MODELLING:
RELATED INVERSE PROBLEMS

QUESTION: WHY AT ALL FOCUS ON THE IDENTIFICATION OF CONDUCTIVITY ?

ANSWER :

I) Flow equation in 1 spatial dimension.

$$\frac{\partial}{\partial x}\left[a(x)\,\frac{\partial}{\partial x}u(x,\,t)\right] = \frac{\partial}{\partial t}u(x,\,t) + f(x,\,t)$$

II) Time independent flow eqns. in 2 spatial dimensions:

either $-\nabla\cdot(\underset{\uparrow\ \text{matrix}}{\mathbf{A}}\nabla u) = f$ or $-\nabla\cdot(\underset{\uparrow\ \text{scalar}}{a}\nabla u) = f$

III) Inverse problem for the zero energy Schroedinger eqn.:

$$\underset{\nwarrow\ \text{kinetic energy}}{\triangle w} - \underbrace{\frac{\triangle\sqrt{a}}{a}}_{\nwarrow\ \text{potential energy}} w = 0 \quad ; \quad \text{Dirichlet BC}\ .$$

PARADIGMS TO OVERCOME PARADOXES:
PRIOR KNOWLEDGE AND *REGULARIZATION*

I) Through *prior knowledge* achieve the *uniqueness* of e.g., a , given *existence*.

II) Through *regularization* assess the *stability* of the unique a.

III) Through *regularization* provide *existence* conditions.

Guidelines and objectives:
translate realistic situations in hydrogeology,
provide computable stability estimates,
design new identification algorithms or improve existing ones.

ASIDE: What is *stability* (of the unique solution) ?

Given two different coefficients, $a(u, f)$, $b(v, f)$ from different potentials

I) find suitable Banach spaces, $\mathbb{X}$ (potential data) and $\mathbb{Z}$ (parameters)

II) try to relate $\| v - u \|_{\mathbb{X}}$ to $\| b - a \|_{\mathbb{Z}}$ by

$$\| b - a \|_{\mathbb{Z}} \leq const. \, \| v - u \|_{\mathbb{X}}$$

SOME RECENT RESULTS IN THE INVERSE TRANSMISSIVITY PROBLEM

1 – Uniqueness Conditions

Consider

$$\frac{\partial}{\partial x}\left[a(x)\,\frac{\partial}{\partial x}u(x,\,t)\right] = \frac{\partial}{\partial t}u(x,\,t) + f(x,\,t),\quad x\in(x_0\,,\,x_1),\ t\in(0,\,T]$$

@ fixed t , this is a 1st order ODE w.r. to a.

Require of u, f the least possible regularity.

Assume $\exists$ at least one positive, bounded $a(u\,;f)$.

Uniqueness of a <=> Cauchy datum.

Cauchy problem → regular ($u' \neq 0,\ \forall\ x \in \bar{D}$)

Cauchy problem → singular ($\exists$ isolated $x_u \in \bar{D} \ni u'(x_u) = 0$

Classification of uniqueness conditions (omitted).

2 – STABILITY ESTIMATES

MAIN RESULT

The type of stability estimate is controlled by the type of Cauchy problem.

Regular problems lead to uniform (L^{∞}) estimates,
singular problems to L^p (i.e., integral) estimates, $1 \leq p < \infty$.

POTENTIAL $\mathbb{X}$

COEFFICIENT $\mathbb{Z}$

$W^{1,\infty}$ — Cauchy problem

→ regular L^{∞}

→ singular L^{p}

$$constant \; \| v - u \|_{\mathbb{X}} \quad \geq \quad \| b - a \|_{\mathbb{Z}}$$

EXAMPLE:

UNIQUENESS AND STABILITY FROM AN ISOLATED *CRITICAL* POINT

(*Singular* Cauchy Problem)

Hp. $f \in H^{-1}(D)$;

$\exists\ a, b \in C^0(\bar{D});\ 0 < a_L \leq a \leq a_H\ ,\ a_L \leq b\ , \forall\ x \in \bar{D}$;

$(au')' =_{\text{a.e.}} f\ ;\ (bv')' =_{\text{a.e.}} f$;

$u', v' \in C^0(\bar{D});\ u'(x_u) = 0\ ,\ v'(x_v) = 0$;

$\exists\ p,\ 1 \leq p < \infty \cdot\!\ni\!\cdot (\frac{1}{u'})\ (\frac{1}{v'}) \in L^p(D)$ *and* $\|\frac{1}{v'}\|_{0,p} \leq c_V$

Th. $\exists!\ a$ (*if* $f \in L^2$ *then* $a = \dfrac{F - F(x_u)}{u'}\ ,\ b = \dfrac{F - F(x_v)}{v'}$)

$\|\, b - a \,\|_{0,p} \leq c_V\ [1 + 2a_H]\ \|v - u\|_{1,\infty}$

SOME RECENT RESULTS, END

3 – CONTINUOUS FLOWS AND CONSTRUCTIVE ALGORITHMS IN 2 D

$\mathbf{A} = \begin{pmatrix} a(\mathbf{x}) & 0 \\ 0 & b(\mathbf{x}) \end{pmatrix}$; $\nabla\cdot(\mathbf{A}\nabla u) = f$ in D.

Def. (*equation error*) $V(\mathbf{A}) := \frac{1}{2}\int_D |\nabla\cdot(\mathbf{A}\nabla u) - f|^2 d\mathbf{x} = \frac{1}{2}\| r \|_2^2$

Scope: relate identification algorithms to dynamical systems, by letting $\mathbf{A} = \mathbf{A}(\mathbf{x}, t) \Rightarrow r = r(\mathbf{x}, t)$

Example: (*continuous flow counterpart of the unconstrained steepest descent algorithm*)

$$\mathbf{A}(\mathbf{x}, t{=}0) = \mathbf{A}^0(\mathbf{x}) \; ; \; \dot{a}(\mathbf{x}, t) = \partial_x r \; \partial_x u \; , \; \dot{b} = \partial_y r \; \partial_y u \; .$$

If $\nabla_{\mathbf{A}} V \in L^2 \; \forall t \geq 0$, *then* $\dot{V} = -\| \nabla_{\mathbf{A}} V \|_2^2$ (*energy decay rate*) .

Proposition: *Let* $\partial_x u$ *and* $\partial_y u \neq 0, \forall x \in \bar{D}$ *and let* $p(\mathbf{x}, t)$ *solve the elliptic* BVP

$$\nabla\cdot(\mathbf{A}(\mathbf{x}, t)\nabla p) = f \; ; \; p\,|_{\partial D} = u\,|_{\partial D} \; ,$$

then

$$\mathbf{A}(\mathbf{x}, t{=}0) = \mathbf{A}^0(\mathbf{x}) \; ; \; \dot{a} = -a\left(1 - \frac{\partial_x p}{\partial_x u}\right) , \; \dot{b} = -b\left(1 - \frac{\partial_y p}{\partial_y u}\right) ,$$

is a gradient flow, which yields the energy decay rate

$$\dot{V} = -2V.$$