ON THE AXIOMS OF RESIDUATED STRUCTURES: INDEPENDENCE, DEPENDENCIES AND ROUGH APPROXIMATIONS

D. CIUCCI

Abstract. Several residuated algebras are taken into account. The set of axioms defining each structure is reduced with the aim to obtain an independent axiomatization. Further, the relationship among all the algebras is studied and their dependencies outlined. Finally, rough approximation spaces are introduced in residuated lattices with involution and their algebraic structure outlined.

Introduction

An algebra, based on a set A, is said *residuated* if on A a partial order and a pair of binary operators $(*, \to)$, called respectively multiplication and residuation, are defined such that the following hold $\forall a, b, c \in A$:

$$a * c \leq b \quad \text{implies} \quad c \leq a \to b$$
$$a * (a \to b) \leq b$$

Trivially, these two properties can be equivalently expressed by the following *adjointness condition*:

$$c \leq (a \to b) \quad \text{iff} \quad a * c \leq b$$

In the last years, residuated structures became popular in computer science since it was understood that they play a fundamental role in fuzzy logics. In 1998 Hájek introduced Basic Logic (BL) algebras [18] as a residuated structure able to describe all many valued logics arising from continuous t–norms [13]. Indeed, any t–norm on $[0, 1]$ defines a BL algebra and a strong completeness theorem hold for this structure [18]. Well known and important logics can be seen as enrichment of BL algebras, among them Wajsberg algebras [27, 28], which axiomatize Łukasiewicz logic and Heyting algebras [22, 23], which axiomatize intuitionistic logic.
Hence, starting from the basic notion of residuated lattices [29, 5], several structures have been defined and studied in literature, both for pure mathematical interest and inspired by soft computing techniques. As a general picture, we can say that two lines have been followed. First, new axioms to better characterize the residuation operator have been introduced. Apart from the aforementioned Basic Logic algebra, the derived notion of Strict Basic Logic algebra has been defined [15] with the aim to characterize only the particular class of t–norms without non–trivial zero divisors [19]. Thus, Heyting algebras are a model of SBL algebras, whereas Wajsberg algebras are not. On the other hand, new structures can be obtained adding an involutive negation to residuated lattices. We recall that a unary operation n on a set A is called an involution if it is a non–decreasing function satisfying the

double negation law $n(n(a)) = a$ [19]. The weakest structure of this kind, named extended residuated lattice [25], has been introduced in order to give an algebraic approach to fuzzy rough sets [24]. Other two different residuated structures with involution, which will be taken into account in our analysis, are involutive residuated lattices [17] and SBL$_\neg$ algebras [15].
As said above, BL algebras give a theoretical approach to fuzzy sets–fuzzy logics and ER–lattices to fuzzy rough sets. Other paradigms of soft computing can be taken into account in this context. Intuitionistic fuzzy sets, for instance, are a model of symmetric Heyting algebras [7], i.e., Heyting algebras with an involutive negation [23]. Further, classical (Pawlak) rough sets have a structure of Heyting Wajsberg (HW) algebra [9], i.e., of an algebra pasting together Heyting algebras and Wajsberg algebras [10]. The same structure of HW algebra can be given also to shadowed sets [8]. It is thus clear that a study of all these algebras and their dependencies can give some insight to all these disciplines (fuzzy sets, rough sets, etc.) and can be useful in uncertain reasoning.
In [10, 11] we studied some residuated algebras with a strong structure, analyzing the dependencies among them and showing how it is possible to define a rough approximation space in an abstract environment. Here, we focus our attention to weaker residuated algebras. Starting from residuated lattices, we will construct a hierarchy of algebras, the richest one will be SBL$_\neg$ algebra, clarifying the dependencies among them. Further, for any introduced algebra, we will try to reduce its set of axioms in order to obtain an independent axiomatization. This operation has of course an intrinsic mathematical interest and has been performed, for instance, in the case of MV algebras and Heyting algebras. Indeed, the original Chang's definition consisted of twenty axioms, then, it has been successively simplified until an independent axiomatization of five axioms has been given [20, 12]. Similarly, in [22], Monteiro gives and independent axiomatization of Heyting algebras (which, in this paper, are called Brouwer algebras). Further, an independent axiomatization makes it possible to understand which properties are necessary and which not. This aspect also reflects on a better knowledge about the models of the algebra in question. For instance, in the last section we will ask which properties are necessary to define a rough approximation in residuated lattices with involution and we will see that the solution of this problem requires the definition of new algebras.
We remark that this work is purely algebraic, so we leave to the interested reader the study of a proper logical axiomatization.

1. Residuated lattices

Let us start our investigation from the bottom of the hierarchy of residuated structures. We introduce different axiomatization of residuated lattices, the last one made of independent properties.

Definition 1.1. [29] A system $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a *residuated lattice* if $\wedge, \vee, *, \to$ are binary operators on A and $0, 1$ are constants such that:

(1) $\langle A, \wedge, \vee, 0, 1\rangle$ is a complete lattice with least element 0 and greatest element 1 with respect to the lattice ordering $a \leq b$ iff $a \wedge b = a$;
(2) The following are satisfied for all $a, b, c \in A$:
  (a) $a \to b = 1$ iff $a \leq b$
  (b) $a \leq b$ implies $c \to a \leq c \to b$, $b \to c \leq a \to c$
  (c) $c \to (b \to a) = b \to (c \to a)$

(d) $c \to (a \wedge b) = (c \to a) \wedge (c \to b)$
(e) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$
(f) $a * b = b * a$
(g) $(a * b) * c = a * (b * c)$
(h) $a * 1 = a$
(i) $(a \vee b) * c = (a * c) \vee (b * c)$
(j) $(a \to b) * a \leq b$, $b \leq a \to (a * b)$
(k) $a * b \leq c$ iff $b \leq a \to c$

As can be easily seen, axioms (a), (b), (k) are not expressed by equations. However conditions (a)–(e) can be proved from the other axioms [29], so the problem relates only to the *adjointness condition* expressed by axiom (k). An equational definition of this property has been given by Blount and Tsinakis, as shown in the following proposition.

Proposition 1.1. [5] *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ be a structure such that $\wedge, \vee, *, \to$ are binary operators on A and $0, 1$ are constant in A satisfying the following:*

(1) *$\langle A, \wedge, \vee, 0, 1\rangle$ is a complete lattice with least element 0 and greatest element 1 with respect to the lattice ordering $a \leq b$ iff $a \wedge b = a$;*
(2) *$\langle A, *, 1\rangle$ is a monoid, i.e., the binary operation $*$ on A is commutative, associative and for all $a \in A$, $1 * a = a$;*
(3) *the following properties are satisfied by all $a, b, c \in A$*
(A1) $a \wedge (b \to ((a * b) \vee c)) = a$
(A2) $((a \to b) * a) \vee b = b$
(A3) $(a \vee b) * c = (a * c) \vee (b * c)$

Then, $\mathcal{A}$ is a residuated lattice according to Definition 1.1 and vice versa, any residuated lattice satisfies points (1)–(3). Starting from this axiomatization and making explicit all the axioms characterizing lattices and monoids, we can obtain an independent set of axioms defining residuated lattices.

Theorem 1.1. *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0\rangle$ be a structure where $\wedge, \vee,$ and $\to$ are binary operators on A and 0 is a constant on A such that, once defined $1 := 0 \to 0$, the following are satisfied for all $a, b, c \in A$:*

(R1) $a \wedge (b \wedge c) = (b \wedge a) \wedge c$
(R2) $a \vee (b \vee c) = (b \vee a) \vee c$
(R3) $a \vee (b \wedge a) = a$
(R4) $0 \wedge a = 0$
(R5) $a * (b * c) = (b * a) * c$
(R6) $a * 1 = a$
(R7) $a \wedge (b \to ((a * b) \vee c)) = a$
(R8) $((a \to b) * a) \vee b = b$
(R9) $(a \vee b) * c = (a * c) \vee (b * c)$

Then, $\mathcal{A}$ is a residuated lattice and, vice versa, any residuated lattice according to Definition 1.1 satisfies (R1)–(R9).

Proof. Let us first prove that the substructure $\langle A, *, 1\rangle$ is a commutative monoid. It is sufficient to show that $a * b = b * a$. Let us set $c = 1$ in (R5): $a * (b * 1) = (b * a) * 1$. Then, by (R6) we get the commutative property of $*$.

Now we prove that the substructure $\langle A, \wedge, \vee, 0, 1\rangle$ satisfies all the axioms defining a bounded lattice (see for instance [3]).

$a \vee 0 = a$.
By (R4) applied to (R3) with $b = 0$.
$a \vee b = b \vee a$.
Simply, set $c = 0$ in (R2).

Before going on with the other lattice axioms, we prove that $0 * a = 0$.
Setting $b = 0$ in (R8) we get $(a \to 0) * a = 0$. By the last property and (R9) with $a := d \to 0$ and $c := d$, we have $((d \to 0) \vee b) * d = ((d \to 0) * d) \vee (b * d)$ and then $((d \to 0) \vee b) * d = b * d$. Now, if $b := 0$, then $(a \to 0) * a = 0 * a$, that is $0 = 0 * a$.
$a \wedge 1 = a$.
Let us set $c := 0$ in (R7): $a \wedge (b \to (a * b)) = a$. Setting also b to 0 and using $0 * a = 0$, we have $a \wedge (0 \to 0) = a$, i.e., $a \wedge 1 = a$.
$1 \vee a = 1$.
From (R3) with $a := 1$: $1 \vee (a \wedge 1) = 1$ and then $1 \vee a = 1$.

In order to prove the remaining axioms characterizing a lattice, we need to show that $1 \to a = a$ holds.
By setting $b = 1$ in (R9) we obtain, by $x \vee 1 = 1$ and (R6), $c = (a * c) \vee c$. Using this property in (R7) with $c = a$, we have $a \wedge (b \to a) = a$, which combined with (R3) gives

$$(b \to a) \vee a = b \to a \tag{*}$$

On the other hand, setting $a = 1$ in (R8) we get $(1 \to b) \vee b = b$. Finally by the last equation and (*) we get $1 \to a = a$.
$a \wedge (b \vee a) = a$.
Let us set $b = 1$ in (R7): $a \wedge (1 \to ((a * 1) \vee c)) = a$. Then, from $1 \to a = a$ and (R6) we get the thesis.
$a \wedge b = b \wedge a$.
Simply, set $c = 1$ in (R1).

This property concludes the proof that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice and hence that $\langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a residuated lattice. In order to prove the vice versa, it is sufficient to show that in any residuated lattice it holds $0 \to 0 = 1$. This property trivially follows from property (a) of Definition 1.1. □

Theorem 1.2. *The axioms (R1)–(R9) defining residuated lattices according to Theorem 1.1 are independent.*

Proof. For any axiom (Rx), we now define a structure which satisfies all the axioms (R1)–(R9) except (Rx).
(R1). Let us consider the three element set $\{0, a, 1\}$ as the support of the structure of Table 1.

Table 1. (R1)

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	0	1

$\vee$	0	a	1
0	1	a	1
a	a	a	1
1	1	1	1

$*$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

This structure satisfies all axioms but (R1), indeed $1 \wedge (a \wedge 1) = 1 \wedge a = 0 \neq a = a \wedge 1 = (a \wedge 1) \wedge 1$.
(R2). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 2.

TABLE 2. (R2)

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	0
1	1	1

$*$	0	1
0	1	0
1	0	1

$\to$	0	1
0	1	0
1	0	1

Then, (R2) is not satisfied: $0 \vee (1 \vee 0) = 0 \vee 1 = 0 \neq 1 = 1 \vee 0 = (1 \vee 0) \vee 0$.
(R3). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 3.

TABLE 3. (R3)

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	0	1

$*$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	1	1

Then, all axioms but (R3) are satisfied: $1 \vee (0 \wedge 1) = 1 \vee 0 = 0 \neq 1$.
(R4). Let us consider the set $(0, 1]$ with $\wedge, \vee$ defined according to the usual linear order on real numbers. Once defined $* = \wedge$ and $a \to b := b$ if $a > b$ and 1 otherwise, then the structure $\left\langle (0, 1], \wedge, \vee, \wedge, \to, \frac{1}{2} \right\rangle$ clearly satisfies all the axioms except (R4).
(R5). In order to show the independence of (R5), let us consider the set $\{0, a, 1\}$, endowed with the linear order $0 \leq a \leq 1$ and the lattice operators $\wedge, \vee$ defined in the usual manner. The remaining operators are defined according to Table 4.

TABLE 4. (R5)

$*$	0	a	1
0	0	0	0
a	0	1	a
1	0	1	1

$\to$	0	a	1
0	1	1	1
a	0	0	1
1	0	a	1

Then, (R5) is not satisfied: $a * (1 * a) = a * 1 = a \neq 1 = a * a = (a * 1) * a$.
(R6). Let us consider the structure based on $0 \leq 1$, where $*, \to$ are defined as in Table 5.

TABLE 5. (R6)

$*$	0	1
0	0	0
1	0	0

$\to$	0	1
0	1	1
1	1	1

Clearly, (R6) is not satisfied since $0 * 1 = 0 \neq 1$.
(R7). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 6.

Table 6. (R7)

$\wedge$	0	1
0	0	0
1	0	0

$\vee$	0	1
0	0	1
1	1	0

$*$	0	1
0	0	0
1	0	1

$\rightarrow$	0	1
0	1	0
1	0	0

This structure satisfies all axioms except (R7): $1 \wedge (0 \rightarrow (1 * 0) \vee 0) = 1 \wedge (0 \rightarrow 0) = 1 \wedge 1 = 0 \neq 1$.

(R8). Let us consider the structure based on $0 \leq 1$, where $*, \rightarrow$ are defined as in Table 7.

Table 7. (R8)

$*$	0	1
0	0	0
1	0	1

$\rightarrow$	0	1
0	1	1
1	1	1

Then, axiom (R8) is not satisfied: $((1 \rightarrow 0) * 1) \vee 0 = 1 \neq 0$.

(R9). Let us consider the structure based on $0 \leq 1$, where $*, \rightarrow$ are defined as in Table 8.

Table 8. (R9)

$*$	0	1
0	1	0
1	0	1

$\rightarrow$	0	1
0	1	1
1	0	1

Then, axiom (R9) is not satisfied: $(0 \vee 1) * 0 = 0 \neq 1 = (0 * 0) \vee (1 * 0)$. □

2. Hoops

In this section we introduce the notion of hoop [4]. We prove that the definition of hoop given in [16] is made of independent equations. Further, we show the relation occurring between hoops and residuated lattices. The structure of hoops will then be used in the next section in order to give an independent axiomatization of Basic Logic algebras.

Definition 2.1. [4] A structure $\langle A, *, \rightarrow, 1 \rangle$ is a *hoop* if

(1) the substructure $\langle A, *, 1 \rangle$ is a commutative monoid;
(2) the following identities hold for all $a, b, c \in A$:
  (a) $a \rightarrow a = 1$
  (b) $(a \rightarrow b) * a = (b \rightarrow a) * b$
  (c) $(a * b) \rightarrow c = a \rightarrow (b \rightarrow c)$

A hoop is *bounded* if there exists an element $0 \in A$ such that $0 \to a = 1$.

Let us remark that residuated lattices and hoops are incomparable. Indeed, not all hoops are residuated lattices since in general a hoop has not a lattice structure but a meet semi–lattice one with respect to the meet operator $a \wedge b := a * (a \to b)$ [4]. Vice versa, not all residuated lattices are hoops as can be seen in the following counterexample.

Example 2.1. Let us consider the totally ordered set made of the four elements $0 \leq a \leq b \leq 1$ and define $*$ and $\to$ according to Table 9.

TABLE 9. An example of residuated lattice which is not a hoop

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then, this structure is a residuated lattice [29] and not a hoop:

$$(a \to b) * a = 1 * a = a \neq 0 = a * b = (b \to a) * b.$$

Now, we give an independent axiomatization of hoops.

Proposition 2.1. [16]. *The structure $\langle A, \to, *, 1\rangle$ where $\to, *$ are binary operators on A and for all $a, b, c \in A$ the following are satisfied:*

(H1) $a * b = b * a$
(H2) $1 * a = a$
(H3) $a \to a = 1$
(H4) $a * (a \to b) = b * (b \to a)$
(H5) $(a * b) \to c = a \to (b \to c)$

is a hoop and vice versa.

Theorem 2.1. *Axioms (H1)–(H5) are independent.*

Proof. (H1). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 10.

TABLE 10. (H1)

$*$	0	1
0	0	1
1	0	1

$\to$	0	1
0	1	1
1	1	1

Then, (H1) is not satisfied: $0 * 1 = 1 \neq 0 = 1 * 0$.

(H2). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 11.

Then, all axioms but (H2) are satisfied since $1 * 1 = 0 \neq 1$.

(H3). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 12.

TABLE 11. (H2)

$*$	0	1
0	0	0
1	0	0

$\to$	0	1
0	1	1
1	1	1

TABLE 12. (H3)

$*$	0	1
0	0	0
1	0	1

$\to$	0	1
0	0	0
1	0	0

Then, all axioms but (H3) are satisfied since $0 \to 0 = 0 \neq 1$.
(H4). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 13.

TABLE 13. (H4)

$*$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	1	1

Then, axiom (H4) is not satisfied: $0*(0 \to 1) = 0*1 = 0 \neq 1 = 1*1 = 1*(1 \to 0)$.
(H5). Finally, the system described in Table 14 is an example of the independence of (H5).

TABLE 14. (H5)

$*$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	0
1	0	1

In effect, all axioms but (H5) are satisfied since $(0 * 0) \to 0 = 0 \to 0 = 1 \neq 0 = 0 \to 1 = 0 \to (0 \to 0)$. □

3. Basic Logic algebras

Basic Logic algebras are a strengthening of residuated lattices, and, as explained in the introduction, they play a fundamental role in the algebraic approach to many valued logics. Main task of this section is to give an independent set of axioms defining Basic Logic algebras.

Definition 3.1. [18]. A system $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a *Basic Logic (BL) algebra* if it is a residuated lattice satisfying the following further properties

(1) $a \wedge b = a * (a \to b)$
(2) $(a \to b) \vee (b \to a) = 1$

Another formulation of BL algebras can be obtained starting from hoops. This new set of axioms is made of less and independent properties, as we are going to show.

Proposition 3.1. [21, 1]. *Any bounded hoop satisfying also the property:*

$$((a \to b) \to c) \to (((b \to a) \to c) \to c) = 1$$

is a BL algebra and vice versa.

Proposition 3.2. *Let $\mathcal{A} = \langle A, \to, *, 0\rangle$ be a structure where $\to$ and $*$ are binary operators on A, $0 \in A$, such that, once defined $1 := 0 \to 0$, the following are satisfied for all $a, b, c \in A$:*

(BLH1) $a * b = b * a$
(BLH2) $1 * a = a$
(BLH3) $a * (a \to b) = b * (b \to a)$
(BLH4) $(a * b) \to c = a \to (b \to c)$
(BLH5) $0 \to a = 1$
(BLH6) $((a \to b) \to c) \to (((b \to a) \to c) \to c) = 1$

Then, $\mathcal{A}$ is a BL algebra and vice versa.

Proof. Due to Proposition 3.1, we simply need to show that from axioms (BLH1)–(BLH6) property (H3) $a \to a = 1$ can be proved. As a first step, let us show the following

$$a \to 1 = 1 \tag{1a}$$

$$(1 \to a) \to a = 1 \tag{1b}$$

Setting $b := a \to 0$ in (BLH4) we get $(a * (a \to 0)) \to c = a \to ((a \to 0) \to c)$. Then, by (BLH3) $(0 * (0 \to a)) \to c = a \to ((a \to 0) \to c)$. Now, using (BLH5) and (BLH2) we get $1 = a \to ((a \to 0) \to c)$. Applying the last property to itself with $c := (a \to 0) \to a$ we get $a \to 1 = 1$.
Setting $a = 1$ and $b = 0$ in (BLH6), $((1 \to 0) \to c) \to (((0 \to 1) \to c) \to c = 1$. Using (BLH2) and (BLH3) we obtain $((0 * (0 \to 1)) \to c) \to (((0 \to 1) \to c) \to c = 1$. Applying (BLH5) and (BLH2) we get $1 \to ((1 \to a) \to a) = 1$ and finally by (BLH4) and (BLH2) we have Equation (1b).
Now, setting $b := (a \to 1)$ and $c := a$ in (BLH4), $(a * (a \to 1)) \to a = a \to ((a \to 1) \to a)$. Using axioms (BLH3) and (BLH2) we get $(1 \to a) \to a = a \to ((a \to 1) \to a)$ and by Equations (1b) and (1a) we get $a \to (1 \to a) = 1$. On the other hand, setting $b := 1$ and $c := a$ in (BLH4), $a \to a = a \to (1 \to a)$. Hence, $a \to a = 1$. □

Before proving the independence of the above axioms, we introduce a stronger structure than BL algebras. SBL algebras were first defined in [15] in order to obtain as models based on the unit interval $[0, 1]$ the structures where $*$ is a triangular norm without non–trivial zero divisors [19]. The original definition, based on the one of BL algebra, is the following one.

Definition 3.2. [15] A structure $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a *Strict Basic Logic (SBL)* algebra if it is a BL algebra and satisfies the further axiom

$$(a * b) \to 0 = (a \to 0) \vee (b \to 0) \tag{SBL}$$

Clearly, due to Proposition 3.2 an equivalent definition of SBL algebras is based on the notion of hoop.

Definition 3.3. A structure $\mathcal{A} = \langle A, *, \to, 0\rangle$ is a SBL algebra if, once defined:

$$a \wedge b := a * (a \to b) \tag{2a}$$

$$a \vee b := [(a \to b) \to b] \wedge [(b \to a) \to a] \tag{2b}$$

it satisfies (BLH1)–(BLH6) and (SBL). This new set of axioms turns out to be an independent axiomatization of SBL algebras.

Theorem 3.1. *The axioms characterizing SBL algebras according to Definition 3.3, i.e., (BLH1)–(BLH6) and (SBL), are independent.*

Proof. (BLH1). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 15.

Table 15. (BLH1)

$*$	0	1
0	0	1
1	0	1

$\to$	0	1
0	1	1
1	1	1

Then, all axioms but (BLH1) are satisfied since $0 * 1 = 1 \neq 0 = 1 * 0$.
(BLH2). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 16.

Table 16. (BLH2)

$*$	0	1
0	0	1
1	1	1

$\to$	0	1
0	1	1
1	1	1

Then, all axioms but (BLH2) are satisfied since $1 * 0 = 1 \neq 0$.
(BLH3). Let us consider the two element set $\{0, 1\}$ as the support of the structure of Table 17.

Table 17. (BLH3)

$*$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	1	1

Then, all axioms but (BLH3) are satisfied since $0 * (0 \to 1) = 0 * 1 = 0 \neq 1 = 1 * 1 = 1 * (1 \to 0)$.
(BLH4). In order to show the independence of (BLH4), let us consider the set $\{0, a, 1\}$, endowed with the linear order $0 \leq a \leq 1$ and the lattice operators $\wedge, \vee$ defined in the usual manner. The remaining operators are defined according to Table 18.

Then, (BLH4) is not satisfied: $(a * a) \to a = a \neq 1 = a \to (a \to a)$.
(BLH5). Let us consider the set $C_\infty = \{1 = a^0, a, a^2, a^3, \ldots\}$ ordered by $a^i \leq a^j$ if

TABLE 18. (BLH4)

$*$	0	a	1
0	0	0	0
a	0	1	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$j \geq i$. On this set, let us define the operations $a^n * a^m := a^{n+m}$ and $a^n \to a^m := a^{\max\{m-n,0\}}$. Then, the structure $\langle C_\infty, *, \to, 1\rangle$ is a hoop according to Definition 2.1 (see [4]). Thus, $\langle C_\infty, *, \to, a\rangle$, once defined $1 := a \to a$, satisfies (BLH1)–(BLH4). Let us show that $\langle C_\infty, *, \to, a\rangle$ satisfies also (BLH6) and (SBL) but not (BLH5). Let us suppose that $n < m$. Then

$$((a^m \to a^n) \to a^l) \to (((a^n \to a^m) \to a^l) \to a^l) = (1 \to a^l) \to (((a^{m-n}) \to a^l) \to a^l)$$

Now, let us set $m - n = c \geq 1$. Then, we have two subcases $l \leq c$ and $l > c$. If $l \leq c$, then, $a^l \to ((a^c \to a^l) \to a^l) = a^l \to ((1 \to a^l) \to a^l) = 1$. If $l > c$, then, $a^l \to ((a^c \to a^l) \to a^l) = a^l \to a^{l-(l-c)} = a^l \to a^c = 1$ since, by hypothesis, $l > c$. The last case, i.e., $m \leq n$ is trivial, so (BLH6) is satisfied. Now, let us suppose that at least one between m, n is not equal to 0. Then, both the left and right hand side of (SBL) are equal to 1. On the other hand, if $m = n = 0$ then, $1 \to a = (1 \to a) \vee (1 \to a)$. Finally, (BLH5) is not satisfied since $a \to a^2 = a \neq 1$.
(BLH6). Let us consider the structure based on the set $\{0, a, 1\}$ whose operators are defined in Table 19.

TABLE 19. (BLH6)

$\wedge$	0	a	1
0	0	0	0
a	0	1	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	1	a
1	1	a	1

$*$	0	a	1
0	0	0	0
a	0	1	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	a	1
1	0	a	1

In this structure, (BLH6) does not hold:

$$((0 \to 0) \to a) \to (((0 \to 0) \to a) \to a) = (1 \to a) \to ((1 \to a) \to a) = a \neq 1.$$

(SBL). It is well known that not all BL algebras are also SBL algebras (see [15]). □

Trivially, as a corollary, we obtain the independence of BL axioms.

Corollary 3.1. *The axioms defining BL algebras according to Proposition 3.2, i.e., (BLH1)–(BLH6), are independent.*

Thus, due to this corollary and to the results of [13], it is possible to conclude that properties (BLH1)–(BLH6) are the necessary and sufficient ones to characterize a continuous t–norm and its residuum $(*, \to)$ on the unit interval $[0, 1]$.

Summarizing, we simplified the axiomatization of BL algebras and residuated lattices, showed the independence of their axioms and also of the ones defining Hoops. The relationship among all these structure is shown in the following diagram

$$\begin{tikzcd}
& SBL \arrow[d] & \\
& BL \arrow[dl] \arrow[dr] & \\
Hoop & & RL
\end{tikzcd}$$

We are now going to consider some enrichment of these algebras with a further, involutive negation.

4. Residuated lattices with involution

In literature, several approaches to residuated lattices with an involutive operator have been proposed. The more general notion of residuated lattices with involution we are going to consider is the one of extended residuated lattices as given in [25]. A strengthening of this structure can be found in [17], where a very general notion of residuated algebra is taken into consideration and the possibility to add an involution to it is investigated. To our purposes, we will only consider the structure of (bounded) involutive residuated lattice. Finally, following [15] an involutive negation is added to SBL algebras. The axioms defining this new structure are quite strong and beyond the notion of involutive operator. Here, we simply study this algebra, even if, it will also be of some interest the study of a SBL algebra with a "pure" involutive negation, in particular, in order to analyze the possibility to define a rough approximation. Indeed, at the end of this section we will introduce and discuss rough approximation operators in SBL$_\neg$ algebras and ER–lattices. We will see that SBL$_\neg$ algebras are powerful enough to define an abstract rough approximation space whereas ER–lattices are not. Finally, it is showed that the collection of all rough approximations on a SBL$_\neg$ algebra has an Heyting Wajsberg algebraic structure.

4.1. **Extended residuated lattices.** In [25] an antitone involution is added to residuated lattices in order to give an algebraic approach to fuzzy rough sets [24]. Here, we give an equational and independent axiomatization of this new algebra.

Definition 4.1. [25] A structure $\langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1\rangle$ is an *Extended Residuated lattice* (ER–lattice) if the substructure $\langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a residuated lattice and $\neg : A \rightarrow A$ a unary operation such that:

(ER1) $\neg\neg a = a$
(ER2) $a \leq b$ implies $\neg b \leq \neg a$
(ER3) $\neg 0 = 1$

Let us slightly modify the above definition giving an equational set of axioms characterizing ER–lattices.

Proposition 4.1. *A structure* $\langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1\rangle$ *such that* $\langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ *is a residuated lattice and* $\neg$ *satisfies the following properties:*

(ER1) $\neg\neg a = a$
(ER2a) $\neg(a \vee b) = \neg a \wedge \neg b$

is an ER–lattice and vice versa.

Proof. We prove that under condition (ER1), properties (ER2) and (ER2a) are mutually equivalent.

From $\{a, b\} \leq a \vee b$ and (ER2) we have $\neg(a \vee b) \leq \{\neg a, \neg b\}$ and so $\neg(a \vee b) \leq \neg a \wedge \neg b$. Moreover, $\neg a \wedge \neg b \leq \{\neg a, \neg b\}$. Then, from (ER1) we get $\{a, b\} = \{\neg\neg a, \neg\neg b\} \leq \neg(\neg a \wedge \neg b)$ and so $a \vee b \leq \neg(\neg a \wedge \neg b)$ from which it follows $\neg a \wedge \neg b \leq \neg(a \vee b)$ concluding that $\neg(a \vee b) = \neg a \wedge \neg b$, that is (ER2a).
Vice versa, let (ER2a) be true and suppose $a \leq b$. Then, $b = a \vee b$ and $\neg b = \neg(a \vee b) = \neg a \wedge \neg b$, that is $\neg b \leq \neg a$.

Finally, let us show that (ER3) holds. Trivially, $\neg 0 \leq 1$. By $0 \leq \neg 1$ and (ER2) we get $\neg\neg 1 \leq \neg 0$, that is $1 \leq \neg 0$. □

Finally, we give an independent axiomatization of ER–lattices, which consists in all the axioms defining Residuated Lattices (Theorem 1.1) except (R2) and (ER1), (ER2a).

Proposition 4.2. *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \neg, 0\rangle$ be a structure where $\wedge, \vee,$ and $\rightarrow$ are binary operators on A, $\neg$ is a unary operator on A and 0 is a constant on A such that, once defined $1 := 0 \rightarrow 0$, the following are satisfied for all $a, b, c \in A$:*

(R1) $a \wedge (b \wedge c) = (b \wedge a) \wedge c$
(R3) $a \vee (b \wedge a) = a$
(R4) $0 \wedge a = 0$
(R5) $a * (b * c) = (b * a) * c$
(R6) $a * 1 = a$
(R7) $a \wedge (b \rightarrow ((a * b) \vee c)) = a$
(R8) $((a \rightarrow b) * a) \vee b = b$
(R9) $(a \vee b) * c = (a * c) \vee (b * c)$
(ER1) $\neg\neg a = a$
(ER2a) $\neg(a \vee b) = \neg a \wedge \neg b$

Then, $\mathcal{A}$ is an ER lattice and vice versa.

Proof. The only thing to show is that axiom (R2): $a \vee (b \vee c) = (b \vee a) \vee c$ holds. By (R1) we get $\neg a \wedge (\neg b \wedge \neg c) = (\neg b \wedge \neg a) \wedge \neg c$. Applying (ER2a) two times, we obtain $\neg(a \vee (b \vee c)) = \neg((b \vee a) \vee c)$. Finally, by double negation property (ER1), we have the thesis. □

Theorem 4.1. *The axioms defining ER–lattices according to Proposition 4.2 are independent.*

Proof. (R1). Let us consider the two element set $\{0, 1\}$ as the support of the structure where $\neg a = a$ and the other operators are defines as in Table 20.

Table 20. (R1)

$\wedge$	0	1
0	0	0
1	1	1

$\vee$	0	1
0	0	0
1	1	1

$*$	0	1
0	1	0
1	0	1

$\rightarrow$	0	1
0	1	0
1	0	1

Then, all axioms but (R1) are satisfied since $0 \wedge (1 \wedge 0) = 0 \wedge 1 = 0 \neq 1 = (1 \wedge 0) \wedge 0$.
(R3). Let us consider the set $\{0, a, 1\}$ and define the operators according to Table 21.

TABLE 21. (R3)

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	0	0
a	0	a	1
1	0	1	1

$*$	0	a	1
0	0	0	a
a	a	a	a
1	0	a	1

$\rightarrow$	0	a	1
0	1	a	a
a	1	1	1
1	1	a	1

x	0	a	1
$\neg x$	0	1	a

This structure does not satisfy (R3): $1 \vee (0 \wedge 1) = 1 \vee 0 = 0 \neq 1$.
(R4). Let us consider the set $A = (-\infty, +\infty)$ and define $\wedge$ and $\vee$ according to the usual order on real numbers, that is, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. If we define $a * b := a + b$, $a \rightarrow b := b - a$ and $\neg a := -a$, then, it is easy to check that the structure $\langle A, \wedge, \vee, *, \rightarrow, 0\rangle$ satisfies all axioms but (R4).

In case of axioms (R5),(R6), (R8) and (R9) the examples are exactly as in Theorem 1.1 with $\neg a$ defined as $\neg 0 = 1$, $\neg 1 = 0$ and $\neg a = a$.
(R7). Let us consider the two element $\{0, 1\}$ and define the lattice operator according to the trivial order $0 \leq 1$, $*$ and $\rightarrow$ according to Table 22 and $\neg 0 = 1$.

TABLE 22. (R7)

$*$	0	1
0	0	0
1	0	1

$\rightarrow$	0	1
0	1	0
1	0	0

The property (R7) in not satisfied in this example since $1 \wedge (0 \rightarrow ((1 * 0) \vee 1)) = 0 \rightarrow 1 = 0 \neq 1$.
(ER1). Let us consider the set $\{0, 1\}$ and define the operators according to Table 23.

TABLE 23. (ER1)

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	0
1	1	1

$*$	0	1
0	1	0
1	0	1

$\rightarrow$	0	1
0	1	0
1	0	1

x	0	1
$\neg x$	0	0

Clearly, $\neg\neg 1 = 0 \neq 1$.
(ER2). If we consider the same structure as the one of the previous counterexample except $\neg$, which is defined as $\neg a = a$, then $\neg(1 \vee 0) = 1 \neq 0 = \neg 1 \wedge \neg 0$. $\square$

4.2. **Involutive Residuated Lattices.** A stronger structure than ER–lattices, where the residuum $\rightarrow$ and the involutive negation $\neg$ are strictly linked, has been defined in [17].

Definition 4.2. [17] A structure $\langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1\rangle$ is an *involutive residuated lattice* (IR–lattice) if the substructure $\langle A, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ is a residuated lattice and $\neg : A \rightarrow A$ a unary operation such that:

(IR1) $\neg\neg a = a$
(IR2) $a \to b = \neg b \to \neg a$

Proposition 4.3. *Any involutive residuated lattice is also an extended residuated lattice.*

Proof. We need to show that in any IR–lattice it holds property (ER2). Let us suppose that $a \leq b$. Then, $a \to b = 1$ and by (IR2) $a \to b = 1 = \neg b \to \neg a$, i.e., $\neg b \leq \neg a$. □

The converse of the above proposition does not hold, i.e., there exists an ER–lattice which is not an IR–lattice.

Example 4.1. Let us consider the ER–lattice whose Hasse diagram is shown in Figure 1 and $*$ and $\to$ operators are defined according to Table 24.

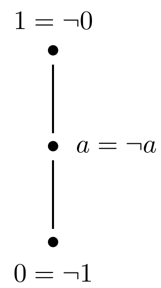

FIGURE 1

TABLE 24. An example of ER–lattice which is not an IR–lattice

$*$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

In this structure, property (IR2) does not hold:

$$1 \to a = a \neq 0 = a \to 0 = \neg a \to \neg 1.$$

Further, IR–lattices have not an hoop structure. Indeed, if in Example 2.1 we define $\neg$ as in Table 25 then, the resulting structure is an IR–lattice (and of course not a hoop as proved in Section 2).

TABLE 25. An example of IR–lattice which is not a hoop

a	0	a	b	1
¬a	1	b	a	0

As a corollary of this result, we can deduce also the following facts:

(1) ER–lattices have not a hoop structure (otherwise, also IR–lattices will have);
(2) IR–lattices have not a BL nor SBL structure (otherwise, they will have a hoop structure).

In the following, an independent set of axioms for IR–lattices is given.

Proposition 4.4. *A structure $\langle A, \wedge, \vee, *, \to, 0, 1\rangle$ satisfying axioms (R1)–(R9) and (IR2) and once defined for all $a \in A$, $\neg a := a \to 0$, is an involutive IR–lattice and vice versa.*

Proof. We have to prove that (IR1) holds.
By $1 \to a = a$ and (IR2) we get $\neg a \to \neg 1 = a$, using axioms (R7) and (R8) we get $a \wedge ((a \to b) \to b) = a$. If, in the last property, we set $a := \neg c$ and $b := \neg 1$, then $\neg c \wedge ((\neg c \to \neg 1) \to \neg 1) = \neg c$ and then $\neg c \wedge (c \to \neg 1) = \neg c$. Now, if $c := \neg d$, we have $\neg\neg d \wedge d = \neg\neg d$, i.e., $\neg\neg d \leq d$. The last property also implies that $\neg\neg d \to d = 1$ and then $\neg\neg\neg d \to \neg d = 1$. Using (IR2) we get, $d \to \neg\neg d = 1$, i.e., $d \leq \neg\neg d$ concluding the first part of the proof.

To show the vice versa, it is needed to prove that in any IR–lattice it holds $\neg a := a \to 0$. This result follows from [17]. □

Theorem 4.2. *The axioms defining IR–lattices according to Proposition 4.4 are independent.*

Proof. (R1). Let us consider the set $\{0, a, 1\}$ and define the operators according to Table 26.

TABLE 26. (R1)

∧	0	a	1
0	0	0	0
a	0	1	0
1	0	a	a

∨	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

∗	0	a	1
0	0	0	0
a	0	a	1
1	0	0	a

→	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Clearly, (R1) is not satisfied, since $1 \wedge (a \wedge 1) = 0 \neq a = (a \wedge 1) \wedge 1$.
(R2). Let us define a structure based on the two element set $\{0, 1\}$ and with operators defined as in Table 2 and $\neg a = a$.

Then, as also proved in Theorem 1.1, (R2) is not satisfied: $0 \vee (1 \vee 0) = 0 \vee 1 = 0 \neq 1 = 1 \vee 0 = (1 \vee 0) \vee 0$.
(R3). Let us consider the two element set $\{0, 1\}$ as the support of the structure where $\neg a$ is defined as $\neg a = 0$ and remaining operators are defined in Table 3 .

Then, all axioms but (R3) are satisfied: $1 \vee (0 \wedge 1) = 1 \vee 0 = 0 \neq 1$.
(R4). Let us consider the set $A = (-\infty, +\infty)$ and define the operators as in Theorem 4.1. Then, it is easy to check that the structure $\langle A, \wedge, \vee, *, \to, 0\rangle$ satisfies also axiom (IR2).
(R5). In order to show the independence of (R5), let us consider the set $\{0, a, 1\}$, endowed with the linear order $0 \leq a \leq 1$ and the lattice operators $\wedge, \vee$ defined in the usual manner. The remaining operators are defined according to Table 27.

In this structure, (R5) in not satisfied: $1*(a*a) = 1*0 = 0 \neq 1 = 1*a = (1*a)*a$.
In case of axioms (R6)–(R9), let us consider the same structures and counterexamples of Theorem 1.1, where the operator $\neg$ is defined in case of axiom (R6) and

TABLE 27. (R5)

$*$	0	a	1
0	0	0	0
a	0	1	0
1	0	1	1

$\to$	0	a	1
0	1	1	1
a	a	a	1
1	0	a	1

a	0	a	1
$\neg$a	1	a	0

(R8) as $\neg a = 0$, in case of (R7) as $\neg a = a$ and for (R9) as $0 = \neg 1, 1 = \neg 0$ and $a = \neg a$.
(R9) same as in Theorem 4.1.
(IR2) Let us consider the residuated lattice based on the two elements $0 \leq 1$. Once define $\neg a = a$, (IR2) is the only axiom not satisfied. $\square$

4.3. **SBL$_\neg$ algebra.** Finally, a new unary operator is added to SBL algebras. The axioms characterizing it are developed in order to define a unary operator ν as the algebraic counterpart of the Baaz operator Δ [2].

Definition 4.3. [15] A system $\mathcal{A} = \langle A, \wedge, \vee, *, \to, \neg, 0, 1\rangle$ is a *SBL$_\neg$ algebra* if

1. $\langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a *SBL* algebra;
2. $\neg$ is a unary operator such that, once defined $\sim a := a \to 0$ and $\nu(a) := \sim\neg a$, the following are satisfied:
   - (SBL$_\neg$1) $\neg\neg a = a$
   - (SBL$_\neg$2) $\sim a \leq \neg a$
   - (SBL$_\neg$3) $\nu(a \to b) = \nu(\neg b \to \neg a)$
   - (SBL$_\neg$4) $\nu(a) \vee \sim\nu(a) = 1$
   - (SBL$_\neg$5) $\nu(a \vee b) \leq \nu(a) \vee \nu(b)$
   - (SBL$_\neg$6) $\nu(a) * (\nu(a \to b)) \leq \nu(b)$

Proposition 4.5. *Any SBL$_\neg$ algebra is an ER–lattice.*

Proof. Trivially, any SBL$_\neg$ algebra is a residuated lattice, (ER1) is property (SBL$_\neg$1) and (ER2) holds in all SBL$_\neg$ algebras as proved in [15]. $\square$

The vice versa does not hold since in Subsection 4.2 we showed that ER–lattices do not even have an hoop structure. Further, SBL$_\neg$ algebras and IR lattices are incomparable. In fact, we have already showed that IR–lattices are not SBL (and hence SBL$_\neg$) algebras, that the vice versa does not hold is proved in the following counterexample.

Example 4.2. Let us consider the *SBL$_\neg$* algebra based on the real unit interval $[0, 1]$, where the lattice operators are defined according to the usual order on real numbers, and the other operators are defined as:

$$a * b := a \wedge b$$

$$a \to_G b := \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

$$\neg a := 1 - a$$

That is, $*$ is the minimum t–norm and $\to$ the induced Gödel implication.
Trivially, $\langle [0, 1], \wedge, \vee, \wedge, \to_G, \neg, 0\rangle$ is a SBL$_\neg$ algebra [15]. However, it is not an

IR–lattice, since (IR2) is not satisfied:

$$0.7 \to_G 0.4 = 0.4 \neq 0.3 = 0.6 \to_G 0.3 = \neg 0.4 \to_G \neg 0.7\,.$$

We now give some properties of SBL algebras which will be useful to simplify the axiomatization of SBL$_\neg$ algebras.

Lemma 4.1. *Let $\mathcal{A}$ be a SBL algebra. Then, the following properties hold for all $a, b, c \in A$:*

(1) $a \to (b \to c) = b \to (a \to c)$
(2) $a \to ((a \to b) \to c) = b \to ((b \to a) \to c)$
(3) $\sim a \vee \sim b = a \to \sim b$
(4) $a \to (b \to a) = 1$
(5) $a \to \sim a = \sim a$
(6) $a * \sim a = 0$
(7) $\sim a \to (a \to b) = 1$

Proof. (1) It follows from (BLH4) and the commutative property of $*$.

(2) By BLH4 we have $a \to ((a \to b) \to c) = (a * (a \to b)) \to c$. Using (BLH3), $a \to ((a \to b) \to c) = (b * (b \to a)) \to c$ and again by (BLH4), $a \to ((a \to b) \to c) = b \to ((b \to a) \to c)$.
(3) By (SBL) and (BLH4) we have $\sim a \vee \sim b = (a * b) \to 0 = a \to \sim a$.
(4) It easily follows from (1): $a \to (b \to a) = b \to (a \to a) = b \to 1 = 1$.
(5) Setting $b := a$ in (3): $\sim a = \sim a \vee \sim a = a \to \sim a$.
(6) Let us set $b := 0$ in (BLH3): $a * (a \to 0) = 0 * (0 \to a)$. By (BLH5) and definition of $\sim$ we get $a * \sim a = 0 * 1 = 0$.
(7) Setting $a := \sim d$ and $b := d$ in (BLH4) we have $(\sim d * d) \to c = \sim d \to (d \to c)$. By the above property (6) and (BLH5) we obtain the thesis: $\sim d \to (d \to c) = 0 \to c = 1$.

□

Theorem 4.3. *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \to, \neg, 0, 1\rangle$ be a structure such that*

(1) *$\langle A, \wedge, \vee, *, \to, 0, 1\rangle$ is a SBL algebra;*
(2) *$\neg$ is a unary operator such that, once defined $\sim a = a \to 0$ and $\nu(a) = \sim\neg a$, the following are satisfied:*
 (SBL$_\neg$1) $\neg\neg a = a$
 (SBL$_\neg$3) $\nu(a \to b) = \nu(\neg b \to \neg a)$
 (SBL$_\neg$6) $\nu(a) * (\nu(a \to b)) \leq \nu(b)$

Then, $\mathcal{A}$ is a SBL$_\neg$ algebra and vice versa.

Proof. We have to prove that from (1) and (2) properties (SBL$_\neg$2), (SBL$_\neg$4) and (SBL$_\neg$5) can be proved.

(SBL$_\neg$4) Setting $a := \nu(c)$, $b := \neg c$ in Lemma 4.1(3) we get $\sim\nu(c) \vee \nu(c) = \nu(c) \to \nu(c) = 1$.

(SBL$_\neg$5) We show that $\nu(a \vee b) = \nu(a) \vee \nu(b)$. By Lemma 4.1(3) with $a := \neg c$ and $b := \neg d$ we have $\nu(c) \vee \nu(d) = \neg c \to \nu(d)$. In order to obtain the thesis we are going to prove that $\neg c \to \nu(d) = \nu(c \vee d)$.
By Lemma 4.1(4) and $\nu(1) = 1$ we have $\nu(a \to (b \to a)) = 1$. Using axioms (SBL$_\neg$1) and (SBL$_\neg$3): $\nu(\neg(b \to \neg a) \to a) = 1$. By this property and (SBL¬6) with $a := \neg(c \to \neg b)$ we obtain $\sim (c \to \neg b) \vee \nu(b) = \nu(b)$ and

using Lemma 4.1(3) we get

(*) $$(a \to \neg b) \to \nu(b) = \nu(b)$$

Further, by Lemma 4.1(5) and Lemma 4.1(1) we can derive

(**) $$\neg a \to (b \to \nu(a)) = b \to \nu(a)$$

By Lemma 4.1(2) we easily derive $a \to ((a \to \neg b) \to \nu(b)) = \neg b \to ((\neg b \to a) \to \nu(b))$ and applying (*) to the left hand side and (**) to the right hand side we obtain $a \to \nu(b) = (\neg b \to a) \to \nu(b)$. Applying Lemma 4.1(1) we get $\neg b \to ((\neg b \to a) \to 0) = a \to \nu(b)$ and from Lemma 4.1(2)

(***) $$a \to ((a \to \neg b) \to 0) = a \to \nu(b)$$

Now, by $\wedge$ and $\vee$ definitions we get $\neg(a \vee b) = \neg a \wedge \neg b = \neg a * (\neg a \to \neg b)$. By (BLH4) $\neg(a \vee b) \to 0 = (\neg a * (\neg a \to \neg b)) \to 0 = \neg a \to ((\neg a \to \neg b) \to 0)$. Finally, by (***), we get $\neg a \to \nu(b) = \nu(a \vee b)$.

(SBL$_\neg$2) Setting $b := c \to d$ and $a := {\sim}c$ in (SBL$_\neg$6): $(\nu({\sim}c) * \nu({\sim}c \to (c \to d)) \vee \nu(c \to d) = \nu(c \to d)$. Applying Lemma 4.1(7) we get $(\nu({\sim}c) * \nu(1)) \vee \nu(c \to d) = \nu(c \to d)$ and then $\nu({\sim}c) \vee \nu(c \to d) = \nu(c \to d)$. Now, remembering that ${\sim}c = c \to 0$ and applying SBL$_\neg$3 we have $\nu(c \to d) = \nu(c \to d) \vee \nu(1 \to \neg c)$, that is $\nu(c \to d) = \nu(c \to d) \vee {\sim}c$ and by Lemma 4.1(3) with $b := \neg(c \to d)$ we obtain

(*) $$\nu(c \to d) = c \to \nu(c \to d)$$

Further, from Lemma 4.1(7) with $a := \neg c$ we get $\nu(\nu(c) \to (\neg c \to d)) = \nu(1) = 1$ and applying (*) and (SBL$_\neg$3): $1 = \nu(c) \to \nu(\neg c \to b) = \nu(c) \to \nu(\neg b \to c) = \nu(c) \to \nu(d \to c) =$. By this last property and (*) we have:

(**) $$\nu(\nu(a) \to a) = \nu(a) \to \nu(\nu(a) \to a) = 1$$

By axioms (SBL) and (BLH4) we get $(a \to 0) \vee (b \to 0) = a \to (b \to 0)$ and clearly, when $a = b$, ${\sim}a = a \to {\sim}a$ and then, setting $a := \neg b$, $\nu(b) = \neg b \to \nu(b)$. On the other hand, by Lemma 4.1(6) with $a := \neg b$, we obtain $\neg b * \nu(b) = 0$. Now, setting $a := \neg c$ and $b := \nu(c)$ in (BLH3): $\neg a \to \nu(a)) * \neg a = \nu(a) * (\nu(a) \to \neg a)$. Thus, by the above properties, we have $\nu(a) * \neg a = \nu(a) * (\nu(a) \to \neg a)$ and then $\nu(a) * (\nu(a) \to \neg a) = 0$. Setting $a := \nu(b) \to b$ we get $\nu(\nu(b) \to b) * (\nu(\nu(b) \to (b)) \to \neg(\nu(b) \to b)) = 0$ and using (**), $1 * (1 \to \neg(\nu(b) \to b)) = 0$, that is $\neg(\nu(b) \to b) = 0$ or equivalently $\nu(b) \to b = 1$. If $b := \neg a$ then we obtain the thesis: ${\sim}b \to \neg b = 1$, that is ${\sim}b \leq \neg b$.

□

The axiomatization of SBL$_\neg$ algebra given in Theorem 4.3 is the best one we can give at the moment, but we are not able to say if it is an independent one or not. Indeed, we are able to show that all the axioms are independent from the others except axiom (SBL$_\neg$6) which we are neither able to derive from the others. This is clearly an interesting open problem.

4.4. **Rough Approximation Spaces.** In all the structures studied in this section there are simultaneously two negations: the primitive (involutive) one $\neg$, and the derived one $\sim a := a \to 0$. This distinction is not meaningful in IR–lattices since as already said, we have $\sim a := a \to 0 = \neg a$. But in the other two cases, it is worthwhile to consider also the two operators

$$\nu(a) := \sim\neg a$$
$$\mu(a) := \neg \sim a$$

4.4.1. *Rough approximation in SBL$_\neg$ algebras.* We have already seen that, in SBL$_\neg$ algebras, ν has some interesting properties. Taking into account also the operator μ, it is possible to define on this structure a rough approximation space, as we are going to explain [10].

Let $\langle A, \wedge, \vee, *, \to, \neg, 0, 1\rangle$ be a SBL$_\neg$ algebra. The operator $\nu : A \to A$, that associates to any element a from A its *interior* $\nu(a) := \sim\neg a$ is a *topological interior operator*, i.e., the following conditions hold, for any $a, b \in A$:

(I1)	$\nu(1) = 1$	(normalization)
(I2)	$\nu(a) \leq a$	(decrease)
(I3)	$\nu(a) = \nu(\nu(a))$	(idempotent)
(I4)	$\nu(a \wedge b) = \nu(a) \wedge \nu(b)$	(multiplicative)

Dually, the operator $\mu : A \to A$, that associates to any element a from A its *closure* $\mu(a) := \neg \sim a$ is a *topological closure operator*, i.e., the following conditions hold, for any $a, b \in A$:

(C1)	$\mu(0) = 0$	(normalization)
(C2)	$a \leq \mu(a)$	(increase)
(C3)	$\mu(a) = \mu(\mu(a))$	(idempotent)
(C4)	$\mu(a \vee b) = \mu(a) \vee \mu(b)$	(multiplicative)

We can define the subset of *open elements*, resp., *closed elements*, as the collection of all elements which are equal to their interior, resp., closure:

$$\mathbb{O}(A) = \{a \in A : a = \nu(a)\}$$
$$\mathbb{C}(A) = \{a \in A : a = \mu(a)\}$$

It can be proved [10] that these two sets of elements coincide: $\mathbb{O}(A) = \mathbb{C}(A)$. Thus, if an element is closed, it is also open and vice versa. In the sequel, we will simply call an open (closed) element *exact* or *sharp* and denote their collection as

$$A_e = \{a \in A : a = \mu(a)\} = \{a \in A : a = \nu(a)\}.$$

These operators give a rough approximation of any element $a \in A$ by sharp definable elements. In fact, $\nu(a)$ turns out to be the best approximation from the bottom of a by sharp elements. To be precise, for any element $a \in A$ the following holds:

(L1) $\nu(a)$ is a sharp element ($\nu(a) \in A_e$).
(L2) $\nu(a)$ is an *inner* (*lower*) approximation of a ($\nu(a) \leq a$).
(L3) $\nu(a)$ is the best inner approximation of a by sharp elements (let $e \in A_e$ be such that $e \leq a$, then $e \leq \nu(a)$).

By properties (L1)–(L3), it follows that the interior of an element a can be expressed in the following form:

$$\nu(a) = \max\{x \in A_e : x \leq a\}$$

Dually, $\mu(a)$ is the best approximation from the top of a by exact elements:

(U1) $\mu(a)$ is a sharp element ($\mu(a) \in A_e$).
(U2) $\mu(a)$ is an *outer* (*upper*) approximation of a ($a \leq \mu(a)$).
(U3) $\mu(a)$ is the best outer approximation of a by sharp elements (let $f \in A_e$ be such that $a \leq f$, then $\mu(a) \leq f$).

By properties (U1)–(U3), it follows that the closure of an element a can be expressed in the following form:

$$\mu(a) = \min\{y \in A_e : a \leq y\}$$

Definition 4.4. Given a SBL$_\neg$ algebra $\mathcal{A}$ the induced *rough approximation space* is the structure $\langle A, A_e, \nu, \mu\rangle$ consisting of the set A of all approximable elements, the set A_e of all *exact* elements, and the *inner* (resp., *outer*) *approximation map* $\nu : A \to A_e$ (resp., $\mu : A \to A_e$).
For any element $a \in A$, its *rough approximation* is defined as the pair of exact (sharp) elements:

$$r(a) := \langle \nu(a), \mu(a)\rangle \quad \text{with} \quad \nu(a) \leq a \leq \mu(a).$$

So the map $r : A \to A_e \times A_e$ approximates an unsharp (fuzzy) element by a pair of sharp (crisp, exact) ones representing its inner and outer sharp approximation, respectively. Equivalently, it is possible to identify the rough approximation of a with the (*inner, outer*) pair:

$$r_\perp(a) := \langle \nu(a), \neg\mu(a)\rangle = \langle \nu(a), {\sim}a\rangle.$$

The operators ν and μ can also be considered as modal–like operators on a non–Boolean algebra. Indeed, they satisfy all the axioms of S_5 modal systems and hence ν can be interpreted as the *necessity* and μ as the *possibility* (see [10]). Consequently, for any element $a \in A$, $\neg\mu(a) = {\sim}a$ represents the *impossibility* of a.

Let us, now, consider the collection of all rough approximations $r_\perp(a)$ on a SBL$_\neg$ algebra $\mathcal{A}$:

$$\mathbb{R}(A) = \{\langle \nu(a), {\sim}a\rangle : a \in A\} \tag{3}$$

It is possible to endow $\mathbb{R}(A)$ with the structure of an HW algebra [6, 10], i.e., a structure pasting together Heyting and Wajsberg algebras.

Definition 4.5. A system $\mathcal{A} = \langle A, \to_L, \to_G, 0\rangle$ is an *Heyting Wajsberg* (*HW*) *algebra* if A is a non empty set, $0 \in A$, and $\to_L, \to_G$ are binary operators such that, if one defines

$$1 := 0 \to_L 0 \tag{4a}$$

$$\neg a := a \to_L 0 \tag{4b}$$

$$\sim a := a \to_G 0 \tag{4c}$$

$$a \wedge b := \neg((\neg a \to_L \neg b) \to_L \neg b) \tag{4d}$$

$$a \vee b := (a \to_L b) \to_L b \tag{4e}$$

then, the following are satisfied whatever be $a, b, c \in A$:

(HW1) $a \rightarrow_G a = 1$
(HW2) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
(HW3) $a \wedge (a \rightarrow_G b) = a \wedge b$
(HW4) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
(HW5) $1 \rightarrow_L a = a$
(HW6) $a \rightarrow_L (b \rightarrow_L c) = \neg(a \rightarrow_L c) \rightarrow_L \neg b$
(HW7) $\neg \sim a \rightarrow_L \sim\sim a = 1$
(HW8) $(a \rightarrow_G b) \rightarrow_L (a \rightarrow_L b) = 1$

In order to show that $\mathbb{R}(A)$ has an HW structure let us set, for the sake of simplicity, $a_i := \nu(a)$ and $a_e := \sim a$.

Proposition 4.6. *Let $\mathcal{A}$ be a SBL$_\neg$ algebra and $\mathbb{R}(A)$ the collection of all rough approximation on A according to equation* (3). *Once defined,*

$$\langle a_i, a_e \rangle \Rightarrow_L \langle b_i, b_e \rangle := \langle (\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, a_i \wedge b_e \rangle$$
$$\langle a_i, a_e \rangle \Rightarrow_G \langle b_i, b_e \rangle := \langle (\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, \neg a_e \wedge b_e \rangle$$

then, the structure $\langle \mathbb{R}(A), \Rightarrow_L, \Rightarrow_G, (0,1) \rangle$ is an HW algebra.

Proof. The proof can be indirectly obtained considering that SBL$_\neg$ algebras are a model of so called de Morgan Brouwer Zadeh (BZ^{dM}) lattices [10] and that the collection of all rough approximations definable in any BZ^{dM} lattice exactly as $\langle \nu(a), \sim a \rangle$, has an HW algebraic structure [9]. □

Clearly, $\Rightarrow_L, \Rightarrow_G$ behaves, respectively, as a Łukasiewicz and a Gödel implication. From these primitive operators it is also possible to derive, according to Definition 4.5, the lattice operators

$$\langle a_i, a_e \rangle \sqcap \langle b_i, b_e \rangle := \langle a_i \wedge b_i, a_e \vee b_e \rangle$$
$$\langle a_i, a_e \rangle \sqcup \langle b_i, b_e \rangle := \langle a_i \vee b_i, a_e \wedge b_e \rangle$$

and the two negations, the fuzzy and the intuitionistic one, of an element $\langle a_i, a_e \rangle$ are, respectively:

$$-\langle a_i, a_e \rangle := \langle a_i, a_e \rangle \Rightarrow_L \langle 0, 1 \rangle = \langle a_e, a_i \rangle = r_\perp(\neg a)$$
$$\approx \langle a_i, a_e \rangle := \langle a_i, a_e \rangle \Rightarrow_G \langle 0, 1 \rangle = \langle a_e, \neg a_e \rangle = r_\perp(\sim a)$$

4.4.2. *Rough approximation in ER–lattices.* Since we have proved that ER–lattices are a weakening of SBL$_\neg$ algebras, it is natural to ask which properties are preserved by ν and μ and if it is possible to define a rough approximation space also in ER–lattices.

We are going to show that the results holding for SBL$_\neg$ algebras can only partially be extended to ER–lattices. In particular, the operators ν and μ do not define a rough approximation space in the sense of Definition 4.4.

Proposition 4.7. *Let $\langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1 \rangle$ be an ER–lattice. Once defined $\sim a := a \rightarrow 0$, $\nu(a) := \sim\neg a$ and $\mu(a) := \neg \sim a$, properties (I1), (I4), (O1), (O4) are satisfied.*

Proof. Trivial, since in any residuated lattice it holds $\sim (a \vee b) = \sim a \wedge \sim b$. □

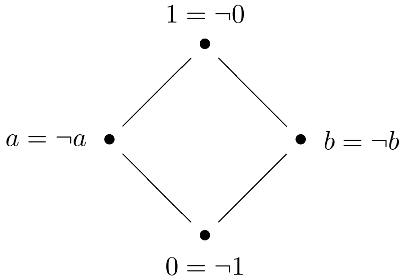

FIGURE 2. An example of ER–lattice: the lattice structure

On the other hand, properties (I2), (I3) and (C2), (C3) are not generally satisfied as can be seen in the following counterexample.

Example 4.3. Let us consider the four element $\{0, a, b, 1\}$ ER–lattice whose Hasse diagram is represented in Figure 2 and operators in Table 28.

TABLE 28. An example of ER–lattice: the operators

$*$	0	a	b1	
0	0	0	0	0
a	0	a	0	1
b	0	0	b	1
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

In this structure, (I2) and (C2) are not satisfied:

$$\nu(a) = \neg a \rightarrow 0 = a \rightarrow 0 = b \not\leq a$$
$$\mu(a) = \neg(a \rightarrow 0) = \neg b = b \not\geq a$$

Further, also (I3) and (C3) are not satisfied:

$$\nu(\nu(a)) = \nu(b) = \neg b \rightarrow 0 = b \rightarrow 0 = a \neq b = \nu(a)$$
$$\mu(\mu(a)) = \mu(b) = \neg(b \rightarrow 0) = \neg a = a \neq b = \mu(a)$$

Properties (I2) and (O2) have a fundamental role in order to define a rough approximation. Indeed, if the semantics of the lower approximation is to approximate a vague element a with a sharp one, $\nu(a)$, representing its degree of "necessity", then $\nu(a)$ must imply a or, in some sense, must be included in a. Dually, a must imply its upper approximation representing its degree of "possibility". Thus, due to the lack of the order chain $\nu(a) \leq a \leq \mu(a)$, in ER–lattices it is not possible to define a rough approximation in the sense of Definition 4.4.

We note that Radzikowska and Kerre define a rough approximation on the ER–lattice of L–fuzzy sets [25] which satisfy properties (I2) and (O2). However, their rough approximation is not based purely on ER–lattice operators but it uses knowledge external to ER–lattices. In particular a rough approximation on a L–fuzzy set is constructed using a given L–fuzzy relation.

So, in order to define a rough approximation in an algebraic way in residuated lattices with involution, it is necessary to understand which axioms are needed to obtain properties (I1)–(I4) and (O1)–(O4). This work will require the definition of new algebras between ER–lattices and SBL$_\neg$ algebras, as well as an independent axiomatization of SBL$_\neg$ algebras. We will tackle this interesting open problem in a future work.

5. Conclusions

We considered several residuated algebras which can be found in literature. In each case we reduced the number of the axioms trying to give an independent axiomatization of the involved structure. This operation has been successful for all algebras except SBL$_\neg$ algebras, whose number of axioms have been reduced even if it is still an open problem their independence.

Furthermore, the relationship among all the considered structures has been studied and the results are drawn in the following diagram:

$$
\begin{tikzcd}
 & SBL_\neg \arrow[dl] \arrow[ddr] & \\
SBL \arrow[d] & & IR \arrow[d] \\
BL \arrow[d] \arrow[dr] & & ER \arrow[dl] \\
Hoop & RL &
\end{tikzcd}
$$

We also remark that in case of ER–lattices, an equational definition of the involution has been given.

Finally, we considered rough approximation spaces in residuated lattices with involution, showing that on SBL$_\neg$ algebras it is possible to define a rough approximation space whereas it is not possible on ER–lattices. This fact leads to the open problem of defining a minimal structure able to characterize a rough approximation. Another open problem is an independent axiomatic scheme for SBL$_\neg$ algebras and, of course, there are other structures that could be taken into account, both at the bottom of our hierarchy, for instance, the general notions of left and right residuated lattices [5], and at the top, for instance, product algebra [18] and its strengthening.

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Dipartimento di Informatica, Sistemistica e Comunicazione, Università di Milano–Bicocca, Via Bicocca degli Arcimboldi 8, I–20126 Milano (Italy)

E-mail address: `ciucci@disco.unimib.it`