Then $\psi\in\mathscr{B}(S,\mathbf{R})$ and $\psi_{(a,b)}=\varphi(a)=$ const. for each fixed $(a,b)\in S$. Consequently,

$$M(\psi)=M(\psi_{(a,b)})=\varphi(a),\qquad a\in S,$$

which is impossible, since we have chosen a nonconstant φ. The nonexistence of a left invariant mean in $\mathscr{B}(S,\mathbf{R})^*$ may be shown in a similar way.

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A multiplier theorem for H-type groups

by

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Abstract. We prove an L^p-boundedness result for a convolution operator with rough kernel supported on a hyperplane of a group of Heisenberg type.

In recent years, several results have been proved in which the Calderón–Zygmund theory of singular integrals has been extended to the more general setting of nilpotent Lie groups (see e.g. [8], [15], [19]). In particular, F. Ricci ([15]) showed that the classical theory of Calderón–Zygmund kernels on $\mathbf{R}^n$ has very natural extensions to kernels on nilpotent Lie groups.

As a further generalization, more singular convolution operators, for instance convolution with distributions which are extensions of Calderón–Zygmund kernels supported on submanifolds, have been considered.

In this context, we would like to mention some results concerning Hilbert transforms along homogeneous curves in Stein and Wainger ([19]) and Christ ([2]). Subsequently, Geller and Stein ([7]) studied smooth homogeneous kernels supported by a hyperplane of the $(2n+1)$-dimensional Heisenberg group; such operators arise in the study of the $\bar{\partial}$-Neumann problem on the Siegel upper half-space. Müller ([13], [14]) showed that Theorem 1.1 in [7] has rather general extensions for more general homogeneous Lie groups and for an even larger class of submanifolds.

Recently, Ricci and Stein unified and extended in a series of papers ([16]–[18]) some results of these previous works; particularly, in ([17]), they considered singular integral operators on homogeneous Lie groups defined by smooth kernels supported on lower-dimensional analytic submanifolds and having the critical degree of homogeneity. To prove the L^p-boundedness, they make a strong use of the smoothness assumption.

In this paper we work on groups of Heisenberg type; we study convolution operators with rough kernels carried by the subspace complementary to the center. We improve Geller and Stein's result, by proving the L^p-boundedness of the convolution operator under some minimal assumptions on the regularity of

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the kernel. Our proof combines the interpolation methods from [7] with a strong use of the twisted convolution.

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1. Notation. Let G be a group of type H, that is, a connected simply connected real Lie group whose Lie algebra is of type H: following A. Kaplan ([11]), we say that the Lie algebra $\mathfrak{g}$ is of *type H* if it is the direct sum $\mathfrak{v}\oplus\mathfrak{z}$ of real Euclidean spaces, with a Lie algebra structure such that $\mathfrak{z}$ is the center of $\mathfrak{g}$ and, for all $V\in\mathfrak{v}$ of length one, the map $\mathbf{ad}(V)$ is a surjective isometry of the orthogonal complement $\mathfrak{v}\ominus\ker\operatorname{ad}(V)$ onto $\mathfrak{z}$. For such an algebra, we define a linear map $j\colon \mathfrak{z}\to\operatorname{End}(\mathfrak{v})$ by the formula

$$\langle j(Z)V, V'\rangle = \langle Z, [V, V']\rangle.$$

It is easy to show that (see e.g. [11])

$$|j(Z)V| = |Z|\,|V| \qquad \forall V\in\mathfrak{v},\ \forall Z\in\mathfrak{z},$$

$$j(Z)^2 = -|Z|^2 I \qquad \forall Z\in\mathfrak{v};$$

in particular, if $|Z| = 1$, then $j(Z)$ defines a complex structure on $\mathfrak{v}$. We shall denote by $2n$ and r the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. If $r = 1$ then G is isomorphic to the Heisenberg group $\mathbf{H}^n$; if $r > 1$, n is always an even integer (see [12]). Hereafter, for a group of type H with a Lie algebra $\mathfrak{g} = \mathfrak{v}\oplus\mathfrak{z}$, we write, using lower case rather than upper case letters,

$$(v, z) = \exp(v+\tfrac{1}{4}z) \qquad \forall v\in\mathfrak{v},\ \forall z\in\mathfrak{z}.$$

Then, by the Campbell–Hausdorff formula, we have

$$(v, z)(v', z') = (v+v', z+z'+2[v, v']),$$

where $[\ ,\ ]$ denotes the Lie bracket in $\mathfrak{g}$.

We define the Haar measure on G by

$$\int_G f(x)dx = \int_{\mathfrak{v}}\int_{\mathfrak{z}} f(v, z)\,dv\,dz.$$

On a group of type H there is a natural gauge defined by

$$N(g) = (|v|^4+|z|^2)^{1/4}$$

for $g = \exp(v+z/4)$, $v\in\mathfrak{v}$, $z\in\mathfrak{z}$. Given any other element $g' = \exp(v'+z'/4)$, we have

$$N(gg') \leqslant N(g)+N(g')$$

(see [5]). G becomes a homogeneous Lie group of dimension $Q = 2n+2r$ with respect to the family of dilations $\{\delta_t\}_{t>0}$ defined by

$$\delta_t(v, z) = (tv, t^2 z).$$

If f is a function defined on G, we set

$$f_t(x) = t^{-Q} f(t^{-1}x), \qquad f^t(x) = f(tx).$$

2. Twisted convolution. Suppose that f and g are integrable functions on $\mathfrak{v}$ and λ is in $\mathfrak{z}^*\setminus\{0\}$, that is, the space of all linear functionals on $\mathfrak{z}$; as usual, we shall identify $\mathfrak{z}^*$ with $\mathfrak{z}$. We define the *λ-twisted convolution* of f and g by

$$f\times_\lambda g(v) = \int_{\mathfrak{v}} f(v-v')g(v')e^{-i\langle j(\lambda)v, v'\rangle}\,dv'. \tag{2.1}$$

Then we have the following results.

PROPOSITION 2.1. *Suppose that f is in $L^2(\mathfrak{v})$. Then*

$$\|1\times_\lambda f\|_2 = (2\pi)^{-n}|\lambda|^{-n}\|f\|_2. \tag{2.2}$$

Proof. Notice that $1\times_\lambda f(v) = \hat{f}(j(\lambda)v)$, and $|\det j(\lambda)| = |\lambda|^{2n}$; therefore, changing variables, we get

$$\|1\times_\lambda f\|_2 = (2\pi)^{-n}\|\hat{f}\circ j(\lambda)\|_2 = (2\pi)^{-n}|\lambda|^{-n}\|\hat{f}\|_2 = (2\pi)^{-n}|\lambda|^{-n}\|f\|_2.$$

PROPOSITION 2.2. *If f and g are functions in $L^p(\mathfrak{v})$, $1\leqslant p\leqslant 2$, then*

$$\|f\times_\lambda g\|_p \leqslant (2\pi)^{-n/p'}|\lambda|^{-n/p'}\|f\|_p\|g\|_p, \tag{2.3}$$

where p' denotes the index conjugate to p.

Proof. Set $\lambda' = \lambda/|\lambda|$. Then a simple calculation shows that

$$f\times_\lambda g = (f^{|\lambda|^{-1/2}}\times_{\lambda'} g^{|\lambda|^{-1/2}})^{|\lambda|^{1/2}}, \qquad \lambda\in\mathfrak{z}.$$

Hence we may assume that λ has unit length. Changing variables, we are led to the study of the standard twisted convolution

$$f\times g(v) = \int_{\mathfrak{v}} f(v-v')g(v')e^{-i\langle Jv, v'\rangle}\,dv', \tag{2.4}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the canonical nonsingular skew-symmetric $2n\times 2n$ matrix (see [9], Ch. 9, § 2).

If $p = 1$, (2.3) is trivial. The case $p = 2$ can be found in [10]. The case $1 < p < 2$ easily follows from a general interpolation theorem (see [1], Th. 4.4.1, p. 96) where we consider the bilinear operator T defined by $T(f, g) = f\times g$.

Now suppose that f is in $L^2(G)$; the partial Fourier transform of f with respect to the central variable is given by

(2.5) $$\mathfrak{F}_\lambda f(v) = \int_{\mathfrak{z}} f(v, z) e^{-i\langle\lambda, z\rangle} dz, \qquad \lambda \in \mathfrak{z}^*.$$

By the Plancherel formula, it is easy to see that $\mathfrak{F}_\lambda f$ belongs to $L^2(\mathfrak{v})$ for a.e. λ; moreover, if g is in $L^2(G)$, an easy computation shows that

(2.6) $$\|f * g\|^2_{L^2(G)} = \int_{\mathfrak{z}} \|\mathfrak{F}_\lambda(f) \times_\lambda \mathfrak{F}_\lambda(g)\|^2_{L^2(\mathfrak{v})} d\lambda.$$

Consider the Laplacian operator Δ on $\mathfrak{z}$, and denote by $\Delta^k\delta$, $k \in \mathbf{N}$, the distribution on $\mathfrak{z}$ defined as follows:

$$\langle \Delta^k\delta, \psi\rangle = \langle\delta, \Delta^k\psi\rangle$$

for every $\psi \in C_0^\infty(\mathfrak{z})$. We may extend $\Delta^k\delta$ to a distribution $1 \otimes \Delta^k\delta$ on G by

$$\langle 1 \otimes \Delta^k\delta, \phi\rangle = \langle\Delta^k\delta, \phi_v\rangle$$

where ϕ is in $C_0^\infty(G)$ and $\phi_v(z) = \phi(v, z)$.

From now on, by abuse of language, all universal constants (i.e. functions of the dimensions n and r only) will be denoted by the same letter C, whenever no ambiguity could arise. We have the following

PROPOSITION 2.3. *The convolution operator on G with kernel $1 \otimes \Delta^{n/2}\delta$ is bounded on $L^2(G)$ if $r > 1$; if $G = \mathbf{H}^n$, we take $1 \otimes (\partial^n/\partial t^n)\delta(t)$, where t is the only central variable.*

Proof. By (2.6) it suffices to prove that

$$\|\mathfrak{F}_\lambda(1 \otimes \Delta^{n/2}\delta) \times_\lambda f\|_2 \leqslant C\|f\|_2$$

for every f in $L^2(\mathfrak{v})$ and C independent of λ. We have

$$\mathfrak{F}_\lambda(1 \otimes \Delta^{n/2}\delta) = \int_{\mathfrak{z}} \Delta^{n/2}\delta(z) e^{-i\langle\lambda, z\rangle} dz = \int_{\mathfrak{z}} d\delta(z) |\lambda|^n e^{-i\langle\lambda, z\rangle} = |\lambda|^n.$$

Thus, using (2.2), we get

$$\|\mathfrak{F}_\lambda(1 \otimes \Delta^{n/2}\delta) \times_\lambda f\|_2 = \| |\lambda|^n \times_\lambda f\|_2 = (2\pi)^{-n}|\lambda|^n|\lambda|^{-n}\|f\|_2 = (2\pi)^{-n}\|f\|_2,$$

thereby concluding the proof.

3. The main result. On the group G with coordinates $(v, z) \in \mathfrak{v} \oplus \mathfrak{z}$, let $\delta(z)$ denote the δ distribution in the z variable. Also, let $k(v)$ be a distribution on $\mathfrak{v}$. Define the distribution K on G by $K(v, z) = k(v)\delta(z)$; thus, if ϕ is a test function,

$$\langle K, \phi\rangle = \langle k, \phi_0\rangle$$

where $\phi_0(v) = \phi(v, 0)$. Let $*$ denote the group convolution. We have the following

THEOREM 3.1. *Let k be a distribution defined on $\mathfrak{v}$ and denote by m its Fourier transform. Assume that*

$$\left|\frac{\partial^\alpha}{\partial\xi_1^{\alpha_1}\ldots\partial\xi_{2n}^{\alpha_{2n}}} m(\xi)\right| \leqslant C|\xi|^{-|\alpha|}$$

for every $\xi \in \mathfrak{v}^*\setminus\{0\}$, $\alpha \in \mathbf{N}^{2n}$, $|\alpha| \leqslant n+1$. Then the operator T_k: $C_0^\infty(G) \to C^\infty(G)$ given by $T_k f = K * f$ extends to a bounded operator on $L^p(G)$ if $1 < p < \infty$.

(For $p = 2$ the result of the theorem is contained in [13], Th. 6.1.)

In order to prove this result, we need some preliminary lemmata. Suppose that G is a homogeneous Lie group, that is, a connected simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_t\}_{t>0}$. Following a common abuse of notation, we shall also denote by $\{\delta_t\}_{t>0}$ the induced family of dilations on G. We shall write tx instead of $\delta_t x$. Assume that G is equipped with a homogeneous norm and denote by Q its homogeneous dimension. Now, suppose that $\{\phi_j\}_{j\in\mathbf{Z}}$ is a family of functions on G satisfying

(3.1) for every j in $\mathbf{Z}$, $\int_G \phi_j(x)\,dx = 0$;

(3.2) there exists a positive ε such that for every j in $\mathbf{Z}$

$$\int_G |\phi_j(x)|(1+|x|^\varepsilon)\,dx \leqslant C;$$

(3.3) there exists a positive ε' such that for every j in $\mathbf{Z}$

$$\int_G |\phi_j(xy) - \phi_j(x)|\,dx \leqslant C|y|^{\varepsilon'}, \qquad j \in \mathbf{Z}.$$

We then have the following

PROPOSITION 3.2. *Let $\{\phi_j\}_{j\in\mathbf{Z}}$ be as above. Set*

$$K = \sum_{j\in\mathbf{Z}} (\phi_j)_{2^{-j}}.$$

Then K defines a distribution on G and the convolution operator with kernel K is bounded on $L^2(G)$. Furthermore, K agrees on $G\setminus\{0\}$ with a function, denoted again by K, satisfying

$$\int_{|x|\geqslant 2|y|} |K(xy^{-1}) - K(x)|\,dx \leqslant C$$

for every $y \in G\setminus\{0\}$.

Proof. A slight modification of the proof of Lemma 2.1 in [7] gives the result.

Throughout this paper, Ψ will denote a radial function in $C_0^\infty(\mathfrak{v}^*)$ such that $\operatorname{supp}\Psi \subseteq \{\xi \in \mathfrak{v}^*\colon 1/2 \leqslant |\xi| \leqslant 4\}$ and $\Psi = 1$ if $1 \leqslant |\xi| \leqslant 2$. We shall assume that $\sum_{j\in\mathbf{Z}} \Psi^{2^j} = 1$ on $\mathfrak{v}^*\setminus\{0\}$. $\check{\Psi}$ will denote the inverse Fourier transform of Ψ.

LEMMA 3.3. *Let k_1 be the distribution on $\mathfrak{v}$ defined by*

$$k_1 = \sum_{j\in\mathbf{Z}} c_j(\check{\Psi})_{2^{-j}},$$

where $\{c_j\}_{j\in\mathbf{Z}}$ is a bounded sequence of real numbers. Then the convolution operator on G with kernel $K_1(v, z) = k_1(v)\delta(z)$ is bounded on $L^p(G)$, $1 < p < \infty$, and its norm depends on $\sup|c_j|$.

Proof. Notice that $\check{\Psi}$ is a radial function and the family $\{\check{\Psi}_{2^{-j}}\}_{j\in\mathbf{Z}}$ satisfies the assumptions of Proposition 3.2; let $\tilde{k}_1$ be the even function on $\mathbf{R}$ defined by $\tilde{k}_1(\varrho) = k_1(v)$, where $|x| = \varrho$, $v \neq 0$. Changing variables and integrating in polar coordinates, we get

$$\begin{aligned}(K_1 * f)(v, z) &= \int_{\mathfrak{v}}\int_{\mathfrak{z}} k_1(v-v')\delta(z-z'-\tfrac{1}{2}[v, v'])f(v', z')\,dv'\,dz'\\ &= \int_{\mathfrak{v}}\int_{\mathfrak{z}} k_1(v')\delta(z-z'+\tfrac{1}{2}[v, v'])f(v-v', z')\,dv'\,dz'\\ &= \int_{\mathfrak{v}} k_1(v')f(v-v', z+\tfrac{1}{2}[v, v'])\,dv'\\ &= \tfrac{1}{2}\int_{S^{2n-1}} d\sigma \int_{\mathbf{R}} |\varrho|^{2n-1}\tilde{k}_1(\varrho)f(v-\varrho\sigma, z+\tfrac{1}{2}\varrho[v, \sigma])\,d\varrho,\end{aligned}$$

where $\varrho\sigma = v'$, $|\sigma| = 1$, $\varrho\in\mathbf{R}$ and S^{2n-1} is the unit sphere in $\mathfrak{v}$. Denote by $(\check{\Psi})^{\sim}$ the even function defined on $\mathbf{R}\setminus\{0\}$ by $(\check{\Psi})^{\sim}(\varrho) = \check{\Psi}(v)$, where $|v| = \varrho$, and consider the Calderón–Zygmund kernel on $\mathbf{R}$

$$\tilde{k}_1(\varrho)|\varrho|^{2n-1} = \sum_{j\in\mathbf{Z}} c_j\,[(\check{\Psi})^{\sim}(\varrho)|\varrho|^{2n-1}]_{2^{-j}}.$$

For every $\sigma\in S^{2n-1}$, define the representation π^σ of $\mathbf{R}$ in $L^p(G)$ by

$$(\pi^\sigma(\varrho))f(v, z) = f(v+\varrho\sigma, z-\tfrac{1}{2}\varrho[v, \sigma]).$$

Then π^σ is a unitary representation on $L^p(G)$ for all p, $1 \leqslant p \leqslant \infty$; moreover,

$$(K_1 * f)(v, z) = \tfrac{1}{2}\int_{S^{2n-1}} d\sigma \int_{\mathbf{R}} \tilde{k}_1(\varrho)|\varrho|^{2n-1}(\pi^\sigma(-\varrho))f(v, z)\,d\varrho.$$

By transference and the rotation method of Calderón ([3], [20], Ch. VI, §2), we get

$$\begin{aligned}\|K_1 * f\|_p &\leqslant \tfrac{1}{2}\int_{S^{2n-1}} d\sigma\,\Big\|\int_{\mathbf{R}} \tilde{k}_1(\varrho)|\varrho|^{2n-1}(\pi^\sigma(-\varrho))f\,d\varrho\Big\|_p\\ &\leqslant C\int_{S^{2n-1}} \|f\|_p\,d\sigma = C\|f\|_p.\end{aligned}$$

This proves the lemma.

Suppose now that k is a distribution on $\mathfrak{v}$ and denote by m its Fourier transform. Set $\Psi_j = \Psi^{2^{-j}}$. We may decompose k and m according to the chosen partition of unity. Set

$$\mu_j = (m\Psi_j)^{2^j} \tag{3.5}$$

and let k_j be the inverse Fourier transform of μ_j. Then

$$m = \sum_{j\in\mathbf{Z}} (\mu_j)^{2^{-j}}, \tag{3.6}$$

$$k = \sum_{j\in\mathbf{Z}} (k_j)_{2^{-j}}. \tag{3.7}$$

Proof of Theorem 3.1. We shall prove the theorem by adapting to the present situation a rather standard complex interpolation argument (see [7]). In view of Lemma 3.3, we may assume that

$$\int_{\mathfrak{v}} \mu_j(v)\,dv = 0 \tag{3.8}$$

for every $j\in\mathbf{Z}$. Choose a positive function η in $C_0^\infty(\mathfrak{z})$ such that $\eta(z) = 1$ if $0 \leqslant |z| \leqslant 1/2$ and $\operatorname{supp}\eta \subset \{z\in\mathfrak{z}\colon |z| \leqslant 1\}$. Define the analytic family of kernels $\{K_\alpha\}_{-n\leqslant\mathrm{Re}\,\alpha<1}$ on G by

$$K_\alpha(v, z) = \sum_{j\in\mathbf{Z}} \left(k_j(v)\eta(z)\frac{2|z|^{-r+\alpha}}{\Omega_r\Gamma(\alpha/2)}\right)_{2^{-j}}, \tag{3.9}$$

where Ω_r denotes the measure of the unit sphere in $\mathfrak{z}$. Notice that

$$\left.\frac{2}{\Omega_r}\frac{|z|^{-r+\alpha}}{\Gamma(\alpha/2)}\right|_{\alpha=-2k} = (-1)^k\frac{\Delta^k\delta(z)}{2^k\ldots(r+2k-2)} \tag{3.10}$$

(see, for instance, [6], Ch. 7, §3.9). In particular, for $k = 0$ we get

$$\left.\frac{2}{\Omega_r}\frac{|z|^{-r+\alpha}}{\Gamma(\alpha/2)}\right|_{\alpha=0} = \delta(z). \tag{3.11}$$

By (3.10) we have $K_0(v, z) = K(v, z)$.

We shall denote by $\{T_\alpha\}_{-n\leqslant\mathrm{Re}\,\alpha<1}$ the family of operators defined by $T_\alpha\phi = K_\alpha * \phi$ for every $\phi\in C_0^\infty(G)$. We shall prove that

(i) if $0 < \mathrm{Re}\,\alpha < 1$, then T_α is bounded on $L^p(G)$, $1 < p < \infty$;
(ii) for every $\gamma\in\mathbf{R}$, $T_{-n+i\gamma}$ is bounded on $L^2(G)$.

The desired conclusion will follow from an application of Stein's complex interpolation theorem.

Denote by h_j the function on G given by

$$h_j(v, z) = k_j(v)\eta(z)\frac{2|z|^{-r+\alpha}}{\Omega_r\Gamma(\alpha/2)}.$$

We claim that the family $\{h_j\}_{j\in\mathbf{Z}}$ satisfies on G the assumptions of Proposition 3.2, with $\varepsilon=\varepsilon'=\operatorname{Re}\alpha$. Using Hölder's inequality and the fact that $\|k_j|v|^\alpha\|_2\leqslant C$ for every $\alpha\in\mathbf{N}^{2n}$, $|\alpha|\leqslant n+1$ (which is a direct consequence of the hypothesis made on k) it is not hard to show that the family $\{k_j\}_{j\in\mathbf{Z}}$ satisfies (3.1)–(3.3) on $\mathfrak{v}$ for every $\varepsilon,\ \varepsilon'\in[0,1)$. Since $\hat{\mu}(0)=0$ we have $\int_G h_j=0$. We need to show that

$$\int_{\mathfrak{v}}\int_{\mathfrak{z}}|h_j(v,z)|\big(1+|(v,z)|^\varepsilon\big)\,dv\,dz\leqslant C\qquad\forall j\in\mathbf{Z}$$

for some positive ε. In the sequel, we shall denote by $C(\alpha)$ a constant which depends only on n, r and α. Since $|z|\leqslant 1$, there exists a positive constant C such that $|(v,z)|^\varepsilon\leqslant C(|v|^\varepsilon+1)$. Using the fact that $\{k_j\}_{j\in\mathbf{Z}}$ satisfies (3.2) on $\mathfrak{v}$, we easily get the required estimate with $C(\alpha)=C'/|\Gamma(\alpha/2)|$, where C' is independent of α. Finally, we need to estimate the integral

$$I(v',z')=\int_{\mathfrak{v}}\int_{\mathfrak{z}}|h(v-v',\,z-z'-\tfrac{1}{2}[v,v'])-h(v,z)|\,dv\,dz \tag{3.12}$$

where (v',z') is a fixed element of G. We may assume that $|(v',z')|\leqslant 1$. We have

$$\begin{aligned}I(v',z')&\leqslant C(\alpha)\Big\{\int_{\mathfrak{v}}\int_{\mathfrak{z}}|k_j(v-v')|\\&\qquad\times\big|\eta(z-z'-\tfrac{1}{2}[v,v'])|z-z'-\tfrac{1}{2}[v,v']|^{-r+\alpha}-\eta(z)|z|^{-r+\alpha}\big|\,dv\,dz\\&\qquad+\int_{\mathfrak{v}}\int_{\mathfrak{z}}|k_j(v-v')-k_j(v)|\eta(z)|z|^{-r+\operatorname{Re}\alpha}\,dv\,dz\Big\}\\&=I_1+I_2.\end{aligned}$$

Notice that if $\operatorname{Re}\alpha<1$ then $\eta(z)|z|^{-r+\operatorname{Re}\alpha}$ is in $L^{1,\operatorname{Re}\alpha}(\mathfrak{z})$; moreover, by (1.1) and the fact that $|(v',z')|\leqslant 1$, we have

$$|z'+\tfrac{1}{2}[v,v']|^{\operatorname{Re}\alpha}\leqslant|(v,0)(v',z')|^{\operatorname{Re}\alpha}\leqslant|(v,0)|^{\operatorname{Re}\alpha}|(v',z')|^{\operatorname{Re}\alpha}\leqslant(1+|v|^{\operatorname{Re}\alpha})|(v',z')|^{\operatorname{Re}\alpha}.$$

Therefore, changing variables, we get

$$\begin{aligned}I_1&\leqslant C(\alpha)\int_{\mathfrak{v}}|k_j(v-v')|\,|z'+\tfrac{1}{2}[v,v']|^{\operatorname{Re}\alpha}\,dv\\&\leqslant C(\alpha)\int_{\mathfrak{v}}|k_j(v)|(1+|v|^{\operatorname{Re}\alpha})|(v',z')|^{\operatorname{Re}\alpha}\,dv\leqslant C(\alpha)|(v',z')|^{\operatorname{Re}\alpha},\end{aligned}$$

for every $\alpha\in\mathbf{C}$, $0<\operatorname{Re}\alpha<1$ and $C(\alpha)=C'/|\Gamma(\alpha/2)|$. Also,

$$\begin{aligned}I_2&\leqslant C(\alpha)\int_{\mathfrak{v}}|k_j(v-v')-k_j(v)|\,dv\int_{\mathfrak{z}}\eta(z)|z|^{-r+\operatorname{Re}\alpha}dz\\&\leqslant C(\alpha)|v'|^\varepsilon\leqslant C(\alpha)|(v',z')|^\varepsilon,\end{aligned}$$

for every $\varepsilon\in[0,1)$. In particular, we may choose $\varepsilon=\operatorname{Re}\alpha$. Thus, for every α, $0<\operatorname{Re}\alpha<1$, T_α is a bounded operator on $L^p(G)$, $1<p<\infty$, whose norm does not exceed $C'/|\Gamma(\alpha/2)|$.

We now turn to the proof of (ii). Assume first that $\alpha=-n$ and $r>1$; the case $r=1$ can be proved in the same way. If $r>1$, by (3.10) we have

$$\frac{|z|^{-r-n}}{\Gamma(-n/2)}=C_{r,n}\Delta^{n/2}\delta(z).$$

Therefore the kernel K_{-n} is given by

$$K_{-n}(v,z)=F(v)\Delta^{n/2}\delta(z),$$

where

$$F(v)=C_{r,n}\sum_{j\in\mathbf{Z}}(k_j)^{2^j}(v). \tag{3.13}$$

By (2.6) we need to show that the twisted convolution operator $f\mapsto\mathfrak{F}_\lambda(K_{-n})\times_\lambda f$, $\lambda\in\mathfrak{z}^*\setminus\{0\}$, is bounded on $L^2(\mathfrak{v})$ with norm independent of λ. By (2.2), we may equivalently prove that

$$\|\,|\lambda|^n\hat{F}\times_\lambda f\|_2\leqslant C\|f\|_2.$$

Assume that $|\lambda|=1$; the case $|\lambda|\neq 1$ will follow easily from this by a dilation argument. By (3.13) we have

$$\hat{F}=C_{r,n}\sum_{j\in\mathbf{Z}}(\mu_j)_{2^j}.$$

It can be readily seen that $\hat{F}$ is a Calderón–Zygmund kernel on $\mathfrak{v}$ (it is here that assumption (3.8) comes into effect). Thus, the L^2-boundedness of the operator $f\mapsto\hat{F}_1\times_\lambda f$, where $\hat{F}_1=C_{r,n}\sum_{j\leqslant 0}(\mu_j)_{2^j}$, follows from Cowling's result ([4]). Furthermore, it is easy to see that $\hat{F}_2=\hat{F}-\hat{F}_1$ is in $L^2(\mathfrak{v})$. Hence, by Proposition 2.2, we get

$$\|\hat{F}_2\times_\lambda f\|_2\leqslant C\|f\|_2,$$

thereby concluding the proof of (ii) in the case $\alpha=-n$.

Assume now that $\alpha=-n+i\gamma$, $\gamma\in\mathbf{R}\setminus\{0\}$. We split the kernel $K_{-n+i\gamma}$ into two parts as follows:

$$\begin{aligned}K_{-n+i\gamma}(v,z)&=\sum_{j\in\mathbf{Z}}\Big(k_j(v)(\eta(z)-1)\frac{2|z|^{-r-n+i\gamma}}{\Omega_r\Gamma((-n+i\gamma)/2)}\Big)_{2^{-j}}\\&\quad+\sum_{j\in\mathbf{Z}}\Big(k_j(v)\frac{2|z|^{-r-n+i\gamma}}{\Omega_r\Gamma((-n+i\gamma)/2)}\Big)_{2^{-j}}\\&=K^1_{-n+i\gamma}(v,z)+K^2_{-n+i\gamma}(v,z).\end{aligned}$$

Dealing with $K^2_{-n+i\gamma}$ as we did for K_{-n}, we get

$$\|\mathfrak{F}_\lambda(K^2_{-n+i\gamma})\times_\lambda f\|_2\leqslant C(\gamma)\|f\|_2,\qquad f\in L^2(\mathfrak{v}),$$

where $C(\gamma)=\big|\sin((-r-n+i\gamma)\pi/2)\Gamma(-n+i\gamma)/\Gamma((-n+i\gamma)/2)\big|$. Set

$$h'_j(v,z)=\frac{2}{\Omega_r\Gamma((-n+i\gamma)/2)}\big(k_j(v)(\eta(z)-1)|z|^{-r-n+i\gamma}\big)_{2^{-j}}.$$

We shall show that $K^1_{-n+i\gamma}$ is a Calderón–Zygmund kernel by proving that the family of functions $\{h'_j\}_{j\in\mathbf{Z}}$ satisfies the assumptions of Proposition 3.2. Taking into account the fact that $(\eta(z)-1)|z|^{-r-n+i\gamma}$ is in $L^p(\mathfrak{z})$ for every $p \geqslant 1$, for every $\varepsilon\in[0, 1)$ we get

$$\begin{aligned} C'(\gamma)\int_{\mathfrak{v}}\int_{\mathfrak{z}} |k_j(v)|(1-\eta(z))|z|^{-r-n}(1+|(v, z)|^{\varepsilon})\,dv\,dz &\leqslant C'(\gamma)\int_{\mathfrak{v}}\int_{\mathfrak{z}} |k_j(v)|(1-\eta(z))|z|^{-r-n}(1+|v|^{\varepsilon}+|z|^{\varepsilon/2})\,dv\,dz \\ &\leqslant C'(\gamma)\int_{\mathfrak{v}} |k_j(v)|(1+|v|^{\varepsilon})\,dv \int_{\mathfrak{z}} (1-\eta(z))|z|^{-r-n}\,dz \\ &\quad + C'(\gamma)\int_{\mathfrak{v}} |k_j(v)|\,dv \int_{\mathfrak{z}} (1-\eta(z))|z|^{-r-n+\varepsilon/2}\,dz \\ &\leqslant C'(\gamma), \end{aligned}$$

with $C(\gamma) = C'/\Gamma(-n+i\gamma/2)$.

Finally, since $(1-\eta(z))|z|^{-r-n+i\gamma}\in L^{1,1}(\mathfrak{z})$, we can easily prove that

$$C'(\gamma)\int_{\mathfrak{v}}\int_{\mathfrak{z}} |h'_j(v+v', z+z'+\tfrac{1}{2}[v, v'])-h'_j(v, z)|\,dv\,dz \leqslant C'(\gamma)|(v', z')|^{\varepsilon}$$

for every $\varepsilon\in[0, 1]$.

To apply Stein's complex interpolation theorem, we need to estimate the growth of the operator norm $|||T_\alpha|||_p$ in the cases: $1<p<\infty$, $0<\operatorname{Re}\alpha<1$ and $p=2$, $\operatorname{Re}\alpha = -n$. A careful reading of the proof shows that

$$|||T_\alpha|||_p \leqslant C_p/|\Gamma(\alpha/2)|$$

if $0<\operatorname{Re}\alpha<1$, $1<p<\infty$, and

$$|||T_{-n+i\gamma}|||_2 \leqslant \frac{C\left|\sin\left(\dfrac{-r-n+i\gamma}{2}\right)\pi\right||\Gamma(-n+i\gamma)|}{|\Gamma((-n+i\gamma)/2)|}$$

for every $\gamma\in\mathbf{R}$. Easy computations, taking into account Stirling's formula, give the desired result.

Given an H-type group G, consider, for every $\lambda\in\mathfrak{z}^*$ and for every p, $1\leqslant p\leqslant\infty$, the representation π^λ of G in $L^p(\mathfrak{v})$ defined by

$$\pi^\lambda(g)f(v) = f(v+x)e^{-i\langle\lambda,t\rangle}e^{-i\langle J(\lambda)x,v\rangle},$$

where $g=(x, t)$. π^λ is an isometric representation of G for every p, $1\leqslant p\leqslant\infty$. Using the transference result (see [3]), we shall prove the following

COROLLARY 3.4. *Suppose that k is a distribution defined on $\mathfrak{v}$, satisfying the assumptions of Theorem 3.1. Then the operator T^λ: $C_0^\infty(\mathfrak{v})\to C^\infty(\mathfrak{v})$ defined by*

$$T^\lambda(f) = k\times_\lambda f$$

is bounded on $L^p(\mathfrak{v})$ for every p, $1<p<\infty$.

Proof. Set, as usual, $K(g) = k(v)\delta(t)$. Then, for every f in $C_0^\infty(\mathfrak{v})$ we have

$$k\times_\lambda f = \int_G K(g)\pi^\lambda(g^{-1})f(v)\,dg.$$

By Theorem 3.1 and using the transference result we easily get the assertion.

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