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A GAME THEORY DERIVATION OF A CLASSICAL FINANCIAL EQUILIBRIUM MODEL

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Abstract. We consider a financial equilibrium model which deals with m economic sectors and n financial instruments. In the classical derivation the equilibrium prices of the financial instruments are exogenous, since the maximization of the utility functions of the sectors is performed with respect to the assets and liabilities only and, as a consequence, the associated KKT system does not yield the equilibrium prices, which are subsequently fixed with the help of an independent economic argument. Instead, we consider both the sectors and the instruments as players of a game whose Nash equilibria provide assets, liabilities and prices. We investigate the Nash equilibria using the variational inequality associated to the pseudogradient of the game. Since the pseudogradient is monotone, but not strictly monotone, we expect multiple solutions and under additional assumptions we find out a relationship between any two solutions. We then compute the solution whose price vector has minimum norm, and also study the price of anarchy of our game. At last, we perform a scenario analysis based on different taxation regimes.

Keywords. Financial equilibrium; Variational inequalities; Nash equilibrium; pseudogradient; price of anarchy.

1. Introduction

In this paper we provide a Game Theory formulation of a financial equilibrium problem previously proposed by Nagureny et al. (see e.g. [10, 12] and [11, Chapter 8]). In the mentioned references, the authors develop a variational inequality approach to a classical model of financial markets. Their work was inspired by the influential papers of Markowitz [8] and Sharpe [15], but instead of using standard optimization tools their approach was based on the theory of variational inequalities. The model consists of a certain number of economic sectors which buy different financial instruments, so as to maximize their utilities, while satisfying the balance law. The equilibrium price of each instrument is considered as given and its value is inserted in the KKT system corresponding to the maximization of the utility of each sector. A variational inequality formulation is then proposed, which is proved to be equivalent to the KKT system, augmented with an additional equilibrium condition involving the prices. In our approach, we put forward a Game Theory model where the players are both the economic sectors and the instruments, and look for the Nash equilibria of the game. While the variational inequality associated to the pseudogradient of the game is the same as in [13], the merit of our formulation is that the equilibrium prices are not exogenous, but are a component of the Nash equilibria of the game, that is the game provides an explanation of the *price formation mechanism*. Within this framework, it is natural to investigate the so called *price of anarchy* [14], and under mild assumptions

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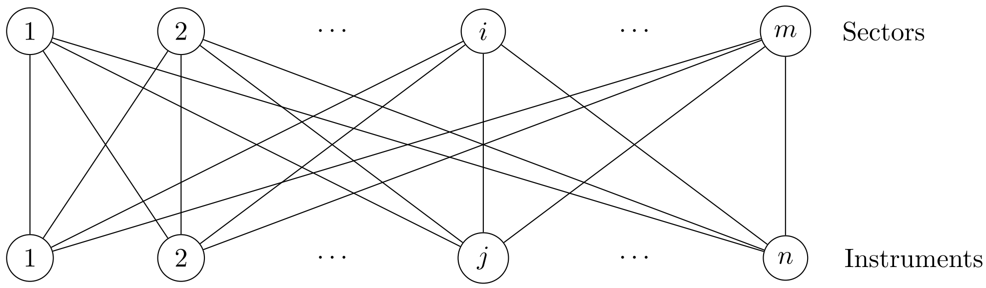

FIGURE 1. Bipartite graph which illustrates the relationship between sectors and instruments.

we prove that the price of anarchy is equal to 1. Moreover, we deal with the issue of non-uniqueness of equilibrium prices and provide a strategy to find the solution whose price vector has minimum norm.

This paper is organized as follows. In Section 2, we outline the economic model, provide some notation, and briefly describe the previous formulation. In Section 3 we put forward our Game Theory formulation whose properties are investigated in the subsequent Section 4, which includes a theorem on the non-uniqueness of the equilibrium prices, a theorem on the price of anarchy, and a Tikhonov regularization procedure which yields the minimum norm equilibrium price. Section 5 is then devoted to some numerical examples, where we also perform a scenario analysis of different taxation regimes. In the concluding section we summarize our findings and outline future research perspectives.

2. The Economic Model

The economic model under investigation consists of m economic sectors and n financial instruments. We denote with x_{ij} the volume of instrument j which is present in the portfolio of sector i as an asset, and with y_{ij} the volume of instrument j in the portfolio of sector i as liability. The relationship between sectors and instruments is depicted in Figure 1. The assets and liabilities of each sector i are grouped into vectors $x^i \in \mathbb{R}^n$ and $y^i \in \mathbb{R}^n$, respectively, while the vectors $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^m)$ describe the assets and liabilities of all sectors. The uncertainty about the future financial values is embodied, for each sector i, in a variance-covariance matrix Q^i corresponding to the assets and liabilities of sector i. We denote with r_j the price of instrument j and with $r \in \mathbb{R}^n$ the price vector of all financial instruments. The scalar product between any two vectors a and b is denoted with $a^\top b$, while if A is a matrix, A_l will denote its l-th column. Each sector wishes to maximize its assets and minimize its liabilities, while reducing the associated risk which is described by the term:

$$\begin{pmatrix} x^i & y^i \end{pmatrix} Q^i \begin{pmatrix} x^i \\ y^i \end{pmatrix}.$$

Thus, for each fixed price vector r, every sector i wants to solve the problem:

$$\min_{x^i, y^i} U_i(x^i, y^i, r) = \begin{pmatrix} x^i & y^i \end{pmatrix} Q^i \begin{pmatrix} x^i \\ y^i \end{pmatrix} - \sum_{j=1}^{n} r_j (x_{ij} - y_{ij}) \tag{2.1}$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad \sum_{j=1}^{n} y_{ij} = s_i, \qquad i = 1, 2, \dots, m, \tag{2.2}$$

$$x_{ij} \geq 0, \qquad y_{ij} \geq 0, \qquad i = 1, \dots, m, j = 1, 2, \dots, n, \tag{2.3}$$

where constraint (2.2) represents the well known accounting identity of sector i, and s_i denote the total financial volume of sector i. In the sequel, it will be convenient to split Q^i into four blocks:

$$Q^i = \begin{bmatrix} Q^i_{11} & Q^i_{12} \\ Q^i_{21} & Q^i_{22} \end{bmatrix}$$

and denote with $\left(q^i_{ab}\right)_{lj}$ the element (l, j) of matrix $Q^i_{ab},\ a, b = 1, 2$. Moreover, being Q^i a variance-covariance matrix, it is symmetric and positive definite, which also implies that Q^i_{11} and Q^i_{22} are symmetric, while $(Q^i_{12})^\top = Q^i_{21}$. Thus, if we set

$$A^i = \begin{pmatrix} x^i & y^i \end{pmatrix} \ Q^i \begin{pmatrix} x^i \\ y^i \end{pmatrix},$$

we can write the Lagrangian and the KKT system for each sector, for a fixed price. Thus, let

$$\begin{aligned} &A^i := {x^i}^T Q^i_{11} x^i + y^i Q^i_{21} x^i + x^i Q^i_{12} y^i + {y^i}^T Q^i_{22} y^i, \\ &\mathcal{L}^i = \begin{pmatrix} x^i & y^i \end{pmatrix} Q^i \begin{pmatrix} x^i \\ y^i \end{pmatrix} - \sum_{j=1}^n r_j \left(x_{ij} - y_{ij}\right) - \mu^1_i \left(\sum_{j=1}^n x_{ij} - s_i\right) - \mu^2_i \left(\sum_{j=1}^n y_{ij} - s_i\right) \\ &\qquad - \sum_{j=1}^n \lambda_{ij} x_{ij} - \sum_{j=1}^n \gamma_{ij} y_{ij}. \end{aligned}$$

For $l = 1, \dots, n$ and for each fixed i we get:

$$\begin{aligned} \frac{\partial A}{\partial x_{il}} &= \sum_{j=1}^n \left(q^i_{11}\right)_{lj} x_{ij} + \sum_{j=1}^n \left(q^i_{11}\right)_{jl} x_{ij} + \sum_{j=1}^n \left(q^i_{21}\right)_{jl} y_{ij} + \sum_{j=1}^n \left(q^i_{12}\right)_{lj} y_{ij} \\ &= 2\sum_{j=1}^n \left(q^i_{11}\right)_{lj} x_{ij} + 2\sum_{j=1}^n \left(q^i_{12}\right)_{lj} y_{ij} = 2\sum_{j=1}^n \left(q^i_{11}\right)_{lj} x_{ij} + 2\sum_{j=1}^n \left(q^i_{21}\right)_{jl} y_{ij} \\ &= 2\left(Q^i_{11}\right)_l^\top x^i + 2\left(Q^i_{21}\right)_l^\top y^i. \end{aligned}$$

$$\begin{aligned} \frac{\partial A}{\partial y_{rl}} &= \sum_{j=1}^n \left(q^i_{22}\right)_{lj} y_{ij} + \sum_{j=1}^n \left(q^i_{22}\right)_{jl} y_{ij} + \sum_{j=1}^n \left(q^i_{21}\right)_{jl} x_{ij} + \sum_{j=1}^n \left(q^i_{12}\right)_{lj} x_{ij} \\ &= 2\sum_{j=1}^n \left(q^i_{22}\right)_{lj} y_{ij} + 2\sum_{j=1}^n \left(q^i_{21}\right)_{lj} x_{ij} = 2\sum_{j=1}^n \left(q^i_{22}\right)_{lj} y_{ij} + 2\sum_{j=1}^n \left(q^i_{12}\right)_{jl} x_{ij} \\ &= 2\left(Q^i_{22}\right)_l y^i + 2\left(Q^i_{12}\right)_l^\top x^i. \end{aligned}$$

The KKT system corresponding to problem (2.1)–(2.3) satisfied by (x^*, y^*) is then:

$$\begin{cases} 2\left(Q^i_{11}\right)_l^\top x^{i^*} + 2\left(Q^i_{21}\right)_l^\top y^{i^*} - r^*_l - \mu^1_i \geq 0 \\ 2\left(Q^i_{22}\right)_l^\top y^{i^*} + 2\left(Q^i_{12}\right)_l^\top x^{i^*} + r^*_l - \mu^2_i \geq 0 \\ x^*_l \left[2\left(Q^i_{11}\right)_l^\top x^{i^*} + 2\left(Q^i_{21}\right)_l^\top y^{i^*} - r^*_l - \mu^1_i\right] = 0 \\ y^{i^*}_l \left[2\left(Q^i_{22}\right)_l^\top y^{i^*} + 2\left(Q^i_{12}\right)_l^\top x^i + r^*_l - \mu^2_i\right] = 0. \end{cases} \tag{2.4}$$

Let us remark that the price r^* has been fixed before solving the system, which means that it is considered exogenous. The authors in [10] provide an additional condition that must be satisfied at equilibrium and that they consider simultaneously with the KKT system for each $j = 1, \dots, n$:

$$\sum_{i=1}^{m} \left(x_{ij}^* - y_{ij}^*\right) \begin{cases} = 0, & \text{if } r_j^* > 0, \\ \geq 0, & \text{if } r_j^* = 0. \end{cases} \tag{2.5}$$

The KKT system, complemented with the additional condition (2.5) is then reformulated as a variational inequality. In the following section, we formulate a game whose solutions coincide with (x^*, y^*, r^*), thus proving that the equilibrium price can be though of as the outcome of a competition between the sectors and the financial instruments.

3. The Game Theory Model

For each sector $i = 1, \dots, m$, consider the problem:

$$\min_{x^i, y^i} U_i(x^i, y^i, r) = \begin{pmatrix} x^i & y^i \end{pmatrix} Q^i \begin{pmatrix} x^i \\ y^i \end{pmatrix} - \sum_{j=1}^{n} r_j (x_{ij} - y_{ij}) \tag{3.1}$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad \sum_{j=1}^{n} y_{ij} = s_i, \qquad i = 1, 2, \dots, m, \tag{3.2}$$

$$x_{ij} \geq 0, \qquad y_{ij} \geq 0, \qquad i = 1, 2, \dots, m, \; j = 1, 2, \dots, n. \tag{3.3}$$

Furthermore, for any instrument $j = 1, \dots, n$ consider the problem

$$\min_{r_j \geq 0} V_j(x, y, r_j) = r_j \sum_{i=1}^{m} (x_{ij} - y_{ij}). \tag{3.4}$$

Each instrument j minimizes the function V_j, with respect to $r_j \geq 0$, for each value of x_{ij} and y_{ij} which is compatible with (3.2), (3.3).

We now consider a game where the players are both the sectors and the instruments, and write the related pseudogradient:

$$F(x, y, r) = \left(\nabla_{(x^1, y^1)} U_1, \dots, \nabla_{(x^m, y^m)} U_m, \frac{\partial V_1}{\partial r_1}, \dots, \frac{\partial V_n}{\partial r_n} \right), \tag{3.5}$$

where we have made a slight abuse of a notation, writing $F(x, y, r) = F(x^1, y^1, x^2, y^2, \dots, x^m, y^m, r)$, that is reordering the variables.

Definition 3.1. A vector (x^*, y^*, r^*) is a Nash equilibrium if $r^* \geq 0$, x^*, y^* satisfy (3.2), (3.3), and for every $i = 1, \dots, m$ $j = 1, \dots, n$:

$$U_i(x^{i*}, y^{i*}, r^*) \leq U_i(x^i, y^i, r^*), \qquad \forall \, (x^i y^i) \text{ such that (3.2), (3.3) hold}, \tag{3.6}$$

$$V_j(x^*, y^*, r_j^*) \leq V_j(x^*, y^*, r_j), \qquad \forall \, r_j \geq 0. \tag{3.7}$$

It is well known that Nash equilibrium problems, under standard differentiability and convexity assumptions, can be formulated by means of an equivalent variational inequality [2, 7, 9, 13]. The interested reader can consult [4] or [11], for the variational inequality approach to equilibrium problems. Thus, (3.6)–(3.7) can be solved investigating the variational inequality $VI(F, K)$, where F is the pseudogradient defined in (3.5) and K is the closed, convex set defined as

$$K = \left\{ (x, y, r) \in \mathbb{R}_+^{2mn+n} : \sum_{j=1}^{n} x_{ij} = s_i, \; \sum_{j=1}^{n} y_{ij} = s_i, \; \forall i = 1, \dots, m, \right\}.$$

Thus, the variational inequality $VI(F,K)$ consists in finding $(x^*, y^*, r^*) \in K$ such that

$$\begin{aligned} &\sum_{i=1}^m \nabla_{x^i} U_i(x^{i*}, y^{i*}, r^*)^\top (x^i - x^{i*}) + \sum_{i=1}^m \nabla_{y^i} U_i(x^{i*}, y^{i*}, r^*)^\top (y^i - y^{i*}) \\ &+ \sum_{j=1}^n \frac{\partial V_j(x^*, y^*, r_j^*)}{\partial r_j}(r_j - r_j^*) \geq 0, \qquad \forall (x,y,r) \ \in K. \end{aligned} \tag{3.8}$$

Let us mention that it has been proved in [10] that variational inequality (3.8) is equivalent to conditions (2.4)–(2.5).

The previous model can be extended in some directions. First of all, under financial regulations, a maximum value $\bar{r}_j$ can be imposed on the price of each instrument. This is a natural assumption that will be kept throughout the paper. Moreover, the government can decide to impose a tax $\tau_{ij} \in [0,1)$ on the revenue of instrument j of sector i. At last, instead of considering the quadratic expression in (3.1), we can assume that the utility of each sector i (still denoted by U_i) is given by

$$U_i(x^i, y^i, r) = u_i(x^i, y^i) + \sum_{j=1}^n (1 - \tau_{ij}) r_j (x_{ij} - y_{ij}) \tag{3.9}$$

where u_i is a strictly concave and continuously differentiable function and the constraints on assets and liabilities are described by a general convex and compact set P_i. Thus, each sector i wishes to solve the problem

$$\max_{(x^i,y^i)\in P_i} \ U_i(x^i, y^i, r), \tag{3.10}$$

while each financial instrument j wishes to solve the problem

$$\min_{r_j \in [0,\bar{r}_j]} \ V_j(x,y,r_j) = r_j \sum_{i=1}^m (1-\tau_{ij})(x_{ij} - y_{ij}). \tag{3.11}$$

The solution concept of our general Game Theory model is given by the following definition.

Definition 3.2. A vector $(x^*, y^*, r^*) \in \prod_{i=1}^m P_i \times \prod_{j=1}^n [0, \bar{r}_j]$ is a Nash equilibrium if

$$U_i(x^{i*}, y^{i*}, r^*) \geq U_i(x^i, y^i, r^*), \qquad \forall \ (x^i y^i) \in P_i, \quad \forall \ i = 1, \dots, m, \tag{3.12}$$

$$V_j(x^*, y^*, r_j^*) \leq V_j(x^*, y^*, r_j), \qquad \forall \ r_j \in [0, \bar{r}_j], \quad \forall \ j = 1, \dots, n. \tag{3.13}$$

Having written for the U_i a maximum problem, the corresponding components of the pseudogradient will be negative. We can easily compute:

$$\begin{aligned} &\nabla_{x^i} U_i(x^i, y^i, r) = \nabla_{x^i} u_i(x^i, y^i) + \nabla_{x^i} \sum_{j=1}^n (1-\tau_{ij}) r_j (x_{ij} - y_{ij}), \\ &\nabla_{y^i} U_i(x^i, y^i, r) = \nabla_{y^i} u_i(x^i, y^i) + \nabla_{x^i} \sum_{j=1}^n (1-\tau_{ij}) r_j (x_{ij} - y_{ij}), \\ &\frac{\partial}{\partial x_{il}} \sum_{j=1}^n (1-\tau_{ij}) r_j (x_{ij} - y_{ij}) = (1-\tau_{il}) r_l, \end{aligned}$$

whence

$$\nabla_{x^i} \sum_{j=1}^n (1-\tau_{ij}) r_j (x_{ij} - y_{ij}) = [(1-\tau_{i1}) r_1, \dots, (1-\tau_{il}) r_l, \dots, (1-\tau_{in}) r_n] = (I - \tau_i) r$$

where $\tau_i = diag(\tau_{i1}, \ldots, \tau_{in})$. Therefore,

$$\nabla_{x^i} U_i(x^i, y^i, r) = \nabla_{x^i} u_i(x^i, y^i) + (I - \tau_i) r,$$
$$\nabla_{y^i} U_i(x^i, y^i, r) = \nabla_{i} u_i(x^i, y^i) - (I - \tau_i) r.$$

The components of the pseudogradient associated with the financial instruments are given by

$$\frac{\partial V_j}{\partial r_j} = \sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij} - y_{ij}).$$

If we denote the set of feasible prices by

$$Q = \{r \in \mathbb{R}^n : \ 0 \leq r \leq \bar{r}\},$$

then the new feasible set of assets liabilities and prices is

$$C = \prod_{i=1}^{m} P_i \times Q.$$

We continue to denote with F the pseudogradient of this general game. The variational inequality $VI(F, C)$ reads as follows: find $(x^*, y^*, r^*) \in C$ such that

$$\begin{aligned} &- \sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i*}, y^{i*}) + r^{*\top}(I - \tau_i)]^\top (x^i - x^{i*}) \\ &- \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i*}, y^{i*}) - r^{*\top}(I - \tau_i)]^\top (y^i - y^{i*}) \\ &+ \sum_{j=1}^{n} \Big[\sum_{i=1}^{m} (1 - \tau_{ij})(x^{i*} - y^{i*}) \Big] (r_j - r_j^*) \geq 0, \qquad \forall (x, y, r) \in C. \end{aligned} \tag{3.14}$$

Since the pseudogradient F is continuous and the set C is compact and convex, the existence of solutions of (3.14) is ensured (see, e.g., [6]).

We conclude this section by briefly recalling the concept of *price of anarchy* [14]. Assume we are given a noncooperative game in standard form, and compute the Welfare function W, which is the sum of the utility functions of all players. Let us also assume that the game has a unique Nash equilibrium z^N and that the Welfare function has a unique maximum point z^O (also called social optimum). The price of anarchy, defined as

$$\gamma = \frac{W(z^N)}{W(z^O)}, \tag{3.15}$$

is a measure of how much the selfish behavior of players affect the social welfare. In the following we also show that even if our model yields multiple Nash equilibria, the price of anarchy is still well defined.

4. Features of the Game Theory Model

The utility functions in (3.9) and (3.11) produce a pseudogradient that is monotone, but not strictly monotone (see [10]):

$$[F(x', y', r') - F(x'', y'', r'')]^\top [(x', y', r') - (x'', y'', r'')] \geq 0, \quad \forall (x', y', r'), (x'', y'', r'') \in C,$$

so that we can expect that the solution is not unique. We recall that F is called strictly monotone if in the above inequality, the equality sign only holds for $(x', y', r') = (x'', y'', r'')$. The structure of the solution set is investigated in the following theorem.

Theorem 4.1 (Equilibrium prices)**.** *The following statements hold:*

a) If (x^*, y^*, r^*) and (x^{**}, y^{**}, r^{**}) are two different Nash equilibria, then $r^* \neq r^{**}$.

b) Suppose the taxes only depend on the sectors, i.e., $\tau_{ij} = \sigma_i$ for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$. If (x^*, y^*, r^*) is a Nash equilibrium, then (x^*, y^*, r^{**}) is a Nash equilibrium as well, where

$$r_j^{**} = \begin{cases} 0 & \text{if } r_j^* = 0, \\ \bar{r}_j & \text{if } r_j^* = \bar{r}_j, \\ r_j^* + \alpha & \text{if } r_j^* \in (0, \bar{r}_j), \end{cases}$$

for any $j = 1, \ldots, n$ and

$$-\min\{r_j^* : \ r_j^* \in (0, \bar{r}_j)\} \leq \alpha \leq \min\{\bar{r}_j - r_j^* : \ r_j^* \in (0, \bar{r}_j)\}. \tag{4.1}$$

c) Suppose the taxes only depend on the sectors, i.e., $\tau_{ij} = \sigma_i$ for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$. If (x^*, y^*, r^*) and (x^*, y^*, r^{**}) are Nash equilibria and there is a sector i such that $x_{ij}^* > 0$ holds for any $j = 1, \ldots, n$ or $y_{ij}^* > 0$ holds for any $j = 1, \ldots, n$, then there exists $\alpha \in \mathbb{R}$ such that $r_j^* - r_j^{**} = \alpha$ for any $j = 1, \ldots, n$.

Proof. a) Assume, by contradiction, that $r^* = r^{**}$. Since (x^*, y^*, r^*) and (x^{**}, y^{**}, r^{**}) are different Nash equilibria, there exists $k \in \{1, \ldots, m\}$ such that $x^{k*} \neq x^{k**}$ or $y^{k*} \neq y^{k**}$. Moreover, (x^*, y^*, r^*) and (x^{**}, y^{**}, r^{**}) solve $VI(F, C)$, hence the following inequalities hold:

$$\begin{aligned} &-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i*}, y^{i*}) + r^{*\top}(I - \tau_i)]^\top (x^{i**} - x^{i*}) \\ &\qquad - \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i*}, y^{i*}) - r^{*\top}(I - \tau_i)]^\top (y^{i**} - y^{i*}) \geq 0, \\ &-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i**}, y^{i**}) + r^{*\top}(I - \tau_i)]^\top (x^{i*} - x^{i**}) \\ &\qquad - \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i**}, y^{i**}) - r^{*\top}(I - \tau_i)]^\top (y^{i*} - y^{i**}) \geq 0. \end{aligned}$$

If we sum the latter inequalities, we get

$$\begin{aligned} &-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i*}, y^{i*}) - \nabla_{x^i} u_i(x^{i**}, y^{i**})]^\top (x^{i**} - x^{i*}) \\ &\qquad - \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i*}, y^{i*}) - \nabla_{y^i} u_i(x^{i**}, y^{i**})]^\top (y^{i**} - y^{i*}) \geq 0. \end{aligned} \tag{4.2}$$

On the other hand, each function u_i is strictly concave, hence the operator $-\nabla u_i$ is strictly monotone and we have

$$\begin{aligned} &[\nabla_{x^i} u_i(x^{i*}, y^{i*}) - \nabla_{x^i} u_i(x^{i**}, y^{i**})]^\top (x^{i**} - x^{i*}) \\ &\qquad + [\nabla_{y^i} u_i(x^{i*}, y^{i*}) - \nabla_{y^i} u_i(x^{i**}, y^{i**})]^\top (y^{i**} - y^{i*}) \geq 0 \qquad \forall\, i \neq k, \\ &[\nabla_{x^k} u_k(x^{k*}, y^{k*}) - \nabla_{x^k} u_k(x^{k**}, y^{k**})]^\top (x^{k**} - x^{k*}) \\ &\qquad + [\nabla_{y^k} u_k(x^{k*}, y^{k*}) - \nabla_{y^k} u_k(x^{k**}, y^{k**})]^\top (y^{k**} - y^{k*}) > 0. \end{aligned}$$

Therefore, we get

$$\sum_{i=1}^{m}[\nabla_{x^i} u_i(x^{i*}, y^{i*}) - \nabla_{x^i} u_i(x^{i**}, y^{i**})]^\top (x^{i**} - x^{i*}) \\ + \sum_{i=1}^{m}[\nabla_{y^i} u_i(x^{i*}, y^{i*}) - \nabla_{y^i} u_i(x^{i**}, y^{i**})]^\top (y^{i**} - y^{i*}) > 0,$$

that contradicts (4.2). Therefore, we proved that $r^* \neq r^{**}$.

b) Since problems (3.10),(3.11) are concave and convex, respectively, (x^*, y^*, r^*) is a Nash equilibrium if and only if it solves the following KKT system:

$$\begin{aligned}
& -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^* + \lambda_i^* - \nu_{ij}^* = 0 && \forall\, i, j, \\
& -\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^* + \mu_i^* - \rho_{ij}^* = 0 && \forall\, i, j, \\
& \sum_{i=1}^{m}(1-\sigma_i)(x_{ij}^* - y_{ij}^*) - \beta_j^* + \gamma_j^* = 0 && \forall\, j, \\
& \nu_{ij}^* x_{ij}^* = 0 && \forall\, i, j, \\
& \rho_{ij}^* y_{ij}^* = 0 && \forall\, i, j, \\
& \beta_j^* r_j^* = 0 && \forall\, j, \\
& \gamma_j^* (r_j^* - \bar{r}_j) = 0 && \forall\, j, \\
& \sum_{j=1}^{n} x_{ij}^* = s_i, \qquad \sum_{j=1}^{n} y_{ij}^* = s_i && \forall\, i, \\
& x_{ij}^* \geq 0, \qquad y_{ij}^* \geq 0, \qquad \nu_{ij}^* \geq 0, \qquad \rho_{ij}^* \geq 0 && \forall\, i, j, \\
& 0 \leq r_j^* \leq \bar{r}_j, \qquad \beta_j^* \geq 0, \qquad \gamma_j^* \geq 0 && \forall\, j,
\end{aligned}$$

where multipliers λ_i^* and μ_i^* are associated to constraints (2.2), ν_{ij}^* and ρ_{ij}^* to constraints (2.3), while β_j^* and γ_j^* to the constraints on r_j. The latter system is equivalent to the following system of equalities and inequalities:

$$-\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^* + \lambda_i^* \geq 0 \qquad \forall\, i, j, \tag{4.4a}$$

$$-\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^* + \mu_i^* \geq 0 \qquad \forall\, i, j, \tag{4.4b}$$

$$\sum_{i=1}^{m}(1-\sigma_i)(x_{ij}^* - y_{ij}^*) + \gamma_j^* \geq 0 \qquad \forall\, j, \tag{4.4c}$$

$$x_{ij}^* \left[-\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^* + \lambda_i^*\right] = 0 \qquad \forall\, i, j, \tag{4.4d}$$

$$y_{ij}^* \left[-\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^* + \mu_i^*\right] = 0 \qquad \forall\, i, j, \tag{4.4e}$$

$$r_j^* \left[\sum_{i=1}^{m}(1-\sigma_i)(x_{ij}^* - y_{ij}^*) + \gamma_j^*\right] = 0 \qquad \forall\, j, \tag{4.4f}$$

$$\gamma_j^* (r_j^* - \bar{r}_j) = 0 \qquad \forall\, j, \tag{4.4g}$$

$$\sum_{j=1}^{n} x_{ij}^* = s_i, \qquad \sum_{j=1}^{n} y_{ij}^* = s_i \qquad \forall\, i, \tag{4.4h}$$
$$x_{ij}^* \geq 0, \qquad y_{ij}^* \geq 0 \qquad \forall\, i, j, \tag{4.4i}$$
$$0 \leq r_j^* \leq \bar{r}_j, \qquad \gamma_j^* \geq 0 \qquad \forall\, j. \tag{4.4j}$$

We prove that (x^*, y^*, r^{**}) is a Nash equilibrium since it solves system (4.4) with multipliers

$$\begin{aligned} \lambda_i^{**} &= \lambda_i^* + (1-\sigma_i)\alpha && \forall\, i = 1, \dots, m, \\ \mu_i^{**} &= \mu_i^* - (1-\sigma_i)\alpha && \forall\, i = 1, \dots, m, \\ \gamma_j^{**} &= \gamma_j^* && \forall\, j = 1, \dots, n. \end{aligned}$$

In fact, we have

$$\begin{aligned} -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^{**} + \lambda_i^{**} &= -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^* - (1-\sigma_i)\alpha + \lambda_i^* + (1-\sigma_i)\alpha \\ &= -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) - (1-\sigma_i) r_j^* + \lambda_i^*, \end{aligned}$$

thus (4.4a) and (4.4d) hold. Moreover, we have

$$\begin{aligned} -\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^{**} + \mu_i^{**} &= -\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^* + (1-\sigma_i)\alpha + \mu_i^* - (1-\sigma_i)\alpha \\ &= -\frac{\partial u_i}{\partial y_{ij}}(x^*, y^*) + (1-\sigma_i) r_j^* + \mu_i^*, \end{aligned}$$

thus (4.4b) and (4.4e) hold. If $r_j^{**} = 0$, then (4.4f) trivially holds, while if $r_j^{**} > 0$, then, by definition, we have $r_j^* > 0$, thus $\sum_{i=1}^m (1-\sigma_i)(x_{ij}^* - y_{ij}^*) + \gamma_j^* = 0$ and hence (4.4f) holds. Moreover, if $r_j^{**} = \bar{r}_j$, then (4.4g) holds, while if $r_j^{**} < \bar{r}_j$, then, by definition, $r_j^* < \bar{r}_j$, thus $\gamma_j^* = 0$ and (4.4g) holds. Finally, condition (4.1) guarantees that $r^{**} \in [0, \bar{r}]$.

c) Since (x^*, y^*, r^*) and (x^*, y^*, r^{**}) are Nash equilibria, they satisfy system (4.4) with multipliers $(\lambda^*, \mu^*, \gamma^*)$ and $(\lambda^{**}, \mu^{**}, \gamma^{**})$, respectively. Suppose that $x_{ij}^* > 0$ holds for any $j = 1, \dots, n$ and define $\alpha = (\lambda_i^* - \lambda_i^{**})/(1-\sigma_i)$. Then, we get from (4.4d) the following equations:

$$\begin{aligned} (1-\sigma_i) r_j^* &= -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) + \lambda_i^* && \forall\, j = 1, \dots, n, \\ (1-\sigma_i) r_j^{**} &= -\frac{\partial u_i}{\partial x_{ij}}(x^*, y^*) + \lambda_i^{**} && \forall\, j = 1, \dots, n, \end{aligned}$$

hence $r_j^* - r_j^{**} = \alpha$ for any $j = 1, \dots, n$. If $y_{ij}^* > 0$ holds for any $j = 1, \dots, n$, then we can get the thesis with $\alpha = (\mu_i^{**} - \mu_i^*)/(1-\sigma_i)$ by exploiting (4.4e).

□

Theorem 4.2 (Price of Anarchy)**.** *Consider the general model where the utility functions of the sectors are given as in* (3.9)*, those of the financial instruments by* (3.11)*, and the price of each financial instrument j is nonnegative and bounded from above by $\bar{r}_j$. If (x^*, y^*, r^*) is a Nash equilibrium and $r_j^* < \bar{r}_j$ for any $j = 1, \dots, n$, then the Price of Anarchy is equal to 1.*

Proof. Let us notice that in our model the welfare function is given by:

$$W(x, y, r) = \sum_{i=1}^{m} u_i(x^i, y^i).$$

It is also convenient to consider the function:

$$\widetilde{W}(x,y) = \sum_{i=1}^{m} u_i(x^i, y^i).$$

Since $\widetilde{W}$ is strictly concave and continuous in the compact set $\prod_i P_i$, we get that it exists a unique point $(\bar{x}, \bar{y})$ such that:

$$\widetilde{W}(\bar{x},\bar{y}) = \max_{(x,y)\in\prod_i P_i} \widetilde{W}(x,y). \tag{4.5}$$

Under our convexity and differentiability hypotheses, the necessary and sufficient condition for (4.5) is that for all $(x,y) \in \prod_{i=1}^{n} P_i$:

$$-\sum_{i=1}^{m} [\nabla_{x^i} u_i(\bar{x},\bar{y})]^\top \cdot (x^i - \bar{x}^i) - \sum_{i=1}^{m} [\nabla_{y^i} u_i(\bar{x},\bar{y})]^\top \cdot (y^i - \bar{y}^i) \geq 0. \tag{4.6}$$

Let us now consider the further set of constraints:

$$S = \left\{ (x,y) \in \prod_{i=1}^{n} P_i : \sum_{i=1}^{m} (1-\tau_{ij})(x_{ij} - y_{ij}) \geq 0,\ \forall j = 1,\dots,n \right\}$$

and let (x^*, y^*, r^*) be a Nash equilibrium of the game. Thus, (x^*, y^*, r^*) satisfies (3.14), where we can choose $r_j = r_j^*$ for any $j = 1,\dots,n$, so as to obtain:

$$\begin{aligned} &-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i*}, y^{i*}) + r^{*\top}(I - \tau_i)]^\top (x^i - x^{i*}) \\ &-\sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i*}, y^{i*}) - r^{*\top}(I - \tau_i)]^\top (y^i - y^{i*}) \geq 0, \quad \forall (x,y,r) \in C, \end{aligned}$$

which yields:

$$\begin{aligned} &-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^{i*}, y^{i*})]^\top (x^i - x^{i*}) - \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^{i*}, y^{i*})]^\top (y^i - y^{i*}) \\ &\geq r^{*\top}(I - \tau_i) \cdot (x^i - y^i) - r^{*\top}(I - \tau_i) \cdot (x^{i*} - y^{i*}) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} r_j^* (1-\tau_{ij})(x_{ij} - y_{ij}) - \sum_{j=1}^{n} \sum_{i=1}^{m} r_j^* (1-\tau_{ij})(x_{ij}^* - y_{ij}^*). \end{aligned} \tag{4.7}$$

We now investigate the sign of the second summand of the right-hand side, i.e., $\sum_{j=1}^{n} \sum_{i=1}^{m} r_j^*(1 - \tau_{ij})(x_{ij}^* - y_{ij}^*)$. If in (3.14) we choose $x^i = x^{i*}, y^i = y^{i*}$ for any $i = 1,\dots,m$ and, if $0 < r_k^* < \bar{r}_k$, $r_k = r_k^* \pm \varepsilon$ with $\varepsilon > 0$ small enough and $r_j = r_j^*$ for any $j \neq k$, then we obtain

$$\sum_{i=1}^{m} (1-\tau_{ik})(x_{ik}^* - y_{ik}^*) = 0.$$

On the other hand, if $r_k^* = 0$, we get $\sum_{i=1}^{m} (1-\tau_{ik})(x_{ik}^* - y_{ik}^*) \geq 0$. We can then conclude that $(x^*, y^*) \in S$ and

$$-\sum_{i=1}^{m} [\nabla_{x^i} u_i(x^*, y^*)]^\top (x^i - x^{i*}) - \sum_{i=1}^{m} [\nabla_{y^i} u_i(x^*, y^*)]^\top (y^i - y^{i*}) \geq 0, \qquad \forall (x,y) \in S, \tag{4.8}$$

which is the necessary and sufficient condition for (x^*, y^*) to be the (unique) maximum point of $\widetilde{W}$ in S. Thus, $x^* = x^S$ and $y^* = y^S$, where (x^S, y^S) is the unique maximum point of $\widetilde{W}$in S, which entails that any two Nash equilibria can differ only for the price components, provided that none of them reaches its upper bound. We have also proved that, if the Welfare function W is considered in $S \times [0, \bar{r}]$, instead of $\prod_{i=1}^{m} P_i \times [0, \bar{r}]$, the price of anarchy is 1. □

Corollary 4.3 (Tikhonov regularization)**.** *Let I be the identity matrix in $\mathbb{R}^{2mn+n}$, $\varepsilon > 0$, and consider the variational inequality $VI(F+\varepsilon I, C)$. Denote with $(x_\varepsilon^*, y_\varepsilon^*, r_\varepsilon^*)$ its unique solution. We then have that, for $\varepsilon \to 0$, $(x_\varepsilon^*, y_\varepsilon^*, r_\varepsilon^*)$ converges to a point $(x^*, y^*, r^*) \in C$, which represents the solution of $VI(F, C)$ with the price vector of minimum norm, among all the solutions (x', y', r') which satisfy $r' < \bar{r}$.*

Proof. The well known elliptic (or Tikhonov) regularization procedure for monotone variational inequalities (see, e.g., [6]) ensures that that if F is monotone then for each $\varepsilon > 0$, $F + \varepsilon I$ is a strictly monotone operator on the compact set C and hence, there is a unique solution to $VI(F+\varepsilon I, C)$ which converges to the minimum norm solution of $VI(F, C)$. However, according to the previous theorem, every two solutions of $VI(F, C)$ with $r < \bar{r}$ may only differ for the price vector. It then follows that the solution thus obtained is the solution which yields a price vector of minimum norm, among all the solutions (x', y', r') which satisfy $r' < \bar{r}$. □

5. Numerical Experiments

In this section we report some numerical experiments on the game theory model described in Section 3. The variational inequality (3.14) has been numerically solved by implementing in MATLAB the optimization reformulation proposed in [1]. We consider the same setting as in the example described in [10]: there are two sectors ($m = 2$) and three financial instruments ($n = 3$), with $s_1 = 1$, $s_2 = 2$ and the variance-covariance matrices defined as follows:

$$Q^1 = \begin{pmatrix} 1 & 0.15 & 0.3 & -0.2 & -0.1 & 0 \\ 0.15 & 1 & 0.1 & -0.1 & -0.2 & 0 \\ 0.3 & 0.1 & 1 & -0.3 & 0 & -0.1 \\ -0.2 & -0.1 & -0.3 & 1 & 0 & 0.3 \\ -0.1 & -0.2 & 0 & 0 & 1 & 0.2 \\ 0 & 0 & -0.1 & 0.3 & 0.2 & 1 \end{pmatrix},$$

$$Q^2 = \begin{pmatrix} 1 & 0.4 & 0.3 & -0.1 & -0.1 & 0 \\ 0.4 & 1 & 0.5 & 0 & -0.05 & 0 \\ 0.3 & 0.5 & 1 & 0 & 0 & -0.1 \\ -0.1 & 0 & 0 & 1 & 0.5 & 0 \\ -0.1 & -0.05 & 0 & 0.5 & 1 & 0.2 \\ 0 & 0 & -0.1 & 0 & 0.2 & 1 \end{pmatrix}.$$

First, we show that the assumption $r^* < \bar{r}$ in Theorem 4.2 is necessary for the Nash equilibrium (x^*, y^*, r^*) to have the price of anarchy equal to 1. In fact, if we set $\bar{r}_j = 1$ for any $j = 1, \ldots, 3$ and the taxes τ_{ij} are defined as follows:

$$\begin{array}{lll} \tau_{11} = 0.28, & \tau_{12} = 0.27, & \tau_{13} = 0.28, \\ \tau_{21} = 0.30, & \tau_{22} = 0.30, & \tau_{23} = 0.22, \end{array}$$

then

$$\begin{array}{lll} x_{11}^* = 0.3136, & x_{12}^* = 0.3774, & x_{13}^* = 0.3091, \\ x_{21}^* = 0.8420, & x_{22}^* = 0.4364, & x_{23}^* = 0.7216, \\ y_{11}^* = 0.4475, & y_{12}^* = 0.3792, & y_{13}^* = 0.1733, \\ y_{21}^* = 0.7186, & y_{22}^* = 0.4345, & y_{23}^* = 0.8469, \\ r_1^* = 1.0000, & r_2^* = 0.9843, & r_3^* = 0.8717 \end{array}$$

is a Nash equilibrium such that $r_1^* = \bar{r}_1$ and (x^*, y^*) does not belong to the set S since

$$(1 - \tau_{11})\,(x_{11}^* - y_{11}^*) + (1 - \tau_{21})\,(x_{21}^* - y_{21}^*) = -0.01 < 0.$$

Therefore, (x^*, y^*, r^*) cannot be the maximizer of the welfare function W in $S \times [0, \bar{r}]$. Moreover, if W is considered in $\prod_{i=1}^m P_i \times [0, \bar{r}]$, we have

$$W(x^*, y^*, r^*) = -4.2557, \qquad \max_{\prod_{i=1}^m P_i \times [0,\bar{r}]} W(x, y, r) = -4.0134,$$

thus the price of anarchy at the equilibrium (x^*, y^*, r^*) is equal to 1.06.

Subsequently, we consider 100 different randomly generated scenarios, with the upper bounds $\bar{r}_j = 5$ for any $j = 1, \ldots, 3$ and each tax rate τ_{ij} defined as a uniformly distributed random number between 0.2 and 0.3. The mean values and standard deviations of the Nash equilibria obtained by applying the Tikhonov regularization method for each scenario are shown in Table 1 (columns 2-3).

Moreover, since the Nash equilibria of the game satisfy special properties in the case where the taxes depend only on the sectors (see Theorem 4.1), we consider 100 additional randomly generated scenarios, with $\bar{r}_j = 5$ for any $j = 1, \ldots, 3$, and $\tau_{ij} = \sigma_i$, for any $i = 1, 2$, and $j = 1, \ldots, 3$, where σ_i are uniformly distributed random numbers between 0.2 and 0.3. The mean values and standard deviations of the Nash equilibria obtained by applying the Tikhonov regularization method for each scenario are shown in Table 1 (columns 4-5). Notice that the variability of equilibrium prices is much higher when taxes τ_{ij} are independent of each other than when they depend only on sectors (in accordance with the results of Theorem 4.1 and 4.2).

Table 1. Mean values and standard deviations of the Nash equilibria obtained by applying the Tikhonov regularization method to 100 randomly generated scenarios in taxes.

	General τ_{ij}		$\tau_{ij} = \sigma_i$	
Variable	Mean values	Std deviation	Mean values	Std deviation
x_{11}	0.3267	0.0275	0.3117	0.0026
x_{12}	0.3707	0.0164	0.3684	0.0005
x_{13}	0.3026	0.0254	0.3199	0.0032
x_{21}	0.8354	0.0261	0.8522	0.0012
x_{22}	0.4415	0.0187	0.4499	0.0005
x_{23}	0.7230	0.0346	0.6979	0.0017
y_{11}	0.4369	0.0257	0.4529	0.0029
y_{12}	0.3862	0.0164	0.3879	0.0004
y_{13}	0.1769	0.0265	0.1592	0.0033
y_{21}	0.7261	0.0259	0.7106	0.0012
y_{22}	0.4271	0.0180	0.4304	0.0003
y_{23}	0.8468	0.0170	0.8591	0.0008
r_1	1.9112	1.9505	0.0934	0.0086
r_2	1.9167	1.9813	0.0687	0.0055
r_3	1.8361	1.9708	0.0000	0.0000

6. Conclusions

In this paper we have proved that a Game Theory formulation of a previously proposed economic model can provide a mechanism which explains the equilibrium price formation in the market. Since our problem has multiple solutions, we have investigated the relationship among them. Moreover, we have computed the price of anarchy of the game and proved that it is equal to 1 provided that the equilibrium prices are all below their upper bound. At last, we have explained how to compute the solution whose price vector has minimum norm, and illustrated our findings with a numerical example with different taxation scenarios. Future research will deal with the extension of this approach to more complex market structures, in particular, taking into account financial intermediaries. Another interesting research perspective is to consider random perturbations of the taxes and, instead of generating different scenarios, frame the problem within the theory of stochastic variational inequality [3].

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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