\end{array}\right.
\tag{1.3.10}
$$

Below, the following elementary inequality is of use: if $a < b$, then $e^a - e^b < -(b-a)e^a$.

Lemma 1.3.4 *There exists a positive δ such that if an ε-approximate wave front tracking solution $u = u(t,x)$ has been defined up to time $\bar t$, $\Upsilon\left(u(\bar t-)\right) < \delta$ and an interaction takes place at time $\bar t$, then the ε-solution can be extended beyond time $\bar t$ and $\Upsilon\left(u(\bar t+)\right) < \Upsilon\left(u(\bar t-)\right)$.*

Proof. Thanks to (1.3.10) and Lemma 1.3.1, the standard interaction estimates, see [16, Lemma 7.2], ensure that Υ decreases at any interaction taking place in the interior of I_j, for any $j = 0, \ldots, n$.

Consider now an interaction at x_j. In the case of 1 in Lemma 1.3.2,

$$
\begin{aligned}
& \Delta Q \\
\leq \;& \sum_{(\sigma_1^+, \sigma_{i,\alpha}) \in \mathcal{A}} \left|\sigma_1^+ \sigma_{i,\alpha}\right| + \sum_{(\sigma_2^+, \sigma_{i,\alpha}) \in \mathcal{A}} \left|\sigma_{i,\alpha}\right| \left(\left|\sigma_2^+\right| - \left|\sigma_2^-\right| \right) \\
\leq \;& \left(K_1 \left|a_j - a_{j-1}\right| \sum_{i,\alpha} \left|\sigma_{i,\alpha}\right| + \left(e^{K_2 \left|a_j - a_{j-1}\right|} - 1 \right) \sum_{i,\alpha} \left|\sigma_{i,\alpha}\right| \right) \left|\sigma_2^-\right| \\
\leq \;& (K_1 + K_2)\, \Upsilon(\bar t-) \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| \\
\leq \;& (K_1 + K_2)\, \delta \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| . \\
& \Delta V \\
\leq \;& e^{C \sum_{h=1}^{j-1} |a_h - a_{h-1}|} \left|\sigma_1^+\right| + e^{C \sum_{h=j}^{n-1} |a_{h+1} - a_h|} \left|\sigma_2^+\right| - e^{C \sum_{h=j-1}^{n-1} |a_{h+1} - a_h|} \left|\sigma_2^-\right| \\
\leq \;& e^{C \sum_{h=1}^{j-1} |a_h - a_{h-1}|} \left(K_1 \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| \right) \\
+ \;& \left(e^{C \sum_{h=j}^{n-1} |a_{h+1} - a_h|} e^{K_2 |a_j - a_{j-1}|} - e^{C \sum_{h=j-1}^{n-1} |a_{h+1} - a_h|} \right) \left|\sigma_2^-\right| \\
\leq \;& \left(K_1 |a_j - a_{j-1}| e^{C \sum_{h=1}^{j-1} |a_h - a_{h-1}|} \right) \left|\sigma_2^-\right| \\
+ \;& e^{C \sum_{h=j}^{n-1} |a_{h+1} - a_h|} \left(e^{K_2 |a_j - a_{j-1}|} - e^{C |a_j - a_{j-1}|} \right) \left|\sigma_2^-\right| \\
\leq \;& \left(K_1 |a_j - a_{j-1}| e^{C \sum_{h=1}^{j-1} |a_h - a_{h-1}|} \right) \left|\sigma_2^-\right| \\
& -(C - K_2) \left|a_j - a_{j-1}\right| e^{K_2 |a_j - a_{j-1}|} \, e^{C \sum_{h=j}^{n-1} |a_{h+1} - a_h|} \left|\sigma_2^-\right| \\
\leq \;& \left((K_1 + K_2) \left(1 + e^{K_2 |a^+ - a^-|} \right) e^{C \mathrm{TV}(a)} - C \right) \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| . \\
& \Delta \Upsilon \\
\leq \;& \left((K_1 + K_2) \left(1 + e^{K_2 |a^+ - a^-|} \right) e^{C \mathrm{TV}(a)} + (K_1 + K_2)\delta - C \right) \left|a_j - a_{j-1}\right| \left|\sigma_2^-\right| .
\end{aligned}
$$

Choosing now, for instance,

$$\delta < 1\,,\quad C = \frac{1}{\mathrm{TV}(a)}\,,\quad \left|a^+ - a^-\right| \le \frac{\ln 2}{K_2} \quad \text{and} \quad \mathrm{TV}(a) < \frac{1}{4(K_1+K_2)e} \tag{1.3.11}$$

the monotonicity of Υ in this first case is proved.

Consider an interaction as in 2. of Lemma 1.3.2. Then, similarly,

$$\begin{aligned}
\Delta Q &\le \sum_{(\sigma^+,\sigma_{i,\alpha})\in\mathcal{A}} \left|\sigma_{i,\alpha}\right| \left(\left|\sigma^+\right| - \left|\sigma^-\right|\right)\\
&\le \left(e^{K_3\left|a_j - a_{j-1}\right|} - 1\right) \sum_{i,\alpha} \left|\sigma_{i,\alpha}\right| \left|\sigma^-\right|\\
&\le K_3\,\Upsilon(\bar t-)\left|a_j - a_{j-1}\right| \left|\sigma^-\right|\\
&\le K_3\,\delta\left|a_j - a_{j-1}\right| \left|\sigma^-\right|.\\
\Delta V &\le e^{C\sum_{h=j}^{n-1}\left|a_{h+1}-a_h\right|}\left|\sigma^+\right| - e^{C\sum_{h=j-1}^{n-1}\left|a_{h+1}-a_h\right|}\left|\sigma^-\right|\\
&\le \left(e^{C\sum_{h=j}^{n-1}\left|a_{h+1}-a_h\right|} e^{K_3\left|a_j-a_{j-1}\right|} - e^{C\sum_{h=j-1}^{n-1}\left|a_{h+1}-a_h\right|}\right)\left|\sigma^-\right|\\
&\le e^{C\sum_{h=j}^{n-1}\left|a_{h+1}-a_h\right|}\left(e^{K_3\left|a_j-a_{j-1}\right|} - e^{C\left|a_j-a_{j-1}\right|}\right)\left|\sigma^-\right|\\
&\le (K_3 - C)\left|a_j - a_{j-1}\right| e^{K_3\left|a_j-a_{j-1}\right|} e^{C\sum_{h=j}^{n-1}\left|a_{h+1}-a_h\right|}\left|\sigma^-\right|.\\
\Delta\Upsilon &\le \left(K_3 e^{K_3\left|a^+-a^-\right|} e^{C\,\mathrm{TV}(a)} + K_3\delta - C\right)\left|a_j - a_{j-1}\right|\left|\sigma^-\right|
\end{aligned}$$

and the choice $\delta < 1$ and $C > 2K_3$ ensures that $\Delta\Upsilon < 0$. □

Proof of Theorem 1.2.4. First, observe that the construction of the stationary solution $\hat u$ directly follows from an iterated application of Lemma 1.2.1. The bound (1.2.12) follows from the Lipschitz continuity of the map T defined in Lemma 1.2.1. Define

$$\tilde{\mathcal{D}} = \left\{u \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0) \colon u \in \mathbf{PC} \text{ and } \Upsilon(u) \le \delta\right\},$$

where $\mathbf{PC}$ denotes the set of piecewise constant functions with finitely many jumps. It is immediate to prove that there exists a suitable $C_1 > 0$ such that $\frac{1}{C_1}\mathrm{TV}(u)(t) \le V(t) \le C_1 \mathrm{TV}(u)(t,\cdot)$ for all $(u) \in \tilde{\mathcal{D}}$. Any initial data in $\tilde{\mathcal{D}}$ yields an approximate solution to (1.1.2) attaining values in $\tilde{\mathcal{D}}$ by Lemma 1.3.4.

We pass now to the $\mathbf{L^1}$-Lipschitz continuous dependence of the approximate solutions from the initial datum. Consider two wave front tracking approximate solutions u_1 and u_2 and define the functional

$$\Phi\left(u_1, u_2\right) = \sum_{j=1}^{n}\sum_{i=1}^{2}\int_0^{+\infty}\left|s_i^j(x)\right| W_i^j(x)\,dx\,, \tag{1.3.12}$$

where $s_i^j(x)$ measures the strengths of the i-th shock wave in the j-th pipe at point x (see [16, Chapter 8]) and the weights W_i^j are defined by

$$W_i^j(x) = 1 + \kappa_1 A_i^j(x) + \kappa_1 \kappa_2 \left(\Upsilon(u_1) + \Upsilon(u_2)\right)$$

for suitable positive constants κ_1, κ_2 chosen as in [16, formula (8.7)]. Here Υ is the functional defined in (1.3.9), while the A_i^j are defined by

$$\begin{aligned} A_i^j(x) &= \sum \left\{ \left|\sigma^j_{k_\alpha,\alpha}\right| : \begin{array}{l} x_\alpha < x,\, i < k_\alpha \le 2 \\ x_\alpha > x,\, 1 \le k_\alpha < i \end{array} \right\} \\ &+ \left\{ \begin{array}{ll} \sum \left\{ \left|\sigma^j_{i,\alpha}\right| : \begin{array}{l} x_\alpha < x,\, \alpha \in \mathcal{J}_j(u_1) \\ x_\alpha > x,\, \alpha \in \mathcal{J}_j(u_2) \end{array} \right\} & \text{if } s_i^j(x) < 0, \\ \sum \left\{ \left|\sigma^j_{i,\alpha}\right| : \begin{array}{l} x_\alpha < x,\, \alpha \in \mathcal{J}_j(u_2) \\ x_\alpha > x,\, \alpha \in \mathcal{J}_j(u_1) \end{array} \right\} & \text{if } s_i^j(x) \ge 0; \end{array} \right. \end{aligned}$$

see [16, Chapter 8]. Here, as above, $\sigma^j_{i,\alpha}$ is the wave belonging to the i-th family, sited at x^α, with $x^\alpha \in I_j$. For fixed κ_1, κ_2 the weights $W_i^j(x)$ are uniformly bounded. Hence the functional Φ is equivalent to $\mathbf{L^1}$ distance:

$$\frac{1}{C_2} \cdot \|u_1 - u_2\|_{\mathbf{L^1}} \le \Phi\left(u_1, u_2\right) \le C_2 \cdot \|u_1 - u_2\|_{\mathbf{L^1}}$$

for a positive constant C_2. The same calculations as in [16, Chapter 8] show that, at any time $t > 0$ when an interaction happens neither in u_1 or in u_2,

$$\frac{d}{dt}\Phi\left(u_1(t), u_2(t)\right) \le C_3\, \varepsilon$$

where C_3 is a suitable positive constant depending only on a bound on the total variation of the initial data.

If $t > 0$ is an interaction time for u_1 or u_2, then, by Lemma 1.3.4, $\Delta\left[\Upsilon\left(u_1(t)\right) + \Upsilon\left(u_2(t)\right)\right] < 0$ and, choosing κ_2 large enough, we obtain

$$\Delta\Phi\left(u_1(t), u_2(t)\right) < 0\,.$$

Thus, $\Phi\left(u_1(t), u_2(t)\right) - \Phi\left(u_1(s), u_2(s)\right) \le C_2\, \varepsilon\, (t - s)$ for every $0 \le s \le t$. The proof is now completed using the standard arguments in [16, Chapter 8].

The proof that in the limit $\varepsilon \to 0$ the semigroup trajectory does indeed yield a Ψ-solution to (1.1.2) and, in particular, that (1.2.11) is satisfied on the traces, is exactly as that of [19, Proposition 5.3], completing the proof of 1.–5.

Due to the *local* nature of the conditions (2.1.8)–(2.1.9) and to the finite speed of propagation of (1.1.2), the uniqueness conditions 6. and 7. are proved exactly as in Theorem 1.2.3. □

Proof of estimate (1.2.14). We first compute $\partial_{a^+}\Sigma$, with Σ defined in (1.2.19). To this aim, by 2. in Proposition 1.2.7 (in Paragraph 2.3), we may choose

$$a(x) = \begin{cases} a^- & \text{if } x \in \left]-\infty, -X\right[, \\ \dfrac{a^+ - a^-}{2X}(x+X) + a^- & \text{if } x \in [-X, X], \\ a^+ & \text{if } x \in \left]X, +\infty\right[, \end{cases}$$

so that we may change variable in the integral in (1.2.19) to obtain

$$\partial_{a^+}\Sigma = \partial_{a^+}\left(\int_{a^-}^{a^+} p\left(R^a(\alpha, u)\right)\, d\alpha\right) = p(\rho) + O(\Delta a)\,. \tag{1.3.13}$$

Now, estimate (1.2.14) directly follows inserting (1.2.13) and (1.3.13) in (1.3.11) and (1.3.8). □

Proof of estimates (1.2.15)–(1.2.16). Refer to the notation in Figure 1.1, where the pipe's section is given by

$$a(x) = \begin{cases} a & \text{if } x \in \left]-\infty, l\right[, \\ a + \Delta a & \text{if } x \in [l, 2l], \\ a & \text{if } x \in \left]2l, +\infty\right[, \end{cases}$$

where $\Delta a > 0$. The wave σ_2^+ arises from the interaction with the first junction and hence satisfies (1.3.7). Using the pressure law (1.2.13) and (1.3.13), we obtain

$$\begin{aligned} \sigma_2^+ &= \big(1 + \psi(u, a)\,\Delta a\big)\,\sigma_2^-\,, \qquad \text{where} \\ \psi(a, u) &= -\frac{1}{a}\left(1 - \frac{1/2}{1 - (v/c)^2}\right). \end{aligned}$$

Now we iterate the previous bound to estimate the wave σ_2^{++} which arises from the interaction with the second junction, i.e.

$$\sigma_2^{++} = \Big(1 - \psi(a + \Delta a, u^+)\,\Delta a\Big)\,\sigma_2^+,$$

where, by (1.3.5),

$$\psi(a + \Delta a, u^+) = \psi\left(a + \Delta a, \left(1 + \frac{1}{1 - \left(\frac{v}{c}\right)^2}\,\frac{\Delta a}{a}\right)\rho, \left(1 - \frac{\Delta a}{a}\right) q\right).$$

Introduce $\eta = 1/\left(1-(v/c)^2\right)$ and $\vartheta = \Delta a/a$ to get the estimate

$$
\begin{aligned}
\sigma_2^{++} &= \left(1+\Big(\psi(a,u,)-\psi(a+\Delta a,u^+)\Big)\,\Delta a\right)\sigma_2^-\\
&= \left(1+\frac{\Delta a}{a}\left(-1+\frac{\eta}{2}\right)+\frac{\Delta a}{a+\Delta a}\left(1-\frac{1/2}{1-\left(\frac{1-\vartheta}{1+\eta\vartheta}\frac{v}{c}\right)^2}\right)\right)\sigma_2^-\\
&= \left(1+\frac{\Delta a}{a}\left(-1+\frac{\eta}{2}+\frac{1}{1+\vartheta}\left(1-\frac{1/2}{1-\left(\frac{1-\vartheta}{1+\vartheta\eta}\frac{v}{c}\right)^2}\right)\right)\right)\sigma_2^-
\end{aligned}
$$

and a further expansion to the leading term in Δa gives (1.2.15)–(1.2.16).
□

1.3.3 Proofs Related to Section 1.2.3

Proof of Lemma 1.2.6. If $a \in \mathbf{C^1}\left(\mathbb{R};[a^-,a^+]\right)$ and u is a weak entropy solution of (1.1.4). Then,

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}\rho\\ q\end{bmatrix}\partial_t\varphi+\begin{bmatrix}q\\ P(u)\end{bmatrix}\partial_x\varphi-\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix}\varphi\right)dx\,dt\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}a\rho\\ aq\end{bmatrix}\partial_t\frac{\varphi}{a}+\begin{bmatrix}aq\\ aP(u)\end{bmatrix}\frac{1}{a}\partial_x\varphi-\begin{bmatrix}aq\\ a\frac{q^2}{\rho}\end{bmatrix}\frac{\varphi}{a^2}\,\partial_x a\right)dx\,dt\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\left(\begin{bmatrix}a\rho\\ aq\end{bmatrix}\partial_t\frac{\varphi}{a}+\begin{bmatrix}aq\\ aP(u)\end{bmatrix}\partial_x\frac{\varphi}{a}+\begin{bmatrix}0\\ p(\rho)\partial_x a\end{bmatrix}\frac{\varphi}{a}\right)dx\,dt
\end{aligned}
$$

showing that (1.2.17) holds. Concerning the entropy inequality, compute preliminarily

$$
\begin{aligned}
\nabla(aE(u))\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix} &= a\left[-\frac{q^2}{2\rho^2}+\int_\rho^{\rho_*}\frac{p(r)}{r^2}dr+\frac{p(\rho)}{\rho},\quad \frac{q}{\rho}\right]\begin{bmatrix}\frac{q}{a}\partial_x a\\ \frac{q^2}{a\rho}\partial_x a\end{bmatrix}\\
&= \left(-\frac{q^3}{2\rho^2}+q\int_\rho^{\rho_*}\frac{p(r)}{r^2}dr+\frac{q}{\rho}p(\rho)+\frac{q^3}{\rho^2}\right)\partial_x a\\
&= \frac{q}{\rho}\big(E(u)+p(\rho)\big)\,\partial_x a\\
&= F(u)\,\partial_x a\,.
\end{aligned}
$$

Consider now the entropy condition for (1.1.4) and, by the above equality,

$$
\begin{aligned}
0 \;&\leq\; \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(E(u)\,\partial_t\varphi + F(u)\,\partial_x\varphi - \nabla E(u) \left[\begin{array}{c} \frac{q}{a}\partial_x a \\ \frac{q^2}{a\rho}\partial_x a \end{array} \right] \varphi \right) dx\,dt \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Bigg(aE(u)\,\partial_t\frac{\varphi}{a} + aF(u)\,\partial_x\frac{\varphi}{a} \\
&\qquad + \left(F(u)\partial_x a - \nabla(aE(u)) \left[\begin{array}{c} \frac{q}{a}\partial_x a \\ \frac{q^2}{a\rho}\partial_x a \end{array} \right] \right) \frac{\varphi}{a} \Bigg) dx\,dt \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Bigg(aE(u)\,\partial_t\frac{\varphi}{a} + aF(u)\,\partial_x\frac{\varphi}{a} \\
&\qquad + \big(F(u)\partial_x a - F(u)\partial_x a\big)\,\frac{\varphi}{a} \Bigg) dx\,dt \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(aE(u)\,\partial_t\frac{\varphi}{a} + aF(u)\,\partial_x\frac{\varphi}{a} \right) dx\,dt,
\end{aligned}
$$

showing that (1.2.18) holds. The extension to $a \in \mathbf{W^{1,1}}$ is immediate. □

Proof of Proposition 1.2.7. The regularity condition **(Σ0)** follows from the theory of ordinary differential equations. Condition **(Σ1)** is immediate.

Consider now the item 2. If a_1 and a_2 both satisfy **(A1)**, are strictly monotone, smooth and have the same range, then $a_1 = a_2 \circ \varphi$ for a suitable strictly monotone φ with, say $\varphi' \geq 0$, the case $\varphi' \leq 0$ is entirely similar. Note that if $u = \big(R_i(x;u^-), Q_i(x;u^-)\big)$ solves (1.1.5) with $a = a_i$, then direct computations show that $R_1(x,u^-) = R_2\left(\varphi(x), u^-\right)$ and $Q_1(x,u^-) = Q_2\left(\varphi(x), u^-\right)$. Hence

$$
\begin{aligned}
\Sigma_1(a^-,a^+;u^-) &= \int_{-X}^{X} p\left(R_1(x;u^-)\right) a_1'(x)\,dx \\
&= \int_{-X}^{X} p\left(R_2\left(\varphi(x);u^-\right)\right) a_2'\left(\varphi(x)\right)\,\varphi'(x)\,dx \\
&= \int_{-X}^{X} p\left(R_2(\xi;u^-)\right) a_2'(\xi)\,d\xi \\
&= \Sigma_2(a^-,a^+;u^-)\,.
\end{aligned}
$$

Having proved **(Σ0)** and **(Σ1)**, we use the map T defined in Lemma 1.2.1. We first prove that Σ satisfies $\Sigma(a^-,a^+;u^-)+\Sigma\left(a^+,a^-;T(a^+,a^-;u^-)\right) = 0$, given a satisfying **(A1)**, strictly monotone and with $a(-X) = a^-$, $a(X) = a^+$, let $\tilde a(x) = a^- + a^+ - a(x)$. Then, using 2. proved above, and integrating (1.1.5) backwards, we have

$$
\Sigma\left(a^+,a^-;T(a^-,a^+;u^-)\right) \;=\; \int_{-X}^{X} p\left(\tilde R(x;T(a^-,a^+;u^-)\right) \tilde a'(x)\,dx
$$

$$
\begin{aligned}
&= -\int_{-X}^{X} p\left(R(x;a^-,a^+;u^-)\right) a'(x)\,dx \\
&= -\Sigma(a^-,a^+;u^-)\,.
\end{aligned}
$$

Finally, condition **(Σ2)** follows from the the flow property of R and the additivity of the integral. Indeed, by 2. and 3. we may assume without loss of generality that $a^- < a^0 < a^+$. Then, let $q = Q(x;u^-)$ be the q component in the solution to (1.1.5) with initial condition $u(0) = u^-$. Then, if T is the map defined in Lemma 1.2.1, we have

$$
T(a^-,a^+;u^-) = \left(R(a^{-1}(a^+);u^-), Q(a^{-1}(a^+);u^-)\right)
$$

so that

$$
\begin{aligned}
& \Sigma(a^-,a^+;u^-) \\
= & \int_{-X}^{X} p\left(R(x,u^-)\right) a'(x)\,dx \\
= & \int_{-X}^{a^{-1}(a^0)} p\left(R(x,u^-)\right) a'(x)\,dx + \int_{a^{-1}(a^0)}^{X} p\left(R(x,u^-)\right) a'(x)\,dx \\
= & \int_{-X}^{a^{-1}(a^0)} p\left(R(x,u^-)\right) a'(x)\,dx \\
& + \int_{a^{-1}(a^0)}^{X} p\left(R\left(x, R(a^{-1}(a^0),u^-), Q(a^{-1}(a^0),u^-)\right)\right) a'(x)\,dx \\
= & \Sigma(a^-,a^0;u^-) + \Sigma\left(a^0,a^+;T(a^-,a^+;u^-)\right)
\end{aligned}
$$

proving 1. □

Proof of Theorem 1.2.8. Fix $\bar a > 0$, and $\bar u \in A_0$. Choose M, Δ, L, δ as in Theorem 1.2.4. With reference to these quantities, let a satisfy **(A1)**. For $n \in \mathbb{N}$, let a_n, α_n, β_n be as in Proposition 1.2.9. Note that α_n is piecewise linear and continuous. By (iii), we have that $\alpha_n \to a$ and $a_n \to a$ in $\mathbf{L^1}$. Moreover, $\mathrm{TV}(\alpha_n) \le M$ and $\mathrm{TV}(a_n) \le M$ and, for n sufficiently large, $a_n(\mathbb{R}) \subseteq \,]\bar a - \Delta, \bar a + \Delta[$. Hence, for n large, a_n satisfies **(A0)**. Call S^n the semigroup constructed in Theorem 1.2.4 and denote by $\mathcal{D}^n$ its domain.

Let u_n^0 be a sequence of initial data in $\mathcal{D}^n$. The S^n are uniformly Lipschitz in time and $S_t^n u_n^0$ have total variation in x uniformly bounded in t. Hence, by [16, Theorem 2.4], a subsequence of $u_n(t) = S_t^n u_n^0$ converges pointwise a.e. to a limit, say, u. For any $\varphi \in \mathbf{C_c^1}(\mathring{\mathbb{R}}^+ \times \mathbb{R};\mathbb{R})$ and for any fixed n, let $\varepsilon > 0$ be sufficiently small and introduce a $\mathbf{C_c^\infty}(\mathbb{R};\mathbb{R})$ function η_ε such that

$$
\begin{aligned}
\eta_\varepsilon(x) = 0 \quad & \text{for all} \quad x \in \textstyle\bigcup_{j=1}^{m_n-1}[x_n^j - \varepsilon, x_n^j + \varepsilon]\,, \\
\eta_\varepsilon(x) = 1 \quad & \text{for all} \quad x \in \textstyle\bigcup_{j=1}^{m_n-2}[x_n^j + 2\varepsilon, x_n^{j+1} - 2\varepsilon]\,.
\end{aligned}
$$

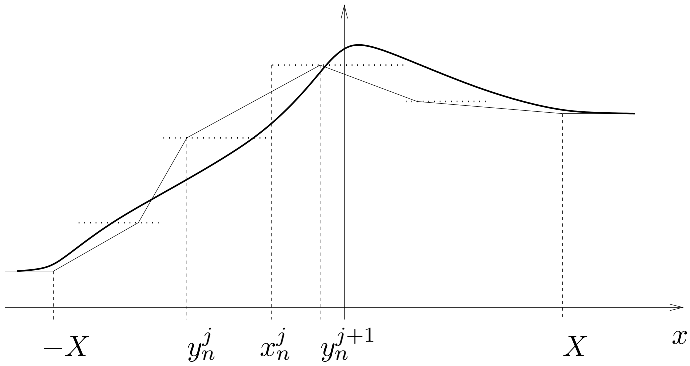


Figure 1.2: The thick line is the graph of $a = a(x)$, the dotted line represents a_n while the polygonal line is α_n

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n \rho_n \\ a_n q_n \end{bmatrix} \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x \varphi \right) dx\, dt \\
= & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n \rho_n \\ a_n q_n \end{bmatrix} \eta_\varepsilon\, \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \eta_\varepsilon\, \partial_x \varphi \right) dx\, dt \\
= & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n \rho_n \\ a_n q_n \end{bmatrix} \partial_t (\eta_\varepsilon\, \varphi) + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x (\eta_\varepsilon\, \varphi) \right) dx\, dt \\
& - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \varphi\, \partial_x \eta_\varepsilon\, dx\, dt\,.
\end{aligned}$$

The first summand in the latter term above vanishes by Definition 1.2.2 applied in a neighborhood of each x_n^j. The second summand, by the **BV** regularity of u_n, converges as follows:

$$\begin{aligned}
& -\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(\begin{bmatrix} a_n \rho_n \\ a_n q_n \end{bmatrix} \partial_t \varphi + \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \partial_x \varphi \right) dx\, dt \\
= & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \begin{bmatrix} a_n q_n \\ a_n P(u_n) \end{bmatrix} \varphi\, \partial_x \eta_\varepsilon\, dx\, dt \\
= & \sum_{j=1}^{m_n - 1} \int_{\mathbb{R}^+} \begin{bmatrix} a_n(x_n^j+) q_n(x_n^j+) - a_n(x_n^j-) q_n(x_n^j-) \\ a_n(x_n^j+) P_n(x_n^j+) - a_n(x_n^j-) P_n(x_n^j-) \end{bmatrix} \varphi(t, x_n^j)\, dt \\
= & \sum_{j=1}^{m_n - 1} \int_{\mathbb{R}^+} \begin{bmatrix} 0 \\ \Sigma\left(a_n(x_n^j-), a_n(x_n^j+), u(t, x_n^j-) \right) \end{bmatrix} \varphi(t, x_n^j)\, dt\,.
\end{aligned}$$

We proceed now considering only the second component. Using the map

$$\begin{aligned}\varphi_n(t,x) &= \varphi(t,x)\,\chi_{]-\infty,y_n^1[}(x) + \sum_{j=1}^{m_n-1} \varphi(t,x_n^j)\,\chi_{[y_n^j,y_n^{j+1}[}(x)\\ &\quad +\varphi(t,x)\,\chi_{]y_n^{m_n},+\infty[}(x)\,,\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{j=1}^{m_n-1}\int_{\mathbb{R}^+}\Sigma\left(a_n(x_n^j-),a_n(x_n^j+),u(t,x_n^j-)\right)\varphi(t,x_n^j)\,dt\\
&= \sum_{j=1}^{m_n-1}\int_{\mathbb{R}^+}\Sigma\left(a_n(y_n^j),a_n(y_n^{j+1}),u(t,x_n^j-)\right)\varphi(t,x_n^j)\,dt\\
&= \sum_{j=1}^{m_n-1}\int_{\mathbb{R}^+}\Sigma\left(\alpha_n(y_n^j),\alpha_n(y_n^{j+1}),u(t,x_n^j-)\right)\varphi(t,x_n^j)\,dt\\
&= \sum_{j=1}^{m_n-1}\int_{\mathbb{R}^+}\int_{y_n^j}^{y_n^{j+1}} p\Big(R^{\alpha_n}\left(x;u_n(t,x_n^j-)\right)\Big)\,\alpha_n'(x)\,dx\;\varphi(t,x_n^j)\,dt\\
&= \int_{\mathbb{R}^+}\sum_{j=1}^{m_n-1}\int_{y_n^j}^{y_n^{j+1}} p\Big(R^{\alpha_n}\left(x;u_n(t,x_n^j-)\right)\Big)\,\alpha_n'(x)\,dx\;\varphi(t,x_n^j)\,dt\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\sum_{j=1}^{m_n-1} p\Big(R^{\alpha_n}\left(x;u_n(t,x_n^j-)\right)\Big)\\
&\qquad\qquad\qquad\times\alpha_n'(x)\,\varphi(t,x_n^j)\,\chi_{[y_n^j,y_n^{j+1}[}(x)\,dx\,dt\\
&\to \int_{\mathbb{R}^+}\int_{\mathbb{R}} p\left(\rho(x)\right)\,\partial_x a(x)\,\varphi(t,x_n^j)\,dx\,dt \qquad \text{as } n\to+\infty\,,\end{aligned}$$

where we used (i) in the choice of the approximation α_n.

We thus constructed a solution to (1.1.1), for any initial datum in $\mathcal{D}$. Note that this solution satisfies (1.2.20)–(1.2.21), as can be proved using exactly the techniques in [45, Theorem 8]. Therefore, the whole sequence u_n converges to a unique limit u, which is Lipschitz with respect to time. This uniqueness implies the semigroup property 2. in Theorem 1.2.8. The Lipschitz continuity with respect to the initial datum follows from the uniform Lipschitz regularity of the approximate solutions u_n, completing the proof of 3. Finally, 6. is proved exactly as in [45, Theorem 8]. □

Chapter 2

Balance Laws with Integrable Unbounded Sources

2.1 Introduction

We consider the Cauchy problem for a $n \times n$ strictly hyperbolic system of balance laws

$$\begin{cases} \partial_t u + \partial_x f(u) = g(x,u), \qquad x \in \mathbb{R}, t > 0 \\ u(0,x) = u_o \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n), \\ |\lambda_i(u)| \geq c > 0 \text{ for all } i \in \{1,\dots,n\}, \\ \left\| g(x,\cdot) \right\|_{\mathbf{C}^2} \leq \tilde{M}(x) \in \mathbf{L^1}, \end{cases}$$

each characteristic field being genuinely nonlinear or linearly degenerate. Assuming that the $\mathbf{L}^1$ norm of $\|g(x,\cdot)\|_{\mathbf{C}^1}$ and $\|u_o\|_{\mathbf{BV}(\mathbb{R})}$ are small enough, we prove the existence and uniqueness of global entropy solutions of bounded total variation extending the result in [1] to unbounded (in $\mathbf{L}^\infty$) sources. Furthermore, we apply this result to the fluid flow in a pipe with discontinuous cross sectional area, showing existence and uniqueness of the underlying semigroup. The recent literature offers several results on the properties of gas flows on networks. For instance, in [20, 25, 26, 31] the well posedness is established for the gas flow at a junction of n pipes with constant diameters. The equations governing the gas flow in a pipe with a smooth varying cross section $a = a(x)$ are given by (see for instance [63]):

$$\begin{cases} \partial_t \rho + \partial_x q = -\frac{q}{a}\partial_x a \\ \partial_t q + \partial_x (\frac{q^2}{\rho} + p) = -\frac{q^2}{a\rho}\partial_x a \\ \partial_t e + \partial_x \left(\frac{q}{\rho}(e+p)\right) = -\frac{\left(\frac{q}{\rho}(e+p)\right)}{a}\partial_x a. \end{cases}$$

The well posedness of this system is covered in [1] where an attractive unified approach to the existence and uniqueness theory for quasilinear strictly hyperbolic systems of balance laws is proposed. The case of discontinuous cross sections is considered in the literature inserting a junction with suitable coupling conditions at the junction, see for example [20, 25, 26, 31]. One way to obtain coupling conditions at the point of discontinuity of the cross section a is to take the limit of a sequence of Lipschitz continuous cross sections a^ε converging to a in $\mathbf{L^1}$ (for a different approach see for instance [28]). Unfortunately the results in [1] require $\mathbf{L}^\infty$ bounds on the source term and well posedness is proved on a domain depending on this $\mathbf{L}^\infty$ bound. Since in the previous equations the source term contains the derivative of the cross sectional area one cannot hope to take the limit $a^\varepsilon \to a$. Indeed when a is discontinuous, the $\mathbf{L}^\infty$ norm of $(a^\varepsilon)'$ goes to infinity. Therefore the purpose of this paper is to establish the result in [1] without requiring the $\mathbf{L}^\infty$ bound. More precisely, we consider the Cauchy problem for the following $n \times n$ system of equations

$$\partial_t u + \partial_x f(u) = g(x,u), \qquad x \in \mathbb{R},\ t > 0, \tag{2.1.1}$$

endowed with a (suitably small) initial data

$$u(0,x) = u_o(x), \qquad x \in \mathbb{R}, \tag{2.1.2}$$

belonging to $\mathbf{L^1} \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n)$, the space of integrable functions with bounded total variation (Tot.Var.) in the sense of [72]. Here $u(t,x) \in (\mathbb{R})^n$ is the vector of unknowns, $f : \Omega \to \mathbb{R}^n$ denotes the fluxes, *i.e.* a smooth function defined on Ω which is an open neighbourhood of the origin in $\mathbb{R}^n$. The system (2.1.1) is supposed to be strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate in the sense of Lax [57]. We recall that if zero does not belong to to the domain Ω of definition of f, as in the case of gas dynamics away from vacuum, then a simple translation of the density vector u leads back the problem to the case $0 \in \Omega$. Concerning the source term g, we assume that it satisfies the following Caratheodory–type conditions:

(P_1) $g : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is measurable with respect to (w.r.t.) x, for any $u \in \Omega$, and is $\mathbf{C^2}$ w.r.t. u, for any $x \in \mathbb{R}$;

(P_2) there exists a $\mathbf{L^1}$ function $\tilde{M}(x)$ such that $\left\|g(x,\cdot)\right\|_{\mathbf{C^2}} \le \tilde{M}(x)$;

(P_3) there exists a function $\omega \in \mathbf{L^1}(\mathbb{R})$ such that $\|g(x,\cdot)\|_{\mathbf{C^1}} \le \omega(x)$.

Remark 2.1.1 *Note that the $\mathbf{L^1}$ norm of $\tilde{M}(x)$ does not have to be small but only bounded differently from $\omega(x)$ whose norm has to be small (see Theorem 2.1.4 below). Furthermore condition (P_2) replaces the $\mathbf{L}^\infty$ bound of the $\mathbf{C^2}$ norm of g in [1]. Finally observe that we do not require any $\mathbf{L}^\infty$*

bound on ω. On the other hand we will need the following observation: if we define

$$\tilde{\varepsilon}_h = \sup_{x\in\mathbb{R}} \int_0^h \omega(x+s)\, ds, \tag{2.1.3}$$

by absolute continuity one has $\tilde{\varepsilon}_h \to 0$ as $h \to 0$.

Moreover, we assume that a *nonresonance* condition holds, that is the characteristic speeds of the system (2.1.1) are bounded away from zero:

$$\hat{\lambda} \geq |\lambda_i(u)| \geq c > 0, \qquad \forall\, u \in \Omega,\ i \in \{1,\dots,n\}, \tag{2.1.4}$$

for some $\hat{\lambda} > c > 0$.

Before stating the main theorem of this paper we need to recall the Riemann problem for the homogeneous system associated to (2.1.1):

$$\partial_t u + \partial_x f(u) = 0, \qquad u(o,x) = \begin{cases} u_\ell & \text{if } x < x_0 \\ u_r & \text{if } x > x_0 \end{cases} \tag{2.1.5}$$

and we need following two definitions.

Definition 2.1.2 *Given a BV function $u = u(x)$ and a point $\xi \in \mathbb{R}$, we denote by $U^\sharp_{(u;\xi)}$ the solution of the homogeneous Riemann Problem (2.1.5) with data*

$$u_\ell = \lim_{x\to\xi-} u(x), \qquad u_r = \lim_{x\to\xi+} u(x), \qquad x_o = \xi. \tag{2.1.6}$$

Definition 2.1.3 *Given a BV function $u = u(x)$ and a point $\xi \in \mathbb{R}$, we define $U^\flat_{(u;\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients*

$$\partial_t w + \widetilde{A}\partial_x w = \widetilde{g}(x), \qquad w(0,x) = u(x), \tag{2.1.7}$$

with $\widetilde{A} = \nabla f\left(u(\xi)\right)$, $\widetilde{g}(x) = g\left(x, u(\xi)\right)$.

Theorem 2.1.4 *Assume (P_1)–(P_3) and (2.1.4). If the norm of ω in $\mathbf{L^1}(\mathbb{R})$ is sufficiently small, there exist a constant $L > 0$, a closed domain $\mathcal{D}$ of integrable functions with small total variation and a unique semigroup $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ satisfying*

i) $P_0 u = u, \quad P_{t+s}u = P_t \circ P_s u$ *for all $u \in \mathcal{D}$ and $t,s \geq 0$;*

ii) $\|P_s u - P_t v\|_{\mathbf{L^1}(\mathbb{R})} \leq L\Big(|s-t| + \|u-v\|_{\mathbf{L^1}(\mathbb{R})}\Big)$ *for all $u,v \in \mathcal{D}$ and $t,s \geq 0$;*

iii) for all $u_o \in \mathcal{D}$ the function $u(t,\cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem (2.1.1)–(2.1.2) and for all $\tau > 0$ satisfies the following integral estimates:

(a) For every ξ, one has

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\xi - \vartheta\hat{\lambda}}^{\xi + \vartheta\hat{\lambda}} \left| u(\tau + \vartheta, x) - U^{\sharp}_{(u(\tau);\xi)}(\vartheta, x) \right| dx = 0. \tag{2.1.8}$$

(b) There exists a constant C such that, for every $a < \xi < b$ and $0 < \vartheta < \frac{b-a}{2\hat{\lambda}}$, one has

$$\frac{1}{\vartheta} \int_{a+\vartheta\hat{\lambda}}^{b-\vartheta\hat{\lambda}} \left| u(\tau + \vartheta, x) - U^{\flat}_{(u(\tau);\xi)}(\vartheta, x) \right| dx \leq \; C\Big[\mathrm{TV}\left\{u(\tau); (a,b)\right\} + \int_a^b \omega(x)\, dx\Big]^2. \tag{2.1.9}$$

Conversely let $u : [0,T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L^1}(\mathbb{R}, \mathbb{R}^n)$ and assume that $u(t,x)$ satisfies the integral conditions (a), (b). Then $u(t, \cdot)$ coincides with a trajectory of the semigroup P.

The proof of this theorem is postponed to sections 2.3 and 2.4, where existence and uniqueness are proved. Before these technical details, we state the application of the above result to gas flow in section 2.2. Here we apply Theorem 2.1.4 to establish the existence and uniqueness of the semigroup related to pipes with discontinuous cross sections. Furthermore, we show that our approach yields the same semigroup as the approach followed in [26] in the special case of two connected pipes. The technical details of section 2.2 can be found at the end of the paper in section 2.5.

2.2 Application to gas dynamics

Theorem 2.1.4 provides an existence and uniqueness result for pipes with Lipschitz continuous cross section where the equations governing the gas flow are given by

$$\begin{cases} \partial_t \rho + \partial_x q = -\frac{q}{a}\partial_x a \\ \partial_t q + \partial_x(\frac{q^2}{\rho} + p) = -\frac{q^2}{a\rho}\partial_x a \\ \partial_t e + \partial_x\left(\frac{q}{\rho}(e+p)\right) = -\frac{\left(\frac{q}{\rho}(e+p)\right)}{a}\partial_x a. \end{cases} \tag{2.2.10}$$

Here, as usual, ρ denotes the mass density, q the linear momentum, e is the energy density, a is the area of the cross section of the pipe and p is the pressure which is related to the conserved quantities $u = (\rho, q, e)$ by the equations of state. In most situations, when two pipes of different size have to be connected, the length l of the adaptor is small compared to the length of the pipes. Therefore it is convenient to model these connections as pipes with a jump in the cross sectional area. These discontinuous cross sections however do not fulfil the requirements of Theorem 2.1.4. Nevertheless, we can use this Theorem to derive the existence of solutions to the discontinuous problem by a limit procedure. To this end, we approximate the discontinuous function

$$a(x) = \begin{cases} a^-, & x < 0 \\ a^+, & x > 0 \end{cases} \tag{2.2.11}$$

by a sequence $a_l \in C^{0,1}(\mathbb{R}, \mathbb{R}^+)$ with the following properties

$$a_l(x) = \begin{cases} a^-, & x < -\frac{l}{2} \\ \varphi_l(x), & x \in \left[-\frac{l}{2}, \frac{l}{2}\right] \\ a^+, & x > \frac{l}{2} \end{cases} \tag{2.2.12}$$

where φ_l is any smooth monotone function which connects the two strictly positive constants a^-, a^+. One possible choice of the approximations a_l as well as the discontinuous pipe with cross section a are shown in figure 2.1.

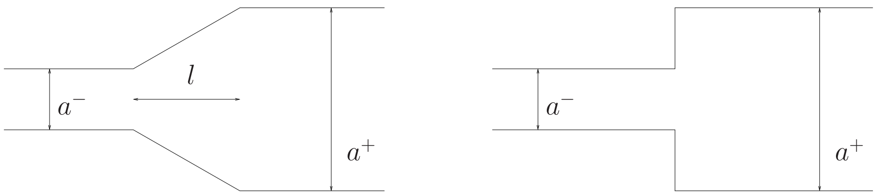


Figure 2.1: Illustration of approximated and discontinuous cross-sectional area

With the position $A_l(x) = \ln a_l(x)$, we can write (2.2.10) in the following form

$$\partial_t u + \partial_x f(u) = A_l'(x) g(u). \tag{2.2.13}$$

Observe that $\|A_l'\|_{\mathbf{L}^1} = |A^+ - A^-| = |\ln a^+ - \ln a^-|$, hence, the smallness of $|a^+ - a^-|$ (away from zero) implies the smallness of $\|A_l'\|_{\mathbf{L}^1}$.

Let $\Phi(a, \bar{u})$ be the solution to the Cauchy problem

$$\begin{cases} \frac{d}{da} u(a) = \left[D_u f(u(a))\right]^{-1} g(u(a)) \\ u(0) = \bar{u}. \end{cases} \tag{2.2.14}$$

and define

$$\Psi\left(u^-, u^+\right) = u^+ - \Phi(A^+ - A^-, u^-). \tag{2.2.15}$$

Definition 2.2.1 *A solution to the Riemann problem with a junction in $x = 0$ described by the function Ψ:*

$$\partial_t u + \partial_x f(u) = 0, \qquad x \neq 0$$

$$u(0,x) = \begin{cases} u_l \ \textit{if } x < x_0 \\ u_r \ \textit{if } x > x_0 \end{cases}$$

$$\Psi(u(t,0-), u(t,0+)) = 0 \tag{2.2.16}$$

is a function $u : [0,+\infty) \times \mathbb{R} \to \mathbb{R}^2$ *such that*

i) the function $(t,x) \to u(t,x)$ *is self similar and in* $x > 0$ *coincides with a fan of Lax entropic waves with positive velocity outgoing from* $(t,x) = (0,0)$ *while in* $x < 0$ *coincides with a fan of a Lax entropic waves of negative velocity outgoing from* $(t,x) = (0,0)$*;*

ii) The traces $u(t,0-)$*,* $u(t,0+)$ *satisfy (2.2.16) for all* $t > 0$*.*

With the help of Theorem 2.1.4 and the techniques used in its proof, we are now able to derive the following Theorem (see also [28] for a similar result obtained with different methods).

Theorem 2.2.2 *Let* $\bar{u}$ *a non sonic state. If* $|a^+ - a^-|$ *is sufficiently small, the semigroups* P^l *(defined on a domain of functions which take value in a small neighborhood of* $\bar{u}$*) related with the smooth section* a_l *converge to a unique semigroup* P*.*

Moreover let $U^\flat$ *be defined for all* ξ *as in Definition 2.1.3 (here the hyperbolic flux* f *is the gas dynamic flux (2.2.10)). Let* $U^\sharp$ *be defined for all* $\xi \neq 0$ *as in Definition 2.1.2 and in the point* $\xi = 0$ *as the solution of the Riemann problem defined in Definition 2.2.1. Then the limit semigroup satisfies and is uniquely identified by the integral estimates (2.1.8), (2.1.9).*

Observe that the same Theorem holds for the 2×2 isentropic system (see Section 2.5)

$$\begin{cases} \partial_t \rho + \partial_x q = -\frac{q}{a}\partial_x a \\ \partial_t q + \partial_x(\frac{q^2}{\rho} + p) = -\frac{q^2}{a\rho}\partial_x a. \end{cases} \tag{2.2.17}$$

In [26] 2×2 homogeneous conservation laws at a junction are considered for given admissible junction conditions. In the 2×2 case, the function Ψ in (2.2.15) depends only on four real variables:

$$\Psi\Big((\rho^-, q^-), (\rho^+, q^+)\Big) = (\rho^+, q^+) - \Phi(A^+ - A^-, (\rho^-, q^-)). \tag{2.2.18}$$

It fulfils the determinant condition in [26, Proposition 2.2] since it satisfies Lemma 2.3.4. Condition (2.2.16) is now given by

$$(\rho^+, q^+) - \Phi(A^+ - A^-, (\rho^-, q^-)) = 0. \tag{2.2.19}$$

The result in [26, Theorem 3.2] for the case of a junction with only two pipes with different cross sections and the function Ψ given by (2.2.19) can be stated in the following way

Theorem 2.2.3 *Given a subsonic state $\bar{u} = (\bar{\rho}, \bar{q})$, there exist a constant $L > 0$, a closed domain $\mathcal{D} \subset \bar{u} + \mathbf{L}^1(\mathbb{R}, \mathbb{R}^2)$ of functions with small total variation and a unique semigroup $S : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ satisfying*

i) $S_0 u = u, \quad S_{t+s} u = S_t \circ S_s u$ *for all* $u \in \mathcal{D}$ *and* $t, s \geq 0$;

ii) $\|S_s u - S_t v\|_{\mathbf{L}^1(\mathbb{R})} \leq L\Big(|s-t| + \|u-v\|_{\mathbf{L}^1(\mathbb{R})}\Big)$ *for all* $u, v \in \mathcal{D}$ *and* $t, s \geq 0$;

iii) for all $u_o \in \mathcal{D}$ *the function* $u(t, \cdot) = S_t u_o \doteq (\rho, q)(t, \cdot)$ *is a weak entropy solution to*

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x (\frac{q^2}{\rho} + p) = 0, \end{cases} \tag{2.2.20}$$

for $x \neq 0$, *equipped with the condition at the junction in* $x = 0$

$$\Psi((\rho, q)(t, 0-); (\rho, q)(t, 0+)) = 0, \tag{2.2.21}$$

for almost every $t > 0$, *where* Ψ *is given by (2.2.19);*

iv) If $u \in \mathcal{D}$ *is piecewise constant, then for* $t > 0$ *sufficiently small,* $S_t u$ *coincides with the juxtaposition of the solutions to homogeneous Riemann Problems centered at the points of jumps of* u *in* $x \neq 0$ *and the solution of the Riemann problem at the junction as defined in Definition 2.2.1 for the point* $x = 0$.

We will prove in Section 2.5 that the semigroup obtained in [26] and recalled in Theorem 2.2.3 satisfies the same integral estimate (see the following proposition) as our limit semigroup hence they coincide.

Proposition 2.2.4 *The semigroup defined in [26] with the junction condition given by (2.2.19) satisfies the integral estimates (2.1.8),(2.1.9) with* $U^\sharp$ *substituted by the solution of the Riemann problem described in Definition 2.2.1.*

The proof is postponed to Section 2.5.

Remark 2.2.5 *Note that Proposition 1 justifies the coupling condition (2.2.19) as well as the condition used in [40] to study the Riemann problem for the gas flow through a nozzle.*

2.3 Existence of BV entropy solutions

Throughout the next two sections, we follow the structure of [1]. We recall some definitions and notations in there, and also the results which do not depend on the $\mathbf{L}^\infty$ boundedness of the source term. We will prove only the results which in [1] do depend on the $\mathbf{L}^\infty$ bound using our weaker hypotheses.

2.3.1 The non homogeneous Riemann-Solver

Consider the stationary equations associated to (2.1.1), namely the system of ordinary differential equations:

$$\partial_x f(v(x)) = g(x, v(x)). \tag{2.3.22}$$

For any $x_o \in \mathbb{R}$, $v \in \Omega$, consider the initial data

$$v(x_o) = v. \tag{2.3.23}$$

As in [1], we introduce a suitable approximation of the solutions to (2.3.22), (2.3.23). Thanks to (2.1.4), the map $u \mapsto f(u)$ is invertible inside some neighbourhood of the origin; in this neighbourhood, for small $h > 0$, we can define

$$\Phi_h(x_o, u) \doteq f^{-1}\left[f(u) + \int_0^h g\left(x_o + s, u\right)\, ds\right]. \tag{2.3.24}$$

This map gives an approximation of the flow of (2.3.22) in the sense that

$$f\left(\Phi_h(x_o, u)\right) - f(u) = \int_0^h g\left(x_o + s, u\right)\, ds. \tag{2.3.25}$$

Throughout the paper we will use the Landau notation $\mathcal{O}(1)$ to indicate any function whose absolute value remains uniformly bounded, the bound depending only on f and $\|\tilde{M}\|_{\mathbf{L}^1}$.

Lemma 2.3.1 *The function $\Phi_h(x_o, u)$ defined in (2.3.24) satisfies the following uniform (with respect to $x_o \in \mathbb{R}$ and to u in a suitable neighbourhood of the origin) estimates.*

$$\begin{gathered}\|\Phi_h(x_o, \cdot)\|_{\mathbf{C}^2} \leq \mathcal{O}(1)\,, \\ \lim_{h\to 0} \sup_{x_o\in\mathbb{R}} |\Phi_h(x_o, u) - u| = 0, \\ \lim_{h\to 0} \|Id - D_u\Phi_h(x_o, u)\| = 0\end{gathered} \tag{2.3.26}$$

Proof. The Lipschitz continuity of f^{-1} and (2.1.3) imply

$$\begin{aligned}\left|\Phi_h(x_o, u) - u\right| &= \left|\Phi_h(x_o, u) - f^{-1}\left(f(u)\right)\right| \leq \mathcal{O}(1)\left|\int_0^h g(x_o + s, u)\, ds\right| \\ &\leq \mathcal{O}(1)\left|\int_0^h \omega(x_o + s)\, ds\right| \leq \mathcal{O}(1)\,\tilde{\varepsilon}_h \to 0,\end{aligned}$$

for $h \to 0$. Next we compute

$$\begin{aligned}D_u\Phi_h(x_o, u) = \quad & Df^{-1}\left[f(u) + \int_0^h g(x_o + s, u)\, ds\right] \\ & \cdot\left(Df(u) + \int_0^h D_u g(x_o + s, u)\, ds\right)\end{aligned}$$

which together to the identity $u = f^{-1}\big(f(u)\big)$ implies

$$\begin{aligned}
\big\|D_u\Phi_h(x_o,u) - Id\big\| &= \Big\|D_u\Phi_h(x_o,u) - Df^{-1}\big(f(u)\big)\Big\| \\
&\leq \left\|Df^{-1}\Big[f(u) + \textstyle\int_0^h g(x_o+s,u)\,ds\Big] - Df^{-1}\big(f(u)\big)\right\| \\
&\quad\cdot\Big(\|Df(u)\| + \textstyle\int_0^h \|D_u g(x_o+s,u)\|\,ds\Big) \\
&\quad+ \Big\|Df^{-1}\big(f(u)\big)\Big\| \cdot \textstyle\int_0^h \|D_u g(x_o+s,u)\|\,ds \\
&\leq \mathcal{O}(1)\,\tilde{\varepsilon}_h \to 0,
\end{aligned}$$

for $h \to 0$. Finally, denoting with D_i the partial derivative with respect to the i component of the state vector and by $\Phi_{h,\ell}$ the ℓ component of the vector Φ_h, we derive

$$\begin{aligned}
D_iD_j\Phi_{h,l}(x_o,u) &= \textstyle\sum_{k,k'}\Bigg(D_kD_{k'}f_\ell^{-1}\Big(f(u) + \int_0^h g(x_o+s,u)\,ds\Big) \\
&\quad\cdot\Big(D_if_k(u) + \textstyle\int_0^h D_ig_k(x_o+s,u)\,ds\Big) \\
&\quad\cdot\Big(D_jf_{k'}(u) + \textstyle\int_0^h D_jg_{k'}(x_o+s,u)\,ds\Big)\Bigg) \\
&\quad+\textstyle\sum_k D_kf_\ell^{-1}\Big(f(u) + \int_0^h g(x_o+s,u)\,ds\Big) \\
&\quad\cdot\Big(D_jD_if_k(u) + \textstyle\int_0^h D_jD_ig_k(x_o+s,u)\,ds\Big)
\end{aligned}$$

so that

$$\Big\|D^2\Phi_h(x_o,u)\Big\| \leq \mathcal{O}(1)\Big(1 + \textstyle\int_0^h \tilde{M}(x_o+s)\,ds\Big) \leq \mathcal{O}(1)\Big(1 + \|\tilde{M}\|_{\mathbf{L}^1}\Big) \leq \mathcal{O}(1)\,.$$

□

For any $x_o \in \mathbb{R}$ we consider the system (2.1.1), endowed with a Riemann initial datum:

$$u(0,x) = \begin{cases} u_\ell & \text{if } x < x_o \\ u_r & \text{if } x > x_o. \end{cases} \tag{2.3.27}$$

If the two states u_ℓ, u_r are sufficiently close, let Ψ be the unique entropic homogeneous Riemann solver given by the map

$$u_r = \Psi(\sigma)(u_\ell) = \psi_n(\sigma_n)\circ\ldots\circ\psi_1(\sigma_1)(u_\ell), \tag{2.3.28}$$

where $\sigma = (\sigma_1,\ldots,\sigma_n)$ denotes the (signed) wave strengths vector in $\mathbb{R}^n$, [57]. Here ψ_j, $j = 1,\ldots,n$ is the shock–rarefaction curve of the j^{th} family, parametrised as in [16] and related to the homogeneous system of conservation laws

$$\partial_t u + \partial_x f(u) = 0. \tag{2.3.29}$$

Observe that, due to (2.1.4), all the simple waves appearing in the solution of (2.3.29), (2.3.27) propagate with *non-zero* speed.

To take into account the effects of the source term, we consider a stationary discontinuity across the line $x = x_o$, that is, a wave whose speed is equal to 0, the so called zero-wave. Now, given $h > 0$, we say that the particular Riemann solution:

$$u(t,x) = \begin{cases} u_\ell & \text{if } x < x_o \\ u_r & \text{if } x > x_o. \end{cases} \qquad \forall\, t \geq 0 \tag{2.3.30}$$

is admissible if and only if $u_r = \Phi_h(x_o, u_\ell)$, where Φ_h is the map defined in (2.3.24). Roughly speaking, we require u_ℓ, u_r to be (approximately) connected by a solution of the stationary equations (2.3.22).

Definition 2.3.2 *Let p be the number of waves with negative speed. Given $h > 0$ suitably small, $x_o \in \mathbb{R}$, we say that $u(t,x)$ is a h–Riemann solver for (2.1.1), (2.1.4), (2.3.27), if the following conditions hold*

(a) there exist two states u^-, u^+ which satisfy $u^+ = \Phi_h(x_o, u^-)$;

(b) on the set $\{t \geq 0,\ x < x_o\}$, $u(t,x)$ coincides with the solution to the homogeneous Riemann Problem (2.3.29) with initial values u_ℓ, u^- and, on the set $\{t \geq 0,\ x > x_o\}$, with the solution to the homogeneous Riemann Problem with initial values u^+, u_r;

(c) the Riemann Problem between u_ℓ and u^- is solved only by waves with negative speed (i.e. of the families $1, \ldots, p$);

(d) the Riemann Problem between u^+ and u_r is solved only by waves with positive speed (i.e. of the families $p+1, \ldots, n$).

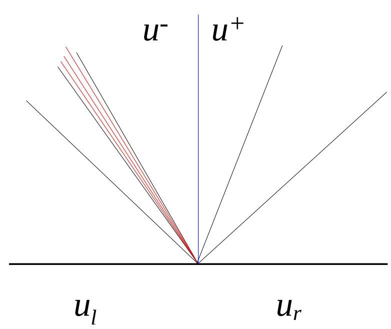


Figure 2.2: Wave structure in an h–Riemann solver.

Lemma 2.3.3 *Let $x_o \in \mathbb{R}$ and $u,\ u_1,\ u_2$ be three states in a suitable neighbourhood of the origin. For h suitably small, one has*

$$\left|\Phi_h(x_o,u) - u\right| \quad = \quad \mathcal{O}(1)\int_0^h \omega(x_o+s)\,ds, \tag{2.3.31}$$

$$\left|\Phi_h(x_o,u_2) - \Phi_h(x_o,u_1) - (u_2-u_1)\right| \quad = \tag{2.3.32}$$

$$\mathcal{O}(1)\,|u_2-u_1| \int_{x_o}^{x_o+h} \omega(s)ds.$$

The proof can be found in [1, Lemma 1].

Lemma 2.3.4 *For any $M > 0$ there exist $\delta_1',\ h_1' > 0$, depending only on M and the homogeneous system (2.3.29), such that for all maps $\varphi \in \mathbf{C^2}(\mathbb{R}^n,\mathbb{R}^n)$ satisfying*

$$\|\varphi\|_{\mathbf{C^2}} \le M, \qquad |\varphi(u) - u| \le h_1', \qquad \|I - D\varphi(u)\| \le h_1'$$

and for all $u_\ell \in B(0,\delta_1')$, $u_r \in B(\varphi(0),\delta_1')$ there exist $n{+}2$ states $w_0,\dots,w_{n+1}$ and n wave sizes $\sigma_1,\dots,\sigma_n$, depending smoothly on $u_\ell,\ u_r$ which satisfy, with the previous notations:

i) $w_0 = u_\ell,\ w_{n+1} = u_r$;

ii) $w_i = \Psi_i(\sigma_i)(w_{i-1}), \qquad i = 1,\dots,p$;

iii) $w_{p+1} = \varphi(w_p)$;

iv) $w_{i+1} = \Psi_i(\sigma_i)(w_i), \qquad i = p+1,\dots,n.$

Here p is the number of waves with negative velocity as in Definition 2.3.2.

The proof can be found in [1, Lemma 3].

The next lemma establishes existence and uniqueness for the h–Riemann solvers (see Fig.2.2).

Lemma 2.3.5 *There exist $\delta_1,\ h_1 > 0$ such that for any $x_o \in \mathbb{R}$, $h \in [0,h_1]$, $u_\ell,\ u_r \in B(0,\delta_1)$, there exists a unique h–Riemann solver in the sense of Definition 2.3.2.*

Proof. By Lemma 2.3.1 if $h_1 > 0$ is chosen sufficiently small then for any $h \in [0,h_1]$, $x_o \in \mathbb{R}$ the map $u \mapsto \Phi_h(x_o,u)$ meets the hypotheses of Lemma 2.3.4. Finally taking h_1 eventually smaller we can obtain that there exists $\delta_1 > 0$ such that $B(0,\delta_1) \subset B(0,\delta_1') \cap B(\Phi_h(x_o,0),\delta_1')$, for any $h \in [0,h_1]$. □

In the sequel, E stands for the implicit function given by Lemmas 2.3.4 and 2.3.5:

$$\sigma \doteq E(h, u_\ell, u_r; x_o),$$

which plays the role of a wave–size vector. We recall that, by Lemma 2.3.4, E is a $\mathbf{C^2}$ function with respect to the variables u_ℓ, u_r and its $\mathbf{C^2}$ norm is bounded by a constant independent of h and x_o.

In contrast with the homogeneous case, the wave–size σ in the h–Riemann solver is not equivalent to the jump size $|u_\ell - u_r|$; an additional term appears coming from the "Dirac source term" (see the special case $u_\ell = u_r$).

Lemma 2.3.6 *Let δ_1, h_1 be the constants in Lemma 2.3.5. For $u_\ell, u_r \in B(0, \delta_1)$, $h \in [0, h_1]$, set $ß = E[h, u_\ell, u_r; x_o]$. Then it holds:*

$$\begin{aligned} |u_\ell - u_r| &= \mathcal{O}(1)\left(|\sigma| + \int_0^h \omega(x_o + s)\, ds\right), \\ |\sigma| &= \mathcal{O}(1)\left(|u_\ell - u_r| + \int_0^h \omega(x_o + s)\, ds\right). \end{aligned} \tag{2.3.33}$$

The proof can be found in [1, Lemma 4].

2.3.2 Existence of a Lipschitz semigroup

Note that as shown in [1] we can identify the sizes of the zero waves with the quantity

$$\sigma = \int_0^h \omega(jh + s)\, ds. \tag{2.3.34}$$

With this definition all the Glimm interaction estimates continue to hold with constants that depend only on f and on $\|\tilde{M}\|_{\mathbf{L^1}}$, therefore all the wave front tracking algorithm can be carried out obtaining the existence of ε, h-approximate solutions. Here we first give a precise definition of ε, h-approximate solutions and then outline the algorithm which allows to obtain them. The detailed proof that the algorithm can be carried out for all times $t \geq 0$ is in [1].

Definition 2.3.7 *Given $\varepsilon, h > 0$, we say that a continuous map*

$$u^{\varepsilon,h} : [0, +\infty) \to \mathbf{L^1}_{loc}(\mathbb{R}, \mathbb{R}^n)$$

is an ε, h–approximate solution of (2.1.1)–(2.1.2) if the following holds:

- *As a function of two variables, $u^{\varepsilon,h}$ is piecewise constant with discontinuities occurring along finitely many straight lines in the x, t plane. Only finitely many wave-front interactions occur, each involving exactly two wave-fronts, and jumps can be of four types: shocks (or contact discontinuities), rarefaction waves, non-physical waves and zero-waves: $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{Z}$.*

– *Along each shock (or contact discontinuity)* $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{S}$, *the values of* $u^- = u^{\varepsilon,h}(t, x_\alpha-)$ *and* $u^+ = u^{\varepsilon,h}(t, x_\alpha+)$ *are related by* $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$ *for some* $k_\alpha \in \{1, ..., n\}$ *and some wave-strength* σ_α. *If the* ka^{th} *family is genuinely nonlinear, then the Lax entropy admissibility condition* $\sigma_\alpha < 0$ *also holds. Moreover, one has*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+, u^-)| \leq \varepsilon$$

where $\lambda_{k_\alpha}(u^+, u^-)$ *is the speed of the shock front (or contact discontinuity) prescribed by the classical Rankine-Hugoniot conditions.*

– *Along each rarefaction front* $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{R}$, *one has* $u^+ = \psi_{k_\alpha}(\sigma_\alpha)(u^-)$, $0 < \sigma_\alpha \leq \varepsilon$ *for some genuinely nonlinear family* k_α. *Moreover, we have:* $|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+)| \leq \varepsilon$.

– *All non-physical fronts* $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ *travel at the same speed* $\dot{x}_\alpha = \hat{\lambda} > \sup_{u,i} |\lambda_i(u)|$. *Their total strength remains uniformly small, namely:*

$$\sum_{\alpha \in \mathcal{NP}} |u^{\varepsilon,h}(t, x_\alpha+) - u^{\varepsilon,h}(t, x_\alpha-)| \leq \varepsilon, \qquad \forall t > 0.$$

– *The zero-waves are located at every point* $x = jh$, $j \in (-\frac{1}{h\varepsilon}, \frac{1}{h\varepsilon}) \cap \mathbb{Z}$. *Along a zero-wave located at* $x_\alpha = j_\alpha h$, $\alpha \in \mathcal{Z}$, *the values* $u^- = u^{\varepsilon,h}(t, x_\alpha-)$ *and* $u^+ = u^{\varepsilon,h}(t, x_\alpha+)$ *satisfy* $u^+ = \Phi_h(x_\alpha, u^-)$ *for all* $t > 0$ *except at the interaction points.*

– *The total variation in space* $\mathrm{TV}u^{\varepsilon,h}(t, \cdot)$ *is uniformly bounded for all* $t \geq 0$. *The total variation in time* $\mathrm{TV}\left\{u^{\varepsilon,h}(\cdot, x); [0, +\infty)\right\}$ *is uniformly bounded for* $x \neq jh$, $j \in \mathbb{Z}$.

Finally, we require that $\|u^{\varepsilon,h}(0, .) - u_o\|_{\mathbf{L^1}(\mathbb{R})} \leq \varepsilon$.

Outline of the wave front tracking algorithm (see Figure 2.3) For notational convenience, we shall drop hereafter the ε, h superscripts as there will be no ambiguity. Now, given any small value for these parameters, let us build up such an approximate solution.

• In order for our approximate solutions to be piecewise constant, we need to discretize the rarefactions; following [16], for a fixed small parameter δ, each rarefaction of size σ, when it is created for the first time, is divided into $m = \left[\frac{\sigma}{\delta}\right] + 1$ wave-fronts, each one with size $\sigma/m \leq \delta$.

• Given the initial data u_o, we can define a piecewise constant approximation $u(0, \cdot)$ satisfying the requirements of Definition 2.3.7; moreover, it is possible to guarantee that

$$\mathrm{TV}u(0, \cdot) \leq \mathrm{TV}u_o.$$

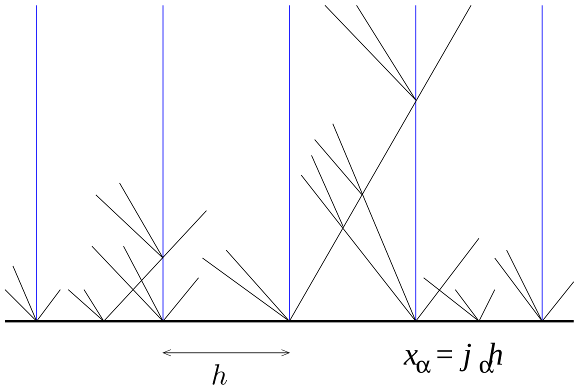


Figure 2.3: Illustration of the wave front tracking algorithm

Then, $u(t,x)$ is constructed, for small t, by applying the h–Riemann solver at every point $x = jh$ with $j \in \left(-\frac{1}{h\varepsilon}, \frac{1}{h\varepsilon}\right) \cap \mathbb{Z}$ (even if in that point $u(0,\cdot)$ does not have a discontinuity), and by solving the remaining discontinuities in $u(0,\cdot)$ using a classical homogeneous Riemann solver for (2.3.29).

• At every interaction point, a new Riemann problem arises. Notice that because of their fixed speed, two non-physical fronts cannot interact with each other; neither can the zero-waves. Moreover, by a slight modification of the speed of some waves (only among shocks, contact discontinuities and rarefactions), it is possible to achieve the property that not more than two wave-fronts interact at a point.

After an interaction time, the number of wave-fronts may well increase. In order to prevent this number to become infinite in finite time, a particular treatment has been proposed for waves whose strength is below a certain threshold value ρ by means of a simplified Riemann solver [16]. We shall use similar trick in our construction, in the same spirit as in the homogeneous case;

Suppose that two wave-fronts of strengths σ, σ' interact at a given point (t,x). If $x \neq x_\alpha$, for any $\alpha \in \mathcal{Z}$, we use the classical accurate or simplified homogeneous Riemann solver as in [16]. Assume now that $x = x_\alpha = j_\alpha h$ with $\alpha \in \mathcal{Z}$, different situations can occur:

– if the wave approaching the zero wave is physical and it holds $|\sigma\sigma'| \geq \rho$ we use the (accurate) h–Riemann solver;

– if the wave incoming to the zero wave is physical and it holds $|\sigma\sigma'| < \rho$ we use a simplified one. Assume that the wave-front on the right is a zero–wave (the other case being similar): let u_ℓ, $u_m = \psi_j(\sigma)(u_\ell)$, $u_r = \Phi_h(x, u_m)$ be the states before the interaction.
We define the auxiliary states

$$\tilde{u}_m = \Phi_h(x, u_\ell), \qquad \tilde{u}_r = \psi_j(\sigma)(\tilde{u}_m).$$

Then three fronts propagate after interaction: the zero-wave $(u_\ell, \tilde{u}_m)$, the physical front $(\tilde{u}_m, \tilde{u}_r)$ and the non-physical one $(\tilde{u}_r, u_r)$.

– Suppose now that the wave on the left belongs to $\mathcal{NP}$. Again we use a simplified solver: let $u_\ell,\ u_m,\ u_r = \Phi_h(x, u_m)$ be the states before the interaction, and define the new state $\tilde{u}_\ell = \Phi_h(x, u_\ell)$.
  After the interaction time, only two fronts propagate: the zero–wave $(u_\ell, \tilde{u}_\ell)$ and the non-physical $(\tilde{u}_\ell, u_r)$.

Keeping $h > 0$ fixed, we are about to first let ε tend to zero. Hence we shall drop the superscript h for notational clarity.

Theorem 2.3.8 *Let u^ε be a family of ε, h–approximate solutions of (2.1.1)–(2.1.2). There exists a subsequence u^{ε_i} converging as $i \to +\infty$ in $\mathbf{L}^1_{loc}\left((0,+\infty)\times\mathbb{R}\right)$ to a function u which satisfies for any $\varphi \in \mathbf{C}^1_c\left((0,+\infty)\times\mathbb{R}\right)$:*

$$\int_0^\infty \int_{\mathbb{R}} \left[u\varphi_t + f(u)\varphi_x\right] dxdt + \int_0^\infty \sum_{j\in\mathbb{Z}} \varphi(t, jh) \left(\int_0^h g\left[jh + s, u(t, jh-)\right] ds\right) dt = 0. \tag{2.3.35}$$

Moreover $\mathrm{TV}u(t,\cdot)$ *is uniformly bounded and u satisfies the Lipschitz property*

$$\int_{\mathbb{R}} \left|u(t',x) - u(t'',x)\right| dx \le C'|t' - t''|, \qquad t', t'' \ge 0; \tag{2.3.36}$$

Now we are in position to prove [1, Theorem 4] with our weaker hypotheses. As in [1] we can apply Helly's compactness theorem to get a subsequence u^{h_i} converging to some function u in $\mathbf{L^1}_{loc}$ whose total variation in space is uniformly bounded for all $t \ge 0$. Moreover, working as in [2, Proposition 5.1], one can prove that $u^{h_i}(t,\cdot)$ converges in $\mathbf{L^1}$ to $u(t,\cdot)$, for all $t \ge 0$.

Theorem 2.3.9 *Let u^{h_i} be a subsequence of solutions to Equation (2.3.35) with uniformly bounded total variation converging as $i \to +\infty$ in $\mathbf{L^1}$ to some function u. Then u is a weak solution to the Cauchy problem (2.1.1)–(2.1.2).*

We omit the proofs of Theorem 2.3.8 and 2.3.9 since they are very similar to the proofs of [1, Theorem 3 and 4]. We only observe that, in those proofs, the computations which rely on the $\mathbf{L}^\infty$ bound on the source term have to be substituted by the following estimates.

- Concerning the proof of Theorem 2.3.8:

$$\begin{aligned} &\textstyle\int_0^h \left|g\left(jh+s, u^\varepsilon(t, jh-)\right) - g\left(jh+s, u(t, jh-)\right)\right| \, ds \\ &\qquad \le \textstyle\int_0^h \left\|g(jh+s,\cdot)\right\|_{\mathbf{C^1}} \cdot \left|u^\varepsilon(t, jh-) - u(t, jh-)\right| \, ds \\ &\qquad\qquad \le \tilde{\varepsilon}_h \cdot \left|u^\varepsilon(t, jh-) - u(t, jh-)\right| . \end{aligned}$$

- Concerning the proof of Theorem 2.3.9:

$$\int_0^h \left| g\left(jh+s, u^h(t,jh-)\right)\right| \, ds \le \int_0^h \left\| g(jh+s,\cdot)\right\|_{\mathbf{C^1}} \, ds \le \tilde{\varepsilon}_h$$

  and

$$\begin{aligned}
&\int_0^h \left| g\left(jh+s, u^h(t,jh-)\right) - g\left(jh+s, u(t,jh+s)\right)\right| \, ds \\
&\quad \le \int_0^h \left\| g(jh+s,\cdot)\right\|_{\mathbf{C^1}} \cdot \left| u^h(t,jh-) - u(t,jh+s)\right| \, ds \\
&\quad \le \tilde{\varepsilon}_h \cdot \mathrm{TV}\left\{ u^h(t,\cdot), [(j-1)h,(j+1)h]\right\} \\
&\qquad + \int_{jh}^{(j+1)h} \omega(x) \left| u^h(t,x) - u(t,x)\right| .
\end{aligned}$$

We observe that all the computations done in [1, Section 4] rely on the source g only through the amplitude of the zero waves and on the interaction estimates. Therefore the following two theorems still hold in the more general setting.

Theorem 2.3.10 *There exists $\delta > 0$ such that if $\|\omega\|_{\mathbf{L^1}(\mathbb{R})}$ is sufficiently small, then for any (small) $h > 0$ there exist a non empty closed domain $\mathcal{D}_h(\dot{)}$ and a unique uniformly Lipschitz semigroup $P^h : [0,+\infty) \times \mathcal{D}_h(\delta) \to \mathcal{D}_h(\delta)$ whose trajectories $u(t,.) = P^h_t u_o$ solve (2.3.35) and are obtained as limit of any sequence of ε, h–approximate solutions as ε tends to zero with fixed h. In particular the semigroups P^h satisfy for any $u_o, v_o \in \mathcal{D}_h(\delta)$, $t, s \ge 0$*

$$P^h_0 u_o = u_o, \qquad P^h_t \circ P^h_s u_o = P^h_{s+t} u_o, \tag{2.3.37}$$

$$\left\| P^h_t u_o - P^h_s v_o \right\|_{\mathbf{L^1}(\mathbb{R})} \le L\left[\|u_o - v_o\|_{\mathbf{L^1}(\mathbb{R})} + |t-s|\right] \tag{2.3.38}$$

for some $L > 0$, independent on h.

Theorem 2.3.11 *If $\|\omega\|_{\mathbf{L^1}(\mathbb{R})}$ is sufficiently small, there exist a constant $L > 0$, a non empty closed domain $\mathcal{D}$ of integrable functions with small total variation and a semigroup $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ with the following properties*

i) $P_0 u = u, \quad \forall u \in \mathcal{D}; \qquad P_{t+s} u = P_t \circ P_s u, \quad \forall u \in \mathcal{D},\ t,s \ge 0.$

ii) $\|P_s u - P_t v\|_{\mathbf{L^1}(\mathbb{R})} \le L\Big(|s-t| + \|u-v\|_{\mathbf{L^1}(\mathbb{R})}\Big), \quad \forall u,\, v \in \mathcal{D},\ t,s \ge 0.$

iii) For all $u_o \in \mathcal{D}$, the function $u(t,\cdot) = P_t u_o$ is a weak entropy solution of system (2.1.1).

iv) For all $h > 0$ small enough $\mathcal{D} \subset \mathcal{D}_h$.

v) There exists a sequence of semigroups P^{h_i} such that $P_t^{h_i}u$ converges in $\mathbf{L^1}$ *to $P_t u$ as $i \to +\infty$ for any $u \in \mathcal{D}$.*

Remark 2.3.12 *Looking at [1, (4.6)] and the proof of [1, Theorem 7] one realises that the invariant domains $\mathcal{D}_h$ and $\mathcal{D}$ depend on the particular source term $g(x,u)$. On the other hand estimate [1, (4.4)] shows that all these domains contain all integrable functions with sufficiently small total variation. Since the bounds $\mathcal{O}(1)$ in Lemma 2.3.6 depend only on f and on $\|\tilde{M}\|_{\mathbf{L^1}}$, also the constant C_1 in [1, (4.4)] depends only on f and on $\|\tilde{M}\|_{\mathbf{L^1}}$. Therefore there exists $\tilde{\delta} > 0$ depending only on f and on $\|\tilde{M}\|_{\mathbf{L^1}}$ such that $\mathcal{D}_h$ and $\mathcal{D}$ contain all integrable functions $u(x)$ with* $\mathrm{TV}\{u\} \leq \tilde{\delta}$.

2.4 Uniqueness of BV entropy solutions

The proof of uniqueness in [1] strongly depends on the boundedness of the source, therefore we have to consider it in a more careful way.

2.4.1 Some preliminary results

As in [1] we shall make use of the following technical lemmas whose proofs can be found in [16].

Lemma 2.4.1 *Let (a,b) a (possibly unbounded) open interval, and let $\hat{\lambda}$ be an upper bound for all wave speeds. If $\bar{u}$, $\bar{v} \in \mathcal{D}_h$ then for all $t \geq 0$ and $h > 0$, one has*

$$\int_{a+\hat{\lambda}t}^{b-\hat{\lambda}t} \left| \left(P_t^h \bar{u}\right)(x) - \left(P_t^h \bar{v}\right)(x) \right| dx \leq L \int_a^b \left|\bar{u}(x) - \bar{v}(x)\right| dx. \tag{2.4.39}$$

Remark 2.4.2 *Observe that in the ε, h-approximate solutions, the waves do not travel faster than $\hat{\lambda}$, therefore the values of the function $P_t^h \bar{v}$ in the interval $\left(a + \hat{\lambda}t, b - \hat{\lambda}t\right)$ depend only on the values that $\bar{v}$ assumes in the interval (a,b). Therefore, estimate (2.4.39) is only a localisation of (2.3.38), in particular the Lipschitz constant L is the same and depends on g only through $\|\tilde{M}\|_{\mathbf{L^1}}$.*

Lemma 2.4.3 *Given any interval $I_0 = [a,b]$, define the interval of determinacy*

$$I_t = [a + \hat{\lambda}t, b - \hat{\lambda}t], \qquad t < \frac{b-a}{2\hat{\lambda}}. \tag{2.4.40}$$

For every Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}_h$ and $h > 0$:

$$\left\| w(t) - P_t^h w(0) \right\|_{\mathbf{L^1}(I_t)} \leq L \int_0^t \left\{ \liminf_{\eta \to 0} \frac{\left\| w(s+\eta) - P_\eta^h w(s) \right\|_{\mathbf{L^1}(I_{s+\eta})}}{\eta} \right\} ds. \tag{2.4.41}$$

Remark 2.4.4 *Lemmas 2.4.1, 2.4.3 hold also substituting P^h with the operator P. In this case we have obviously to substitute the domains $\mathcal{D}_h(\delta)$ with the domain $\mathcal{D}$ of Theorem 2.3.11.*

Let now u_ℓ, u_r be two nearby states and $\lambda < \hat{\lambda}$; we consider the function

$$v(t,x) = \begin{cases} u_\ell & \text{if} \quad x < \lambda t + x_o \\ u_r & \text{if} \quad x \geq \lambda t + x_o. \end{cases} \tag{2.4.42}$$

Lemma 2.4.5 *Call $w(t,x)$ the self-similar solution given by the standard homogeneous Riemann Solver with the Riemann data (2.3.27).*

(i) In the general case, one has

$$\frac{1}{t}\int_{-\infty}^{+\infty} \big|v(t,x) - w(t,x)\big|\, dx = \mathcal{O}(1)\,|u_\ell - u_r|; \tag{2.4.43}$$

(ii) Assuming the additional relations $u_r = R_i(\sigma)(u_\ell)$ and $\lambda = \lambda_i(u_r)$ for some $\sigma > 0$, $i = 1, \ldots, n$ one has the sharper estimate

$$\frac{1}{t}\int_{-\infty}^{+\infty} \big|v(t,x) - w(t,x)\big|\, dx = \mathcal{O}(1)\,\sigma^2; \tag{2.4.44}$$

(iii) Let $u^* \in \Omega$ and call $\lambda_1^* < \ldots < \lambda_n^*$ the eigenvalues of the matrix $A^* = \nabla f(u^*)$. If for some i it holds $A^*(u_r - u_\ell) = \lambda_i^*(u_r - u_\ell)$ and $\lambda = \lambda_i^*$ in (2.4.42), then one has

$$\frac{1}{t}\int_{-\infty}^{+\infty} \big|v(t,x) - w(t,x)\big|\, dx = \mathcal{O}(1)\,|u_\ell - u_r|\Big(|u_\ell - u^*| + |u^* - u_r|\Big); \tag{2.4.45}$$

The proof can be found in [16, Lemma 9.1].

We now prove the next result which is directly related to our h-Riemann solver.

Lemma 2.4.6 *Call $w(t,x)$ the self-similar solution given by the h–Riemann Solver in x_o with the Riemann data (2.3.27).*

(i) In the general case one has

$$\frac{1}{t}\int_{-\infty}^{+\infty} \big|v(t,x) - w(t,x)\big|\, dx = \mathcal{O}(1)\left(|u_\ell - u_r| + \int_0^h \omega(x_o + s)\, ds\right); \tag{2.4.46}$$

(ii) Assuming the additional relation

$$u_r = u_\ell + [\nabla f]^{-1}(u^*) \int_0^h g(x_o + s, u^*) ds$$

with $\lambda = 0$ *in (2.4.42) one has the sharper estimate*

$$\begin{aligned} & \frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx \qquad (2.4.47) \\ = \; & \mathcal{O}(1) \left(\int_0^h \omega(x_o + s)\, ds + |u_\ell - u^*| \right) \cdot \int_0^h \omega(x_o + s)\, ds. \end{aligned}$$

Proof. Suppose $\lambda \geq 0$ (the other case being similar) and compute

$$\begin{aligned} & \frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx \\ = \; & \frac{1}{t} \int_{-\hat\lambda t}^{0} \left| u_\ell - w(t,x) \right| dx + \frac{1}{t} \int_0^{\lambda t} \left| u_\ell - w(t,x) \right| dx + \frac{1}{t} \int_{\lambda t}^{\hat\lambda t} \left| u_r - w(t,x) \right| dx \\ = \; & \mathcal{O}(1) \sum_{\iota=1}^{p} |\sigma_\iota| + \frac{1}{t} \int_0^{\lambda t} \left| u_\ell - w(t,x) \right| dx + \mathcal{O}(1) \sum_{\iota=p+1}^{n} |\sigma_\iota| \\ = \; & \mathcal{O}(1) |\sigma| + \frac{1}{t} \int_0^{\lambda t} \left| u_\ell - w(t,x) \right| dx, \end{aligned}$$

where $\sigma = E[h, u_\ell, u_r; x_o]$. Since $w(t,x)$ is the solution to the h–Riemann problem, Lemma 2.3.6 implies $|u_\ell - w(t,x)| = \mathcal{O}(1) \left(|\sigma| + \int_0^h \omega(x_o + s)\, ds \right)$. Therefore using again Lemma 2.3.6 we get *(i)*.

Let us prove now *(ii)*. Setting $\lambda = 0$ in the previous computation gives

$$\frac{1}{t} \int_{-\infty}^{+\infty} \left| v(t,x) - w(t,x) \right| dx = \mathcal{O}(1) |\sigma|.$$

This leads to

$$\begin{aligned} |ß| &= \left| E[h, u_\ell, u_r; x_o] - E\left[h, u_\ell, \Phi_h(x_o, u_\ell); x_o\right] \right| \\ &= \mathcal{O}(1) \left| u_r - \Phi_h(x_o, u_\ell) \right|. \end{aligned}$$

To estimate this last term, we define $b(y, u) = f^{-1}\left(f(u) + y\right)$ and compute for some y_1, y_2:

$$\begin{aligned} \left| u_\ell + [\nabla f]^{-1}(u^*) y_1 - b(y_2, u_\ell) \right| \; \leq \; & \mathcal{O}(1) |y_1| \cdot |u^* - u_\ell| + \mathcal{O}(1) |y_1 - y_2| \\ & + \left| u_\ell + [\nabla f]^{-1}(u_\ell) y_2 - b(y_2, u_\ell) \right|. \end{aligned}$$

The function $z(y_2) = u_\ell + [\nabla f]^{-1}(u_\ell)y_2 - b(y_2, u_\ell)$ satisfies $z(0) = 0$, $D_{y_2}z(0) = 0$, hence we have the estimate

$$\left|u_\ell + [\nabla f]^{-1}(u^*)y_1 - b(y_2, u_\ell)\right| \leq \mathcal{O}(1)\left[|y_1| \cdot |u^* - u_\ell| + |y_1 - y_2| + |y_2|^2\right].$$

If in this last expression we substitute

$$y_1 = \int_0^h g(x_o + s, u^*)ds, \qquad y_2 = \int_0^h g(x_o + s, u_\ell)ds$$

then, we get

$$\left|u_r - \Phi_h(x_o, u_\ell)\right| = O(1)\Big(\int_0^h \omega(x_o + s)\, ds + |u_\ell - u^*|\Big)\int_0^h \omega(x_o + s)\, ds$$

which proves (2.4.47). $\square$

2.4.2 Characterisation of the semigroup's trajectories

In this section we are about to give necessary and sufficient conditions for a function $u(t, \cdot) \in \mathcal{D}$ to coincide with a semigroup's trajectory. To this end, we prove the uniqueness of the semigroup P and the convergence of all the sequence of semigroups P^h towards P as $h \to 0$.

We will need the following approximations of $U^\flat_{(u;\xi)}$ (see Definition 2.1.3). Let $\overline{v}$ be a piecewise constant function. We will call w^h the solution of the following Cauchy problem:

$$\partial_t(w^h) + \partial_x\widetilde{A}(w^h) = \sum_{j\in\mathbb{Z}} \delta(x - jh)\int_0^h \widetilde{g}(jh + s)\, ds, \qquad w^h(0, x) = \overline{v}(x).$$

Define $u^* \doteq u(\xi)$ and let $\lambda_i = \lambda_i(u^*)$, $r_i = r_i(u^*)$, $l_i = l_i(u^*)$ be respectively the i^{th} eigenvalue, the i^{th} right/left eigenvectors of the matrix $\widetilde{A}$ (see Definition 2.1.3). As in [1] w and w^h have the following explicit representation

$$\begin{aligned} w(t, x) &= \sum_{i=1}^n \left\{\left\langle l_i, u\left(x - \lambda_i t\right)\right\rangle + \frac{1}{\lambda_i}\int_{x-\lambda_i t}^x \left\langle l_i, \widetilde{g}(x')\right\rangle dx'\right\} r_i, \\ w^h(t, x) &= \sum_{i=1}^n \left\{\left\langle l_i, \overline{v}\left(x - \lambda_i t\right)\right\rangle + \frac{1}{\lambda_i}\left\langle l_i, G^h(t, x)\right\rangle\right\} r_i, \quad (2.4.48) \end{aligned}$$

where the function $G^h(t, x) = \sum_{i=1}^n G_i^h(t, x) r_i$ is defined by

$$G_i^h(t, x) = \begin{cases} \displaystyle\sum_{j:\, jh\in(x-\lambda_i t, x)} \int_0^h \left\langle l_i, \widetilde{g}\left(jh + s\right)\right\rangle ds & \text{if } \lambda_i > 0 \\ \displaystyle -\sum_{j:\, jh\in(x, x-\lambda_i t)} \int_0^h \left\langle l_i, \widetilde{g}\left(jh + s\right)\right\rangle ds & \text{if } \lambda_i < 0. \end{cases} \quad (2.4.49)$$

Using (2.1.3) we can compute

$$\left| G_i^h(t,x) - \int_{x-\lambda_i t}^{x} \langle l_i, \widetilde{g}(x') \rangle \, dx \right| = \mathcal{O}(1)\, \tilde{\varepsilon}_h. \tag{2.4.50}$$

Hence, for any $a, b \in \mathbb{R}$ with $a < b$, we have the error estimate

$$\int_{a+\hat{\lambda}t}^{b-\hat{\lambda}t} \left| w(t,x) - w^h(t,x) \right| dx \leq \mathcal{O}(1) \left[\int_a^b \left| u(x) - \overline{v}(x) \right| dx + (b-a)\tilde{\varepsilon}_h \right]. \tag{2.4.51}$$

From (2.4.48), (2.4.49), it is easy to see that $w^h(t,x)$ is piecewise constant with discontinuities occurring along finitely many lines on compact sets in the (t,x) plane for $t \geq 0$. Only finitely many wave front interactions occur in a compact set, and jumps can be of two types: contact discontinuities or zero waves. The zero waves are located at the points jh, $j \in \mathbb{Z}$ and satisfy

$$w^h(t, jh+) - w^h(t, jh-) = [\nabla f]^{-1}(u^*) \int_{jh}^{(j+1)h} \widetilde{g}\,(jh+s)\, ds. \tag{2.4.52}$$

Conversely a contact discontinuity of the i^{th} family located at the point $x_\alpha(t)$ satisfies $\dot{x}_\alpha(t) = \lambda_i(u^*)$ and

$$w^h(t, x_\alpha(t)+) - w^h(t, x_\alpha(t)-) = \sigma r_i(u^*) \tag{2.4.53}$$

for some $\sigma \in \mathbb{R}$.

Now, we apply a technique introduced in [15] to state and prove the uniqueness result in our more general setting.

Theorem 2.4.7 *Let $P : \mathcal{D} \times [0, +\infty) \to \mathcal{D}$ be the semigroup of Theorem 2.3.11, let $\hat{\lambda}$ be an upper bound for all wave speeds (see (2.1.4)) and let $U^\sharp$ and $U^\flat$ the functions defined in Definitions 2.1.2 and 2.1.3. Then every trajectory $u(t,\cdot) = P_t u_0$, $u_0 \in \mathcal{D}$, satisfies the integral estimates (2.1.8) and (2.1.9) at every $\tau \geq 0$.*

Viceversa let $u : [0,T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L^1}(\mathbb{R}, \mathbb{R}^n)$ and assume that the conditions (2.1.8), (2.1.9) hold at almost every time τ. Then $u(t,\cdot)$ coincides with a trajectory of the semigroup P.

Remark 2.4.8 *The difference with respect to the result in [1] is the presence of the integral in the right hand side of formula (2.1.9). If ω is in $\mathbf{L}^\infty$, the integral can be bounded by $\mathcal{O}(1)\,(b-a)$ and we recover the estimates in [1]. Note also that the quantity*

$$\mu\left((a,b)\right) = \mathrm{TV}\left\{u(\tau); (a,b)\right\} + \int_a^b \omega(x)\, dx$$

is a uniformly bounded finite measure and this is what is needed for proving the sufficiency part of the above Theorem.

Proof. **Part 1: Necessity** Given a semigroup trajectory $u(t,\cdot) = P_t\bar{u}$, $\bar{u} \in \mathcal{D}$ we now show that the conditions *(2.1.8)*, *(2.1.9)* hold for every $\tau \geq 0$.

As in [1] we use the following notations. For fixed h, ϑ, $\varepsilon > 0$ we define $J_t = J_t^- \cup J_t^o \cup J_t^+$ with

$$J_t^- = \Big(\xi - (2\vartheta - t + \tau)\,\hat{\lambda}, \xi - (t-\tau)\hat{\lambda}\Big)\,; \tag{2.4.54}$$

$$J_t^o = \Big[\xi - (t-\tau)\hat{\lambda}, \xi + (t-\tau)\hat{\lambda}\Big]\,; \tag{2.4.55}$$

$$J_t^+ = \Big(\xi + (t-\tau)\hat{\lambda}, \xi + (2\vartheta - t + \tau)\,\hat{\lambda}\Big)\,. \tag{2.4.56}$$

Let $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(\vartheta, x)$ be the piecewise constant function obtained from $U^{\sharp}_{(u(\tau);\xi)}(\vartheta, x)$ dividing the centered rarefaction waves in equal parts and replacing them by rarefaction fans containing wave fronts whose strength is less than ε. Observe that:

$$\frac{1}{t}\int_{-\infty}^{+\infty} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(\vartheta, x) - U^{\sharp}_{(u(\tau);\xi)}(\vartheta, x) \right| \, dx = \mathcal{O}(1)\,\varepsilon. \tag{2.4.57}$$

Applying estimate (2.4.41) to the function $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ we obtain

$$\int_{J_{\tau+\vartheta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(\vartheta, x) - \left(P^h_\vartheta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(0)\right)(x) \right| dx \tag{2.4.58}$$

$$\leq L \int_\tau^{\tau+\vartheta} \liminf_{\eta\to 0} \frac{\left\| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta) - P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right\|_{\mathbf{L}^1\left(J_{t+\eta}\right)}}{\eta} dt.$$

The discontinuities of $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ do not cross the Dirac comb for almost all times $t \in (\tau, \tau+\vartheta)$. Therefore we compute for such a time t:

$$\frac{1}{\eta}\int_{J_{t+\eta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)\right)(x) \right| dx \tag{2.4.59}$$

$$\frac{1}{\eta}\int_{J^-_{t+\eta}\cup J^o_{t+\eta}\cup J^+_{t+\eta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)\right)(x) \right| dx.$$

Define $\mathcal{W}_t$ the set of points in which $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)$ has a discontinuity while $\mathcal{Z}_h$ is the set of points in which the zero waves are located. If η is sufficiently small, the solutions of the Riemann problems arising at the discontinuities of $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)$ do not interact, therefore

$$\frac{1}{\eta}\int_{J^o_{t+\eta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)\right)(x) \right| dx$$

$$= \left(\sum_{x \in J_t^o \cap \mathcal{W}_t} + \sum_{x \in J_t^o \cap \mathcal{Z}_h} \right)$$
$$\frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, y) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right)(y) \right| dy$$

Note that the shock are solved exactly both in $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ and in $P^h U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ therefore they make no contribution in the summation. To estimate the approximate rarefactions we use the estimate (2.4.44) hence

$$\begin{aligned} &\sum_{x \in J_t^o \cap \mathcal{W}_t} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right)(x) \right| dx \\ &\leq \mathcal{O}(1) \sum_{\substack{x \in J_t^o \cap \mathcal{W}_t \\ rarefaction}} |\sigma|^2 \leq \mathcal{O}(1)\, \varepsilon \mathrm{TV} \left\{ U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau); J_t^0 \right\} \qquad (2.4.60) \\ &\leq \mathcal{O}(1)\, \varepsilon \left| u(\tau,\xi+) - u(\tau,\xi-) \right| \end{aligned}$$

Concerning the zero waves, recall that t is chosen such that $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ is constant there, and P^h is the exact solution of an h–Riemann problem, hence we can apply (2.4.46) with $u_\ell = u_r$ and obtain

$$\begin{aligned} &\sum_{x \in J_t^o \cap \mathcal{Z}_h} \frac{1}{\eta} \int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right)(x) \right| dx \\ \leq \; &\mathcal{O}(1) \sum_{jh \in J_t^o} \int_0^h \omega(jh+s)\, ds \leq \mathcal{O}(1) \left(\int_{J_t^o} \omega(x)\, dx + \tilde{\varepsilon}_h \right) \qquad (2.4.61) \end{aligned}$$

Finally using (2.4.61) and (2.4.60) we get in the end

$$\begin{aligned} &\frac{1}{\eta} \int_{J^o_{t+\eta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right)(x) \right| dx \qquad (2.4.62) \\ &\qquad\qquad = \mathcal{O}(1) \left\{ \int_{J_t^0} \omega(x)\, dx + \tilde{\varepsilon}_h + \varepsilon \right\}. \end{aligned}$$

Moreover, following the same steps as before and using (2.4.43) and (2.4.46) with $u_\ell = u_r$ we get

$$\begin{aligned} &\frac{1}{\eta} \int_{J^+_{t+\eta}} \left| U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta, x) - \left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau) \right)(x) \right| dx \qquad (2.4.63) \\ &\qquad\qquad = \mathcal{O}(1) \left\{ \int_{J_t^+} \omega(x)\, dx + \tilde{\varepsilon}_h \right\}. \end{aligned}$$

Note that here there is no total variation of $U^{\sharp,\varepsilon}_{(u(\tau);\xi)}$ since in J_t^+ it is constant. A similar estimate holds for the interval $J^-_{t+\eta}$. Putting together (2.4.59), (2.4.62), (2.4.63), one has

$$\frac{1}{\eta}\int_{J_{t+\eta}}\left|U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau+\eta,x)-\left(P^h_\eta U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(t-\tau)\right)(x)\right|dx$$
$$=\mathcal{O}(1)\left(\int_{J_\tau}\omega(x)\,dx+\tilde{\varepsilon}_h+\varepsilon\right).$$

Hence, setting $\tilde{v}=U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(0)=U^{\sharp}_{(u(\tau);\xi)}(0)$ by (2.4.58), we have

$$\begin{aligned}&\int_{J_{\tau+\vartheta}}\left|U^{\sharp,\varepsilon}_{(u(\tau);\xi)}(\vartheta,x)-\left(P^h_\vartheta\tilde{v}\right)(x)\right|dx \\ =\;&\mathcal{O}(1)\,(\vartheta\Big(\int_{J_\tau}\omega(x)\,dx+\tilde{\varepsilon}_h+\varepsilon\Big).\end{aligned} \tag{2.4.64}$$

Finally we take the sequence P^{h_i} converging to P. Using (2.4.39) we have

$$\begin{aligned}\frac{1}{\vartheta}\left\|P^{h_i}_\vartheta u(\tau)-P^{h_i}_\vartheta\tilde{v}\right\|_{\mathbf{L^1}\left(J_{\tau+\vartheta}\right)} &\leq \frac{1}{\vartheta}L\left\|u(\tau)-\tilde{v}\right\|_{\mathbf{L^1}(J_\tau)}\\ &= \frac{L}{\vartheta}\int_{\xi-2\hat{\lambda}\vartheta}^{\xi+2\hat{\lambda}\vartheta}\left|u(\tau,x)-\tilde{v}(x)\right|dx\\ &\doteq \bar{\varepsilon}_\vartheta,\end{aligned} \tag{2.4.65}$$

where $\bar{\varepsilon}_\vartheta$ tends to zero as ϑ tends to zero due to the fact that $u(\tau)$ has right and left limit at any point: for any given $\varepsilon>0$ if ϑ is sufficiently small $|u(\tau,x)-\tilde{v}(x)|=|u(\tau,x)-u(\tau,\xi-)|\leq\varepsilon$ for $x\in(\xi-2\hat{\lambda}\vartheta,\xi)$.

Therefore by (2.4.57), (2.4.64), we derive:

$$\begin{aligned}&\frac{1}{\vartheta}\int_{\xi-\vartheta\hat{\lambda}}^{\xi+\vartheta\hat{\lambda}}\left|u(\tau+\vartheta,x)-U^{\sharp}_{(u(\tau);\xi)}(\vartheta,x)\right|dx\\ =\;&\frac{\left\|P_\vartheta u(\tau)-P^{h_i}_\vartheta u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R})}}{\vartheta}+\bar{\varepsilon}_\vartheta+\mathcal{O}(1)\left[\int_{J_\tau}\omega(x)\,dx+\tilde{\varepsilon}_{h_i}\right].\end{aligned}$$

The left hand side of the previous estimate does not depend on ε and h_i, hence

$$\frac{1}{\vartheta}\int_{\xi-\vartheta\hat{\lambda}}^{\xi+\vartheta\hat{\lambda}}\left|u(\tau+\vartheta,x)-U^{\sharp}_{(u(\tau);\xi)}(\vartheta,x)\right|dx=\mathcal{O}(1)\int_{J_\tau}\omega(x)\,dx+\bar{\varepsilon}_\vartheta.$$

Note that the intervals J_τ depend on ϑ (see 2.4.55). So taking the limit as $\vartheta\to 0$ in the previous estimate yields (2.1.8).

To prove *(2.1.9)* let $\vartheta > 0$ and a point (τ, ξ) be given together with an open interval (a, b) containing ξ. Fix $\varepsilon > 0$ and choose a piecewise constant function $\bar{v} \in \mathcal{D}$ satisfying $\bar{v}(\xi) = u(\tau, \xi)$ together with

$$\int_a^b \left|\bar{v}(x) - u(\tau, x)\right| \, dx \leq \varepsilon, \quad \mathrm{TV}\left\{\bar{v}; (a, b)\right\} \leq \mathrm{TV}\left\{u(\tau); (a, b)\right\} \quad (2.4.66)$$

Let now w^h be defined by (2.4.48) ($u^* = \bar{v}(\xi) = u(\tau, \xi)$). From (2.4.51), (2.4.66) we have the estimate

$$\int_{a+\vartheta\hat{\lambda}}^{b-\vartheta\hat{\lambda}} \left|U^{\flat}_{(u(\tau);\xi)}(\vartheta, x) - w^h(\vartheta, x)\right| dx \leq \mathcal{O}(1) \left(\varepsilon + \tilde{\varepsilon}_h(b-a)\right). \quad (2.4.67)$$

Using (2.4.40), (2.4.41) we get

$$\begin{aligned} & \int_{a+\vartheta\hat{\lambda}}^{b-\vartheta\hat{\lambda}} \left|w^h(\vartheta, x) - \left(P^h_\vartheta w^h(0)\right)(x)\right| dx \qquad (2.4.68) \\ \leq \; & L \int_\tau^{\tau+\vartheta} \liminf_{\eta\to 0} \frac{\left\|w^h(t-\tau+\eta) - P^h_\eta w^h(t-\tau)\right\|_{\mathbf{L^1}(\tilde{I}_{t+\eta})}}{\eta} dt \end{aligned}$$

where we have defined $\tilde{I}_{t+\eta} = I_{t-\tau+\eta}$. Let $t \in (\tau, \tau+\vartheta)$ be a time for which there is no interaction in w^h; in particular, discontinuities which travel with a non-zero velocity do not cross the Dirac comb (this happens for almost all t). We observe that by the explicit formula (2.4.48):

$$\mathrm{TV}\left\{w^h(t-\tau); \tilde{I}_t\right\} = \mathcal{O}(1)\left(\mathrm{TV}\left\{\bar{v}; (a, b)\right\} + \int_a^b \omega(x)\, dx + \tilde{\varepsilon}_h\right) \quad (2.4.69)$$

$$\left|w^h(t-\tau, x) - \bar{v}(\xi)\right| = \mathcal{O}(1)\left(\mathrm{TV}\left\{\bar{v}; (a, b)\right\} + \int_a^b \omega(x)\, dx + \tilde{\varepsilon}_h\right). \quad (2.4.70)$$

As before for η sufficiently small we can split homogeneous and zero waves

$$\begin{aligned} & \frac{1}{\eta}\int_{\tilde{I}_{t+\eta}} \left|w^h(t-\tau+\eta, x) - \left(P^h_\eta w^h(t-\tau)\right)(x)\right| dx \\ = \; & \left(\sum_{x\in\tilde{I}_t\cap\mathcal{W}_t} + \sum_{x\in\tilde{I}_t\cap\mathcal{Z}_h}\right) \frac{1}{\eta}\int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left|w^h(t-\tau+\eta, x) - \left(P^h_\eta w^h(t-\tau)\right)(x)\right| dx \end{aligned}$$

The homogeneous waves in w^h satisfy (2.4.53), with $\bar{v}(\xi)$ in place of u^*, hence we can apply (2.4.45) which together with (2.4.69), (2.4.70) leads to

$$\sum_{x\in\tilde{I}_t\cap\mathcal{W}_t} \frac{1}{\eta}\int_{x-\hat{\lambda}\eta}^{x+\hat{\lambda}\eta} \left|w^h(t-\tau+\eta, x) - \left(P^h_\eta w^h(t-\tau)\right)(x)\right| dx$$

$$
\begin{aligned}
&\le \mathcal{O}(1) \sum_{x\in\tilde I_t\cap\mathcal{W}_t} |\Delta w^h(t-\tau,x)| \Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)\\
&\le \mathcal{O}(1)\,\mathrm{TV}\left\{w^h(t-\tau),\tilde I_t\right\}\Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)\\
&\le \mathcal{O}(1)\Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)^2
\end{aligned}
$$

where $\Delta w^h(t-\tau,x)$ denotes the jump of $w^h(t-\tau)$ at x.

The zero waves in w^h satisfy (2.4.52), hence we can apply (2.4.47) which together with (2.4.70) leads to

$$
\begin{aligned}
&\sum_{x\in\tilde I_t\cap\mathcal{Z}_h} \frac{1}{\eta}\int_{x-\hat\lambda\eta}^{x+\hat\lambda\eta} \left| w^h(t-\tau+\eta,x) - \Big(P^h_\eta w^h(t-\tau)\Big)(x)\right| dx\\
\le\;& \mathcal{O}(1) \sum_{x\in\tilde I_t\cap\mathcal{Z}_h} \int_0^h \omega(x+s)\,ds\cdot\Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)\\
\le\;& \mathcal{O}(1)\left(\int_{\tilde I_t} \omega(x)\,dx + \tilde\varepsilon_h\right)\Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)\\
\le\;& \mathcal{O}(1)\Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_h\Big)^2
\end{aligned}
$$

Let now P^{h_i} be the subsequence converging to P. Since $w^h(0)=\bar v$ using (2.4.67), (2.4.68), (2.4.66), and the last estimates we get

$$
\begin{aligned}
&\frac{1}{\vartheta}\int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda} \left| u(\tau+\vartheta,x) - U^\flat_{(u(\tau);\xi)}(\vartheta,x)\right| dx\\
\le\;& \frac{\|P_\vartheta u(\tau) - P^{h_i}_\vartheta u(\tau)\|_{\mathbf{L^1}(\mathbb{R})}}{\vartheta} + L\frac{\|u(\tau)-\bar v\|_{\mathbf{L^1}(\mathbb{R})}}{\vartheta}\\
&+\mathcal{O}(1)\left\{\frac{\varepsilon+\tilde\varepsilon_{h_i}\cdot(b-a)}{\vartheta} + \Big(\mathrm{TV}\left\{\bar v;(a,b)\right\} + \int_a^b \omega(x)\,dx + \tilde\varepsilon_{h_i}\Big)^2\right\}.
\end{aligned}
$$

So for $\varepsilon,\ h_i\to 0$ we obtain the desired inequality.

Part 2: Sufficiency By Remark 2.4.4 we can apply (2.4.41) to P and hence the proof for the homogeneous case presented in [16], which relies on the property recalled in Remark 2.4.8, can be followed exactly for our case, hence it will be not repeated here. □

□

Proof of Theorem 2.1.4 It is now a direct consequence of Theorems 2.3.11 and 2.4.7. □

2.5 Proofs related to Section 2.2

Consider the equation
$$u_t + f(u)_x = a' g(u)$$
for some $a \in \mathbf{BV}$. Equation (2.2.10) is comprised in this setting with the substitution $a \mapsto \ln a$. For this kind of equations we consider the exact stationary solutions instead of approximated ones as in (2.3.24). Therefore call $\Phi(a, \bar{u})$ the solution of the following Cauchy problem:

$$\left\{ \begin{array}{l} \frac{d}{da} u(a) = \left[D_u f(u(a)) \right]^{-1} g(u(a)) \\ u(0) = \bar{u} \end{array} \right. \tag{2.5.71}$$

If a is sufficiently small, the map $u \mapsto \Phi(a, u)$ satisfies Lemma 2.3.4. We call a-Riemann problem the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = a' g(u) \\ (a, u)(0, x) = \left\{ \begin{array}{ll} (a^-, u_l) & \text{if } x < 0 \\ (a^+, u_r) & \text{if } x > 0 \end{array} \right. \end{array} \right. \tag{2.5.72}$$

its solution will be the function described in Definition 2.3.2 using the map $\Phi(a^+ - a^-, u^-)$ instead of the Φ_h in there. Observe that if $a^+ = a^-$ the a-Riemann solver coincides with the usual homogeneous Riemann solver.

Definition 2.5.1 *Given a function $u \in \mathbf{BV}$ and two states a^-, a^+, we define $\bar{U}^\sharp_u(t, x)$ as the solution of the a-Riemann solver (2.5.72) with $u_l = u(0-)$ and $u_r = u(0+)$.*

Proof of Theorem 2.2.2: Since $\|a_l'\|_{\mathbf{L^1}} = |a^+ - a^-|$, hypothesis (P_2) is satisfied uniformly with respect to l, moreover the smallness of $|a^+ - a^-|$ ensures that the $\mathbf{L^1}$ norm of ω in (P_3) is small. Therefore the hypotheses of Theorem 2.1.4 are satisfied uniformly with respect to l.

Let P^l be the semigroup related with the smooth section a_l. By Remark 2.3.12, if $\mathrm{TV}\{u\}$ is sufficiently small, u belongs to the domain of P^l for every $l > 0$. Since the total variation of $P^l_t u$ is uniformly bounded for a fixed initial data u, Helly's theorem guarantees that there is a converging subsequence $P^{l_i}_t u$. By a diagonal argument one can show that there is a converging subsequence of semigroups converging to a limit semigroup P defined on an invariant domain (see [1, Proof of Theorem 7]).

For the uniqueness we are left to prove the integral estimate (2.1.8) in the origin with $U^\sharp$ substituted by $\bar{U}^\sharp$.

Therefore we have to show that the quantity

$$\frac{1}{\vartheta} \int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}} \left| u(\tau + \vartheta, x) - \bar{U}^\sharp_{u(\tau)}(\vartheta, x) \right| dx \tag{2.5.73}$$

converges to zero as ϑ tends to zero. We will estimate (2.5.73) in several steps. First define $\bar{v} = \bar{U}^{\sharp}_{u(\tau)}(0, x)$ and compute

$$\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}} \left|(P_\vartheta u(\tau))(x) - (P_\vartheta \bar{v})(x)\,(\vartheta, x)\right| dx \leq \bar{\varepsilon}_\vartheta. \tag{2.5.74}$$

as in (2.4.65). Then we consider the approximating sequence P^{l_i} corresponding to the source term a_{l_i} and the semigroups $P^{l_i,h}$ which converge to P^{l_i} in the sense of Theorem 2.3.11. Hence we have

$$\lim_{i\to\infty}\lim_{h\to 0}\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}} \left|(P_\vartheta^{l_i,h}\bar{v})(x) - (P_\vartheta\bar{v})(x)\right| dx = 0$$

For notational convenience we skip the subscript i in l_i. As in (2.4.57) we approximate rarefactions in $\bar{U}^{\sharp}_{u(\tau)}$ introducing the function $\bar{U}^{\sharp,\varepsilon}_{u(\tau)}$. Then we define (see Figure 2.4)

$$\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau, x) = \begin{cases} \bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau, x+\frac{l}{2}) & x < -l/2 \\ \widetilde{U}(x) & -l/2 \leq x \leq l/2 \\ \bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau, x-\frac{l}{2}) & x > l/2 \end{cases}$$

where $\widetilde{U}(x)$ is piecewise constant with jumps in the points jh satisfying $\widetilde{U}(jh+) = \Phi(jh, \widetilde{U}(jh-))$. Furthermore $\widetilde{U}(-l/2-) = \bar{U}^{\sharp,\varepsilon}_{(u;\tau)}(t-\tau, 0-)$ and

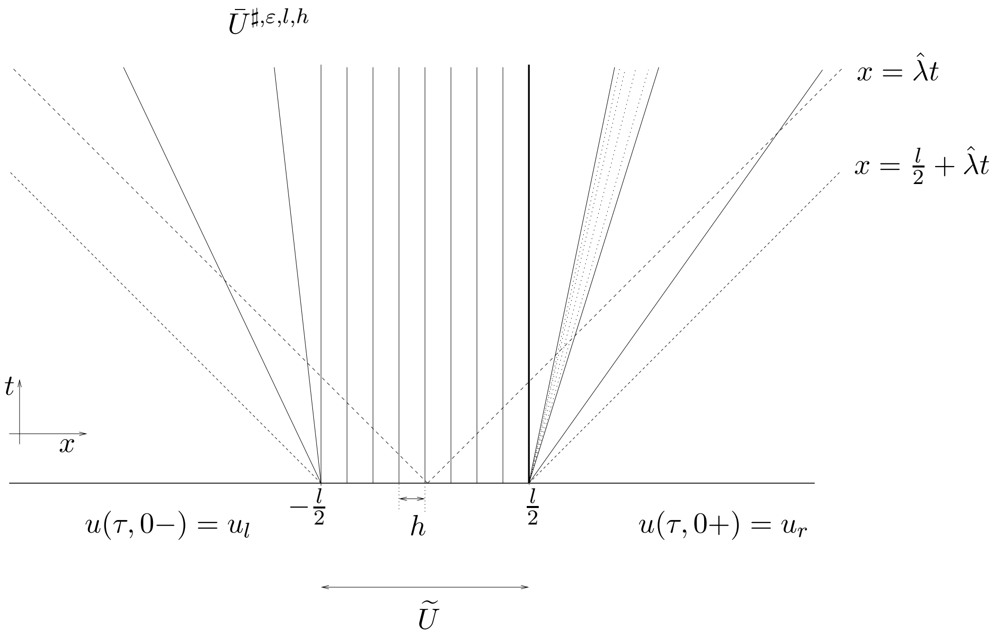


Figure 2.4: Illustration of $\bar{U}^{\sharp,\varepsilon,l,h}$ in the (t, x) plane

Φ is defined as in (2.3.24) using the source term $g(x,u) = a_l'(x)g(u)$. Observe that the jump between $\widetilde{U}(l/2-)$ and $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau, l/2+)$ does not satisfy any jump condition, but as $\widetilde{U}(x)$ is an "Euler" approximation of the ordinary differential equation $f(u)_x = a_l' g(u)$, this jump is of order $\tilde{\varepsilon}_h$. Since $\bar{U}^{\sharp,\varepsilon}_{u(\tau)}$ and $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}$ have uniformly bounded total variation we have the estimate

$$\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}} \left| \bar{U}^{\sharp,\varepsilon}_{u(\tau)}(\vartheta,x) - \bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(\vartheta,x)\right| dx \le \mathcal{O}(1)\,\frac{l}{\vartheta}$$

the bound $\mathcal{O}(1)$ not depending on h. We apply Lemma 2.4.3 on the remaining term

$$\begin{aligned}&\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}} \left| (P^{l,h}_{\vartheta}\bar{v})(x) - \bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(\vartheta,x)\right| dx \\ &\le L\int_{\tau}^{\tau+\vartheta} \liminf_{\eta\to 0} \frac{\|\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau+\eta) - P^{l,h}_{\eta}\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau)\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta}\end{aligned}$$

To estimate this last term we proceed as before. Observe that $P^{l,h}$ does not have zero waves outside the interval $[-\frac{l}{2}-h, \frac{l}{2}+h]$ since outside the interval $[-\frac{l}{2}, \frac{l}{2}]$ the function a_l' is identically zero. If η is small enough, the waves in $P^{l,h}_{\eta}\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau)$ do not interact, therefore the computation of the $\mathbf{L}^1$ norm in the previous integral, as before can be splitted in a summation on the points in which there are zero waves in $P^{l,h}$ or jumps in $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau)$. Observe that the jumps of $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau+\eta)$ in the interval $(-\frac{l}{2}, +\frac{l}{2})$, are defined exactly as the zero waves in $P^{l,h}$ so we have no contribution to the summation from this interval. Outside the interval $[-\frac{l}{2}-h, \frac{l}{2}+h]$, P^h coincides with the homogeneous semigroup, hence we have only the second order contribution from the approximate rarefactions in $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau)$ as in (2.4.60). Furthermore we might have a zero wave in the interval $[-\frac{l}{2}-h, -\frac{l}{2}]$ and a discontinuity of $\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}$ in the point $x = \frac{l}{2}$ of order $\tilde{\varepsilon}_h$. Using (2.4.46) for the zero wave and (2.4.43) for the discontinuity (since P^h is equal to the homogeneous semigroup in $x = \frac{l}{2}$), we get

$$\liminf_{\eta\to 0} \frac{\|\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau+\eta) - P^{l,h}_{\eta}\bar{U}^{\sharp,\varepsilon,l,h}_{u(\tau)}(t-\tau)\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta} \le \mathcal{O}(1)\,(\varepsilon + \tilde{\varepsilon}_h)$$

Which completes the proof if we first let ε tend to zero, then let h tend to zero, then l tend to zero and finally ϑ tend to zero. As in the previous proof, the sufficiency part can be obtained following the proof for the homogeneous case presented in [16]. $\square$

Proof of Proposition 2.2.4: Call S the semigroup defined in [26]. The estimates for this semigroup outside the origin are equal to the ones for the Standard Riemann Semigroup see [16]. Concerning the origin we first observe that the choice (2.2.19) implies that the solution to the Riemann problem in [26, Proposition 2.2] coincides with $\bar{U}^{\sharp}_{u(\tau)}$. We need to show that

$$\lim_{\vartheta\to 0}\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}}\left|u(\tau+\vartheta,x)-\bar{U}^{\sharp}_{u(\tau)}(\vartheta,x)\right|dx=0. \tag{2.5.75}$$

with $u(t,x)=(S_t u_o)(x)$. As before, we first approximate $\bar{U}^{\sharp}_{u(\tau)}$ with $\bar{U}^{\sharp,\varepsilon}_{u(\tau)}$ and $u(\tau)$ with $\bar{U}^{\sharp}_{u(\tau)}(0)\doteq\bar{v}$ then we apply Lemma 2.4.3 (which holds also for the semigroup S) and compute

$$\begin{aligned}
&\frac{1}{\vartheta}\int_{-\vartheta\hat{\lambda}}^{+\vartheta\hat{\lambda}}\left|(S_\vartheta\bar{v})(x)-\bar{U}^{\sharp,\varepsilon}_{u(\tau)}(\vartheta,x)\right|dx\\
&\le L\frac{1}{\vartheta}\int_{\tau}^{\tau+\vartheta}\liminf_{\eta\to 0}\frac{\|\bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau+\eta)-S_\eta\bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau)\|_{\mathbf{L}^1(J_{t+\eta})}}{\eta}
\end{aligned}$$

The discontinuities of $\bar{U}^{\sharp,\varepsilon}_{u(\tau)}$ are solved by S_η with exact shock or rarefaction for $x\neq 0$ and with the a–Riemann solver in $x=0$ therefore the only difference between $\bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau+\eta)$ and $S_\eta\bar{U}^{\sharp,\varepsilon}_{u(\tau)}(t-\tau)$ are the rarefactions solved in an approximate way in the first function and in an exact way in the second. Recalling (2.4.44) we know that this error is of second order in the size of the rarefactions.

To show that (2.5.75) holds, proceed as in (2.4.60). □

Chapter 3

Coupling Conditions for the 3×3 Euler System

3.1 Introduction

We consider Euler equations for the evolution of a fluid flowing in a pipe with varying section $a = a(x)$, see [74, Section 8.1] or [40, 64]:

$$\begin{cases} \partial_t(a\rho) + \partial_x(aq) = 0 \\ \partial_t(aq) + \partial_x\left[aP(\rho, q, E)\right] = p(\rho, e)\ \partial_x a \\ \partial_t(aE) + \partial_x\left[aF(\rho, q, E)\right] = 0 \end{cases} \tag{3.1.1}$$

where, as usual, ρ is the fluid density, q is the linear momentum density and E is the total energy density. Moreover

$$E(\rho, q, E) = \frac{1}{2}\frac{q^2}{\rho} + \rho e, \quad P(\rho, q, E) = \frac{q^2}{\rho} + p, \quad F(\rho, q, E) = \frac{q}{\rho}(E + p), \tag{3.1.2}$$

with e being the internal energy density, P the flow of the linear momentum density and F the flow of the energy density. The above equations express the conservation laws for the mass, momentum, and total energy of the fluid through the pipe. Below, we will often refer to the standard case of the ideal gas, characterized by the relations

$$p = (\gamma - 1)\rho e, \qquad S = \ln e - (\gamma - 1)\ln\rho, \tag{3.1.3}$$

for a suitable $\gamma > 1$. Note however, that this particular equation of state is necessary only in case **(p)** of Proposition 3.3.1 and has been used in the examples in Section 3.4. In the rest of this work, the usual hypothesis [70, formula (18.8)], that is $p > 0$, $\partial_\tau p(\tau, S) < 0$ and $\partial^2_{\tau\tau} p(\tau, S) > 0$, are sufficient.

The case of a sharp discontinuous change in the pipe's section due to a junction sited at, say, $x = 0$, corresponds to $a(x) = a^-$ for $x < 0$ and

$a(x) = a^+$ for $x > 0$. Then, the motion of the fluid can be described by

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x P(\rho, q, E) = 0 \\ \partial_t E + \partial_x F(\rho, q, E) = 0, \end{cases} \tag{3.1.4}$$

for $x \neq 0$, together with a *coupling condition* at the junction of the form:

$$\Psi\left(a^-, (\rho, q, E)(t, 0-); a^+, (\rho, q, E)(t, 0+)\right) = 0. \tag{3.1.5}$$

Above, we require the existence of the traces at $x = 0$ of (ρ, q, E). Various choices of the function Ψ are present in the literature, see for instance [10, 22, 26, 28] in the case of the p-system and [31] for the full 3×3 system (3.1.4). Here, we consider the case of a general coupling condition which comprises all the cases found in the literature. Within this setting, we prove the well posedness of the Cauchy problem for (3.1.4)–(3.1.5). Once this result is obtained, the extension to pipes with several junctions and to pipes with a $\mathbf{W^{1,1}}$ section is achieved by the standard methods considered in the literature. For the analytical techniques to cope with networks having more complex geometry, we refer to [37].

The above statements are global in time and local in the space of the thermodynamic variables (ρ, q, E). Indeed, for any fixed (subsonic) state $(\bar\rho, \bar q, \bar E)$, there exists a bound on the total variation $\mathrm{TV}(a)$ of the pipe's section, such that all sections below this bound give rise to Cauchy problems for (3.1.4)–(3.1.5) that are well posed in $\mathbf{L^1}$. We show the necessity of this bound in the conditions found in the current literature. Indeed, we provide explicit examples showing that a wave can be arbitrarily amplified through consecutive interactions with the pipe walls, see Figure 3.1.

This chapter is organized as follows. The next section is divided into three parts, the former one deals with a single junction and two pipes, then we consider n junctions and $n+1$ pipes, the latter part presents the case of a $\mathbf{W^{1,1}}$ section. Section 3.3 is devoted to different specific choices of coupling conditions (3.1.5). In Section 3.4, an explicit example shows the necessity of the bound on the total variation of the pipe's section. All proofs are gathered in Section 3.5.

3.2 Basic Well Posedness Results

Throughout, we let $u = (\rho, q, E)$. We denote by $\mathbb{R}^+$ the real halfline $[0, +\infty[$, while $\mathring{\mathbb{R}}^+ =]0, +\infty[$. Following various results in the literature, such as [10, 11, 22, 26, 28, 31, 45], we limit the analysis in this paper to the *subsonic* region given by $\lambda_1(u) < 0 < \lambda_3(u)$ and $\lambda_2(u) \neq 0$, where λ_i is the $i-$th eigenvalue of (3.1.4), see (3.5.1). Without any loss of generality, we further restrict to

$$A_0 = \left\{u \in \mathring{\mathbb{R}}^+ \times \mathbb{R}^+ \times \mathring{\mathbb{R}}^+ \colon \lambda_1(u) < 0 < \lambda_2(u)\right\} . \tag{3.2.6}$$

Note that we fix *a priori* the sign of the fluid speed v, since $\lambda_2(u) = q/\rho = v > 0$.

3.2.1 A Junction and two Pipes

We now give the definition of weak $\Psi-$solution to the Cauchy Problem for (3.1.4) equipped with the condition (3.1.5), extending [22, Definition 2.1] and [28, Definition 2.2] to the 3×3 case (3.1.4) and comprising the particular case covered in [31, Definition 2.4].

Definition 3.2.1 *Let* $\Psi\colon (\mathring{\mathbb{R}}^+ \times A_0)^2 \to \mathbb{R}^3$, $u_o \in \mathbf{BV}(\mathbb{R}; A_0)$ *and two positive sections* a^-, a^+ *be given. A* Ψ*-solution to (3.1.4) with initial datum* u_o *is a map*

$$\begin{array}{rcll} u & \in & \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{L^1_{loc}}(\mathbb{R}^+; A_0)\right) & \\ u(t) & \in & \mathbf{BV}(\mathbb{R}; A_0) & \textit{for a.e. } t \in \mathbb{R}^+ \end{array} \tag{3.2.7}$$

such that

1. *for* $x \neq 0$, u *is a weak entropy solution to (3.1.4);*

2. *for a.e.* $x \in \mathbb{R}$, $u(0,x) = u_o(x)$*;*

3. *for a.e.* $t \in \mathbb{R}^+$*, the coupling condition (3.1.5) at the junction is met.*

Below, extending the 2×2 case of the p-system, see [10, 20, 22, 26, 28], we consider some properties of the coupling condition (3.1.5), which we rewrite here as

$$\Psi(a^-, u^-; a^+, u^+) = 0 \,. \tag{3.2.8}$$

($\boldsymbol{\Psi}$0) Regularity: $\Psi \in \mathbf{C^1}\left((\mathring{\mathbb{R}}^+ \times A_0)^2; \mathbb{R}^3\right)$.

($\boldsymbol{\Psi}$1) No-junction case: for all $a > 0$ and all $u^-, u^+ \in A_0$, then

$$\Psi(a, u^-; a, u^+) = 0 \Longleftrightarrow u^- = u^+ \,.$$

($\boldsymbol{\Psi}$2) Consistency: for all positive a^-, a^0, a^+ and all $u^-, u^0, u^+ \in A_0$,

$$\begin{array}{l} \Psi(a^-, u^-; a^0, u^0) = 0 \\ \Psi(a^0, u^0; a^+, u^+) = 0 \end{array} \Longrightarrow \Psi(a^-, u^-; a^+, u^+) = 0 \,.$$

Moreover, by an immediate extension of [28, Lemma 2.1], (**Ψ0**) ensures that (3.2.8) implicitly defines a map

$$u^+ = T(a^-, a^+; u^-) \tag{3.2.9}$$

in a neighborhood of any pair of subsonic states u^-, u^+ and sections a^-, a^+ that satisfy $\Psi(a^-, u^-; a^+, u^+) = 0$.

The technique in [24] allows to prove the following well posedness result.

Theorem 3.2.2 *Assume that Ψ satisfies conditions (**Ψ0**)-(**Ψ2**). For every $\bar{a} > 0$ and $\bar{u} \in A_0$ such that*

$$\det \left[D_{u^-}\Psi \cdot r_1(\bar{u}) \quad D_{u^+}\Psi \cdot r_2(\bar{u}) \quad D_{u^+}\Psi \cdot r_3(\bar{u}) \right] \neq 0 \tag{3.2.10}$$

there exist positive δ, L such that for all a^-, a^+ with $\left|a^+ - \bar{a}\right| + \left|a^- - \bar{a}\right| < \delta$ there exists a semigroup $S: \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$ with the following properties:

1. *$\mathcal{D} \supseteq \left\{u \in \bar{u} + \mathbf{L^1}(\mathbb{R}; A_0) : \mathrm{TV}(u) < \delta\right\}$.*

2. *For all $u \in \mathcal{D}$, $S_0 u = u$ and for all $t, s \geq 0$, $S_t S_s u = S_{s+t} u$.*

3. *For all $u, u' \in \mathcal{D}$ and for all $t, t' \geq 0$,*

$$\left\|S_t u - S_{t'} u'\right\|_{\mathbf{L^1}} \leq L \cdot \left(\left\|u - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right| \right)$$

4. *If $u \in \mathcal{D}$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at the junction at $x = 0$.*

5. *For all $u_o \in \mathcal{D}$, the orbit $t \to S_t u_o$ is a Ψ-solution to (3.1.4) with initial datum u_o.*

The proof is postponed to Section 3.5. Above $r_i(u)$, with $i = 1, 2, 3$, are the right eigenvectors of $Df(u)$, see (3.5.1). Moreover, by *solution to the Riemann Problems at the points of jump* we mean the usual Lax solution, see [16, Chapter 5], whereas for the definition of *solution to the Riemann Problems at the junction* we refer to [26, Definition 2.1].

3.2.2 n Junctions and $n + 1$ Pipes

The same procedure used in [28, Paragraph 2.2] allows now to construct the semigroup generated by (3.1.4) in the case of a pipe with piecewise constant section

$$a = a_0\, \chi_{]-\infty, x_1]} + \sum_{j=1}^{n-1} a_j\, \chi_{[x_j, x_{j+1}[} + a_n\, \chi_{[x_n, +\infty[}$$

with $n \in \mathbb{N}$. In each segment $\left]x_j, x_{j+1}\right[$, the fluid is modeled by (3.1.4). At each junction x_j, we require condition (3.1.5), namely

$$\Psi(a_{j-1}, u_j^-; a_j, u_j^+) = 0 \qquad \begin{array}{l} \text{for all } j = 1, \ldots, n, \text{ where} \\ u_j^\pm = \lim_{x \to x_j \pm} u_j(x)\,. \end{array} \tag{3.2.11}$$

We omit the formal definition of Ψ-solution to (3.1.4)–(3.1.5) in the present case, since it is an obvious iteration of Definition 3.2.1. The natural extension of Theorem 3.2.2 to the case of (3.1.4)–(3.2.11) is the following result.

Theorem 3.2.3 *Assume that Ψ satisfies conditions ($\mathbf{\Psi 0}$)-($\mathbf{\Psi 2}$). For any $\bar a > 0$ and any $\bar u \in A_0$, there exist positive $M, \Delta, \delta, L, \mathcal{M}$ such that for any pipe's profile satisfying*

$$a \in \mathbf{PC}\left(\mathbb{R};]\bar a - \Delta, \bar a + \Delta[\right) \text{ with } \mathrm{TV}(a) < M \tag{3.2.12}$$

there exists a piecewise constant stationary solution

$$\hat u = \hat u_0 \chi_{]-\infty, x_1[} + \sum_{j=1}^{n-1} \hat u_j \chi_{]x_j, x_{j+1}[} + \hat u_n \chi_{]x_n, +\infty[}$$

to (3.1.4)–(3.2.11) satisfying

$$\begin{aligned} &\hat u_j \in A_0 \text{ with } \left|\hat u_j - \bar u\right| < \delta \text{ for } j = 0, \ldots n \\ &\Psi\left(a_{j-1}, \hat u_{j-1}; a_j, \hat u_j\right) = 0 \text{ for } j = 1, \ldots, n \\ &\mathrm{TV}(\hat u) \le \mathcal{M}\, \mathrm{TV}(a) \end{aligned} \tag{3.2.13}$$

and a semigroup $S^a : \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

1. *$\mathcal{D}^a \supseteq \left\{u \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0) : \mathrm{TV}(u - \hat u) < \delta\right\}$.*
2. *S_0^a is the identity and for all $t, s \ge 0$, $S_t^a S_s^a = S_{s+t}^a$.*
3. *For all $u, u' \in \mathcal{D}^a$ and for all $t, t' \ge 0$,*

$$\left\|S_t^a u - S_{t'}^a u'\right\|_{\mathbf{L^1}} \le L \cdot \left(\left\|(u) - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right|\right).$$

4. *If $u \in \mathcal{D}^a$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at each junction x_j.*
5. *For all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is a weak Ψ-solution to (3.1.4)–(3.2.11).*

We omit the proof, since it is based on the natural extension to the present 3×3 case of [28, Theorem 2.4]. Remark that, as in that case, δ and L depend on a only through $\bar a$ and $\mathrm{TV}(a)$. In particular, all the construction above is independent from the number of points of jump in a.

3.2.3 A Pipe with a $\mathbf{W}^{1,1}$ Section

In this paragraph, the pipe's section a is assumed to satisfy

$$\begin{cases} a \in \mathbf{W}^{\mathbf{1,1}}\left(\mathbb{R};]\bar{a}-\Delta, \bar{a}+\Delta[\right) \text{ for suitable } \Delta > 0,\ \bar{a} > \Delta \\ \mathrm{TV}(a) < M \text{ for a suitable } M > 0 \\ a'(x) = 0 \text{ for a.e. } x \in \mathbb{R} \setminus [-X, X] \text{ for a suitable } X > 0\,. \end{cases} \tag{3.2.14}$$

The same procedure used in [28, Theorem 2.8] allows to construct the semigroup generated by (3.1.1) in the case of a pipe which satisfies (3.2.14). Indeed, thanks to Theorem 3.2.3, we approximate a with a piecewise constant function a_n. The corresponding problems to (3.1.4)–(3.2.11) generate semigroups S_n defined on domains characterized by uniform bounds on the total variation and that are uniformly Lipschitz in time. Here, uniform means also independent from the number of junctions. Therefore, we prove the pointwise convergence of the S_n to a limit semigroup S, along the same lines in [28, Theorem 2.8].

3.3 Coupling Conditions

This section is devoted to different specific choices of (3.2.8).

(S)-Solutions We consider first the coupling condition inherited from the smooth case. For smooth solutions and pipes' sections, system (3.1.1) is equivalent to the 3×3 balance law

$$\begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a}\,\partial_x a \\ \partial_t q + \partial_x P(\rho, q, E) = -\dfrac{q^2}{a\rho}\,\partial_x a \\ \partial_t E + \partial_x F(\rho, q, E) = -\dfrac{F}{a}\,\partial_x a. \end{cases} \tag{3.3.1}$$

The stationary solutions to (3.1.1) are characterized as solutions to

$$\begin{cases} \partial_x (a(x)\, q) = 0 \\ \partial_x \big(a(x)\, P(\rho, q, E)\big) = p(\rho, e)\,\partial_x a \\ \partial_x \big(a(x)\, F(\rho, q, E)\big) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \partial_x q = -\frac{q}{a}\,\partial_x a \\ \partial_x P\,(\rho, q, E) = -\frac{q^2}{a\rho}\,\partial_x a \\ \partial_x F\,(\rho, q, E) = -\frac{F}{a}\,\partial_x a\,. \end{cases} \tag{3.3.2}$$

As in the 2×2 case of the p-system, the smoothness of the sections induces a unique choice for condition (3.2.8), see [28, (2.3) and (2.19)], which reads

$$\textbf{(S)} \quad \Psi = \begin{bmatrix} a^+ q^+ - a^- q^- \\ a^+ P(u^+) - a^- P(u^-) + \displaystyle\int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right) a'(x)\mathrm{d}x \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix} \tag{3.3.3}$$

where $a = a(x)$ is a smooth monotone function satisfying $a(-X) = a^-$ and $a(X) = a^+$, for a suitable $X > 0$. $\mathcal{R}^a, \mathcal{E}^a$ are the ρ and e component in the solution to (3.3.2) with initial datum u^- assigned at $-X$. Note that, by the particular form of (3.3.3), the function Ψ is independent both from the choice of X and from that of the map a, see [28, 2. in Proposition 2.7].

(P)-Solutions The particular choice of the coupling condition in [31, Section 3] can be recovered in the present setting. Indeed, conditions **(M)**, **(E)** and **(P)** therein amount to the choice

$$\textbf{(P)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+q^+ - a^-q^- \\ P(u^+) - P(u^-) \\ a^+F(u^+) - a^-F(u^-) \end{bmatrix}, \tag{3.3.4}$$

where a^+ and a^- are the pipe's sections. Consider fluid flowing in a horizontal pipe with an elbow or kink, see [51]. Then, it is natural to assume the conservation of the total linear momentum along directions dependent upon the geometry of the elbow. As the angle of the elbow vanishes, one obtains the condition above, see [31, Proposition 2.6].

(L)-Solutions We can extend the construction in [10, 11, 20] to the 3×3 case (3.1.4). Indeed, the conservation of the mass and linear momentum in [20] with the conservation of the total energy for the third component lead to the choice

$$\textbf{(L)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+q^+ - a^-q^- \\ a^+P(u^+) - a^-P(u^-) \\ a^+F(u^+) - a^-F(u^-) \end{bmatrix}, \tag{3.3.5}$$

where a^+ and a^- are the pipe's sections. The above is the most immediate extension of the standard definition of Lax solution to the case of the Riemann problem at a junction.

(p)-Solutions Following [10, 11], motivated by the what happens at the hydrostatic equilibrium, we consider a coupling condition with the conservation of the pressure $p(\rho)$ in the second component of Ψ. Thus

$$\textbf{(p)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+\,q^+ - a^-\,q^- \\ p(\rho^+, e^+) - p(\rho^-, e^-) \\ a^+F(u^+) - a^-F(u^-) \end{bmatrix}, \tag{3.3.6}$$

where a^+ and a^- are the pipe's sections.

Proposition 3.3.1 *For every $\bar{a} > 0$ and $\bar{u} \in A_0$, each of the coupling conditions Ψ in (3.3.3), (3.3.4), (3.3.5), (3.3.6) satisfies the requirements (**Ψ0**)-(**Ψ2**) and (3.2.10). In the case of (3.3.6), we also require that the fluid is perfect, i.e. that (3.1.3) holds.*

The proof is postponed to Section 3.5. Thus, Theorem 3.2.2 applies, yielding the well posedness of (3.1.4)–(3.1.5) with each of the particular choices of Ψ in (3.3.3), (3.3.4), (3.3.5), (3.3.6).

3.4 Blow-Up of the Total Variation

In the previous results a key role is played by the bound on the total variation $\mathrm{TV}(a)$ of the pipe's section. This requirement is intrinsic to problem (3.1.4)–(3.1.5) and not due to the technique adopted above. Indeed, we show below that in each of the cases (3.3.3), (3.3.4), (3.3.5), (3.3.6), it is possible to choose an initial datum and a section $a \in \mathbf{BV}(\mathbb{R}; [a^-, a^+])$ with $a^+ - a^-$ arbitrarily small, such that the total variation of the corresponding solution to (3.1.4)–(3.1.5) becomes arbitrarily large.

Consider the case in Figure 3.1. A wave σ_3^- hits a junction where the

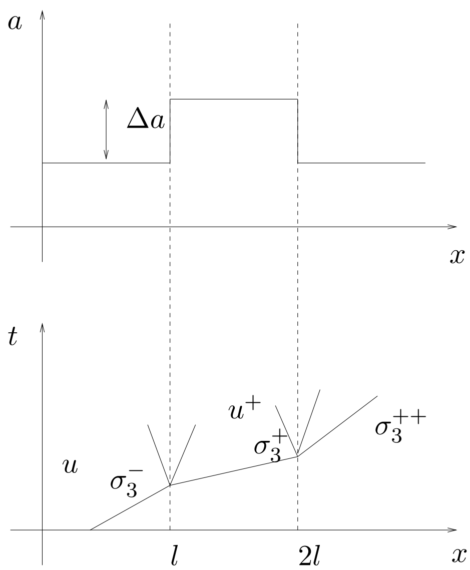


Figure 3.1: A wave σ_3^- hits a junction where the pipe's section increases by Δa. From this interaction, the wave σ_3^+ arises, which hits a second junction, where the pipe section decreases by Δa.

pipe's section increases by, say, $\Delta a > 0$. The fastest wave arising from this interaction is σ_3^+, which hits the second junction where the section diminishes by Δa.

Solving the Riemann problem at the first interaction amounts to solve the system

$$L_3\left(L_2\left(T\left(L_1(u;\sigma_1^+)\right);\sigma_2^+\right);\sigma_3^+\right) = T\left(L_3(u;\sigma_3^-)\right), \tag{3.4.1}$$

where $u \in A_0$, see Figure 3.2 for the definitions of the waves' strengths σ_i^+ and σ_3^-. Above, T is the map defined in (3.2.9), which in turn depends from the specific condition (3.2.8) chosen. In the expansions below, we use the (ρ, q, e) variables, thus setting $u = (\rho, q, e)$ throughout this section. Differently from the case of the 2×2 p-system in [28], here we need to

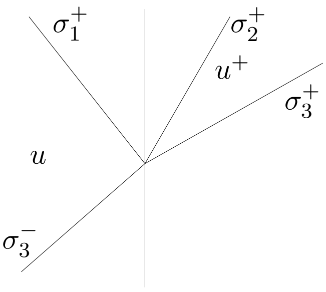


Figure 3.2: Notation used in (3.4.1) and (3.4.4).

consider the second order expansion in $\Delta a = a^+ - a^-$ of the map T; that is

$$T(a, a+\Delta a; u) = u + H(u)\frac{\Delta a}{a} + G(u)\left(\frac{\Delta a}{a}\right)^2 + o\left(\frac{\Delta a}{a}\right)^2 \tag{3.4.2}$$

The explicit expressions of H and G in (3.4.2), for each of the coupling conditions (3.3.3), (3.3.4), (3.3.5), (3.3.6), are in Section 3.5.2.

Inserting (3.4.2) in the first order expansions in the wave's sizes of (3.4.1), with $\tilde{r}_i$ for $i = 1, 2, 3$ as in (3.5.3), we get a linear system in $\sigma_1^+, \sigma_2^+, \sigma_3^+$. Now, introduce the fluid speed $v = q/\rho$ and the adimensional parameter

$$\vartheta = \left(\frac{v}{c}\right)^2 = \frac{v^2}{\gamma(\gamma-1)e},$$

a sort of *"Mach number"*. Obviously, $\vartheta \in [0, 1]$ for $u \in A_0$. We thus obtain an expression for σ_3^+ of the form

$$\sigma_3^+ = \left(1 + f_1(\vartheta)\frac{\Delta a}{a} + f_2(\vartheta)\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^-. \tag{3.4.3}$$

The explicit expressions of f_1 and f_2 in (3.4.3) are in Section 3.5.2.

Remark that the present situation is different from that of the 2×2 p-system considered in [28]. Indeed, for the p-system $f_2(\vartheta) = f_2(\vartheta^+) = 0$, while here it is necessary to compute the second order term in $(\Delta a)/a$.

Concerning the second junction, similarly, we introduce the parameter $\vartheta^+ = (v^+/c^+)^2$ which corresponds to the state u^+. Recall that u^+ is defined by $u^+ = L_3^-\left(T\left(L_3(u;\sigma_3^-);\sigma_3^+\right)\right)$, see Figure 3.2 and Section 3.5.2 for the explicit expressions of ϑ^+. We thus obtain the estimate

$$\sigma_3^{++} = \left(1 - f_1(\vartheta^+)\,\frac{\Delta a}{a} + f_2(\vartheta^+)\,\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^+, \tag{3.4.4}$$

where $\vartheta^+ = \vartheta^+\left(\vartheta, \sigma_3^-, (\Delta a)/a\right)$. Now, at the second order in $(\Delta a)/a$ and at the first order in σ_3^-, (3.4.3) and (3.4.4) give

$$\begin{aligned}\sigma_3^{++} &= \left(1 - f_1(\vartheta^+)\,\frac{\Delta a}{a} + f_2(\vartheta^+)\,\left(\frac{\Delta a}{a}\right)^2\right)\\ &\qquad\times\left(1 + f_1(\vartheta)\,\frac{\Delta a}{a} + f_2(\vartheta)\,\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^-\\ &= \left(1 + \chi(\vartheta)\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^-\,. \end{aligned}\tag{3.4.5}$$

Indeed, computations show that $f_1(\vartheta) - f_1\left(\vartheta^+\right)$ vanishes at the first order in $(\Delta a)/a$, as in the case of the p-system. The explicit expressions of χ are in Section 3.5.2.

It is now sufficient to compute the sign of χ. If it is positive, then repeating the interaction in Figure 3.1 a sufficient number of times leads to an arbitrarily high value of the refracted wave σ_3 and, hence, of the total variation of the solution u.

Below, Section 3.5 is devoted to the computations of χ in the different cases (3.3.3), (3.3.4), (3.3.5) and (3.3.6). To reduce the formal complexities of the explicit computations below, we consider the standard case of an ideal gas characterized by (3.1.3) with $\gamma = 5/3$.

The results of these computations are in Figure 3.3. They show that in all the conditions (3.1.5) considered, there exists a state $u \in A_0$ such that $\chi(\vartheta) > 0$, showing the necessity of condition (3.2.12). However, in case **(L)**, it turns out that χ is negative on an non trivial interval of values of ϑ. If $\bar{u}$ is chosen in this interval, the wave σ_3 in the construction above is not magnified by the consecutive interactions. The computations leading to the diagrams in Figure 3.3 are deferred to Section 3.5.2.

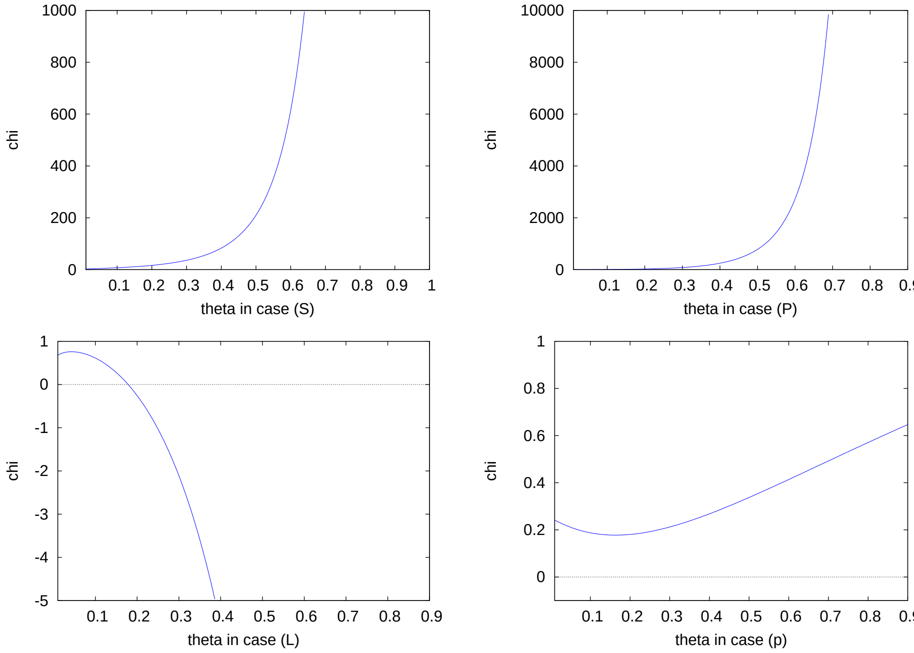


Figure 3.3: Plots of χ as a function of ϑ. Top, left, case **(S)**; right, case **(P)**; bottom, left, case **(L)**; right, case **(p)**. Note that in all four cases, χ attains strictly positive values, showing the necessity of the requirement (3.2.12).

3.5 Technical Details

We recall here basic properties of the Euler equations (3.1.1), (3.1.4). The characteristic speeds and the right eigenvectors have the expressions

$$\begin{array}{lll} \lambda_1 = \frac{q}{\rho} - c & \lambda_2 = \frac{q}{\rho} & \lambda_3 = \frac{q}{\rho} + c \\ r_1 = \begin{bmatrix} -\rho \\ \rho c - q \\ qc - E - p \end{bmatrix} & r_2 = \begin{bmatrix} \rho \\ q \\ E + p - \frac{\rho^2 c^2}{\partial_e p} \end{bmatrix} & r_3 = \begin{bmatrix} \rho \\ q + \rho c \\ E + p + qc \end{bmatrix} \end{array} \tag{3.5.1}$$

whose directions are chosen so that $\nabla \lambda_i \cdot r_i > 0$ for $i = 1, 2, 3$. In the case of an ideal gas, the sound speed $c = \sqrt{\partial_\rho p + \rho^{-2}\, p\, \partial_e p}$ becomes

$$c = \sqrt{\gamma(\gamma - 1)e}\,. \tag{3.5.2}$$

The shock and rarefaction curves curves of the first and third family are:

$$S_1(u_o, \sigma) = \left\{ \begin{array}{l} \rho = -\sigma + \rho_o \\ v = v_o - \sqrt{-\left(p - p_o\right)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}\left(p + p_o\right)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) \end{array} \right. \quad \text{for} \quad \begin{array}{rcl} \sigma & \leq & 0 \\ \rho & \geq & \rho_o \\ v & \leq & v_o \\ S & \geq & S_o \end{array}$$

$$S_3(u_o, \sigma) = \left\{ \begin{array}{l} \rho = \sigma + \rho_o \\ v = v_o - \sqrt{-\left(p - p_o\right)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}\left(p + p_o\right)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) \end{array} \right. \quad \text{for} \quad \begin{array}{rcl} \sigma & \leq & 0 \\ \rho & \leq & \rho_o \\ v & \leq & v_o \\ S & \leq & S_o \end{array}$$

$$R_1(u_o, \sigma) = \left\{ \begin{array}{l} \rho = -\sigma + \rho_o \\ v = v_o - \displaystyle\int_{p_o}^{p} [(\rho\, c)(p, S_o)]^{-1}\, dp \\ S(\rho, e) = S(\rho_o, e_o) \end{array} \right. \quad \text{for} \quad \begin{array}{rcl} \sigma & \geq & 0 \\ \rho & \leq & \rho_o \\ v & \geq & v_o \\ e & \leq & e_o \end{array}$$

$$R_3(u_o, \sigma) = \left\{ \begin{array}{l} \rho = \sigma + \rho_o \\ v = v_o + \displaystyle\int_{p_o}^{p} [(\rho\, c)(p, S_o)]^{-1}\, dp \\ S(\rho, e) = S(\rho_o, e_o) \end{array} \right. \quad \text{for} \quad \begin{array}{rcl} \sigma & \geq & 0 \\ \rho & \geq & \rho_o \\ v & \geq & v_o \\ e & \geq & e_o \end{array}$$

The 1,2,3-Lax curves have the expressions

$$L_1(\sigma; \rho_o, q_o, E_o) = \left\{ \begin{array}{ll} S_1(\sigma; \rho_o, q_o, E_o), & \sigma < 0 \\ R_1(\sigma; \rho_o, q_o, E_o), & \sigma \geq 0 \end{array} \right.$$

$$L_2(\sigma; \rho_o, q_o, E_o) = \left\{ \begin{array}{l} \rho = \sigma + \rho_o \\ v = v_o \\ p(\rho, e) = p(\rho_o, e_o) \end{array} \right.$$

$$L_3(\sigma; \rho_o, q_o, E_o) = \left\{ \begin{array}{ll} S_3(\rho; \rho_l, q_l, E_l), & \sigma < 0 \\ R_3(\sigma; \rho_o, q_o, E_o), & \sigma \geq 0 \end{array} \right.$$

Their reversed counterparts are

$$S_1^-(u_o,\sigma) \quad = \quad \begin{cases} \rho = \sigma + \rho_o \\ v = v_o + \sqrt{-\left(p-p_o\right)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}\left(p+p_o\right)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right) \end{cases} \text{for} \begin{array}{ccc} \sigma & \leq & 0 \\ \rho & \leq & \rho_o \\ v & \geq & v_o \\ S & \leq & S_o \end{array}$$

$$S_3^-(u_o,\sigma) \quad = \quad \begin{cases} \rho = -\sigma + \rho_o \\ v = v_o + \sqrt{-\left(p-p_o\right)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}\left(p+p_o\right)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right) \end{cases} \text{for} \begin{array}{ccc} \sigma & \leq & 0 \\ \rho & \geq & \rho_o \\ v & \geq & v_o \\ S & \geq & S_o \end{array}$$

$$R_1^-(u_o,\sigma) \quad = \quad \begin{cases} \rho = \sigma + \rho_o \\ v = v_o - \displaystyle\int_{p_o}^{p} [(\rho\, c)(p,S_o)]^{-1}\, dp \\ S(\rho,e) = S(\rho_o,e_o) \end{cases} \quad \text{for} \begin{array}{ccc} \sigma & \geq & 0 \\ \rho & \geq & \rho_o \\ v & \leq & v_o \\ e & \geq & e_o \end{array}$$

$$R_3^-(u_o,\sigma) \quad = \quad \begin{cases} \rho = -\sigma + \rho_o \\ v = v_o + \displaystyle\int_{p_o}^{p} [(\rho\, c)(p,S_o)]^{-1}\, dp \\ S(\rho,e) = S(\rho_o,e_o) \end{cases} \quad \text{for} \begin{array}{ccc} \sigma & \geq & 0 \\ \rho & \leq & \rho_o \\ v & \leq & v_o \\ e & \leq & e_o \end{array}$$

and

$$L_1^-(\sigma;\rho_o,q_o,E_o) \quad = \quad \begin{cases} S_1^-(\sigma;\rho_o,q_o,E_o), & \sigma < 0 \\ R_1^-(\sigma;\rho_o,q_o,E_o), & \sigma \geq 0 \end{cases}$$

$$L_2^-(\sigma;\rho_o,q_o,E_o) \quad = \quad \begin{cases} \rho = -\sigma + \rho_o \\ v = v_o \\ p(\rho,e) = p(\rho_o,e_o) \end{cases}$$

$$L_3^-(\sigma;\rho_o,q_o,E_o) \quad = \quad \begin{cases} S_3^-(\sigma;\rho_o,q_o,E_o), & \sigma < 0 \\ R_3^-(\sigma;\rho_o,q_o,E_o), & \sigma \geq 0\,. \end{cases}$$

In the (ρ,q,e) space, for a perfect ideal gas, the tangent vectors to the Lax curves are:

$$\tilde{r}_1 = \begin{bmatrix} -1 \\ -\frac{q}{\rho} - \sqrt{\gamma(\gamma-1)e} \\ -(\gamma-1)\frac{e}{\rho} \end{bmatrix} \quad \tilde{r}_2 = \begin{bmatrix} 1 \\ \frac{q}{\rho} \\ -\frac{e}{\rho} \end{bmatrix} \quad \tilde{r}_3 = \begin{bmatrix} 1 \\ \frac{q}{\rho} - \sqrt{\gamma(\gamma-1)e} \\ (\gamma-1)\frac{e}{\rho} \end{bmatrix} \tag{3.5.3}$$

3.5.1 Proofs of Section 3.2

The following result will be of use in the proof of Proposition 3.2.2.

Proposition 3.5.1 *Let $\sigma_i \mapsto L_i(u_0, \sigma_i)$ be the i-th Lax curve and $\sigma_i \mapsto L_i^-(u_0, \sigma_i)$ be the reversed i-th Lax curve through u_0, for $i = 1, 2, 3$. The following equalities hold:*

$$\frac{\partial L_1}{\partial \sigma_1}_{|\sigma_1=0} = \begin{pmatrix} 1 \\ \lambda_1(u_o) \\ \dfrac{E_o + p_o}{\rho_o} - \dfrac{q_o}{\rho_o} c_o \end{pmatrix}, \quad \frac{\partial L_2}{\partial \sigma_2}_{|\sigma_2=0} = \begin{pmatrix} 1 \\ \lambda_2(u_o) \\ \dfrac{E_o + p_o}{\rho_o} - \dfrac{\rho_o \, c_o^2}{\partial_e p_o} \end{pmatrix},$$

$$\frac{\partial L_3}{\partial \sigma_3}_{|\sigma_3=0} = \begin{pmatrix} 1 \\ \lambda_3(u_o) \\ \dfrac{E_o + p_o}{\rho_o} + \dfrac{q_o}{\rho_o} c_o \end{pmatrix},$$

$$\textit{for } i = 1,2,3 \qquad \frac{\partial L_i^-}{\partial \sigma_i}_{|\sigma_i=0} = -\frac{\partial L_i}{\partial \sigma_i}_{|\sigma_i=0},$$

$$\frac{\partial L_i}{\partial \rho_o}_{|\sigma_i=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial L_i}{\partial q_o}_{|\sigma_i=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{\partial L_i}{\partial E_o}_{|\sigma_i=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$+\frac{\partial L_i^-}{\partial \rho_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial \rho_o}_{|\sigma_i=0}, \quad \frac{\partial L_i^-}{\partial q_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial q_o}_{|\sigma_i=0}, \quad \frac{\partial L_i^-}{\partial E_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial E_o}_{|\sigma_i=0}.$$

The proof is immediate and, hence, omitted.

Proof of Theorem 3.2.2. Following [25, Proposition 4.2], the 3×3 system (3.1.4) defined for $x \in \mathbb{R}$ can be rewritten as the following 6×6 system defined for $x \in \mathbb{R}^+$:

$$\left\{ \begin{array}{lrcl} \partial_t U + \partial_x \mathcal{F}(U) = 0 & (t,x) & \in & \mathbb{R}^+ \times \mathbb{R}^+ \\ b\big(U(t, 0+)\big) = 0 & t & \in & \mathbb{R}^+ \end{array} \right. \tag{3.5.4}$$

the relations between U and $u = (\rho, q, E)$, between $\mathcal{F}$ and the flow in (3.1.4) being

$$U(t,x) = \begin{bmatrix} \rho(t,-x) \\ q(t,-x) \\ E(t,-x) \\ \rho(t,x) \\ q(t,x) \\ E(t,x) \end{bmatrix} \quad \text{and} \quad \mathcal{F}(U) = \begin{bmatrix} U_2 \\ P(U_1,U_2,U_3) \\ F(U_1,U_2,U_3) \\ U_5 \\ P(U_4,U_5,U_6) \\ F(U_4,U_5,U_6) \end{bmatrix}$$

with $x \in \mathbb{R}^+$ and E, P, F defined in (3.1.2); whereas the boundary condition in (3.5.4) is related to (3.1.5) by

$$b(U) = \Psi\Big(a^-, (U_1, U_2, U_{;3}); a^+, (U_4, U_5, U_6)\Big)$$

for fixed sections a^- and a^+.

The thesis now follows from [24, Theorem 2.2]. Indeed, the assumptions **(γ)**, **(b)** and **(f)** therein are here satisfied. More precisely, condition **(γ)** follows from the choice (3.2.6) of the subsonic region A_0. Simple computations show that condition **(b)** reduces to

$$\begin{aligned}
&\det \left[D_{u^-}\Psi \cdot \frac{\partial L_1}{\partial \sigma_1}_{|\sigma_1=0} \quad D_{u^+}\Psi \cdot \frac{\partial L_2^-}{\partial \sigma_2}_{|\sigma_2=0} \quad D_{u^+}\Psi \cdot \frac{\partial L_2^-}{\partial u^+}_{|\sigma_2=0} \cdot \frac{\partial L_3^-}{\partial \sigma_3}_{|\sigma_3=0} \right] \\
&= \det \left[D_{u^-}\Psi \cdot r_1(\bar u) \quad - D_{u^+}\Psi \cdot r_2(\bar u) \quad - D_{u^+}\Psi \cdot r_3(\bar u) \right] \\
&= \det \left[D_{u^-}\Psi \cdot r_1(\bar u) \quad D_{u^+}\Psi \cdot r_2(\bar u) \quad D_{u^+}\Psi \cdot r_3(\bar u) \right],
\end{aligned}$$

which is non zero for assumption if $\bar u \in A_0$ and $\bar a > 0$. Condition **(f)** needs more care. Indeed, system (3.5.4) is *not* hyperbolic, for it is obtained gluing two copies of the Euler equations (3.1.4). Nevertheless, the two systems are coupled only through the boundary condition, hence the whole wave front tracking procedure in the proof of [24, Theorem 2.2] applies, see also [25, Proposition 4.5]. □

Proof of Proposition 3.3.1. It is immediate to check that each of the coupling conditions (3.3.3), (3.3.4), (3.3.5), (3.3.6) satisfies the requirements (**Ψ0**) and (**Ψ1**).

To prove that (**Ψ2**) is satisfied, we use an *ad hoc* argument for condition **(S)**. In all the other cases, note that the function Ψ admits the representation $\Psi(a^-, u^-; a^+, u^+) = \psi(a^-, u^-) - \psi(a^+, u^+)$. Therefore, (**$\Psi$2**) trivially holds.

We prove below (3.2.10) in each case separately. Note however that for any of the considered choices of Ψ,

$$D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} = -D_{u^-}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \quad (3.5.5)$$

so that (3.2.10) reduces to

$$\begin{aligned}
&\det \left[D_{u^-}\Psi \cdot r_1(\bar u) \quad D_{u^+}\Psi \cdot r_2(\bar u) \quad D_{u^+}\Psi \cdot r_3(\bar u) \right] \\
= \;& -\det D_{u^+}\Psi \cdot \det \left[r_1(\bar u) \quad r_2(\bar u) \quad r_3(\bar u) \right].
\end{aligned}$$

Thus, it is sufficient to prove that $\det D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \neq 0$.

(S)-solutions To prove that the coupling condition (3.3.3) satisfies (**Ψ2**), simply use the additivity of the integral and the uniqueness of the solution to the Cauchy problem for the ordinary differential equation (3.3.2).

Next, we have

$$D_u \left(\int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right) \, a'(x) \, \mathrm{d}x \right)_{|u=\bar u,\, a=\bar a} = 0 \,,$$

since $a'(x) = 0$ for all x, because $a^- = a^+ = \bar{a}$. Thus, Ψ in (3.3.3) satisfies

$$
\begin{aligned}
&D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar{u},\, a=\bar{a}} \\
&= \bar{a}^3 \det \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\bar{q}^2}{\bar{\rho}^2} + \partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) & \frac{\bar{q}}{\bar{\rho}}\left(2 - \frac{\partial_e \bar{p}}{\bar{\rho}}\right) & \frac{\partial_e \bar{p}}{\bar{\rho}} \\ -\frac{\bar{q}}{\bar{\rho}}\left(\partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) - \frac{\bar{E}+\bar{p}}{\bar{\rho}}\right) & \frac{\bar{E}+\bar{p}}{\bar{\rho}} - \frac{\partial_e \bar{p}}{\bar{\rho}}\frac{\bar{q}^2}{\bar{\rho}^2} & -\frac{\bar{q}}{\bar{\rho}}\left(\frac{\partial_e \bar{p}}{\bar{\rho}} + 1\right) \end{bmatrix} \\
&= -\bar{a}^3\, \lambda_1(\bar{u})\, \lambda_2(\bar{u})\, \lambda_3(\bar{u}),
\end{aligned}
$$

which is non zero if $\bar{u} \in A_0$.

(P)-solutions Concerning condition (3.3.4), we have

$$
\begin{aligned}
&D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar{u},\, a=\bar{a}} \\
&= \det \begin{bmatrix} 0 & \bar{a} & 0 \\ -\frac{\bar{q}^2}{\bar{\rho}^2} + \partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) & \frac{\bar{q}}{\bar{\rho}}\left(2 - \frac{\partial_e \bar{p}}{\bar{\rho}}\right) & \frac{\partial_e \bar{p}}{\bar{\rho}} \\ -\bar{a}\frac{\bar{q}}{\bar{\rho}}\left(\partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) - \frac{\bar{E}+\bar{p}}{\bar{\rho}}\right) & \bar{a}\frac{\bar{E}+\bar{p}}{\bar{\rho}} - \bar{a}\frac{\partial_e \bar{p}}{\bar{\rho}}\frac{\bar{q}^2}{\bar{\rho}^2} & -\bar{a}\frac{\bar{q}}{\bar{\rho}}\left(\frac{\partial_e \bar{p}}{\bar{\rho}} + 1\right) \end{bmatrix} \\
&= -\bar{a}^2 \lambda_1(\bar{u})\lambda_2(\bar{u})\lambda_3(\bar{u}),
\end{aligned}
$$

which is non zero if $\bar{u} \in A_0$.

(L)-solution For condition (3.3.5) the computations very similar to the above case:

$$
D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar{u},\, a=\bar{a}} = -\bar{a}^3 \lambda_1(\bar{u})\lambda_2(\bar{u})\lambda_3(\bar{u}),
$$

which is non zero if $\bar{u} \in A_0$.

(p)-solution Finally, concerning condition (3.3.6),

$$
\begin{aligned}
&D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar{u},\, a=\bar{a}} \\
&= \det \begin{bmatrix} 0 & \bar{a} & 0 \\ \partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) & -\frac{\bar{q}}{\bar{\rho}^2}\partial_e \bar{p} & \frac{\partial_e \bar{p}}{\bar{\rho}} \\ -\bar{a}\frac{\bar{q}}{\bar{\rho}}\left(\partial_\rho \bar{p} + \frac{\partial_e \bar{p}}{\bar{\rho}}\left(\frac{\bar{q}^2}{\bar{\rho}^2} - \frac{\bar{E}}{\bar{\rho}}\right) - \frac{\bar{E}+\bar{p}}{\bar{\rho}}\right) & \bar{a}\frac{\bar{E}+\bar{p}}{\bar{\rho}} - \bar{a}\frac{\partial_e \bar{p}}{\bar{\rho}}\frac{\bar{q}^2}{\bar{\rho}^2} & -\bar{a}\frac{\bar{q}}{\bar{\rho}}\left(\frac{\partial_e \bar{p}}{\bar{\rho}} + 1\right) \end{bmatrix} \\
&= \bar{a}^2 \lambda_2(\bar{u})\left(c^2 + \lambda_2^2(\bar{u})\frac{\partial_e \bar{p}}{\bar{\rho}}\right),
\end{aligned}
$$

which is non zero if $\bar{u} \in A_0$ and if the fluid is perfect, i.e. (3.1.3) holds. $\square$

3.5.2 Computation of χ in (3.4.5)

The Case of Condition (S) Let Ψ be defined in (3.3.3) and set

$$\Sigma(a^-, a^+, u) = \int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right) \, a'(x) dx$$

where the functions $\mathcal{R}^a, \mathcal{E}^a$ have the same meaning as in (3.3.3). A perturbative method allows to compute the solution to (3.3.2) with a second order accuracy in $(\Delta a)/a$. Then, long elementary computations allow to get explicitly the terms H and G in (3.4.2) of the second order expansion of T:

$$\begin{array}{rcl} H(\rho, q, e) & = & \left[\begin{array}{l} -\dfrac{\vartheta^3 - 4\,\vartheta^2 + 5\,\vartheta - 2}{\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1}\,\rho \\ -q \\ -\dfrac{2\left(-\vartheta^3 + 2\,\vartheta^2 - \vartheta\right)}{3\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\,e \end{array} \right] \\ G(\rho, q, e) & = & \left[\begin{array}{l} -\dfrac{4\left(\vartheta^3 - 2\,\vartheta^2\right)}{3\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\,\rho \\ q \\ -\dfrac{70\,\vartheta^4 - 257\,\vartheta^3 + 342\,\vartheta^2 - 207\,\vartheta + 36}{18\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\,e \end{array} \right]. \end{array}$$

Moreover, the coefficients f_1, f_2 in (3.4.3) read

$$\begin{array}{rcl} f_1(\vartheta) & = & -\dfrac{-3\,\vartheta + (\vartheta - 3)\,\sqrt{\vartheta} - 3}{6\,\sqrt{\vartheta}\,(\vartheta - 1) - 6\,(\vartheta - 1)} \\ f_2(\vartheta) & = & \dfrac{\sqrt{\vartheta}\,\left(126\,\vartheta^4 - 505\,\vartheta^3 + 758\,\vartheta^2 - 489\,\vartheta + 270\right)}{72\left(\sqrt{\vartheta}\,\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right) - \vartheta^3 + 3\,\vartheta^2 - 3\,\vartheta + 1\right)} \\ & & +\dfrac{42\,\vartheta^4 - 183\,\vartheta^3 + 278\,\vartheta^2 + 33\,\vartheta + 54}{72\left(\sqrt{\vartheta}\,\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right) - \vartheta^3 + 3\,\vartheta^2 - 3\,\vartheta + 1\right)}. \end{array}$$

Next, χ is given by

$$\chi = \frac{\sqrt{\vartheta}\,\left(126\,\vartheta^4 - 506\,\vartheta^3 + 773\,\vartheta^2 - 480\,\vartheta + 279\right) + 42\,\vartheta^4 - 174\,\vartheta^3 + 311\,\vartheta^2 + 96\,\vartheta + 45}{36\left(\sqrt{\vartheta}\,\left(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right) - \vartheta^3 + 3\,\vartheta^2 - 3\,\vartheta + 1\right)}.$$

The Case of Condition (P) Let Ψ be defined in (3.3.4). With reference to (3.4.2), we show below explicitly the terms H and G in (3.4.2) of the second order expansion of T,

$$H(\rho,q,e) \;=\; \begin{bmatrix} \dfrac{8\left(-\vartheta^3+2\,\vartheta^2-\vartheta\right)}{3\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)}\,\rho \\ -q \\ -\dfrac{2\left(5\,\vartheta^4-7\,\vartheta^3-\vartheta^2+3\,\vartheta\right)}{9\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)}\,e \end{bmatrix}$$

$$G(\rho,q,e) \;=\; \begin{bmatrix} \dfrac{64\left(\vartheta^3+3\,\vartheta^2\right)}{27\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)}\,\rho \\ q \\ -\dfrac{565\,\vartheta^4-1599\,\vartheta^3+927\,\vartheta^2-405\,\vartheta}{81\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)}\,e \end{bmatrix}.$$

Moreover, the coefficients f_1, f_2 in (3.4.3) read

$$\begin{aligned} f_1(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(9\,\vartheta^2+2\,\vartheta-27\right)+3\,\vartheta^2-42\,\vartheta-9}{18\sqrt{\vartheta}\,(\vartheta-1)-18\,(\vartheta-1)} \\ f_2(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(154\,\vartheta^5+931\,\vartheta^4-4416\,\vartheta^3+6570\,\vartheta^2+990\,\vartheta+891\right)}{324\left(\sqrt{\vartheta}\,\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)} \\ &\quad+\frac{86\,\vartheta^5-311\,\vartheta^4-752\,\vartheta^3+7038\,\vartheta^2+1026\,\vartheta+81}{324\left(\sqrt{\vartheta}\,\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)}. \end{aligned}$$

Next, χ is given by

$$\begin{aligned} \chi &= \frac{\sqrt{\vartheta}\,\left(407\,\vartheta^5+1931\,\vartheta^4-7858\,\vartheta^3+14766\,\vartheta^2+1179\,\vartheta+1863\right)}{324\left(\sqrt{\vartheta}\,\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)} \\ &\quad+\frac{-23\,\vartheta^5+141\,\vartheta^4+2002\,\vartheta^3+15714\,\vartheta^2+2565\,\vartheta+81}{324\left(\sqrt{\vartheta}\,\left(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1\right)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)}. \end{aligned}$$

The Case of Condition (L) Let Ψ be defined in (3.3.5). Then,

$$H(\rho,q,e) = \begin{bmatrix} -\rho \\ -q \\ 0 \end{bmatrix} \qquad \text{and} \qquad G(\rho,q,e) = \begin{bmatrix} -\dfrac{4\,\vartheta}{3\,(\vartheta-1)}\,\rho \\ q \\ -\dfrac{35\,\vartheta^2-9\,(4\,\vartheta-1)}{9\,(\vartheta-1)}\,e \end{bmatrix}.$$

The coefficients f_1, f_2 in (3.4.3) read

$$\begin{aligned} f_1(\vartheta) &= 0 \\ f_2(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(63\,\vartheta^2-106\,\vartheta+27\right)+21\,\vartheta^2-78\,\vartheta+9}{36\left(\sqrt{\vartheta}\,(\vartheta-1)-\vartheta+1\right)}, \end{aligned}$$

so that χ is

$$\chi = \frac{\sqrt{\vartheta}\,\left(63\,\vartheta^2 - 106\,\vartheta + 27\right) + 21\,\vartheta^2 - 78\,\vartheta + 9}{18\left(\sqrt{\vartheta}\,(\vartheta - 1) - \vartheta + 1\right)}.$$

The Case of Condition (p) Let Ψ be defined in (3.3.6). With reference to (3.4.2),

$$\begin{aligned} H(\rho, q, e) &= \begin{bmatrix} -\dfrac{2\left(4\,\vartheta^3 + 12\,\vartheta^2 + 9\,\vartheta\right)}{4\left(2\,\vartheta^3 + 9\,\vartheta^2\right) + 27\,(2\,\vartheta + 1)}\,\rho \\ -q \\ \dfrac{2\left(4\,\vartheta^3 + 12\,\vartheta^2 + 9\,\vartheta\right)}{4\left(2\,\vartheta^3 + 9\,\vartheta^2\right) + 27\,(2\,\vartheta + 1)}\,e \end{bmatrix} \\ G(\rho, q, e) &= \begin{bmatrix} -\dfrac{4\left(\vartheta^3 + 3\,\vartheta^2\right)}{4\left(2\,\vartheta^3 + 9\,\vartheta^2\right) + 27\,(2\,\vartheta + 1)}\,\rho \\ q \\ \dfrac{12\left(\vartheta^3 + 2\,\vartheta^2\right)}{4\left(2\,\vartheta^3 + 9\,\vartheta^2\right) + 27\,(2\,\vartheta + 1)}\,e \end{bmatrix}, \end{aligned}$$

with f_1 and f_2 given by

$$\begin{aligned} f_1(\vartheta) &= \frac{-2\,\vartheta^2 + 4\,\vartheta^{\frac{3}{2}} + 3\,\vartheta - 9}{2\left(4\,\vartheta^2 + 12\,\vartheta + 9\right)} \\ f_2(\vartheta) &= \frac{32\,\vartheta^4 + 8\,\sqrt{\vartheta}\,\left(4\,\vartheta^3 + 9\,\vartheta^2 - 9\,\vartheta\right) + 316\,\vartheta^3 + 558\,\vartheta^2 + 216\,\vartheta + 81}{6\left(16\,\vartheta^4 + 96\,\vartheta^3 + 216\,\vartheta^2 + 216\,\vartheta + 81\right)}, \end{aligned}$$

so that

$$\chi = \frac{60\,\vartheta^4 + 96\,\sqrt{\vartheta}\,\left(\vartheta^3 + \vartheta^2 - 3\,\vartheta\right) + 700\,\vartheta^3 + 1107\,\vartheta^2 - 54\,\vartheta + 81}{6\left(16\,\vartheta^4 + 96\,\vartheta^3 + 216\,\vartheta^2 + 216\,\vartheta + 81\right)}.$$

$$
\begin{aligned}
\vartheta^{+} \;&=\; \vartheta - \frac{\sqrt{\vartheta}\,\left(\left(\sigma_3^- + 6\right)\,\vartheta^2 + 18\,\vartheta - 9\,\sigma_3^-\right) + 2\left(3 - 2\,\sigma_3^-\right)\,\vartheta^2 + 6\left(2\,\sigma_3^- + 3\right)\,\vartheta}{9\sqrt{\vartheta}\,(\vartheta - 1) + 9\,(\vartheta - 1)}\,\frac{\Delta a}{a} \\
&\;-\; \frac{\sqrt{\vartheta}\,\left(14\left(11\,\sigma_3^- - 30\right)\,\vartheta^5 + \left(990 - 301\,\sigma_3^-\right)\,\vartheta^4 + 3\left(25\,\sigma_3^- - 236\right)\,\vartheta^3 + \left(111\,\sigma_3^- - 18\right)\,\vartheta^2 + 45\left(12 - \sigma_3^-\right)\,\vartheta - 378\,\sigma_3^-\right)}{108\left(\sqrt{\vartheta}\,(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1) + \vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\left(\frac{\Delta a}{a}\right)^2 \\
&\;-\; \frac{4\left(217\,\sigma_3^- - 105\right)\,\vartheta^5 + 2\left(495 - 1489\,\sigma_3^-\right)\,\vartheta^4 + 4\left(928\,\sigma_3^- - 177\right)\,\vartheta^3 - 2\left(1077\,\sigma_3^- - 9\right)\,\vartheta^2 + 24\left(26\,\sigma_3^- + 15\right)\,\vartheta}{108\left(\sqrt{\vartheta}\,(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1) + \vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\left(\frac{\Delta a}{a}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\vartheta^{+} \;&=\; \vartheta - \frac{\sqrt{\vartheta}\,\left(\left(11\,\sigma_3^- - 30\right)\,\vartheta^3 + \left(-19\,\sigma_3^- - 108\right)\,\vartheta^2 + 9\left(5\,\sigma_3^- - 6\right)\,\vartheta + 27\,\sigma_3^-\right) + 2\left(31\,\sigma_3^- - 15\right)\,\vartheta^3 - 36\left(\sigma_3^- + 3\right)\,\vartheta^2 - 18\left(5\,\sigma_3^- + 3\right)\,\vartheta}{27\sqrt{\vartheta}\,(\vartheta - 1) + 27\,(\vartheta - 1)}\,\frac{\Delta a}{a} \\
&\;-\; \frac{\sqrt{\vartheta}\,\left(2\left(233\,\sigma_3^- - 300\right)\,\vartheta^6 + \left(1279\,\sigma_3^- - 5310\right)\,\vartheta^5 + \left(5400 - 2543\,\sigma_3^-\right)\,\vartheta^4 + 6\left(677\,\sigma_3^- + 1242\right)\,\vartheta^3 + 36\left(109 - 297\,\sigma_3^-\right)\,\vartheta^2 + 729\left(2 - 5\,\sigma_3^-\right)\,\vartheta - 1215\,\sigma_3^-\right)}{486\left(\sqrt{\vartheta}\,(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1) + \vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\left(\frac{\Delta a}{a}\right)^2 \\
&\;-\; \frac{4\left(601\,\sigma_3^- - 150\right)\,\vartheta^6 + 2\left(3521\,\sigma_3^- - 2655\right)\,\vartheta^5 + 8\left(225 - 3814\,\sigma_3^-\right)\,\vartheta^4 + 36\left(655\,\sigma_3^- + 207\right)\,\vartheta^3 + 108\left(53\,\sigma_3^- + 36\right)\,\vartheta^2 + 162\left(9\,\sigma_3^- + 25\right)\,\vartheta}{486\left(\sqrt{\vartheta}\,(\vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1) + \vartheta^3 - 3\,\vartheta^2 + 3\,\vartheta - 1\right)}\left(\frac{\Delta a}{a}\right)^2
\end{aligned}
$$

$$
\vartheta^{+} \;=\; \vartheta - \frac{\sqrt{\vartheta}\,\left(\left(77\,\sigma_3^- - 210\right)\,\vartheta^3 + \left(-13\,\sigma_3^- - 36\right)\,\vartheta^2 + 27\left(\sigma_3^- + 2\right)\,\vartheta - 27\,\sigma_3^-\right)2 + \left(217\,\sigma_3^- - 105\right)\,\vartheta^3 - 4\left(147\,\sigma_3^- + 9\right)\,\vartheta^2 + 18\left(5\,\sigma_3^- + 3\right)\,\vartheta}{54\sqrt{\vartheta}\,(\vartheta - 1) + 54\,(\vartheta - 1)}\left(\frac{\Delta a}{a}\right)^2
$$

$$
\begin{aligned}
\vartheta^{+} \;&=\; \vartheta + \frac{-2\left(\sigma_3^- + 6\right)\,\vartheta^3 + \sqrt{\vartheta}\,\left(10\,\sigma_3^-\,\vartheta^2 + 27\,\sigma_3^-\,\vartheta + 27\,\sigma_3^-\right) + 3\left(\sigma_3^- - 18\right)\,\vartheta^2 - 9\left(\sigma_3^- + 6\right)\,\vartheta}{3\,(4\,\vartheta^2 + 12\,\vartheta + 9)}\,\frac{\Delta a}{a} \\
&\;-\; \frac{-48\left(\sigma_3^- + 5\right)\,\vartheta^5 + \sqrt{\vartheta}\,\left(144\,\sigma_3^-\,\vartheta^4 + 780\,\sigma_3^-\,\vartheta^3 + 2214\,\sigma_3^-\,\vartheta^2 + 1944\,\sigma_3^-\,\vartheta + 1215\,\sigma_3^-\right)}{48\,(2\,\vartheta^4 + 12\,\vartheta^3 + 27\,\vartheta^2 + 27\,\vartheta + 12)}\left(\frac{\Delta a}{a}\right)^2 \\
&\;-\; \frac{-4\left(91\,\sigma_3^- + 342\right)\,\vartheta^4 - 54\left(11\,\sigma_3^- + 56\right)\,\vartheta^3 - 216\left(2\,\sigma_3^- + 15\right)\,\vartheta^2 - 81\left(5\,\sigma_3^- + 18\right)\,\vartheta}{48\,(2\,\vartheta^4 + 12\,\vartheta^3 + 27\,\vartheta^2 + 27\,\vartheta + 12)}\left(\frac{\Delta a}{a}\right)^2
\end{aligned}
$$

Above are the values of ϑ^+ in the cases **(S)**, **(P)**, **(L)** and **(p)**.

Chapter 4

A 2–Phase Traffic Model Based on a Speed Bound

4.1 Introduction

We present here a new macroscopic traffic model displaying 2 phases, based on a non-smooth 2×2 system of conservation laws. We extend the classical LWR traffic model allowing different maximal speeds to different vehicles. Then, we add a uniform bound on the traffic speed.

Several observations of traffic flow result in underlining two different behaviors, sometimes called *phases*, see [18, 36, 39, 53]. At low density and high speed, the flow appears to be reasonably described by a function of the (mean) traffic density. On the contrary, at high density and low speed, flow is not a single valued function of the density.

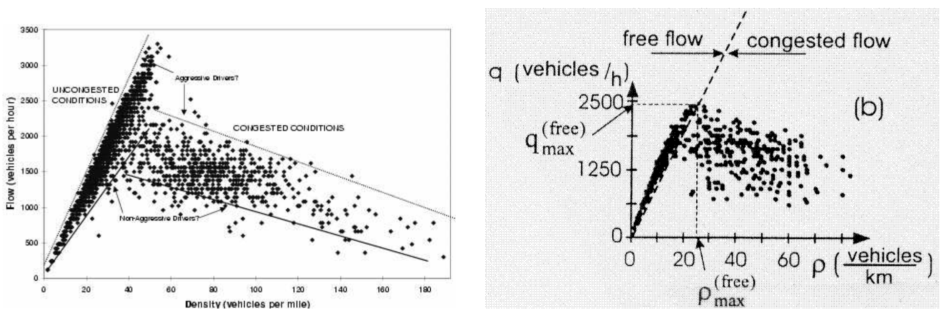


Figure 4.1: Experimental fundamental diagrams. Left, [55, Figure 1] and, right, [53, Figure 1], (see also [48]).

Here we present a model providing an explanation to this phenomenon, its two key features being:

1. At a given density, different drivers may choose different velocities;

2. There exists a uniform bound on the speed.

By *"bound"*, here we do not necessarily mean an official speed limit. On the contrary, we assume that different drivers may have different speeds at the same traffic density. Nevertheless, there exists a speed $V_{\max}$ that no driver exceeds. As a result from this postulate, we obtain a fundamental diagram very similar to those usually observed, see Figure 4.1 and Figure 4.2, left. Besides, the evolution prescribed by the model so obtained is reasonable and coherent with that of other traffic models in the literature. In particular, we verify that the minimal requirements stated in [6, 34] are satisfied.

Recall the classical Lighthill-Whitham [62] and Richards [67] (LWR) model

$$\partial_t \rho + \partial_x (\rho V) = 0 \tag{4.1.1}$$

for the traffic density ρ. Assume that the speed V is not the same for all drivers. More precisely, different drivers differ in their *maximal* speed w, so that $V = w\,\psi(\rho)$, with $w \in [\check{w}, \hat{w}]\,, \check{w} > 0$, being transported along the road at the mean traffic speed V. We identify the different behaviors of the different drivers by means of their maximal speed, see also [12, 13]. One is thus lead to study the equations

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{array}\right. \qquad \text{with} \qquad v = w\,\psi(\rho)\,. \tag{4.1.2}$$

Here, the role of the second equation is to let the maximal velocity w be propagated with the traffic speed. Indeed, w is a specific feature of every single driver, in other words is a Lagrangian marker. Therefore this model falls into the class of models introduced in [6], and later on extended in [60], see also [8, formula (1.2)].

Introducing a uniform bound $V_{\max}$ on the speed, we obtain the model

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{array}\right. \qquad \text{with} \qquad v = \min\left\{V_{\max},\, w\,\psi(\rho)\right\}\,. \tag{4.1.3}$$

We choose to reformulate the above quasilinear system in conservation form, similarly to [54, formula (1)], [7, formula (2.2)], [60, formula (1)], see also [71], as follows:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \big(\rho\, v(\rho,\eta)\big) = 0 \\ \partial_t \eta + \partial_x \big(\eta\, v(\rho,\eta)\big) = 0 \end{array}\right. \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\} \tag{4.1.4}$$

see the Remark 4.5.3 for further comments on this choice. This model consists of a 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ but not $\mathbf{C^1}$ flow. Note in fact that $\eta/\rho = w \in [\check{w}, \hat{w}]$. A 2×2 system of conservation laws with a flow having a similar $\mathbf{C^{0,1}}$ regularity is presented in [46] and studied in [3].

From the traffic point of view, we remark that, under mild reasonable assumptions on the function ψ, the flow in (4.1.4) may vanish if and only if $\rho = 0$, i.e the road is empty, or $\rho = R$, i.e. the road is fully congested. It is also worth noting the agreement between experimental fundamental diagrams often found in the literature and the one related to (4.1.4), see Figure 4.2, left.

From the analytical point of view, we can extend the present treatment to the more general case of a maximal speed $V_{\max}$ that depends on ρ, i.e. $V_{\max} = V_{\max}(\rho)$. However, we prefer to highlight the main features of the model (4.1.4) in its simplest analytical framework.

As we already said, the model studied here, inspired from [18], falls into the class of "Aw-Rascle" models. So we could use the approach and the theoretical results of [5], which should apply here with minor modifications.

However, our approach is different: here, contrarily to the above reference, we establish *directly* a connection between the Follow-The-Leader model in Section 4.4 and the macroscopic system (4.1.4), *without* viewing both systems as issued from a same fully discrete system (Godunov scheme) with different limits, and *without* passing in Lagrangian coordinates. For related works, including vacuum, see also [4, 41, 42].

This chapter is arranged in the following way: in the next section we study the Riemann Problem for (4.1.4) and present the qualitative properties of this model from the point of view of traffic. In Section 4.3 we compare the present model with others in the current literature and in Section 4.4 we establish the connection with a microscopic Follow-The-Leader model based on ordinary differential equations. We also show *rigorously* that the macroscopic model (4.1.4) can be viewed as the limit of the microscopic model as the number of vehicles increases to infinity. All proofs are gathered in the last section.

4.2 Notation and Main Results

We assume throughout the following hypotheses:

a. $R, \check{w}, \hat{w}, V_{\max}$ are positive constants, with $\check{w} < \hat{w}$.

b. $\psi \in \mathbf{C^2}\left([0,R];[0,1]\right)$ is such that

$$\begin{array}{rclcrcll} \psi(0) & = & 1, & & \psi(R) & = & 0, & \\ \psi'(\rho) & \leq & 0, & & \dfrac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right) & \leq & 0 & \text{for all } \rho \in [0,R]\,. \end{array}$$

c. $\check{w} > V_{\max}$.

Here, R is the maximal possible density, typically $R = 1$ if ρ is normalized as in Section 4.4; $\check{w}$, respectively $\hat{w}$, is the minimum, respectively maximum, of the maximal speeds of each vehicle; $V_{\max}$ is the overall uniform upper bound on the traffic speed. At **b.**, the first three assumptions on Ψ are the classical conditions usually assumed on speed laws, while the fourth one is technically necessary in the proof of Theorem 4.2.1. The latter condition means that all drivers do feel the presence of the speed limit.

Moreover, we introduce the notation

$$F = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = V_{\max}\right\} \tag{4.2.1}$$
$$C = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = w\, \psi(\rho)\right\} \tag{4.2.2}$$

to denote the *Free* and the *Congested* phases. Note that F and C are closed sets and $F \cap C \neq \emptyset$. Note also that F is 1–dimensional in the $(\rho, \rho v)$ plane of

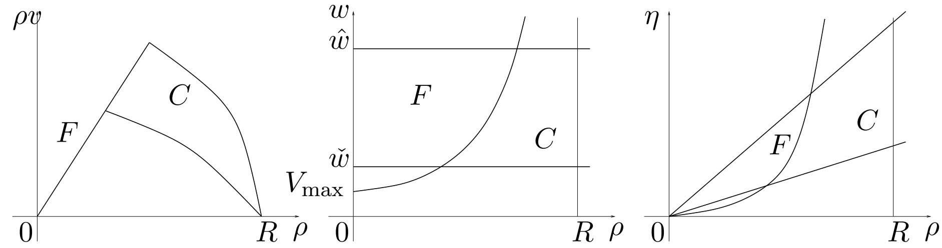


Figure 4.2: The phases F and C in the coordinates, from left to right, $(\rho, \rho v)$, (ρ, w) and (ρ, η).

the fundamental diagram, while it is 2–dimensional in the (ρ, w) and (ρ, η) coordinates, see Figure 4.2. See also Figure 4.3 to have a vision in three dimensions.

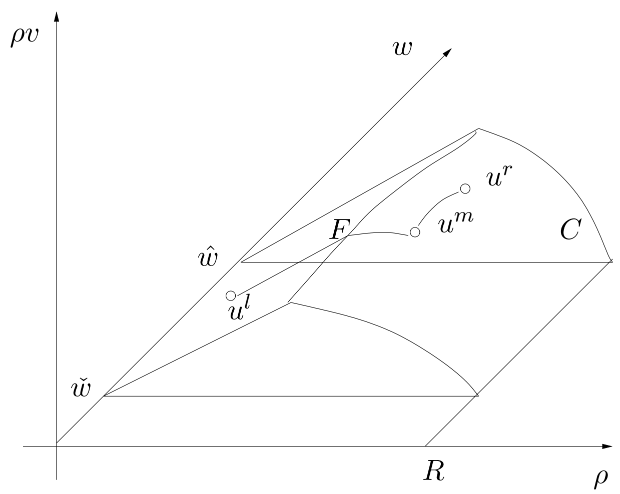


Figure 4.3: The phases F and C in the coordinates $(\rho, \rho v, w)$. Note that F is contained in a plane. This figure shows an example of Riemann Problem when $u^l = (\rho^l, \rho^l v^l, w^l) \in F$ and $u^r = (\rho^r, \rho^r v^r, w^r) \in C$.

Let ρ_* be the maximum of the points of maximum of the flow, i.e. $\rho_* = \max\left\{\rho \in [0,R] \colon \rho\,\psi(\rho) = \max_{r\in[0,R]} r\,\psi(r)\right\}$. Then, the condition

$$\hat{w}\psi\left(\rho_*\right) \geq V_{\max} \tag{4.2.3}$$

is a further reasonable assumption. Indeed, it means that the maximum flow is attained in the free phase, coherently with the *capacity drop* phenomenon, see for instance [47]. However, (4.2.3) is not necessary in the following results.

Our next goal is to study the Riemann Problem for (4.1.4).

Theorem 4.2.1 *Under the assumptions* ***a.***, ***b.*** *and* ***c.***, *for all states* (ρ^l,η^l), $(\rho^r,\eta^r) \in F \cup C$, *the Riemann problem consisting of (4.1.4) with initial data*

$$\rho(0,x) = \left\{ \begin{array}{lcl} \rho^l & \text{if } x & < & 0 \\ \rho^r & \text{if } x & > & 0 \end{array} \right. \qquad \eta(0,x) = \left\{ \begin{array}{lcl} \eta^l & \text{if } x & < & 0 \\ \eta^r & \text{if } x & > & 0 \end{array} \right. \tag{4.2.4}$$

admits a unique self similar weak solution $(\rho,\eta) = (\rho,\eta)(t,x)$ *constructed as follows:*

(1) If $(\rho^l,\eta^l), (\rho^r,\eta^r) \in F$, *then*

$$(\rho,\eta)(t,x) = \left\{ \begin{array}{llcl} (\rho^l,\eta^l) & \text{if} & x < & V_{\max}t \\ (\rho^r,\eta^r) & \text{if} & x > & V_{\max}t\,. \end{array} \right. \tag{4.2.5}$$

(2) If $(\rho^l,\eta^l), (\rho^r,\eta^r) \in C$, *then* (ρ,η) *consists of a* 1*–Lax wave (shock or rarefaction) between* (ρ^l,η^l) *and* (ρ^m,η^m), *followed by a* 2*–contact discontinuity between* (ρ^m,η^m) *and* (ρ^r,η^r). *The middle state* (ρ^m,η^m) *is in* C *and is uniquely characterized by the two conditions* $\eta^m/\rho^m = \eta^l/\rho^l$ *and* $v(\rho^m,\eta^m) = v(\rho^r,\eta^r)$.

(3) If $(\rho^l,\eta^l) \in C$ *and* $(\rho^r,\eta^r) \in F$, *then the solution* (ρ,η) *consists of a rarefaction wave separating* (ρ^r,η^r) *from a state* (ρ^m,η^m) *and by a linear wave separating* (ρ^m,η^m) *from* (ρ^l,η^l). *The middle state* (ρ^m,η^m) *is in* $F \cap C$ *and is uniquely characterized by the two conditions* $\eta^m/\rho^m = \eta^r/\rho^r$ *and* $v(\rho^m,\eta^m) = V$.

(4) If $(\rho^l,\eta^l) \in F$ *and* $(\rho^r,\eta^r) \in C$, *then* (ρ,η) *consists of a shock between* (ρ^l,η^l) *and* (ρ^m,η^m), *followed by a contact discontinuity between* (ρ^m,η^m) *and* (ρ^r,η^r). *The middle state* (ρ^m,η^m) *is in* C *and is uniquely characterized by the two conditions* $\eta^m/\rho^m = \eta^l/\rho^l$ *and* $v(\rho^m,\eta^m) = v(\rho^r,\eta^r)$.

(If $\frac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right)$ vanishes, then the words *"shock"* and *"rarefaction"* above have to be understood as *"contact discontinuities").*

We now pass from the solution to single Riemann problems to the properties of the *Riemann Solver*, i.e. of the map $\mathcal{R}\colon (F \cup C)^2 \to \mathbf{BV}(\mathbb{R}; C \cup F)$ such that $\mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right)$ is the solution to (4.1.4)–(4.2.4) computed at time, say, $t = 1$.

To this aim, recall the following definition, see [18]:

Definition 4.2.2 *A Riemann Solver $\mathcal{R}$ is* consistent *if the following two conditions hold for all (ρ^l,η^l), (ρ^m,η^m), $(\rho^r,\eta^r) \in F \cup C$, and $\bar{x} \in \mathbb{R}$:*

(C1) *If $\mathcal{R}\left((\rho^l,\eta^l),(\rho^m,\eta^m)\right)(\bar{x}) = (\rho^m,\eta^m)$ and $\mathcal{R}\left((\rho^m,\eta^m),(\rho^r,\eta^r)\right)(\bar{x}) = (\rho^m,\eta^m)$, then*

$$\mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right) = \begin{cases} \mathcal{R}\left((\rho^l,\eta^l),(\rho^m,\eta^m)\right) & \text{,if } x < \bar{x}\,, \\ \mathcal{R}\left((\rho^m,\eta^m),(\rho^r,\eta^r)\right) & \text{,if } x \geq \bar{x}\,, \end{cases}$$

(C2) *If $\mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right)(\bar{x}) = (\rho^m,\eta^m)$, then*

$$\mathcal{R}\left((\rho^l,\eta^l),(\rho^m,\eta^m)\right) = \begin{cases} \mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right)\,, & \text{if } x \leq \bar{x}\,, \\ (\rho^m,\eta^m)\,, & \text{if } x > \bar{x}\,, \end{cases}$$

$$\mathcal{R}\left((\rho^m,\eta^m),(\rho^r,\eta^r)\right) = \begin{cases} (\rho^m,\eta^m)\,, & \text{if } x < \bar{x}\,, \\ \mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right)\,, & \text{if } x \geq \bar{x}\,. \end{cases}$$

Essentially, **(C1)** states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem. Condition **(C2)** is the vice-versa.

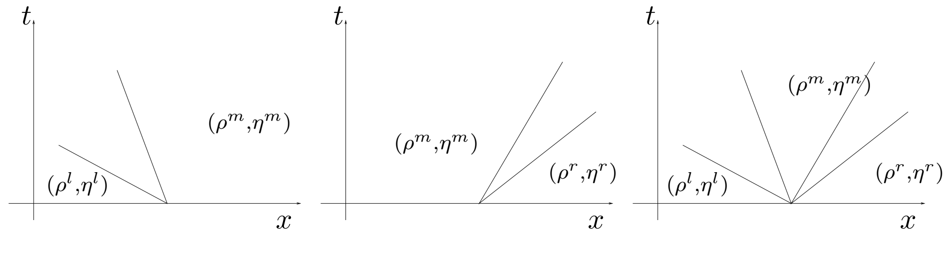


Figure 4.4: The conditions **(C1)** and **(C2)**.

The next result characterizes the Riemann Solver defined above.

Proposition 4.2.3 *Let the assumptions **a.**, **b.** and **c.** hold. The Riemann Solver $\mathcal{R}$ defined in Theorem 4.2.1 enjoys the following three conditions*

1. *It is consistent in the sense of Definition 4.2.2.*

2. *If* $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$, *then* $\mathcal{R}\left((\rho^l, \eta^l), (\rho^r, \eta^r)\right)$ *is the standard solution to the linear system*

$$\begin{cases} \partial_t \rho + \partial_x (\rho V_{\max}) = 0 \\ \partial_t \eta + \partial_x (\eta V_{\max}) = 0, \end{cases} \tag{4.2.6}$$

3. *If* $(\rho^l, \eta^l) \in F \cup C$ *and* $(\rho^r, \eta^r) \in C$, *then* $\mathcal{R}\left((\rho^l, \eta^l), (\rho^r, \eta^r)\right)$ *is the standard Lax solution to*

$$\begin{cases} \partial_t \rho + \partial_x \big(\eta\, \psi(\rho)\big) = 0 \\ \partial_t \eta + \partial_x \left(\frac{\eta^2}{\rho}\, \psi(\rho)\right) = 0\,. \end{cases} \tag{4.2.7}$$

Moreover, the conditions ***(C1)****, 2. and 3. uniquely characterize the Riemann Solver* $\mathcal{R}$.

The above properties are of use, for instance, in using model (4.1.4) on traffic networks, according to the techniques described in [37].

The next result presents the relevant qualitative properties of the Riemann Solver defined in Theorem 4.2.1 from the point of view of traffic.

Proposition 4.2.4 *Let the assumptions* ***a.****,* ***b.*** *and* ***c.*** *hold. Then, the Riemann Solver* $\mathcal{R}$ *enjoys the following properties:*

1. *If the initial datum attains values in* F, C, *or* $F \cup C$ *then, respectively, the solution attains values in* F, C, *or* $F \cup C$.

2. *Traffic density and speed are uniformly bounded.*

3. *Traffic speed vanishes if and only if traffic density is maximal.*

4. *No wave in the solution to (4.1.4)–(4.2.4) may travel faster than traffic speed, i.e. information is carried by vehicles.*

4.3 Comparison with Other Macroscopic Models

This section is devoted to compare the present model (4.1.4) with a sample of models from the literature. In particular, we consider differences in the number of free parameters and functions, in the fundamental diagram and in the qualitative structures of the solutions. Recall that the evolution described by model (4.1.4) and the corresponding invariant domain depends on the function ψ and on the parameters $V_{\max}$, R, $\check{w}$ and $\hat{w}$. The fundamental diagram of (4.1.4) is in Figure 4.2, left.

4.3.1 The LWR Model

In the LWR model (4.1.1), a suitable speed law has to be selected, analogous to the choice of ψ in (4.1.4). Besides, in (4.1.4) we also have to set $V_{\max}$, R and the two geometric positive parameters $\check{w}$ and $\hat{w}$.

The fundamental diagram of (4.1.4) seems to better agree with experimental data than that of (4.1.1), shown in Figure 4.5, left. Indeed, compare

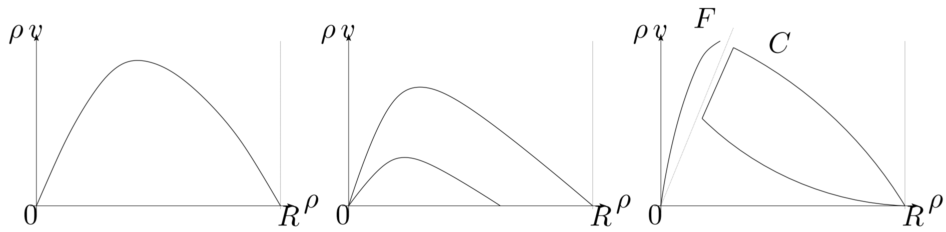


Figure 4.5: Fundamental diagrams, from left to right, of the (LWR) model (4.1.1), of the (AR) model (4.3.1) and of the 2-phase model (4.3.2).

Figure 4.2, left with the measurements in Figure 4.1.

As long as the data are in F, the solutions to (4.1.4) are essentially the same as those of (4.1.1). In the congested phase, the solutions to (4.1.4) obviously present a richer structure, for they generically contain 2 waves instead of 1. In particular, the (LWR) model (4.1.1) may not describe the *"homogeneous-in-speed"* solutions, i.e. a type of synchronized flow, see [53, Section 2.2] and [48, 73], which is described by the 2-waves in (4.1.4).

Finally, note that if in (4.1.4) the two geometric parameters $\check{w}$ and $\hat{w}$ coincide, then we recover the LWR (4.1.1) model with $V(\rho) = \min\{V_{\max}, \hat{w}\,\psi(\rho)\}$.

4.3.2 The Aw-Rascle Model

Consider now the Aw–Rascle (AR) model

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho\, v(\rho, y)\right] = 0 \\ \partial_t y + \partial_x \left[y\, v(\rho, y)\right] = 0 \end{cases} \qquad v(\rho, y) = \frac{y}{\rho} - p(\rho) \tag{4.3.1}$$

introduced in [6] and successively refined in several papers, see for instance [5, 8, 39, 42, 43, 44, 49, 66, 69] and the references therein.

Note that w in (4.1.4) plays a role analogous to that of $v+p\,(\rho)$ in (4.3.1).

In the (AR) model, R and the *"pressure"* function need to be selected, similarly to R and ψ in (4.1.4). No other parameter appears in (4.3.1), but the definition of an invariant domain requires two parameters, with a role similar to that of $\check{w}$ and $\hat{w}$. Indeed, an invariant domain for (4.3.1) is

$$\Big\{(\rho, y)\colon \rho \in [0, R] \text{ and } y \in \Big[\rho\left(v_- + p(\rho)\right), \rho\left(v_- + p(\rho)\right)\Big]\Big\}$$

see Figure 4.5, center, and depends on the speeds v_- and v_+. More recent versions of (4.3.1) contain also a suitable relaxation source term in the right hand side of the second equation; in this case one more arbitrary function needs to be selected. The original (AR) model does not distinguish between a free and a congested phase. However, it was extended to describe two different phases in [39]. Further comments on (4.3.1) are found in [59].

Concerning the analytical properties of the solutions, the Riemann solver for the (AR) model suffers lack of continuous dependence at vacuum, see [6, Section 4]. However, existence of solutions attaining also the vacuum state was proved in [42], while the 2-phase construction in [39] also displays continuous dependence.

A qualitative difference between the (AR) model and the present one is property 3. in Proposition 4.2.4. Indeed, solutions to (4.3.1) may well have zero speed while being at a density strictly lower than the maximal one.

4.3.3 The Hyperbolic 2-Phase Model

Recall the model presented in [18], with a notation similar to the present one:

$$\begin{array}{ll}
\text{Free flow: } (\rho, q) \in F, & \text{Congested flow: } (\rho, q) \in C, \\
\partial_t \rho + \partial_x \left[\rho \cdot v_F(\rho)\right] = 0, & \left\{\begin{array}{l} \partial_t \rho + \partial_x \left[\rho \cdot v_C(\rho, q)\right] = 0 \\ \partial_t q + \partial_x \left[(q - q_*) \cdot v_C(\rho, q)\right] = 0 \end{array}\right. \\
v_F(\rho) = \left(1 - \frac{\rho}{R}\right) \cdot V & v_C(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}
\end{array} \tag{4.3.2}$$

the phases being defined as

$$\begin{array}{rcl}
F & = & \{(\rho, q) \in [0, R] \times \mathbb{R}^+ \colon v_f(\rho) \geq V_f,\ q = \rho \cdot V\}, \\
C & = & \left\{(\rho, q) \in [0, R] \times \mathbb{R}^+ \colon v_c(\rho.q) \leq V_c,\ \frac{q - q_*}{\rho} \in \left[\frac{Q_1 - q_*}{R}, \frac{Q_2 - q_*}{R}\right]\right\}.
\end{array}$$

In (4.3.2) no function can be selected, on the other hand the evolution depends on the parameters V, R and q_* while the invariant domains F and C depend on V_f, V_c, Q_1 and Q_2. A geometric construction of the solutions to (4.3.2) in the congested phase is in [61].

The main difference between fundamental diagrams of (4.3.2), see Figure 4.5, right, and that of (4.1.4) is that (4.3.2) requires the two phases to be *disconnected*: there is a *gap* between the free and the congested phase. This restriction is necessary for the well posedness of the Riemann problem for (4.3.2) and can be hardly justified on the basis of experimental data. More recently, the global well-posedness of the model (4.3.2) was proved in [23].

Note that in both models, as well as in that presented in [39], the free phase is one dimensional, while the congested phase is bidimensional.

The model (4.3.2) allows for the description of *wide jams*, i.e. of persistent waves in the congested phase moving at a speed *different* from that of traffic. Here, as long as $\frac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right) < 0$, persistent phenomena can be described only through waves of the second family, which move at the mean traffic speed. We refer to [59] for further discussions on (4.3.2) and comparisons with other macroscopic models.

4.3.4 A Kinetic Model

Recall, with a notation adapted to the present case, the kinetic model introduced in [13, Section 1]:

$$\partial_t r(t,x;w) + \partial_x \left[w\, r(t,x;w)\, \psi\left(\int_{\check{w}}^{\hat{w}} r(t,x;w')\, dw' \right) \right] = 0\,. \tag{4.3.3}$$

The function ψ and the speed w play the same role as here. The unknown $r = r(t,x;w)$ is the probability density of vehicles having maximal speed w that at time t are at point x.

In (4.3.3) there is one function to be specified, ψ, as in (4.1.4). The parameters are R (which is normalized to 1 in [13]), $\check{w}$ and $\hat{w}$, similarly to (4.1.4). Since no limit speed is there defined, no parameter in (4.3.3) has the same role as here $V_{\max}$.

Being of a kinetic nature, there is no real equivalent to a fundamental diagram for (4.3.3).

From the analytical point of view, the existence of solutions to (4.3.3) has not been proved, yet. The main result in [13] only states that (4.3.3) can be rigorously obtained as the limit of systems of $n \times n$ conservation laws describing n populations of vehicles, each characterized by their maximal speed.

Let the measure r solve (4.3.3) and be such that for suitable functions ρ and w

$$r(t,x;\cdot) = \rho(t,x)\, \delta_{w(t,x)} \tag{4.3.4}$$

where δ is the usual Dirac measure. Then, formally, (ρ, w) solves (4.1.4). Indeed, for the first equation simply substitute (4.3.4) in (4.3.3) and integrate; for the second equation substitute (4.3.4) in (4.3.3), multiply by w and integrate over $[\check{w}, \hat{w}]$.

Remark that (4.3.4) suggests a further interpretation of the quantity ρ in (4.1.4). Indeed, in the present model, at (t,x) vehicles of only one species are present, namely those with maximal speed $w(t,x)$.

4.4 Connections with a Follow-The-Leader Model

Within the framework of (4.1.3), a single driver starting from $\tilde{p}$ at time $t = 0$ follows the *particle path* $p = p(t)$ that solves the Cauchy problem

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left(\left(t, p(t)\right)\right)\right) \\ p(0) = \tilde{p} \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\, w\,\psi(\rho)\right\}, \tag{4.4.1}$$

refer to [30] for the well posedness of the particle path for the LWR model (see also [5]). Recall now that w is a specific feature of every single driver, i.e. $w\left(t, p(t)\right) = w(0, \tilde{p})$ for all $\tilde{p}$. On the other hand, from a microscopic point of view, if n drivers are distributed along the road, then ρ is approximated by $l/(p_{i+1} - p_i)$, where l is a standard length of a car.

We fix $L > 0$ and assume that n drivers are distributed along $[-L, L]$. Then, the natural microscopic counterpart to (4.1.4) is therefore the *Follow-The-Leader* (FTL) model defined by the Cauchy problem

$$\begin{cases} \dot{p}_i = v\left(\frac{l}{p_{i+1} - p_i}, w_i\right) & i = 1, \ldots, n \\ \dot{p}_{n+1} = V_{\max} & \\ p_i(0) = \tilde{p}_i & i = 1, \ldots, n+1 \end{cases} \tag{4.4.2}$$

where $\tilde{p}_1 = -L$ and $\tilde{p}_{n+1} = L - l$. Proposition 4.4.1 shows that (4.4.2) admits a unique global solution defined for every $t \geq 0$ and such that $p_{i+1} - p_i \geq l$ for all $t \geq 0$.

Proposition 4.4.1 *Let* **a.**, **b.** *and* **c.** *hold. Fix $L > 0$. For any $n \in \mathbb{N}$, with $n \geq 2$, choose initial data $\tilde{p}_i^n$ for $i = 1, \ldots, n$ satisfying $\tilde{p}_{i+1}^n - \tilde{p}_i^n \geq l$. Then, the Cauchy problem (4.4.2) admits a unique solution $p_i^n = p_i^n(t)$, for $i = 1, \ldots, n+1$, defined for all $t \geq 0$ and satisfying $p_{i+1}^n(t) - p_i^n(t) \geq l$ for all $t \geq 0$ and for $i = 1, \ldots, n$.*

The proof is postponed to Section 4.5.

Our next aim is to rigorously show that in the limit $n \to +\infty$ with $n\,l = \text{constant} > 0$, the microscopic model in (4.4.2) yields the macroscopic one in (4.1.4). Given the position p^i of every single vehicle and its maximal speed w_i, for $i = 1, \ldots, n+1$, the macroscopic variables ρ, w are given by

$$\rho(x) = \sum_{i=1}^{n} \frac{l}{p_{i+1}^n - p_i^n}\, \chi_{[p_i^n, p_{i+1}^n[}(x) \quad \text{and} \quad w(x) = \sum_{i=1}^{n} w_i^n\, \chi_{[p_i^n, p_{i+1}^n[}(x)\,.$$

Note that necessarily $p_{i+1}^n - p_i^n \geq l$.

On the contrary, given $(\rho, w) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0,1] \times [\check{w}, \hat{w}])$, with $\operatorname{supp} \rho$, $\operatorname{supp} w \subseteq [-L, L]$, we reconstruct a microscopic description defining $l =$

$\left(\int_{\mathbb{R}} \rho(x)\,dx\right)/n$ and

$$\begin{aligned} p_{n+1}^n &= L - l \\ p_i^n &= \max\left\{p \in [-L, L]\colon \int_p^{p_{i+1}} \rho(x)\,dx = l\right\} \qquad \text{for } i = 1, \ldots, n \\ w_i^n &= w(p_i^n+) \quad \text{for } i = 1, \ldots, n+1\,. \end{aligned}$$

Note that $\int_{\mathbb{R}} \rho(x)\,dx = nl > 0$. Now we are able to rigorously show that, as the number of vehicles increases to infinity, the microscopic model in (4.4.2) yields the macroscopic one in (4.1.4).

Proposition 4.4.2 *Let **a.**, **b.** and **c.** hold. Fix $T > 0$. Choose $(\tilde\rho, \tilde w) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0,1] \times [\check w, \hat w])$ with $\operatorname{supp}\tilde\rho$, $\operatorname{supp}\tilde w \subseteq [-L, L]$. Construct the initial data for the microscopic model setting $l = \left(\int_{\mathbb{R}} \tilde\rho(x)\,dx\right)/n$ and*

$$\begin{aligned} \tilde p_{n+1}^n &= L - l \\ \tilde p_i^n &= \max\left\{p \in [-L, L]\colon \int_p^{\tilde p_{i+1}} \tilde\rho(x)\,dx = l\right\} \qquad \textit{for } i = 1, \ldots, n \\ \tilde w_i^n &= \tilde w(p_i^n+) \quad \textit{for } i = 1, \ldots, n+1\,. \end{aligned}$$

Let $p_i^n(t)$, for $i = 1, \ldots, n$, be the corresponding solution to (4.4.2). Define

$$\rho^n(t,x) = \sum_{i=1}^n \frac{l}{p_{i+1}^n(t) - p_i^n(t)}\,\chi_{[p_i^n(t), p_{i+1}^n(t)[}(x) \tag{4.4.3}$$

$$w^n(t,x) = \sum_{i=1}^n \tilde w_i^n\,\chi_{[p_i^n(t), p_{i+1}^n(t)[}(x)\,. \tag{4.4.4}$$

If there exists a pair $(\rho, w) \in \mathbf{L}^\infty\big([0,T]; \mathbf{L^1}(\mathbb{R}; [0,1] \times [\check w, \hat w]\big)$ such that

$$\lim_{n\to+\infty} (\rho^n, w^n)(t,x) = (\rho, w)(t,x) \qquad \textit{p.a.e.}$$

then, the pair $(\rho, \rho w)$ is a weak solution to (4.1.4) with initial datum $(\tilde\rho, \tilde\rho\tilde w)$.

The proof is postponed to Section 4.5.

4.5 Technical Details

We first prove an elementary consequence of our assumption **b**.

Lemma 4.5.1 *Let ψ satisfy **b**. Then,*

$$\exists\,\bar\rho \in [0, R[\ \textit{such that} \left\{\begin{array}{l} \psi \textit{ is constant on } [0, \bar\rho], \\ \psi \textit{ is strictly decreasing on } [\bar\rho, R]. \end{array}\right.$$

Proof. Call $q(\rho) = \rho\,\psi(\rho)$. If ψ is strictly monotone, then $\bar\rho = 0$ and the proof is completed. Otherwise, assume that $\psi(\rho_1) = \psi(\rho_2) = c$ for suitable $\rho_1, \rho_2 \in \,]0, R]$ and $\rho_1 \neq \rho_2$. Then, by **b.**, for all $\rho \in [\rho_1, \rho_2]$ we have $\psi(\rho) = c$ and $q(\rho) = c\rho$. If $\psi(0) = c$, then the proof is completed. Otherwise, note that $q'(0) = \psi(0) > c$ contradicts the convexity of q. $\square$

Corollary 4.5.2 *Let ψ satisfy **b.** and **c.** Then,*

$$\bar\rho < \min\left\{\rho \in [0, R] \colon \exists w \in [\check w, \hat w] \text{ such that } (\rho, w) \in C\right\} .$$

The proof is immediate and, hence, omitted.

In the sequel, for the basic definitions concerning the standard theory of conservation laws we refer to [16].

Proof of Theorem 4.2.1. We consider different cases, depending on the phase of the data (4.2.4).

1. $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$.

In this case, (4.1.4) reduces to the degenerate linear system (4.2.6) so that the problem (4.1.4)–(4.2.4) is solved by (4.2.5). Remark, for later use, that the characteristic speed is $\lambda^F = V_{\max}$.

2. $(\rho^l, \eta^l), (\rho^r, \eta^r) \in C$.

Now, $v(\rho, \eta) = \eta\,\psi(\rho)/\rho$. We show that C is invariant with respect to the 2×2 system of conservation laws (4.2.7). To this aim, we compute the eigenvalues, right eigenvectors and the Lax curves in C:

$$\begin{array}{rclrcl}
\lambda_1(\rho,\eta) & = & \eta\,\psi'(\rho) + v(\rho,\eta) & \lambda_2(\rho,\eta) & = & v(\rho,\eta) \\
r_1(\rho,\eta) & = & \begin{bmatrix} -\rho \\ -\eta \end{bmatrix} & r_2(\rho,\eta) & = & \begin{bmatrix} 1 \\ \eta\left(\frac{1}{\rho} - \frac{\psi'(\rho)}{\psi(\rho)}\right) \end{bmatrix} \\
\nabla\lambda_1 \cdot r_1 & = & -\dfrac{d^2}{d\rho^2}\left[\rho\,\psi(\rho)\right] & \nabla\lambda_2 \cdot r_2 & = & 0 \\
\mathcal{L}_1(\rho;\rho_o,\eta_o) & = & \eta_o \dfrac{\rho}{\rho_o} & \mathcal{L}_2(\rho;\rho_o,\eta_o) & = & \dfrac{\rho\, v(\rho_o,\eta_o)}{\psi(\rho)}, \ \rho_o < R.
\end{array}$$

When $\rho_o = R$, the 2–Lax curve through (ρ_o, η_o) is the segment $\rho = R$, $\eta \in [R\check w, R\hat w]$.

Shock and rarefaction curves of the first characteristic family coincide by [7, Lemma 2.1], see also [16, Problem 1, Chapter 5]. The second characteristic field is linearly degenerate. Hence, (4.2.7) is a Temple system and C is invariant, since its boundary consists of Lax curves, see [50, Theorem 3.2].

Thus, the solution to (4.1.4) is as described in *(2)* and attains values in C.

3. $(\rho^l, \eta^l) \in C$, $(\rho^r, \eta^r) \in F$.

Let ρ^m satisfy $\psi(\rho^m) = V_{\max}\rho^r/\eta^r$. Note that such ρ^m exists in $]0, 1[$ by **b** and **c.**, it is unique by Corollary 4.5.2. Define $\eta^m = (\rho^m/\rho^r)\eta^r$ and note that (ρ^l, η^l), (ρ^m, η^m) are connected by a 1–rarefaction wave of (4.2.7) having maximal speed of propagation $\lambda_1(\rho^m, \eta^m) < V_{\max}$. Hence, a linear wave, solution to (4.2.6), can be juxtaposed connecting (ρ^m, η^m) to (ρ^l, η^l) and the solution to (4.1.4) is as described at *(3)*.

4. $(\rho^l, \eta^l) \in F$, $(\rho^r, \eta^r) \in C$ (see Figure 4.3).

Note that system (4.2.7) can be considered on the whole of $F \cup C$. Also this set is invariant for (4.2.7), by [50, Theorem 3.2]. Then, in this case, we let (ρ, η) be the standard Lax solution to (4.2.7), as described at *(4)*. □

Proof of Proposition 4.2.3. We consider different cases depending on the phase of the data (4.2.4).

If $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$, then $\mathcal{R}\left((\rho^l, \eta^l), (\rho^r, \eta^r)\right)$ coincides with the Riemann solver of a linear system, which satisfies **(C1)**. Condition **(C2)** is immediate since no nontrivial middle state is available.

Similarly, if $(\rho^l, \eta^l), (\rho^r, \eta^r) \in C$, then $\mathcal{R}\left((\rho^l, \eta^l), (\rho^r, \eta^r)\right)$ coincides with the standard Riemann solver of a 2×2 system, which is consistent. The consistency of $\mathcal{R}$ then follows by the invariance of C, by **2.** in the proof of Theorem 4.2.1.

By the same argument, also the case $(\rho^l, \eta^l) \in F$ and $(\rho^r, \eta^r) \in C$ is proved. Indeed, in **(C2)**, note that the only possible nontrivial middles states are in C.

Finally, if $(\rho^l, \eta^l) \in C$ and $(\rho^r, \eta^r) \in F$, then $\mathcal{R}\left((\rho^l, \eta^l), (\rho^r, \eta^r)\right)$ takes values in $F \cup C$ and is the juxtaposition of 2 consistent Riemann problems, hence **(C1)** holds. Concerning **(C2)**, note that the the only possible nontrivial middles states are in C, and **(C2)** follows by the consistency of the standard Riemann solver for (4.2.7).

Thus 1. is proved. Assertions 2. and 3. are immediate consequences of the construction of Theorem 4.2.1.

Assume now that $\mathcal{R}$ satisfies 2 and 3. Then all Riemann problems with data $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$, $(\rho^l, \eta^l) \in F$, $(\rho^r, \eta^r) \in C$ and $(\rho^l, \eta^l), (\rho^r, \eta^r) \in C$ are uniquely solved. The solution to Riemann problems with $(\rho^l, \eta^l) \in C$ and $(\rho^r, \eta^r) \in F$ is then uniquely constructed through **(C1)**. □

Proof of Proposition 4.2.4. Consider the different statements separately.

1. The invariance of F, C and $F \cup C$ is shown in the proof of Theorem 4.2.1.

2. By the invariance of $F \cup C$, it is sufficient to observe that on the compact set $F \cup C$, the density ρ, respectively the speed v, is uniformly bounded by R, respectively $V_{\max}$.

3. It is immediate, see for instance Figure 4.2, left.

4. In phase C we have

$$\lambda_1(\rho, \eta) = \eta\, \psi'(\rho) + v(\rho, \eta) \leq v(\rho, \eta) \quad \text{and} \quad \lambda_2(\rho, \eta) \leq v(\rho, \eta).$$

In the free phase the wave speed is $V_{\max} = v(\rho, \eta)$. The only case left is that of a phase boundary with left state in F and right state, say (ρ^r, η^r), in C. Then, the speed Λ of the phase boundary clearly satisfies $\Lambda \leq \lambda_1(\rho^r, \eta^r) < v(\rho^r, \eta^r)$. $\square$

Proof of Proposition 4.4.1. Note first that the functions $\rho \to v(\rho, w_i)$ in (4.4.2) are uniformly bounded and Lipschitz continuous for $i = 1, \ldots, n$. We extend them to functions with the same properties and defined on $[0, +\infty[$ setting

$$u_i(\rho) = \begin{cases} V_{\max} & if \quad \rho < 0 \\ v(\rho, w_i) & if \quad \rho \in [0,1] \\ 0 & if \quad \rho > 1. \end{cases} \tag{4.5.1}$$

We also note that, for $i = 1, \ldots, n$, the composite applications $\delta \to u_i(l/\delta)$, can be extended to uniformly bounded and Lipschitz continuous functions on $[0, +\infty[$. Now we consider the Cauchy problem

$$\begin{cases} \dot{p}_i^n = u_i \left(\frac{l}{p_{i+1}^n - p_i^n} \right) & i = 1, \ldots, n \\ \dot{p}_{n+1}^n = V_{\max} & \\ p_i^n(0) = \tilde{p}_i & i = 1, \ldots, n+1\,. \end{cases} \tag{4.5.2}$$

Note that $\tilde{p}_i^n$, for $i = 1, \ldots, n+1$ are defined in Proposition 4.4.2 and satisfy the condition $\tilde{p}_{i+1}^n - \tilde{p}_i^n \geq l > 0$, for every $i = 1, \ldots, n$.

By the standard ODE theory, there exists a $\mathbf{C^1}$ solution p_i^n defined as long as $p_{i+1}^n - p_i^n > 0$ for all $i = 1, \ldots, n$. We now prove that in fact $p_{i+1}^n(t) - p_i^n(t) \geq l$ for every $t \geq 0$. To this aim we assume by contradiction that there exist positive $\bar{t}$ and t^*, with $\bar{t} < t^*$, such that $p_{i+1}^n(\bar{t}) - p_i^n(\bar{t}) = l$ and $p_{i+1}^n(t) - p_i^n(t) < l$ for every $t \in \left]\bar{t}, t^*\right]$. Then,

$$p_i^n(t) = p_i^n(\bar{t}) + \int_{\bar{t}}^t \dot{p}_i(s)\, ds = p_i^n(\bar{t}) + \int_{\bar{t}}^t u_i \left(\frac{l}{p_{i+1}^n(s) - p_i^n(s)} \right) ds = p_i^n(\bar{t}).$$

This yields a contradiction, since for every $t \in \left]\bar{t}, t^*\right]$

$$p_{i+1}^n(t) - p_i^n(t) \geq p_{i+1}^n(\bar{t}) - p_i^n(\bar{t}) = l\,,$$

completing the proof. $\square$

Proof of Proposition 4.4.2. Recall first the definition of weak solution to (4.1.4): for all $\varphi \in \mathbf{C_c^\infty}$, setting $v(\rho, w) = \min\{V_{\max},\, w\,\psi(\rho)\}$,

$$\int_0^T \int_{\mathbb{R}} \left(\begin{bmatrix} \rho \\ \rho\, w \end{bmatrix} \partial_t \varphi + \begin{bmatrix} \rho\, v(\rho, w) \\ \rho\, w\, v(\rho, w) \end{bmatrix} \partial_x \varphi \right) dx\, dt + \int_{\mathbb{R}} \begin{bmatrix} \tilde{\rho} \\ \tilde{\rho}\, \tilde{w} \end{bmatrix} \varphi(0, x)\, dx = 0$$

and consider the two components separately.

Below, $\mathcal{O}(1)$ denotes a constant that uniformly bounds from above the modulus of φ and all its derivatives up to the second order. Insert first (4.4.3) in the above equality and obtain:

$$
\begin{aligned}
I^n &:= \int_0^T \int_{\mathbb{R}} \left(\rho^n \partial_t \varphi + \rho^n \, v(\rho^n, w^n) \, \partial_x \varphi\right) \, dx \, dt + \int_{\mathbb{R}} \tilde{\rho} \, \varphi(0,x) \, dx \\
&= \sum_{i=1}^{n} \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \left[\partial_t \varphi + v\left(\frac{l}{p_{i+1}^n(t) - p_i^n(t)}, w_i^n\right) \partial_x \varphi \right] dt \\
&\qquad + \int_{\mathbb{R}} \rho^n(0,x) \, \varphi(0,x) \, dx + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \, \varphi(0,x) \, dx \\
&= \sum_{i=1}^{n} \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \left(\partial_t \varphi(t,x) + \dot{p}_i^n(t) \partial_x \varphi(t,x)\right) dx \, dt \\
&\qquad + \sum_{i=1}^{n} \frac{l}{\tilde{p}_{i+1}^n - \tilde{p}_i^n} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \varphi(0,x) dx + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \varphi(0,x) \, dx.
\end{aligned}
$$

Approximating $\varphi\,(t,x)$ with $\varphi\left(t, p_i^n(t)\right)$ for every x in $[p_i^n(t), p_{i+1}^n(t)]$, we obtain:

$$
\begin{aligned}
I^n &= \sum_{i=1}^{n} \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \frac{d}{dt} \varphi\left(t, p_i^n(t)\right) \, dx \, dt \\
&\qquad + \sum_{i=1}^{n} \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \mathcal{O}(1) \left(p_{i+1}^n(t) - p_i^n(t)\right) \, dx \, dt \\
&\qquad + \sum_{i=1}^{n} \frac{l}{\tilde{p}_{i+1}^n - \tilde{p}_i^n} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \varphi(0,x) \, dx + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \varphi(0,x) \, dx \\
&= l \sum_{i=1}^{n} \int_0^T \frac{d}{dt} \varphi\left(t, p_i^n(t)\right) dt + \Delta x \sum_{i=1}^{n} \int_0^T \mathcal{O}(1) \left(p_{i+1}^n(t) - p_i^n(t)\right) dx \, dt \\
&\qquad + \sum_{i=1}^{n} \frac{l}{\tilde{p}_{i+1}^n - \tilde{p}_i^n} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \varphi(0,x) dx + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \varphi(0,x) \, dx \\
&= \sum_{i=1}^{n} \frac{l}{\tilde{p}_{i+1}^n - \tilde{p}_i^n} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \left[\varphi(0,x) - \varphi(0, \tilde{p}_i^n)\right] \, dx \\
&\qquad + \mathcal{O}(1) \; l \left(p_{n+1}^n(T) - p_1^n(T)\right) + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \, \varphi(0,x) \, dx \\
&= \mathcal{O}(1) \; l \, (2L + V_{\max} T) + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \, \varphi(0,x) \, dx
\end{aligned}
$$

and both terms in the latter quantity clearly vanish as $n \to +\infty$.

The computations related to the other component are entirely similar, since w is constant along any set of the form

$$\left\{(t,x) \in [0,T] \times \mathbb{R}\colon x \in \left[p_i^n(t), p_{i+1}^n(t)\right[\right\}$$

and the proof is completed. □

Remark 4.5.3 *System (4.1.2) is not in conservation form. As far as smooth solutions are concerned, it is equivalent to infinitely many 2×2 systems of conservation laws. Indeed, introduce a strictly monotone function $f \in \mathbf{C}^2\left([\check{w}, \hat{w}];]0, +\infty[\right)$. Then, elementary computations show that, as long as smooth solutions are concerned, system (4.1.2) is equivalent to*

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \left(\rho\, \psi(\rho)\, g(\eta/\rho)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, \psi(\rho)\, g(\eta/\rho)\right) = 0 \end{array}\right. \qquad \textit{where} \qquad \begin{array}{l} \eta = \rho\, f(w) \ \textit{and} \\ g\left(f(w)\right) = w \end{array} \tag{4.5.3}$$

Clearly, different choices of f yield different weak solutions to (4.5.3), but they are all equivalent when written in terms of ρ and w.

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