ALGEBRAIC STRUCTURES RELATED TO MANY VALUED LOGICAL SYSTEMS
PART II: EQUIVALENCE AMONG SOME WIDESPREAD STRUCTURES

G. CATTANEO, D. CIUCCI, R. GIUNTINI, AND M. KONIG

Abstract. Several algebraic structures (namely HW, BZMV^{dM}, Stonean MV and MV$_\Delta$ algebras) related to many valued logical systems are considered and their equivalence is proved. Four propositional calculi whose Lindenbaum–Tarski algebra corresponds to the four equivalent algebraic structures are axiomatized and their semantical completeness is given.

Introduction

Any propositional calculus gives a algebra (of equivalent propositions) which belongs to a certain variety. For instance, the Lindenbaum-Tarski algebra of classical (intuitionistic, resp.) logic belongs to the variety generated by all Boolean (Heyting, resp.) algebras. As shown by C.C.Chang [9], the Lindenbaum-Tarski algebra of Łukasiewicz $\aleph_0$-logic belongs to the variety of all MV-algebras. Conversely, let us consider a variety K whose members are algebraic structures (of the same type) conveniently equipped with a finite set of operations. The variety K can be thought as the algebraic counterpart of a logical calculus whose Lindenbaum-Tarski algebra belongs to K.

In this paper, after recalling the definition of HW-algebra (introduced in part I) we will define three classes of algebraic structure (MV$_\Delta$ algebras, Stonean MV algebras and BZMV^{dM}) which we prove to be term-equivalent to HW-algebras.

In the final part we investigate the logical systems associated to these algebraic structures.

Thus, the results obtained for one of these term-equivalent structures, can be applied, mutas mutandi, to the others. In particular, Hajek's completeness for MV$_\Delta$ algebras [12] can be reported in terms of the logical systems based on HW-algebra, Stonean MV algebra and BZMV^{dM}-algebra.

As a corollary of our equivalence theorems we obtain that these logical calculi turn out to have, up to syntactical translation, the same set of tautologies.

1. Heyting Wajsberg algebras and equivalent structures

We introduce and study the new structure of Heyting Wajsberg (HW) algebra [4, 5]. Its originality consists in the presence of two implication connectives as primitive operators. Further some other structures are defined; in the next section they are proved to be equivalent to HW algebras.

1.1. **HW algebras.**

Definition 1.1. A system $\mathcal{A} = \langle A, \to_L, \to_G, 0\rangle$ is a *Heyting Wajsberg* (HW) algebra if A is a non empty set, $0 \in A$ and $\to_L, \to_G$ are binary operators, such that, once defined

$$1 := \neg 0 \tag{1a}$$

$$\neg a := a \to_L 0 \tag{1b}$$

$$\sim a := a \to_G 0 \tag{1c}$$

$$a \wedge b := \neg((\neg a \to_L \neg b) \to_L \neg b) \tag{1d}$$

$$a \vee b := (a \to_L b) \to_L b \tag{1e}$$

the following are satisfied:

(HW1) $a \to_G a = 1$
(HW2) $a \to_G (b \wedge c) = (a \to_G c) \wedge (a \to_G b)$
(HW3) $a \wedge (a \to_G b) = a \wedge b$
(HW4) $(a \vee b) \to_G c = (a \to_G c) \wedge (b \to_G c)$
(HW5) $1 \to_L a = a$
(HW6) $a \to_L (b \to_L c) = \neg(a \to_L c) \to_L \neg b$
(HW7) $\neg \sim a \to_L \sim\sim a = 1$
(HW8) $(a \to_G b) \to_L (a \to_L b) = 1$

Proposition 1.1. [5] *In any HW algebra we can define a partial order relation in one of the following mutually equivalent ways:*

$$a \leq b \quad \stackrel{def}{\Longleftrightarrow} \quad a \to_L b = 1 \tag{2a}$$

$$\Longleftrightarrow \quad a \to_G b = 1 \tag{2b}$$

$$\Longleftrightarrow \quad a \wedge b = a \tag{2c}$$

Moreover, the substructure $\mathcal{A}_L := \langle A, \wedge, \vee, 0\rangle$ involving the two binary operators (1d) *and* (1e) *is a distributive lattice where $a \wedge b$ (resp., $a \vee b$) turns out to be the glb (resp., lub) of the generic pair $a, b \in A$ with respect to the partial order relation defined by* (2)*. This lattice is bounded by the least element* 0 *and the greatest element* 1*:*

$$\forall a \in A : 0 \leq a \leq 1$$

Finally, we recall that in any HW algebra $\mathcal{A}$, it is possible to introduce the modal–like operators of necessity and possibility [5], defined, for all $a \in A$, respectively as $\nu(a) := \sim\neg a$ and $\mu(a) := \neg \sim a$.

1.2. **Stonean MV algebras.**

We first recall the definition of a MV algebra, in the independent axiomatization given in [8], which is equivalent to the original Chang's definition [9].

Definition 1.2. A structure $\mathcal{A} = \langle A, \oplus, \neg, 0\rangle$ where $\oplus$ in a binary operator on A, $\neg$ in a unary operator on A, is a MV algebra if the following axioms are satisfied:

(MV1) $(a \oplus b) \oplus c = b \oplus (c \oplus a)$
(MV2) $a \oplus 0 = a$
(MV3) $a \oplus \neg 0 = \neg 0$
(MV4) $\neg(\neg a) = a$

(MV5) $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$

From the $\oplus$ operator, it is possible to derive a meet and join lattice operators as follows:

$$a \vee b := \neg(\neg a \oplus b) \oplus b \tag{3}$$

$$a \wedge b := \neg(\neg(a \oplus \neg b) \oplus \neg b) \tag{4}$$

Of course, also the partial order relation can be induced by the $\oplus$ operator as:

$$a \leq b \quad \text{iff} \quad \neg a \oplus b = 1 \quad \text{iff} \quad a \wedge b = a \tag{5}$$

The minimum element with respect to this order is 0 and the greatest is $1 := \neg 0$. Finally, the following two binary operators are introduced in MV algebras:

$$a \odot b := \neg(\neg a \oplus \neg b) \tag{6a}$$

$$a \to_L b := \neg a \oplus b \tag{6b}$$

Let $\mathcal{A}$ be an MV algebra. The set of all the $\oplus$–idempotent elements of A is called the set of *Boolean elements* of A and it is denoted by $A_{e,\oplus}$:

$$A_{e,\oplus} := \{x \in A \mid x \oplus x = x\}$$

Proposition 1.2. [9] *Let $\mathcal{A}$ be a MV algebra. Then, the collection $A_{e,\oplus}$ of all its Boolean elements satisfies the following properties:*

(1) The lattice connectives coincide with the MV ones:

$$\forall e, f \in A_{e,\oplus}, \quad e \wedge f = e \odot f \quad \textit{and} \quad e \vee f = e \oplus f$$

(2) The structure $\mathcal{A}_e = \langle A_{e,\oplus}, \wedge, \vee, \neg, 0\rangle$ is a Boolean lattice (algebra), which is the largest MV subalgebra of $\mathcal{A}$ that is at the same time a Boolean algebra with respect to the same operations $\wedge(= \odot)$, $\vee(= \oplus)$.

Definition 1.3. A MV algebra $\mathcal{S} = \langle A, \oplus, \neg, 0, 1\rangle$ is said to be *Stonean* (SMV for short) iff $\mathcal{S}$ satisfies the following condition:

$$\forall a \in A : \exists e_a \in A_{e,\oplus} : \{b \in A : a \wedge b = 0\} = \{b \in A : b \leq e_a\}.$$

It can be easily observed that for any $a \in A$, the corresponding Boolean element e_a of the above definition is trivially unique.

1.3. **Brouwer Zadeh Many Valued algebras.** By a pasting of BZ lattices and MV algebras one obtains so–called BZMV algebras [6, 7].

Definition 1.4. A *Brouwer Zadeh Many Valued* (BZMV) algebra is a system $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$, where A is a non empty set, 0 is a constant, $\neg$ and $\sim$ are unary operators, $\oplus$ is a binary operator, obeying the following axioms:

(BZMV1) $(a \oplus b) \oplus c = (b \oplus c) \oplus a$
(BZMV2) $a \oplus 0 = a$
(BZMV3) $\neg(\neg a) = a$
(BZMV4) $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$
(BZMV5) $\sim a \oplus \sim\sim a = \neg 0$
(BZMV6) $a \oplus \sim\sim a = \sim\sim a$
(BZMV7) $\sim\neg[(\neg(\neg a \oplus b) \oplus b)] = \neg(\sim a \oplus \sim\sim b) \oplus \sim\sim b$

A *de Morgan* BZMV (BZMV^{dM}) algebra, is a BZMV algebra $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ where axiom (BZMV7) is replaced by the following:

(BZMV7') $\sim\neg[(\neg(a \oplus \neg b) \oplus \neg b)] = \neg(\sim\sim a \oplus \neg\sim\sim b) \oplus \neg\sim\sim b$

Remark 1. Analogously, it is possible to define BZW and BZW^{dM} algebras as a pasting of BZ (resp., BZ^{dM}) lattices and Wajsberg algebras [3]. Clearly BZW and BZMV are equivalent structures.

In [6], it has been shown the following result.

Proposition 1.3. *Let $\langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV algebra, then the substructure $\langle A, \oplus, \neg, 0\rangle$ is a MV algebra according to Definition 1.2.* Thus in any BZMV algebra, it is defined a lattice structure, which is, moreover, a BZ lattice as stated in the next theorem.

Proposition 1.4. [6]. *Let $\mathcal{A}$ be a BZMV (resp., BZMV^{dM}) algebra. The substructure $\langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ is a distributive BZ (resp., BZ^{dM}) lattice with respect to the partial order relation defined in* (5).

Turning our attention to sharp elements (see [5]), we note that property $a \oplus (\nu(\mu(a)) = (\nu(\mu(a)))$ holds but in general it is not valid the stronger idempotence law $a \oplus a = a$.

Example 1.1. Let us consider the BZMV^{dM} algebra based on the unit interval: $\langle [0, 1], \oplus, \neg, \sim, 0\rangle$, where the operators are defined as usual as [5]

$$a \oplus b = \min\{1, a + b\}$$
$$\neg a = 1 - a$$
$$\sim a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, for all elements $a \in (0, 1)$, $a \oplus a \neq a$.

So, it is possible to consider the collection of MV Boolean elements $A_{e,\oplus}$, i.e., the idempotent elements with respect to the $\oplus$ operator, as a new set of sharp elements. The relation among this set of exact elements and the ones introduced in [5] is the following:

Proposition 1.5. [6]. *Let $\mathcal{A}$ be a BZMV algebra. Then*

$$A_{e,B} = A_{e,M} \subseteq A_{e,\oplus} = A_{e,\neg}.$$

Let $\mathcal{A}$ be a BZMV^{dM} algebra. Then

$$A_{e,B} = A_{e,M} = A_{e,\oplus} = A_{e,\neg}.$$

We now list a set of properties of BZMV^{dM} algebras which will be useful later.

Lemma 1.1. [7]. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then, the following hold for all $a, b \in A$:*

(i) $\sim(a \wedge b) = \sim a \vee \sim b$.
(ii) $a \wedge b = 0$ *iff* $a \leq \sim b$.
(iii) $a \wedge \sim b = a \odot \sim b$; $a \vee \sim b = a \oplus \sim b$.
(iv) $a \wedge b = 0$ *implies* $\sim\sim a \wedge b = 0$.
(v) $\sim(a \oplus b) = \sim a \odot \sim b$.

Lemma 1.2. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then the following hold:*

(i) $a \wedge \sim(a \odot \neg b) \leq b$.

(ii) $b \wedge \sim\sim(a \odot \neg b) \leq a$.

Proof. (i) We prove $\neg[a \wedge \sim(a \odot \neg b)] \oplus b = 1$. Then, by an MV property, we can conclude $a \wedge \sim(a \odot \neg b) \leq b$.

$$\begin{aligned}
\neg[a \wedge \sim(a \odot \neg b)] \oplus b &= [\neg a \vee \sim\sim(a \odot \neg b)] \oplus b \\
&= \neg a \oplus \sim\sim(a \odot \neg b) \oplus b && \text{Lemma 1.1(iii)} \\
&= \neg(a \odot \neg b) \oplus \sim\sim(a \odot \neg b) \\
&\geq \sim(a \odot \neg b) \oplus \sim\sim(a \odot \neg b) \\
&= 1
\end{aligned}$$

(ii) By an MV property, $(a \odot \neg b) \wedge (\neg a \odot b) = 0$. By Lemma 1.1(ii):

$$(\neg a \odot b) \leq \sim(a \odot \neg b) \quad (*)$$

We want to show that $\neg[b \wedge \sim\sim(a \oplus \neg b)] \oplus a = 1$.

$$\begin{aligned}
\neg[b \wedge \sim\sim(a \oplus \neg b)] \oplus a &= \neg[\neg b \vee \sim\sim(a \odot \neg b)] \oplus a \\
&= \neg b \oplus \sim(a \odot \neg b) \oplus a && \text{Lemma 1.1(iii)} \\
&= \neg(\neg a \odot b) \oplus \sim(a \odot \neg b) \\
&\geq \neg(\neg a \odot b) \oplus (\neg a \odot b) && (*) \\
&= 1
\end{aligned}$$

□

1.4. **MV$_\Delta$ algebras.** We have shown that BZMV algebras can be seen as a strengthening of MV algebras. In literature one can also find another structure, namely MV$_\Delta$ algebra [12], which is a MV algebra with a further unary operator, Δ.

Definition 1.5. A MV$_\Delta$ algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, 0, 1, \Delta\rangle$ such that $\langle A, \oplus, \neg, 0, 1\rangle$ is a MV algebra and once defined $\odot$ and $\rightarrow_L$ as in Equations (6), Δ is a unary operation on A satisfying the following axioms:

(Δ1) $\Delta a \vee \neg\Delta a = 1$
(Δ2) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$
(Δ3) $\Delta a \leq a$
(Δ4) $\Delta a \leq \Delta\Delta a$
(Δ5) $\Delta a \odot \Delta(a \rightarrow_L b) \leq \Delta b$
(Δ6) $\Delta 1 = 1$

In order to show the equivalence between BZMV^{dM} algebras and MV$_\Delta$ algebras we give some properties of MV$_\Delta$ algebras.

Lemma 1.3. *Let $\mathcal{A}$ be a MV$_\Delta$ algebra. Then for all $a, b \in A$:*

(i) $a \leq b$ *implies* $\Delta a \leq \Delta b$
(ii) $\Delta(a \vee b) = \Delta a \vee \Delta b$
(iii) $\neg a \wedge \Delta a = 0$
(iv) $a \wedge b = 0$ *implies* $a \leq \Delta(\neg b)$
(v) $\Delta a \oplus \Delta a = \Delta a$

Proof. (i) Suppose that $a \wedge b = a$. Then:

$$\begin{aligned}
\Delta(a) &= \Delta a \odot 1 && (\Delta 6) \\
&= \Delta a \odot \Delta 1 && \neg a \oplus (a \vee b) = 1 \\
&= \Delta a \odot \Delta(\neg a \oplus (a \vee b)) && a \leq b \\
&= \Delta a \odot \Delta(\neg a \oplus b) && (\Delta 5) \\
&\leq \Delta b
\end{aligned}$$

(ii) By ($\Delta 2$) it holds $\Delta(a \vee b) \leq \Delta a \vee \Delta b$. Now, $a, b \leq a \vee b$. Then, by (i) $\Delta a, \Delta b \leq \Delta(a \vee b)$. Thus, $\Delta a \vee \Delta b \leq \Delta(a \vee b)$.
(iii) By axiom ($\Delta 3$), $a \geq \Delta a$. Thus, $a \vee \neg(\Delta a) \geq \Delta a \vee \neg(\Delta a) = 1$ (Axiom ($\Delta 1$). Hence: $\Delta a \vee \neg(\Delta a) = 1$. Therefore, $\neg a \wedge \Delta a = 0$.
(iv) Suppose $a \wedge b = 0$.

$$\begin{aligned}
a \wedge \Delta(\neg b) &= (a \wedge \Delta(\neg b)) \vee 0 && (iii) \\
&= (a \wedge \Delta(\neg b)) \vee (a \wedge \Delta(\neg a)) && \text{distributivity of } \wedge \text{ over } \vee \\
&= a \wedge [\Delta(\neg b) \vee \Delta(\neg a)] && (ii) \\
&= a \wedge [\Delta(\neg b \vee \neg a)] \\
&= a \wedge [\Delta(\neg(a \wedge b))] && \text{Hypothesis} \\
&= a \wedge \Delta 1 && (\Delta 6) \\
&= a \wedge 1 = a
\end{aligned}$$

(v) It follows form the MV property $a \wedge \neg a = 0$ iff $a = a \oplus a$, and from Axiom ($\Delta 1$). ☐

2. Equivalence theorems

In this section, the equivalence among all the algebras introduced in Section 1 is proved. Let us recall that the equivalence between SMV and BZMV^{dM} algebras has already been proved in [7]. Here, we prove the equivalence between MV$_\Delta$ and BZMV^{dM} algebras and BZMV^{dM} and HW algebras.

2.1. Equivalence among BZMV^{dM} and MV$_\Delta$ algebras.

Lemma 2.1. *Let $\mathcal{A}$ be a MV$_\Delta$ algebra. Then:*

$$\forall b \in A: \quad \{a |\ a \wedge b = 0\} = \{a |\ a \leq \Delta(\neg b)\}$$

Proof. Suppose $a \wedge b = 0$. By Lemma 1.3(iv), $a \leq \Delta(\neg b)$.
Suppose $a \leq \Delta(\neg b)$. Then

$$\begin{aligned}
a \wedge b &= (a \wedge \Delta(\neg b)) \wedge b \\
&= a \wedge [\Delta(\neg b) \wedge b] && \text{Lemma 1.3(iii)} \\
&= a \wedge 0 = 0
\end{aligned}$$

☐

Proposition 2.1. *Let $\mathcal{A}$ be a MV$_\Delta$ algebra. Then $\mathcal{A}$ is a Stonean MV algebra.*

Proof. By Lemma 1.3(v), $\forall a \in A: \quad \Delta a = \Delta a \oplus \Delta a$. Thus, by Lemma 2.1, we can conclude that A is a Stonean MV algebra. ☐

In [7] a proof of the equivalence of Stonean MV and BZMV^{dM} algebras is given. As a consequence, the following theorem holds.

Theorem 2.1. *Let $\mathcal{A} = \langle A, \oplus, \neg, 0, \Delta\rangle$ be a MV$_\Delta$ algebra and define $\sim a := \Delta(\neg a)$. Then the structure $\mathcal{A}_{(BZMV^{dM})} = \langle A, \oplus, \neg, \sim, 0\rangle$ is a BZMV^{dM} algebra.* Moreover, also the other direction of the equivalence can be proved:

Theorem 2.2. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Once set $\Delta a := \sim\neg a = \nu(a)$, then the structure $\mathcal{A}_{MV_\Delta} = \langle A, \oplus, \neg, 0, \Delta\rangle$ is a MV$_\Delta$ algebra.*

Proof. We prove only ($\Delta 5$) because the proves for the other axioms are trivial. We have to prove that $\sim\neg a \odot \sim\neg(\neg a \oplus b) \leq \sim\neg b$. Now, since $\sim\neg a \leq a$, we have $\sim\neg a \odot \sim\neg(\neg a \oplus b) \leq a \odot \sim(a \odot \neg b)$. By Lemma 1.1(iii), we get $a \odot \sim(a \odot \neg b) = a \wedge \sim(a \odot \neg b)$, and by Lemma 1.2(i), $a \wedge \sim(a \odot \neg b) \leq b$. By a BZMV property $\sim\neg[a \wedge \sim(a \odot \neg b)] = \sim\neg a \wedge (\sim(a \odot \neg b))$. Thus, $\sim\neg(\sim\neg a \odot \sim\neg(\neg a \oplus b)) \leq \sim\neg a \wedge \sim(a \odot \neg b) \leq \sim\neg b$ and by Lemma 1.1(v) we have the thesis. □

Further, the two structures are term equivalent.

Proposition 2.2. *(1) Given any BZMV^{dM} algebra $\mathcal{A}$ there holds*

$$\mathcal{A} = (\mathcal{A}_{MV_\Delta})_{BZMV^{dM}}$$

(2) Conversely, given any MV$_\Delta$ algebra $\mathcal{A}$, there holds:

$$\mathcal{A} = (\mathcal{A}_{BZMV^{dM}})_{MV_\Delta}$$

Proof. It trivially follows by Theorems 2.1 and 2.2. □

2.2. Equivalence between BZMV^{dM} and HW algebras.

Theorem 2.3. *Let $\mathcal{A}$ be a HW algebra. Then, once defined $a \oplus b := \neg a \rightarrow_L b$ the structure $\mathcal{A}_{BZMV^{dM}} = \langle A, \oplus, \neg, \sim, 1\rangle$ is a BZMV^{dM} algebra.*

Proof. In [5] it has been shown that any HW algebra has as a substructure a Wajsberg algebra. It is well known that Wajsberg algebras are equivalent to Chang's MV algebras [18, p. 45], hence axioms (BZMV1)–(BZMV4) are satisfied. Axiom (BZMV5) can easily be obtained by axiom (HW7). We show, now, that (BZMV6) holds.

Using the non contradiction law for the Brouwer negation, we have:

$$\begin{aligned} 0 &= a \wedge \sim a && \text{Def. } \wedge \\ &= \neg((\neg a \rightarrow_L \neg\sim a) \rightarrow_L \neg\sim a) && (in) \\ &= \neg((\neg a \rightarrow_L \sim\sim a) \rightarrow_L \sim\sim a) \end{aligned}$$

And now, it is sufficient to apply $\neg\neg a = a$, $0 = \neg 1$ and the definition of $\oplus$. Finally, as proved in [6], under conditions (BZMV1)–(BZMV6), axiom (BZMV7') is equivalent to the following two:

$$\neg\sim a = \sim\sim a$$
$$\sim(a \wedge b) = \sim a \vee \sim b$$

which have been proved in [5]. □

Now, we show how HW algebras can be obtained as a substructure of BZMV^{dM} algebras. First, we give some results concerning BZMV^{dM} algebras which will be useful in proving the equivalence theorem.

Theorem 2.4. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then, $\forall a, b, x \in A$ the following condition holds:*

$$x \wedge a \leq b \quad \textit{iff} \quad x \leq \sim(a \odot \neg b) \oplus b.$$

Proof. $\Rightarrow$. Suppose $x \wedge a \leq b$. Then $\neg b \leq \neg x \vee \neg a$. Thus, using MV properties, $x \odot \neg b \leq x \odot (\neg x \vee \neg a) = (x \odot \neg x) \vee (x \odot \neg a) = x \odot \neg a$. By an MV property, $(x \odot \neg a) \wedge (\neg x \odot a) = 0$. Thus, by Lemma 1.1(ii) $(x \odot \neg a) \leq \sim(\neg x \odot a)$. Hence: $x \odot \neg b \leq \sim(\neg x \odot a)$. If we prove that $\sim(\neg x \odot a) \leq \sim(a \odot \neg b)$, then, we can conclude that $(x \odot \neg b) \leq \sim(a \odot \neg b)$ and therefore $\neg x \oplus b \oplus \sim(a \odot \neg b) = 1$, so that $x \leq b \oplus \sim(a \odot \neg b)$. We prove now $\sim(\neg x \odot a) \leq \sim(a \odot \neg b)$. By Lemma 1.2(i), $a \wedge \sim(a \odot \neg x) \leq x$. Thus, by hypothesis, $a \wedge \sim(a \odot \neg x) \leq a \wedge x \leq b$. Therefore,

$$\begin{aligned}
0 &= [a \wedge \sim(a \odot \neg x)] \odot b && \text{Lemma 1.1(iii)}\\
&= [a \odot \sim(a \odot \neg x)] \odot b &&\\
&= (a \odot \neg b) \odot \sim(a \odot \neg x) && \text{Lemma 1.1(iii)}\\
&= (a \odot \neg b) \wedge \sim(a \odot \neg x)
\end{aligned}$$

By Lemma 1.1(ii), $\sim(a \odot \neg x) \leq \sim(a \odot \neg b)$.
$\Leftarrow$. Suppose $x \leq \sim(a \odot \neg b) \oplus b$. Then, by Lemma 1.1(iii), $x \wedge a \leq a \wedge [\sim(a \odot \neg b) \oplus b] = a \wedge [\sim(a \odot \neg b) \vee b]$ and by distributivity properties $a \wedge [\sim(a \odot \neg b) \vee b] = [a \wedge \sim(a \odot \neg b)] \vee (a \wedge b)$. By Lemma 1.2(i), $a \wedge \sim(a \odot \neg b) \leq b$. Hence: $[a \wedge \sim(a \odot \neg b)] \vee (a \wedge b) \leq b \vee (a \wedge b) = b$. Thus, $x \wedge a \leq b$. □

Corollary 2.1. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then:*

$$\sim(a \odot \neg b) \oplus b = \vee\{x \in A | a \wedge x \leq b\}$$

Proof. By Theorem 2.4, $\sim(a \odot \neg b) \oplus b$ is an upper bound of $\{x \in A | a \wedge x \leq b\}$. If we prove that $\sim(a \odot \neg b) \oplus b \in \{x \in A | a \wedge x \leq b\}$, then we can conclude that $\sim(a \oplus \neg b) \oplus b$ is the least upper bound of $\{x \in A | a \wedge x \leq b\}$. We show $a \wedge [\sim(a \odot \neg b) \oplus b] \leq b$. As a first step, we apply Lemma 1.1(iii) to $a \wedge [\sim(a \odot \neg b) \oplus b]$:

$$\begin{aligned}
a \wedge [\sim(a \odot \neg b) \oplus b] =& a \wedge [\sim(a \odot \neg b) \vee b] && \text{Distributivity of } \wedge \text{ over } \vee\\
=& [a \wedge \sim(a \odot \neg b)] \vee (a \wedge b) && \text{Lemma 1.2(i)}\\
=& b \vee (a \wedge b)\\
=& b
\end{aligned}$$

□

Theorem 2.5. *Let $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Let*

$$a \to_L b := \neg a \oplus b$$
$$a \to_G b := \sim(a \odot \neg b) \oplus b = \sim\neg(\neg a \oplus b) \oplus b$$

Then, the structure $\mathcal{A}_{HW} = \langle A, \to_G, \to_L, 0\rangle$ is a HW algebra.

Proof. Clearly, $1 = 0 \to_L 0 = \neg 0$, $a \to_L 0 = \neg a$ and

$$a \to_G 0 = \sim(a \odot 1) \oplus 0 = \sim a.$$
$$a \wedge b := \neg((\neg a \to_L \neg b) \to_L \neg b) = (a \oplus \neg b) \odot b.$$
$$a \vee b := (a \to_L b) \to_L b = (a \odot \neg b) \oplus b.$$

By Corollary 2.1 and MV properties, the structure $\langle A, \to_G, \wedge, \vee, 0\rangle$ is a pseudo-Boolean algebra [16] and therefore a Heyting algebra [14, 15]. Thus, Axioms $(HW1)$–$(HW4)$ are satisfied.
(HW5). $1 \to_L a = \neg(0 \to_L 0) \oplus a = a$.
(HW6). It follows from the associative property of $\oplus$.
(HW7). It is Axiom (BZMV5).
(HW8). We prove $(a \to_G b) \odot \neg(a \to_L b) = 0$.

$$\begin{aligned}(a \to_G b) \odot \neg(a \to_L b) &= [\sim(a \odot \neg b) \oplus b] \odot \neg(\neg a \oplus b) \\ &= [\sim(a \odot \neg b) \vee b] \odot (a \odot \neg b) && \text{Lemma 1.1(iii)} \\ &= [(a \odot \neg b) \odot \sim(a \odot \neg b)] \vee [(a \odot \neg b) \odot b] && \text{distributivity of } \odot \text{ over } \vee \\ &= 0 \vee 0 = 0\end{aligned}$$

□

Theorem 2.6. *(1) For any BZMV^{dM} algebra $\mathcal{A}$, there holds*

$$\mathcal{A} = (\mathcal{A}_{HW})_{BZMV^{dM}}$$

(2) Conversely, for any HW algebra $\mathcal{A}$, there holds:

$$\mathcal{A} = (\mathcal{A}_{BZMV^{dM}})_{HW}$$

Proof. (1)
Let $\mathcal{A} =< A, \oplus, \neg, \sim, 0 >$ be a BZMV^{dM}. In Theorem 2.5 we have obtained that the structure $\mathcal{A}_{HW} =< A, \to_L, \to_G, 0 >$ is a HW algebra. Moreover, if we define $\forall a, b \in A$:

$$a' := a \to_L 0$$
$$a^* := a \to_G 0$$
$$a \oplus^\star b := a' \to_L b$$

we have introduced a new structure which by Theorem 2.3 is a BZMV^{dM}. Trivially, we have that $\forall a, b \in A$, $a' = \neg a$ and $a \oplus^\star b = a \oplus b$. We have only to show that $a^* = \sim a$:

$$a^* = a \to_G 0 = \sim(a \odot \neg 0) \oplus 0 = \sim a.$$

(2)
Let $\mathcal{A} =< A, \to_L, \to_G, 0 >$ be a HW algebra. We recall that $a \leq b$ iff $a \to_L b = 1$. By Theorem 2.3 the structure $\mathcal{A}_{BZMV^{dM}} = \langle A, \oplus, \neg, \sim, 0\rangle$ defined $\forall a, b \in A$ by:

$$\neg a := a \to_L 0$$
$$\sim a := a \to_G 0$$
$$a \oplus b := \neg a \to_L b$$

is a BZMV^{dM} with its own order relation $\preceq$:

$$\forall a, b \in A, \quad a \preceq b \quad \text{iff} \quad a \vee b = b \quad \text{iff} \quad a \wedge b = a.$$

Starting from this structure, by Theorem 2.5, we rebuild a HW algebra $\mathcal{A}^* =< A, \to_L^*, \to_G^*, 0 >$ where $\forall a, b \in A^*$

$$a \to_L^* b := \neg a \oplus b = (a \to_L 0) \oplus b = ((a \to_L 0) \to_L 0) \to_L b = a \to_L b.$$

Then, its partial order $\leq^*$ is defined such that:

$$\forall a, b \in A^* \quad a \leq^* b \quad \text{iff} \quad a \to_L^* b = 1 \quad \text{iff} \quad a \to_L b = 1$$

and hence $a \leq^* b$ iff $a \wedge^* b = a$ iff $a \vee^* b = b$ iff $a \wedge b = a$ iff $a \vee b = b$. By a well known property of Heyting algebra [2, p.45] in $\mathcal{A}^*$ $\forall a, b \in A^*$ there always exists the sup of all elements $x \in A^*$ s.t. $a \wedge^* x$ precedes or is equal to b and it defines a residuum $\to_G^*$ s.t. $a \to_G^* b = 1$. We have shown that both $\leq$ and $\leq^*$ are defined in the same way in term of $\to_L$. Thus we have proved that $\to_L = \to_L^*$ and $\leq = \leq^*$.

On the other hand by property of Heyting algebra and definition of residuum we have:

$$a \to_G b = \bigvee \{x \mid a \wedge x \leq b\}$$

$$a \to_G^* b = \bigvee^* \{x \mid a \wedge^* x \leq^* b\}$$

Since $\leq = \leq^*$ we can conclude that $\to_G = \to_G^*$. □

Clearly, we have as a corollary of Proposition 2.2 and Theorem 2.6 that HW algebras and MV$_\Delta$ algebras are term–equivalent.

Another important result of the above proved theorems is that it is impossible to have a HW algebra which does not satisfy the Dummett condition.

Corollary 2.2. *Let $\mathcal{A}$ be a HW algebra then it holds the Dummett condition of the intuitionistic implication, i.e.:*

$$(a \to_G b) \vee (b \to_G a) = 1. \tag{D}$$

Proof. By Theorems 2.5 and 2.6 it is sufficient to show:

$$[\sim(a \odot \neg b) \oplus b] \vee [\sim(b \odot \neg a) \oplus a] = 1.$$

Applying Lemma 1.1(iii) to $[\sim(a \odot \neg b) \oplus b] \vee [\sim(b \odot \neg a) \oplus a]$, we get

$$\begin{aligned}
[\sim(a \odot \neg b) \oplus b] \vee [\sim(b \odot \neg a) \oplus a] &= [\sim(a \odot b) \vee b] \vee [\sim(\neg a \odot b) \vee a] \\
&= \sim(a \odot b) \vee \sim(b \odot \neg a) \vee a \vee b && \text{Lemma 1.1(i)} \\
&= \sim[(a \odot b) \wedge (b \odot \neg a)] \vee a \vee b && \text{MV property} \\
&= \sim 0 \vee a \vee b \\
&= 1
\end{aligned}$$

□

2.3. **Representation and completeness theorems.** A subdirect representation theorem as well as both a weak form and a strong one of a completeness theorem can be proved for MV$_\Delta$ algebras [12] and for SMV algebras [13]. We report below these two completeness results.

Theorem 2.7. [12, th. 2.3.22]. *Let ϕ and ψ be well defined terms, in the traditional way, on the language of MV$_\Delta$ algebra. An identity $\phi = \psi$ holds in all linearly ordered MV$_\Delta$ algebras iff it holds in all MV$_\Delta$ algebras.*

Theorem 2.8. [12, th. 3.2.13]. *Let ϕ and ψ be well defined terms, in the traditional way, on the language of MV$_\Delta$ algebra. An identity $\phi = \psi$ holds in $[0, 1]$–model iff it holds in all MV$_\Delta$ algebras.* Due to their equivalence, these two last results applies also to HW, SMV and BZMV^{dM} algebras. On the other hand, they cannot be extended to BZMV algebras. In fact, on a linearly ordered structure the Gödel negation is defined as:

$$\sim a := \begin{cases} 1 & \text{if} \quad a = 0 \\ 0 & \text{otherwise} \end{cases}$$

and consequently the $\wedge$ de Morgan property $\sim(a \wedge b) = \sim a \vee \sim b$ is always satisfied. So, there exists a property which holds in all linearly ordered BZMV algebras and not in all BZMV algebras, hence a result similar to Theorem 2.7 cannot be proved for BZMV algebras.

Finally, we show that $\mathrm{SBL}_\neg$ algebras are a substructure of HW algebras.

Proposition 2.3. *Let $\langle A, \to_L, \to_G, 0\rangle$ be a HW algebra. Once defined $\neg a := a \to_L 0$, $a * b := a \wedge b = \neg((\neg a \to_L \neg b) \to_L \neg b)$, $a \vee b := (a \to_L b) \to_L b$, the structure $\langle A, \wedge, \vee, *, \to_G, \neg, 0, 1\rangle$ is a $SBL_\neg$ algebra.*

Proof. Clearly, HW algebras are Gödel algebras, hence as showed in [11] are also SBL algebras. Trivially, properties ($\mathrm{SBL}_\neg$ 1), ($\mathrm{SBL}_\neg$ 2), ($\mathrm{SBL}_\neg$ 4), ($\mathrm{SBL}_\neg$ 5) hold.

We prove that axiom ($\mathrm{SBL}_\neg$ 3) holds.
By [12, Lemma 2.4.8] we know that in all linearly ordered MV_Δ algebras the operator Δ behaves as follows:

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

By equivalence of MV_Δ and HW algebras, this property holds also in HW algebras. Now, let us suppose that $a \to_G b = 1$. Then $a \leq b$ and by contraposition $\neg b \leq \neg a$, that is $\neg b \to_G \neg a = 1$. Vice versa, if $a \to_G b \neq 1$ then $\Delta(a \to_G b) = 0$. Further, since we are considering a totally ordered algebra, then $a > b$. Again, by contraposition $\neg b > \neg a$ and $\neg b \to_G \neg a \neq 1$. So, $\Delta(\neg b \to_G \neg a) = 0$. Thus, we proved that in all linearly ordered HW algebras it holds axiom ($\mathrm{SBL}_\neg$ 3) and by Theorem 2.7 and the equivalence of HW and MV_Δ algebras it holds in all HW algebras.

We show that also ($\mathrm{SBL}_\neg$ 6) hold. By lattice properties: $a \wedge b \leq b$. Then, one has $\sim\neg(a \wedge b) \leq \sim\neg b$ and by axiom (HW3), $\sim\neg(a \wedge (a \to_G b)) \leq \sim\neg b$. Finally, using de Morgan properties $\sim\neg a \wedge \sim\neg(a \to_G b) \leq \sim\neg b$. □

Thus, $\mathrm{SBL}_\neg$ algebras are a substructure also of MV_Δ and BZMV^{dM} algebras.

3. Axiomatizations

We introduce in this section the three logical propositional calculi whose Lindenbaum algebra are the three algebraic structures we have proved above to be equivalent to MV_Δ algebra. A propositional calculus whose Lindenbaum algebra is a MV_Δ algebra has been studied by P. Hájek and can be found in [12, p. 57 and p. 63].

We start from the logical system corresponding to Stonean MV algebra.

Definition 3.1. The *Propositional Calculus SMVL* has a denumerable set of propositional variables $p_1, p_2, \ldots, p_n, \ldots$ and connectives $\to$, $\neg$ and $\sim$. Each propositional variable is a *formula* and if α and β are formulas then $\alpha \to \beta$, $\neg\alpha$, $\sim\alpha$ are formulas.

Further connectives are defined as follows:

$$\alpha \wedge \beta := \neg(\alpha \to \neg(\alpha \to \beta))$$
$$\alpha \vee \beta := (\beta \to \alpha) \to \alpha$$
$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$

Definition 3.2. Once set $\top := \neg(\alpha \to \neg(\alpha \to \beta))$, the following formulas are axioms of SMVL:
(S1) $\neg(\alpha \to \neg(\alpha \to \beta))$
(S2) $(\neg\alpha \to \neg\top) \leftrightarrow \alpha$
(S3) $(\neg\alpha \to \top) \leftrightarrow \top$
(S4) $\alpha \wedge \beta \leftrightarrow \beta \wedge \alpha$
(S5) $\neg\neg\alpha \leftrightarrow \alpha$
(S6) $(\alpha \to \beta) \to (\sim\beta \to \sim\alpha)$
(S7) $\neg(\alpha \wedge \sim\alpha)$
(S8) $(\sim\alpha \vee \sim\beta) \leftrightarrow \sim(\alpha \wedge \beta)$
(S9) $(\sim\alpha \wedge \sim\beta) \leftrightarrow \sim(\alpha \vee \beta)$
(S10) $((\neg\alpha \to \alpha) \leftrightarrow \alpha) \leftrightarrow (\neg\alpha \leftrightarrow \sim\alpha)$
(S11) $(\neg \sim\alpha \to \sim\alpha) \leftrightarrow \sim\alpha$

The only *deduction rule* of SMVL is Modus Ponens: $\frac{\alpha, \alpha \to \beta}{\beta}$

The Lindenbaum algebra of SMVL is trivially a Stonean MV-algebra because (S1)-(S11) correspond to the properties introduced by L.P. Belluce to define this kind of MV-algebra in [1].

Now, we introduce the propositional calculus BZMVL whose Lindenbaum algebra is a BZMV^{dM} algebra. Syntax and language of BZMVL coincide with syntax and language of definition 6.1 for SMVL. We present below the axiomatization of BZMVL.

Definition 3.3. Once set $\top := \neg(\alpha \to \neg(\alpha \to \beta))$, the following formulas are axioms of BZMVL:
(Z1) $\neg(\alpha \to \neg(\alpha \to \beta))$
(Z2) $(\neg\alpha \to \neg\top) \leftrightarrow \alpha$
(Z3) $\alpha \wedge \beta \leftrightarrow \beta \wedge \alpha$
(Z4) $\neg\neg\alpha \leftrightarrow \alpha$
(Z5) $\neg \sim\alpha \to \sim\sim\alpha$
(Z6) $(\neg\alpha \to \sim\sim\alpha) \leftrightarrow \sim\sim\alpha$
(Z7) $\sim\neg((\neg\alpha \to \neg\beta) \to \neg\beta) \leftrightarrow ((\neg \sim\sim\alpha \to \neg \sim\sim\beta) \to \neg \sim\sim\beta)$
The only *deduction rule* of BZMVL is Modus Ponens: $\frac{\alpha, \alpha \to \beta}{\beta}$

The Lindenbaum algebra of the following propositional calculus is trivially an HW algebra.

Definition 3.4. The *Propositional Calculus HWL* has a denumerable set of propositional variables $p_1, p_2, \ldots, p_n, \ldots$ and connectives $\to_L$, $\to_G$. Each propositional variable is a *formula* and if α and β are formulas then $\alpha \to_L \beta$ and $\alpha \to_G \beta$ are formulas.

Further connectives are defined as follows:

$$\sim\alpha := \alpha \to_G \neg(\alpha \to_G \alpha)$$
$$\alpha \wedge \beta := \neg((\neg\alpha \to_L \neg\beta) \to_L \neg\beta)$$
$$\alpha \vee \beta := (\beta \to_L \alpha) \to_L \alpha$$
$$\alpha \leftrightarrow \beta := (\alpha \to_L \beta) \wedge (\beta \to_L \alpha)$$

Definition 3.5. The following formulas are axioms of HWL:
(H1) $\alpha \to_G \alpha$
(H2) $\alpha \to_G (\beta \wedge \gamma) \leftrightarrow (\alpha \to_G \gamma) \wedge (\beta \to_G \gamma)$
(H3) $\alpha \wedge (\alpha \to_G \beta) \leftrightarrow \alpha \wedge \beta$
(H4)$(\alpha \vee \beta) \to_G \gamma \leftrightarrow (\alpha \to_G \gamma) \wedge (\beta \to_G \gamma)$
(H5) $\alpha \to_G \alpha) \to_L \alpha \leftrightarrow \alpha$
(H6) $\alpha \to_L (\beta \to_L \gamma) \leftrightarrow \neg(\alpha \to_L \gamma) \to_L \neg\beta$
(H7) $\neg\sim\alpha \to_L \sim\sim\alpha$
(H8) $(\alpha \to_G \beta) \to_L (\alpha \to_L \beta)$

Deduction rules of HWL are $\frac{\alpha, \alpha \to_L \beta}{\beta}$ and $\frac{\alpha, \alpha \to_G \beta}{\beta}$

Definition 3.6. Let $FORM(SMVL/BZMVL)$ be the set of wff of SMVL and BZMVL. An *evaluation* on SMVL/BZMVL is a mapping $e_s : FORM(SMVL/BZMVL) \mapsto [0,1]$ s.t.:

$$e_s(\neg\alpha) = 1 - e_s(\alpha)$$
$$e_s(\sim\alpha) = \begin{cases} 0 & \text{if } e_s(\alpha) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$
$$e_s(\alpha \to \beta) = \min\{1, 1 - e_s(\alpha) + e_s(\beta)\}$$

Definition 3.7. Let $FORM(HW)$ be the set of wff of HWL. An *evaluation* on HWL is a mapping $e_h : FORM(HW) \mapsto [0,1]$ s.t.:

$$e_h(\alpha \to_L \beta) = \min\{1, 1 - e_h(\alpha) + e_h(\beta)\}$$
$$e_h(\alpha \to_G \beta) = \begin{cases} 1 & \text{if } e_h(\alpha) \leq e_h(\beta) \\ e_h(\beta) & \text{otherwise} \end{cases}$$

Definition 3.8. A formula $\tau \in$ FORM(SMVL/BZMVL) is a 1-*tautology* of SMVL/BZMVL iff $e_s(\tau) = 1$ for any evaluation e_s.

Definition 3.9. A formula $\upsilon \in$ FORM(HWL) is a 1-*tautology* of HWL iff $e_h(\upsilon) = 1$ for any evaluation e_h.

Theorem 3.1. *Let α be a formula of SMVL/BZMVL/HWL. Then, SMVL/BZMVL/HWL $\vdash \alpha$ iff α is a 1-tautology on SMVL/BZMVL/HWL.*

Proof. ($\Rightarrow$:) It can be easily verified that axioms (S1)-(S11), (Z1)-(Z7) and (H1)-(H9) are 1-tautologies and that deduction rules cannot decrease the evaluation of an inferred formula.
($\Leftarrow$:) If α is a 1-tautology then $e_{s/h}(\alpha) = 1$ is satisfied in $[0,1]$–models (SMV/BZMV^{dM}/HW) for each $e_{s/h}$. Each evaluation maps to a set of terms t_α, meant in the traditional way [10, p. 21], related to formulas. Then we have $t_\alpha = 1$ in $[0,1]$–models. By the strong algebraic completeness expressed in theorem 8 if $t_\alpha = 1$ is satisfied in $[0,1]$ it is satisfied in any model and thus $[\alpha]_\equiv$ is the top element of the Lindenbaum algebra of SMVL/BZMVL/HWL. Hence SMVL/BZMVL/HWL $\vdash \alpha$. □

Corollary 3.1. *HWL, SMVL, BZMVL and Lukasiewicz logic with Δ axiomatized by Hájek in* [12] *produce, up to syntactical translation, the same set of 1-tautologies.*

Proof. By Theorem 3.1 and equivalence Theorems of chapter 5 among HW-algebras, Stonean MV-algebras, BZMV^{dM}-algebras and MVΔ-algebras. $\square$

4. Conclusions

The original structure of HW algebra has been proved equivalent to other well known structures. Some weakening of HW algebras have been considered and their lattice structure has been studied. Taking into account the diagram of the conclusions in [5], in the following diagram the relationship existing among all the discussed algebraic structures is summarized.

$$\begin{tikzcd}
& HW \equiv BZMV^{dM} \equiv MV_\Delta \equiv SMV \arrow[dl] \arrow[dr] & \\
BZ^{dM} \arrow[d] & & SBL_\neg \arrow[ll] \arrow[d] \\
Brouwer^{dM} \arrow[d] & & SBL \arrow[ll] \arrow[d] \\
pre - Brouwer^{dM} & & BL \arrow[ll]
\end{tikzcd}$$

An open problem that needs more investigation is the relationship between SBL$_\neg$ and HW algebras (or equivalent structures). Indeed, we know that SBL$_\neg$ algebras are a substructure of HW algebras, but from the lattice point of view they both have as substructure a BZ^{dM} lattice. The question is if there exist a lattice property satisfied by HW and not by SBL$_\neg$ algebras. This question is also related to which conditions are sufficient and necessary for a SBL$_\neg$ algebra to be an HW algebra.

References

[1] Belluce, L. P.: Generalized fuzzy connectives on MV–algebras, *J. Math. Anal. Appl.*, **206**(1), 1997, 485–499.

[2] Birkhoff, G.: *Lattice Theory*, vol. XXV of *American Mathematical Society Colloquium Publication*, Third edition, American Mathematical Society, Providence, Rhode Island, 1967.

[3] Cattaneo, G., Ciucci, D.: BZW algebras for an abstract approach to roughness and fuzziness, *IPMU 2002, July 1–5 2002, Annecy, France, Proceedings*, ESIA – Université de Savoie, 2002.

[4] Cattaneo, G., Ciucci, D.: Heyting Wajsberg algebras as an abstract environment linking fuzzy and rough sets, *Lecture Notes in Artificial Intelligence*, **2475**, 2002, 77–84.

[5] Cattaneo, G., Ciucci, D., Giuntini, R., Konig, M.: Algebraic Structures Related to Many Valued Logical Systems. Part I: Heyting Wajsberg Algebras, *Fundamenta Informaticae*, 2004, Accepted.

[6] Cattaneo, G., Dalla Chiara, M. L., Giuntini, R.: Some Algebraic Structures for Many-Valued Logics, *Tatra Mountains Mathematical Publication*, **15**, 1998, 173–196, Special Issue: Quantum Structures II, Dedicated to Gudrun Kalmbach.

[7] Cattaneo, G., Giuntini, R., Pilla, R.: BZMV^{dM} and Stonian MV algebras (Applications to fuzzy sets and rough approximations), *Fuzzy Sets Syst.*, **108**, 1999, 201–222.

[8] Cattaneo, G., Lombardo, F.: Independent axiomatization of MV-algebras, *Tatra Mountains Mathematical Publication*, **15**, 1998, 227–232, Special Issue: Quantum Structures II, Dedicated to Gudrun Kalmbach.

[9] Chang, C. C.: Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88**, 1958, 467–490.

[10] Cignoli, R., D'Ottaviano, I., Mundici, D.: *Algebraic foundations of many–valued reasoning*, Kluwer academic, Dordrecht, 1999.

[11] Esteva, F., Godo, L., Hájek, P., Navara, M.: Residuated Fuzzy logics with an involutive negation, *Archive for Mathematical Logic*, **39**, 2000, 103–124.

[12] Hájek, P.: *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.

[13] Konig, M.: Many-valued logics enriched with a Stonean negation:a direct proof of representation and completeness, 2004, Pre–print.

[14] Monteiro, A.: Axiomes Independants pour les Algebres de Brouwer, *Revista de la Unión matematica Argentina y de la Asociación Fisica Argentina*, **17**, 1955, 149–160.

[15] Monteiro, A.: Sur Les Algèbres de Heyting symétriques, *Portugaliae Mathematica*, **39**, 1980, 1–237.

[16] Rasiowa, H., Sikorski, R.: *The Mathematics of Metamathematics*, vol. 41 of *Monografie Matematyczne*, Third edition, Polish Scientific Publishers, Warszawa, 1970.

[17] Surma, S.: *Logical Works*, Polish Academy of Sciences, Wroclaw, 1977.

[18] Turunen, E.: *Mathematics Behind Fuzzy Logic*, Physica–Verlag, Heidelberg, 1999.

Dipartimento di Informatica, Sistemistica e Comunicazione, Università di Milano–Bicocca, Via Bicocca degli Arcimboldi 8, I–20126 Milano (Italy)

E-mail address: {cattang,ciucci}@disco.unimib.it

Dipartimento di Scienze Pedagogiche e Filosofiche, Università di Cagliari, Via Is Mirrionis 1, 09123 Cagliari (Italia)

E-mail address: giuntini@unica.it, martinvk@iol.it