first of all let me thank the conference organizers for accepting my contribution -

this is the one dimensional heat equation, which is defined in this space-time domain and shall be understood in the distribution sense.

If we were interested in determining the temperature u from knowledge of the position dependent conductivity a and the source term f, then we would also need some boundary and initial condition. On the other hand the problem consists of determining conductivity from interior measurements of both temperature and the source term.

the source term shall belong to this class, whereas the potential data shall give rise to continuous trajectories s.t. the spatial derivative is ~~of~~ bounded ~~variation~~ for $\forall t$ in T closure and the time derivative shall be in $L^2(D)$.

the set of admissible conductivities consists of bounded, measurable functions, which are bounded away from zero and above by these constants; moreover they shall be continuous at the left end pt of the interval.

we shall always assume that a potential, source term pair gives rise to at least one admissible conductivity.

In the following, reference will be often made to the defect equation, which relates differences, V, between the second potential, v and the reference potential u on one hand to differences, B, between the second conductivity b and the reference conductivity a.

the defect eqn. shall be also understood as in the distribution sense - As well as the state equation, it is an ordinary differential equation w.r. to conductivity

The scope of this talk is to provide a unified view over some uniqueness conditions and the corresponding stability estimates.
Uniqueness conditions can be classified according to 3 criteria
the type of available inform. may be either local or non-local.
Examples of local conditions are Cauchy problems for either the state or the defect equation.
Cauchy problems, on their turn may be either regular or singular. The latter arise when information is given at a critical point s.t., the potential is spatially stationary.
Finally, the required information may be supplied at a specific instant of time or in the whole of T.

In order to state the regular Cauchy problem for the defect equation, I have to introduce the defect (r) and define this set Bad made of bounded conduct differences, which are continu @ x_0 and vanish there. This is the Cauchy datum for the Defect equation.
If these conditions are met, in particular if at some instant τ the derivative v_x vanishes almost nowhere in D, the uniqueness result holds:
if V vanishes and V_t is zero a.e in D then B is also zero a.e. in D.
Regular Cauchy problems arise in a number of situations, which I do not describe in detail.

Singular Cauchy problems are based on the definition of ~~this set~~: $E_u(t)$ ~~is the~~ closure of the set of critical points for the potential u at time t.
Let there exist two admissible conductivities which give rise to the same potential.
If either condition is met, then b coincides with a a.e. in D.
The first condition requires $E_u(t)$ to be non empty and have zero Lebesgue measure.
The other requires the meas of this intersection to vanish.
This statement generalises a well-known result by K+N, which is 15 years old.

A non local condition is the following. Let me skip these hypotheses, which are intended to generalize the set of admissible data. Let me focus on these two lines.
If the admissible conductivity satisfies this condition; i.e. the domain average of flux is zero, then said conductivity is unique and given by this relationship.
The converse is also true.
Here $g_0^{(-1)}$ stands for the distributional derivative of g which by hy is continuous at x_0 and by definition vanishes there.

Stability estimates now rely on integrating the defect equation.
In this slide I summarize the comparison between the ~~regular and~~ stability results affecting solutions, which are unique because of a regular and resp. singular Cauchy problem.

The following list of properties shall be met at one instant of time, τ.

In the regular Cauchy case these reciprocals shall be bounded above by a given constant.

An admissible reference conductivity shall exist, whereas the 2nd conductivity shall yield a B in B_{ad}.

~~the corresponding inequality estimate is represented by an inequality in a uniform~~

Stability in this case is uniform; the L^∞ norm of B is bounded above by the product of these constants times the $X(\tau)$ norm of V - the latter is defined as this sum -

~~Here $\mathcal{V}_x$ is the total variation of V_x, which can be expressed by the L^∞ norm of a distributional anti-derivative~~

On the other hand, if uniqueness was due to a singular Cauchy problem I have to introduce an closed interval I, require that at τ the potential derivatives be continuous in I, moreover they comply with the first uniqueness condition.

Both sets of critical points shall be contained in I / Hp for the

moreover, the L^q norms of these reciprocals must be bounded by a given constant. Note that q ~~larger~~ can take on any finite value from 1 onwards -

The reference conductivity shall be continuous in $C^0(I)$

If all of these Hp are met, then a stability estimate for the L^q norm of B holds - Note that formally the rhs of the inequality is the same as for the regular case - Of course c_V has a different meaning.

As a corollary, a stability estimate in L^q (the slide only shows the result in L^1) can be obtained even if meas $E_v(T) > 0$ strictly - Since (b) is not unique there, it is equated to the reference conductivity - A unique (b) is thus obtained, to which the theorem applies.

If uniqueness is due to the second condition, in the singular case, then the result is ~~the same~~ similar, although the hp and the proof are slightly more complicated. The main difference is the $\mathcal{X}$-norm here, on the rhs.

ABRIDGE

It is now interesting to compare the proofs of the two inequalities which apply to the regular and respectively the singular - unique solutions - For both of them the starting point is the defect equation. The regular Cauchy pbm supplies this initial value - the defect equation is integrated term-wise - Since antiderivatives of distributions differ by a constant, ~~the~~ a means shall be found to determine this constant. Since the hp imply that the involved antiderivatives, ~~are conti~~ in addition to being in L^∞, are continuous at x_0, the constant is found. $v_0^{(-1)}$ stands for the antider of v which vanishes at x_0.
When uniqueness is due to a singular Cauchy pbm, the available initial condition is given at the point $\Xi_v(T)$. These are the antiderivatives which vanish at that point.
In the regular case, absolute values are taken and an L^∞ estimate is obtained
In the singular case, integration over the whole domain is needed, which yields L^1 or at most L^q estimates.

It may be also interesting to compare how the solution procedure affects (the final result as well as the regularity requirements).

If Bv_x is kept together, the result is what we have just seen. It requires that V_x be ~~of~~ bounded ~~variation~~ and continuous at x_0^+ and V_t in L^1

On the other hand if we expand the derivative of the product, we must give sense to each of these terms independently. Since we loose information on the functional relationship between terms in this ODE, we need that v_x' be continuous (and nonzero), moreover we need v_{xx} in L^2. ~~We may estimate the growth of B~~. Two steps are needed. First we have to estimate the growth of B by means of a generalization of Gronwall-Bellman inequality, next we have to replace r/v_x ~~by~~, in fact its antiderivative, by a function of V. This only works if the reference conductivity is of bounded variation.

There is no way to generalise the result. Since a is of BV, then this antiderivative is in L^∞ and a bound can be prescribed, which eventually appears in the estimate.

The estimates for conductivities obtained from non-local conditions have a similar structure. Again let me show the two statements in the same slide for the sake of comparison. Data, reference and second conductivity as usual. Let then exist τ, s.t. the domain averages are equal and are known. If this reciprocal can be uniformly bounded, then an L^∞ estimate applies to B. If the reciprocal of V_x satisfies an integral bound, then an L^2 estimate is obtained.

Again here it may be interesting to assess the role of the procedure on the final result.

If we estimate the defect equation by keeping the product Bv_x together, the result is what we have just seen-

Otherwise, if we want to explicitly solve for the two conductivities and then estimate their difference, the final estimate is ~~worse~~ different although it requires less regularity of the potentials.

Why have these stability estimates been sought for at all.?

The main purpose is not assessing the stability of the identified conductivity per se but to assess its eventual impact on control problems, as recalled by prof Fitzpatrick yesterday afternoon.

And this is where my talk actually be ought to fins!!

Anyway: I have provided a unified view over uniqueness and stability. Classification has been made possible by the distinction between local and non local conditions, by regular vs singular problems &c.

When the Cauchy pblm is regular, then uniform estimates have been obtained.

When potentials are stationary, there is no way of obtaining uniform estimates, nor can the continuity requirement be relaxed

All of these results also apply to the time independent case, because the structure of the ODE for conductivity does not change

→ to conclude let me point out that this work is part of these research projects

Also let me acknowledge the frequent flyer programs of these airlines for travel support.
Thank you.