

ON THE NUMBER OF FIXED POINTS OF AUTOMORPHISMS OF VERTEX-TRANSITIVE GRAPHS

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ABSTRACT. The main result of this paper is that, if Γ is a finite connected 4-valent vertex- and edge-transitive graph, then either Γ is part of a well-understood family of graphs, or every non-identity automorphism of Γ fixes at most $1/3$ of the vertices. As a corollary, we get a similar result for 3-valent vertex-transitive graphs.

1. INTRODUCTION

The aim of this paper is to study a graph-theoretical parameter called *fixicity*, defined as the maximal number of vertices that are fixed by a non-trivial automorphism of the graph. Investigation of a group theoretical analogue of this parameter (the maximum number of points fixed by a permutation group) has a long history going back to a classical work of Jordan studying primitive permutation group containing a non-trivial permutation fixing all but a prescribed number of points. His results were later improved significantly by several authors: for example, Babai [1], Liebeck and Saxl [22], Guralnick and Magaard [17], Burnes [6, 7], Liebeck and Shalev [22], to name a few. As a result, all primitive groups G having a non-trivial permutation fixing more than half of the points are known.

To the best of our knowledge, the fixicity of a graph was first studied by Babai [2, 3] and was motivated by the famous graph isomorphism problem [4]. In these papers, Babai shows how fixicity is related to a number of important notions, such as the spectrum of the graph, the order of individual automorphisms and the automorphism group of the graph. While the focus there are strongly regular graphs (that can be thought of as graphs are highly symmetrical through from a purely combinatorial point of view) this paper is devoted to the fixicity of graphs exhibiting a high level of symmetry as measured through their automorphism groups. In particular, we will be interested in connected graphs of valence at most 4 admitting a group of automorphisms G acting transitively on the vertices (G -vertex-transitive graphs), edges (G -edge-transitive graphs) and/or ordered pairs of adjacent vertices (G -arc-transitive graphs).

Our understanding of vertex-transitive graphs is a function of time. The fact that this function has increased so much recently (especially for graphs of valency 3 and 4) is, in our opinion, due to two processes intimately intertwined. On the one hand, theoretical results allow us to get deeper into the structure (both combinatorial and algebraic) of vertex-transitive graphs. These results can often be used to improve our database of vertex-transitive graphs, see [10, 11, 27, 28, 29]. On the other hand, these databases can be used to test open problems or to formulate conjectures; see for example [9, 31, 37]. The spin off of this process is more theoretical work. And the loop starts again, if one can really say that there is a “start” and an “end” in this process.

The pattern described in this paper starts with some computer evidence, found by Gabriel Verret and the first-named author of this paper. By checking the census of connected 3-valent vertex-transitive graphs [27, 28] (which was obtained from the theoretical work in [26]), they observed that (for graphs small enough to be in this list) non-identity automorphisms of a connected 3-valent vertex-transitive graph Γ cannot fix more than $1/3$ of the vertices of Γ , unless Γ is in a very special family or very small. A similar pattern holds for the family of connected 4-valent vertex- and edge-transitive graphs.

Our main results are the following. For not breaking the flow of the argument, we refer the reader to Sections 1.2 and 1.3 for undefined terminology, including the definition of the Praeger-Xu graphs $C(r, s)$ and the Split Praeger-Xu graphs $S(C(r, s))$.

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Theorem 1.1. *Let Γ be a finite connected edge- and vertex-transitive 4-valent graph admitting a non-identity automorphism fixing more than $1/3$ of the vertices. Then Γ is arc-transitive and one of the following holds:*

- (i): $|\text{V}\Gamma| \leq 70$ and Γ is one of the six exceptions Ψ_1, \dots, Ψ_6 , defined in Section 1.2;
- (ii): Γ is isomorphic to a Praeger-Xu graph $C(r, s)$ with $1 \leq s < 2r/3$ and $r \geq 3$.

Theorem 1.2. *Let Γ be a finite connected 3-valent vertex-transitive graph admitting a non-identity automorphism fixing more than $1/3$ of the vertices. Then one of the following holds:*

- (i): $|\text{V}\Gamma| \leq 20$ and Γ is one of the six exceptions $\Lambda_1, \dots, \Lambda_6$, defined in Section 1.2;
- (ii): Γ is isomorphic to a Split Praeger-Xu graph $S(C(r, s))$ with $1 \leq s < 2r/3$ and $r \geq 3$.

Observe that every primitive permutation group $G \leq \text{Sym}(\Omega)$ acts as an edge- and vertex-transitive group of automorphisms on each of its non-trivial orbital graphs on Ω . The results of this paper can thus be considered as an attempt to generalise the theorems about the maximal number of fixed points of a non-identity element in a primitive permutation group; see for example [17, 20, 22]. One of the key ingredients in our proof of Theorems 1.1 is the following recent result [25]. Since its proof depends heavily on the classification of finite simple groups, so do the proofs of Theorems 1.1 and 1.2.

Theorem 1.3. [25, Theorem 1.1] *Let G be a transitive permutation group on Ω containing no non-trivial normal subgroups of order a power of 2 (that is, $\mathbf{O}_2(G) = 1$) and let $\omega \in \Omega$ with G_ω being a 2-group. Then $|\{\delta \in \Omega \mid \delta^g = \delta\}| \leq |\Omega|/3$, for every $g \in G \setminus \{1\}$.*

The bound $1/3$ in the Theorems 1.1 and 1.2 is sharp in the sense that there exists an infinite family (in each case) meeting this bound. To see this, consider the graph DW_m with vertex-set $\mathbb{Z}_m \times \mathbb{Z}_3$ and the edge-set $\{(x, i), (x+1, j) \mid x \in \mathbb{Z}_m, i, j \in \mathbb{Z}_3, i \neq j\}$. The graph DW_m is clearly connected, 4-valent and arc-transitive. Moreover, it admits an automorphism which fixes every vertex of the form $(x, 0)$ while swapping the vertices in each pair $\{(x, 1), (x, 2)\}$, $x \in \mathbb{Z}_m$. In a similar way as 4-valent arc-transitive Praeger-Xu graphs yield 3-valent vertex-transitive Split Praeger-Xu graphs (see Section 1.4), one can apply the splitting “operation” to obtain a family of 3-valent vertex-transitive graphs $S(DW_m)$ of fixity exactly $1/3$ of the number of vertices. This suggests the following problem:

Problem 1.4. Determine the connected 4-valent arc-transitive graphs and the connected 3-valent vertex-transitive graphs admitting an automorphism fixing precisely $1/3$ of the vertices.

Theorems 1.1 and 1.2 seem to be suggesting that the proportion of fixed points of a non-identity automorphism of a connected vertex-transitive graph is bounded by a “small” constant unless the graph is either small or rather “special”. Since at this point it is not clear to us what the class of “special” graphs of larger valencies might be and what would be a meaningful “small constant”, we include the definition of “special graphs” and “small constant” as a part of the following problem:

Problem 1.5. For a given positive integer d find a “small constant” c_d and a “well-understood” family of “special graphs” \mathcal{F}_d such that every finite connected d -valent vertex-transitive graph Γ admitting a non-trivial automorphism fixing more than $c_d|\text{V}\Gamma|$ vertices belongs to \mathcal{F}_d .

Finally, we would like to propose the following “edge-fixing” variation of Theorems 1.1 and 1.2:

Problem 1.6. Determine the connected 4-valent arc-transitive graphs and the connected 3-valent vertex-transitive graphs admitting an automorphism fixing more than $1/3$ of the *edges*.

1.1. Basic terminology and notation. A graph in this paper will be viewed as a pair (V, E) where V is a finite non-empty set of *vertices* and E is a set of unordered pairs of V , called *edges*. If $\Gamma := (V, E)$ is a graph, then we let $\text{V}\Gamma := V$ and $\text{E}\Gamma := E$. An s -*arc* of a graph is an $(s+1)$ -tuple of vertices with every two consecutive vertices adjacent and every three consecutive vertices pair-wise distinct. In particular, a 1-arc is also called an *arc*. The set of arcs of a graph Γ is denoted $\text{A}\Gamma$.

We will also need a notion of a *digraph*, which we define to be a pair (V, A) , where V is a finite non-empty set of *vertices* and A is a set of ordered pairs of distinct vertices. Elements of A are called *arcs* of the digraph. An s -*arc* of a digraph is an $(s+1)$ -tuple of vertices such that every two consecutive vertices form an arc. If (u, v) is an arc of a digraph, then we say that v is an *out-neighbour* of u and that u is an *in-neighbour* of v . The *out-valency* (*in-valency*, respectively) of a given vertex is then the number of its in-neighbours (out-neighbours, respectively). If $\vec{\Gamma} := (V, A)$ is a digraph, then the underlying graph of $\vec{\Gamma}$ is the graph

(V, E) with $E := \{\{u, v\} : (u, v) \in A\}$. Note that when $\vec{\Gamma}$ is an *orientation* (that is, when $(u, v) \in A\vec{\Gamma}$ implies $(v, u) \notin A\vec{\Gamma}$), then there is a bijective correspondence between the arcs of $\vec{\Gamma}$ and edges of the underlying graph.

Let Γ be a graph (or a digraph), let $G \leq \text{Aut}(\Gamma)$ and let $v \in V\Gamma$. We denote by G_v the *stabiliser* of the vertex v , by $\Gamma(v) = \{u \in V\Gamma : (v, u) \in A\Gamma\}$ the *neighbourhood* of the vertex v and by $G_v^{\Gamma(v)}$ the permutation group induced by G_v on $\Gamma(v)$. Suppose now that Γ is a G -arc-transitive connected graph. As usual, when $G = \text{Aut}(\Gamma)$, we omit the label G and we simply say that Γ is s -arc-transitive. Observe that a G -arc-transitive graph Γ is $(G, 2)$ -arc-transitive if and only if $G_v^{\Gamma(v)}$ is a 2-transitive permutation group.

An edge- and vertex-transitive group of automorphisms G of a connected graph Γ that is not arc-transitive is called $\frac{1}{2}$ -arc-transitive. Note that in this case G possesses two orbits on arcs, each orbit containing precisely one arc underlying each edge. If A is an orbit of G on the arc-set of Γ , then $(V\Gamma, A)$ is an arc-transitive digraph, denoted $\vec{\Gamma}^{(G)}$, whose underlying graph is Γ . In particular, if Γ has valency 4, then the in-valence and out-valence of every vertex of $\vec{\Gamma}^{(G)}$ is 2.

Given a set Ω , we denote by $\text{Sym}(\Omega)$ and $\text{Alt}(\Omega)$ the symmetric and the alternating group on Ω . When the domain Ω is irrelevant or clear from the context, we write $\text{Sym}(n)$ and $\text{Alt}(n)$ for the symmetric and alternating group of degree n . Given a permutation $g \in \text{Sym}(\Omega)$, we write $\text{Fix}_\Omega(g)$ for the set of *fixed points* $\{\omega \in \Omega \mid \omega^g = \omega\}$ of g and we write $\text{fpr}_\Omega(g)$ for the *fixed-point-ratio* of g , that is

$$\text{fpr}_\Omega(g) := \frac{|\text{Fix}_\Omega(g)|}{|\Omega|}.$$

Given $n \in \mathbb{N} \setminus \{0\}$, we denote by D_n the dihedral group of order $2n$ and we view D_n as a permutation group of degree n ; similarly, we denote by C_n the cyclic group of order n . Similarly, we denote by \mathbb{Z}_n the integers modulo n .

A subgroup G of $\text{Sym}(\Omega)$ is said to be *semiregular* if the identity is the only element of G fixing some point of Ω . Let G be a group and let H be a subgroup of G , we denote by $H \backslash G$ the *set of right cosets* of H in G . Recall that G acts transitively on $H \backslash G$ by right multiplication. If G is a group and $a, b \in G$, we let $[a, b] = a^{-1}b^{-1}ab$ be the commutator of a and b , and $C_G(a) = \{c \in G : ca = ac\}$ be the centraliser of a in G .

1.2. The twelve sporadic graphs from Theorems 1.1 and 1.2. We start by describing the six sporadic examples from Theorem 1.1.

- Ψ_1 The first graph is the complete graph K_5 . The automorphism group of K_5 is $\text{Sym}(5)$. A permutation of $\text{Sym}(5)$ fixing two or three points gives rise to a non-identity automorphism fixing more than a $1/3$ of the vertices.
- Ψ_2 The second graph is the complete bipartite graph minus a complete matching $K_{5,5} - 5K_2$. The automorphism group of this graph is isomorphic to $\text{Sym}(5) \times C_2$. A permutation of $\text{Sym}(5)$ fixing two or three points gives rise to a non-identity automorphism fixing four or six vertices and hence fixing more than a $1/3$ of the vertices. Moreover, $\text{Aut}(\Psi_2)$ contains a vertex-transitive copy of $\text{Sym}(5)$ which fixes four vertices of Ψ_2 .
- Ψ_3 The third graph arises from the Fano plane. This graph is bipartite with bipartition given by the seven points and the seven lines of the Fano plane, where the incidence in the graph is given by the anti-flags in the plane, that is, the point p is adjacent to the line ℓ if and only if $p \notin \ell$. In other words, Ψ_3 is the bipartite complement of the Heawood graph. The automorphism group of this graph is isomorphic to $\text{Aut}(\text{PSL}_3(2)) \cong \text{PGL}_2(7)$. An involution of $\text{PSL}_3(2)$ gives rise to a non-identity automorphism fixing six vertices and hence fixing more than a $1/3$ of the vertices of the graph.
- Ψ_4 The fourth graph is similar to Ψ_3 and arises from the projective plane over the finite field with three elements. This graph is bipartite with bipartition given by the thirteen points and the thirteen lines of the projective plane, where the incidence in the graph is given by the flags in the plane, that is, the point p is adjacent to the line ℓ if and only if $p \in \ell$. The automorphism group of Ψ_4 is isomorphic to $\text{Aut}(\text{PGL}_3(3))$. An involution of $\text{PGL}_3(3)$ gives rise to a non-identity automorphism fixing ten vertices and hence fixing more than $1/3$ of the vertices.
- Ψ_5 The fifth graph is a Kneser graph. This graph has 35 vertices and these are labeled by the 35 subsets of $\{1, \dots, 7\}$ having cardinality 3. Two 3-subsets a and b are declared to be adjacent if and only if $a \cap b = \emptyset$. The automorphism group of this graph is isomorphic to $\text{Sym}(7)$. A transposition of

$\text{Sym}(7)$ gives rise to a non-identity automorphism fixing fifteen vertices and hence fixing more than $1/3$ of the vertices of the graph.

- Ψ_6 The sixth (and last) graph is the standard double cover of Ψ_5 . This graph has 70 vertices and these are labeled by the ordered pairs (v, i) , where v is a vertex of Ψ_5 and $i \in \{0, 1\}$. The vertices $(v, 0)$ and $(w, 1)$ are declared to be adjacent if and only if v and w are adjacent in Ψ_5 . The automorphism group of this graph is isomorphic to $\text{Sym}(7) \times C_2$. A transposition of $\text{Sym}(7)$ gives rise to a non-identity automorphism fixing thirty vertices and hence fixing more than $1/3$ of the vertices of the graph. Similarly as Ψ_2 , $\text{Aut}(\Psi_6)$ also contains a vertex-transitive copy of the symmetric group $\text{Sym}(7)$, however the maximum fixed point-ratio of a non-trivial element in this group is $1/5$.

We now describe the six sporadic examples from Theorem 1.2.

- Λ_1 The first graph is the complete graph K_4 . The automorphism group of this graph is $\text{Sym}(4)$. A transposition of $\text{Sym}(4)$ gives rise to a non-identity automorphism fixing more than $1/3$ of the vertices of the graph.
- Λ_2 The second graph is the complete bipartite graph $K_{3,3}$. The automorphism group of this graph is isomorphic to $\text{Sym}(3) \text{ wr } \text{Sym}(2)$. A transposition from the base group $\text{Sym}(3) \times \text{Sym}(3)$ gives rise to a non-identity automorphism fixing four vertices and hence fixing more than $1/3$ of the vertices.
- Λ_3 The third graph is the 1-skeleton of the cube. This graph is the Hamming graph over the 3-dimensional vector space \mathbb{F}_2^3 over the field \mathbb{F}_2 with two elements. Two vertices (x_1, x_2, x_3) and (y_1, y_2, y_3) are declared to be adjacent if and only if the vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) differ in one, and only one, coordinate. The automorphism group of this graph is isomorphic to $\text{Sym}(2) \text{ wr } \text{Sym}(3) \cong \text{Sym}(4) \times \text{Sym}(2)$. A transposition from $\text{Sym}(4)$ gives rise to a non-identity automorphism fixing four vertices and hence fixing more than $1/3$ of the vertices.
- Λ_4 The fourth graph is the ubiquitous Petersen graph and it is a Kneser graph where the 10 vertices are the subsets of $\{1, \dots, 5\}$ having cardinality 2. Two 2-subsets a and b are declared to be adjacent if and only if $a \cap b = \emptyset$. The automorphism group of this graph is isomorphic to $\text{Sym}(5)$. A transposition from $\text{Sym}(5)$ gives rise to a non-identity automorphism fixing four vertices and hence fixing more than $1/3$ of the vertices of the graph.
- Λ_5 The fifth graph arises from the Fano plane and it is the bipartite complement of Ψ_3 , that is, Λ_5 is the Heawood graph. The automorphism group of this graph is isomorphic to $\text{Aut}(\text{GL}_3(2)) \cong \text{PGL}_2(7)$. An involution of $\text{GL}_3(2)$ gives rise to a non-identity automorphism fixing six vertices and hence fixing more than $1/3$ of the vertices of the graph.
- Λ_6 The sixth (and last) graph is the standard double cover of the Petersen graph. This graph has 20 vertices and these are labeled by the ordered pairs (v, i) , where v is a vertex of the Petersen graph and $i \in \{0, 1\}$. The vertices $(v, 0)$ and $(w, 1)$ are declared to be adjacent if and only if v and w are adjacent in the Petersen graph. The automorphism group of this graph is isomorphic to $\text{Sym}(5) \times C_2$. A transposition of $\text{Sym}(5)$ gives rise to a non-identity automorphism fixing eight vertices and hence fixing more than $1/3$ of the vertices of the graph.

1.3. The Praeger-Xu graphs. We now define the infinite family appearing in Theorem 1.2 (ii). These are the ubiquitous 4-valent *Praeger-Xu graphs* $C(r, s)$, studied in detail by Gardiner, Praeger and Xu in [15, 35], and more recently in [18]. We introduce them through their directed counterparts defined in [34].

Let r be an integer, $r \geq 3$. Then $\vec{C}(r, 1)$ is the the lexicographic product of a directed cycle of length r with an edgeless graph on 2 vertices. In other words, $V\vec{C}(r, 1) = \mathbb{Z}_r \times \mathbb{Z}_2$ with the out-neighbours of a vertex (x, i) being $(x + 1, 0)$ and $(x + 1, 1)$. For $s \geq 2$, let $V\vec{C}(r, s)$ be the set of all $(s - 1)$ -arcs of $\vec{C}(r, 1)$ with the out-neighbours of $(v_0, v_1, \dots, v_{s-1}) \in V\vec{C}(r, s)$ being (v_1, \dots, v_{s-1}, u) and $(v_1, \dots, v_{s-1}, u')$, where u and u' are the two out-neighbours of u in $\vec{C}(r, 1)$. The graph $C(r, s)$ is then defined as the underlying graph of $\vec{C}(r, s)$.

Clearly, $C(r, s)$ is a connected 4-valent graph with $r2^s$ vertices (see [34, Theorem 2.8]). Let us now discuss the automorphisms of the graphs $C(r, s)$. Clearly, every automorphism of $\vec{C}(r, 1)$ ($C(r, 1)$, respectively) acts naturally as an automorphism of $\vec{C}(r, s)$ ($C(r, s)$, respectively) for every $s \geq 2$. For $i \in \mathbb{Z}_r$, let τ_i be the transposition on $V\vec{C}(r, 1)$ swapping the vertices $(i, 0)$ and $(i, 1)$ while fixing every other vertex. This is clearly an automorphism of $\vec{C}(r, 1)$, and thus also of $\vec{C}(r, s)$ for $s \geq 2$. Let

$$(1.1) \quad K := \langle \tau_i \mid i \in \mathbb{Z}_r \rangle$$

and observe that $K \cong C_2^r$. Further, let ρ and σ be the permutations on $V\vec{C}(r, 1)$ defined by

$$(x, i)^\rho := (x + 1, i) \quad \text{and} \quad (x, i)^\sigma := (x, -i).$$

Then ρ is an automorphism of $\vec{C}(r, 1)$, and σ is an automorphism of $C(r, 1)$ (but not of $\vec{C}(r, 1)$). Observe that the group $\langle \rho, \sigma \rangle$ normalises K . Let

$$(1.2) \quad H := K\langle \rho, \sigma \rangle \quad \text{and} \quad H^+ := K\langle \rho \rangle.$$

Then clearly $C_2 \text{ wr } D_r \cong H \leq \text{Aut}(C(r, s))$ and $C_2 \text{ wr } C_r \cong H^+ \leq \text{Aut}(\vec{C}(r, s))$ for every $r \geq 3$ and $s \geq 1$. Moreover, H (H^+ , respectively) acts arc-transitively on $C(r, s)$ ($\vec{C}(r, s)$, respectively) whenever $1 \leq s \leq r - 1$. With three exceptions, the groups H and H^+ are in fact the full automorphism groups of $C(r, s)$ and $\vec{C}(r, s)$, respectively:

Lemma 1.7. ([35, Theorem 2.13] and [34, Theorem 2.8]) *Let r, s, H and H^+ be as above. Then $\text{Aut}(\vec{C}(r, s)) = H^+$ and if $r \neq 4$, then $\text{Aut}(C(r, s)) = H$. Moreover, $|\text{Aut}(C(4, 1)) : H| = 9$, $|\text{Aut}(C(4, 2)) : H| = 3$ and $|\text{Aut}(C(4, 3)) : H| = 2$.*

Remark 1.8. Lemma 1.7 implies that $C(r, s)$ is 2-arc-transitive if and only if $r = 4$ and $s \in \{1, 2\}$.

Let v be a vertex of $\vec{C}(r, s)$ which as an $(s - 1)$ -arc of $\vec{C}(r, 1)$ starts in $(x, 0)$ or $(x, 1)$ for some $x \in \mathbb{Z}_r$. Observe that then $\text{Aut}(\vec{C}(r, s))_v = \langle \tau_i \mid i \in \mathbb{Z}_r \setminus \{x, x + 1, \dots, x + s - 1\} \rangle$, showing that

$$(1.3) \quad K = \langle \text{Aut}(\vec{C}(r, s))_v \mid v \in V\vec{C}(r, s) \rangle = \langle (H^+)_v \mid v \in V\vec{C}(r, s) \rangle = \langle K_v \mid v \in V\vec{C}(r, s) \rangle.$$

The following result explains the restriction on r and s in Theorem 1.1 (ii) and characterises the automorphisms of $C(r, s)$ that fix more than $1/3$ of the vertices.

Lemma 1.9. *The graph $C(r, s)$ with $r \geq 3$ and $1 \leq s \leq r - 1$ contains a non-identity automorphism fixing more than $1/3$ of the vertices if and only if $s < 2r/3$. In this case all such automorphisms belong to the group K defined by (1.1).*

Proof. Let $\Gamma = C(r, s)$. For $r \leq 6$ the claim of the lemma can be verified by inspecting the 14 graphs $C(r, s)$, $3 \leq r \leq 6$, $1 \leq s \leq r - 1$, with a computer algebra system such as Magma [5]. We may thus assume that $r \geq 7$. By Lemma 1.7, we see that $\text{Aut}(\Gamma) = H$.

Let g be an arbitrary automorphism of $\text{Aut}(\Gamma)$ fixing more than $1/3$ of the vertices. Then $g = \tau\rho^i\sigma^\epsilon$ for some $\tau \in K$, $i \in \mathbb{Z}_r$ and $\epsilon \in \{0, 1\}$. For $x \in \mathbb{Z}_r$, let Δ_x be the set of $(s - 1)$ -arcs of $\vec{C}(r, 1)$ that start at a vertex $(x, 0)$ or $(x, 1)$, and consider the elements of Δ_x as vertices of Γ . Observe that each Δ_x is an orbit of the action of K on $V\Gamma$, that $(\Delta_x)^\rho = \Delta_{x+1}$ and that $(\Delta_x)^\sigma = \Delta_{-x-s+1}$. Consequently, $(\Delta_x)^g$ is either Δ_{x+i} if $\epsilon = 0$ or $\Delta_{-x-i-s+1}$ if $\epsilon = 1$. In particular, unless $i = 0$ and $\epsilon = 0$, g preserves at most 2 out of r orbits Δ_x , $x \in \mathbb{Z}_r$. Since $r \geq 7$ and since g fixes more than $1/3$ vertices of Γ , this implies that $i = 0$, $\epsilon = 0$, and thus that $g \in K$.

Finally, note that τ_i moves precisely those $(s - 1)$ -arcs of $\vec{C}(r, 1)$ that pass through one of the vertices $(i, 0)$ or $(i, 1)$. Therefore, τ_i , as an automorphism of $C(r, s)$, fixes all but $s2^s$ vertices. Similarly, an element $\prod_{i \in J} \tau_i \in K$ moves precisely those $(s - 1)$ -arcs of $\vec{C}(r, 1)$ that pass through at least one of the vertices $\{(i, \epsilon) : \epsilon \in \{0, 1\}, i \in J\}$, implying that such an element fixes at most as many elements as a single τ_i . Hence

$$\frac{1}{3} < \text{fpr}_{V\Gamma}(g) \leq \text{fpr}_{V\Gamma}(\tau_i) = \frac{(r - s)2^s}{r2^s} = \frac{r - s}{r}$$

and the result follows. \square

The Praeger-Xu graphs can be characterised by an existence of an abelian normal subgroup not acting semiregularly on the vertices. The following result appeared as Theorem 1 in [35] for the case of G acting arc-transitively, and as Theorem 2.9 in [34] for the $\frac{1}{2}$ -arc-transitive case.

Lemma 1.10. (see [35, Theorem 1] and [34, Theorem 2.9]) *Let Γ be a connected 4-valent graph and let G be an edge- and vertex-transitive group of automorphisms of Γ . If G has an abelian normal subgroup which is not semiregular on the vertices of Γ , then $\Gamma \cong C(r, s)$ with $r \geq 3$ and $1 \leq s \leq r - 1$.*

The following lemma is a generalisation of [15, Lemma 3.1] from the case of G being arc-transitive group to the case of an arbitrary edge- and vertex-transitive group G . The proof closely follows that of [15, Lemma 3.1].

Lemma 1.11. *Let Γ be a connected 4-valent graph, let G be an edge- and vertex-transitive group of automorphisms of Γ and let N be a minimal normal subgroup of G . If N is a 2-group and Γ/N is a cycle, then $\Gamma \cong C(r, s)$ for some r and s .*

Proof. If N does not act semiregularly on $V\Gamma$, then the result follows by Lemma 1.10. We may thus assume that N is semiregular. If G is arc-transitive, then result follows directly from [15, Lemma 3.1]. We may thus assume that G is $\frac{1}{2}$ -arc-transitive.

Let K be the kernel of the action of G on the vertex-set $\{u^N : u \in V\Gamma\}$ of Γ/N and let $C = C_K(N)$ be the centraliser of N in K . Note that both K and C are normal in G . Since N is abelian, we have $N \leq C$. The stabiliser C_u clearly fixes every vertex of u^N , showing that the group C^{u^N} induced by the action of C on u^N is regular, implying that $C^{u^N} = N^{u^N}$ and so C^{u^N} is abelian. But every permutation group is isomorphic to a subgroup of the direct product of the permutation groups it induces on its orbits. In particular, C is isomorphic to a subgroup $C^{u_1^N} \times \cdots \times C^{u_m^N}$ where u_1^C, \dots, u_m^C are the C -orbits on $V\Gamma$, implying that C is abelian. If $C_u \neq 1$ for some $u \in V\Gamma$, then by Lemma 1.10, $\Gamma \cong C(r, s)$. We may thus assume that $C_u = 1$, and therefore that $C = N$.

Let $v \in V\Gamma$ and let u, w be the out-neighbours of v in the digraph $\vec{\Gamma}^{(G)}$. Since G acts arc-transitively on $\vec{\Gamma}^{(G)}$, there is an element $h \in G_v$ swapping the vertices u and w . Moreover, such an element clearly preserves each N orbit, implying that $g \in K$. In particular, $K_v \neq 1$, and so N is a proper subgroup of K . On the other hand, since $v^N = v^K$, we have $K = NK_v$, implying that K is a 2-group. The action of K on the set $N \setminus \{1\}$ of odd cardinality by conjugation thus has at least one fixed point, say $x \in N$. But then x is centralised by every element in K and thus $x \in \mathbf{Z}(K)$. In particular $\mathbf{Z}(K) \cap N$ is a non-trivial normal subgroup of G contained in G . By minimality of N , it follows that $N \leq \mathbf{Z}(K)$. But then $C = K$, contradicting the fact that $C_v = 1$. This contradiction shows that $\Gamma \cong C(r, s)$ as claimed. \square

1.4. Split Praeger-Xu graphs. We now define the family of the *Split Praeger-Xu graphs* $S(C(r, s))$, featuring in Theorem 1.2 (ii). This family is obtained from the Praeger-Xu graphs via the *splitting operation* $S(-)$, which was introduced in [27, Construction 9]. Rather than defining $S(-)$ in its full generality here, we only describe it in the special case of the Praeger-Xu graphs $C(r, s)$. We refer the reader to [27, Section 4] for more information on this operator.

Split each vertex v of $\vec{\Gamma} := \vec{C}(r, w)$ into two copies, denoted v_+ and v_- , and let v_- be adjacent to v_+ and to u_+ whenever u is an in-neighbour of v in $\vec{\Gamma}$. Similarly, let v_+ be adjacent to v_- and to w_- whenever w is an out-neighbour of v in Γ . The resulting 3-valent graph is then called the *Split Praeger-Xu graph* and denoted $S(C(r, s))$. Observe that the automorphism group $\text{Aut}(\vec{C}(r, s))$ ($= H$) acts faithfully as a vertex- but not arc-transitive group of automorphisms of $S(C(r, s))$ and that for every $g \in H$ we have $\text{fpr}_{V S(C(r, s))}(g) = \text{fpr}_{V C(r, s)}(g)$.

1.5. Normal quotients. The proofs of the main theorems are inductive with the induction step using the notion of a normal quotient of a graph introduced in [36, Section 4].

Definition 1.12. Let Γ be a connected graph (or digraph) and let $N \leq \text{Aut}(\Gamma)$. The *normal quotient* Γ/N is the graph (or digraph) whose vertices are the N -orbits on $V\Gamma$ with two distinct such N -orbits v^N and u^N forming an arc (v^N, u^N) of Γ/N whenever there is a pair of vertices $v' \in v^N$ and $u' \in u^N$ such that (v', u') is an arc of Γ .

If the group N is normalised by some group $G \leq \text{Aut}(\Gamma)$, then G/N acts (possibly unfaithfully) on Γ/N as a group of automorphisms. If G is vertex-, edge- or arc-transitive on Γ , then so is G/N on Γ/N . Suppose now that Γ is a 4-valent graph and that G is arc-transitive. Then the valency of Γ/N is either 0 (when N is transitive on $V\Gamma$), 1 (when N has 2 orbits on $V\Gamma$), 2 (when Γ/N is a cycle) or 4. In the latter case, G/N acts faithfully on $V\Gamma$ and hence Γ/N is a connected 4-valent G/N -arc-transitive graph with vertex-stabiliser $(G/N)_{v^N} = G_v N/N$ in G/N isomorphic to G_v . Moreover, if Γ/N is 4-valent, $N_v = 1$ for every vertex $v \in V\Gamma$.

The following lemma follows almost immediately from the theory of lifting automorphisms along covering projections, as developed in [23]. However, in order to avoid leading the reader astray with introducing these methods, we decided to provide a straightforward, though longer proof.

Lemma 1.13. *Let Γ be a connected 4-valent G -arc-transitive graph and let N be a semiregular normal subgroup of G such that the normal quotient Γ/N is a cycle of length $r \geq 3$. Let K be the kernel of the action of G on the N -orbits on $V\Gamma$. Then K_v is an elementary abelian 2-group.*

Proof. Let $\Delta_0, \Delta_1, \dots, \Delta_{r-1}$ be the orbits of N in its action on $V\Gamma$. Since Γ/N is a cycle, we may assume that Δ_i is adjacent to Δ_{i-1} and Δ_{i+1} with indices computed modulo r . Since N is normal in G , the orbits of N on the edge-set $E\Gamma$ form a G -invariant partition of $E\Gamma$. Since N acts semiregularly on $V\Gamma$, no two edges incident to a fixed vertex of Γ belong to the same N -edge-orbit. Moreover, since G is arc-transitive, every vertex $v \in \Delta_i$ is adjacent to two vertices in Δ_{i-1} and two vertices in Δ_{i+1} , implying that the edges between Δ_i and Δ_{i+1} are partitioned into precisely two N -edge-orbits; let's call these two orbits Θ_i and Θ'_i .

Clearly, an element of K can map an edge in Θ_i only to an edge in Θ_i or to an edge in Θ'_i . On the other hand, for every vertex $v \in \Theta_i$ there is an element $g \in G_v$ which maps an edge of Θ_i incident to v to the edge of Θ'_i incident to v ; and this element g is clearly an element of K . This shows that the orbits of K on $E\Gamma$ are precisely the sets $\Theta_i \cup \Theta'_i$, $i \in \mathbb{Z}_r$. In other words, each orbit of the induced action of K on the set $E\Gamma/N = \{e^N : e \in E\Gamma\}$ has length 2. Consequently, if X denotes the kernel of the action of K on $E\Gamma$, then K/X embeds into $\text{Sym}(2)^r$ and is therefore an elementary abelian 2-group.

Let us now show that $X = N$. Clearly, $N \leq X$. Let $v \in \Delta_0$. Since N is transitive on Δ_0 , it follows that $X = NX_v$. Suppose that X_v is non-trivial and let g be a non-trivial element of X_v . Further, let w be a vertex which is closest to v among all the vertices not fixed by g , and let $v = v_0 \sim v_1 \sim \dots \sim v_m = w$ be a shortest path from v to w . Then v_{m-1} is fixed by g . Since g fixes each N -edge-orbit set-wise and since every vertex of Γ is incident to at most one edge in each N -edge-orbit, it follows that g fixes all the neighbours of v_{m-1} , thus also v_m . This contradicts our assumptions and proves that X_v is a trivial group, and hence that $X = N$.

Thus K/N is an elementary abelian 2-group. Now, since N is semiregular, we see that $K_v \cong K_v/(N \cap K_v) \cong K_v N/N = K/N$ (the latter equality following from the fact that N is transitive on v^K). Hence, K_v is an elementary abelian 2-group, as claimed. \square

1.6. Miscellanea. We now present several auxiliary results that will come useful when proving Theorems 1.1 and 1.2.

Lemma 1.14. *Let Γ be a connected k -valent graph admitting an abelian group of automorphisms N having at most two orbits on $V\Gamma$ and let $v \in V\Gamma$. Then either $N_v \neq 1$ and there exist two distinct vertices u and u' with $\Gamma(u) = \Gamma(u')$, or $N_v = 1$, N has a generating set consisting of at most $2k - 2$ elements and $|V\Gamma| = 2|N|$.*

Proof. Observe that since N is abelian, $N_v = N_u$ whenever $u \in v^N$. Hence, if N is transitive on $V\Gamma$, then Γ is a Cayley graph on the group N , and N is generated by the k -element set $\{x \in N : v^x \sim_\Gamma v\}$.

Suppose now that N has two orbits on $V\Gamma$ and let u be a neighbour of v . If $N_v = 1$, then Γ is isomorphic to a bi-Cayley graph $\text{BiCay}(N; L, R, S)$ (see, for example, [8] for the definition of bi-Cayley graphs and basic properties) where $S, L, R \subseteq N$, $1 \in S$, $|S| + |L| = |S| + |R| = k$ and $\langle S \cup L \cup R \rangle = N$. In particular, N has a generating set of size at most $2k - 2$. If $N_v \neq 1$, then N_v and N_u are kernels of the action of N on v^N and u^N , respectively, and thus $N_v \cap N_u = \emptyset$. Hence, if x is a non-trivial element of N_v , then $u^x \neq u$ and $\Gamma(u) = \Gamma(u^x)$. \square

Lemma 1.15. *Let Γ be one of the exceptional graphs Ψ_1, \dots, Ψ_6 or $C(r, s)$ with $r \geq s$ and $1 \leq s \leq r - 1$ and let G be an edge- and vertex-transitive group of automorphisms of Γ containing a non-trivial element g with $\text{fpr}_{V\Gamma}(g) > 1/3$. Then exactly one of the following happens:*

- (1) G is 2-arc-transitive, or
- (2) $\Gamma \cong C(r, s)$ with $1 \leq s \leq 2r/3$ and G is $\text{Aut}(\Gamma)$ -conjugate to a subgroup of H defined in (1.2).

Proof. Suppose first that $\Gamma \cong C(r, s)$. Then Lemma 1.9 implies that $s < 2r/3$. If $r \neq 4$, then, by Lemma 1.7, we see that $\text{Aut}(\Gamma) = H$ and the result follows (note that H is not 2-arc-transitive). Suppose now that $r = 4$, and thus $s \in \{1, 2\}$. Since $|H| = r2^{r+1}$, we see that H is a 2-group, and by Lemma 1.7, $|\text{Aut}(\Gamma) : H| = 3$ or 9. Hence H is a Sylow 2-subgroup of $\text{Aut}(\Gamma)$. If G is not 2-arc-transitive, then G_v is a 2-group and since $|V\Gamma| = 8$ or 16, we see that G is a 2-group, implying that G is conjugate to a subgroup of the Sylow 2-subgroup H . For $\Gamma \cong \Psi_i$ with $i \in \{1, \dots, 6\}$ we verified the claim of the lemma by using the algebra computational system MAGMA [5]. \square

Lemma 1.16. *Let G be a group acting transitively on the set Ω and let Σ be a G -invariant partition of Ω . For $g \in G$, let g^Σ be the permutation of Σ induced by g . Then*

$$\text{fpr}_\Omega(g) \leq \text{fpr}_\Sigma(g^\Sigma).$$

In particular, if N is a normal subgroup of G , then $\text{fpr}_\Omega(g) \leq \text{fpr}_{\Omega/N}(Ng)$.

Proof. Observe that if $\omega \in \text{Fix}_\Omega(g)$ and $[\omega]$ is the element of Σ containing ω , then $[\omega] \in \text{Fix}_{\Omega/N}(Ng)$. Hence

$$|\text{Fix}_\Omega(g)| = \sum_{B \in \text{Fix}_\Sigma(g^\Sigma)} |B \cap \text{Fix}_\Omega(g)| \leq b |\text{Fix}_\Sigma(g^\Sigma)|,$$

where b is the cardinality of an arbitrary element of Σ . Note that $b|\Sigma| = |\Omega|$. The claim of the lemma then follows by dividing the above inequality by $|\Omega|$ and observing that Ω/N is a G -invariant partition of Ω . \square

The following lemma was proved in [25, Lemma 2.2] and is just a slight generalisation of [22, Lemma 2.5].

Lemma 1.17. [25, Lemma 2.2] *Let X be a group acting on a set Ω , let Y be a normal subgroup of X , let $\omega \in \Omega$ and let $x \in X_\omega$. Then*

$$(1.4) \quad \text{fpr}_{\omega^Y}(x) = \frac{|x^Y \cap X_\omega|}{|x^Y|} = \frac{|x^Y \cap X_\omega|}{|Y : \mathbf{C}_Y(x)|},$$

where $x^Y := \{x^y \mid y \in Y\}$ is the Y -conjugacy class of the element x and $\mathbf{C}_Y(x)$ the centraliser of x in Y .

If $B \leq G \leq \text{Sym}(\Omega)$ and $\omega \in \Omega$, we let $[a, B] = \{[a, b] : b \in B\}$ and $[a, B]_\omega = [a, B] \cap G_\omega$. Note that $[a, B]$ is not necessarily a subgroup of G . Observe that if B is semiregular and normalised by a , then $[a, B]_\omega = 1$.

Lemma 1.18. *Let $G \leq \text{Sym}(\Omega)$, let $\omega \in \Omega$, let $g \in G_\omega$, and let X be a normal subgroup of G such that $[g, X]_\omega = 1$. Then*

$$\text{Fix}_{\omega^X}(g) = \omega^{\mathbf{C}^X(g)} \quad \text{and} \quad \text{fpr}_\Omega(g) \leq \text{fpr}_{\omega^X}(g) = \frac{1}{|X : \mathbf{C}_X(g)|}.$$

Proof. Observe first that $1 = [g, X]_\omega = \{[g, x] : x \in X, [g, x] \in G_\omega\} \supseteq \{[g, x] : x \in X_\omega\}$, implying that every $x \in X_\omega$ centralises g and so $X_\omega = \mathbf{C}_X(g)_\omega$. Let $\delta \in \text{Fix}_{\omega^X}(g)$. Then there exists $x \in X$ with $\delta^x = \omega$ and $\omega^{g^{-1}x^{-1}g} = (\omega^{x^{-1}})^g = \delta^g = \delta = \omega^{x^{-1}}$, implying that $\omega^{[g, x]} = \omega$. Since $[g, X]_\omega = 1$, this shows that $x \in \mathbf{C}_X(g)$ and hence that $\text{Fix}_{\omega^X}(g) = \omega^{\mathbf{C}^X(g)}$, as claimed. Therefore $|\text{Fix}_{\omega^X}(g)| = |\mathbf{C}_X(g) : \mathbf{C}_X(g)_\omega|$ and so

$$\text{fpr}_{\omega^X}(g) = \frac{|\text{Fix}_{\omega^X}(g)|}{|\omega^X|} = \frac{|\mathbf{C}_X(g)| |X_\omega|}{|X| |\mathbf{C}_X(g)_\omega|} = \frac{1}{|X : \mathbf{C}_X(g)|},$$

as claimed. In particular, $\text{fpr}_{\omega^g}(g) = \text{fpr}_{\delta^g}(g)$ for every $\delta \in \text{Fix}_\Omega(g)$. Now choose a set $\{\delta_1, \dots, \delta_m\}$ of representatives of those orbits δ^X for which $\text{Fix}_{\delta^X}(g) \neq \emptyset$. Without loss of generality, we may assume that all δ_i are fixed by g . The set $\text{Fix}_\Omega(g)$ is then the disjoint union of the sets $\text{Fix}_{\delta_i^X}(g)$ for $i \in \{1, \dots, m\}$. Since $|\Omega| = |\Omega/X| |\omega^X| \geq m |\omega^X|$, we see that

$$\text{fpr}_\Omega(g) = \frac{|\text{Fix}_\Omega(g)|}{|\Omega|} \leq \frac{|\text{Fix}_\Omega(g)|}{m |\omega^X|} = \frac{1}{m} \sum_{i=1}^m \frac{|\text{Fix}_{\delta_i^X}(g)|}{|\omega^X|} = \frac{1}{m} \sum_{i=1}^m \text{fpr}_{\delta_i^X}(g) = \text{fpr}_{\omega^X}(g),$$

completing the proof. \square

2. PROOF OF THEOREM 1.1 FOR Γ NOT 2-ARC-TRANSITIVE

This section is devoted to the proof of Theorem 1.1 in the case where Γ is not a 2-arc-transitive graph. Throughout this section, we work under the following assumption:

Hypothesis 2.1. Let Γ be a connected 4-valent graph, let G be a subgroup of $\text{Aut}(\Gamma)$ acting transitively on $V\Gamma$ and on $E\Gamma$ and let g be a non-trivial element of G such that $\text{fpr}_{V\Gamma}(g) > 1/3$.

Lemma 2.2. *Let Γ , G and g be as in Hypothesis 2.1. If G is not 2-arc-transitive, then either $\Gamma \cong C(r, s)$ for some integers r and s , or G contains a minimal normal subgroup N of order a power of 2 such that Γ/N is a 4-valent graph.*

Proof. First, observe that since G is not 2-arc-transitive, the vertex-stabiliser G_v is a 2-group (for example, see [26] for the arc-transitive case and [30] for the $\frac{1}{2}$ -arc-transitive case). If G possesses no non-trivial normal 2-subgroups, then Theorem 1.3 yields a contradiction. We may thus assume that G has a minimal normal subgroup N which is a 2-group (and hence an elementary abelian 2-group). If $N_v \neq 1$, then by Lemma 1.10, we have $\Gamma \cong C(r, s)$. We may thus assume that $N_v = 1$. If N has at most two orbits on $V\Gamma$, then by Lemma 1.14, we see that $|V\Gamma| \leq 128$ and the validity of the claim can be checked computationally by inspecting the candidate graphs from the list of all 4-valent arc-transitive graphs of small order (see [27] or [33]) or small 4-valent $\frac{1}{2}$ -transitive graphs [28]. We may thus assume that N has at least three orbits on $V\Gamma$, and therefore Γ/N is either a cycle C_r for some $r \geq 3$ or a 4-valent graph. In the former case, Lemma 1.11 implies that $\Gamma \cong C(r, s)$, and the result follows. \square

We now prove a result that will enable us to reduce the proof of Theorem 1.1 to the case where G is 2-arc-transitive.

Lemma 2.3. *Let Γ , G and g be as in Hypothesis 2.1. Suppose that G contains a minimal normal subgroup N of order a power of 2 such that Γ/N is isomorphic to $C(r, s)$ for some r and s . If $G/N \leq H^+$ where H^+ is as in the formula (1.2) of Section 1.3, then $\Gamma \cong C(r', s')$ with $1 \leq s' \leq 2r'/3$.*

Proof. By Lemma 1.9, we have $1 \leq s \leq 2r/3$. Moreover, by way of contradiction, we may assume that $\Gamma \not\cong C(r', s')$ for any r' and s' . Since $G/N \leq H^+$, the group G/N is not arc-transitive and thus neither is G . Let $\vec{\Gamma} := \vec{\Gamma}^{(G)}$ be a digraph induced by the $\frac{1}{2}$ -arc-transitive action of G . Then $\vec{\Gamma}$ is a G -arc-transitive digraph of in- and out-valency 2, and thus in view of Lemmas 1.7 and 1.10, we may assume that:

$$(2.1) \quad \text{Every abelian normal subgroup of } G \text{ acts semiregularly on } V\vec{\Gamma}.$$

Note that $\vec{\Gamma}/N \cong \vec{C}(r, s)$. We will now follow the ideas developed in the proof of [26, Theorem 3.10] as well as in [30], which in turn draw heavily from the classical work of Tutte [39], Djokovović [14] and Sims [38].

Identify $\vec{\Gamma}/N$ with $\vec{C}(r, s)$ and let the automorphisms $\tau_i, \rho \in \text{Aut}(\vec{\Gamma})$ and the group $K = \langle \tau_i \mid i \in \mathbb{Z}_r \rangle \cong C_2^2$ be as in Section 1.3. Recall that $H^+ = K\langle \rho \rangle$ and that $K = \langle (H^+)_x : x \in V\vec{C}(r, s) \rangle = \langle K_x : x \in V\vec{C}(r, s) \rangle$ (see formula (1.3)). Since $G/N \leq H^+$ this implies that $(G/N)_{vN} \leq K_{vN}$ and hence $(G/N)_{vN} = (G/N) \cap K_v$. Now let

$$(2.2) \quad E = \langle G_u : u \in V\vec{\Gamma} \rangle$$

and observe that $E = (G_v)^G$, the normal closure of G_v in G . We now see that

$$(2.3) \quad EN/N = (G_v N)^G/N = (G_v N/N)^{G/N} = ((G/N)_{vN})^{G/N} = ((G/N) \cap K_v)^{G/N} = (G/N) \cap K.$$

By minimality of N , it follows that either $N \leq G$ or $N \cap E = 1$. If $N \cap E = 1$, we see that $E \cong EN/N \leq K$; in particular, E is an abelian normal subgroup of G not acting semiregularly on $V\vec{\Gamma}$, contradicting (2.1).

We may thus assume that $N \leq E$. Then $E/N = EN/N = (G/N) \cap K$. Hence E/N is an elementary abelian 2-group, implying that E is a 2-group. Moreover, since $E_u \cong E_u/(N \cap E_u) \cong (E_u N/N) = (E/N)_{uN}$, we see that

$$(2.4) \quad E_u \text{ is an elementary abelian 2-group for every } u \in V\Gamma.$$

Now consider the Frattini subgroup $\Phi(E)$ and the derived subgroup $[E, E]$ of E . Being characteristic in E , they are both normal in G . Recall that $\Phi(E)$ (respectively, $[E, E]$) is the smallest normal subgroup of E with respect to which the quotient group is elementary abelian (respectively, abelian). In particular, since E/N is elementary abelian, we see that $[E, E] \leq \Phi(E) \leq N$. By the minimality of N , it follows that either $[E, E] = 1$ or $[E, E] = \Phi(E) = N$. If $[E, E] = 1$, then E is abelian, which contradicts (2.1). We may thus assume that

$$(2.5) \quad [E, E] = \Phi(E) = N.$$

Moreover, since E and N are 2-groups, the action of E on $N \setminus \{1\}$ by conjugation must have at least one fixed point, implying that N intersects the centre $\mathbf{Z}(E)$ non-trivially. The minimality of N then implies that $\mathbf{Z}(E) \geq N$. If $(\mathbf{Z}(E))_v \neq 1$, then $\mathbf{Z}(E)$ is a normal elementary abelian 2-subgroup of G not acting semiregularly on $V\Gamma$, which contradicts (2.1), showing that $\mathbf{Z}(E)$ acts semiregularly on $V\Gamma$.

We will now set up a standard notation typically used when studying the structure of a vertex-stabiliser G_v in a G -arc-transitive digraph of out-valence 2 (see, for example, [30, Section 2.3]). Let t be the largest

integer such that G acts transitively on the t -arcs of $\vec{\Gamma}$. Note that G then acts regularly on the set of all t -arcs of $\vec{\Gamma}$ and that t is the largest integer such that G_v (which clearly equals E_v) acts transitively on the t -arcs starting at v . Let a be any element of G such that (v^a, v) is an arc of $\vec{\Gamma}$ and let

$$(2.6) \quad v_i := v^{a^{-i}} \text{ for } i \in \mathbb{Z}.$$

Note that, for every $i \geq 0$, the $(i+1)$ -tuple (v_0, v_1, \dots, v_i) is an i -arc of $\vec{\Gamma}$. Observe also that $Ea \in G/E$ acts as a one-step rotation of $\vec{\Gamma}/E \cong \vec{C}_r$, implying that $G = E\langle a \rangle$.

Now consider the stabiliser $G_{(v_0, \dots, v_i)}$ of this i -arc in G ; since $v_0 = v$ and $G_v = E_v$, we see that $G_{(v_0, \dots, v_i)} = E_{(v_0, \dots, v_i)}$. By the definition of t , it follows that $G_{(v_0, \dots, v_t)} = 1$ and that $G_{(v_0, \dots, v_{t-1})} = \langle x_0 \rangle$, where x_0 is the unique element of G which fixes the $t-1$ arc $(v_0, v_1, \dots, v_{t-1})$ but moves the vertex v_t . For $i \geq 1$, let

$$(2.7) \quad x_i := x^{a^i} \text{ and } E_i := \langle x_0, \dots, x_{i-1} \rangle, \quad E_0 := 1.$$

It is not difficult to deduce (see [30, Section 2.3] for the proof) that

$$(2.8) \quad \text{for every } i \in \{0, \dots, t\} : E_i = G_{(v_0, \dots, v_{t-i})} \text{ and } |E_i| = 2^i.$$

Moreover, by [30, Section 2.3], there exists a positive integer e with the following properties:

- e is the smallest integer such that $E_{t+e} = E_{t+e+1}$;
- e is the smallest integer such that $E_{t+e} = E$.

Recall that $E/N = (G/N) \cap K$ is an elementary abelian 2-group. Let us now show that

$$(2.9) \quad |E/N| = 2^{t+e} \text{ and } E/N = \langle Nx_0, Nx_1, \dots, Nx_{t+e-1} \rangle.$$

Indeed: Suppose that for some $e' \leq e$ we have $E_{t+e'}N = E_{t+e'+1}N$. Since $(E_i)^a = \langle x_1, \dots, x_i \rangle$, it follows that $E_{t+e'+2}N = \langle E_{t+e'+1}N, (E_{t+e'+1}N)^a \rangle = \langle E_{t+e'}N, (E_{t+e'}N)^a \rangle = E_{t+e'+1}N = E_{t+e'}N$, and thus by induction we see that $E = E_{t+e}N = E_{t+e'}N$. Since $N = \Phi(E)$, the set of non-generators of E , it follows that $E_{t+e'} = E$, and thus $e = e'$. In particular, e is the smallest integer such that $E_{t+e}N = E_{t+e+1}N$. Hence

$$(2.10) \quad N = E_0N < E_1N < E_2N < \dots < E_{t+e-1}N < E_{t+e}N = E.$$

In particular, $|E| \geq 2^{t+e}|N|$ and thus $|E/N| \geq 2^{t+e}$. On the other hand, $E/N = \langle Nx_0, Nx_1, \dots, Nx_{t+e-1} \rangle$, and since E/N is elementary abelian, we see that $|E| \leq 2^{t+e}$, proving the claim (2.9).

Recall that $\mathbf{Z}(E)$ acts semiregularly on VT . Since $E_v = E_t = \langle x_0, \dots, x_{t-1} \rangle$ is abelian (see (2.4)), it follows that $E_t^{a^{t-1}} = \langle x_{t-1}, \dots, x_{2t-2} \rangle$ is also abelian. Therefore x_{t-1} is central in $\langle E_t, (E_t)^{a^{t-1}} \rangle = \langle x_0, \dots, x_{2t-2} \rangle = E_{2t-1}$. Since $x_{t-1} \in E_v$ and $\mathbf{Z}(E) \cap E_v = 1$, we get $E_{2t-1} < E = E_{t+e}$ and hence $2t-1 < t+e$ from which it follows that

$$(2.11) \quad e \geq t.$$

We now prove a technical result (its relevance will become apparent later in the proof). Let h be an arbitrary element of $E_{v_0} \setminus E_{v_{-1}}$ and let $\beta = \min\{b \in \mathbb{N} : h \in G_{(v_0, \dots, v_{t-b})}\}$. Then:

$$(2.12) \quad 1 \leq \beta \leq t \text{ and for every } j \in \{1, \dots, e\} \text{ and every } x \in E \text{ we have } h^{a^{t-\beta+j}x} \notin E_v \text{ and } h^{a^{-1}x} \notin E_v.$$

Since $G_{(v_0, v_1, \dots, v_t)} = 1$, we see that $\beta \geq 1$ and since $h \in E_{v_0} = G_{v_0}$, we see that $\beta \leq t$. Recall that $G_{(v_0, v_1, \dots, v_{t-i})} = \langle x_0, \dots, x_{i-1} \rangle$ and that e is the smallest integer such that $E = \langle x_0, \dots, x_{t-1+e} \rangle$. By definition, h is an automorphism of $\vec{\Gamma}$ fixing the $(t-\beta)$ -arc $(v_0, v_1, \dots, v_{t-\beta})$ and moving the vertices v_{-1} and $v_{t-\beta+1}$. Since $v_0 = v$ and $E_v = G_v = \langle x_0, \dots, x_{t-1} \rangle$, we may write $h = x_\alpha x_{\alpha+1}^{\varepsilon_{\alpha+1}} \dots x_{\gamma-1}^{\varepsilon_{\gamma-1}} x_\gamma$, for some $0 \leq \alpha < \gamma < t$ and $\varepsilon_i \in \{0, 1\}$. Then $h \in \langle x_0, \dots, x_\gamma \rangle = G_{(v_0, \dots, v_{t-\gamma-1})}$ and thus by the definition of β we see that $\gamma = \beta - 1$. Further, since $E_{v_{-1}} = E_{v^a} = (E_v)^a = \langle x_1, \dots, x_t \rangle$ and since $h \notin E_{v_{-1}}$, we see that $\alpha = 0$. Therefore $h = x_0 x_1^{\varepsilon_1} \dots x_{\beta-2}^{\varepsilon_{\beta-2}} x_{\beta-1}$ and thus

$$h^{a^{t-\beta+j}x} = x^{-1} x_{t-\beta+j} x_{t-\beta+j+1}^{\varepsilon_1} \dots x_{t+j-2}^{\varepsilon_{\beta-2}} x_{t+j-1} x.$$

Suppose now that $h^{a^{t-\beta+j}x} \in E_v = \langle x_0, \dots, x_{t-1} \rangle$. Since E/N is an abelian group, then the above equality, when considered modulo the group N , implies

$$Nx_{t-\beta+j} Nx_{t-\beta+j+1}^{\varepsilon_1} \dots Nx_{t+j-2}^{\varepsilon_{\beta-2}} Nx_{t+j-1} \in \langle Nx_0, \dots, Nx_{t-1} \rangle.$$

Since $j \geq 1$ and $t-\beta \geq 0$, we then see that

$$Nx_{t+j-1} \in \langle Nx_0, \dots, Nx_{t+j-2} \rangle.$$

But since $j \leq e$ this contradicts the fact that $\{Nx_0, Nx_1, \dots, Nx_{t+e-1}\}$ is a minimal generating set for E/N ; see (2.9). This contradiction shows that $h^{a^{t-\beta+j}x} \notin E_v$, as claimed. Similarly, if $h^{a^{-1}x} \in E_v$, then $h \in (E_v)^{x^{-1}a}$ and so $h \in x^a \langle x_1, \dots, x_t \rangle (x^a)^{-1}$, showing that $Nx_0 \in \langle Nx_1, \dots, Nx_t \rangle$, again contradicting (2.9).

Now let g be a non-trivial element of G with $\text{fpr}_{\text{v}\Gamma}(g) > \frac{1}{3}$ and let $u \in \text{Fix}_{\text{v}\Gamma}(g)$. Since $g \in G_u = E_u$, we see that $[g, E]_u \leq [E, E]_u = N_u = 1$. In particular, since $[g, E_u] \leq [g, E]_u$, it follows that $[g, E_u] = 1$, and thus $\mathbf{C}_E(g)_u = E_u$ for every $u \in \text{Fix}_{\text{v}\Gamma}(g)$. We may now apply Lemma 1.18 with $\text{v}\Gamma$ in place of Ω , with E in place of X and with u in place of ω , to conclude that

$$(2.13) \quad \text{Fix}_{uE}(g) = u^{\mathbf{C}_E(g)} \quad \text{and} \quad \frac{1}{3} < \text{fpr}_{\text{v}\Gamma}(g) \leq \text{fpr}_{uE}(g) = \frac{1}{|E : \mathbf{C}_E(g)|}$$

and so $|E : \mathbf{C}_E(g)| \leq 2$. If $E = \mathbf{C}_E(g)$, then $g \in \mathbf{Z}(E) \cap E_v = (\mathbf{Z}(E))_v = 1$, a contradiction. Therefore $|E : \mathbf{C}_E(g)| = 2$ and thus

$$(2.14) \quad \text{fpr}_{uE}(g) = \begin{cases} \frac{1}{|E : \mathbf{C}_E(g)|} = \frac{1}{2} & \text{if } \text{Fix}_{uE}(g) \neq \emptyset, \\ 0 & \text{if } \text{Fix}_{uE}(g) = \emptyset. \end{cases}$$

Now assume without loss of generality that $v \in \text{Fix}_{\text{v}\Gamma}(g)$. Recall that $E = E_{t+e} = \langle x_0, \dots, x_{t+e-1} \rangle$. Since $\text{Fix}_{\text{v}\Gamma}(g) \neq \emptyset$ it follows by (2.14) that $|E : \mathbf{C}_E(g)| = 2$ and thus there exists the smallest integer $i \in \{0, \dots, t+e-1\}$ such that $x_i \notin \mathbf{C}_E(g)$. If $i < t$, then $x_i \in E_v = \mathbf{C}_E(g)_v$ (see (2.8)), contradicting the choice of i . Hence we have

$$(2.15) \quad E = \mathbf{C}_E(g) \cup x_i \mathbf{C}_E(g) \quad \text{for some } i \in \{t, t+1, \dots, t+e-1\}.$$

Since the automorphism a maps a vertex to its neighbour in Γ , the connectivity of Γ implies that $G = \langle G_v, a \rangle \leq \langle E, a \rangle$. Now observe that $\vec{\Gamma}/E \cong (\vec{\Gamma}/N)/(E/N) \cong \vec{C}(r, s)/((G/N) \cap K) \cong \vec{C}_r$; in particular, the E -orbits on $\text{v}\vec{\Gamma}$ can be labelled by Δ_i , $i \in \mathbb{Z}_r$, in such a way that every arc of $\vec{\Gamma}$ starting in some Δ_i ends in Δ_{i+1} . We may assume without loss of generality that $v \in \Delta_0$. Since the automorphism a maps the vertex v to its in-neighbour (see (2.6)), it follows that $(\Delta_j)^a = \Delta_{j-1}$ for every $j \in \mathbb{Z}_r$, and thus

$$\Delta_j = v^{Ea^{-j}} = v^{a^{-j}E} = (v_j)^E.$$

Using (2.15), we can split this E -orbit into two halves, that is,

$$\Delta_j = \Delta'_j \cup \Delta''_j \quad \text{where } \Delta'_j = v^{a^{-j}\mathbf{C}_E(g)}, \quad \Delta''_j = v^{a^{-j}x_i\mathbf{C}_E(g)} \quad \text{and } \Delta'_j \cap \Delta''_j = \emptyset.$$

Let us call the E -orbit Δ_j *blue* provided that $\text{Fix}_{\Delta_j}(g) = \emptyset$, *red* if $g \in E_{v_j}$, and *pink* if $g \in E_{v_j^{x_i}}$. By (2.13) we see that Δ_j is red if and only if $\text{Fix}_{\Delta_j}(g) = \Delta'_j$ and that it is pink if and only if $\text{Fix}_{\Delta_j}(g) = \Delta''_j$. In particular, if Δ_j is not blue, then it is either red or pink. Moreover, if Δ_j is red, then $g \in E_{v_j} = (E_v)^{a^{-j}}$, and if it is pink, then $g \in E_{(v_j)^{x_i}} = (E_v)^{a^{-j}x_i} = (E_v)^{x_i+j a^{-j}}$. This immediately implies that:

$$(2.16) \quad \Delta_j \text{ is red} \iff g^{a^j} \in E_v \quad \text{and} \quad \Delta_j \text{ is pink} \iff g^{a^j x_i+j} \in E_v.$$

If, for a colour $X \in \{\text{red, pink, blue}\}$, an element $k \in \mathbb{Z}_r$ and a positive integer ℓ the orbits $\Delta_k, \dots, \Delta_{k+\ell-1}$ are all of colour X while Δ_{k-1} and $\Delta_{k+\ell}$ are of a colour different than X , we say that $S := \{k, \dots, k+\ell-1\}$ is a *strip of colour X* and of length $\ell(S) := \ell$. Let $S := \{k, \dots, k+\ell-1\}$ be a strip. Then the strip containing $k-1$ is said to *precede* S and the strip containing $k+\ell$ *follows* the strip S .

Let S be a strip preceded by a strip S^- and followed by a strip S^+ . We will now show that the following holds:

$$(2.17) \quad \text{If } S \text{ is red or pink, then } S^+ \text{ and } S^- \text{ are blue, } \ell(S) \leq t \text{ and } \ell(S^+) \geq e.$$

Suppose first that $S = \{k, \dots, k+\ell-1\}$ is a red strip. Let $h := g^{a^k}$. Then, by (2.16), we see that $h^{a^j} \in E_v$ for every $j \in \{0, \dots, \ell-1\}$, while $h^{a^{-1}}, h^{a^\ell} \notin E_v$. In other words, $h \in E_{v_0} \setminus E_{v_{-1}}$, $h \in E_{(v_0, \dots, v_{\ell-1})}$ while $h \notin E_{v_\ell}$. Hence $\beta := \min\{b \in \mathbb{N} : h \in G_{(v_0, \dots, v_{t-b})}\} = t - \ell + 1$. We may now apply (2.12) to conclude that $1 \leq t - \ell + 1 \leq t$ (implying $\ell \leq t$, as required) and that neither of the elements $h^{a^{-1}x}$ and $h^{a^{\ell-1+j}x}$ for $j \in \{1, \dots, e\}$ and $x \in E$ belongs to E_v . In view of (2.16), this implies that neither of the orbits Δ_{k-1} and $\Delta_{k+\ell}, \dots, \Delta_{k+\ell+e-1}$ are red or pink, showing that the strips S^- and S^+ are blue and that $\ell(S^+) \geq e$. This completes the proof of the claim (2.17).

Since $e \geq t$ (see (2.11)), the claim (2.17) implies that the number of blue orbits is greater or equal to the number of red and pink orbits combined. Since g has no fixed points in blue orbits and fixes precisely half of the points in each red or pink orbit, this shows that g fixes at most $1/4$ of the vertices of Γ . This contradiction completes the proof of the theorem in the case where $\Gamma/N \cong C(r, s)$ for some r and s . \square

We can now prove Theorem 1.1 under the assumption that G is not 2-arc-transitive.

Theorem 2.4. *Let Γ be a connected 4-valent graph, let G be an edge- and vertex-transitive but not 2-arc-transitive group of automorphisms of Γ and let g be a non-trivial element of G with $\text{fpr}_{V\Gamma}(g) > 1/3$. Then $\Gamma \cong C(r, s)$ for some positive integers r and s with $1 \leq s < 2r/3$.*

Proof. Suppose that the theorem is false and let Γ be a counterexample with the smallest number of vertices. Moreover, among all groups G satisfying the assumptions of the theorem, choose one of smallest order.

By Lemma 2.2, there exists a minimal normal subgroup N of G of order a power of 2 acting semiregularly on $V\Gamma$ such that Γ/N is a 4-valent graph. Then G/N acts edge- and vertex-transitively on Γ/N but not 2-arc-transitively, and by Lemma 1.16, we see that $\text{fpr}_{V\Gamma}(Ng) > 1/3$. The minimality of Γ then implies that $\Gamma/N \cong C(r', s')$ for some r' and s' with $1 \leq s' < 2r'/3$. By Lemma 1.15, it follows that G/N is $\text{Aut}(\Gamma/N)$ -conjugate to a subgroup of H . Without loss of generality we may thus assume that $G/N \leq H$. Furthermore, by Lemma 1.9, we see that $Ng \in K \leq H^+$. Now consider the group $X := G/N \cap H^+$. Since $|H : H^+| = 2$, we see that $|G/N : X| \leq 2$ and X is a $\frac{1}{2}$ -arc-transitive group of automorphisms of Γ/N . Let G^+ be the preimage of X with respect to the quotient projection $G \rightarrow G/N$. Then $G^+/N \cong X \leq H^+$, G^+ is $\frac{1}{2}$ -arc-transitive and since $Ng \in X$, we see that $g \in G^+$. By our choice of G this implies that $G = G^+$, and hence $G/N \leq H^+$. The result now follows from Lemma 2.3. \square

3. PROOF OF THEOREM 1.2 FOR Γ NOT ARC-TRANSITIVE

We now move our attention to 3-valent vertex- but not arc-transitive graphs. As observed in [26, 27], the 3-valent graph admitting a vertex- but not arc-transitive group of automorphisms are closely related to the family of 4-valent graph admitting an arc- but not 2-arc-transitive group of automorphisms. This will enable us to reduce the proof of Theorem 3.1 below to the situation covered by Theorem 1.1.

In the proof of Theorem 3.1 we need to refer to two special families of cubic vertex-transitive graphs: the *prisms* Pr_n , that can be defined as the Cayley graphs $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(0, 1), (1, 0), (-1, 0)\})$ for $n \geq 3$, and the *Möbius ladders* Mb_n , defined as the Cayley graphs $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1, n\})$ for some $n \geq 2$.

Theorem 3.1. *Let Γ be a connected 3-valent graph admitting a vertex-transitive but not arc-transitive group of automorphisms G . Let $g \in G$ be a non-trivial element of smallest order such that $\text{fpr}_{V\Gamma}(g) > 1/3$. Then Γ is either a Split Praeger-Xu graph $S(C(r, s))$ with $1 \leq s \leq 2r/3$, $r \geq 3$, or isomorphic to Λ_1 (the complete graph K_4) or Λ_3 (the skeleton of the cube).*

Proof. By consulting the database [27] of 3-valent vertex-transitive graphs on at most 1280 vertices, we checked that Theorem 1.2 holds if $|V\Gamma| \leq 1280$. We may thus assume that $|V\Gamma| > 1280$ (in fact, we will only use $|V\Gamma| > 140$). Observe the vertex-stabiliser G_v is a 2-group (and thus the order $o(g)$ of g is 2) whose action upon $\Gamma(v)$ has two orbits, one of length 2 and one of length 1.

For a vertex $w \in V(\Gamma)$ let w' be the neighbour of w such that $\{w\}$ is the orbit of G_w of length 1. Then clearly $w'' = w$ and $G_w = G'_{w'}$. Hence, the set $\mathcal{M} := \{\{w, w'\} : w \in V\Gamma\}$ is a complete matching of Γ , while edges outside \mathcal{M} form a 2-factor \mathcal{F} . The group G preserves both \mathcal{F} and \mathcal{M} and acts transitively on the arcs of each of these two sets. Let $\tilde{\Gamma}$ be the graph with vertex-set \mathcal{M} and two vertices $e_1, e_2 \in \mathcal{M}$ adjacent if and only if they are (as edges of Γ) at distance 1 in Γ . The graph $\tilde{\Gamma}$ is then called the *merge* of Γ . We may also think of Γ as being obtained by contracting all the edges in \mathcal{M} . The group G clearly acts as an arc-transitive group of automorphisms on $\tilde{\Gamma}$. Moreover, the connected components of the the 2-factor \mathcal{F} gives rise to a decomposition \mathcal{C} of $E\tilde{\Gamma}$ into cycles.

If $\Gamma \cong \text{Pr}_n$ or Mb_n for some $n \geq 3$, then it is easy to see that a non-trivial automorphism of Γ can fix at most 4 vertices, which, together with the assumption $\text{fpr}_{V\Gamma}(g) > 1/3$ implies that $|V\Gamma| < 12$, contradicting our assumption on Γ . We may thus assume that Γ is neither a prism nor a Möbius ladder. As was shown in [27, Lemma 9 and Theorem 10], this implies that $\tilde{\Gamma}$ is 4-valent. Moreover, the action of G on $V\tilde{\Gamma}$ is faithful, arc-transitive but not 2-arc-transitive. Observe also that $\text{fpr}_{V\tilde{\Gamma}}(g) \geq \text{fpr}_{V\Gamma}(g) > 1/3$. By Theorem 1.1, it thus follows that $\tilde{\Gamma} \cong C(r, s)$ with $1 \leq s < 2r/3$, $r \geq 3$. In view of [27, Theorem 12], the graph Γ can then be

uniquely reconstructed from $\tilde{\Gamma}$ and the decomposition \mathcal{C} of $E\tilde{\Gamma}$ arising from the 2-factor \mathcal{F} via the *splitting operation* defined in [27, Construction 11]. In short, Γ can be obtained from $\tilde{\Gamma}$ by splitting each vertex v of $\tilde{\Gamma}$ into two adjacent vertices v', v'' , each of them retaining two neighbours of v in $\tilde{\Gamma}$, that together with v form a part of a cycle in \mathcal{C} . It is then straightforward to see that Γ is the Split Praeger-Xu graph $S(C(r, s))$; or, which is equivalent, that the merging operation applied to $S(C(r, s))$ yields the graph $C(r, s)$. \square

4. GRAPH-THEORETICAL CONSIDERATION

In this section we make a digression into purely graph-theoretical considerations. We begin with an easy observation about 3-valent vertex-transitive graphs, and then prove the in the 4-valent arc-transitive case we may assume that a non-trivial element fixing more than $1/3$ vertices fixes an arc of the graph.

Lemma 4.1. *Let Γ be a connected 3-valent vertex-transitive graph. If there exist two distinct vertices u and u' of Γ such that $\Gamma(u) = \Gamma(u')$, then $\Gamma \cong K_{3,3}$.*

Proof. Let $\Gamma(u) = \Gamma(u') = \{v_1, v_2, v_3\}$. Since Γ is vertex-transitive, there exist $v'_1 \in V\Gamma$ such that $\Gamma(v_1) = \Gamma(v'_1)$. But then $v'_1 \in \Gamma(u)$, implying that v'_1 is one of the vertices v_2 or v_3 , say $v'_1 = v_2$. By applying the same argument to v_3 in place of v_1 , we see that $\Gamma(v_1) = \Gamma(v_2) = \Gamma(v_3)$. But then connectivity of Γ yields $\Gamma \cong K_{3,3}$. \square

Theorem 4.2. *Let $k \in \{3, 4\}$ and let Γ be a connected k -valent arc-transitive graph admitting a non-trivial automorphism g fixing no arc of Γ and satisfying $\text{fpr}_{V\Gamma}(g) > 1/3$. Then $k = 4$ and $\Gamma \cong C(r, 1)$ for some positive integer r , $r \geq 3$, or $k = 3$ and $\Gamma \cong K_{3,3}$.*

Proof. Let us first consider the case $k = 3$. Let d be the minimal distance between two vertices fixed by g . Since g fixes no arcs of Γ , we see that $d \geq 2$. If $d \geq 3$, then every vertex v in $F' := V\Gamma \setminus \text{Fix}_{V\Gamma}(g)$ is adjacent to at most one vertex in $F := \text{Fix}_{V\Gamma}(g)$, while every vertex $u \in F$ is adjacent to three vertices in F' . Therefore, $|F'| \geq 3|F|$ and thus

$$\text{fpr}_{V\Gamma} = \frac{|F|}{|F| + |F'|} \leq \frac{|F|}{|F| + 3|F|} = \frac{1}{4},$$

a contradiction. Hence $d = 2$. Let v and w be two vertices at distance 2 fixed by g . If $\Gamma(v) = \Gamma(w)$, then by Lemma 4.1, $\Gamma \cong K_{3,3}$. If $|\Gamma(v) \cap \Gamma(w)| = 1$, then the vertex in $\Gamma(v) \cap \Gamma(w)$ is also fixed by g , contradicting $d = 2$. Therefore $\Gamma(v) \cap \Gamma(w) = \{u_1, u_2\}$ with $u_1 \neq u_2$. But then g fixes the vertex in $\Gamma(v) \setminus \{u_1, u_2\}$, contradicting $d = 2$. This complete the proof in the case $k = 3$.

Let us now assume that $k = 4$. We divide the proof into several steps. We start by recalling that a connected 4-valent arc-transitive graph containing two distinct vertices w and w' with $\Gamma(w) = \Gamma(w')$ is isomorphic to $C(r, 1)$ for some $r \geq 3$; for the proof, see [32, Lemma 4.3], for instance. For the rest of the argument, we may thus assume that Γ has no two distinct vertices with the same neighbourhood.

Step 1: For every four distinct vertices $v_1, v_2, v_3, v_4 \in \text{Fix}_{V\Gamma}(g)$, we have $\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(v_4) = \emptyset$.

We argue by contradiction and we suppose that there exist four distinct vertices $v_1, v_2, v_3, v_4 \in \text{Fix}_{V\Gamma}(g)$ with $\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(v_4) \neq \emptyset$. Let $w \in \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(v_4)$. Observe that $w^g \in \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(v_4)$ because v_1, v_2, v_3, v_4 are fixed by g , and $w^g \neq w$ because g fixes no arc of Γ . Thus $\Gamma(w) = \{v_1, v_2, v_3, v_4\} = \Gamma(w^g)$, which is a contradiction.

Step 2: For every three distinct vertices $v_1, v_2, v_3 \in \text{Fix}_{V\Gamma}(g)$, we have $\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) = \emptyset$.

We argue by contradiction and we suppose that there exist three distinct vertices $v_1, v_2, v_3 \in \text{Fix}_{V\Gamma}(g)$ with $\Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \neq \emptyset$. Let $w \in \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)$. Arguing as in STEP 1, $w^g \in \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)$ and $w^g \neq w$. Thus

$$(4.1) \quad w \text{ and } w^g \text{ have three neighbours in common.}$$

From this point onwards one could follow the proof of the Subcase II.A of [32, Theorem 3.3] to conclude that then $\Gamma \cong K_{5,5} - 5K_2$ (yielding a contradiction). However, for the sake of completeness, we provide an independent proof of Step 2.

If w is adjacent to w^g in Γ , then from the arc-transitivity of Γ we deduce Γ is isomorphic to the complete graph K_5 . Since g fixes no arc of Γ , we have $|\text{Fix}_{V\Gamma}(g)| \leq 1$ and hence $\text{fpr}_{V\Gamma}(g) \leq 1/5 < 1/3$ and (??) holds. Thus, we may suppose for the rest of the proof of this step that w is not adjacent to w^g .

Let us now prove that $w^{g^2} = w$. If that were not the case, then w, w^g and w^{g^2} are all adjacent to v_1, v_2 and v_3 . Moreover, since v_1 and v_2 cannot have all neighbours in common, we also see that $w^{g^3} = w$. Let u_1 be the fourth neighbour of v_1 other than w, w^g and w^{g^2} . Since g fixes no arcs, $u_1^g \neq u_1$, and hence u_1^g , being adjacent to v_1 , is one of $w = w^{g^3}, w^g$ and w^{g^2} . But then $u_1 \in \{w^{g^2}, w, w^g\}$, yielding a contradiction. This shows that $w^{g^2} = w$, as claimed.

Let $v_4 \in \text{VF}$ with $\Gamma(w) = \{v_1, v_2, v_3, v_4\}$. If $v_4 \in \text{Fix}_{\text{VF}}(g)$, then $w \in \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(v_4)$ and $\Gamma(w) = \{v_1, v_2, v_3, v_4\} = \Gamma(w^g)$, that is, w and w^g are two distinct vertices with the same neighbourhood, contradicting our assumption. Therefore v_4 is not fixed by g . Thus $\Gamma(w^g) = \{v_1, v_2, v_3, v_4^g\}$ and $v_4 \neq v_4^g$. Note that since $w^{g^2} = w$, we have $v_4^{g^2} = v_4$. For the next two paragraphs Figure 4.1 might be of some help for following the argument. Since Γ is vertex-transitive, (4.1) yields that for each of $v_i, i \in \{1, 2, 3\}$, there

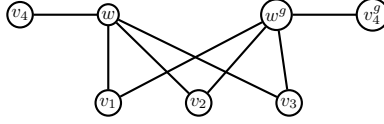


FIGURE 4.1. Graph for the proof of Theorem 4.2, I

exists $v'_i \in \text{VF}$ with v_i and v'_i having three neighbours in common.

Our next claim is that each of $v_i, i \in \{1, 2, 3\}$, has three neighbours in common with v_4 and three neighbours in common with v_4^g . Due to the symmetry conditions, it suffices to show that v_1 has three neighbours in common with v_4 .

Since $w, w^g \in \Gamma(v_i)$, by the pigeonhole principle, either w or w^g is a common neighbour of v_1 and v'_1 . Without loss of generality, we may assume that $w \in \Gamma(v_1) \cap \Gamma(v'_1)$. As $\Gamma(w) = \{v_1, v_2, v_3, v_4\}$, we deduce $v'_1 \in \{v_2, v_3, v_4\}$.

We first suppose that $v'_1 \in \{v_2, v_3\}$. Without loss of generality, we may assume that $v'_1 = v_2$. Let us call v_5 the third vertex in common to v_1 and v_2 . Clearly, v_5 cannot be fixed by g , otherwise g fixes the arc (v_1, v_5) . Since g fixes v_1 and v_2 , we obtain that v_5^g is a neighbour of both $v_1^g = v_1$ and $v_2^g = v_2$. Thus $\Gamma(v_1) = \{w, w^g, v_5, v_5^g\} = \Gamma(v_2)$, contradicting the fact that Γ has no two distinct vertices with the same neighbourhood. This paragraph shows that $v'_1 \notin \{v_2, v_3\}$ and hence $v'_1 = v_4$. Since v_1 has three neighbours in common with v_4 , we deduce that $v_1^g = v_1$ has three neighbours in common with v_4^g .

By symmetry, the argument in the previous four paragraphs can be applied also to the vertex v_2 and v_3 . Therefore, we deduce that each of v_1, v_2 and v_3 has three neighbours in common with v_4 and three neighbours in common with v_4^g .

Since v_1 has three neighbours in common with v_4 and three neighbours in common with v_4^g , we deduce that v_1, v_4 and v_4^g must have at least two neighbours in common. These vertices cannot be w or w^g , otherwise we contradicting Figure 4.1. Thus, let us call v_5 one of the two neighbours in common to v_1, v_4 and v_4^g . As g fixes no arcs, we have $v_5^g \neq v_5$. Thus v_5^g is a neighbour in common to $v_1^g = v_1, v_4^g$ and $(v_4^g)^g = v_4$. The left side of Figure 4.2 might be of some help for following the rest of the argument. As v_3 has three neighbours

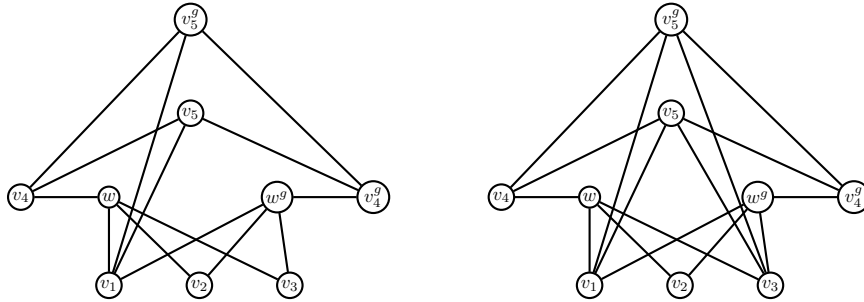


FIGURE 4.2. Graphs for the proof of Theorem 4.2, II

in common with v_4 and three neighbours in common with v_4^g , we may apply the argument of the previous paragraph with the vertex v_1 replaced by v_3 . We deduce that v_3 , v_4 and v_4^g have at least two neighbours in common, which cannot be neither w nor w^g . From the graph of the left side of Figure 4.2, we see that these mutually common neighbours are v_5 and v_5^g , otherwise we contradict the fact that v_4 and v_4^g have valency 4. See now the graph on the right side of Figure 4.2. When we apply the argument in the previous paragraph to the vertex v_3 replaced by the vertex v_2 , we deduce that v_5 and v_5^g are neighbours of v_2 , contradicting the fact that Γ has valency 4.

Step 3: $\text{fpr}_{\text{VT}}(g) \leq 1/3$.

For simplicity, set $F := \text{Fix}_{\text{VT}}(g)$ and $F' := \text{VT} \setminus \text{Fix}_{\text{VT}}(g)$. Since g fixes no arc of Γ , for every $v \in F$, we have $\Gamma(v) \subseteq F'$. Moreover, from STEP 2, we see that, for every $v \in F'$, we have $|\Gamma(v) \cap F| \leq 2$. Thus, by counting the edges between F and F' , we obtain $4|F| \leq 2|F'|$. As $|F| + |F'| = |\text{VT}|$, it follows

$$\text{fpr}_{\text{VT}}(g) = \frac{|F|}{|F| + |F'|} \leq \frac{|F|}{|F| + 2|F|} \leq \frac{1}{3},$$

which contradicts our assumptions. □

5. THE 2-ARC-TRANSITIVE CASE

In this section we complete the proofs of Theorems 1.1 and 1.2, by considering the remaining cases of 4-valent 2-arc-transitive graphs and 3-valent arc-transitive graphs. These cases are considered in [21] in a more general context of arc-transitive locally quasiprimitive graphs (that is, graphs, where the stabiliser of a vertex acts quasiprimitively on the neighbourhood) for which the order of the vertex-stabiliser is bounded by some constant depending only on the valence of the graph. There it is proved that for every constant c there are only finitely many graphs in such a family that admit a non-trivial automorphism fixing more than $1/c$ vertices. Since the order of $\text{Aut}(\Gamma)_v$ is bounded by 11 664 if Γ is a connected 2-arc-transitive 4-valent graph [41], and by 48 if Γ is a connected arc-transitive 3-valent graph [39], the result proved in [21] implies that there can be only a finite number of counterexamples to Theorems 1.1 and 1.2 (however, with the bound on their order being too large to be practical). The analysis carried out in this section is thus aimed at a finite number of graphs only.

We first prove two reduction results simultaneously for the 4-valent and the 3-valent case and split our analysis later. Note that by Theorem 4.2, the element g fixing more than $\frac{1}{3}$ vertices fixes an arc. Moreover, if Γ is a connected 3-valent graph with $G \leq \text{Aut}(\Gamma)$ acting arc-transitive but not 2-arc-transitively, then the arc-stabiliser G_{vw} is trivial. This shows that in our analysis we may assume that the graph Γ is 2-arc-transitive not only in the 4-valent case but also when Γ is 3-valent case. If a group of automorphisms G of a graph Γ acts transitively on the 2-arcs of Γ , we say that Γ is $(G, 2)$ -arc-transitive.

Lemma 5.1. *Let Γ be a connected k -valent $(G, 2)$ -arc-transitive graph with $k \in \{3, 4\}$ and let g be a nontrivial element of G with $\text{fpr}_{\text{VT}}(g) > 1/3$. Suppose that G contains a minimal normal subgroup N such that Γ/N is isomorphic to one of the graphs $\Psi_1, \dots, \Psi_6, \mathbf{C}(r, s)$, $1 \leq s \leq 2r/3$, $r \geq 3$ (if $k = 4$); or to one of the graphs $\Lambda_1, \dots, \Lambda_6$ (if $k = 3$). Then Γ itself is isomorphic to one of these graphs and is therefore not a counterexample to Theorem 1.1 or Theorem 1.2.*

Proof. If $|\text{VT}| \leq 768$ and $k = 4$ or if $|\text{VT}| \leq 10\,000$ and $k = 3$, the claim can be checked with a computer assisted computation using the census of connected 4-valent 2-arc-transitive graphs of order at most 768 [24] and the census of connected 3-valent arc-transitive graphs of order at most 10 000 [10]. We may therefore assume that $|\text{VT}|$ exceeds these bounds.

Since Γ/N is of the same valency as Γ , it follows that $N_v = 1$ for every $v \in \text{VT}$ and that G/N acts faithfully on VT/N . In particular, $Ng \in G/N$ is a non-trivial automorphism of Γ/N fixing more than $1/3$ of the vertices. Furthermore, G/N acts transitively on the arcs of Γ/N and if $k = 4$ then it also acts transitively on the 2-arcs of Γ/N . Consequently, if $\Gamma/N \cong \mathbf{C}(r, s)$, then by Remark 1.8, $r = 4$ and $s \in \{1, 2\}$. By applying Lemma 1.18 with N in place of X and VT in place of Ω , we conclude that $1/3 < \text{fpr}_{\text{VT}}(g) \leq 1/|N : \mathbf{C}_N(g)|$, implying that $|N : \mathbf{C}_N(g)| \leq 2$.

Suppose first that N is an elementary abelian 2- or 3-group. Since N is a minimal normal subgroup of G , the action of G/N on N by conjugation endows N with the structure of a G/N -irreducible module over

a field \mathbb{F} of size 2 or 3. In this case the proof can be completed by a straightforward computation with the computer algebra system, such as MAGMA [5], in a way which we now describe.

For every graph $\Delta \in \{\Psi_1, \dots, \Psi_6, C(r, 1), C(r, 2), \Lambda_1, \dots, \Lambda_6\}$ we consider every 2-arc-transitive subgroup H of $\text{Aut}(\Delta)$ and contains a non-trivial element h fixing more than $1/3$ of the vertices of Δ . Thus (Δ, H, h) is our putative triple $(\Gamma/N, G/N, Ng)$.

Next, we compute all the irreducible $\mathbb{F}H$ -modules V . Since $|N : \mathbf{C}_N(g)| \leq 2$, the element g either centralises N , or $p = 2$ and g acts as a transvection on N . Among all irreducible $\mathbb{F}H$ -modules V , we select those with $\mathbf{C}_H(V) \neq 0$ or (in the case $p = 2$) those admitting an element h of H with $|V : \mathbf{C}_V(h)| = 2$. Thus, in this refined family, V is our putative N .

When $k = 3$, a direct computation shows that all such modules V satisfy $|\mathbf{V}\Gamma/N| \cdot p^{\dim V} \leq 10\,000$, contradicting our assumption that $|\mathbf{V}\Gamma| > 10\,000$.

In the case $(k, p) = (4, 2)$ we have checked that

- $|\mathbf{V}\Gamma/N| \cdot 2^{\dim V} \leq 640$, or
- $\Gamma/N \cong \Psi_5$, $H \cong \text{Sym}(7)$, $\dim_{\mathbb{F}_2}(V) = 6$ and there is only one choice for V , or
- $\Gamma/N \cong \Psi_6$, $H \cong \text{Sym}(7) \times C_2$, $\dim_{\mathbb{F}_2}(V) = 6$ and there is only one choice for V .

Since $|\mathbf{V}\Gamma/N||N| = |\mathbf{V}\Gamma| > 640$, we may consider only the last two possibilities. For these cases, we have computed the cohomology module of H over V and we have obtained the corresponding first and second cohomology groups. These groups have dimension zero and hence G splits over N and N has a unique conjugacy class of complements in G . Thus G is isomorphic to a subgroup of $\mathbb{F}_2^6 \rtimes \text{Sym}(7)$ when $\Gamma/N \cong \Psi_5$ and G is isomorphic to a subgroup of $\mathbb{F}_2^6 \rtimes (\text{Sym}(7) \times C_2)$ when $\Gamma/N \cong \Psi_6$. In these cases, we have constructed the abstract group G and we have considered all the permutation representations of G of the relevant degree (of degree $2^6 \cdot 35$ when $\Gamma/N \cong \Psi_5$ and of degree $2^6 \cdot 70$ when $\Gamma/N \cong \Psi_6$). Finally, we have checked that none of these permutation groups acts arc-transitively on a connected 4-valent graph.

In the case $(k, p) = (4, 3)$ we know that g centralises N , and hence we may consider only those $\mathbb{F}H$ -modules V with $\mathbf{C}_H(V) \neq 0$. The computation in this case is similar to the case $p = 2$, and again none of the modules V yields an appropriate group G .

We may thus assume that N is not an elementary abelian 2- or 3-group. Since $|N : \mathbf{C}_N(g)| \leq 2$ and since N has no index-2 subgroups in this case, we deduce $g \in \mathbf{C}_G(N)$ and hence $C := \mathbf{C}_G(N)$ is a normal subgroup of G not acting semiregularly on $\mathbf{V}\Gamma$.

Suppose $v^N \subseteq v^C$. Then, for every $n \in N$, there exists $c \in C$ with $v^{nc} = v$, that is, $nc \in G_v$. Since n and c commute, the order $o(nc)$ of nc equals $\text{lcm}\{o(n), o(c)\}$. Since G_v is a $\{2, 3\}$ -group, we thus see that $o(nc)$ is a power of 2 times a power of 3. Thus N is a $\{2, 3\}$ -group. From Burnside's $p^\alpha q^\beta$ -theorem, N is solvable and hence elementary abelian, contradicting our assumption.

We may thus assume that $v^N \not\subseteq v^C$. Observe that $G_v^{\Gamma(v)}$ is a primitive group, implying that $C_v^{\Gamma(v)}$ is either transitive or trivial. In the latter case, it follows that $C_v = 1$ contradicting the fact that $g \in C_v$. Hence C_v acts transitively on $\Gamma(v)$, implying that C is either transitive on $\mathbf{V}\Gamma$, or Γ is bipartite with bipartition given by the orbits of C on $\mathbf{V}\Gamma$. As $v^N \not\subseteq v^C$, we have $v^C \neq \mathbf{V}\Gamma$ and hence C is not transitive on $\mathbf{V}\Gamma$; thus Γ is bipartite with bipartition given by the C -orbits. As $v^N \not\subseteq v^C$, N contains permutations interchanging the two parts of the bipartition of Γ . Thus N contains a subgroup having index 2, which is a contradiction because N is not a 2-group. \square

Let us now assume that Theorem 1.1 or Theorem 1.2 fails due to a 4-valent 2-arc-transitive graph or due to a 3-valent arc-transitive graph, respectively. Let us consider the minimal counterexample, that is, let us work under the following assumption:

Hypothesis 5.2. Let $k \in \{3, 4\}$ and let Γ be a smallest connected k -valent 2-arc-transitive graph not isomorphic to any of the exceptional graphs $\Psi_1, \dots, \Psi_6, \Lambda_1, \dots, \Lambda_6$ or $C(r, s)$ with $1 \leq s \leq 2r/3$, $r \geq 3$, but admitting a non-trivial automorphism fixing more than $1/3$ of the vertices. Among such automorphisms, pick one of smallest order. In view of Theorem 4.2, it follows that such an automorphism fixes an arc (v, w) of Γ . Let G be a smallest 2-arc-transitive subgroup of $\text{Aut}(\Gamma)$ containing g . Since G_{vw} is a 2-group if $k = 3$, and is a $\{2, 3\}$ -group if $k = 4$, we see that the order $o(g)$ of g satisfies $o(g) \in \{2, 3\}$ if $k = 4$ and $o(g) = 2$ if $k = 3$. Since the validity of Theorems 1.1 and 1.2 was checked for the graphs in the census of 4-valent 2-arc-transitive graphs of order at most 768 [24] and the census of 3-valent arc-transitive graphs of order at most 10 000, we assume that $|\mathbf{V}\Gamma| > 768$ if $k = 4$ and $|\mathbf{V}\Gamma| > 10\,000$ if $k = 3$.

Lemma 5.3. *Assuming Hypothesis 5.2, it follows that G has a unique normal subgroup, which is non-abelian, has at most 2 orbits on $V(\Gamma)$ and does not act semiregularly on $V\Gamma$.*

Proof. Suppose that G contains a minimal normal subgroup N having at least 3 orbits on $V\Gamma$. By [36, Theorem 4.1] it then follows that N is semiregular, and Γ/N is 4-valent with G/N acting faithfully as an s -arc-transitive group of automorphisms. By Lemma 1.16, $Ng \in G/N$ is a non-trivial automorphism of Γ/N with $\text{fpr}_{V\Gamma/N}(Ng) > 1/3$. The minimality of Γ now implies that Γ/N is one of the exceptional graphs $\Psi_1, \dots, \Psi_6, \Lambda_1, \dots, \Lambda_6$ or $C(r, s)$ for some r in s . But then, by Lemma 5.1, Γ is not a counterexample to Theorem 1.1 or Theorem 1.2, contradicting Hypothesis 5.2. We have thus shown that every minimal normal subgroup of G has at most two orbits on $V\Gamma$. Moreover, if a minimal normal subgroup N has two orbits, then Γ is bipartite with $\{v^N, w^N\}$ being the bipartition of Γ .

Suppose now that G contains an abelian minimal normal subgroup N . By Lemma 1.14, either $N_v \neq 1$ and there exist two distinct vertices $u, u' \in V\Gamma$ such that $\Gamma(u) = \Gamma(u')$, or $N_v = 1$ and $|V\Gamma| \leq 2|N| \leq 2^{2k-1}$. The latter case contradicts our assumption on the order of Γ , so we may assume that former case happens. If $k = 3$, then Lemma 4.1 yields $\Gamma \cong K_{3,3}$, while if $k = 4$, then it is easy to see that $\Gamma \cong C(r, 1)$ (see [32, Lemma 4.3] for a proof). We may therefore assume that no minimal normal subgroup of G is abelian.

Suppose now that a minimal normal subgroup N of G acts semiregularly on $V\Gamma$. By Lemma 1.18, we see that $|N : \mathbf{C}_N(g)| = 1$ or 2 . If $|N : \mathbf{C}_N(g)| = 2$, then N is abelian, a contradiction. Hence g centralises N and since $g \in G_{vw}$, we see that g fixes every element in $v^N \cup w^N = V\Gamma$. This contradiction shows that none of the minimal normal subgroups of G acts semiregularly on $V\Gamma$.

Suppose now that G contains two distinct minimal normal subgroups N and M . Let K_N and K_M be the kernels of the actions of G on $V\Gamma/N$ and $V\Gamma/M$ respectively. Suppose that $N \leq K_M$. Then $v^N \subseteq v^{K_M} = v^M$. Let $n \in N$ be an element of prime order at least 5. We have $v^n \in v^M$ and hence $v^n = v^m$, for some $m \in M$. This gives $nm^{-1} \in G_v$. Since $o(nm^{-1}) = \text{lcm}\{o(n), o(m)\}$, it follows that G_v contains an element of order divisible by a prime number at least 5. This contradiction shows that $N \not\leq K_M$. This yields that N acts faithfully as a group of automorphisms of the graph Γ/M . However, since M is not semiregular, Γ/M has valency at most 2; thus the automorphism group of Γ/M is soluble and hence so is N . However, this contradicts the fact that N is non-abelian, and thus shows that our initial assumption on the existence of two minimal normal subgroups of G was false. \square

We now continue our analysis under the assumption of Hypothesis 5.2. Let N be the unique minimal normal subgroup of G . Since N is non-abelian, we see that for some non-abelian simple group T we have:

$$(5.1) \quad N \cong T_1 \times T_2 \times \dots \times T_\ell \text{ with } T_i \cong T \text{ for every } i, \text{ and } G \lesssim \text{Aut}(T) \text{ wr Sym}(\ell);$$

where by $X \lesssim Y$ we indicate that X is a group isomorphic to a subgroup of Y . Observe also that $\mathbf{C}_G(N) = 1$. For $h \in G$, let σ_h denote the permutation of $\{1, \dots, \ell\}$ mapping i to j if and only if $(T_i)^h = T_j$. Then $\sigma : G \rightarrow \text{Sym}(\ell)$, $h \mapsto \sigma_h$, is a homomorphism whose kernel equals

$$(5.2) \quad M := G \cap \text{Aut}(T)^\ell \leq \text{Aut}(T) \text{ wr Sym}(\ell).$$

Note that every element $h \in G$ can now be written uniquely as $(y_1, \dots, y_\ell)\sigma_h$ for some $y_1, \dots, y_\ell \in \text{Aut}(T)$. In particular, let $x_1, \dots, x_\ell \in \text{Aut}(T)$ be such that

$$(5.3) \quad g = (x_1, \dots, x_\ell)\sigma_g.$$

Let $K \in \{N, M\}$. Since $K \trianglelefteq G$, we see that $K_v \trianglelefteq G_v$. Moreover, since $K_v \neq 1$, the connectivity of Γ implies that $K_v^{\Gamma(v)}$ is a non-trivial normal subgroup of the 2-transitive group $G_v^{\Gamma(v)}$. Hence $K_v^{\Gamma(v)}$ is transitive. Since G_{vw} is the stabiliser of the action of G_v on $\Gamma(v)$, we thus see that $G_v = G_{vw}K_v$. Since $K_v^{\Gamma(v)}$ is transitive, the quotient Γ/K has valence 0 or 1 and $|K_v : K_{vw}| = k$. In the first case, K is transitive on $A\Gamma$, implying that $G = G_{vw}K$, while in the second case, K is edge-transitive and has two orbits on $A\Gamma$ and $V\Gamma$, the latter forming the bipartition of Γ . In both cases, we see that

$$(5.4) \quad K \text{ is transitive on } E\Gamma,$$

and thus $G = KG_{\{v,w\}}$ with $|G : KG_v| = |G : KG_{vw}| = 1$ or 2 , depending of whether Γ/K has valence 0 or 1, respectively. In particular, since K is contained in the kernel of σ , this implies that

$$(5.5) \quad \sigma(G) = \sigma(G_{\{v,w\}}) \text{ and thus } \sigma(G_{\{v,w\}}) \leq \text{Sym}(\ell) \text{ is transitive.}$$

The structure of the vertex- arc- and edge-stabiliser in a group G acting 2-arc-transitively on a connected k -valent graph with $k \in \{3, 4\}$ was first studied by Tutte in his seminar work [39] for the case $k = 3$, and by Weiss [41] for the case $k = 4$. It follows from their work that $|G_v| \leq 48$ if $k = 3$ and $|G_v| \leq 11\,664$ if $k = 4$. Furthermore, the triples $(G_v, G_{vw}, G_{\{v,w\}})$ were completely determined (up to isomorphism of triples of groups) by Conder and Lorimer in [12] for $k = 3$, and by the first-named author of this paper in [24, Table 1] for $k = 4$. In Table 5.1 we gather some information about these triples that will be frequently used in what follows. In particular, for each of the nine triples, we give the number of elements of order 2 and (if $k = 4$) of order 3 in G_v . In the last column, the information on the minimal order of an element $h \in G_{\{v,w\}} \setminus G_{vw}$ is also provided.

k	type	$ G_v $	$ \{x \in G_v : o(x) = 2\} $	$ \{x \in G_v : o(x) = 3\} $	$o(h)$
3	G_5	48	19		2
3	G_4^1, G_4^2	24	9		2
3	G_3	12	7		2
3	G_2^2	6	3		4
3	G_2^1	6	3		2
4	7-AT	11 664	405	890	2
4	4-AT	432	45	80	2
4	$S_3 \times S_4$	144	39	26	2
4	$C_3 \rtimes S_4^*$	72	21	26	4
4	$C_3 \rtimes S_4$	72	21	26	2
4	$C_3 \times A_4$	36	3	26	2
4	S_4	24	9	8	2
4	A_4 and A_4 s	12	3	8	2

TABLE 5.1. Vertex-stabilisers of groups G acting 2-arc-transitively on connected 4-valent graphs.

With the information provided in Table 5.1 we can now obtain a series of useful bounds. For example, by applying Lemma 1.17 with (G, N, v) in place of (X, X, ω) we see that

$$(5.6) \quad |N : \mathbf{C}_N(g)| \leq |g^G| < 3|g^G \cap G_v| \leq \begin{cases} 3 \cdot 19 = 57 & \text{if } k = 3 \\ 3 \cdot 405 = 1215 & \text{if } k = 4 \text{ and } o(g) = 2 \\ 3 \cdot 890 = 2670 & \text{if } k = 4 \text{ and } o(g) = 3 \end{cases}$$

We now split the analysis into two case, depending on whether $\sigma_g = 1$ (or equivalently, $g \in M$) or not.

SUPPOSE $\sigma_g \neq 1$.

Let κ be the length of a longest cycle in σ_g . In particular, $\kappa = o(g) \in \{2, 3\}$. Without loss of generality, we may assume that $\sigma_g = (12 \cdots \kappa)\sigma'$, for some $\sigma' \in \text{Sym}(\{\kappa + 1, \dots, \ell\})$. Since $g^\kappa = 1$, we see that $x_1 x_2 \cdots x_\kappa = 1$. Now consider the element

$$h := (1, x_1^{-1}, (x_1 x_2)^{-1}, \dots, (x_1 x_2 \cdots x_{\kappa-1})^{-1}, 1, 1, \dots, 1) \in \text{Aut}(T)^\ell,$$

and observe that

$$h^{-1}gh = (1, \dots, 1, x_{\kappa+1}, x_{\kappa+2}, \dots, x_\ell)(12 \cdots \kappa)\sigma'.$$

Replacing the graph Γ with the graph $\Gamma^h := (\text{VF}, (\text{EF})^h)$, the group G with G^h and hence g with g^h , we may assume that $x_1 = x_2 = \cdots = x_\kappa = 1$. A calculation in T^κ gives that $\mathbf{C}_{T^\kappa}((12 \cdots \kappa))$ is the diagonal subgroup $\{(t, \dots, t) \mid t \in T\}$ of T^κ . Thus $|\mathbf{C}_{T^\kappa}((12 \cdots \kappa))| = |T|$ and $|\mathbf{C}_N(g)| \leq |T| \cdot |T|^{\ell-\kappa} = |T|^{\ell-\kappa+1}$. Hence $|N : \mathbf{C}_N(g)| = |T|^\ell / |\mathbf{C}_N(g)| \geq |T|^\ell / |T|^{\ell-\kappa+1} = |T|^{\kappa-1}$. As $|T| \geq 60$, we can now deduce from (5.6) that

$$(5.7) \quad k = 4, \kappa = o(g) = 2, \text{ and thus } |N : \mathbf{C}_N(g)| = |g^N| \leq |g^G| < 1215.$$

Assume that σ has more than one cycle of length 2. Without loss of generality we may assume that $\sigma = (12)(34)\sigma''$, for some $\sigma'' \in \{5, \dots, \ell\}$. As above, replacing g by a suitable $\text{Aut}(T)^\ell$ -conjugate, we may

assume $x_3 = x_4 = 1$. A computation gives $|\mathbf{C}_{T^4}((12)(34))| = |\{(t, t, t', t') \mid t, t' \in T\}| = |T|^2$ and hence $1215 > |N : \mathbf{C}_N(g)| \geq |T|^2 \geq 3600$, which is a contradiction. Thus $\sigma = (12)$,

$$(5.8) \quad g = (1, 1, x_3, \dots, x_\ell)(12) \quad \text{and} \quad |N : \mathbf{C}_N(g)| = |T| |T : \mathbf{C}_T(x_3)| \cdots |T : \mathbf{C}_T(x_\ell)|.$$

Therefore $|T| \leq |N : \mathbf{C}_N(g)| \leq 3 \cdot 405 = 1215$, implying that

$$(5.9) \quad T \in \{\text{Alt}(5), \text{Alt}(6), \text{PSL}_2(7), \text{PSL}_2(8), \text{PSL}_2(11), \text{PSL}_2(13)\}.$$

Let $V := \langle g^x \mid x \in G_{\{v,w\}} \rangle$ and observe that $V \leq G_{vw}$. Let Δ be the graph defined by $V\Delta := \{1, \dots, \ell\}$ and $E\Delta := \{\{r, t\} : (rt) \in \sigma(V)\}$. Since $(rt), (t, s) \in \sigma(V)$ implies $(rs) = (rt)^{(srt)} = (rt)^{(ts)(rt)} \in \sigma(V)$, we see that every connected component of Δ is a complete graph. Let W_1, \dots, W_k be the vertices of the connected components of Δ . Then for each $i \in \{1, \dots, k\}$, the group $\sigma(V)$ contains all the transpositions (rt) with $r, t \in W_i$, implying that $\text{Sym}(W_1) \times \dots \times \text{Sym}(W_k) \leq \sigma(V)$. Now observe that the group $\sigma(G_{\{v,w\}})$ preserves $E\Delta$ and hence $\sigma(G_{\{v,w\}})$ is a subgroup of $\text{Aut}(\Delta)$, which by (5.5) acts vertex-transitively. In particular, Δ is vertex-transitive and thus $|W_i| = m$ for some $m \geq 2$ dividing ℓ and every $i \in \{1, \dots, k\}$. Hence

$$(5.10) \quad \text{Sym}(m)^{\ell/m} \leq \sigma(V) \leq \sigma(G_{vw}) \leq \sigma(G_{\{v,w\}}) \leq \text{Aut}(\Delta) = \text{Sym}(m) \text{ wr } \text{Sym}(\ell/m).$$

Since $|G_{vw}|$ divides $2^2 \cdot 3^6$, this implies that either $m = 3$ and $\ell \in \{3, 6\}$ or $m = 2$ and $\ell \in \{2, 4\}$.

Suppose first that $(m, \ell) = (2, 4)$. Since $\sigma(G_{\{u,v\}})$ is transitive, (5.10) implies that $\sigma(G) = \sigma(G_{\{v,w\}}) = \text{Sym}(2) \text{ wr } \text{Sym}(2) \cong D_4$, and hence $\sigma(V) = \sigma(G_{vw}) = C_2^2$. In particular, $|G_{vw}|$ is divisible by 4, implying that G_v is of type 7-AT, 4-AT or $S_3 \times S_4$. Moreover, the kernel M_{vw} of the restriction of σ to G_{vw} must be a group of odd order. Since M_v is transitive on $\Gamma(v)$, we see that $|M_v| = 4|M_{vw}|$. However, a direct computation shows that if G_v is of type 4-AT or 7-AT, then G_v contains no normal subgroup of order 4 times an odd integer, implying that G_v is of type $S_3 \times S_4$. In view of (5.6), (5.8) and Table 5.1, we see that $|T| |T : \mathbf{C}_T(x_3)| |T : \mathbf{C}_T(x_4)| = |N : \mathbf{C}_N(g)| \leq 3 \cdot 39$ and thus $T \cong \text{Alt}(5)$ and $x_3 = x_4 = 1$. Hence $T^4 \leq G \leq \text{Aut}(T) \text{ wr } D_4$, with $T = \text{Alt}(5)$, $\sigma(G) = D_4$, and $|g^G| < 3 \cdot 39$ with $g = (12)$. We have checked with MAGMA [5], that no such group G exists.

Suppose now that $(m, \ell) = (2, 2)$. Then $\sigma(G_{vw}) = \sigma(G_{\{v,w\}}) = \text{Sym}(2)$, $T^2 \leq G \leq \text{Aut}(T) \text{ wr } \text{Sym}(2)$ with T as in (5.9), $\sigma(G) = \text{Sym}(2)$, $g = (12)$ and $|g^G| < 1215$. If G_v is of type 7-AT, then $2^4 \cdot 3^6 = 11\,664 = |G_v|$ divides $|G|$, which in turn divides $2|\text{Aut}(T)|^2$. By inspecting the groups in (5.9), we see that only $T = \text{PSL}(2, 8)$ satisfies this condition. A computer assisted computation showed that in this case there are two groups G satisfying the above conditions, however none of the contains a subgroup isomorphic to the vertex-stabiliser of type 7-AT. Hence G_v is not of type 7-AT. But then, in view of (5.6) and Table 5.1, we have $|g^G| < 3 \cdot 45$. Checking the groups in (5.9) and all the groups G satisfying $T^2 \leq G \leq \text{Aut}(T) \text{ wr } \text{Sym}(2)$ and $\sigma(G) = \text{Sym}(2)$, we see that $|g^G| < 3 \cdot 45$ holds only when $T = \text{Alt}(5)$ with $|g^G| = 60$ or 120 , implying that G_v is of type $C_2 \times S_4$, $C_2 \times S_4^*$, $S_3 \times S_4$ or 4-AT. In particular, $|G_v| \geq 72$, and since $|G| \leq 2|\text{Sym}(5)|^2 = 28\,000$, we see that $|\text{VT}| \leq 400$. However, all 2-arc-transitive graphs of order at most 512 are known (see [24]) and it can be easily checked that none of these graphs, with the exception of Ψ_1, \dots, Ψ_6 and $C(4, s)$ with $s \in \{1, 2\}$, has a non-trivial automorphism fixing more than $1/3$ of the vertices.

Suppose now that $(m, \ell) = (3, 3)$. Then

$$(5.11) \quad \sigma(G) = \sigma(G_{\{v,w\}}) = \text{Sym}(3), \quad T^3 \leq G \leq \text{Aut}(T) \text{ wr } \text{Sym}(3), \quad T \text{ as in (5.9), and } g = (1, 1, x_3)(12).$$

If $x_3 \neq 1$, then in view of (5.8), we have $60 \leq |T| < |G_v|/|T : \mathbf{C}_T(x_3)|$. By inspecting the centralisers of involutions of the simple group in (5.9), we see that $T = \text{Alt}(5)$ and G_v is of type 7-AT. However, $|\text{Aut}(T) \text{ wr } \text{Sym}(3)|$ is not divisible by $|G_v| = 11\,664$ in this case, yielding a contradiction. Hence $x_3 = 1$ and thus $g = (12)$. If G_v is of type 7-AT, then the divisibility condition $|G_v| \mid |\text{Aut}(T) \text{ wr } \text{Sym}(3)|$ yields $T \in \{\text{Alt}(6), \text{PSL}(2, 8)\}$. If $T = \text{Alt}(6)$, then no group G satisfying (5.11) is such that $|g^G| \leq 1215$. If $T = \text{PSL}(2, 8)$, then there are 25 groups G satisfying (5.11), with the minimum value of $|g^G|$ being 1080. Now observe that g is not a square of any element in $\text{Aut}(T) \text{ wr } \text{Sym}(3)$. A direct inspection of the vertex-stabiliser of type 7-AT reveals that there are only 324 involutions in G_v that are non-squares, implying that $|g^G \cap G_v| \leq 3 \cdot 324$, which contradicts the fact that $1080 \leq |g^G| \leq |g^G \cap G_v|$. Hence G_v is not of type 7-AT. By (5.6) and Table 5.1, it follows that $|g^G| \leq 3 \cdot 45$ and $|T| = |N : \mathbf{C}_N(g)| \leq 3|g^G \cap G_v| \leq 3 \cdot 45$, forcing $T = \text{Alt}(5)$. However, direct computation shows that no group G satisfying (5.11) such that $|g^G| \leq 3 \cdot 45$ exists in this case.

Suppose finally that $(m, \ell) = (3, 6)$. Then $\sigma(G_{uv})$ contains a subgroup isomorphic to $\text{Sym}(3) \times \text{Sym}(3)$. Inspecting the orders of the arc-stabilisers in Table 5.1, we see that $(G_v, G_{vw}, G_{\{v,w\}})$ is of type 7-AT, 4-AT or

$S_3 \times S_4$ and that $\sigma(V) = \sigma(G_{vw}) = \text{Sym}(3) \times \text{Sym}(3)$. Similarly as in the case $(m, \ell) = (2, 4)$, we see that M_{vw} has odd order and thus G_v contains a subgroup of order 4 times an odd number, which rules out the types 4-AT and 7-AT. But then, in view of (5.8), we see that $|N : \mathbf{C}_N(g)| = |T||T : \mathbf{C}_T(x_3)| \cdots |T : \mathbf{C}_T(x_6)| \leq 3 \cdot 39$, implying that $T \cong \text{Alt}(5)$ and $x_3 = \dots = x_6 = 1$. Now let h be an element of minimal order in $G_{\{v,w\}} \setminus G_{vw}$. According to Table 5.1, we see that $o(h) = 2$. Consider the group $L := \langle M, g, h \rangle$. Since M is transitive on $\text{E}\Gamma$ (see (5.4)) and since h swaps the arc (v, w) , we see that L is an arc-transitive subgroup of G containing g . By Theorem 2.4, L is 2-arc-transitive, and by Hypothesis 5.2, it follows that $G = L$. Now, since $G_{\{v,w\}} = G_{vw}\langle h \rangle$ and since $\sigma(G_{\{v,w\}})$ is transitive on $\{1, \dots, 6\} = \text{V}\Delta$, we see that $\sigma(h)$ swaps the two connected component W_1 and W_2 of Δ . By construction, one connected component of Δ contains the vertices 1 and 2, and without loss of generality, we may assume that $W_1 = \{1, 2, 3\}$ and $W_2 = \{4, 5, 6\}$ and hence that $\sigma(h) = (1\ 4)(2\ 5)(3\ 6)$. But then we see that $\langle T_1, T_2, T_4, T_5 \rangle$ is normalised by M, g and h and thus by $G = \langle M, g, h \rangle$, which contradict the assumption that N is a minimal normal subgroup of G .

SUPPOSE $\sigma_g = 1$.

Then $g = (x_1, x_2, \dots, x_\ell) \in M$, where M is as in (5.2). Let h be an element of $G_{\{v,w\}} \setminus G_{vw}$ of minimal possible order. From the information given in Table 5.1, it follows that $o(h) \in \{2, 4\}$; moreover, $o(h) = 4$ if and only if $k = 4$ and $(G_v, G_{vw}, G_{\{v,w\}})$ is of type $C_3 \rtimes S_4^*$, or $k = 3$ and $(G_v, G_{vw}, G_{\{v,w\}})$ is of type G_2^2 . Now observe that $\langle M, h \rangle = M\langle h \rangle \leq G$ acts arc-transitively on Γ . Since G is a smallest arc-transitive group of Γ containing the element g , it follows that $G = M\langle h \rangle$. Since M is the kernel of the homomorphism $\sigma : G \rightarrow \text{Sym}(\ell)$, we see that $\langle h \rangle = \sigma(\langle h \rangle) = \sigma(G) = \sigma(G_{\{u,v\}})$, which is by (5.5) a transitive subgroup of $\{1, \dots, \ell\}$. Hence $\ell \in \{1, 2, 4\}$.

For a finite simple group X , embedded as the group of inner automorphisms into $\text{Aut}(X)$, and an integer $r \geq 2$ such that X , (respectively, $\text{Aut}(X)$) contains an element of order r , let

$$\begin{aligned} \iota(X, r) &:= \min\{|X : \mathbf{C}_X(x)| : x \in X, o(x) = r\}; \\ \iota_*(X, r) &:= \min\{|X : \mathbf{C}_X(x)| : x \in \text{Aut}(X), o(x) = r\}; \\ m(X) &:= \min\{|X : H| : H \leq X, H \neq X\}. \end{aligned}$$

Note that $m(X) \leq \iota_*(X, r) \leq \iota(X, r)$ and that $m(X)$ equals the minimal degree of a faithful transitive permutation representation of X .

Now observe that $|N : \mathbf{C}_N(g)| = |T : \mathbf{C}_T(x_1)| |T : \mathbf{C}_T(x_2)| \dots |T : \mathbf{C}_T(x_\ell)|$. Let $\alpha := \{i \in \{1, \dots, \ell\} : x_i \neq 1\}$ and observe that $\alpha \geq 1$. Inequality (5.6) and Table 5.1 now imply that

$$(5.12) \quad m(T)^\alpha \leq \iota_*(T, o(g))^\alpha < |g^G \cap G_v| \leq \begin{cases} 57; & \text{if } k = 3; \\ 1215; & \text{if } k = 4 \text{ and } o(g) = 2; \\ 2670; & \text{if } k = 4 \text{ and } o(g) = 3. \end{cases}$$

The values of $m(T)$ for finite simple groups T are known and can be found, for example, in [16, Table 4] for the groups of Lie type (this table takes in account the corresponding table in [19, Table 5.2A] together with the corrections of Mazurov and Vasil'ev in [40]) and in [42] or [43], for sporadic groups. In Table 5.2, containing all non-abelian simple groups T with $m(T) < 2670$, we summarise the relevant information; note that the last two columns give a condition for the group in the corresponding row satisfies $m(T) < 2670$ and $m(T) < 117$, respectively (the meaning of the bound 117 will become apparent later). We will now consider the possible values of ℓ case by case and show that cases $\ell = 4$ and $\ell = 2$ lead to a contradiction.

Suppose first that $\ell = 4$. Recall that in this case $o(h) = 4$ and $(G_v, G_{vw}, G_{\{v,w\}})$ is of type $C_3 \rtimes S_4^*$ if $k = 4$ or of type G_2^2 if $k = 3$. If $k = 3$, then $m(T) \leq \iota_*(T, 2) \leq 8$, implying that T embeds into $\text{Sym}(n)$ for some $n \in \{5, 6, 7, 8\}$. But then T embeds into $\text{Sym}(m)$ for $m \leq 8$. Hence either $T = \text{Alt}(n)$ for $n \in \{5, \dots, 8\}$ or $T = \text{PSL}(3, 2)$. However, a closer inspection of these groups shows that none of them satisfies $\iota_*(T, 3) \leq 8$. We may therefore assume that $k = 4$ and that $(G_v, G_{vw}, G_{\{v,w\}})$ is of type $C_3 \rtimes S_4^*$. From the information given in [24, Table 1], we see that $|G_v| = 2^3 \cdot 3^2$, $|G_{vw}| = 2 \cdot 3^2$, $|G_{\{v,w\}}| = 2^2 \cdot 3^2$, the Sylow 3-subgroup P of G_{vw} is normal in G_{vw} , $P \cong C_3^2$, and h^2 inverts every element of P . Since $G = MG_{\{v,w\}}$, we see that $G/M \cong G_{\{v,w\}}/(M \cap G_{\{v,w\}}) = G_{\{v,w\}}/M_{\{v,w\}}$. Without loss of generality, $h = (y_1, y_2, y_3, y_4)(1\ 2\ 3\ 4)$ for some $y_i \in \text{Aut}(T)$, implying that $G/M \cong C_4$. But then $|M_{\{v,w\}}| = 3^2$ and since $|M_{\{v,w\}} : M_{vw}| \leq 2$, we see that $M_{\{v,w\}} = M_{vw}$ and thus $M_{vw} = P$; in particular, $o(g) = 3$ and $g^{h^2} = g^{-1}$. Now, $h^2 = (y_4 y_1, y_1 y_2, y_2 y_3, y_3 y_4)(1\ 3)(2\ 4)$, and thus $(x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}) = (x_3^{y_2 y_3}, x_4^{y_3 y_4}, x_1^{y_4 y_1}, x_2^{y_1 y_2})$. Since $g \neq 1$, this implies that at least two of the elements x_1, \dots, x_4 are non-trivial. In view of (5.12) we see that

Group T	$m(T)$	$m(T) < 2670$	$m(T) < 117$
$\text{Alt}(n)$	n	$5 \leq n \leq 2669$	$5 \leq n \leq 116$
$\text{PSL}_2(q)$	$q + 1$	$8 \leq q \leq 2663, q \neq 9, 11$	$8 \leq q \leq 113, q \neq 9, 11$
$\text{PSL}_3(q)$	$(q^3 - 1)/(q - 1)$	$2 \leq q \leq 49$	$2 \leq q \leq 9$
$\text{PSL}_4(q)$	$(q^4 - 1)/(q - 1)$	$3 \leq q \leq 13$	$3 \leq q \leq 4$
$\text{PSL}_5(q)$	$(q^5 - 1)/(q - 1)$	$2 \leq q \leq 5$	$q = 2$
$\text{PSL}_d(2)$	$2^d - 1$	$6 \leq d \leq 11$	$d = 6$
$\text{PSL}_2(11)$	11	true	true
$\text{PSL}_6(3), \text{PSL}_7(3), \text{PSL}_6(4)$	364, 1093, 1365	true	false
$\text{PSP}_4(q)$	$(q^4 - 1)/(q - 1)$	$4 \leq q \leq 13$	$q = 4$
$\text{PSP}_{2m}(2)$	$2^{m-1}(2^m - 1)$	$3 \leq m \leq 6$	$m = 3$
$\text{PSP}_6(3), \text{PSP}_6(4)$	364, 1365	true	false
$\text{PSU}_3(q)$	$q^3 + 1$	$3 \leq q \leq 13, q \neq 5$	$3 \leq q \leq 4$
$\text{PSU}_4(q)$	$(q + 1)(q^3 + 1)$	$2 \leq q \leq 5$	$2 \leq q \leq 3$
$\text{PSU}_3(5)$	50	true	true
$\text{PSU}_5(2), \text{PSU}_6(2), \text{PSU}_5(3)$	165, 672, 2440	true	false
$\text{P}\Omega_{2m}^+(2),$ $\text{P}\Omega_{2m}^-(2)$	$2^{m-1}(2^m - 1)$ $2^m(2^{m-1} - 1)$	$4 \leq m \leq 6$ $4 \leq m \leq 6$	false $m = 4$
$\text{P}\Omega_7(3), \text{P}\Omega_8^-(3), \text{P}\Omega_8^+(3)$	351, 1066, 1080	true	false
$G_2(3), G_2(4)$	351, 416	true	false
${}^2B_2(8),$ ${}^3D_4(2), {}^2B_2(32), {}^2F_4(2)'$	65 819, 1025, 1755	true true	true false
$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$	11, 12, 22, 23, 24	true	true
J_1, McL	266, 275	true	false
J_2, HS	100, 100	true	true
$\text{Co}_2, \text{Co}_3, \text{Suz}, \text{He}$	2300, 276, 1782, 2058	true	false

TABLE 5.2. Simple groups T with $m(T) < 2670$.

$m(T) \leq \iota_*(T, 3)^2 < 3 \cdot 26 = 78$, and hence $\iota_*(T, 3) \leq 8$. However, as we have shown in case $k = 3$, no simple group T satisfies this condition. This contradiction shows that $\ell \neq 4$.

Suppose now that $\ell = 2$. Then $h = (y_1, y_2)(1\ 2)$ for some $y_1, y_2 \in \text{Aut}(T)$, implying that $G_{\{v,w\}}/M_{\{v,w\}} \cong G/M \cong C_2$. Since $g \in M$, the minimality of G then implies that M is not arc-transitive, showing that Γ is bipartite with $\{v^M, w^M\}$ being the bipartition, and that $|M_v| = |M|/|v^M| = |G|/|\text{V}\Gamma| = |G_v|$. In particular, $M_v = G_v$ and M is the kernel of the action of G on the bipartition. Consider the groups $L_1 := M \cap (\text{Aut}(T_1) \times \{1\})$ and $L_2 := M \cap (\{1\} \times \text{Aut}(T_2))$. Note that both L_1 and L_2 are normal in M , that $L_1 \cap L_2 = 1$, and that conjugation by h swaps L_1 with L_2 . Hence $L := \langle L_1, L_2 \rangle \cong L_1 \times L_2$ is a normal subgroup of $G = M \langle h \rangle$. Moreover, since $T_1 \times T_2 = N \leq M$, we see that $T_i \leq L_i$ for $i \in \{1, 2\}$.

Suppose that g is contained in one of the group L_1 or L_2 . Without loss of generality, we may assume that $g \in L_1$, and thus $(L_1)_{vw} \neq 1$. Since L_1 is normal in M and since $v^M \cup w^M = \text{V}\Gamma$, we see that $(L_1)_u \neq 1$ for every $u \in \text{V}\Gamma$. The connectivity of Γ then implies that $(L_1)_u^{\Gamma(u)} \neq 1$, and since $G_u^{\Gamma(u)}$ is primitive, we see that $(L_1)_u^{\Gamma(u)}$ is transitive for every $u \in \text{V}\Gamma$. Hence $\Gamma/L_1 \cong K_2$, implying that $v^{L_1} = v^M$. Therefore $M = L_1 M_v$ and thus $M/L_1 \cong L_1 M_v/L_1 \cong M_v/(L_1)_v$. Since M_v is soluble, so is M/L_1 ; however, M/L_1 contains a subgroup isomorphic to L_2 , which is non-soluble since it contains T_2 .

This contradiction shows that g is contained neither in L_1 nor in L_2 and thus $g = (x_1, x_2)$ with both x_1 and x_2 nontrivial. In view of inequality (5.12) (where we may assume $\alpha \geq 2$) and Table 5.1, we thus see that

$$\begin{aligned} \iota_*(T, 2) &\leq 7 \quad \text{if } k = 3 \\ \iota_*(T, 2) &\leq 34 \quad \text{or } \iota_*(T, 3) \leq 51 \quad \text{if } k = 4 \text{ and } G_v \text{ is of type 7-AT,} \\ \iota_*(T, 2) &\leq 11 \quad \text{or } \iota_*(T, 3) \leq 15 \quad \text{if } k = 4 \text{ and } G_v \text{ is not of type 7-AT.} \end{aligned}$$

We have already seen that no non-abelian simple group T satisfies $\iota_*(T, 2) \leq 7$. We may thus assume that $k = 4$. If G_v is not of type 7-AT, then one can easily use a computer algebra system, such as MAGMA [5], to

check that none of the groups T in Table 5.2 with $m(T) \leq 15$ satisfies the second of the above conditions. Similarly, if G_v is of type 7-AT, then $|G_v| = 11664$ and since $G \leq \text{Aut}(T) \text{ wr } \text{Sym}(T)$, we see that 11664 divides $2|\text{Aut}(T)|^2$. By first checking the groups T in Table 5.2 with $m(T) \leq 51$ against this divisibility condition and then, for the remaining groups, directly computing the values $\iota_*(T, r)$, $r \in \{2, 3\}$, one sees that no groups T satisfying the first of the above conditions exists either. This shows that $\ell \neq 2$.

We may thus assume for the rest of the proof that $\ell = 1$; that is $T \leq G \leq \text{Aut}(T)$ where T is the unique minimal normal subgroup of G and $\mathbf{C}_G(T) = 1$. If $T_v = 1$, then Lemma 1.18 implies that $|T : \mathbf{C}_T(g)| = 1/\text{fpr}_{v,T}(g) < 3$, implying that g centralises T , contradicting the fact that $\mathbf{C}_G(T) = 1$. Since T is normal in G , we thus see that T_v is transitive on $\Gamma(v)$ and $\Gamma/T \cong K_2$ or K_1 . We will now split our analysis depending on the valence of Γ .

Suppose first that $k = 4$. Let $H := \langle T, h \rangle = T\langle h \rangle$ and observe that H is arc-transitive. Moreover, $T = H$ (which happens if $\Gamma/T \cong K_1$) or T has index 2 in H (which happens if $\Gamma/T \cong K_2$). In both cases we have $T_v = H_v$, implying that T_v is isomorphic to one of the nine possible vertex-stabilisers of 4-valent, 2-arc-transitive graphs given in Table 5.1. Now, observe that the vertex-stabiliser of type 4-AT or 7-AT contains no proper normal subgroup isomorphic to one of the stabilisers in Table 5.1. This implies that either $T_v = G_v$ (and thus $g \in T_v$ and $|G : T| \leq 2$) or G_v is not of type 4-AT or 7-AT. Having in mind that $g \in T$ implies that the expression $\iota_*(T, o(g))$ in (5.12) can be substituted with $\iota(T, o(g))$ and using the information from Table 5.1, we can now conclude that one of the following holds (here part (b) corresponds to the case when G_v is of type 4-AT and part (c) to the case when G_v is of type 7-AT):

- (a) $\iota_*(T, 2) < 117$ or $\iota_*(T, 3) < 78$;
- (b) $|T|$ is divisible by 432, and $\iota(T, 2) < 135$ or $\iota(T, 3) < 240$;
- (c) $|T|$ is divisible by 11664, and $\iota(T, 2) < 1215$ or $\iota(T, 3) < 2670$.

Non-abelian simple groups T satisfying one of the above conditions can now be determined using purely theoretical argument or in combination with computer assisted computations. For example, for the alternating groups $\text{Alt}(n)$, $n \geq 5$, it is well-known and easy to see that:

$$\iota(\text{Alt}(n), 3) = \iota_*(\text{Alt}(n), 3) = 2 \binom{n}{3}, \quad \iota_*(\text{Alt}(n), 2) = \binom{n}{2}, \quad \iota(\text{Alt}(n), 2) = 3 \binom{n}{4} \text{ for } n \neq 8, \quad \iota(\text{Alt}(8), 2) = 105.$$

From this we see that $\text{Alt}(n)$ satisfies (a) if and only if $5 \leq n \leq 15$, that it never satisfies (b), and that it satisfies (c) if and only if $15 \leq n \leq 16$. To determine the non-alternating groups T satisfying (a), we have considered all the groups T in Table 5.2 satisfying $m(T) < 117$ (see the last column of the table), and then compute the values $\iota_*(T, 2)$ and $\iota_*(T, 3)$ directly with MAGMA. The groups T satisfying conditions (b) and (c) were determined by first checking divisibility conditions on $|T|$ and then checking the bounds on $\iota(T, r)$ directly with MAGMA. This computations resulted in the following list of groups T satisfying at least one of the conditions (a), (b) and (c):

$$\text{Alt}(n) \text{ with } 5 \leq n \leq 16, \text{ PSL}_2(8), \text{ PSL}_2(11), \text{ PSL}_2(13), \text{ PSL}_2(16), \text{ PSL}_2(25), \text{ PSL}_3(2), \text{ PSL}_3(3), \text{ PSL}_4(3), \\ \text{PSU}_3(3), \text{PSU}_4(2), \text{PSU}_4(3), \text{PSU}_5(2), \text{PSU}_6(2), \text{PSp}_6(2), \text{PSp}_6(3), \text{PSp}_{10}(2), \text{P}\Omega_7(3), G_2(3)$$

To deal with these possible groups T and corresponding groups G with $T \leq G \leq \text{Aut}(T)$, consider a chain

$$G_v := X_1 < X_2 < \dots < X_{m-1} < X_m := G$$

such that each X_i , $i \in \{1, \dots, m-1\}$, is a maximal subgroup of X_{i+1} . Let k be the smallest index such that $T \leq X_k$. Then, for each $i \in \{1, \dots, k-1\}$, the action of G by right multiplication on the cosets of X_i in G is faithful and in view of Lemma 1.16, we see that g is a non-trivial permutation of $X_i \backslash G$ with $\text{fpr}_{X_i \backslash G}(g) > 1/3$. This observation allows us to use the following naive algorithm which finishes the proof of Theorem 1.1.

Let T be one of the groups satisfying a condition (a), (b) or (c) and let G be such that $T \leq G \leq \text{Aut}(G)$. Initialise the procedure by letting $\mathcal{Y} := \{G\}$. Now construct a set \mathcal{Z} by going through all the group $Y \in \mathcal{Y}$ and then through all the maximal subgroups M of Y (modulo conjugation in Y). Put M into \mathcal{Z} if and only if either $T \leq M$ or there exists an element $g \in G$ with $o(g) \in \{2, 3\}$ such that $\text{fpr}_{M \backslash G}(g) > 1/3$ (this can be checked by determining the set $M \cap g^G$ of elements in M that are conjugate in G to g , dividing its size by $|g^G| = |G : \mathbf{C}_G(g)|$, and checking if ratio is larger than $1/3$). In the latter case, check if M is isomorphic to a possible vertex-stabiliser of a connected 4-valent 2-arc-transitive graph in [24, Table 1], and if it is, check if any of the orbital graphs of G acting on $M \backslash G$ is a connected 4-valent graph with G acting 2-arc-transitively

on it. If there is such a graph, store it. Finally, we repeat this procedure with \mathcal{Z} in place of \mathcal{Y} , until the set \mathcal{Y} becomes empty.

This computation might seem very time and memory consuming but for most groups T the procedure stops after the first few iteration. The resulting graphs are: $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5$, arising from $T = \text{Alt}(5), \text{Alt}(5), \text{PSL}(3, 2), \text{PSL}(3, 3)$ and $\text{Alt}(7)$, respectively. This finishes the case $k = 4$ and thus proves Theorem 1.1.

Let us now assume that $k = 3$. By (5.12) we see that $m(T) \leq \iota_*(T, 2) \leq 56$, and if G_v is not of type G_5 , then we obtain that $m(T) \leq \iota_*(T, 2) \leq 27$. Similarly as in the case $k = 4$, a computer assisted inspection of the groups in Table 5.2 yields that the only non-abelian simple groups T satisfying $\iota_*(T, 2) \leq 26$ are $\text{Alt}(5), \text{Alt}(6), \text{Alt}(7)$ and $\text{PSL}(3, 2)$. Since $|\text{Aut}(T)|/6 \leq 840$, we see that all graphs arising from a 2-arc-transitive action of G have order at most 840, contradicting our assumption that $|\text{VT}| > 10\,000$. We may thus assume that G_v is of type G_5 , and thus that $|\text{Aut}(T)|$ is divisible by 48 and that $|\text{Aut}(T)|/48 > 10\,000$. The only group from Table 5.2 satisfying these restrictions together with $\iota_*(T, 2) \leq 56$ is $\text{Alt}(11)$. Using the algorithm described at the end of the case $k = 4$ reveals that no graph satisfying Hypothesis 5.2 arises in this case. This completes the proof of Theorem 1.2.

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