

Deontic Defeasible Description Logic

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Abstract

We introduce an extension of Description Logics (DL), appropriate for modelling and reasoning about deontic notions. Moving from *defeasible ALC*, an extension of the DL *ALC* which allows us to model and reason about defeasible information (rules that admit exceptions), we show how to modify the semantics in order to also represent and manage conditional *obligations* and *permissions*, and we present the correspondent, and easily implementable, decision procedures. Also, we define a semantics combining reasoning about expectations and reasoning about norms, pairing it with corresponding decision procedures.

Keywords

Description Logics, Deontic Reasoning, Defeasible Reasoning

1. Introduction

Deontic logic [1, 2] and Description Logics (DLs) [3] have been developed under very different circumstances. Deontic logic started off as a branch of philosophical logic to formalize relations between deontic concepts like norms, obligations and permissions, whereas DLs were developed as a branch of knowledge representation to reason about taxonomies and ontologies. Nevertheless, there are striking similarities between the two formalisms. For example, both use a *conditional* or *rule-based* representation, and consider various kinds of inferences.

Deontic and description logics have both dealt with defeasible reasoning. The former dates back to the eighties, when *prima facie* norms and moral dilemmas became a topic of study, using techniques from non-monotonic logic [4, 5, 6]. Over the past fifteen years, various defeasible DLs were also introduced to reason about exceptions and conflicts [7, 8, 9, 10, 11]. In the philosophical literature, it is common to restrict the attention to propositional modal logic, whereas in the application of deontic logic to natural language semantics, it is common to use full first-order logic. In the area of legal informatics fragments of first-order logic have been used with good computational properties, such as defeasible logics [12]. Deontic defeasible description logics aim at a trade-off between expressive power and good computational properties, still being able to tackle the main reasoning challenges of deontic logic. Classical DLs have already been extended to formalize constitutive norms [13], but defeasible logics are used regularly to model constitutive norms as well. In our formalism we consider defeasible constitutive and regulative norms, which are both described and reasoned about using defeasible DLs.

In Section 2 we briefly introduce dyadic deontic logic and defeasible DLs. Section 3 presents the language, semantics, and an entailment relation for deontic description logics. Section 4 introduces a framework combining expectations and norms in our reasoning.

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2. Background

We first provide a very basic introduction to defeasible deontic logic and defeasible description logics. *Deontic Logic*: Let L be a classical logical language (e.g. propositional or first-order logic) and $\rightarrow_C$, $\rightarrow_O$, and $\rightarrow_P$ be conditional implications. A normative system is a tuple $\{\mathcal{C}, \mathcal{R}, \mathcal{A}\}$ containing a set $\mathcal{C} = \{A_i \rightarrow_C B_i\}$ of *constitutive norms*, with $A_i, B_i \in L$, describing the (legal) notions read as “ A_i counts as B_i ,” a set $\mathcal{R} = \{A_i \rightarrow_X B_i\}$ of *regulative norms*, with $A_i, B_i \in L$ and $X \in \{O, P\}$, describing the regulative part of the normative system; $A \rightarrow_O B$ is read as “if A then it is obligatory that B ,” $A \rightarrow_P B$ is read as “if A then B is permitted,” and $\mathcal{A} \subseteq L$. Reasoning in this setting consists of deriving propositions from $\mathcal{C}$ (*institutional facts*), and from $\mathcal{C}$ and $\mathcal{R}$ together (*obligations* and *permissions*).

In deontic logic, conflicts may arise (i) between constitutive norms, expressing exceptions in the legal ontology; (ii) between regulative norms, expressing moral dilemmas or contrary-to-duty reasoning; or (iii) between constitutive and regulative norms, expressing violations. These are managed using techniques from dyadic deontic logic [14, 4, 15, 16, 17, 18, 19, 12, 20, 21]. We follow a similar approach.

Defeasible Description Logic: We assume basic knowledge of the DL $\mathcal{ALC}$, and refer to [3] for the details. In the last 15 years DLs have been extended to handle defeasible knowledge, especially working with preferential semantics [22, 23, 24, 25, 8, 26, 27, 28, 29, 10, 30, 31]. Here we refer in particular to the framework by Britz et al.. *Defeasible $\mathcal{ALC}$* extends $\mathcal{ALC}$ with *defeasible concept inclusions* of the form $C \sqsubseteq_E D$ expressing that ‘if an individual is a C , it *usually* also is a D .’ A *defeasible KB* is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, where $\mathcal{T}$ is a TBox and $\mathcal{D}$ is a finite set of defeasible inclusions called a *DBox*.

The semantics is based on *ranked interpretations*, obtained through a *preference relation* $\prec_E^{\mathcal{R}}$ over the interpretation domain, where $a \prec_E^{\mathcal{R}} b$ ($a, b \in \Delta^{\mathcal{I}}$) means that ‘the object a is more typical than the object b .’ Given a set X and an order $\prec$, $\min_{\prec}(X)$ denotes the set of the minimal elements of X w.r.t. $\prec$; i.e., $\min_{\prec}(X) = \{x \in X \mid \text{for every } y \in X, y \not\prec x\}$.

Definition 1 (Ranked interpretation). *Given a set X , the binary relation $\prec \subseteq X \times X$ is a ranked order if there is some mapping $h : X \rightarrow \mathbb{N}$ (the height function) such that i) for every $i \in \mathbb{N}$ and $0 \leq j < i$, if $h(x) = i$ for some $x \in X$, then there is a $y \in X$ for which $h(y) = j$; and ii) for every $x, y \in X$, $x \prec y$ iff $h(x) < h(y)$. A ranked interpretation is a triple $R = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle$, where $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ is a DL interpretation, and $\prec_E^{\mathcal{R}}$ is a ranked order over $\Delta^{\mathcal{R}}$. We indicate with $h_{\prec_E^{\mathcal{R}}}$ the associated height function.*

The height function $h_{\prec_E^{\mathcal{R}}}$ can be extended to concepts by considering the lowest height of its members; that is, $h_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) = \min\{h_{\prec_E^{\mathcal{R}}}(x) \mid x \in \Delta^{\mathcal{R}} \text{ and } x \in C^{\mathcal{R}}\}$.

Definition 2 (Satisfaction). *Let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle$ be a ranked interpretation. $\mathcal{R}$ satisfies (is a model of) $C \sqsubseteq_E D$ (denoted $\mathcal{R} \models C \sqsubseteq_E D$) iff $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$. $\mathcal{R}$ satisfies (is a model of) $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ iff $\mathcal{R} \models \alpha$ for all the statements $\alpha \in \mathcal{T} \cup \mathcal{D}$. The KB $\mathcal{K}$ is rank-satisfiable iff it has a ranked model. Two KBs $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ and $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}' \rangle$ are rank-equivalent iff they have exactly the same ranked models.*

This semantics can model defeasible information in the form of inclusion axioms allowing for exceptions: $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$ means that all the most typical objects in C are in D , allowing for the existence of more exceptional objects in C which are not in D .

Example 1. *In classical $\mathcal{ALC}$, the TBox $\mathcal{T} = \{PhD \sqsubseteq Ad, PhD \sqsubseteq \neg W, Ad \sqsubseteq W\}$ where PhD , Ad , and W refer to the concepts ‘PhD student,’ ‘Adult,’ and ‘Worker’ respectively, implies that PhD students cannot exist. Instead we want to indicate that they are exceptional adults who are not workers. To achieve this, it suffices to change the GCI $Ad \sqsubseteq W$ into the defeasible inclusion $Ad \sqsubseteq_E W$ (adults are usually workers). A ranked interpretation $\mathcal{R}$ in which all typical adults (the objects in $\min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}})$) are workers ($\min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}}) \subseteq W^{\mathcal{R}}$), while PhD students are atypical adults ($PhD^{\mathcal{R}} \subseteq (Ad^{\mathcal{R}} \setminus \min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}}))$) is a ranked model of the KB. We want also to enforce the possibility that some PhD students could be particularly young, or that they are working students, hence, we end up with an ontology $\langle \mathcal{T}, \mathcal{D} \rangle$ where $\mathcal{T}$ is empty and $\mathcal{D} = \{PhD \sqsubseteq_E Ad, PhD \sqsubseteq_E \neg W, Ad \sqsubseteq_E W\}$.*

It is possible to define multiple kinds of defeasible entailment for DLs [33, 26, 34, 35, 36, 37]. Here we refer exclusively to Rational Closure (RC), which has been characterised in various inferentially-equivalent ways [38, 39, 40, 41, 10, 31, 32], and we are going to refer to one DL-specific characterisation [42, 32]. We briefly introduce the *big ranked model*, the semantic construction used to characterise the RC of a KB $\mathcal{K}$. We refer the reader to the work of Britz et al. (2021) for the details. The construction is based on the *ranked union*, a variant of the *disjoint union* of DL interpretations. We generalise the original definition by Britz et al. to consider models with possibly many ranked orders defined over the domain.

Definition 3 (Ranked Union). Let $\mathfrak{R}$ be a set of interpretations $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_1^{\mathcal{R}}, \dots, \prec_n^{\mathcal{R}} \rangle$ ($n \geq 0$), with each $\prec_j^{\mathcal{R}}$ being a ranked order defined over the domain $\Delta^{\mathcal{R}}$.

The Ranked Union of $\mathfrak{R}$ is an interpretation $\mathcal{R}^{\mathfrak{R}} := \langle \Delta^{\mathfrak{R}}, \cdot^{\mathfrak{R}}, \prec_1^{\mathfrak{R}}, \dots, \prec_n^{\mathfrak{R}} \rangle$ where:

- $\Delta^{\mathfrak{R}} := \coprod_{\mathcal{R} \in \mathfrak{R}} \Delta^{\mathcal{R}}$, i.e., the disjoint union of the domains from $\mathfrak{R}$, where each $\mathcal{R} \in \mathfrak{R}$ has the elements $x, y, \dots$ of its domain renamed as $x_{\mathcal{R}}, y_{\mathcal{R}}, \dots$ so that they are all distinct in $\Delta^{\mathfrak{R}}$;
- $x_{\mathcal{R}} \in A^{\mathfrak{R}}$ iff $x \in A^{\mathcal{R}}$;
- $(x_{\mathcal{R}}, y_{\mathcal{R}}) \in r^{\mathfrak{R}}$ iff $\mathcal{R} = \mathcal{R}'$ and $(x, y) \in r^{\mathcal{R}'}$;
- for all i ($1 \leq i \leq n$), $h_{\prec_i^{\mathfrak{R}}}(x_{\mathcal{R}}) = h_{\prec_i^{\mathcal{R}}}(x)$ (By Definition 1, $x \prec_i^{\mathfrak{R}} y$ iff $h_{\prec_i^{\mathfrak{R}}}(x) < h_{\prec_i^{\mathfrak{R}}}(y)$).

To define the *big ranked model* for a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, we fix a countably infinite domain Δ and we consider the set $Mod_{\Delta}(\mathcal{K})$ of the ranked models of $\mathcal{K}$ with domain Δ : $Mod_{\Delta}(\mathcal{K}) := \{ \mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle \mid \mathcal{R} \models \mathcal{K} \text{ and } \mathcal{R} \text{ is ranked} \}$. We define our entailment relation, modelling RC for DLs, on the model obtained by the ranked union (with $n = 1$) of the models in $Mod_{\Delta}(\mathcal{K})$.

Definition 4 (Big ranked model; Minimal Entailment). The big ranked model $\mathcal{O}$ of a KB $\mathcal{K}$ is the ranked union of all the models in $Mod_{\Delta}(\mathcal{K})$. An inclusion $\alpha ::= C \sqsubseteq D \mid C \sqsubseteq_E D$ is minimally entailed by $\mathcal{K}$ ($\mathcal{K} \models_{\min} \alpha$) iff $\mathcal{O} \models \alpha$.

The intuition underlying RC is to enforce the *presumption of typicality* [44]: we reason assuming that every object behaves in the most typical/expected way according to the information available. Semantically, this can be modelled by considering the ranked models of the KB in which the objects are interpreted as low as possible in the ranks: we assume they are as typical as possible modulo the satisfaction of the KB. Britz et al. (2021) prove that the big ranked model and the associated minimal entailment are a correct semantic representation of RC. They also describe a procedure to decide minimal entailment of defeasible inclusions $C \sqsubseteq_E D$ based on three main points:

Exceptionality: A concept D is *exceptional* w.r.t. a KB $\mathcal{K}$ iff all models of $\mathcal{K}$ satisfy the inclusion $\top \sqsubseteq_E \neg D$; i.e., there is no model $\mathcal{R}$ of $\mathcal{K}$ in which an object in $\min_{\prec_{\mathcal{R}}}(\top)$ is in $D^{\mathcal{R}}$. The statement $C \sqsubseteq_E D$ is exceptional iff C is exceptional. For example, $Penguin \sqsubseteq_E \neg Fly$ represents an exceptional conditional w.r.t. a KB containing also the inclusions $Penguin \sqsubseteq Bird$ and $Bird \sqsubseteq_E Fly$.

Ranking: Iterating exceptionality yields a *ranking* of all defeasible inclusions in $\mathcal{D}$, starting from rank 0, and moving to more specific ones at every rank. We could, e.g., have a defeasible inclusion $Bird \sqsubseteq_E Fly$ at rank 0, an exceptional inclusion $Penguin \sqsubseteq_E \neg Fly$ at rank 1, and a more exceptional inclusion $PenguinWithJetpack \sqsubseteq_E Fly$ at rank 2. The ranking process shows some strict information that is potentially ‘hidden’ in D . E.g., if $\mathcal{T} = \{D \sqsubseteq E\}$, the inclusions in $\mathcal{D} = \{C \sqsubseteq_E D, C \sqsubseteq_E \neg E\}$ convey strict information, since the KB is satisfied only by ranked models satisfying $C \sqsubseteq \perp$.

Rational Closure: Given the ranking of $\mathcal{D}$, one can decide whether an inclusion $C \sqsubseteq_E D$ is minimally entailed by $\mathcal{K}$: informally, we calculate the rank i of C , associate to C only the defeasible inclusions in $\mathcal{D}$ of rank $j \geq i$, and check whether we can derive $C \sqsubseteq_E D$.

These three points are implemented by three algorithms:

- **Exceptional**($\langle \mathcal{T}, \mathcal{D} \rangle$) returns the *exceptional* defeasible concept inclusions in $\mathcal{D}$ w.r.t. $\langle \mathcal{T}, \mathcal{D} \rangle$.
- **ComputeRanking**($\langle \mathcal{T}, \mathcal{D} \rangle$) returns, through some calls to **Exceptional**: (i) a rank-equivalent KB $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$ obtained by moving the strict information possibly present in $\mathcal{D}$ into $\mathcal{T}$; (ii) a sequence $\mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n)$ s.t. $\mathcal{E}_0 = \mathcal{D}^*$ and, for each $i < n$, $\mathcal{E}_{i+1} \subset \mathcal{E}_i$, where $\mathcal{E}_{i+1}$ contains the defeasible inclusions in $\mathcal{E}_i$ that are exceptional w.r.t. $\langle \mathcal{T}^*, \mathcal{E}_i \rangle$.

- $\text{RationalClosure}(\langle \mathcal{T}, \mathcal{D} \rangle, C \sqsubseteq_E D)$ returns, based on ComputeRanking , true if $C \sqsubseteq_E D$ is minimally entailed by $\langle \mathcal{T}, \mathcal{D} \rangle$, and false otherwise.

3. Deontic Reasoning in $\mathcal{ALC}$

We use the framework presented before to introduce deontic notions in $\mathcal{ALC}$, reasoning about conditional obligations $C \sqsubseteq_{\mathcal{N}} D$ ('if something is a C , then it ought to be a D ') and conditional permissions $C \sqsubseteq_{\mathcal{P}} D$ ('if something is a C , then it is allowed to be a D '). In this section we consider knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, where $\mathcal{T}$ is a classical TBox, and $\mathcal{N}$ (resp., $\mathcal{P}$) is a finite set of conditional obligations (resp., permissions). In this section we do not consider defeasible inclusions $\mathcal{D}$ modeling expectations.

Semantically, we formalise deontic reasoning by adapting ranked interpretations to deontic conditionals through a ranked order $\prec_N^{\mathcal{R}}$ that models the preferences of the agent.

Definition 5 (D-interpretation). A D-interpretation (deontic interpretation) is a tuple $\mathcal{R} := \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$, with $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ a DL interpretation, and $\prec_N^{\mathcal{R}}$ a ranked order over $\Delta^{\mathcal{R}}$.

The relation $x \prec_N^{\mathcal{R}} y$ indicates that the object x is preferred than y . In the deontic logic community it is customary, given a pair $x \prec_N^{\mathcal{R}} y$, to consider y the preferred one; we rely on the inverse reading since it is more common in preferential semantics. Obligations are treated as defeasible inclusions except that the preference relation $\prec_N^{\mathcal{R}}$ ranks the objects in the domain according to their conformance to our normative values, and an object in C ought to be in D iff all the preferred objects in C are in D ($\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$). We also consider *permissions*, which are considered dual to obligations [5]: something is permitted if its negation is not mandatory. I.e., $C \sqsubseteq_{\mathcal{P}} D$ holds iff $C \sqsubseteq_{\mathcal{N}} \neg D$ does not hold.

Definition 6 (Satisfaction). Let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$ be a D-interpretation.

$$\mathcal{R} \models C \sqsubseteq D \text{ iff } C^{\mathcal{R}} \subseteq D^{\mathcal{R}}; \quad \mathcal{R} \models C \sqsubseteq_{\mathcal{N}} D \text{ iff } \min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}; \quad \mathcal{R} \models C \sqsubseteq_{\mathcal{P}} D \text{ iff } \begin{cases} \mathcal{R} \not\models C \sqsubseteq_{\mathcal{N}} \neg D; \text{ or} \\ \mathcal{R} \models C \sqsubseteq \perp. \end{cases}$$

$\mathcal{R}$ satisfies (is a model of) $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ iff $\mathcal{R} \models \alpha$ for every $\alpha \in \mathcal{T} \cup \mathcal{N} \cup \mathcal{P}$. A deontic KB $\mathcal{K}$ is consistent iff there is at least a D-interpretation $\mathcal{R}$ that satisfies it.

Note that $\mathcal{R} \not\models C \sqsubseteq_{\mathcal{N}} \neg D$ is semantically equivalent to $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \cap D^{\mathcal{R}} \neq \emptyset$. The interpretation of permissions as negated obligations was already formalised in von Wright's seminal work (1951), and in dyadic deontic logic such a duality has simply been reformulated in the conditional form. It works also in our DL framework: $C \sqsubseteq_{\mathcal{N}} D$ interpreted as $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$ means that 'knowing *only* that an object is in C , according to our normative system it ought to be also in D '; consequently, $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \cap D^{\mathcal{R}} \neq \emptyset$ means that 'knowing *only* that an object is in C , it is consistent with our normative system that it is also in D ', since there is a $\prec_N^{\mathcal{R}}$ -preferred objects in $C^{\mathcal{R}}$, that is, an object in C abiding to all the rules that our system associates to C , that is also in D . That is, 'knowing *only* that an object is in C , it is permitted for it to be also in D '. Modelling permissions as negated obligations in our framework immediately recalls the definition of RC from [40] based on propositional conditionals ($\alpha \vdash \beta$) and negated conditionals ($\alpha \not\vdash \beta$). Our semantic construction is indeed inspired by it, despite a few differences, due to the expressivity of the language and a slightly different notion of consistency.

Example 2. We still consider the class of the PhD students, but for now we put aside expectations and focus on the normative side. Consider a KB $\mathcal{K} = \langle \emptyset, \mathcal{N}, \mathcal{P} \rangle$ where

$$\mathcal{N} = \{PhD \sqsubseteq_{\mathcal{N}} \neg \exists \text{pay}.T, PhD \sqcap W \sqsubseteq_{\mathcal{N}} \exists \text{pay}.T, PhD \sqsubseteq_{\mathcal{N}} \neg PT\}, \quad \mathcal{P} = \{PhD \sqcap W \sqsubseteq_{\mathcal{P}} PT\},$$

meaning that PhD students do not have to pay Taxes (T), but if they work they should; and they are not allowed to be part-time (PT), except if they work.

We consider the derivation of obligations and permissions. We adapt the *big ranked model*, from Section 2 to the case of deontic reasoning. As a result, we model a kind of deontic entailment based on what we could generally call *presumption of non-exceptionality*: we reason assuming that things are as least exceptional as possible, and apply to them the most general obligations and permissions consistent with the available information. The main difference w.r.t. the entailment of expectations from Section 2 is that we consider two kinds of conditionals, obligations and permissions, in our semantics and decision procedures. Since permissions are negated obligations, we need to reformulate the semantic definition of the minimal model and the corresponding decision procedure accordingly.

Given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a fixed countably infinite domain Δ , we consider the set $Mod_{\Delta}(\mathcal{K})$ of the D-models of $\mathcal{K}$ with domain Δ : $Mod_{\Delta}(\mathcal{K}) := \{ \mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle \mid \mathcal{R} \models \mathcal{K} \text{ and } \mathcal{R} \text{ is ranked} \}$, and, again, we define our entailment relation on the ranked union of the models in $Mod_{\Delta}(\mathcal{K})$.

Definition 7 (Big D-model; Minimal Entailment). *The big D-model $\mathcal{O}_{\mathcal{N}}$ of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is the ranked union of all the models in $Mod_{\Delta}(\mathcal{K})$. An inclusion $\alpha ::= C \sqsubseteq D \mid C \sqsubseteq_{\mathcal{N}} D \mid C \sqsubseteq_{\mathcal{P}} D$ is minimally entailed by $\mathcal{K}$ ($\mathcal{K} \models_{\min} \alpha$) iff $\mathcal{O}_{\mathcal{N}} \models \alpha$.*

As in defeasible $\mathcal{ALC}$, we can prove that, for any consistent deontic KB $\mathcal{K}$, $\mathcal{O}_{\mathcal{N}}$ is a *minimal* model of $\mathcal{K}$; i.e., for every concept C , the height of C in $\mathcal{O}_{\mathcal{N}}$ is minimal w.r.t. all the models of $\mathcal{K}$.

Proposition 1. *Let $\mathcal{K}$ be a consistent deontic KB and $\mathcal{O}_{\mathcal{N}} := \langle \Delta^{\mathcal{O}_{\mathcal{N}}}, \cdot^{\mathcal{O}_{\mathcal{N}}}, \prec_N^{\mathcal{O}_{\mathcal{N}}} \rangle$ its Big D-model. Then $\mathcal{O}_{\mathcal{N}} \models \mathcal{K}$, and, for any $C \in \mathcal{L}$, $h_{\mathcal{O}_{\mathcal{N}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \models \mathcal{K}\}$.*

We now present sound and complete decision procedures for minimal entailment in this deontic framework, obtained by modifying the procedures from Section 2 to handle obligations and permissions. Our procedures have the same features as that of Britz et al.: (i) they reduce decision problems in non-monotonic reasoning to a series of calls to classical $\mathcal{ALC}$ reasoners, making the implementation of such procedures particularly easy; and (ii) the runtime behaviour of the overall decision procedure stays in the same complexity class as classical $\mathcal{ALC}$. To reduce all the decision problems to a series of classical $\mathcal{ALC}$ decision problems, we use two transformations: given any set Φ of conditionals $C \sqsubseteq D$ ($\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}\}$), $\bar{\Phi} := \{\neg C \sqcup D \mid C \sqsubseteq D \in \Phi\}$; and $\Phi^{\neg} := \{\neg C \mid C \sqsubseteq D \in \Phi\}$. The main procedures refer to three points that are analogous to the ones presented at the end of Section 2.

Exceptionality: concepts that are exceptional w.r.t. preferences, based on the obligations and permissions from the KB. A concept C is *exceptional* w.r.t. $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ iff all models of $\mathcal{K}$ satisfy the inclusion $\top \sqsubseteq_{\mathcal{N}} \neg C$. That is, there is no model $\mathcal{R}$ of $\mathcal{K}$ in which an object violating no obligations (i.e., an object in $\min_{\prec_N^{\mathcal{R}}}(\top)$) is in $C^{\mathcal{R}}$. A conditional obligation $C \sqsubseteq_{\mathcal{N}} D$ (resp., permission $C \sqsubseteq_{\mathcal{P}} D$) is exceptional iff C is exceptional. We can, e.g., have one norm that killing is forbidden ($\top \sqsubseteq_{\mathcal{N}} \neg Kill$), and another norm that, if you kill, you should do it gently ($Kill \sqsubseteq_{\mathcal{N}} GentleKill$) and that if you kill you have the right to defend yourself in court ($Kill \sqsubseteq_{\mathcal{P}} Defend$). $Kill \sqsubseteq_{\mathcal{N}} GentleKill$ and $Kill \sqsubseteq_{\mathcal{P}} Defend$ are exceptional inclusions, since they refer to a situation ($Kill$) incompatible with the preferred objects.

The algorithm `NormExceptional` is used to decide exceptionality. From a TBox $\mathcal{T}$, an NBox Φ , and a PBox Ψ , it outputs two (possibly empty) sets Φ' and Ψ' , containing the exceptional conditionals in Φ and Ψ , respectively. The main difference w.r.t. defeasible $\mathcal{ALC}$ is that exceptional permissions $C \sqsubseteq_{\mathcal{P}} D$ impose that the most preferred objects fall under the concept $\neg C$, adding it to the set $\mathcal{E}$.

Example 3. *Running `NormExceptional`($\mathcal{T}, \mathcal{N}, \mathcal{P}$) on the KB from Example 2 yields $\mathcal{T} = \emptyset$, $\mathcal{E} = \emptyset$, and $\bar{\Phi} = \{\neg PhD \sqcup \neg \exists pay.T, \neg(PhD \sqcap W) \sqcup (\exists pay.T), \neg PhD \sqcup \neg PT\}$. Since $\models \bar{\Phi} \sqsubseteq \neg(PhD \sqcap W \sqcap PT)$, we conclude that $PhD \sqcap W \sqsubseteq_{\mathcal{P}} PT$ is an exceptional permission ($\Psi' = \{PhD \sqcap W \sqsubseteq_{\mathcal{P}} PT\}$) and $\mathcal{E} = \{\neg(PhD \sqcap W)\}$. We also have $\not\models \bar{\Phi} \sqcap \bar{\mathcal{E}} \sqsubseteq \neg PhD$, and $\models \bar{\Phi} \sqcap \bar{\mathcal{E}} \sqsubseteq \neg(PhD \sqcap W)$, identifying the obligation $\Phi' = \{PhD \sqcap W \sqsubseteq_{\mathcal{N}} \exists pay.T\}$ as exceptional.*

Ranking: As with defeasible $\mathcal{ALC}$, iteratively applying the notion of exceptionality yields a *ranking* of all the defeasible inclusions. The ranking process may identify some pieces of defeasible information in the NBox and the PBox that correspond to strict GCIs. For example, the statements $C \sqsubseteq_{\mathcal{N}} D$ in the

Procedure NormExceptional($\mathcal{T}, \Phi, \Psi$)

Input: $\mathcal{T}, \Phi \subseteq \mathcal{N}$ and $\Psi \subseteq \mathcal{P}$ **Output:** $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that Φ' and Ψ' are sets of exceptional axioms w.r.t. $\langle \mathcal{T}, \Phi, \Psi \rangle$ $\Phi' := \emptyset; \Psi' := \emptyset; \mathcal{E} := \emptyset;$ **repeat** $\mathcal{F} := \mathcal{E};$ **foreach** $C \sqsubseteq_{\mathcal{P}} D \in \Psi$ **do****if** $\mathcal{T} \models \Box \bar{\Phi} \Box \Box \mathcal{E} \sqsubseteq \neg(C \Box D)$ **then** $\Psi' := \Psi' \cup \{C \sqsubseteq_{\mathcal{P}} D\};$ $\mathcal{E} := \mathcal{E} \cup \{\neg C\};$ **until** $\mathcal{E} = \mathcal{F};$ **foreach** $C \sqsubseteq_{\mathcal{N}} D \in \Phi$ **do****if** $\mathcal{T} \models \Box \bar{\Phi} \Box \Box \mathcal{E} \sqsubseteq \neg C$ **then** $\Phi' := \Phi' \cup \{C \sqsubseteq_{\mathcal{N}} D\};$ **return** Φ', Ψ'

Procedure ComputeNormRanking($\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$)

Input: $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ **Output:** Ontology $\langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$, exceptionality ranking $R_{\mathcal{N}}$ of $\mathcal{N}^\circ$, exceptionality ranking $R_{\mathcal{P}}$ of $\mathcal{P}^\circ$ $\mathcal{T}^\circ := \mathcal{T};$ **repeat** $i := 0;$ $\Phi_0 := \mathcal{N};$ $\Psi_0 := \mathcal{P};$ $\Phi_1, \Psi_1 := \text{NormExceptional}(\mathcal{T}^*, \Phi_0, \Psi_0);$ **repeat** $i := i + 1;$ $\Phi_{i+1}, \Psi_{i+1} := \text{NormExceptional}(\mathcal{T}^*, \Phi_i, \Psi_i);$ **until** $\Phi_{i+1} = \Phi_i$ and $\Psi_{i+1} = \Psi_i;$ $\mathcal{N}_\infty := \Phi_i;$ $\mathcal{P}_\infty := \Psi_i;$ $\mathcal{N}^\circ := \mathcal{N} \setminus \mathcal{N}_\infty;$ $\mathcal{P}^\circ := \mathcal{P} \setminus \mathcal{P}_\infty;$ $\mathcal{T}^\circ := \mathcal{T}^\circ \cup \{E \sqsubseteq F \mid E \sqsubseteq_{\mathcal{N}} F \in \mathcal{N}_\infty\} \cup \{E \sqsubseteq \perp \mid E \sqsubseteq_{\mathcal{P}} F \in \mathcal{P}_\infty\};$ **until** $\mathcal{N}_\infty = \emptyset$ and $\mathcal{P}_\infty = \emptyset;$ $R_{\mathcal{N}} := (\Phi_0, \dots, \Phi_{i-1});$ $R_{\mathcal{P}} := (\Psi_0, \dots, \Psi_{i-1});$ **return** $\langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle, R_{\mathcal{N}}, R_{\mathcal{P}};$

NBox and $C \sqsubseteq_{\mathcal{P}} \neg D$ in the PBox express that it is impossible to have any object in the concept C . That is, an interpretation is a model of $C \sqsubseteq_{\mathcal{N}} D$, $C \sqsubseteq_{\mathcal{P}} \neg D$ iff it is a model of $C \sqsubseteq \perp$.

ComputeNormRanking implements the KB ranking. It calls NormExceptional iteratively, until reaching a fixed point, and returns two objects: (1) a KB $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$ that is rank-equivalent to $\mathcal{K}$, obtained by moving all the strict information that was ‘hidden’ in $\mathcal{N}$ and $\mathcal{P}$ into $\mathcal{T}$; and (2) two sequences: $R_{\mathcal{N}} = (\Phi_0, \dots, \Phi_n)$ s.t. $\Phi_0 = \mathcal{N}^\circ$ and, for every $i < n$, $\Phi_{i+1} \subseteq \Phi_i$, since Φ_{i+1} contains the obligations in Φ_i which are exceptional w.r.t. $\langle \mathcal{T}^\circ, \Phi_i, \Psi_i \rangle$; and $R_{\mathcal{P}} = (\Psi_0, \dots, \Psi_n)$ s.t. $\Psi_0 = \mathcal{P}^\circ$ and, for every $i < n$, $\Psi_{i+1} \subseteq \Psi_i$, contains the permissions in Ψ_i which are exceptional w.r.t. $\langle \mathcal{T}^\circ, \Phi_i, \Psi_i \rangle$.

Example 4. We run ComputeNormRanking($\emptyset, \mathcal{N}, \mathcal{P}$) on the KB from Example 2; the single run of NormExceptional($\emptyset, \mathcal{N}, \mathcal{P}$) seen in Example 3 is sufficient to give us the ranking. We end up with $\mathcal{T}^\circ = \emptyset$; $R_{\mathcal{N}} = \{\Phi_0, \Phi_1\}$, with $\Phi_0 = \mathcal{N}$ and $\Phi_1 = \{PhD \Box W \sqsubseteq_{\mathcal{N}} \exists pay.T\}$; and $R_{\mathcal{P}} = \{\Psi_0, \Psi_1\}$, with $\Psi_0 = \Psi_1 = \mathcal{P} = \{PhD \Box W \sqsubseteq_{\mathcal{P}} PT\}$.

Minimal Entailment. Given the rankings $R_{\mathcal{N}}$ and $R_{\mathcal{P}}$, one can decide whether some obligation $C \sqsubseteq_{\mathcal{N}} D$ or permission $C \sqsubseteq_{\mathcal{P}} D$ is minimally entailed by $\mathcal{K}$ using (i) N-DeonticClosure($\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D$), to decide whether $C \sqsubseteq_{\mathcal{N}} D$ is minimally entailed by $\mathcal{K}$; and (ii) P-DeonticClosure($\mathcal{K}, C \sqsubseteq_{\mathcal{P}} D$), to decide whether $C \sqsubseteq_{\mathcal{P}} D$ is minimally entailed by $\mathcal{K}$. Informally, N-DeonticClosure($\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D$)

computes the rank i of C , associates to C only the defeasible obligations in rank i , plus the negations of all the antecedents of the permissions that have rank $i + 1$, and checks whether it can derive $C \sqsubseteq_{\mathcal{N}} D$.

Example 5. To see whether PhD students with family (F) cannot be part time, we run $\text{N-DeonticClosure}(\mathcal{K}, \text{PhD} \sqcap F \sqsubseteq_{\mathcal{N}} \neg PT)$, with $\mathcal{K}$ from Example 2. $\mathcal{T}^\circ$, $R_{\mathcal{N}}$ and $R_{\mathcal{P}}$ are indicated in Example 4. Since $\mathcal{T}^\circ \not\models \sqcap \overline{\Phi_0} \sqcap \sqcap \Psi_1 \sqcap (\text{PhD} \sqcap F) \sqsubseteq \perp$, the rank i of $\text{PhD} \sqcap F$ is 0. The reader can check that $\mathcal{T}^\circ \models \sqcap \overline{\Phi_0} \sqcap \sqcap \Psi_1 \sqcap (\text{PhD} \sqcap F) \sqsubseteq \neg PT$. That is, with only the rules in the KB, we must conclude that PhD students with families cannot be part time. This example shows that minimal entailment implements the presumption of non-exceptionality: based on our KB, PhD students with families do not have any special rule applying to them, and consequently they are not treated as exceptional, at least until extra rules about them are introduced in the KB.

The procedure $\text{P-DeonticClosure}(\mathcal{K}, C \sqsubseteq_{\mathcal{P}} D)$ decides conditional permissions using Definition 6. It first calls $\text{N-DeonticClosure}(\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D)$ to check whether $\mathcal{K}$ minimally entails $C \sqsubseteq_{\mathcal{N}} \neg D$. If that is not the case, then $\mathcal{K}$ minimally entails $C \sqsubseteq_{\mathcal{P}} D$. If, instead, $\mathcal{K}$ minimally entails $C \sqsubseteq_{\mathcal{N}} \neg D$, the procedure checks whether $C \sqsubseteq \perp$ is minimally entailed by $\mathcal{K}$, and, if not, concludes that $C \sqsubseteq_{\mathcal{P}} D$ is not minimally entailed by $\mathcal{K}$. Let $\vdash_{\mathcal{N}}$ be a derivation relation that refers to these procedures.

Definition 8 (Relation $\vdash_{\mathcal{N}}$). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ be a deontic KB, $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle = \text{ComputeNormRanking}(\mathcal{K})$, $C, D \in \mathcal{L}$, and $\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}\}$. $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq D$ iff

- $\sqsubseteq = \sqsubseteq$ and $\mathcal{T}^\circ \models C \sqsubseteq D$;
- $\sqsubseteq = \sqsubseteq_{\mathcal{N}}$ and $\text{N-DeonticClosure}(\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D) = \text{true}$;
- $\sqsubseteq = \sqsubseteq_{\mathcal{P}}$ and $\text{P-DeonticClosure}(\mathcal{K}, C \sqsubseteq_{\mathcal{P}} D) = \text{true}$.

It can be proved that the procedures are sound and complete w.r.t. minimal entailment.

Theorem 1. For any deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and inclusion $C \sqsubseteq D$ ($\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}\}$), we have $\mathcal{K} \models_{\min} C \sqsubseteq D$ iff $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq D$.

An easy consequence is that to decide whether a deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is consistent, it suffices to check the TBox $\mathcal{T}^\circ$: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is ranked-inconsistent iff $\mathcal{T}^\circ \models \top \sqsubseteq \perp$.

The computational costs of the overall decision procedure stay is not higher than for classical $\mathcal{ALC}$.

Proposition 2. Checking minimal entailment for a Deontic KB is EXPTIME-complete.

4. Deontic Defeasible DL

Now we move to the task of modelling *deontic defeasible reasoning*; i.e., combining the epistemic and deontic modalities, merging reasoning about expectations and about obligations/permissions in order to determine what agents should do in front of uncertain information. The goal is to derive conditionals of the kind ‘presumably, a C ought/is permitted to be a D .’ To this end, we introduce ranked models in which the two ranked orders, $\prec_E^{\mathcal{R}}$ modelling expectations and $\prec_N^{\mathcal{R}}$ modelling norms, interact.

Procedure $\text{N-DeonticClosure}(\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle, C \sqsubseteq_{\mathcal{N}} D)$

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a query $C \sqsubseteq_{\mathcal{N}} D$.

Output: **true** if $C \sqsubseteq_{\mathcal{N}} D$ is in the Deontic Closure of $\mathcal{K}$, **false** otherwise

$(\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle, R_{\mathcal{N}} = (\Phi_0, \dots, \Phi_n), R_{\mathcal{P}} = (\Psi_0, \dots, \Psi_n)) := \text{ComputeNormRanking}(\mathcal{K})$;

$i := 0$;

$\Psi_{n+1} := \emptyset$;

while $\mathcal{T}^\circ \models \sqcap \overline{\Phi_i} \sqcap \sqcap \Psi_{i+1} \sqcap C \sqsubseteq \perp$ **and** $i \leq n$ **do**

$i := i + 1$;

if $i \leq n$ **then**

return $\mathcal{T}^\circ \models \sqcap \overline{\Phi_i} \sqcap \sqcap \Psi_{i+1} \sqcap C \sqsubseteq D$;

else

return $\mathcal{T}^\circ \models C \sqsubseteq D$;

Procedure P-DeonticClosure($\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle, C \sqsubseteq_{\mathcal{P}} D$)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a query $C \sqsubseteq_{\mathcal{P}} D$.

Output: true if $C \sqsubseteq_{\mathcal{P}} D$ is in the Deontic Closure of $\mathcal{K}$, false otherwise

if N-DeonticClosure($\mathcal{K}, C \sqsubseteq_{\mathcal{N}} \neg D$) = false **then**

return true;

else if $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$ **then**

return true;

else

return false;

Definition 9 (DD-Interpretation). A DD-interpretation (deontic defeasible interpretation) is a tuple $\mathcal{R} := \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$, with $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ a DL interpretation, and $\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$ ranked orders over $\Delta^{\mathcal{R}}$.

The relations $x \prec_E^{\mathcal{R}} y$, $x \prec_N^{\mathcal{R}} y$ indicate that the object x is more in line with our expectations or deontic preferences than y , respectively. The orders $\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$ are independent from each other, each associated to a different modality and with correspondent height functions $h_{\prec_E^{\mathcal{R}}}$ and $h_{\prec_N^{\mathcal{R}}}$, respectively. Given an object $x \in \Delta^{\mathcal{R}}$, $h_{\prec_E^{\mathcal{R}}}(x) = 0$ indicates that x satisfies all the expectations, and the height is proportional to the exceptionality of its properties w.r.t our expectations. Analogously, $h_{\prec_N^{\mathcal{R}}}(x) = 0$ indicates that x satisfies all our normative constraints, height is inversely connected to norm respect.

We consider KBs $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ containing classical GCIs $C \sqsubseteq D$, conditional expectations $C \sqsubseteq_E D$, conditional obligations $C \sqsubseteq_N D$, and conditional permissions $C \sqsubseteq_P D$; and introduce two new kinds of defeasible inclusions, that can be derived from a KB: *presumptive obligations* $C \sqsubseteq_{dN} D$ and *presumptive permissions* $C \sqsubseteq_{dP} D$. In fact, we will derive the following new inclusions: $C \sqsubseteq D$ (An individual in C is also in D); $C \sqsubseteq_E D$ (Presumably, an individual in C is in D); $C \sqsubseteq_N D$ (An individual in C should be in D); $C \sqsubseteq_P D$ (An individual in C is allowed to be in D); $C \sqsubseteq_{dN} D$ (Presumably, an individual in C should be in D); $C \sqsubseteq_{dP} D$ (Presumably, an individual in C is allowed to be in D). We put aside for one moment the newly introduced presumptive obligations and permissions, since they are only derivable and not in the KB. Regarding the inclusions we have considered until now, the satisfaction conditions are preserved from the previous sections.

Definition 10 (Satisfaction). Let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$ be a DD-interpretation. The satisfaction relation $\models$ is defined as: $\mathcal{R} \models C \sqsubseteq D$ iff $C^{\mathcal{R}} \subseteq D^{\mathcal{R}}$; $\mathcal{R} \models C \sqsubseteq_E D$ iff $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$; $\mathcal{R} \models C \sqsubseteq_N D$ iff $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$; and $\mathcal{R} \models C \sqsubseteq_P D$ iff either $\mathcal{R} \not\models C \sqsubseteq_N \neg D$ or $\mathcal{R} \models C \sqsubseteq \perp$.

$\mathcal{R}$ satisfies (is a model of) $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ iff $\mathcal{R} \models \alpha$ for all statements $\alpha \in \mathcal{T} \cup \mathcal{D} \cup \mathcal{N} \cup \mathcal{P}$. A KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ is consistent iff it is satisfied by a DD-interpretation.

In this multi-modal context, we aim at models in which everything is considered as *least exceptional as possible*, w.r.t. $\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$, compatibly with $\mathcal{K}$. Hence, we implement once again the same construction seen above, based on a fixed countably infinite domain and the ranked union (Definition 3). Given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and fixed any countably infinite domain Δ , we consider the set $Mod_{\Delta}(\mathcal{K})$ of the DD-models of $\mathcal{K}$ with domain Δ $Mod_{\Delta}(\mathcal{K}) := \{ \mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle \mid \mathcal{R} \models \mathcal{K} \text{ and } \mathcal{R} \text{ is a DD-interpretation} \}$. We define our entailment relation on the model obtained by the ranked union of the models in $Mod_{\Delta}(\mathcal{K})$.

Definition 11 (Big DD-model & Minimal Entailment). The big DD-model $\mathcal{O}_{\mathcal{DN}}$ of a KB $\mathcal{K}$ is the ranked union of all the models in $Mod_{\Delta}(\mathcal{K})$. An inclusion α ($\alpha ::= C \sqsubseteq D \mid C \sqsubseteq_E D \mid C \sqsubseteq_N D \mid C \sqsubseteq_P D$) is minimally entailed by $\mathcal{K}$ ($\mathcal{K} \models_{\min} \alpha$) iff $\mathcal{O}_{\mathcal{DN}} \models \alpha$.

As before, we can prove that, for any consistent deontic KB $\mathcal{K}$, $\mathcal{O}_{\mathcal{DN}}$ is a *minimal* model of $\mathcal{K}$, meaning that for every concept C , the height of C in $\mathcal{O}_{\mathcal{DN}}$ is minimal w.r.t. all the models of $\mathcal{K}$.

Proposition 3. Let $\mathcal{K}$ be a consistent deontic KB and $\mathcal{O}_{\mathcal{DN}} := \langle \Delta^{\mathcal{O}_{\mathcal{DN}}}, \cdot^{\mathcal{O}_{\mathcal{DN}}}, \prec_E^{\mathcal{O}}, \prec_N^{\mathcal{O}} \rangle$ its Big DD-model. Then $\mathcal{O}_{\mathcal{DN}} \models \mathcal{K}$, and, for any $C \in \mathcal{L}$, $h_{\prec_N^{\mathcal{O}}}(C) = \min\{h_{\prec_N^{\mathcal{R}}}(C) \mid \mathcal{R} \models \mathcal{K}\}$ and $h_{\prec_E^{\mathcal{O}}}(C) = \min\{h_{\prec_E^{\mathcal{R}}}(C) \mid \mathcal{R} \models \mathcal{K}\}$.

Procedure CombinedRanking($\langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$)

Input: $\langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$

Output: $\langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle$, and rankings $\mathcal{E} = \{\mathcal{E}_0, \dots, \mathcal{E}_n\}$ for $\mathcal{D}^*$, $R_{\mathcal{N}} = \{\Phi_0, \dots, \Phi_m\}$ for $\mathcal{N}^*$, and $R_{\mathcal{P}} = \{\Psi_0, \dots, \Psi_m\}$ for $\mathcal{P}^*$

$\mathcal{T}^* := \mathcal{T}; \mathcal{D}^* := \mathcal{D}; \mathcal{N}^* := \mathcal{N}; \mathcal{P}^* := \mathcal{P};$

repeat

$\mathcal{T}^{in} := \mathcal{T}^*;$

$\mathcal{D}^{in} := \mathcal{D}^*;$

$\mathcal{N}^{in} := \mathcal{N}^*;$

$\langle \mathcal{T}^*, \mathcal{D}^* \rangle, \mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n) := \text{ComputeRanking}(\langle \mathcal{T}^*, \mathcal{D}^* \rangle);$

$\langle \mathcal{T}^*, \mathcal{N}^*, \mathcal{P}^* \rangle, R_{\mathcal{N}} = (\Phi_0, \dots, \Phi_m), R_{\mathcal{P}} = (\Psi_0, \dots, \Psi_m) := \text{ComputeNormRanking}(\langle \mathcal{T}^*, \mathcal{N}^*, \mathcal{P}^* \rangle);$

until $\mathcal{T}^* = \mathcal{T}^{in};$

return $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle, \mathcal{E}, R_{\mathcal{N}}, R_{\mathcal{P}};$

Returning to the new conditionals introduced in this section, the idea to model ‘presumably, if something is C then it ought to be D ’ is that $C \sqsubseteq_{d\mathcal{N}} D$ is satisfied by a DD-interpretation iff we consider all the expectations that we associate with C , that are represented by the objects in $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})$, and check if, according to $\prec_N^{\mathcal{R}}$, we associate to such expectations some obligation that enforces D ; i.e., $\mathcal{R} \Vdash C \sqsubseteq_{d\mathcal{N}} D$ iff $\min_{\prec_N^{\mathcal{R}}}(\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})) \subseteq D^{\mathcal{R}}$. Analogously for $C \sqsubseteq_{d\mathcal{P}} D$, we consider the expectations associated to C and check whether they are compatible, according to $\prec_N^{\mathcal{R}}$, with D . That is, if C is satisfiable in $\mathcal{R}$, $\mathcal{R} \Vdash C \sqsubseteq_{d\mathcal{P}} D$ if $\min_{\prec_N^{\mathcal{R}}}(\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})) \cap D^{\mathcal{R}} \neq \emptyset$, while we maintain that in case $C^{\mathcal{R}}$ is empty, we trivially derive that everything is permitted given C . From this definition it is immediate to see that, also for presumptive permissions and obligations, the complementarity of the two notions persists. That is, $\mathcal{R} \Vdash C \sqsubseteq_{d\mathcal{P}} D$ iff $\mathcal{R} \nVdash C \sqsubseteq_{d\mathcal{N}} \neg D$; or $\mathcal{R} \Vdash C \sqsubseteq \perp$. From the semantic definitions it is easy to check that, also in the case of $\sqsubseteq_{d\mathcal{N}}$ and $\sqsubseteq_{d\mathcal{P}}$, if a concept is defeasibly subsumed by $\perp$, it is also classically subsumed by $\perp$.

Proposition 4. *For any concept C and DD-interpretation, the following inclusions are equivalent: $C \sqsubseteq \perp$, $C \sqsubseteq_E \perp$, $C \sqsubseteq_{\mathcal{N}} \perp$, $C \sqsubseteq_{\mathcal{P}} \perp$, $C \sqsubseteq_{d\mathcal{N}} \perp$, $C \sqsubseteq_{d\mathcal{P}} \perp$.*

The procedures we already have at our disposal are almost everything needed to handle deontic defeasible $\mathcal{ALC}$. We just need to model how the two ranked orders interact, and how to decide $\sqsubseteq_{d\mathcal{N}}$ - and $\sqsubseteq_{d\mathcal{P}}$ -inclusions. To this end, we first use CombinedRanking($\langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$) to model how the two rankings, dealing respectively with $\mathcal{D}$ and $\langle \mathcal{N}, \mathcal{P} \rangle$, interact. The two ranks can interact through the equivalence of the inclusions with $\perp$ on the right (Proposition 4); i.e., a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ can be divided into the part dealing with expectations, $\langle \mathcal{T}, \mathcal{D} \rangle$, and the part dealing with norms, $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. Each of them is in principle “independent,” and can be ranked by ComputeRanking and ComputeNormRanking, respectively. Yet, the TBox is present in both procedures, and each of them can identify some GCIs which are implicit in the defeasible information and then added to the TBox thus affecting each ranking procedure. The function of CombinedRanking is to iterate the two ranking procedures until a fixed point is found and all the GCIs ‘hidden’ in the expectations and in the norms have been moved to the TBox $\mathcal{T}^*$, and two final rankings have been defined.

From the output of CombinedRanking, we can decide $\sqsubseteq_E^-$, $\sqsubseteq_{\mathcal{N}}^-$, and $\sqsubseteq_{\mathcal{P}}^-$ -inclusions running RationalClosure, N-DeonticClosure and P-DeonticClosure, respectively, substituting each call of ComputeRanking and ComputeNormRanking with CombinedRanking. We introduce new procedures N-DD-Closure($\mathcal{K}, C \sqsubseteq_{d\mathcal{N}} D$) and P-DD-Closure($\mathcal{K}, C \sqsubseteq_{d\mathcal{P}} D$) to decide $\sqsubseteq_{d\mathcal{N}}$ - and $\sqsubseteq_{d\mathcal{P}}$ -inclusions.

N-DD-Closure($\mathcal{K}, C \sqsubseteq_{d\mathcal{N}} D$), given the query $C \sqsubseteq_{d\mathcal{N}} D$, and the TBox $\mathcal{T}^*$ plus the rankings of $\mathcal{D}$, $\mathcal{N}$ and $\mathcal{P}$ obtained by CombinedRanking($\mathcal{K}$), (1) identifies the rank i of C w.r.t. the $\prec_E^{\mathcal{O}}$, modelling expectations, and associates to C the concepts $\bar{\mathcal{E}}_i$, representing the expectations of rank i ; then (2) identifies the rank l of $C \sqcap \prod \bar{\mathcal{E}}_i$ w.r.t. the $\prec_N^{\mathcal{O}}$, modelling norms, and associates to C the concepts $\bar{\Phi}_l$ and Ψ_{l+1}^- , representing the norms and permissions at rank l ; and (3) decides whether we can conclude that a C , presumably, should be a D by checking whether C , together $\bar{\mathcal{E}}_i$, $\bar{\Phi}_l$, and Ψ_{l+1}^- , is subsumed by D . P-DD-Closure($\mathcal{K}, C \sqsubseteq_{d\mathcal{P}} D$) is modelled relying on its relation with $C \sqsubseteq_{d\mathcal{N}} \neg D$.

Procedure N-DD-Closure($\mathcal{K}, \alpha$)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$, a query $\alpha = C \sqsubseteq_{d\mathcal{N}} D$.

Output: **true** if $\mathcal{K} \models_{\min} C \sqsubseteq_{d\mathcal{N}} D$, **false** otherwise.

$\langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle, \mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n), R_{\mathcal{N}} = (\Phi_0, \dots, \Phi_m), R_{\mathcal{P}} = (\Psi_0, \dots, \Psi_m) := \text{CombinedRanking}(\mathcal{K});$

$i := 0;$

$\Psi_{m+1} := \emptyset;$

while $\mathcal{T}^* \models \bigwedge \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \perp$ **and** $i \leq n$ **do**

$i := i + 1;$

if $i = n + 1$ **then**

$\mathcal{E}_{n+1} := \emptyset;$

$l := 0;$

while $\mathcal{T}^* \models \bigwedge \overline{\mathcal{E}_i} \sqcap \bigwedge \overline{\Phi_l} \sqcap \bigwedge \Psi_{l+1} \sqcap C \sqsubseteq \perp$ **and** $l \leq m$ **do**

$l := l + 1;$

if $l = m + 1$ **then**

$\Phi_l := \emptyset;$

$\Psi_{l+1} := \emptyset;$

return $\mathcal{T}^* \models \bigwedge \overline{\mathcal{E}_i} \sqcap \bigwedge \overline{\Phi_l} \sqcap \bigwedge \Psi_{l+1} \sqcap C \sqsubseteq D;$

Example 6. Let $\mathcal{K} = \langle \emptyset, \mathcal{D}, \mathcal{N}, \emptyset \rangle$ be a KB where $\mathcal{D} = \{PhD \sqsubseteq_E Ad, PhD \sqsubseteq_E \neg W, Ad \sqsubseteq_E W\}$. On the normative side we consider only obligations, to show the interaction of expectations and norms while preserving simplicity in the example: $\mathcal{N} = \{\neg W \sqsubseteq_{\mathcal{N}} \neg \exists pay.T, W \sqsubseteq_{\mathcal{N}} \exists pay.T\}$ (non-workers should not pay taxes, while workers should). $\text{CombinedRanking}(\mathcal{K})$ calls $\text{ComputeRanking}(\langle \mathcal{T}, \mathcal{D} \rangle)$, and obtains $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1\}$, with $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{PhD \sqsubseteq_E Ad, PhD \sqsubseteq_E \neg W\}$: PhD students are exceptional cases w.r.t. our expectations (since they are non-working adults). $\mathcal{T}$ remains empty. The procedure calls $\text{ComputeNormRanking}(\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle)$. Since $\mathcal{T}$ and $\mathcal{P}$ are both empty, $\text{NormExceptionality}$ is straightforward. $\overline{\Phi} = \{W \sqcup \neg \exists pay.T, \neg W \sqcup \exists pay.T\}$ and, since $\not\models \bigwedge \overline{\Phi} \sqsubseteq W$ and $\not\models \bigwedge \overline{\Phi} \sqsubseteq \neg W$, we do not have exceptional rules: $R_{\mathcal{N}} = \{\mathcal{N}\}$.

We want to know whether presumably PhD students should not pay taxes ($PhD \sqsubseteq_{d\mathcal{N}} \neg \exists pay.T$), and run $\text{N-DD-Closure}(\mathcal{K}, PhD \sqsubseteq_{d\mathcal{N}} \neg \exists pay.T)$. It identifies the rank of PhD w.r.t. $\langle \mathcal{T}, \mathcal{D} \rangle$ (which is 1) and associates the expectations $\mathcal{E}_1$ to it. It then checks the rank of $PhD \sqcap \bigwedge \overline{\mathcal{E}_1}$ w.r.t. $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. It concludes that this rank is 0, and associates Φ_0 to it. It can be checked that $\models PhD \sqcap \bigwedge \overline{\mathcal{E}_1} \sqcap \bigwedge \overline{\Phi_0} \sqsubseteq \neg \exists pay.T$, since $PhD \sqsubseteq_E \neg W \in \mathcal{E}_1$ and $\neg W \sqsubseteq_{\mathcal{N}} \neg \exists pay.T \in \Phi_0$. That is, we conclude $PhD \sqsubseteq_{d\mathcal{N}} \neg \exists pay.T$: presumably PhD students should not pay taxes.

We now check if presumably working PhD students should pay taxes ($PhD \sqcap W \sqsubseteq_{d\mathcal{N}} \exists pay.T$). The concept $PhD \sqcap W$ is exceptional also w.r.t. $\mathcal{E}_1$, hence its rank is 2 and is associated to $\mathcal{E}_2 = \emptyset$ (hence $\bigwedge \overline{\mathcal{E}_2} = \top$ and no need to consider it further). For the deontic ranking, $PhD \sqcap W$ is not exceptional w.r.t. $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, and we associate Φ_0 to it. The reader can check that $\models PhD \sqcap W \sqcap \bigwedge \overline{\Phi_0} \sqsubseteq \exists pay.T$, since $W \sqsubseteq_{\mathcal{N}} \exists pay.T \in \Phi_0$. Thus, $PhD \sqcap W \sqsubseteq_{d\mathcal{N}} \exists pay.T$: presumably working PhD students should pay taxes.

The relation $\vdash_{\mathcal{DN}}$ is introduced in order to refer to the derivations obtained using the new procedures.

Definition 12 (Relation $\vdash_{\mathcal{DN}}$). Let $\mathcal{K}$ be a deontic defeasible KB; $C, D \in \mathcal{L}; \mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle = \text{CombinedRanking}(\mathcal{K})$; and $\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_E, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}, \sqsubseteq_{d\mathcal{N}}, \sqsubseteq_{d\mathcal{P}}\}$. $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq D$ iff

- $\sqsubseteq = \sqsubseteq$ and $\mathcal{T}^* \models C \sqsubseteq D$;
- $\sqsubseteq = \sqsubseteq_E$ and $\text{RationalClosure}^*(\mathcal{K}, C \sqsubseteq_E D) = \text{true}$;
- $\sqsubseteq = \sqsubseteq_{\mathcal{N}}$ and $\text{N-DeonticClosure}^*(\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D) = \text{true}$;
- $\sqsubseteq = \sqsubseteq_{\mathcal{P}}$ and $\text{P-DeonticClosure}^*(\mathcal{K}, C \sqsubseteq_{\mathcal{P}} D) = \text{true}$;
- $\sqsubseteq = \sqsubseteq_{d\mathcal{N}}$ and $\text{N-DD-Closure}(\mathcal{K}, C \sqsubseteq_{d\mathcal{N}} D) = \text{true}$;
- $\sqsubseteq = \sqsubseteq_{d\mathcal{P}}$ and $\text{P-DD-Closure}(\mathcal{K}, C \sqsubseteq_{d\mathcal{P}} D) = \text{true}$.

where RationalClosure^* indicates the procedure obtained from RationalClosure [43] by substituting each call of ComputeRanking with a call of CombinedRanking ; analogously $\text{N-DeonticClosure}^*$ and $\text{P-DeonticClosure}^*$ are obtained substituting calls to $\text{ComputeNormRanking}$ with calls to CombinedRanking .

The procedures are sound and complete w.r.t. minimal entailment.

Theorem 2. For any KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and inclusion $C \sqsubset D$ ($\sqsubset \in \{\sqsubseteq, \sqsubseteq_E, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}, \sqsubseteq_{d\mathcal{N}}, \sqsubseteq_{d\mathcal{P}}\}$), we have $\mathcal{K} \models_{\min} C \sqsubset D$ iff $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubset D$.

Hence, checking the consistency of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ corresponds to checking the TBox $\mathcal{T}^*$.

Corollary 1. $\langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ is inconsistent iff $\mathcal{T}^* \models \top \sqsubseteq \perp$.

The computational costs of the overall decision procedure stay in the same complexity class as $\mathcal{ALC}$.

Proposition 5. Checking minimal entailment for a Deontic Defeasible KB is EXPTIME-complete.

5. Conclusions

We introduced Deontic Defeasible Description Logic. The main contributions of the paper are (i) a system for modelling deontic reasoning in DLs; (ii) a second system combining reasoning about expectations and norms in DLs. Of both entailment relations we have given a semantic characterisation and the correspondent decision procedures, which are easy to implement (built over classical DL reasoners). Deciding entailment stays in the same complexity class of classical $\mathcal{ALC}$. Our work has a more general contribution to the development of non-monotonic reasoning in DLs, showing that it is possible to easily add negated inclusions in the KB (here represented by permissions), and combine different modalities.

Many frameworks for non-monotonic reasoning have been proficently used both for modelling deontic reasoning and defeasible reasoning: beyond the preferential framework used here, that has been extensively used for modeling dyadic deontic logic on one hand and defeasible DLs on the other, there has been relevant work in the framework of *Defeasible Logic* [46, 12], both in the direction of deontic reasoning [47, 48, 49, 50] and of defeasible reasoning for DLs [51, 52], but we are not aware of any work in this framework modeling deontic reasoning specifically for DLs. In Section 4 we combine two forms of non-monotonic reasoning, respectively based on expectations and norms; as far as we know, the only other work on this topic is [53], which is based on sequent calculi and a propositional language.

There are various directions in which the present work can be improved. First of all, we have not considered ABox reasoning. It is quite straightforward to introduce reasoning about individuals in a ranked semantics based on minimality, but, as shown for RC in DLs [8, 42], once we reason about individuals we risk to have a multiplicity of minimal configurations, incompatible with each other, associated to the same ABox, making reasoning more complex and computationally expensive. The same happens in this framework; this is an interesting property to be analysed from the deontic point of view, especially in a multi-agent framework. Our approach to minimal entailment has a known limit regarding the application of defeasible information beyond the roles (e.g., we cannot derive $C \sqsubseteq_E \exists r.E$ from $C \sqsubseteq \exists r.D$ and $D \sqsubseteq_E E$). The proposal in [30] gives a possible refinement of RC for the DL $\mathcal{EL}_{\perp}$ to overcome this limit. In addition, RC has some well-known inferential limits [44], overcome through RC extensions [26, 34, 35, 37], which should be also be considered in our framework. Finally, it would also be worth introducing alternative rule-based approaches to deontic reasoning into the framework of defeasible DLs, such as the Input/Output logic approach [54].

Procedure P-DD-Closure($\mathcal{K}, \alpha$)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$, a query $\alpha = C \sqsubseteq_{d\mathcal{P}} D$.

Output: true if $\mathcal{K} \models_{\min} C \sqsubseteq_{d\mathcal{P}} D$, false otherwise

if N-DD-Closure($\mathcal{K}, C \sqsubseteq_{\mathcal{N}} \neg D$) = false then

 return true;

else if $\mathcal{T}^* \models C \sqsubseteq \perp$ then

 return true;

else

 return false;

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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Deontic Defeasible Description Logic

Appendix: Additional Material & Relevant Proofs

Note: all the propositions labelled with an asterisk are not present in the main article.

A. Procedures in Section 2

In Section 2 we refer to three procedures from [43]: `Exceptional`, `ComputeRanking`, and `RationalClosure`. A full analysis of such procedure can be found in the original paper, but we report them also here for the reviewers' convenience.

Procedure `Exceptional`($\mathcal{T}, \mathcal{E}$)

Input: $\mathcal{T}$ and $\mathcal{E} \subseteq \mathcal{D}$
Output: $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mathcal{E}'$ is a set of exceptional axioms w.r.t. $\langle \mathcal{T}, \mathcal{E} \rangle$
 $\mathcal{E}' := \emptyset$;
foreach $E \sqsubseteq_E F \in \mathcal{E}$ **do**
 if $\mathcal{T} \models \bigwedge \bar{\mathcal{E}} \sqsubseteq \neg C$ **then**
 $\mathcal{E}' := \mathcal{E}' \cup \{E \sqsubseteq_E F\}$;
return $\mathcal{E}'$

Procedure `ComputeRanking`($\langle \mathcal{T}, \mathcal{D} \rangle$)

Input: Ontology $\langle \mathcal{T}, \mathcal{D} \rangle$
Output: Ontology $\langle \mathcal{T}^*, \mathcal{D}^* \rangle$, the exceptionality ranking $\mathcal{E}$ of $\mathcal{D}^*$
 $\mathcal{T}^* := \mathcal{T}$;
 $\mathcal{D}^* := \mathcal{D}$;
repeat
 $i := 0$;
 $\mathcal{E}_0 := \mathcal{D}^*$;
 $\mathcal{E}_1 := \text{Exceptional}(\mathcal{T}^*, \mathcal{E}_0)$;
 while $\mathcal{E}_{i+1} \neq \mathcal{E}_i$ **do**
 $i := i + 1$;
 $\mathcal{E}_{i+1} := \text{Exceptional}(\mathcal{T}^*, \mathcal{E}_i)$;
 $\mathcal{D}_\infty := \mathcal{E}_i$;
 $\mathcal{D}^* := \mathcal{D}^* \setminus \mathcal{D}_\infty$;
 $\mathcal{T}^* := \mathcal{T}^* \cup \{E \sqsubseteq F \mid E \sqsubseteq_E F \in \mathcal{D}_\infty\}$;
until $\mathcal{D}_\infty = \emptyset$;
 $\mathcal{E} := (\mathcal{E}_0, \dots, \mathcal{E}_{i-1})$;
return $(\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle, \mathcal{E})$

Procedure `RationalClosure`($\mathcal{K}, \alpha$)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ and a query $C \sqsubseteq_E D$.
Output: **true** if $\mathcal{K} \models_{\min} C \sqsubseteq_E D$, **false** otherwise
 $(\mathcal{K}^* = \mathcal{T}^* \cup \mathcal{D}^*, \mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n)) := \text{ComputeRanking}(\mathcal{K})$;
 $i := 0$;
while $\mathcal{T}^* \models \bigwedge \bar{\mathcal{E}}_i \sqcap C \sqsubseteq \perp$ **and** $i \leq n$ **do**
 $i := i + 1$;
if $i \leq n$ **then**
 return $\mathcal{T}^* \models \bigwedge \bar{\mathcal{E}}_i \sqcap C \sqsubseteq D$;
else
 return $\mathcal{T}^* \models C \sqsubseteq D$;

B. Proofs for Section 3

Proposition 1. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ be a deontic KB and $\mathcal{O}_{\mathcal{N}} := \langle \Delta^{\mathcal{O}_{\mathcal{N}}}, \cdot^{\mathcal{O}_{\mathcal{N}}}, \prec_N^{\mathcal{O}_{\mathcal{N}}} \rangle$ its Big D-model. Then $\mathcal{O}_{\mathcal{N}} \Vdash \mathcal{K}$, and, for any $C \in \mathcal{L}$, $h_{\mathcal{O}_{\mathcal{N}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

Proof. First, we prove that $\mathcal{O}_{\mathcal{N}}$ is a model of $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. It can be easily proved referring to Lemma 8 and Corollary 1 in [43]. We have to prove that such a result holds also once we consider the conditional permission. Hence, it is sufficient to prove that the satisfaction of conditional permissions is preserved under ranked union.

Let $C \sqsubseteq_{\mathcal{P}} D$ be any conditional permission, $\mathcal{R}$ and $\mathcal{R}'$ be two ranked models of $C \sqsubseteq_{\mathcal{P}} D$, and $\mathcal{R}^{\cup}$ be ranked union of $\mathcal{R}$ and $\mathcal{R}'$. We want to prove that $\mathcal{R}^{\cup} \Vdash C \sqsubseteq_{\mathcal{P}} D$. Assume that is not the case. Hence we have $\min_{\prec_N^{\mathcal{R}^{\cup}}}(C^{\mathcal{R}^{\cup}}) \cap D^{\mathcal{R}^{\cup}} = \emptyset$. We have three possible cases to be considered:

- $\min_{\prec_N^{\mathcal{R}^{\cup}}}(C^{\mathcal{R}^{\cup}}) = \min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}});$
- $\min_{\prec_N^{\mathcal{R}^{\cup}}}(C^{\mathcal{R}^{\cup}}) = \min_{\prec_N^{\mathcal{R}'}}(C^{\mathcal{R}'});$
- $\min_{\prec_N^{\mathcal{R}^{\cup}}}(C^{\mathcal{R}^{\cup}}) = \min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \cup \min_{\prec_N^{\mathcal{R}'}}(C^{\mathcal{R}'}).$

We consider only the first case, since the other two are analogous. We would have $\min_{\prec_N^{\mathcal{R}^{\cup}}}(C^{\mathcal{R}^{\cup}}) \cap D^{\mathcal{R}} = \emptyset$, that would imply $\mathcal{R} \not\Vdash C \sqsubseteq_{\mathcal{P}} D$, against the hypothesis. Hence ranked union preserves also the satisfaction of the conditional permissions. $\mathcal{O}_{\mathcal{N}} \Vdash \mathcal{P}$ is an immediate consequence, and we have proved that $\mathcal{O}_{\mathcal{N}} \Vdash \mathcal{K}$.

Now we prove that $\mathcal{O}_{\mathcal{N}}$ is a minimal model of $\mathcal{K}$. The proof is organised in three steps.

- Given any concept C , there is a finite model $\mathcal{S}$ of $\mathcal{K}$ s.t. $h_{\mathcal{S}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

That is easily provable by taking any model $\mathcal{R}'$ of $\mathcal{K}$ s.t. $h_{\mathcal{R}'}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$, and defining from $\mathcal{R}'$ a finite ranked interpretation $\mathcal{S}$ according to the procedure described in Definition 34 in [43]. The definition of $\mathcal{S}$ such that $\mathcal{S}$ is a model of $\mathcal{K}$ s.t. $h_{\mathcal{S}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

- Given any concept C , there is a model $\mathcal{R}^*$ of $\mathcal{K}$ in $Mod_{\Delta}(\mathcal{K})$ s.t. $h_{\mathcal{R}^*}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

We can define such a model from the finite model $\mathcal{S}$: we make a model with a countably infinite domain by simply creating an enumerable number of copies of $\mathcal{S}$ and making the ranked union of them. Then there is an equivalent model in $Mod_{\Delta}(\mathcal{K})$.

- For any $C \in \mathcal{L}$, $h_{\mathcal{O}_{\mathcal{N}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

By Definition of $\mathcal{O}_{\mathcal{N}}$, it is clear that, for every $C \in \mathcal{L}$, $h_{\mathcal{O}_{\mathcal{N}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \in Mod_{\Delta}(\mathcal{K})\}$. In the previous point we have proved that $\min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \in Mod_{\Delta}(\mathcal{K})\} = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$, and it follows from the ranked union defining $\mathcal{O}_{\mathcal{N}}$ that, for every $C \in \mathcal{L}$, $h_{\mathcal{O}_{\mathcal{N}}}(C) = \min\{h_{\mathcal{R}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

Theorem 1. For any deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and any inclusion $C \sqsubset D$ ($\sqsubset \in \{\sqsubseteq, \sqsubseteq_E, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}\}$), we have

$$\mathcal{K} \models_{\min} C \sqsubset D \text{ iff } \mathcal{K} \vdash_{\mathcal{N}} C \sqsubset D.$$

The proof of Theorem 1 is quite articulated, and needs to be organised in a series of propositions.

The first lemma proves that the procedure `NormExceptional` is correct w.r.t. the semantic notion of exceptionality.

Lemma 1 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ be any deontic KB.

- For any $C \sqsubseteq_{\mathcal{P}} D \in \mathcal{P}$, if $C \sqsubseteq_{\mathcal{P}} D \in \Psi'$ according to $\text{NormExceptional}(\mathcal{T}, \mathcal{N}, \mathcal{P})$, then $C \sqsubseteq_{\mathcal{P}} D$ is exceptional w.r.t. $\mathcal{K}$.
- For any $C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}$, if $C \sqsubseteq_{\mathcal{N}} D \in \Phi'$ according to $\text{NormExceptional}(\mathcal{T}, \mathcal{N}, \mathcal{P})$, then $C \sqsubseteq_{\mathcal{N}} D$ is exceptional w.r.t. $\mathcal{K}$.

Proof. We start by considering the conditionals in $\mathcal{P}$. We proceed by induction on the number of repetitions of the cycle in Procedure NormExceptional .

Let $C \sqsubseteq_{\mathcal{P}} D$ be added to Ψ' in the first iteration of the **repeat – until** cycle. That is,

$$\mathcal{T} \models \bigwedge \overline{\mathcal{N}} \sqsubseteq \neg(C \sqcap D).$$

Looking for a proof by contradiction, assume $C \sqsubseteq_{\mathcal{P}} D$ is not exceptional w.r.t. $\mathcal{K}$. That is, there is a model $\mathcal{R}$ of $\mathcal{K}$ s.t. $\mathcal{R} \not\models \top \sqsubseteq_{\mathcal{N}} \neg C$. That is, $\min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(C^{\mathcal{R}}) \subseteq \min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(\top^{\mathcal{R}})$, and, since $\mathcal{R} \models C \sqsubseteq_{\mathcal{P}} D$ there is an $o \in \min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(C^{\mathcal{R}})$ s.t. $o \in (C \sqcap D)^{\mathcal{R}}$. Also note that such an o , since it is in $\min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(\top^{\mathcal{R}})$, must be in $(\bigwedge \overline{\mathcal{N}})^{\mathcal{R}}$. Let $\mathcal{I}$ be the classical DL interpretation specified by $\mathcal{R}$, that is $\mathcal{I} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$. $\mathcal{I}$ is a model of $\mathcal{T}$, and, since $o \in ((C \sqcap D)^{\mathcal{R}} \cap (\bigwedge \overline{\mathcal{N}})^{\mathcal{R}})$, we cannot have $\mathcal{T} \models \bigwedge \overline{\mathcal{N}} \sqsubseteq \neg(C \sqcap D)$, against our hypothesis. Hence, if $C \sqsubseteq_{\mathcal{P}} D$ is added to Ψ' in the first iteration of the **repeat – until** loop, we have that $C \sqsubseteq_{\mathcal{P}} D$ is exceptional w.r.t. $\mathcal{K}$. Note that, by definition of exceptionality, $C \sqsubseteq_{\mathcal{P}} D \in \Psi'$ implies that $\min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(\top^{\mathcal{R}}) \subseteq \neg C^{\mathcal{R}}$. As a consequence, $\mathcal{R} \models \top \sqsubseteq_{\mathcal{N}} \bigwedge \mathcal{E}$.

Now we apply the induction step: we check the $n + 1$ iteration of the **repeat – until** loop, while assuming that all the conditionals added in Ψ' up to the n th iteration are actually exceptional w.r.t. $\mathcal{K}$, and $\mathcal{R} \models \top \sqsubseteq_{\mathcal{N}} \bigwedge \mathcal{E}$ holds for any model $\mathcal{R}$ of $\mathcal{K}$. Again, we proceed by contradiction, assuming that, for some $C \sqsubseteq_{\mathcal{P}} D \in \Psi$,

$$\mathcal{T} \models \bigwedge \overline{\mathcal{N}} \sqcap \bigwedge \mathcal{E} \sqsubseteq \neg(C \sqcap D),$$

while $C \sqsubseteq_{\mathcal{P}} D$ is not exceptional w.r.t. $\mathcal{K}$. The proof proceeds exactly as for the first iteration, we have just to consider that $\mathcal{E}$ is not empty.

We have proved that the procedure is correct in the definition of the set of exceptional permissions $\mathcal{E}$. As a consequence, we know that $\mathcal{R} \models \top \sqsubseteq_{\mathcal{N}} \bigwedge \mathcal{E}$ holds for any model $\mathcal{R}$ of $\mathcal{K}$.

Now we have to check that also the procedure applied to each $C \sqsubseteq_{d\mathcal{N}} D \in \Phi$ to define the set Φ' is correct, that is, Φ' contains only exceptional conditionals in $\mathcal{N}$.

It suffices to prove that if there is a model $\mathcal{R}$ of $\mathcal{K}$ s.f. $\mathcal{R} \not\models \top \sqsubseteq_E \neg C$, then $\mathcal{T} \models \bigwedge \overline{\mathcal{D}} \sqcap \bigwedge \mathcal{E} \sqsubseteq \neg C$ does not hold. $\mathcal{R} \not\models \top \sqsubseteq_E \neg C$ implies that there is an $o \in \min_{\prec_{\mathcal{N}}}^{\mathcal{R}}(\top^{\mathcal{R}})$ such that $o \in C^{\mathcal{R}}$. Let $\mathcal{I}$ be the classical interpretation associated with $\mathcal{R}$. It follows that $\mathcal{I}$ is a model of $\mathcal{T}$ and that $x \in ((\bigwedge \overline{\mathcal{D}})^{\mathcal{I}} \cap (\bigwedge \mathcal{E})^{\mathcal{I}})$, but that $x \notin (\neg C)^{\mathcal{I}}$. That is, $\mathcal{T} \models \bigwedge \overline{\mathcal{D}} \sqcap \bigwedge \mathcal{E} \not\models \neg C$.

We have proved that the procedure $\text{NormExceptional}(\mathcal{T}, \mathcal{N}, \mathcal{P})$ is sound, that is, the sets Φ' and Ψ' contain only exceptional conditionals.

We introduce some notation that is used in the following proofs, but does not appear in the paper:

- $Cn_{\mathcal{N}}$ is the closure operation that corresponds to $\vdash_{\mathcal{N}}$ (Definition 8). That is,

$$Cn_{\mathcal{N}}(\mathcal{K}) = \{C \sqsubseteq D \mid \mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq D\},$$

with $\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_{\mathcal{N}}, \sqsubseteq_{\mathcal{P}}\}$.

- A *defeasible subsumption relation* $\sqsubseteq : \mathcal{L} \times \mathcal{L}$ is a binary relation between concepts. It is called *rational* iff it satisfies certain specific closure properties. We refer the reader to Section 3.1 in [43] for the detailed definitions.
- $rk_{\mathcal{K}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C the rank i of the concept C w.r.t. $\mathcal{K}$ according to the **While** loop in Procedure **N – DeonticClosure**. If $i = n + 1$ and $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, then $rk_{\mathcal{K}} = \infty$.

The second lemma proves that the closure operator $Cn_{\mathcal{N}}$ satisfies inclusion and defines a rational subsumption relation.

Lemma 2 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. Then (i) $\mathcal{K} \subseteq Cn_{\mathcal{N}}(\mathcal{K})$ and (ii) $Cn_{\mathcal{N}}(\mathcal{K})$ induces a defeasible subsumption relation $\sqsubseteq_{\mathcal{N}}^{\mathcal{K}_{deon}} := \{(C, D) \mid \mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D\}$ that is rational.*

Proof. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$.

Proving (i): Assume $C \sqsubseteq D \in \mathcal{T}$. $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq D$ iff $\mathcal{T}^{\circ} \models C \sqsubseteq D$; since $\mathcal{T} \subseteq \mathcal{T}^{\circ}$, $\mathcal{T} \subseteq Cn_{\mathcal{N}}(\mathcal{K})$.

Assume that $C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}$. Either $C \sqsubseteq_{\mathcal{N}} D$ ends up in $\mathcal{N}_{\infty}$, or there will be an i ($0 \leq i \leq n$) s.t. $C \sqsubseteq_{\mathcal{N}} D \in \Phi_i$. In the former case, $C \sqsubseteq_{\mathcal{N}} D$ is in $\mathcal{T}^{\circ}$, and so $\mathcal{T}^{\circ} \models C \sqsubseteq D$, i.e., $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D$. In the latter case, $\models \overline{\Phi}_i \sqsubseteq \neg C \sqcup D$, and so $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \mathcal{E}_i \sqcap C \sqsubseteq D$, i.e., $C \sqsubseteq_{\mathcal{N}} D \in Cn_{\mathcal{N}}(\mathcal{K})$. Hence $\mathcal{N} \subseteq Cn_{\mathcal{N}}(\mathcal{K})$.

Assume that $C \sqsubseteq_{\mathcal{P}} D \in \mathcal{P}$. Either $C \sqsubseteq_{\mathcal{P}} D$ ends up in $\mathcal{P}_{\infty}$, or there will be an i ($0 \leq i \leq n$) s.t. $C \sqsubseteq_{\mathcal{P}} D \in \Psi_i$. In the former case, $C \sqsubseteq \perp$ is in $\mathcal{T}^{\circ}$, and so $\mathcal{T}^{\circ} \models C \sqsubseteq D$, i.e., $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{P}} D$. In the latter case, $\models \overline{\Phi}_i \sqcap \overline{\mathcal{E}}_i \sqsubseteq \neg(C \sqcap D)$, and so $\mathcal{T}^{\circ} \not\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \mathcal{E}_i \sqcap C \sqsubseteq \neg D$, i.e., $C \sqsubseteq_{\mathcal{P}} D \in Cn_{\mathcal{N}}(\mathcal{K})$. Hence $\mathcal{P} \subseteq Cn_{\mathcal{N}}(\mathcal{K})$.

Proving (ii): We show $\sqsubseteq_{\mathcal{N}}^{\mathcal{K}_{deon}}$ satisfies all rationality properties.

For (Ref)-(CM) the proof is straightforward, see Lemma 44 in [43]. We just check (RM)

- (RM). If $rk_{\mathcal{K}}(C) = i < \infty$, let $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \overline{\Psi}_i \sqcap C \sqsubseteq E$ and $\mathcal{T}^{\circ} \not\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \overline{\Psi}_i \sqcap C \sqsubseteq \neg D$. As a consequence, $\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \overline{\Psi}_i \sqsubseteq \neg(C \sqcap D)$ does not hold, otherwise we would have $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \overline{\Psi}_i \sqcap C \sqsubseteq \neg D$. Hence we have $C \sqcap D \sqsubseteq_{\mathcal{N}} E \in Cn_{\mathcal{N}}(\mathcal{K})$ since $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \overline{\Psi}_i \sqcap C \sqcap D \sqsubseteq E$. If $rk_{\mathcal{K}}(C) = \infty$, then we have $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, and the proof is trivial.

The next lemma proves that the TBox $\mathcal{T}^{\circ}$, computed by the procedure `ComputeNormRanking`, is sufficient to decide all the inclusion of form $C \sqsubseteq \perp$ (that, semantically, is equivalent to $C \sqsubseteq_{\mathcal{N}} \perp$ and $C \sqsubseteq_{\mathcal{P}} \perp$).

Lemma 3 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and assume $C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}$. Then $\mathcal{R} \models C \sqsubseteq_{\mathcal{N}} \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$.*

Proof. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$.

For the only-if part, $\mathcal{R} \models C \sqsubseteq_{\mathcal{N}} \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ implies that every rational subsumption relation containing $\mathcal{K}$ must satisfy also $C \sqsubseteq_{\mathcal{N}} \perp$. Hence we have that $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} \perp$, since $Cn_{\mathcal{N}}(\mathcal{K})$ induces a rational subsumption relation extending $\mathcal{K}$ (Lemma 2). We know that $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} \perp$ is possible only if C is always negated in the ranking procedure, i.e., $C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}_{\infty}$ or $C \sqsubseteq_{\mathcal{P}} \perp \in \mathcal{P}_{\infty}$ at some iteration of the procedure, that implies $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$.

For the if part, through `ComputeNormRanking` we define from $\mathcal{K}$ a new knowledge base $\mathcal{K}^{\circ} = \langle \mathcal{T}^{\circ}, \mathcal{N}^{\circ}, \mathcal{P}^{\circ} \rangle$, with $\mathcal{T}^{\circ}$ obtained from $\mathcal{T}$ by adding the sets $\{C \sqsubseteq D \mid C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}_{\infty}\}$ and $\{C \sqsubseteq \perp \mid C \sqsubseteq_{\mathcal{P}} D \in \mathcal{P}_{\infty}\}$, where $\mathcal{N}_{\infty}$ and $\mathcal{P}_{\infty}$ are defined at every iteration of the main repeat – until loop in the procedure.

Let us denote with $\mathcal{N}_{\sqsubseteq}^1, \dots, \mathcal{N}_{\sqsubseteq}^n$ and $\mathcal{P}_{\sqsubseteq}^1, \dots, \mathcal{P}_{\sqsubseteq}^n$ such sets. Assume that $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, but there is a ranked model of $\mathcal{K}$ in which C is non-empty. Let $\mathcal{R}$ be such a model, with an object o falling under $C^{\mathcal{R}}$. Since $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, there must be either a GCI $E \sqsubseteq F$ in some $\mathcal{N}_{\sqsubseteq}^i$ or some GCI $E \sqsubseteq \perp$ in some $\mathcal{P}_{\sqsubseteq}^j$ that is not satisfied. That is, given the nature of the GCIs in every $\mathcal{N}_{\sqsubseteq}^n$ ($\mathcal{T}^{\circ} \models E \sqsubseteq \perp$ for every $E \sqsubseteq F$ contained in some $\mathcal{N}_{\sqsubseteq}^n$), this means that there is a subsumption $E \sqsubseteq \perp$ that is not satisfied in $\mathcal{R}$, s.t. either $E \sqsubseteq_{\mathcal{N}} F$ is in some $\mathcal{N}_{\sqsubseteq}^i$ or $E \sqsubseteq \perp$ is in some $\mathcal{P}_{\sqsubseteq}^j$. Therefore, there must be an object o falling under $E^{\mathcal{R}}$.

We have to distinguish two main cases:

- A. The concept E refers to some $E \sqsubseteq F$ in some $\mathcal{N}_{\sqsubseteq}^i$.
- B. The concept E refers to some $E \sqsubseteq \perp$ in some $\mathcal{P}_{\sqsubseteq}^j$.

Case A. $o \in E$ for some $E \sqsubseteq F$ in some $\mathcal{N}_{\sqsubseteq}^i$. This is in contradiction with $\mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j \models \bigcap \{ \neg G \sqcup H \mid G \sqsubseteq_{\mathcal{N}} H \in \mathcal{N}_{\sqsubseteq}^i \} \cap \bigcap \{ \neg G \mid G \sqsubseteq_{\mathcal{P}} H \in \mathcal{P}_{\sqsubseteq}^i \} \sqsubseteq \neg E$. We have 4 possible cases, described below.

Case B. $o \in E$ for some $E \sqsubseteq \perp$ in some $\mathcal{P}_{\sqsubseteq}^i$. We know that $\mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j \models \bigcap \{ \neg G \sqcup H \mid G \sqsubseteq_{\mathcal{N}} H \in \mathcal{N}_{\sqsubseteq}^i \} \cap \bigcap \{ \neg G \mid G \sqsubseteq_{\mathcal{P}} H \in \mathcal{P}_{\sqsubseteq}^i \} \sqsubseteq \neg(E \sqcap F)$, for some $E \sqsubseteq_{\mathcal{P}} F \in \Psi_{i-1}$. Here we have two possible situations regarding o : either $o \in (E \sqcap \neg F)^{\mathcal{R}}$ or $o \in (E \sqcap F)^{\mathcal{R}}$. In the former case, it is necessary that there is another object o' s.t. $o' \in (E \sqcap F)^{\mathcal{R}}$, otherwise the inclusion $E \sqsubseteq_{\mathcal{P}} F$ would be falsified, and $\mathcal{R}$ would not be a model of $\mathcal{K}$. Hence, w.l.o.g., we can assume that $o \in (E \sqcap F)^{\mathcal{R}}$, that is in direct conflict with $\mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j \models \bigcap \{ \neg G \sqcup H \mid G \sqsubseteq_{\mathcal{N}} H \in \mathcal{N}_{\sqsubseteq}^i \} \cap \bigcap \{ \neg G \mid G \sqsubseteq_{\mathcal{P}} H \in \mathcal{P}_{\sqsubseteq}^i \} \sqsubseteq \neg(E \sqcap F)$. Also in this case, we have 4 possible cases, that we are as follows:

- $\mathcal{R} \models \mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j$ and $o \in (G \sqcap \neg H)^{\mathcal{R}}$ for some $G \sqsubseteq H \in \mathcal{N}_{\sqsubseteq}^i$ (Case 1 below);
- $\mathcal{R} \models \mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j$ and $o \in (G)^{\mathcal{R}}$ for some $G \sqsubseteq \perp \in \mathcal{P}_{\sqsubseteq}^i$ (Case 2 below);
- $\mathcal{R} \not\models \mathcal{N}_{\sqsubseteq}^j$ for some $j < i$ (Case 3 below);
- $\mathcal{R} \not\models \mathcal{P}_{\sqsubseteq}^j$ for some $j < i$ (Case 4 below)

Case 1. Since $\mathcal{R} \models \mathcal{K}$, $\mathcal{R}$ is also a model of $G \sqsubseteq_{\mathcal{N}} H$, which is an element of $\mathcal{N}$. Hence there must be an object m such that $m \prec_N^{\mathcal{R}} o$ (remember that $o \in (G \sqcap \neg H)^{\mathcal{R}}$) and $m \in (G \sqcap H)^{\mathcal{R}}$. $G \sqsubseteq H \in \mathcal{N}_{\sqsubseteq}^i$ means that $\mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j \models \bigcap \{ \neg G \sqcup H \mid G \sqsubseteq_{\mathcal{N}} H \in \mathcal{N}_{\sqsubseteq}^i \} \sqsubseteq \neg G$, which is incompatible with the existence of m , since the height of m must be strictly less than i . Since $\mathcal{R} \models \mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j$, there must be a GCI $I \sqsubseteq L \in \mathcal{N}_{\sqsubseteq}^i$ such that $m \in (I \sqcap \neg L)^{\mathcal{R}}$, or a GCI $I \sqsubseteq \perp \in \mathcal{P}_{\sqsubseteq}^i$ such that $m \in (I)^{\mathcal{R}}$. In the former case we fall again in the present case, but the value of the parameter i is lower; in the latter case we move to the second case, again lowering the value of the parameter i .

Case 2. If $G \sqsubseteq \perp \in \mathcal{P}_{\sqsubseteq}^i$ there must be some $G \sqsubseteq_{\mathcal{P}} H \in \mathcal{P}$. Since $o \in (G)^{\mathcal{R}}$, either $o \in (G \sqcap \neg H)^{\mathcal{R}}$, or $o \in (G \sqcap H)^{\mathcal{R}}$. Since $\mathcal{R} \models \mathcal{K}$, $\mathcal{R}$ is also a model of $G \sqsubseteq_{\mathcal{P}} H$ (it is an element of $\mathcal{P}$), in the former case we must conclude that there must be another object m such that $m \prec_N^{\mathcal{R}} o$ and $m \in (G \sqcap H)^{\mathcal{R}}$. Hence, w.l.o.g., we can assume that $o \in (G \sqcap H)^{\mathcal{R}}$, that is in direct conflict with $\mathcal{T} \cup \bigcup_{j=1}^{i-1} \mathcal{N}_{\sqsubseteq}^j \cup \bigcup_{j=1}^{i-1} \mathcal{P}_{\sqsubseteq}^j \models \bigcap \{ \neg G' \sqcup H' \mid G' \sqsubseteq_{\mathcal{N}} H' \in \mathcal{N}_{\sqsubseteq}^i \} \cap \bigcap \{ \neg G' \mid G' \sqsubseteq_{\mathcal{P}} H' \in \mathcal{P}_{\sqsubseteq}^i \} \sqsubseteq \neg(G \sqcap H)$, that, in turn, is a consequence of $G \sqsubseteq \perp \in \mathcal{P}_{\sqsubseteq}^i$. Now the case proceeds as Case 1, with the same outcome.

Case 3. $\mathcal{R} \not\models \mathcal{N}_{\sqsubseteq}^j$ for some $j \leq i$, then $\mathcal{R}$ does not satisfy some $E \sqsubseteq F \in \mathcal{N}_{\sqsubseteq}^j$, and therefore there must be an object falling under $(E \sqcap \neg F)^{\mathcal{R}}$.

Either we move to Case 1, or we move to this same case, but, in both the case, the value of the parameter i must lower.

Case 4. $\mathcal{R} \not\models \mathcal{P}_{\sqsubseteq}^j$ for some $j \leq i$. That is, $\mathcal{R} \not\models G \sqsubseteq \perp$ for some $G \sqsubseteq \perp \in \mathcal{P}_{\sqsubseteq}^j$, and there must be some object falling under $G^{\mathcal{R}}$. Either we move to Case 2, or we move to this same case, but, in both the case, the value of the parameter i must lower.

In all cases, we can proceed moving to another case, but at every step the value of the parameter i must decrease, hence creating an infinitely descending chain. However, since the ranks in a ranked model are not infinitely descending, and moreover the number of antecedents of the conditionals in $\mathcal{N} \cup \mathcal{P}$ is finite too, we cannot descend indefinitely, and we must end up incurring in a contradiction, proving also this direction of our proposition.

An immediate consequence of Lemma 3 binds the consistency of any deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ (that is, existence of a ranked model of $\mathcal{K}$) to classical consistency.

Corollary 2 (*). . Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. Then $\mathcal{K}$ is inconsistent iff $\mathcal{T}^\circ \models \top \sqsubseteq \perp$.

Another immediate consequence is that the analogous of Lemma 3 holds also for permission.

Corollary 3 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and assume $C \lesssim_{\mathcal{P}} D \in \mathcal{P}$. Then $\mathcal{R} \Vdash C \lesssim_{\mathcal{P}} \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^\circ \models C \sqsubseteq \perp$.

Proof. It is easy to check from the definitions that, for any concept C and any ranked model $\mathcal{R}$, $\mathcal{R} \Vdash C \lesssim_{\mathcal{N}} \perp$ iff $\mathcal{R} \Vdash C \lesssim_{\mathcal{P}} \perp$ iff $\mathcal{R} \Vdash C \sqsubseteq \perp$.

We can now prove that the knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$ are ranked-equivalent.

Lemma 4 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and let $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$ be obtained from $\mathcal{K}$ through function `ComputeNormRanking`. Then $\mathcal{K}$ and $\mathcal{K}^\circ$ are ranked equivalent.

Proof. Given $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, the function `ComputeNormRanking`($\mathcal{K}$) outputs a knowledge base $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$, in which every iteration of the main `repeat – until` loop identifies a (possibly empty) set $\{C_1 \lesssim_{\mathcal{N}} D_1, \dots, C_n \lesssim_{\mathcal{N}} D_n\}$ of always exceptional normative subsumptions and a (possibly empty) set $\{C_{n+1} \lesssim_{\mathcal{P}} D_{n+1}, \dots, C_{n+m} \lesssim_{\mathcal{P}} D_{n+m}\}$ of always exceptional permission subsumptions, that are moved, respectively, from $\mathcal{N}$ and $\mathcal{P}$ to $\mathcal{T}$. That is, we have $\mathcal{T}^\circ = \mathcal{T} \cup \{C_1 \sqsubseteq D_1, \dots, C_n \sqsubseteq D_n\} \cup \{C_{n+1} \sqsubseteq \perp, \dots, C_{n+m} \sqsubseteq \perp\}$, $\mathcal{N}^\circ = \mathcal{N} \setminus \{C_1 \lesssim_{\mathcal{N}} D_1, \dots, C_n \lesssim_{\mathcal{N}} D_n\}$, and $\mathcal{P}^\circ = \mathcal{P} \setminus \{C_{n+1} \lesssim_{\mathcal{P}} D_{n+1}, \dots, C_{n+m} \lesssim_{\mathcal{P}} D_{n+m}\}$. It is sufficient to prove that, for every $C_i \lesssim_{\mathcal{N}} D_i$ moved to $\mathcal{T}^\circ$, $C_i \sqsubseteq D_i$ is satisfied in every ranked model of $\mathcal{K}$ while $C_i \lesssim_{\mathcal{N}} D_i$ is satisfied in every model of $\mathcal{K}^\circ$. Analogously for every $C_i \lesssim_{\mathcal{P}} D_i$ moved to $\mathcal{T}^\circ$.

Consider any $C \lesssim_{\mathcal{N}} D \in \mathcal{N}$ s.t. `ComputeNormRanking`($\mathcal{K}$) has eliminated it from $\mathcal{N}^\circ$ and added $C \sqsubseteq D$ to $\mathcal{T}^\circ$. We know that for every $C \sqsubseteq D \in \mathcal{T}^\circ$, $\mathcal{T}^\circ \models C \sqsubseteq \perp$, that, by Lemma 3, implies $\mathcal{R} \Vdash C \lesssim_{\mathcal{N}} \perp$ for every ranked model $\mathcal{R}$ of $\mathcal{K}$. The latter implies $\mathcal{R} \Vdash C \sqsubseteq \perp$, that in turn implies $\mathcal{R} \Vdash C \sqsubseteq D$ for every ranked model $\mathcal{R}$ of $\mathcal{K}$. On the other hand, $\mathcal{T}^\circ \models C \sqsubseteq \perp$ implies that every model $\mathcal{R}$ of $\mathcal{K}^\circ$ satisfies $C \sqsubseteq \perp$, that in turn implies that every model $\mathcal{R}$ of $\mathcal{K}^\circ$ is a model of $C \sqsubseteq \perp$, and, consequently, of $C \sqsubseteq D$. Hence every model of $\mathcal{K}^\circ$ is a model of $\mathcal{N}$.

An analogous proof shows, relying on Corollary 3, that for every $C_i \lesssim_{\mathcal{P}} D_i$ moved to $\mathcal{T}^\circ$, $C_i \sqsubseteq D_i$ is satisfied in every ranked model of $\mathcal{K}$ while $C_i \lesssim_{\mathcal{P}} D_i$ is satisfied in every model of $\mathcal{K}^\circ$.

On one hand we can conclude that every model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is a model of $\mathcal{T}^\circ$, and obviously it is also a model of $\mathcal{N}^\circ$ and $\mathcal{P}^\circ$, since $\mathcal{N}^\circ \subseteq \mathcal{N}$ and $\mathcal{P}^\circ \subseteq \mathcal{P}$.

On the other hand we can conclude that every model of $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$ is a model of $\mathcal{N}$ and $\mathcal{P}$, while obviously it is also a model of $\mathcal{T}$ since $\mathcal{T} \subseteq \mathcal{T}^\circ$.

We can conclude that $\mathcal{K}$ and $\mathcal{K}^\circ$ are satisfied exactly by the same ranked models.

Lemma 5 (*). For every $C \in \mathcal{L}$ and every $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, $\mathcal{R} \Vdash \top \lesssim_{\mathcal{N}} C$ for every model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{K} \vdash_{\mathcal{N}} \top \lesssim_{\mathcal{N}} C$.

Proof. If $\mathcal{K}$ is inconsistent, the proposition is trivial. Let $\mathcal{K}$ be consistent.

Consider the If-direction. If $\mathcal{K}$ is consistent, then $\top$ cannot be exceptional, and, according to the procedure `N – DeonticClosure`, $\mathcal{K} \vdash_{\mathcal{N}} \top \lesssim_{\mathcal{N}} C$ iff $\mathcal{T}^\circ \models \bigwedge \bar{\mathcal{N}}^\circ \sqcap \bigwedge \Psi_1^- \sqsubseteq C$.

Since Lemma 1 has proved that the procedure `NormExceptional` is sound w.r.t. semantic exceptionality, we can easily check that all the objects in the bottom layer of all the models of $\mathcal{K}$ must satisfy the concept $\bigwedge \bar{\mathcal{N}}^\circ \sqcap \bigwedge \Psi_1^-$. On the other hand, Lemma 4 tells us that all the models of $\mathcal{K}$ satisfy $\mathcal{T}^\circ$. Hence, all the objects in $\min_{\prec_{\mathcal{N}}}(\top)$ for all the models $\mathcal{R}$ of $\mathcal{K}$ are in $C^{\mathcal{R}}$. that is, $\mathcal{R} \Vdash \top \lesssim_{\mathcal{N}} C$ for all the models $\mathcal{R}$ of $\mathcal{K}$.

For the Only-If-direction, the proposition is an immediate consequence of Lemma 2.

For the next lemma, we introduce a new function:

- $rank_{\mathcal{K}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C its minimal height in the ranked models of $\mathcal{K}$ (that is, the height of C in $\mathcal{O}_{\mathcal{N}}$). If $\mathcal{R} \models C \sqsubseteq \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ (that is, $\mathcal{O}_{\mathcal{N}} \models C \sqsubseteq \perp$), then $rank_{\mathcal{K}} = \infty$.

And we remind that $rk_{\mathcal{K}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C the rank i of the concept C w.r.t. $\mathcal{K}$ according to the While loop of Procedure N – DeonticClosure. If $i = n+1$ and $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, then $rk_{\mathcal{K}} = \infty$.

The following lemmas compares $rank_{\mathcal{K}}$ and $rk_{\mathcal{K}}$, showing that they are equivalent. That is, the ranking done by the decision procedure corresponds to the ranking defined in the semantics.

Lemma 6 (*). For every $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and every $C \in \mathcal{L}$, $rank_{\mathcal{K}}(C) = \infty$ iff $rk_{\mathcal{K}}(C) = \infty$.

Proof. It is a straightforward consequence of Corollary 3.

Lemma 7 (*). For every defeasible knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and every concept C , $rank_{\mathcal{K}}(C) = rk_{\mathcal{K}}(C)$.

Proof. From Lemma 3, Corollary 3, Lemma 6 and Lemma 4 we can see that, given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, we can define a ranked-equivalent knowledge base $\mathcal{K}^{\circ} = \langle \mathcal{T}^{\circ}, \mathcal{N}^{\circ}, \mathcal{P}^{\circ} \rangle$ such that all the classical information contained in $\mathcal{N}$ and $\mathcal{P}$ is moved into $\mathcal{T}^{\circ}$. $\mathcal{K}^{\circ}$ can be defined identifying the elements of $\mathcal{N}$ and $\mathcal{P}$ that have ∞ as ranking value, and Lemma 6 shows that w.r.t. the value ∞ , $rank_{\mathcal{K}}(\cdot)$ and $rk(\cdot)$ are equivalent, while Lemma 4 telling us that $\mathcal{K}$ and $\mathcal{K}^{\circ}$ are rank-equivalent. Once we have defined $\mathcal{K}^{\circ}$, Lemma 5 implies that a concept $C \in \mathcal{L}$ is exceptional w.r.t. $\mathcal{K}$ iff $\mathcal{K} \vdash_{\mathcal{N}} \top \sqsubseteq_E \neg C$. Hence the two ranking functions $rank_{\mathcal{K}}(\cdot)$ and $rk(\cdot)$ give back exactly the same results.

Now we have all the ingredients to prove the completeness of the procedure, starting from the obligations.

Proposition 6 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{N}} D$.

Proof. Since we have already proven Lemma 7, here we can use $rk_{\mathcal{K}}$ to indicate indifferently the equivalent ranking functions $rank_{\mathcal{K}}$ and $rk_{\mathcal{K}}$.

For the if part, assume $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{N}} D$. That means that either $rk_{\mathcal{K}}(C \sqcap \neg D) > rk_{\mathcal{K}}(C)$ or $rk_{\mathcal{K}}(C) = \infty$. In the first case, it means that there is some i , $0 \leq i \leq n$, such that $\mathcal{T}^{\circ} \not\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq \perp$ and $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap (C \sqcap \neg D) \sqsubseteq \perp$, hence $\mathcal{T}^{\circ} \not\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq D$, i.e., $\mathcal{K} \vdash_{\mathcal{N}} C \not\sqsubseteq_E D$. In the second case, we have $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$, which implies $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D$.

For the only-if part, assume $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D$. Then either there is some i which is the lowest number such that $\mathcal{T}^{\circ} \not\models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq \perp$ (hence $rk_{\mathcal{K}}(C) = i$), or $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$. In the first case, we have also that $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq D$, which implies $\mathcal{T}^{\circ} \models \bigwedge \overline{\Phi}_i \sqcap \bigwedge \Psi_{i+1}^{\neg} \sqcap (C \sqcap \neg D) \sqsubseteq \perp$, i.e., $rk_{\mathcal{K}}(C \sqcap \neg D) > i$. In the second case, $rk_{\mathcal{K}}(C) = \infty$. In both the cases we can conclude $\mathcal{K} \models_{\min} C \sqsubseteq_E D$.

Proposition 7 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{P}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{P}} D$.

Proof. Given Proposition 6, it is an immediate consequence of the correspondence between the semantic definition of the satisfaction of $C \sqsubseteq_{\mathcal{P}} D$ (Definition 6) and Procedure P – DeonticClosure.

Now we can easily prove **Theorem 1**.

Proof of Theorem 1. Propositions 6 and 7 guarantee that $\vdash_{\mathcal{N}}$ is correct and complete w.r.t. conditional obligations and permissions. We have just so check that, for every classical GCI $C \sqsubseteq D$, $\mathcal{K} \models_{\min} C \sqsubseteq D$ iff $\mathcal{T}^{\circ} \models C \sqsubseteq D$.

The proof is easy. $\mathcal{K} \models_{\min} C \sqsubseteq D$ iff $\mathcal{K} \models_{\min} C \sqcap \neg D \sqsubseteq \perp$. Since, for any concept C and ranked model $\mathcal{R}$, $\mathcal{R} \models C \sqsubseteq \perp$ iff $\mathcal{R} \models C \sqsubseteq_{\mathcal{N}} \perp$, we have that $\mathcal{K} \models_{\min} C \sqcap \neg D \sqsubseteq \perp$ iff $\mathcal{K} \models_{\min} C \sqcap \neg D \sqsubseteq_{\mathcal{N}} \perp$. From Proposition 6 we know that $\mathcal{K} \models_{\min} C \sqcap \neg D \sqsubseteq_{\mathcal{N}} \perp$ iff $\mathcal{K} \vdash_{\mathcal{N}} C \sqcap \neg D \sqsubseteq_{\mathcal{N}} \perp$. It is easy to check in procedure N – DeonticClosure that the latter is the case if and only if $\mathcal{T}^{\circ} \models C \sqcap \neg D \sqsubseteq \perp$, that is, $\mathcal{T}^{\circ} \models C \sqsubseteq D$.

Hence, to summarise, $\mathcal{K} \models_{\min} C \sqsubseteq D$ iff $\mathcal{T}^\circ \models C \sqsubseteq D$.

Eventually, we move to the proof regarding the computational complexity.

Proposition 2. *Checking minimal entailment for a Deontic KB is EXPTIME-complete.*

Proof. EXPTIME-hardness is trivial as the logic extends $\mathcal{ALC}$. We show just that the procedure runs in exponential time. Consider `N-DeonticClosure`; it decides at most n standard (EXPTIME) entailments, where n and each entailment is linear on the size of $\mathcal{K}$. It remains to show that `ComputeNormRanking` runs in exponential time as well. The internal `repeat – until` loop runs `NormExceptional` a linear number of times as each run decreases the sets Φ, Ψ . `NormExceptional` is known to run in EXPTIME thus yielding an exponential upper bound for the execution time of each such loop. The external `repeat – until` loop is again executed a linear number of times on $\mathcal{N}$ and $\mathcal{P}$, as in each loop the resulting sets decrease. As argued already, each such loop requires only exponential time to conclude, thus yielding the desired exponential upper bound.

C. Proofs for Section 4

Proposition 3. *Let $\mathcal{K}$ be a consistent deontic KB and $\mathcal{O}_{\mathcal{DN}} := \langle \Delta^{\mathcal{O}_{\mathcal{DN}}}, \cdot^{\mathcal{O}_{\mathcal{DN}}}, \prec_E^{\mathcal{O}}, \prec_N^{\mathcal{O}} \rangle$ its Big DD-model. Then $\mathcal{O}_{\mathcal{DN}} \models \mathcal{K}$, and, for any $C \in \mathcal{L}$, $h_{\prec_N^{\mathcal{O}}}(C) = \min\{h_{\prec_N^{\mathcal{R}}}(C) \mid \mathcal{R} \models \mathcal{K}\}$ and $h_{\prec_E^{\mathcal{O}}}(C) = \min\{h_{\prec_E^{\mathcal{R}}}(C) \mid \mathcal{R} \models \mathcal{K}\}$.*

Proof. The proof works exactly as in Proposition 1. We need only to consider two ranked orders instead of one.

Proposition 4 *For any concept C and DD-interpretation, the following inclusions are equivalent: $C \sqsubseteq \perp$, $C \sqsubseteq_E \perp$, $C \sqsubseteq_N \perp$, $C \sqsubseteq_{\mathcal{P}} \perp$, $C \sqsubseteq_{dN} \perp$, $C \sqsubseteq_{dP} \perp$.*

Proof. It is an straightforward consequence of the fact that for any ranked order $\prec$ in any interpretation $\mathcal{R}$ and any concept C , $\min_{\prec}(C^{\mathcal{R}}) = \emptyset$ iff $C^{\mathcal{R}} = \emptyset$.

Theorem 2 *For any KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and any inclusion $C \sqsubseteq D$ ($\sqsubseteq \in \{\sqsubseteq, \sqsubseteq_E, \sqsubseteq_N, \sqsubseteq_{\mathcal{P}}, \sqsubseteq_{dN}, \sqsubseteq_{dP}\}$), we have*

$$\mathcal{K} \models_{\min} C \sqsubseteq D \text{ iff } \mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq D.$$

The proof of Theorem 2 can be built on top of the completeness theorem for the procedure `RationalClosure` [43, Theorem 5] and the completeness proof for the relation $\vdash_{\mathcal{N}}$ (Theorem 1 above).

The first necessary step is to prove that the procedure `CombinedRanking` correctly ranks the defeasible inclusions and identifies the TBox.

In the following lemma we use *ranked-equivalence* w.r.t. DD-interpretations. That is, two KBs are ranked-equivalent iff they are satisfied exactly by the same DD-interpretations.

Lemma 8 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle$ be obtained from $\mathcal{K}$ through the procedure `CombinedRanking`. Then $\mathcal{K}$ and $\mathcal{K}^*$ are ranked-equivalent.*

Proof. We start applying to $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ the procedure `ComputeRanking`, obtaining $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}', \mathcal{N}, \mathcal{P} \rangle$. Since by Lemma 12 in [43] we know that $\langle \mathcal{T}, \mathcal{D} \rangle$ and $\langle \mathcal{T}', \mathcal{D}' \rangle$ are equivalent w.r.t. ranked interpretations, $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}', \mathcal{N}, \mathcal{P} \rangle$ are equivalent w.r.t. DD-interpretations.

Then we apply to $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}', \mathcal{N}, \mathcal{P} \rangle$ the procedure `ComputeNormRanking`, obtaining $\mathcal{K}'' = \langle \mathcal{T}'', \mathcal{D}', \mathcal{N}'', \mathcal{P}'' \rangle$. By Lemma 4 we know that $\langle \mathcal{T}', \mathcal{N}, \mathcal{P} \rangle$ and $\langle \mathcal{T}'', \mathcal{N}'', \mathcal{P}'' \rangle$ are equivalent w.r.t. D-interpretations, hence $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}', \mathcal{N}, \mathcal{P} \rangle$ and $\mathcal{K}'' = \langle \mathcal{T}'', \mathcal{D}', \mathcal{N}'', \mathcal{P}'' \rangle$ are equivalent w.r.t. DD-interpretations.

That is, every iteration of the `repeat – until` cycle preserves ranked-equivalence: the initial $\mathcal{K}$ and the final $\mathcal{K}^*$ must be ranked-equivalent.

Lemma 9 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and assume $C \sqsubseteq_N D \in \mathcal{N}$. Then $\mathcal{R} \models C \sqsubseteq_N \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^* \models C \sqsubseteq \perp$.*

Proof. For the if-direction, assume that $\mathcal{T}^* \models C \sqsubseteq \perp$. Since, according to Lemma 8, $\mathcal{K}$ and $\mathcal{K}^*$ are equivalent w.r.t. DD-interpretations, $\mathcal{T}^* \models C \sqsubseteq \perp$ implies that for every DD-model $\mathcal{R}$ of $\mathcal{K}$, $\mathcal{R} \Vdash C \sqsubseteq \perp$, that is, $\mathcal{R} \Vdash C \sqsubseteq_{\mathcal{N}} \perp$.

For the only if-direction, $\mathcal{R} \Vdash C \sqsubseteq_{\mathcal{N}} \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}$. That is, by Lemma 8, that $\mathcal{R} \Vdash C \sqsubseteq_{\mathcal{N}} \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}^*$.

Consider just the normative part of $\mathcal{K}^*$, that is, $\mathcal{K}_{\mathcal{N}}^* = \langle \mathcal{T}^*, \mathcal{N}^*, \mathcal{P}^* \rangle$. Lemma 2 tell us that $Cn_{\mathcal{N}}(\mathcal{K})$ induces a defeasible subsumption relation $\sqsubseteq_{\mathcal{N}}^{\mathcal{K}_{\mathcal{N}}^*} := \{(C, D) \mid \mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} D\}$ that is rational and extends $\mathcal{K}^*$. Let $\sqsubseteq_{\mathcal{N}}^{\mathcal{K}} := \{(C, D) \mid \mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{N}} D\}$. Since $\text{N-DeonticClosure}^*(\mathcal{K}, C \sqsubseteq_{\mathcal{N}} D)$ equals $\text{N-DeonticClosure}(\mathcal{K}^*, C \sqsubseteq_{\mathcal{N}} D)$, and by Lemma 8 $\mathcal{K}$ and $\mathcal{K}^*$ are equivalent w.r.t. DD-interpretations, we have that $\sqsubseteq_{\mathcal{N}}^{\mathcal{K}} := \{(C, D) \mid \mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{N}} D\}$ is a rational relation extending $\mathcal{K}$. $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubseteq_{\mathcal{N}} \perp$ is the case only if C is always negated in the ranking procedure, i.e., $C \sqsubseteq_{\mathcal{N}} D \in \mathcal{N}_{\infty}$ or $C \sqsubseteq_{\mathcal{P}} \perp \in \mathcal{P}_{\infty}$ at some iteration of the procedure, that implies $\mathcal{T}^{\circ} \models C \sqsubseteq \perp$.

Lemma 10 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$. Then, for any C , $\mathcal{R} \Vdash C \sqsubseteq_{\mathcal{N}} \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^* \models C \sqsubseteq \perp$.*

Proof. It is straightforward given Lemma 9. Let $\mathcal{K}' = \langle \mathcal{T}, \mathcal{D}, \mathcal{N} \cup \{C \sqsubseteq_{\mathcal{N}} C\}, \mathcal{P} \rangle$. Since for any concept C the inclusion $C \sqsubseteq_{\mathcal{N}} C$ is satisfied by every DD-interpretation, $\mathcal{K}'$ is rank-equivalent to $\mathcal{K}$, and, since C is now among the antecedents of some obligation in the NBox, we can apply Lemma 9.

The following corollaries are immediate consequences of Lemma 10 and Proposition 4.

Corollary 4 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$. Then, for any concept C , $\mathcal{R} \Vdash C \sqsubseteq_{\mathcal{P}} \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^* \models C \sqsubseteq \perp$.*

Corollary 5 (*). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$. Then, for any concept C , $\mathcal{R} \Vdash C \sqsubseteq_E \perp$ for every DD-model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{T}^* \models C \sqsubseteq \perp$.*

Corollary 1 *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$. Then $\mathcal{K}$ is inconsistent iff $\mathcal{T}^* \models \top \sqsubseteq \perp$.*

In what follows we use the following functions:

- $rank_{\mathcal{K}}^{\mathcal{N}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C its minimal height in the ranked models of $\mathcal{K}$ w.r.t. $\prec_N^{\mathcal{R}}$ (that is, the height of C in $\mathcal{O}_{\mathcal{DN}}$ w.r.t. $\prec_N^{\mathcal{O}}$). If $\mathcal{R} \Vdash C \sqsubseteq \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ (that is, $\mathcal{O}_{\mathcal{DN}} \Vdash C \sqsubseteq \perp$), then $rank_{\mathcal{K}}^{\mathcal{N}} = \infty$.
- $rank_{\mathcal{K}}^{\mathcal{D}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C its minimal height in the ranked models of $\mathcal{K}$ w.r.t. $\prec_E^{\mathcal{R}}$ (that is, the height of C in $\mathcal{O}_{\mathcal{DN}}$ w.r.t. $\prec_E^{\mathcal{O}}$). If $\mathcal{R} \Vdash C \sqsubseteq \perp$ for every model $\mathcal{R}$ of $\mathcal{K}$ (that is, $\mathcal{O}_{\mathcal{DN}} \Vdash C \sqsubseteq \perp$), then $rank_{\mathcal{K}}^{\mathcal{D}} = \infty$.
- $rk_{\mathcal{K}}^{\mathcal{N}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C the rank i of the concept C w.r.t. $\mathcal{K}$ according to the while loop in Procedure N-DeonticClosure*. If $i = n + 1$ and $\mathcal{T}^* \models C \sqsubseteq \perp$, then $rk_{\mathcal{K}}^{\mathcal{N}} = \infty$.
- $rk_{\mathcal{K}}^{\mathcal{D}} : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ associates to each concept C the rank i of the concept C w.r.t. $\mathcal{K}$ according to the while loop in Procedure RationalClosure*. If $i = n + 1$ and $\mathcal{T}^* \models C \sqsubseteq \perp$, then $rk_{\mathcal{K}} = \infty$.

At this point, once we have proved that $\mathcal{K}$ and $\mathcal{K}^*$ are equivalent w.r.t. DD-interpretations (Lemma 8) and the TBox $\mathcal{T}^*$ contains all the strict information (Lemma 10 and Corollaries 4 and 5), the proofs regarding $\sqsubseteq_E$, $\sqsubseteq_{\mathcal{N}}$, and $\sqsubseteq_{\mathcal{P}}$ -inclusions are the same as for the simpler systems.

We start with the proofs for $\sqsubseteq_{\mathcal{N}}$, and $\sqsubseteq_{\mathcal{P}}$ -inclusions.

Lemma 11 (*). *For every $C \in \mathcal{L}$ and every $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$, $\mathcal{R} \Vdash \top \sqsubseteq_{\mathcal{N}} C$ for every model $\mathcal{R}$ of $\mathcal{K}$ iff $\mathcal{K} \vdash_{\mathcal{DN}} \top \sqsubseteq_{\mathcal{N}} C$.*

Proof. The proof mirrors the one for Lemma 5. Just substitute N-DeonticClosure with $\text{N-DeonticClosure}^*$.

Lemma 12 (*). For every $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and every $C \in \mathcal{L}$, $\text{rank}_{\mathcal{K}}^{\mathcal{N}}(C) = \infty$ iff $\text{rk}_{\mathcal{K}}^{\mathcal{N}}(C) = \infty$.

Proof.

It is a straightforward consequence of Lemma 10.

Lemma 13 (*). For every defeasible knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and every concept C , $\text{rank}_{\mathcal{K}}^{\mathcal{N}}(C) = \text{rk}_{\mathcal{K}}^{\mathcal{N}}(C)$.

Proof. Follow the proof of Lemma 7, just referring to Lemma 10, Corollary 4, Lemma 12 and Lemma 8.

Now we have all the ingredients to prove the completeness of the procedure, starting from the obligations.

Proposition 8 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{N}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{N}} D$.

Proof. Follow the proof of Proposition 6, just referring to Lemma 13.

Proposition 9 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{P}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{P}} D$.

Proof. Given Proposition 8, it is an immediate consequence of the correspondence between the semantic definition of the satisfaction of $C \sqsubseteq_{\mathcal{P}} D$ (Definition 10) and Procedure P-DeonticClosure*.

For $\sqsubseteq_E$ -inclusions, the proofs are analogous to the correspondent ones in [43].

Lemma 14 (*). For every $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and every $C \in \mathcal{L}$, $\text{rank}_{\mathcal{K}}^{\mathcal{D}}(C) = \infty$ iff $\text{rk}_{\mathcal{K}}^{\mathcal{D}}(C) = \infty$.

Proof. See Lemma 47 in [43].

Lemma 15 (*). For every defeasible knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and every concept C , $\text{rank}_{\mathcal{K}}^{\mathcal{D}}(C) = \text{rk}_{\mathcal{K}}^{\mathcal{D}}(C)$.

Proof. See Lemma 13 in [43].

Proposition 10 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_E D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_E D$.

Proof. See Theorem 5 in [43].

Now we miss only to prove that $\vdash_{\mathcal{DN}}$ is sound and complete w.r.t. the new $\sqsubseteq_{\mathcal{DN}}$ - and $\sqsubseteq_{\mathcal{DP}}$ - conditionals.

Proposition 11 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{DN}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{DN}} D$.

Proof. The structure of the proof is the same as the one for Proposition 6. It is just more tedious since we have to consider both the rankings, hence we need to refer to both Lemma 13 and Lemma 15.

Proposition 12 (*). Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and let $C, D \in \mathcal{L}$. Then $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubseteq_{\mathcal{DP}} D$ iff $\mathcal{K} \models_{\min} C \sqsubseteq_{\mathcal{DP}} D$.

Proof. Given Proposition 11, it is an immediate consequence of the correspondence between the semantic definition of the satisfaction of $C \sqsubseteq_{\mathcal{DP}} D$ and Procedure P-DD-Closure*.

Now we have all the ingredients to prove **Theorem 2**.

Proof of Theorem 2. Propositions 8, 9, 10, 11, 12 guarantee that $\vdash_{\mathcal{DN}}$ is correct and complete w.r.t. $\sqsubseteq_{\mathcal{N}}$ -, $\sqsubseteq_{\mathcal{P}}$ -, $\sqsubseteq_E$ -, $\sqsubseteq_{\mathcal{DN}}$ -, and $\sqsubseteq_{\mathcal{DP}}$ -inclusions. We have still to check that, for every classical GCI $C \sqsubseteq D$, $\mathcal{K} \models_{\min} C \sqsubseteq D$ iff $\mathcal{T}^\circ \models C \sqsubseteq D$, but that is an immediate consequence of Lemma 10, Proposition 4, and the equivalence between $\mathcal{T}^* \models C \sqsubseteq D$ and $\mathcal{T}^* \models C \sqcap \neg D \sqsubseteq \perp$.

D. Reasoning with the ABox

In this section we start investigating how to extend the entailment relation also to the ABox, that is, how to draw defeasible conclusions about individuals. Possible ways of dealing with the ABox reasoning when we consider only the presumptive reasoning (that is, only the DBox and the $\prec_E^{\mathcal{R}}$ -order) are investigated in [8, 55, 36]. Here the framework is more complex, since we are dealing with two preference orders ($\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$), and here we present a preliminary analysis.

First of all we are going to extend the concept language, in order to distinguish the different kinds of conclusions we can draw about individuals. Let $\mathcal{L}' = \mathcal{L} \cup \{C_{\mathcal{N}} \mid C \in \mathcal{L}\}$, where ' $a : C_{\mathcal{N}}$ ' is read '*presumably, a ought to be in C*'.

Next step is to define the relevant model to determine the entailment relation w.r.t. the ABox. We want to interpret the individuals in such a way that first we presume that they behave in the most expected way (w.r.t. $\prec_E^{\mathcal{R}}$), and then we determine their respective duties w.r.t. $\prec_N^{\mathcal{R}}$. Given a $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{D}, \mathcal{N} \rangle$, first we determine $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec_E^{\mathcal{O}}, \prec_N^{\mathcal{O}} \rangle$ w.r.t. $\langle \mathcal{T}, \mathcal{D}, \mathcal{N} \rangle$, and then we consider the possible ways of in which we can extend the interpretation function $\cdot^{\mathcal{O}}$ in order to interpret the individuals named in the $\mathcal{A}$. Let $N_{\mathcal{A}}$ be the set of individuals named in $\mathcal{A}$, and $\mathfrak{D}_{\mathcal{A}}$ be the set of all the models of $\mathcal{K}$ that extend $\mathcal{O}$ in order to satisfy also the ABox $\mathcal{A}$. In order to determine which models in $\mathfrak{D}_{\mathcal{A}}$ are relevant to determine the entailment relation w.r.t. the ABox, we need to define two preference relations, $\leq_C$ and $\leq_N$, over the elements of $\mathfrak{D}_{\mathcal{A}}$:

Definition 13. Let $\mathcal{R}, \mathcal{R}' \in \mathfrak{D}_{\mathcal{A}}$.

- $\mathcal{R} \leq_C \mathcal{R}'$ iff, for every $a \in N_{\mathcal{A}}$, $h^{\prec_E^{\mathcal{R}}}(a) \leq h^{\prec_E^{\mathcal{R}'}}(a)$.
- $\mathcal{R} \leq_N \mathcal{R}'$ iff, for every $a \in N_{\mathcal{A}}$, $h^{\prec_N^{\mathcal{R}}}(a) \leq h^{\prec_N^{\mathcal{R}'}}(a)$.

Given a set $\mathfrak{S} \subseteq \mathfrak{D}_{\mathcal{A}}$,

- $\min_{\leq_{\mathcal{X}}}(\mathfrak{S}) := \{\mathcal{R} \in \mathfrak{S} \mid \text{there is no } \mathcal{R}' \in \mathfrak{S} \text{ s.t. } \mathcal{R}' \leq_{\mathcal{X}} \mathcal{R}\},$

with $\mathcal{X} \in \{C, N\}$.

The intuition is that $\mathcal{R} \leq_C \mathcal{R}'$ iff $\mathcal{R}$ interprets the elements of $N_{\mathcal{A}}$ more in accordance with the expectations expressed in $\mathcal{D}$, while $\mathcal{R} \leq_N \mathcal{R}'$ iff $\mathcal{R}$ interprets the elements of $N_{\mathcal{A}}$ more in accordance with the preferences expressed in $\mathcal{N}$. Using $\leq_C$ and $\leq_N$ we can determine which models in $\mathfrak{D}_{\mathcal{A}}$ we want to refer to in order to determine the entailment relation for the ABox.

Definition 14 (ABox minimal DD-models). Let $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{D}, \mathcal{N} \rangle$ be an ontology, let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec_E^{\mathcal{O}}, \prec_N^{\mathcal{O}} \rangle$ the Big DD-model of $\langle \mathcal{T}, \mathcal{D}, \mathcal{N} \rangle$, and let $\mathfrak{D}_{\mathcal{A}}$ be the set of all the models of $\mathcal{K}$ obtained from $\mathcal{O}$ by extending the interpretation function $\cdot^{\mathcal{O}}$ to all the individual names in $N_{\mathcal{A}}$. The relations $\leq_C$ and $\leq_N$ over the elements of $\mathfrak{D}_{\mathcal{A}}$ are defined according to Definition 13.

$\mathfrak{D}_{\mathcal{A}}^{\min}$ is the set of the ABox minimal DD-models, and it is defined as

$$\mathfrak{D}_{\mathcal{A}}^{\min} := \min_{\leq_N}(\min_{\leq_C}(\mathfrak{D}_{\mathcal{A}})).$$

The rationale behind considering the models in $\mathfrak{D}_{\mathcal{A}}^{\min}$ is on the same line as the satisfaction of defeasible normative inclusion axioms: for any individual a , first of all we consider the interpretation that respects as much as possible our expectations (we minimise the interpretation of the a w.r.t. $\prec_E^{\mathcal{R}}$); then, among the possible models that interpret a in the most typical possible way, we consider those that maximise the satisfaction of the normative information (we consider the models that in which the interpretation of a is minimal w.r.t. $\prec_N^{\mathcal{R}}$).

Once we consider the models in $\mathfrak{D}_{\mathcal{A}}^{\min}$, we decide whether we can conclude $a : C_{\mathcal{N}}$ checking whether such models satisfy $a : C$.

Example 7 (Running example - cont.). As for Example ??, we cannot represent any element of $\mathfrak{D}_A^{\min}$, since they have an infinite domain. However, to have a feeling of what it means to use a minimal configuration for the ABox for entailment, also in this case we can refer to the model shown in Fig. ??: in the ABox $\mathcal{A}$ of the ontology $\mathcal{K}$ the named individuals are $a, b, c, br3, br5$ (see Example ??). In Fig. ?? they are interpreted on objects that are as lower as possible in both the orders: if we lower b to the lower rank w.r.t. $\prec_E^{\mathcal{R}}$ or we lower a to the lower rank w.r.t. $\prec_N^{\mathcal{R}}$ we would falsify the ontology $\mathcal{K}$. From this configuration, for example, we conclude the following: since a 's house has a fence, $a : \forall o.(\neg C \sqcup \neg \exists hasPart.F)$ does not hold and a is exceptional w.r.t. the order $\prec_N^{\mathcal{R}}$. It applies, however, the contrary-to-duty obligation to have a white fence, and in fact $a : \forall o.(\neg(C \sqcap \exists hasPart.F) \sqcup (C \sqcap \exists hasPart.(F \sqcap \exists color.W)))$ is satisfied in the minimal model, that is, we can conclude $a : (\forall o.(\neg(C \sqcap \exists hasPart.F) \sqcup (C \sqcap \exists hasPart.(F \sqcap \exists color.W))))_{\mathcal{N}}$: since she already has a fence, Ann should at least have a white fence around her cottage.

The main issue with the ABox reasoning is that we do not necessarily have a single minimal configuration. Once we consider the individuals in the ABox, it is possible that the same ontology allows for different, incompatible, minimal configurations of the individuals. That is, it could be that $|\mathfrak{D}_A^{\min}| > 1$, as the following example shows.

Example 8. Consider the following scenario: Ann (a) and Bob (b) are both employed in the same company (C), and they are office neighbours ($(a, b) : n$ and, imposing n is a symmetric role, $(b, a) : n$). The safety rules of the company state that everyone should have a fire extinguisher (E) in their own office ($(x, y) : o$ is read as ' y is place in x 's office'); but, in case your office neighbour already has one, then you should not have a fire extinguisher in your office. The ontology $\mathcal{K} = \langle \mathcal{A}, \mathcal{N} \rangle$ is as follows:

- $\mathcal{A} = \{a : C, b : C, (a, b) : n\}$;
- $\mathcal{N} = \{C \sqsubseteq_{\mathcal{N}} \exists o.E, C \sqcap (\exists n.\exists o.E) \sqsubseteq_{\mathcal{N}} \neg \exists o.E\}$.

TBox and DBox are empty. Consider what would happen if we interpret both a and b as minimal elements of $\Delta^{\mathcal{O}}$ w.r.t. $\prec_N^{\mathcal{R}}$: hence we have $\{a^{\mathcal{O}}, b^{\mathcal{O}}\} \subseteq \min_{\prec_N^{\mathcal{R}}}(C)$, and, by $C \sqsubseteq_{\mathcal{N}} \exists o.E$, we must conclude both $\mathcal{O} \models a : \exists o.E$ and $\mathcal{O} \models b : \exists o.E$. But in such a case we also have both $\mathcal{O} \models a : \exists n.\exists o.E$ and $\mathcal{O} \models b : \exists n.\exists o.E$, that is, $\{a^{\mathcal{O}}, b^{\mathcal{O}}\} \subset \min_{\prec_N^{\mathcal{R}}}(C \sqcap (\exists n.\exists o.E))$, and, by $C \sqcap (\exists n.\exists o.E) \sqsubseteq_{\mathcal{N}} \neg \exists o.E$, we must have both $\mathcal{O} \models a : \neg \exists o.E$ and $\mathcal{O} \models b : \neg \exists o.E$. This implies that we would have both $\mathcal{O} \models a : \perp$ and $\mathcal{O} \models b : \perp$, that is inconsistent. Hence it cannot be the case that both a and b are minimal w.r.t. $\prec_N^{\mathcal{O}}$. Actually we have two possible minimal configurations:

1. Ann has a fire extinguisher in her own office ($a : (\exists o.E)_{\mathcal{O}}$), and consequently Bob has an office neighbour with a fire extinguisher and he does not have one (we have $b : (\exists n.\exists o.E)_{\mathcal{O}}$ and $b : (\neg \exists o.E)_{\mathcal{O}}$). Semantically, in this minimal configuration we would have a interpreted a minimal element of $\Delta^{\mathcal{O}}$ w.r.t. $\prec_N^{\mathcal{R}}$, and b as a less preferred object w.r.t. $\prec_N^{\mathcal{R}}$ ($a \prec_N^{\mathcal{R}} b$).
2. The opposite configuration: Bob has a fire extinguisher in his own office ($b : (\exists o.E)_{\mathcal{O}}$), and consequently Ann does not have one ($a : (\exists n.\exists o.E)_{\mathcal{O}}$ and $a : (\neg \exists o.E)_{\mathcal{O}}$), that semantically corresponds to an interpretation assigning b to a minimal element of $\Delta^{\mathcal{O}}$ w.r.t. $\prec_N^{\mathcal{R}}$ and a as a less preferred object w.r.t. $\prec_N^{\mathcal{R}}$ ($b \prec_N^{\mathcal{R}} a$).

The chance of multiple minimal configurations opens the possibility to different forms of entailment, depending of the strategy we want to rely on. Two basic strategies are the following:

- *Skeptical*: we consider the intersection of what holds in each extension. let $\models_{\min}^{sk}$ indicate the entailment relation we associate with this approach. Then we have

$$\mathcal{K} \models_{\min}^{sk} a : C_{\mathcal{N}} \text{ iff } \mathcal{R} \models a : C \text{ for every } \mathcal{R} \in \mathfrak{D}_A^{\min}.$$

In the situation from Example 8 we would not be able to conclude about neither a or b that they should have a fire extinguisher in the office. This is the option that, from the point of view of reasoning only about expectations, has been investigated in [8, 55].

- *Choice*: We just choose one of the possible preferred configurations, and we define an entailment relation $\models_{\min}^{ch}$ only on that one. That is, we choose only one interpretation $\mathcal{R} \in \mathfrak{D}_{\mathcal{A}}^{\min}$, and we have

$$\mathcal{K} \models_{\min}^{ch} a : C_{\mathcal{N}} \text{ iff } \mathcal{R} \models a : C \text{ for every } \mathcal{R} \in \mathfrak{D}_{\mathcal{A}}^{\min}.$$

So, in Example 8, we choose either option (i) or option (ii).

Clearly the second option is inferentially more powerful, that is, $\models_{\min}^{ch} \supseteq \models_{\min}^{sk}$, and it also satisfies interesting formal properties (that we cannot discuss here), but $\models_{\min}^{ch}$ relies on the choice of a specific member of $\mathfrak{D}_{\mathcal{A}}^{\min}$ over the others, choice that should be justified.

Also in the case of the ABox we have implementable procedures, on the line of those presented in [8, 55, 36], that we plan to present in a forthcoming paper.