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Scaling limit of wetting models in 1+1 dimensions pinned to a shrinking strip

Jean-Dominique Deuschel[a], Tal Orenshtein[a,b]

[a] Technische Universität Berlin, Strasse des 17. Juni 135, 10623 Berlin, Germany
[b] Humboldt Universität Berlin, Unter den Linden 6, 10117 Berlin, Germany

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Abstract

We consider wetting models in 1+1 dimensions with a general pinning function on a shrinking strip. We show that under a diffusive scaling, the interface converges in law to the reflected Brownian motion, whenever the strip size is $o(N^{-1/2})$ and the pinning function is close enough to the critical value of the so-called δ-pinning model of Deuschel–Giacomin–Zambotti [10]. As a corollary, the same result holds for the constant pinning strip wetting model at criticality with order $o(N^{-1/2})$ shrinking strip.

Keywords: δ-pinning model; Strip-wetting model; Entropic repulsion; Interface model; Zero-set; Markov renewal process

1. Introduction

1.1. The standard wetting model

Let $(S_k)_{k=0,1,\ldots}$ be a random walk with increments $S_k - S_{k-1}$, $k \geq 1$, which are i.i.d with law $\mathbb{P}$. We assume that $\mathbb{P}$ has a continuous probability density of the form $\rho(x) = \frac{1}{\kappa} e^{-V(x)}$, so that κ is a normalizing constant, V is symmetric and strictly convex (in the sense that V in C^2 and $V''(x) \in [1/c, c]$ for some $c > 1$). As a result

$$\rho(\cdot) \text{ is symmetric and monotonically decreasing on the positive half line.} \qquad (1)$$

E-mail addresses: deuschel@math.tu-berlin.de (J.-D. Deuschel), orenshtein@tu-berlin.de (T. Orenshtein).

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Symmetry then implies that $\mathbb{E}[S_1] = 0$. We assume further that $\mathbb{E}[S_1^2] = 1$. Denote by $\mathbb{P}_x$ the law of S, starting at $x \in \mathbb{R}$, and let $\mathbb{E}_x$ be the corresponding expectation function. For ease of notation we let $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$.

As a convention, throughout the paper expressions of the form $\mathbb{P}_x[A, S_N = y] = \mathbb{E}_x[\mathbb{1}_A \mathbb{1}_{\{y\}}(S_N)]$, are to be read as the density of S_N at y with respect to the measure $\mathbb{P}_x$ on the event A. More explicitly, for a random variable Y,

$$\mathbb{E}_x[Y \mathbb{1}_{\{y\}}(S_N)] := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}_x[Y \mathbb{1}_{[y,y+\epsilon]}(S_N)]. \tag{2}$$

The standard wetting model, also called the δ-pinning model, was introduced in [10]. It is a measure on $\mathbb{R}_+^N$ where two possible boundary conditions are considered, *free* and *constraint*. The constraint case is defined by

$$P_{\beta,N}^c(\mathrm{d}x) = \frac{1}{Z_{\beta,N}^c} \exp\left(-\sum_{i=1}^N V(x_i - x_{i-1})\right) \prod_{i=1}^{N-1} \left(\mathrm{d}x_i \mathbb{1}_{[0,\infty)}(x_i) + e^\beta \delta_0(\mathrm{d}x_i)\right), \tag{3}$$

where $x_0 = x_N = 0$. Analogously, the free case is defined by

$$P_{\beta,N}^f(\mathrm{d}x) = \frac{1}{Z_{\beta,N}^f} \exp\left(-\sum_{i=1}^N V(x_i - x_{i-1})\right) \prod_{i=1}^{N} \left(\mathrm{d}x_i \mathbb{1}_{[0,\infty)}(x_i) + e^\beta \delta_0(\mathrm{d}x_i)\right), \tag{4}$$

where $x_0 = 0$. Here $\mathrm{d}x_i$ is the Lebesgue measure on $\mathbb{R}$, and the partition functions $Z_{\epsilon,N}^c$ and $Z_{\epsilon,N}^f$ are normalizing constants so that $P_{\beta,N}^c$ and $P_{\beta,N}^f$ are probability measures on $\mathbb{R}_+^N$.

A remarkable localization transition was proved in [10] using a renewal structure naturally corresponding to the model. On the heuristic level, conditioning on the contact set, the excursions from zeros are independent and their law is independent of the pinning parameter. Hence one expects to see that under the conditioning, the (appropriately rescaled) excursions converge to the Brownian excursions. To analyze the full path one therefore needs to understand the contact set distribution. Whenever N is large, the contact set looks like a renewal process with inter-arrival distribution expressed in terms of the Green function of the walk.

In particular, making the above intuition accurate and quantitative, in [10] (and tailored for renewal theory techniques in [5]) the authors proved that there exists some $\beta_c \in \mathbb{R}$, explicitly defined in (5) below, so that under the standard diffusive scaling and interpolation to continuous paths on $[0, 1]$ the following limit in distribution holds, with the following laws:

- For $\beta < \beta_c$, the Brownian meander (free case) or the Brownian excursion (constrained case).
- For $\beta > \beta_c$, a mass-one measure on the constant zero function.
- For $\beta = \beta_c$, the reflecting Brownian motion (free case) or the reflecting Brownian bridge (constrained case).

Moreover, β_c is explicit in terms of the random walk density ρ. In particular,

$$e^{-\beta_c} = \sum_{n=1}^\infty f_n, \tag{5}$$

where $f_n := \mathbb{P}_0[\mathcal{C}_n, S_n = 0]$ is the density of S_n at zero on the event $\mathcal{C}_n = \{S_1 \geq 0, \dots, S_n \geq 0\}$ (remember (2)). We remark already at this stage that

$$f_n = \frac{1}{\sqrt{2\pi}} n^{-3/2} + o(n^{-3/2}) \tag{6}$$

and moreover, in the Gaussian case $V(x) = \frac{1}{2}x^2$, the error term is identically zero [10, Lemma 1] (see also (11) and a few lines below it) and in particular $\beta_c = \log\left(\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty} n^{-3/2}\right)$.

1.2. The strip wetting model with general pinning function

The strip wetting model is the analogous family of measures on $\mathbb{R}_0^N$ which we now define. Fix a one-parameter family of functions $\{\varphi_a, a \in (0, a_0]\}$, so that $\varphi_a : \mathbb{R}_+ \to \mathbb{R}$ and $\int_0^a e^{\varphi_a(x)}\mathrm{d}x$ is finite for $0 < a \leq a_0$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{C}_N$ be the event $\{S_1 \geq 0, \ldots, S_N \geq 0\}$. We shall define now $\mathbb{P}^\alpha_{\varphi_a,N}$ for $\alpha \in \{c, f\}$. Whenever we would like to emphasize the pinning functions we also call them *the φ_a-wetting model*. The case of free boundary conditions is defined by the Radon–Nikodym derivative

$$\mathrm{d}\mathbb{P}^f_{\varphi_a,N}(S) = \frac{1}{\mathcal{Z}^f_{\varphi_a,N}} \exp\left(\sum_{n=1}^{N} \varphi_a(S_n)\mathbb{1}_{[0,a]}(S_n)\right) \mathbb{1}_{\mathcal{C}_N}\mathrm{d}\mathbb{P}(S), \tag{7}$$

while the constraint case is defined by the Radon–Nikodym derivative

$$\mathrm{d}\mathbb{P}^c_{\varphi_a,N}(S) = \frac{1}{\mathcal{Z}^c_{\varphi_a,N}} \exp\left(\sum_{n=1}^{N} \varphi_a(S_n)\mathbb{1}_{[0,a]}(S_n)\right) \mathbb{1}_{[0,a]}(S_N)\mathbb{1}_{\mathcal{C}_N}\mathrm{d}\mathbb{P}(S). \tag{8}$$

The normalizing constants $\mathcal{Z}^f_{\varphi_a,N}$ and $\mathcal{Z}^c_{\varphi_a,N}$ are called the partition functions. When we want to specify the initial and ending points, we also define the density at $y \in \mathbb{R}_+$ by

$$\mathcal{Z}^c_{\varphi_a,N}(x, y) = \mathbb{E}_x\left[\exp\left(\sum_{n=1}^{N} \varphi_a(S_n)\mathbb{1}_{[0,a]}(S_n)\right) \mathbb{1}_{\{y\}}(S_N)\mathbb{1}_{\mathcal{C}_N}\right], x \in \mathbb{R}_+,\ \ N \geq 1, \tag{9}$$

so that

$$\mathcal{Z}^c_{\varphi_a,N} = \int_0^a \mathcal{Z}^c_{\varphi_a,N}(0, y)\mathrm{d}y.$$

1.3. Main results

As mentioned in the introduction this paper deals with strip models approximating the critical standard wetting model in a regularizing way. The regularization is due to the fact that we allow the pinning functions φ_a to be smooth. The approximation is due to the fact the strip size a goes to zero with the model size N.

As we shall see in Section 1.6, as an application we prove that the strip wetting model with constant pinning $\beta_c(a_N)$ has the same asymptotic behavior as the critical standard wetting model, whenever the strip size a_N is decaying asymptotically faster than $\frac{1}{\sqrt{N}}$.

We start with some notations. For a path $(S_i)_{i\geq 0}$, let $\tau_0^a = 0$, $\tau_j^a = \inf\{n > \tau_{j-1}^a : S_j \in [0, a]\}$, $\ell_N^a = \sup\{k \leq N : S_k \in [0, a]\}$. Let $\mathcal{A}_N^a = \{\frac{\tau_j}{N} : j \leq \ell_N^a\} \subset [0, 1]$ be the zero-set up to time N. Define now for $A = \{t_1, \ldots, t_{|A|}\}$, $0 =: t_1 < \cdots < t_{|A|} \leq N$,

$$\tilde{\mathbf{p}}^\alpha_{\varphi_a,N}(\mathcal{A}_N^a = A/N) := \mathbb{P}^\alpha_{\varphi_a,N}(\tau_i^a = t_i, i \leq \ell_N^a), \tag{10}$$

and $\tilde{\mathbf{E}}^\alpha_{\varphi_a,N}$, $\alpha \in \{c, f\}$, the corresponding expectation. We shall use $\tilde{\mathbf{p}}^c_{\varphi_a,N}(A)$ and $\tilde{\mathbf{p}}^c_{\varphi_a,N}(\mathcal{A}_N^a = A/N)$ with no distinction. Note that by definition $\tilde{\mathbf{p}}^c_{\varphi_a,N}(A) = 0$ whenever $\ell_N^a(A) < N$.

Definition 1.1. We say that $(\varphi_a)_{0<a<a_0}$ satisfies Condition (A) if there is some $C > 0$ so that

$$|\log \int_0^a e^{\varphi_a(x)-\beta_c} \mathrm{d}x| \leq Ca$$

for all $0 < a < a_0$, where β_c was defined in (5).

Remark 1.2. Note that Condition (A) guarantees that for N fixed, the φ_a-wetting model converges weakly to the critical standard wetting model as a tends to 0.

The content of the next theorem is a scaling limit of the contact sets. For that we shall use the Matheron topology on closed real sets [17]. The basic notions can be found in [14, page 209], [10, Chapter 7], and [5, Appendix B].

Definition 1.3. Let B be a standard one-dimensional Brownian motion (resp. bridge from 0 to 1). We call the random set $\{t \in [0, 1] : B_t = 0\}$ the *Brownian motion (resp. bridge) zero-set*.

Theorem 1.4. *Fix some sequence $a_N = o(N^{-1/2})$. Assume that φ_a satisfies Condition (A) from Definition 1.1. Then under $\tilde{\mathbf{p}}^{\alpha}_{\varphi_{a_N},N}$, seen as a probability measure on the Matheron topological space of closed sets of $[0, 1]$, the set $\mathcal{A}_N$ is converging in distribution to the Brownian motion zero-set for $\alpha = f$, and to the Brownian bridge zero-set for $\alpha = c$.*

We also have a full path scaling limit.

$$X_t^{(N)} := \frac{1}{N^{1/2}} X_{\lfloor Nt \rfloor} + \frac{1}{N^{1/2}} (Nt - \lfloor Nt \rfloor)(X_{\lfloor Nt \rfloor+1} - X_{\lfloor Nt \rfloor}).$$

Theorem 1.5. *If $a_N = o(N^{-1/2})$ then the process $(X_t^{(N)})_{t\in[0,1]}$ under $\mathbb{P}^{\alpha}_{\varphi_{a_N},N}$ converges weakly in $C[0, 1]$ to the reflected Brownian motion on $[0, 1]$ for $\alpha = f$ and to the reflected Brownian bridge on $[0, 1]$ for $\alpha = c$.*

Remark 1.6. As it will become clear from the proof of Theorems 1.4 and 1.5, they hold if we weaken Condition (A) from Definition 1.1 so that the constant C is a function of a, $C = C(a)$, as long as $a + aC(a) \to 0$ as $a \to 0$ faster than $N^{-1/2}$. For example if $C(a) = a^{-\epsilon}$ for some $0 < \epsilon < 1$, then the theorems hold whenever the faster shrinking rate $a_N = o(N^{-1/2(1-\epsilon)})$ holds.

1.4. Examples

1.4.1. Constant pinning

We call the model *the strip wetting model with constant pinning* whenever the pinning function is constant on the strip, i.e., for some $\beta = \beta(a) \in \mathbb{R}$ $\varphi_a(x) = \beta$, $x \in [0, a]$.

This model was suggested in Giacomin's monograph [14, Equation (2.57)] as an open problem, and a major progress was done by Sohier [20,21]. Applications of our results in this case are presented in Section 1.6.

1.4.2. Smooth approximation of the critical standard model

We construct functions $\varphi_a \in C^\infty(\mathbb{R})$ supported on $[0, a]$ so that they satisfy Condition (A) from Definition 1.1. Let

$$f(x) := \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases}$$

It is easy to verify that the derivatives of f at 0 vanish and hence it is $C^\infty(\mathbb{R})$. Choose some $\epsilon(a) \to 0$ as $a \to 0$ with the rate of decay to be specified later on and let $g = g_a$ be defined by

$$g(x) = \epsilon(a) + \frac{1}{a} \frac{f(\frac{a-x}{\epsilon(a)})}{f(1 - \frac{a-x}{\epsilon(a)}) + f(1 + \frac{a-x}{\epsilon(a)})}.$$

It is easy to check that $\epsilon(a) \le g(x) \le 1/a + \epsilon(a)$, $g(x) = 1/a + \epsilon(a)$ if $x \le a - \epsilon(a)$, and $g(x) = \epsilon(a)$ if $x \ge a$. Therefore $(1/a+\epsilon(a))(a-\epsilon(a)) \le \int_0^a g(x)\mathrm{d}x \le (1/a+\epsilon(a))a$. Therefore, choosing $\epsilon(a) \le a^2$ then there is some constant $C > 0$ so that for all a small enough

$$e^{-Ca} \le 1 + a\epsilon(a) - \epsilon(a)/a + \epsilon(a)^2 \le \int_0^a g_a(x)\mathrm{d}x \le 1 + a\epsilon(a) \le e^{Ca}.$$

We remark that $\exp(\beta_c) \equiv \sqrt{2\pi} / \sum_{n\ge1} n^{-3/2} \approx 0.961849$. Set $\varphi_a(x) := (\beta_c + \log g_a(x))\, \mathbb{1}_{\mathbb{R}_+}(x)$, $x \in \mathbb{R}$, where $\epsilon(a) = a^2$. See Fig. 1 for a graphical presentation. Then $\varphi_a \in C^\infty([0, a])$ and satisfies Condition (A) from Definition 1.1.

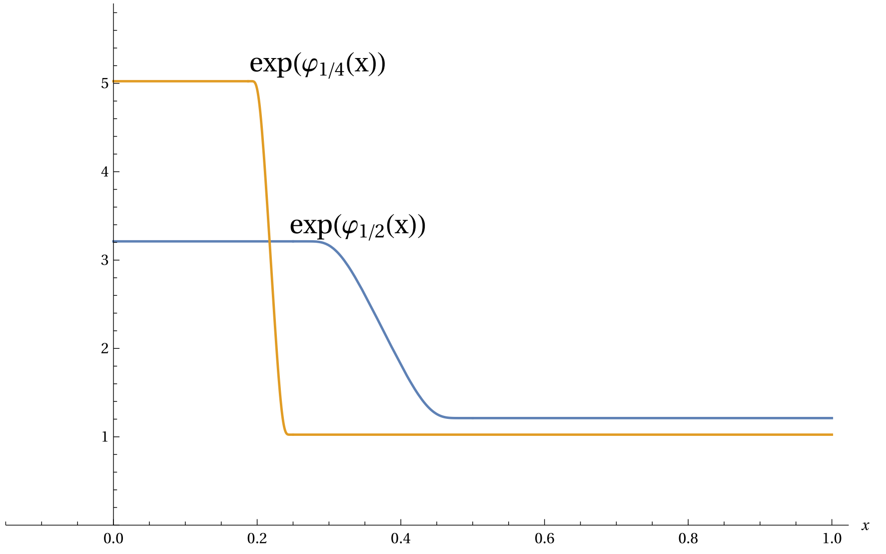

Fig. 1. The graph of $\exp(\varphi_a(x))$, $0 \le x \le 1$, for $a = 1/4$ and $a = 1/2$.

1.5. The critical wetting dynamics problem

Our main motivation for the question discussed in this paper comes from the notorious problem of constructing and studying a continuous dynamics which is reversible with respect to the reflected Brownian motion/bridge. The well-known Nualart–Pardoux type SPDE [18] is a scaling limit of the infinite-dimensional Ornstein–Uhlenbeck process with reflection at zero [13], and is reversible under the Brownian excursion [23]. As mentioned in the introduction the latter is the scaling limit of a subcritical standard wetting model [10].

Therefore a natural candidate for a solution to the problem is the critical wetting dynamics. This process should arise as a scaling limit of the finite volume dynamics which is reversible with respect to the critical wetting measure. However, a canonical dynamics associated with the standard wetting measure is non-trivial due to the presence of an atom in zero. A construction of the dynamics for a given finite size is possible using Dirichlet form techniques [12] (see also Funaki's lecture notes [7, Chapter 15.2]). Taking a limit using this approach seems out of reach. Nevertheless, an integration by parts formula for the reflected Brownian bridge was achieved in [16], where it was formulated in terms of Hida distribution and also a relevant SPDE was conjectured.

Using the result of this paper we can present a straightforward approach based on Skorokhod type equation. More precisely, take $\varphi_a \in C^2$ supported on $[0, a]$, $0 < a < 1$, which satisfies Condition (A) from Definition 1.1 (e.g., the one constructed in Section 1.4.2). Similarly to [13], we can construct a dynamics $X_t(x)$, $t \geq 0$, $x \in I_N := \{0, 1, \ldots, N\}$, for which the measure $\mathbb{P}^c_{\varphi_a,N}$ defined in (8) is a reversible equilibrium. Indeed, consider the Skorokhod type equation

$$X_x(t) = -\int_0^t \partial_x H_N(X(s))\mathrm{d}s + \ell_t(x) + \sqrt{2}W_t(x),\ \ x \in I_N, t \geq 0,$$

with boundary conditions

$$X_0(t) = X_N(t) = 0,\ \ t \geq 0,$$

initial law

$$(X_x(0))_{x\in I_N} \sim \mathbb{P}^c_{\varphi_a,N},$$

so that the local time process ℓ_t satisfies

$$\mathrm{d}\ell_x(t) \geq 0,\ \ t \geq 0, x \in I_N,$$

and

$$\int_0^\infty X_x(t)\mathrm{d}\ell_x(t) = 0,\ \ x \in I_N,$$

$W(x)$, $x \in I_N$, are independent standard Wiener measures, the Hamiltonian

$$H_N(X) := \sum_{x=0}^N \varphi_a(X_x) + \frac{1}{2}\sum_{x=1}^N (X_x - X_{x-1})^2,$$

and

$$\partial_x H_N(X) := \frac{\partial H_N(X)}{\partial X_x} = \varphi_a'(X_x) + \Delta_x(X),$$

where

$$\Delta_x(X) = 2X_x - X_{x-1} - X_{x+1} \text{ whenever } 1 \leq x < N$$

is the discrete Laplacian. Let $X^N(t)$ be the diffusively rescaled and linearly interpolated path given by

$$X^N_y(t) = \frac{1}{N^{1/2}}X_{\lfloor Ny\rfloor}(t) + \frac{1}{N^{1/2}}(Ny - \lfloor Ny\rfloor)(X_{\lfloor Ny\rfloor+1}(t) - X_{\lfloor Ny\rfloor}(t)),\ t \geq 0,\ y \in [0, 1].$$

Theorem 1.5 states that if $a = a_N = o(N^{-1/2})$, then

$$(X^N_y(0))_{y\in[0,1]} \Rightarrow (\beta_y)_{y\in[0,1]} \quad (*)$$

where $(\beta_y)_{y\in[0,1]}$ is the reflected Brownian bridge.

The goal is to get a limiting dynamics as the volume size tends to infinity and simultaneously the strip size tends to zero and then identify the corresponding SPDE which should be the natural reversible dynamics associated with the reflected Brownian bridge.

In a collaboration with Henri Elad Altman we recently proved that

$$\{X^N_y(tN^2),\, y \in [0,1], t \geq 0\}_{N\in\mathbb{N}} \text{ is a tight sequence}$$

in $H^{-1}(0,1)$ [9], thus showing existence of the limit. The reader is invited to consult that last reference for more details on the construction of the dynamics.

Attacking the problem from yet a different angle, a major progress on it was made recently by Elad Altman and Zambotti [11] where they consider a local time mollification of a continuous model to construct the dynamics. They also conjecture the formal (singular) SPDE the dynamics should solve. Showing that any of these models is a solution to the proposed SPDE, in some reasonable sense, is still an open problem.

1.6. Applications to strip wetting with constant pinning at criticality

In order to achieve progress in the open problem of constructing the wetting dynamics which was described in Section 1.5, a natural question would be to choose φ_a to be a constant function on the strip, $\varphi_a \equiv \beta \mathbb{1}_{[0,a]}$. However, in this case the drift term in the dynamics has no derivative at a and therefore an approximating φ_a as in Section 1.4.2 should be taken into account. On the other hand, one might be still interested to understand the constant strip wetting scaling limit per se. In order to use our theorem, condition (A) has to be satisfied. It is easy to see that β independent of a will not work.

Sohier [21] considered the strip wetting model with constant pinning and proved that there is some $\beta_c(a) \in \mathbb{R}$ so that off-criticality, the same path scaling limit results as in the standard wetting model hold true. Namely, in this case the limiting object is

- Brownian meander (free case) or the Brownian excursion (constrained case), whenever $\beta < \beta_c(a)$, and
- a unit mass on the constant zero function, whenever $\beta > \beta_c(a)$.

In particular, he proved also a corresponding statement on the off-critical contact set scaling limits. Moreover, $\beta_c(a)$ is represented in terms of an eigenvalue of a natural Hilbert–Schmidt integral operator, see [21], and Section 5.1.

In [20] the critical contact set with free boundary conditions was considered, for a *fixed size* a of the strip. That paper states that the rescaled contact set converges to a random set with a distribution which is absolutely continuous but not equal to the Brownian motion zero-set. Moreover the Radon–Nikodym derivative is claimed there to be independent of a, which would suggest that the limit as $a \to 0$ is discontinuous.We strongly believe that there is mistake in the argument in Lemma 3.3 of that paper which is the key for proving Theorem 1.5 there. Specifically, the decomposition in lines (70) and (86) is false. E.g., instead of the probability appearing in the second line of (86) one has a probability which is not bounded away from zero (except for the location on the strip, it depends also on the size $N - t_k$ of the last excursion before time N, and vanishes in the limit as $N - t_k \to 0$). Therefore, one cannot use that paper's main result dealing with Markov renewal processes for the sum in (86) in order to bound all of it from below by order $N^{1/2}$. In any case, [20] does not contradict our results, since it deals with a constant strip size a.

The next theorem deals with the critical value $\beta_c(a)$ of the constant pinning model for a small. It states that the critical value β_c of the standard wetting model is well-approximated by $\beta_c(a)$.

Theorem 1.7. *There are constants $C, D > 0$ so that*

$$Da^2 \leq \log a + \beta_c(a) - \beta_c \leq Ca$$

for all $a > 0$ small enough. In particular, the constant function $\varphi_a = \beta_c(a)$ satisfies Condition (A) from Definition 1.1, and moreover $ae^{\beta_c(a)} \to e^{\beta_c}$ as $a \to 0$.

Corollary 1.8. *Theorems 1.4 and 1.5 hold true also for the critical constant pinning models, i.e. whenever $\varphi_a(x) = \beta_c(a)$, $x \in [0, a]$.*

In other words, in the case where $a = a_N = o(N^{-1/2})$ the rescaled path of the strip wetting with critical constant pinning $\beta_c(a)$ converges to reflected Brownian motion. Remark 5.2 for a partial result on the case $a = a_N = O(N^{-1/2})$

Remark 1.9. As discussed in Section 1.5, in this work we are only interested in a wetting model on the strip so that a scaling limit as in Theorem 1.5 is achieved with a smooth pinning function. In particular, the condition on the rate of which the strip size shrinks is not relevant from the SPDE point of view. On the other hand, our results seem to be far from optimal. In particular one can ask what is the optimal rate for which $a \to 0$ as $N \to \infty$ for which we still see the reflected Brownian bridge/motion at criticality, for the constant pinning at criticality $\beta = \beta_c(a)$? We believe that the result stays true even with a *constant* strip size a.

1.7. Organization of the paper

The main argument in the paper is to compare the φ_a-strip wetting model with the critical standard wetting model through a mediator, the near-critical standard wetting model. We first approximate the φ_a-strip wetting model in terms of a near-critical standard wetting. This is the content of Section 2. In Section 3 near-critical standard wetting model is shown to approximate the critical standard wetting models. This is done by the connection of standard wetting models to pinning models on renewal processes and Sohier's result [19] on the latter. Here the $o(N^{-1/2})$ condition appears. In Section 4 we use Condition (A) with the two approximations to prove Theorems 1.4 and 1.5. To conclude, Section 5 deals with the constant pinning case and contains the proof of Theorem 1.7. Appendices A, B, and C contain the proof of some technical Lemmas.

2. Comparing φ_a-wetting model to near-critical wetting model

2.1. Comparing excursion kernels

Define the excursion kernel density

$$f_n^a(x, y) := \mathbb{P}_x[S_1 > a, \ldots, S_{n-1} > a, S_n = y] \tag{11}$$

for $n \geq 2$, $x, y \in \mathbb{R}$, where $f_1^a(x, y) = \mathbb{P}_x[S_1 = y]$ (remember the notation from (2)). Let

$$f_n^a := f_n^a(0, 0),$$

and we omit the upper-case a whenever $a = 0$, that is

$$f_n = f_n^0.$$

The first observation is that the f_n^a approximate the corresponding f_n.

Lemma 2.1. *The following hold:*

- $f_n^a(\cdot, \cdot)$ *is symmetric:* $f_n^a(x, y) = f_n^a(y, x)$ *for all* $x, y \in [0, a]$, $n \geq 1$.
- $f_n^a(x, y)$ *is monotonously increasing in* $x \in [0, a]$ *(and in* $y \in [0, a]$*).*
- $f_n^a(a, a) = f_n$.

In particular,

$$\frac{f_n^a}{f_n} \leq \frac{f_n^a(x, y)}{f_n} \leq 1 \tag{12}$$

for all $x, y \in [0, a]$ *and* $n \geq 1$. *Moreover,* $\frac{f_n^a}{f_n}$ *decreases in* a *and tends to 1 as* $a \to 0$, *for all* n.

Proof. For the first two properties, one uses the assumptions (1) on the following explicit expression for the densities

$$f_{n+1}^a(x, y) = \int_a^\infty \dots \int_a^\infty \rho(s_1 - x)\rho(s_2 - s_1) \cdots \rho(s_n - s_{n-1})\rho(y - s_n)\mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

The last property follows, e.g., by the change of variables $s_i \to s_i + a, i = 1, \dots, n$. □

The main goal of this section is to estimate f_n^a in terms of f_n and a. The next lemma actually supplies upper and lower bounds, but for the results of the paper we shall only use the lower bound.

The next lemma is crucial for the argument. Its proof is rather technical and differed to Appendix A.

Lemma 2.2. *There are constants* C_0, C_1 *and* $0 < a_0$ *so that for all* $0 \leq a \leq a_0$ *and* $n \geq 1$

$$\exp(-C_0 a) \leq f_n^a / f_n \leq \exp(-C_1 a). \tag{13}$$

2.2. *Comparing the partition functions*

Lemma 2.3. *Fix* φ_a *and assume Condition (A) from Definition 1.1 with the constant* C. *Then, there is a constant* C' *and a positive decreasing function* $C'(a)$ *so that* $C'(a) \to 1$ *as* $a \to 0$, *and for all* $N \geq 1$ *we have*

$$Z^c_{\beta_c - C'a, N} \leq \mathcal{Z}^c_{\varphi_a, N} \leq Z^c_{\beta_c + C'a, N}, \tag{14}$$

and

$$C'(a) Z^f_{\beta_c - C'a, N} \leq \mathcal{Z}^f_{\varphi_a, N} \leq Z^f_{\beta_c + C'a, N}. \tag{15}$$

Proof. We start with the constraint case.

$$\mathcal{Z}^c_{\varphi_a,N}(0,y)=\sum_{k=0}^{N-1}\sum_{0=t_0<t_1<\cdots<t_k<N}\int_0^a\cdots\int_0^a\prod_{i=1}^k f^a_{t_i-t_{i-1}}(y_{i-1},y_i)e^{\varphi_a(y_i)}f^a_{N-t_k}(y_k,y)$$
$$\times\ e^{\varphi_a(y)}\mathrm{d}y_i=:(*).$$

Using (12) and Condition (A) we have the following upper bounds.

$$\begin{aligned}(*)&\le\sum_{k=0}^{N-1}\sum_{0=t_0<t_1<\cdots<t_k<N}\int_0^a\cdots\int_0^a f_{N-t_k}\prod_{i=1}^k f_{t_i-t_{i-1}}e^{\varphi_a(y_i)}e^{\varphi_a(y)}\mathrm{d}y_i\\&=e^{\varphi_a(y)}\sum_{k=0}^{N-1}\Big(\int_0^a e^{\varphi_a(z)}\mathrm{d}z\Big)^k\sum_{0=t_0<t_1<\cdots<t_k<N}f_{N-t_k}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&\le e^{\varphi_a(y)}\sum_{k=0}^{N-1}e^{(\beta_c+Ca)k}f_{N-t_k}\sum_{0=t_0<t_1<\cdots<t_k<N}\prod_{i=1}^k f_{t_i-t_{i-1}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{Z}^c_{\varphi_a,N}=\int_0^a\mathcal{Z}^c_{\varphi_a,N}(0,y)\mathrm{d}y&\le\int_0^a e^{\varphi_a(y)}\mathrm{d}y\sum_{k=0}^{N-1}e^{(\beta_c+Ca)k}\sum_{0=t_0<t_1<\cdots<t_k<N}f_{N-t_k}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&\le\sum_{k=0}^{N-1}e^{(\beta_c+Ca)(k+1)}\sum_{0=t_0<t_1<\cdots<t_k<N}f_{N-t_k}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&=\sum_{k=1}^{N}e^{(\beta_c+Ca)k}\sum_{0=t_0<t_1<\cdots<t_k=N}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&=Z^c_{\beta_c+Ca,N}.\end{aligned}$$

Similarly for the lower bound, using (13) and Condition (A), we get

$$(*)\ge e^{\varphi_a(y)}\sum_{k=0}^{N-1}e^{(\beta_c-Ca-C_0a)k}f_{N-t_k}\sum_{0=t_0<t_1<\cdots<t_k<N}\prod_{i=1}^k f_{t_i-t_{i-1}}$$

Hence,

$$\begin{aligned}\mathcal{Z}^c_{\varphi_a,N}=\int_0^a\mathcal{Z}^c_{\varphi_a,N}(0,y)\mathrm{d}y&\ge\int_0^a e^{\varphi_a(y)}\mathrm{d}y\sum_{k=0}^{N-1}e^{(\beta_c-Ca-C_0a)k}\\&\quad\times\sum_{0=t_0<t_1<\cdots<t_k<N}f_{N-t_k}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&\ge\mathrm{d}y\sum_{k=1}^{N}e^{(\beta_c-Ca-C_0a)k}\sum_{0=t_0<t_1<\cdots<t_k=N}\prod_{i=1}^k f_{t_i-t_{i-1}}\\&=Z^c_{\beta_c-(C+C_0)a,N}.\end{aligned}$$

Since $\mathcal{Z}^c_{\varphi_a,N}=\int_0^a\mathcal{Z}^c_{\varphi_a,N}(0,y)\mathrm{d}y$, setting $C'=C+C_0$ we conclude the two bounds.

The free case is done in a similar manner. Indeed summing over the last contact before time N, we have

$$\mathcal{Z}^f_{\varphi_a,N} = \sum_{k=0}^{N} \int_0^a \mathcal{Z}^c_{a,\varphi_a,k}(0,y) P^a_y(N-k)\mathrm{d}y =: (*).$$

Using (35), the line before it, Condition (A), and the constraint case we have the following upper bound.

$$\begin{aligned}(*) &\le \sum_{k=0}^{N} P(N-k) \int_0^a \mathcal{Z}^c_{a,\varphi_a,k}(0,y)\mathrm{d}y \\ &\le \sum_{k=0}^{N} P(N-k) Z^c_{\beta_c+Ca,k} \\ &= Z^f_{\beta_c+Ca,N}.\end{aligned}$$

Similarly for the lower bound, using (35), the line before it, (13) and Condition (A), we get

$$(*) \ge C^a(0)e^{-Ca} Z^f_{\beta_c-(C+C_0)a,N}.$$

Setting $C'(a) := C^a(0)e^{-Ca}$, we are done. □

2.3. *Derivative of φ_a-strip wetting with respect to near-critical standard wetting*

In this section we shall discuss the contact set distribution, and show that the φ_a-strip wetting is approximated by a near-critical standard wetting model. By near-criticality we mean a linear perturbation by a constant multiple of the strip-size of the critical pinning strength.

Remember the definition in (10) with the notations above it. We introduce the analog for the standard wetting model.

$$\mathbf{p}^\alpha_{\beta,N}(\mathcal{A}_N = A/N) := \mathbb{P}^\alpha_{e^\beta,N}(\tau_i = t_i, i \le \ell_N), \tag{16}$$

and $\mathbf{E}^\alpha_{\beta,N}$, $\alpha \in \{c, f\}$, the corresponding expectation. Here as well, with a slight abuse of notation we use $\mathbf{p}^c_{\beta,N}(A)$ and $\mathbf{p}^c_{\beta,N}(\mathcal{A}_N = A/N)$ with no distinction. Note again that by definition $\mathbf{p}^c_{\beta,N}(A) = 0$ whenever $\ell_N(A) < N$.

Lemma 2.4. *Assume φ_a satisfies Condition (A) from Definition 1.1 with the constant C. Remember the definitions from (10). There are some constants c_i, $i = 1, \dots, 6$, so that for $\alpha \in \{c, f\}$*

$$\frac{d\tilde{\mathbf{p}}^\alpha_{\varphi_a,N}}{d\mathbf{p}^\alpha_{\beta_c+c_3a,N}} \le \frac{Z^\alpha_{\beta_c+c_1a,N}}{C^\alpha(a) Z^\alpha_{\beta_c-c_2a,N}}$$

and

$$\frac{d\tilde{\mathbf{p}}^\alpha_{\varphi_a,N}}{d\mathbf{p}^\alpha_{\beta_c-c_6a,N}} \ge \frac{Z^\alpha_{\beta_c-c_4a,N}}{Z^\alpha_{\beta_c+c_5a,N}}.$$

Here $C^c(a) = 1$ and $C^f(a) = C'(a)$ is from (15).

Proof. Assume that $A = \{t_0, \dots, t_k\}$ so that $0 = t_0 < \cdots < t_k = N$. We have

$$\tilde{\mathbf{p}}^c_{\varphi_a,N}(\mathcal{A}^a_N = A/N) = \frac{1}{\mathcal{Z}^c_{\varphi_a,N}} \int_0^a \cdots \int_0^a \prod_{i=1}^k f^a_{t_i - t_{i-1}}(y_{i-1}, y_i) e^{\varphi_a(y_i)} \mathrm{d}y_i =: (*).$$

Using (13), Condition (A), and Lemma 2.3 we have

$$\begin{aligned}
(*) &\le \frac{1}{\mathcal{Z}^c_{\varphi_a,N}} e^{(\beta_c + Ca)k} \prod_{i=1}^k f_{t_i - t_{i-1}} \\
&= \frac{Z^c_{\beta_c + Ca,N}}{\mathcal{Z}^c_{\varphi_a,N}} \mathbf{p}^c_{\beta_c + Ca,N}(\mathcal{A}_N = A/N) \\
&\le \frac{Z^c_{\beta_c + Ca,N}}{Z^c_{\beta_c - C'a,N}} \mathbf{p}^c_{\beta_c + Ca,N}(\mathcal{A}_N = A/N).
\end{aligned}$$

The lower bound is analogous. For the free case, fix $A = \{t_0, \dots, t_k\}$ so that $0 = t_0 < \cdots < t_k < N$.

$$\tilde{\mathbf{p}}^f_{\varphi_a,N}(\mathcal{A}^a_N = A/N) = \frac{1}{\mathcal{Z}^f_{\varphi_a,N}} \int_0^a \cdots \int_0^a \prod_{i=1}^k f^a_{t_i - t_{i-1}}(y_{i-1}, y_i) e^{\varphi_a(y_i)} P^a_{y_k}(N - t_k) \mathrm{d}y_i =: (*).$$

Using (13), Condition (A), and Lemma 2.3 we have

$$\begin{aligned}
(*) &\le \frac{1}{\mathcal{Z}^f_{\varphi_a,N}} e^{(\beta_c + Ca)(k-1)} \prod_{i=1}^k f_{t_i - t_{i-1}} \int_0^a P^a_{y_k}(N - t_k) e^{\varphi_a(y_k)} \mathrm{d}y_k \\
&\le \frac{1}{\mathcal{Z}^f_{\varphi_a,N}} e^{(\beta_c + Ca)k} \prod_{i=1}^k P(N - t_k) f_{t_i - t_{i-1}} \\
&= \frac{Z^f_{\beta_c + Ca,N}}{\mathcal{Z}^f_{\varphi_a,N}} \mathbf{p}^f_{\beta_c + Ca,N}(\mathcal{A}_N = A/N) \\
&\le \frac{Z^f_{\beta_c + Ca,N}}{C'(a) Z^f_{\beta_c - C'a,N}} \mathbf{p}^c_{\beta_c + Ca,N}(\mathcal{A}_N = A/N).
\end{aligned}$$

Similarly for the lower bound, where we should omit the $C'(a)$ in the analogous statement. □

3. Near-critical standard wetting, scaling limit of the contact set

In this section we shall use a result by Julien Sohier on order $N^{-1/2}$ near-critical pinning models defined by a renewal process with free boundary conditions [19] to deduce that for $o(N^{-1/2})$ near-critical standard wetting models, and also for pinning models defined by a renewal process with constraint boundary conditions, the rescaled limiting contact set coincides with the one which is corresponding to the critical pinning model. That is, very roughly speaking, we shall show that in the standard wetting model, the rescaled contact set limit is invariant under $o(N^{-1/2})$ linear perturbation of the critical pinning strength. We now make these statements exact and formal.

First, let us formulate Sohier's result. Let τ be a renewal process on the positive integers with inter-arrival mass function K. More precisely, let $\tau_k = \sum_{i=1}^k l_i$ where l_i are i.i.d. random variables with $\mathbf{P}(l_1 = n) = K(n)$, then τ is the random subset $\tau := \{\tau_i : i \ge 0\} \subset \mathbb{N}$ with respect to $\mathbf{P}$. Let $\mathbf{E}$ be the corresponding expectation.

Assume that $K(n) = \frac{L(n)}{n^{3/2}}$, where L is slowly varying at infinity (i.e. $L(cx)/L(x) \to 1$ as $x \to \infty$ for all $c > 0$).

Let $\mathbf{P}_{\beta,N}$ be a probability measure on subsets of $\{0, \dots, N\}$ and naturally, on subsets of $\mathbb{N}$, defined by

$$\mathrm{d}\mathbf{P}_{\beta,N}(\tau) = \mathrm{d}\mathbf{P}_{\beta,N}(\tau \cap [0, N]) := \frac{1}{\mathbf{Z}_{\beta,N}} \exp(\beta|\tau \cap [0, N]|)\mathrm{d}\mathbf{P}(\tau)$$

so that the partition function is $\mathbf{Z}_{\beta,N} = \mathbf{E}[\exp(\beta|\tau \cap [0, N]|)]$. Let $\mathbf{E}_{\beta,N}$ be the corresponding expectation. We also define $\beta_c^{(K)}$ by the identity $e^{\beta_c^{(K)}} \sum_{n\geq 1} K(n) = 1$. Obviously, one notes that $\beta_c^{(K)} = 0$ whenever $\sum_{\geq 1} K(n) = 1$.

As in Section 1.3, in this section weak convergence of closed random subsets of $[0, 1]$ is with respect to the Matheron topology on closed subsets.

For readability, we exclude some notations which are irrelevant to our argument and we now formulate a special version of Sohier's theorem. For elaborated discussion see Sohier [19, Sections 1 and 3]. See also the monograph [14] for a comprehensive, rich, and approachable analysis of the renewal model.

Theorem 3.1 (*Theorem 3.1.(1) and Part of the Proof of [19] in the Case* $\alpha = \frac{1}{2}$, $L \sim C_K := \frac{1}{\sqrt{2\pi}} e^{\beta_c}$). *Assume* $K(n) = q(n) := \frac{C_K}{n^{3/2}}$, *where* C_K *is defined so that* $\sum_{n\geq 1} q(n) = 1$. *Let* $b = 2\sqrt{\pi} C_K$ *and fix* $\epsilon \in \mathbb{R}$. *Then, under* $\mathbf{P}_{\frac{b}{\sqrt{N}}\epsilon,N}$ *the rescaled contact set* $\mathcal{A}_N := \frac{1}{N}\tau \cap [0, N] := \{\frac{i}{N} : i \in \tau \cap [0, N]\} \subset [0, 1]$ *is converging weakly to a random set* $\mathcal{B}_{1/2}$. *Moreover, the law of* $\mathcal{B}_{1/2}$ *is absolutely continuous with respect to the law of* $\mathcal{A}_{1/2}$, *the set of zeros in* $[0, 1]$ *of the standard Brownian motion, with Radon–Nikodym density* $\frac{\exp(\epsilon L_1)}{\mathbb{E}[\exp(\epsilon L_1)]}$, *where* L_1 *is the local time in* 0 *of the Brownian motion at time* 1 *endowed with probability measure* $\mathbb{P}$ *and expectation* $\mathbb{E}$. *In particular, for every continuous bounded function* $\Phi : \mathcal{F} \to \mathbb{R}$, *where* $\mathcal{F}$ *is the space of closed sets in* $[0, 1]$ *with the Matheron topology, it holds that*

$$\mathbf{E}_{\frac{b\epsilon}{\sqrt{N}},N}[\Phi(\mathcal{A}_N)] = \mathbf{E}\left[\exp\left(b\epsilon \frac{|\tau \cap [0, N]|}{\sqrt{N}}\right) \Phi(\mathcal{A}_N)\right] \to \mathbb{E}[\exp(\epsilon L_1)\Phi(\mathcal{A}_{1/2})], \tag{17}$$

and specifically

$$\mathbf{Z}_{\frac{b\epsilon}{\sqrt{N}},N} = \mathbf{E}\left[\exp\left(b\epsilon \frac{|\tau \cap [0, N]|}{\sqrt{N}}\right)\right] \to \mathbb{E}[\exp(\epsilon L_1)]. \tag{18}$$

Remark 3.2. Following Sohier's notation in lines (3.4) and (3.7) in his paper, in the case $\alpha = \frac{1}{2}$ and $L(x) \sim C_K$, we have $a_n \sim 4\pi C_K^2 n^2$ and $b_n \sim \frac{1}{2\sqrt{\pi} C_K}\sqrt{n}$. We note again that $\beta_c^{(K)} = 0$ since $\sum_{n=1}^{\infty} K(n) = 1$.

Remark 3.3. We note that in the case $K(n) = f_n = \frac{1}{\sqrt{2\pi}} n^{-3/2}$ we have $\beta_c^{(K)} = \beta_c$, the critical wetting model pinning strength, and for $K(n) = e^{\beta_c} f_n$ we have $\beta_c^{(K)} = 0$.

Corollary 3.4. *Fix a sequence* ϵ_N *so that* $\epsilon_N \to 0$ *as* $N \to \infty$. *Let* $K(\cdot) = q(\cdot)$, *as in Theorem 3.1. Then, under* $\mathbf{P}_{\frac{\epsilon_N}{\sqrt{N}},N}$ *the rescaled contact set* $\mathcal{A}_N$ *is converging weakly to* $\mathcal{A}_{1/2}$, *the set of zeros in* $[0, 1]$ *of a standard Brownian motion.*

Sketch of proof. By considering the positive and negative parts of ϵ_N we may assume without loss of generality that they all have the same sign. We consider the case where they are non-negative. The complementary case is similar. First, note that for every $\epsilon > 0$ we have by (18) that

$$1 \le \limsup_{N\to\infty} \mathbf{Z}_{\frac{b\epsilon_N}{\sqrt{N}},N} \le \lim_{N\to\infty} \mathbf{Z}_{\frac{b\epsilon}{\sqrt{N}},N} = \mathbb{E}[\exp(\epsilon L_1)].$$

Hence,

$$1 \le \limsup_{N\to\infty} \mathbf{Z}_{\frac{b\epsilon_N}{\sqrt{N}},N} \le \liminf_{\epsilon\to 0} \mathbb{E}[\exp(\epsilon L_1)] = 1$$

and so

$$\lim_{N\to\infty} \mathbf{Z}_{\frac{b\epsilon_N}{\sqrt{N}},N} = 1. \tag{19}$$

Similarly, it holds that for any measurable bounded function $\Phi : \mathcal{F} \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbf{E}\left[\exp\left(b\epsilon_N \frac{|\tau\cap[0,N]|}{\sqrt{N}}\right)\Phi(\mathcal{A}_N)\right] = \mathbb{E}[\Phi(\mathcal{A}_{1/2})]. \tag{20}$$

The statement of the corollary follows. □

Define $\mathbf{P}^c_{\beta,N}$ similarly to be the constrained version of $\mathbf{P}_{\beta,N}$:

$$d\mathbf{P}^c_{\beta,N}(\tau) = d\mathbf{P}^c_{\beta,N}(\tau\cap[0,N]) := \frac{1}{\mathbf{Z}^c_{\beta,N}} \exp(\beta|\tau\cap[0,N]|)\mathbb{1}_{\{N\in\tau\}} d\mathbf{P}(\tau).$$

One can write

$$\mathbf{Z}^c_{0,N} = \sum_{k=1}^{N} \mathbf{P}_{0,N}(\tau_k = N)\mathbf{Z}_{0,N} = \mathbf{E}_{0,N}(\mathbb{1}_{\{N\in\tau\}})\mathbf{Z}_{0,N},$$

and so it holds

$$\mathbf{P}^c_{0,N}(\cdot) = \frac{\mathbf{P}_{0,N}(\cdot\cap\{N\in\tau\})}{\mathbf{P}_{0,N}(N\in\tau)} = \mathbf{P}_{0,N}(\cdot|N\in\tau)$$

(compare with Giacomin [14, Remark 2.8]).

The next proposition is an analog of Corollary 3.4 in the corresponding constraint case, and moreover for the near-critical *standard wetting model*.

Proposition 3.5. *Let $K(\cdot) = q(\cdot)$, as in Theorem 3.1. Fix a sequence ϵ_N so that $\epsilon_N \to 0$ as $N \to \infty$. The rescaled contact set $\mathcal{A}_N \subset [0,1]$ distributed according to $\mathbf{p}^f_{\beta_c+\frac{\epsilon_N}{\sqrt{N}},N}$, is converging weakly to $\mathcal{A}_{1/2}$, the set of zeros in $[0,1]$ of a standard Brownian motion. Moreover, when distributed according to either $\mathbf{P}^c_{\frac{\epsilon_N}{\sqrt{N}},N}$ or $\mathbf{p}^c_{\beta_c+\frac{\epsilon_N}{\sqrt{N}},N}$, $\mathcal{A}_N$ is converging weakly to $\mathcal{A}^c_{1/2}$, the set of zeros of the Brownian bridge in $[0,1]$. Here $\mathbf{P}^c_{\frac{\epsilon_N}{\sqrt{N}},N}$ is corresponding to K with the same conditions as in Theorem 3.1, and, as before, all sets are considered in the Matheron topology on closed subsets of the real line.*

For the proof we shall essentially imitate the way Proposition 5.2. of [5] was deduced from Lemma 5.3 of that paper (which is partly based on [10]), while performing the necessary changes. In light of Eqs. (19) and (20) the free case is almost the same as in [5]. In the constrained cases we will borrow an estimate from [8].

Proof. First, for the free case, let $A = \{t_1, \dots, t_{|A|}\}$ so that $0 =: t_0 < t_1 < \cdots < t_{|A|} \le N$. Note that $Z_{0,N} = 1$ for all N (see [14, equation (2.17)]), so $\mathbf{P}_{0,N}(A) = \mathbf{P}(A)$. Now

$$\mathbf{P}(A) = \prod_{j=1}^{|A|} q(t_j - t_{j-1}) Q(N - t_{|A|})$$

where $Q(n) = \bar{K}(n+1) = \sum_{t \ge n+1} q(t)$. Also

$$\mathbf{p}^f_{\beta,N}(A) = \frac{1}{Z^f_{\beta,N}} e^{(\beta-\beta_c)|A|} P(N - t_{|A|}) \prod_{j=1}^{|A|} q(t_j - t_{j-1}),$$

where as before $P(n) = P^0(n) := \mathbb{P}[S_1 > 0, \dots, S_n > 0]$. We then have for $\beta_N = \beta_c + \frac{\epsilon_N}{\sqrt{N}}$

$$\frac{\mathbf{p}^f_{\beta_N,N}(A)}{\mathbf{P}(A)} = \exp\left(\frac{\epsilon_N}{\sqrt{N}}|A|\right) \phi_N(\max A),$$

where $\phi_N : [0,1] \to \mathbb{R}_+$ is defined by

$$\phi_N(t) := \frac{1}{Z^f_{\beta_c,N}} \frac{P(N(1-t))}{Q(N(1-t))}.$$

Therefore for every bounded measurable functional Φ we have

$$\mathbf{E}^f_{\beta_N,N}[\Phi(\mathcal{A}_N)] = \mathbf{E}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\mathcal{A}_N|\right) \phi_N(\max A) \Phi(\mathcal{A}_N)\right],$$

It was proved in [5, proof of Proposition 5.2.] that $\phi_N(t) \to 1$ uniformly in $t \in [0, v]$, for every $v \in (0,1)$. Since $\mathbb{P}$-a.s. $0 \notin \mathcal{A}_{1/2}$, it follows from (20) (for general $\epsilon_N \to 0$) that

$$\mathbf{E}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\mathcal{A}_N|\right) \phi_N(\max A) \Phi(\mathcal{A}_N)\right] \to \mathbb{E}[\Phi(\mathcal{A}_{1/2})],$$

and the free case is done. We will now show the constraint case. By definition, for every $A \subset \{1, \dots, N\}$ containing N we have

$$\frac{\mathbf{P}^c_{\beta,N}(A)}{\mathbf{p}^c_{\beta_c+\beta,N}(A)} = \frac{Z^c_{\beta_c+\beta,N}}{\mathbf{Z}^c_{\beta_c,N}}.$$

That is, the ratio of these two probability measures is constant and so they coincide. We shall work with $\mathbf{P}^c_{\frac{\epsilon_N}{\sqrt{N}},N}$. As in the free case we follow the proof of [5, Proposition 5.2.], and accordingly we now consider $\mathcal{A}_N \cap [0, 1/2]$. We have for $\beta_N = \beta_c + \frac{\epsilon_N}{\sqrt{N}}$

$$\mathbf{E}^c_{\beta_N,N}[\Phi(\mathcal{A}_N \cap [0,1/2])] = \mathbf{E}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\mathcal{A}_N \cap [0,1/2]|\right) \phi^c_N(\max \mathcal{A}_N \cap [0,1/2]) \times \Phi(\mathcal{A}_N \cap [0,1/2])\right],$$

where

$$\phi^c_N(t) := \frac{\sum_{n=0}^{N/2} \mathbf{Z}^c_{\frac{\epsilon_n}{\sqrt{n}},n} q(N(1-t)-n)}{\mathbf{Z}^c_{\frac{\epsilon_N}{\sqrt{N}},N} Q(N(1-t))}, \quad t \in [0, 1/2].$$

We remind the reader that here $\mathbf{Z}^c_{\beta,N}$ is the partition function corresponding to $\mathbf{P}^c_{\beta,N}$. Now, since $\phi^c_N(t)$ is defined similarly to $f^c_N(t)$ in the proof of [5, Proposition 5.2.], with the only difference being that all the $\mathbf{Z}^c_{\frac{\epsilon_k}{\sqrt{k}},k}$ are replaced by the corresponding $\mathbf{Z}^c_{0,k}$, and since that proof uses only the asymptotic rates of $\mathbf{Z}^c_{0,\cdot}$, $q(\cdot)$ and $Q(\cdot)$, we are done once we show that

$$\frac{\mathbf{Z}^c_{\frac{\epsilon_N}{\sqrt{N}},N}}{\mathbf{Z}^c_{0,N}} \to 1 \text{ as } N \to \infty. \tag{21}$$

By a direct expansion, one finds that $\mathbf{Z}^c_{\frac{\epsilon_N}{\sqrt{N}},N} = \mathbf{Z}^c_{0,N}\mathbf{E}^c_{0,N}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\tau \cap [0,N]|\right)\right]$. Therefore,

$$\frac{\mathbf{Z}^c_{\frac{\epsilon_N}{\sqrt{N}},N}}{\mathbf{Z}^c_{0,N}} = \mathbf{E}^c_{0,N}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\tau \cap [0,N]|\right)\right] = \mathbf{E}\left[\exp\left(\frac{\epsilon_N}{\sqrt{N}}|\tau \cap [0,N]|\right)|N \in \tau\right].$$

Assume without loss of generality that $\epsilon_N \geq 0$ for all N and fix $\epsilon > 0$. Since for large N the right most expression in last line is smaller than $\mathbf{E}\left[\exp\left(\frac{\epsilon}{\sqrt{N}}|\tau \cap [0,N]|\right)|N \in \tau\right]$, by [8, equation (A.12)] (cf. [22], and [15, Lemma A.2]), there is a constant $C > 0$ bounding the expression. Using Lemma C.1 we deduce that the expression is in fact converging to 1 as $N \to \infty$, and so we have (21). We therefore conclude the proof of the proposition. □

4. Contact set and path scaling limit — proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. First, we note that for $a_N = o(N^{-1/2})$, and $s, r \in \mathbb{R}$, we have by (21) that

$$\frac{Z^c_{\beta_c + ra_N,N}}{Z^c_{\beta_c + sa_N,N}} \to 1.$$

Moreover, by (19) we have

$$Z^f_{\beta_c,N} \to 1 \text{ and } Z^f_{\beta_c + ra_N,N} \to 1 \text{ as } a \to 0.$$

Using Proposition 3.5 with $r\epsilon_N$ instead of ϵ_N we have the desired corresponding scaling limit under $\mathbf{p}^\alpha_{\beta_c + ra_N}$. Using Lemma 2.4 we can now conclude. Indeed, let $\Phi : \mathcal{F} \to \mathbb{R}$ be a measurable bounded function. As before, considering separately the positive and negative parts in the presentation $\Phi = \Phi_+ - \Phi_-$ we can assume without loss of generality that Φ is non-negative. We therefore have by Lemma 2.4

$$\tilde{\mathbf{E}}^\alpha_{\varphi_a,N}[\Phi(\mathcal{A}_N)] \leq R_N \mathbf{E}^\alpha_{\beta_c + c_3 a_N,N}[\Phi(\mathcal{A}_N)] \to \mathbb{E}[\Phi(\mathcal{A}^\alpha_{1/2})]$$

and

$$\tilde{\mathbf{E}}^\alpha_{\varphi_a,N}[\Phi(\mathcal{A}_N)] \geq L_N \mathbf{E}^\alpha_{\beta_c - c_6 a_N,N}[\Phi(\mathcal{A}_N)] \to \mathbb{E}[\Phi(\mathcal{A}^\alpha_{1/2})],$$

where L_N, R_N are positive reals so that $L_N, R_N \to 1$. □

Next, once we have the contact set convergence, Theorem 1.4, to move to the path limit, Theorem 1.5, is by now routine, following the guidelines of [10]. Let us first give a rough sketch.

Tightness will be proved as in [10, Lemma 4] where we need a small linear modification of the oscillation function, and instead of using Propositions 7 and 8 of that paper, we shall use stronger results as follows. The first result is the weak convergence in $C[0,1]$ under $\mathbf{p}^c_{0,N}(x_N, y_N)$ the pinning-free process (i.e. $\varphi_a = 0$) conditioned on the starting and ending points $x_N, y_N \in [0, a_N]$ to the Brownian bridge, which was proved by Caravenna–Chaumont [4]. The second result is the analogous statement on the free case and the Brownian meander which is available by Caravenna–Chaumont [3].

Once we have tightness, we need to prove the finite-dimensional distributions, for that we follow [10, Chapter 8]. Since we know that our contact set converges to the zero-set of the Brownian motion or bridge, then we know that the probability that a fixed finite number of points in [0, 1] are the limiting zero-set is 0, and there is no change of that part of the argument. The only difference in the proof is that we condition not only on the contact indices but also on *their location in the strip*. But since the conditioned processes converge by the last two aforementioned theorems, we can conclude using dominated convergence on the full path as in [10].

Let $A_n^a(y) := \{S_1 > a, \ldots, S_{n-1} > a, S_n = y\}$. We have the following densities comparison bound.

Lemma 4.1. *For every $\gamma > 0$ and $n \in \mathbb{N}$, we have*

$$\mathbb{P}_x\left(\max_{0\le i,j\le n} |S_i - S_j| > \gamma, A_n^a(y)\right) \le \mathbb{P}_0\left(\max_{0\le i,j\le n} |S_i - S_j| > \gamma - a, A_n^0(0)\right) \tag{22}$$

uniformly in $x, y \in [0, a]$. Moreover, the same holds whenever in both sides of the inequality the index set satisfies in addition that $|i - j| \le m$ for some fixed $m > 0$.

Proof. Let $a - x = S_0, S_1, \ldots, S_n = a - y$ so that $S_i \ge 0$, $i = 1, \ldots, n-1$, and $|S_{i_0} - S_{j_0}| = \max_{0\le i,j\le n} |S_i - S_j|$. Then, if $i_0, j_0 \notin \{1, \ldots, n-1\}$, without loss of generality $i_0 = 0$, and so $|S_{i_0} - S_{j_0}| = |S_{j_0} - (a - x)| \le |S_{j_0}| + |a - x| \le |S_{j_0} - 0| + a$. In other words, $\max_{0\le i,j\le n} |S_i - S_j| \le \max_{0\le i,j\le n} |S'_i - S'_j| + a$ where $S'_i = S_i$ for $i = 1, \ldots, n-1$ but $S_0 = S_n = 0$. Therefore, by monotonicity of $\rho(\cdot)$

$$\begin{aligned}
&\mathbb{P}_x\left(\max_{0\le i,j\le n} |S_i - S_j| > \gamma, A_n^a(y)\right)\\
&= \frac{1}{\kappa^n}\int_a^\infty \cdots \int_a^\infty \mathbb{1}_{\max_{i,j\le n}|S_i - S_j| > \gamma}\\
&\times \rho(s_1 - x)\rho(s_2 - s_1)\cdots\rho(s_{n-1} - s_{n-2})\rho(y - s_{n-1})\mathrm{d}s_1\cdots\mathrm{d}s_{n-1}\\
&= \frac{1}{\kappa^n}\int_0^\infty \cdots \int_0^\infty \mathbb{1}_{\max_{0\le i,j\le n}|S_i - S_j| > \gamma}\\
&\times \rho(s_1 - x + a)\rho(s_2 - s_1)\cdots\rho(s_{n-1} - s_{n-2})\rho(y - s_{n-1} - a)\mathrm{d}s_1\cdots\mathrm{d}s_{n-1}\\
&\le \frac{1}{\kappa^n}\int_0^\infty \cdots \int_0^\infty \mathbb{1}_{\max_{0\le i,j\le n}|S_i - S_j| > \gamma - a}\\
&\times \rho(s_1)\rho(s_2 - s_1)\cdots\rho(s_{n-1} - s_{n-2})\rho(s_{n-1})\mathrm{d}s_1\cdots\mathrm{d}s_{n-1}\\
&= \mathbb{P}_0\left(\max_{0\le i,j\le n} |S_i - S_j| > \gamma - a, A_n^0(0)\right).
\end{aligned}$$

The 'moreover' part is similar, we omit its proof. □

We shall now prove that whenever $\varphi = \varphi^0_{a_N}$, i.e. no pinning is present, the scaling limit is a Brownian excursion, for any fixed endpoints $x_N, y_N \in [0, a_N]$. Shifting by a_N, it is equivalent to show that conditioning on starting and ending at $S_0 = x_N - a_N$, $S_N = y_N - a_N$ and S_n non-negative at times $1 \le n \le N - 1$, the rescaled path converges weakly to the Brownian excursion.

The following is a formulation of Theorem 1.1 of Caravenna–Chaumont [4] which shows the same for non-negative endpoints which are $o(\sqrt{N})$ away from the zero line.

Let us first introduce a notation for the conditioning. Define

$$\mathbb{P}^{+,N}_{x,y} := \mathbb{P}_x(\cdot|\mathcal{C}_{N-1}, S_N = y),$$

for any $x, y \in \mathbb{R}$, $N \in \mathbb{N}$.

Theorem 4.2 (*Caravenna–Chaumont [4]*). *Let $(x_N), (y_N)$ be sequences of non-negative real numbers such that $x_N, y_N = o(\sqrt{N})$ as $N \to \infty$. Then under $\mathbb{P}^{+,N}_{x_N,y_N}$, $(X^{(N)}_t)_{t\in[0,1]}$ converges weakly in $C[0,1]$ to the Brownian excursion.*

Theorem 4.2 implies the following theorem.

Theorem 4.3. *Let $(x_N), (y_N)$ be sequences of non-negative real numbers such that $x_N, y_N \le a_N = o(1)$ as $N \to \infty$. Then under $\mathbb{P}^{+,N}_{x_N-a_N,y_N-a_N}$, $(X^{(N)}_t)_{t\in[0,1]}$ converges weakly in $C[0,1]$ to the Brownian excursion.*

We note that the assumption $x_N, y_N \le a_N = o(1)$ is only to make sure that $X^N_0, X^N_1 \to 0$. We will use the theorem under the stronger condition $a_N = o(\frac{1}{\sqrt{N}})$.

Proof. First we prove tightness. For a path $x \in C[0,1]$ define

$$\Gamma(\delta)(x) := \sup_{\{t,s\in[0,1]:|t-s|\le\delta\}} |x_t - x_s|. \tag{23}$$

Using the fact that $f^0_N(x_N - a_N, y_N - a_N) = f^{a_N}_N(x_N, y_N)$, the 'moreover' part of Lemma 4.1 implies that

$$\mathbb{P}^{+,N}_{x_N-a_N,y_N-a_N}\left(\max_{\substack{0\le i,j\le N:\\|i-j|\le\delta n}} |S_i - S_j| > \gamma\right) f^{a_N}_N(x_N, y_N)$$
$$\le \mathbb{P}^{+,N}_{0,0}\left(\max_{\substack{0\le i,j\le N:\\|i-j|\le\delta n}} |S_i - S_j| > \gamma - a_N\right) f^0_N(0,0)$$

for every $\delta, \gamma > 0$ and $n \in \mathbb{N}$, uniformly in $x_N, y_N \in [0, a_N]$. Now, by (12) and (13) we get

$$\mathbb{P}^{+,N}_{x_N-a_N,y_N-a_N}\left(\max_{\substack{0\le i,j\le N:\\|i-j|\le\delta n}} |S_i - S_j| > \gamma\right)$$
$$\le \exp(C_0 a_N)\mathbb{P}^{+,N}_{0,0}\left(\max_{\substack{0\le i,j\le N:\\|i-j|\le\delta n}} |S_i - S_j| > \gamma - a_N\right). \tag{24}$$

Theorem 4.2 implies in particular that $(X^{(N)}_t)_{t\in[0,1]}$ is tight under $\mathbb{P}^{+,N}_{0,0}$, and so by (24), it is also tight under $\mathbb{P}^{+,N}_{x_N-a_N,y_N-a_N}$. Indeed, the standard necessary and sufficient condition for tightness on $C[0,1]$ is Prokhorov's Theorem: for every $\gamma > 0$ $\lim_{\delta\to0}\sup_N \mathbb{P}^{+,N}_{0,0}(\Gamma(\delta) > \gamma) = 0$. To get our tightness, fix $\gamma > 0$. Choose N_0 large enough so that $\gamma - a_N > \gamma/2$ for all $N \ge N_0$. Tightness will hold by considering only $\delta < 1/N_0$.

We shall now prove the convergence of the finite-dimensional distributions. Let $0 < s_1 < \cdots < s_n < 1$. Fix N large enough so that $1/N < s_1 < s_n < 1 - 1/N$. Then $(X^{(N)}_{s_i})_{i=1,\dots,n}$

have the same distribution under both conditional distributions $\mathbb{P}^{+,N}_{0,0}(\cdot|S_1 = x, S_{N-1} = y)$ and $\mathbb{P}^{+,N}_{x_N-a_N,y_N-a_N}(\cdot|S_1 = x, S_{N-1} = y)$, for all $x, y \geq 0$. Since $\frac{x_N}{\sqrt{N}}, \frac{y_N}{\sqrt{N}} \to 0$, the difference between the corresponding expectations on any test function on $(X^{(N)}_{s_i})_{i=1,\dots,n}$ goes to zero as $N \to \infty$. Using Theorem 4.2 again, we conclude by the convergence of the distributions of $(X^{(N)}_{s_i})_{i=1,\dots,n}$ under $\mathbb{P}^{+,N}_{0,0}$. □

The next lemma provides bounds on the oscillations of the φ_a-model conditioned on the contact set and the contact locations in terms of the oscillations of the standard model conditioned on the contact set. For ease of notation we write $i \sim_N j$ whenever $\frac{i}{N} \sim_{X^{(N)}} \frac{j}{N}$.

Lemma 4.4. *It holds that*

$$\mathbb{P}^{\alpha}_{\varphi_a,N}\left(\max_{|i-j|\leq \delta N, i\sim_N j} |S_i - S_j| > \gamma | A, y_1, \dots, y_{|A|}\right)$$
$$\leq \exp(C_0 a|A|)\mathbb{P}^{\alpha}_{\beta_c,N}\left(\max_{|i-j|\leq \delta N, i\sim_N j} |S_i - S_j| > \gamma - a|A\right)$$

where A is the contact set, $y_k \in [0, a]$ are the corresponding values in the strip.

Proof. Note that conditioning on $A = \{t_1, \dots, t_{|A|}\}$ the excursions are independent. Moreover, conditioning on the endpoints the law of the excursions is the same as with respect to $\mathbb{P}^{+,N}_{y_{k-1},y_k}$. To conclude, we use the "moreover" part of Lemma 4.1 with $m = \delta N$ and $n = t_k - t_k - 1$ on each excursion separately. □

Proof of Theorem 1.5. First, we shall prove that $a_N = o(N^{-1/2})$ then the sequence $\left((X^{(N)}_t)_{t\in[0,1]}, \mathbb{P}^{\alpha}_{\varphi_{a_N},N}\right)$ is tight.

We modify the definition (23) as follows. For a path $x \in C[0, 1]$ define the modified δ-oscillation of strip size a by

$$\tilde{\Gamma}^a(\delta)(x) := \sup_{\{t,s\in[0,1]:|t-s|\leq\delta, s\sim_x t\}} |x_t - x_s|, \tag{25}$$

where $s \sim_x t$ if and only if $x_u > a$ for all $u \in (s, t)$ (see [6] for the case $a = 0$).

We naturally extend the definition of $\tilde{\mathbf{p}}^{\alpha}_{\varphi_a,N}$ to include pairs (A, y) where $y \in [0, a]^{|A|}$ the vector of positions at the contact indices. Since $\Gamma(\delta)(x) \leq \tilde{\Gamma}^a(\delta)(x)$, it is enough to show that $\mathbb{P}^{\alpha}_{\varphi_a,N}(\tilde{\Gamma}^a(\delta)(x) > \gamma) \to 0$ as $\delta \to 0$. By Lemma 4.4

$$\begin{aligned}\mathbb{P}^{\alpha}_{\varphi_a,N}(\tilde{\Gamma}^a(\delta) > \gamma) &= \sum_{A\subset\{0,\dots,N\}} \int_0^a \dots \int_0^a \mathbb{P}^{\alpha}_{\varphi_a,N}(\tilde{\Gamma}^a(\delta) > \gamma | A, y_1, \dots, y_{|A|}) \\ &\quad \times \tilde{\mathbf{p}}^{\alpha}_{\varphi_a,N}(A, y_1, \dots, y_{|A|}) \mathrm{d}y_1 \cdots \mathrm{d}y_{|A|} \\ &\leq \sum_{A\subset\{0,\dots,N\}} \exp(C_0 a_N |A|)\mathbb{P}^{\alpha}_{\beta_c,N}(\tilde{\Gamma}(\delta) > \gamma - a_N|A)\tilde{\mathbf{p}}^{\alpha}_{\varphi_a,N}(A).\end{aligned}$$

Now, from Lemma 2.4, using the fact that $a_N \to 0$, we have $C'_N \to 1$ so that

$$\tilde{\mathbf{p}}^{\alpha}_{\varphi_a,N}(A) \leq C'_N \mathbf{p}^{\alpha}_{\beta_c+c_3 a_N,N}(A)$$

The partition functions ratio between pinning perturbation of constant times a_N is going to 1. Hence we have

$$\begin{aligned}&\sum_{A\subset\{0,\dots,N\}} \exp(C_0 a_N|A|)\mathbb{P}^{\alpha}_{\beta_c,N}(\tilde{\Gamma}(\delta)>\gamma-a_N|A)\tilde{\mathbf{p}}^{\alpha}_{\varphi_a,N}(A)\\ &\le C_N \sum_{A\subset\{0,\dots,N\}} \mathbb{P}^{\alpha}_{\beta_c,N}(\tilde{\Gamma}(\delta)>\gamma-a_N|A)\mathbf{p}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}(A),\end{aligned}$$

for some $C_N\to 1$. To conclude, note that the conditioning allows us to change β_c, to get

$$\begin{aligned}&C_N \sum_{A\subset\{0,\dots,N\}} \mathbb{P}^{\alpha}_{\beta_c,N}(\tilde{\Gamma}(\delta)>\gamma-a_N|A)\mathbf{p}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}(A)\\ &\le \tilde{C}_N \sum_{A\subset\{0,\dots,N\}} \mathbb{P}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}(\tilde{\Gamma}(\delta)>\gamma-a_N|A)\mathbf{p}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}(A)\\ &= \tilde{C}_N \mathbb{P}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}(\tilde{\Gamma}(\delta)>\gamma-a_N).\end{aligned}$$

To sum up, tightness follows once we show tightness under $\mathbb{P}^{\alpha}_{\beta_c+(c_3+C_0)a_N,N}$. The latter is a special case of [2, Theorem 3.5].

To prove the convergence of finite-dimensional distributions we follow closely [10, Chapter 8], with the necessary modifications. Let us deal with the constraint case. Let $(\beta_t)_{t\in[0,1]}$ be the Brownian bridge. Let $0<s_1<\cdots<s_n<1$. Remember the law of $\mathcal{A}^{\alpha}_N$ given in (10), where φ_a satisfying Condition A.

To unify the notations denote by $Z(x)$ the zero-set of the path $x\in C[0,1]$. Given a closed set $Z\subset[0,1]$ and $t\in[0,1]$ we let $d_t(Z):=\inf Z\cap[t,1]$, $g_t(Z):=\sup Z\cap[0,t]$, and $\Lambda_t(Z):=d_t-g_t$.

By Theorem 1.4 and the Skorokhod representation Theorem there is a sequence Z_N with laws $\mathcal{A}^c_N$ converging a.s. to $\mathcal{A}^c_{1/2}$, in the Matheron topology defined above.

We define random equivalence relations, with respect to Z_N, on $\{s_1,\dots,s_n\}$ by declaring that $s_i\sim s_j$ if and only if either $d_{s_i}=d_{s_j}$ or $g_{s_i}=g_{s_j}$. In words, $s_i\sim s_j$ if and only if (s_i,s_j) is contained in an excursion of $X^{(N)}$.

Notice that a.s. $(\beta_{s_i})\neq 0$ for all $1\le i\le n$. Since the Matheron topology is also homeomorphic to the Hausdorf metric space (see (29) and (30) in [10]) then $g_{s_i}(Z_N)$ and $d_{s_i}(Z_N)$ converge a.s. to strictly positive random variables, and A^N_k, $k=1,\dots,I^N$, the random equivalent classes of $\{s_1,\dots,s_n\}$ (here $I^N\le n$) are a.s. eventually constant with N (but still random). Denote their eventual a.s. limit by A_k, $k=1,\dots,I$. Let $W^{N,(y^N_{i-1},y^N_i)}_{s_i}$ $i=1,\dots,n$, $y^N_i\in[0,a_N]$ be a set of random variables with values in $C[0,1]$, so that $W^{N,(y^N_{i-1},y^N_i)}_{s_i}$ is distributed as X^N under $\mathbb{P}^{+,N}_{y_{i-1},y_i}$, and is independent of $g_{s_i}(Z_N)$ and $\Lambda_{s_i}(Z_N)$. Theorem 4.3 tells us that $W^{N,(y^N_{i-1},y^N_i)}_{s_i}$ converges weakly to the Brownian excursion $(\mathcal{E}_t)_{t\in[0,1]}$. Set

$$M^N_{s_i}=\sum_{k=1}^{I^N}\mathbb{1}_{s_i\in A^N_k}\sqrt{\Lambda_{A^N_k}}\cdot W^{N,(y^N_{i-1},y^N_i)}_{s_i}\left(\frac{s_i-g_{A_k}}{\Lambda_{A_k}}\right).$$

Then $(M^N_{s_i})_{i=1,\dots,n}$ is distributed at $\mathbb{P}^c_{\varphi_{a_N},N}$ conditioned on the excursions' endpoints $y_1,\dots,y_{I^N}$. Note that the measures $\mathbb{P}((|\beta_{s_i}|)_{i\in A_k}\in \mathrm{d}x)$ and $\mathbb{P}(\sqrt{\Lambda_{A_k}}\cdot(\mathcal{E}_{s_i/\Lambda_{A_k}})_{i\in A_k}\in\mathrm{d}x)$ on $\mathbb{R}^{A_k}$ have the same densities (see [10, Chapter 8]). Using dominated convergence and the Brownian scaling of $(\mathcal{E}_t)_{t\in[0,1]}$, the finite-dimensional distributions for the path conditioned on the endpoints y^N_i have a limiting law $|\beta|$. But since the limit is independent of y^N_i, we conclude. The free case follows analogously. □

5. The strip wetting model with constant pinning

The goal in this chapter is to prove Theorem 1.7.

5.1. The associated Markov renewal process, integral operator, and free energy, and the critical value

To fix notations and for sake of self containment, we shall elaborate on the analysis of the strip wetting model, and follow closely Sohier [21]. We state here the argument mostly without proofs, which can be found in [21]. We remind the reader that in our case $\varphi = \varphi_a^\beta := \beta \mathbb{1}_{[0,a]}$. Here $a \geq 0$ and $\beta \in \mathbb{R}$ are the corresponding parameters. Let us first introduce a notation for the corresponding measures in this case.

$$\mathrm{d}\mathbb{P}^f_{a,\beta,N}(S) = \frac{1}{Z^f_{a,\beta,N}} \exp\left(\beta \sum_{k=1}^N \mathbb{1}_{[0,a]}(S_k)\right) \mathbb{1}_{\mathcal{C}_N} \mathrm{d}\mathbb{P}_0(S), \tag{26}$$

$$\mathrm{d}\mathbb{P}^c_{a,\beta,N}(S) = \frac{1}{Z^c_{a,\beta,N}} \exp\left(\beta \sum_{k=1}^N \mathbb{1}_{[0,a]}(S_k)\right) \mathbb{1}_{[0,a]}(S_N)\mathbb{1}_{\mathcal{C}_N} \mathrm{d}\mathbb{P}_0(S), \tag{27}$$

and the density

$$Z^c_{a,\beta,N}(S)(x,y) = \mathbb{E}_x\left[\exp\left(\beta \sum_{k=1}^N \mathbb{1}_{[0,a]}(S_k)\right) \mathbb{1}_{\mathcal{C}_N} \mathbb{1}_{\{y\}}(S_N)\right]. \tag{28}$$

Remember the density

$$f_n^a(x,y) := \frac{1}{\mathrm{d}y} \mathbb{P}_x[S_1 > a, \dots, S_{n-1} > a, S_n \in \mathrm{d}y]$$

with respect to the Lebesgue measure, where

$$f_1^a(x,y) := \rho(x-y).$$

Define the resolvent kernel density on $[0,a]$

$$b_\lambda^a(x,y) := \sum_{n=1}^\infty e^{-\lambda n} f_n^a(x,y) \mathbb{1}_{[0,a]^2}(x,y) \tag{29}$$

for all $\lambda \geq 0$. The following Lemma is an easy estimate, we defer its proof to Appendix B.

Lemma 5.1. *b_λ^a is a kernel density of a Hilbert–Schmidt integral operator, for all $\lambda \geq 0$. In other words, $\int_0^\infty \int_0^\infty b_\lambda^a(x,y)^2 dxdy < \infty$.*

Let $\delta_a(\lambda)$ be the eigenvalue corresponding to the integral operator defined by the kernel density b_λ^a. We note that since b_λ^a is smooth, strictly positive, and point-wise decreasing with $\lambda \geq 0$, then $\delta_a(\lambda)$ is also decreasing, continuous and moreover, its corresponding left eigenfunction $V_\lambda^a(\cdot)$ is continuous and strictly positive on $[0,a]$. In particular, $\delta_a(\lambda)$ has an inverse function which is also continuous, strictly positive and decreasing $\delta_a^{-1}(\cdot) : [0, \delta_a(0)) \to (0,\infty)$.

Define the free energy by

$$F^a(\beta) := \delta_a^{-1}(e^{-\beta})$$

whenever $\beta \geq \beta_c(a) := -\log(\delta_a(0))$ and set $F^a(\beta) := 0$ if $\beta < \beta_c(a)$. In the critical and super-critical cases, $\beta \geq \beta_c(a)$, we denote the corresponding left eigenfunction by $V_{a,\beta}(\cdot) := V^a_{F^a(\beta)}(\cdot)$ Therefore we have

$$\int_0^a \sum_{n=1}^{\infty} e^{-F^a(\beta)n} f_n^a(x,y) \frac{V_{a,\beta}(y)}{V_{a,\beta}(x)} e^{\beta} \mathrm{d}y = 1 \tag{30}$$

for all $x \in [0, 1]$. Note that by symmetry of f_n^a, the left eigenvalue equals the right eigenvalue and moreover one can check that in this case the measure with density $V^2_{a,\beta}$ is invariant for the Markov process on $[0, a]$ with jump density $\int_0^a \sum_{n=1}^{\infty} e^{-F^a(\beta)n} f_n^a(x,y) \frac{V_{a,\beta}(y)}{V_{a,\beta}(x)} e^{\beta}$.

In the critical case we omit the $\beta_c(a)$ from the notation and write

$$V_a(\cdot) := V^a_{F^a(\beta_c)}(\cdot) = V^a_0(\cdot).$$

(Attention, V_a as well as $V_{a,\beta}$ should not be confused with the potential V discussed in introduction!). In particular,

$$\int_0^a \sum_{n=0}^{\infty} q_n^a(x,y) \mathrm{d}y = 1 \tag{31}$$

for all $x \in [0, a]$, where $q_n^a(x, y) := f_n^a(x,y) \frac{V_a(y)}{V_a(x)} e^{\beta_c(a)}$.

Strip model in terms of Markov renewal

Let $\mathcal{P}^{\beta}$ be measure of a Markov renewal process (τ, J) on $\mathbb{N} \times [0, a]$ with kernel density

$$q_n^{a,\beta}(x,y) := e^{-F^a(\beta)n} f_n^a(x,y) \frac{V_{a,\beta}(y)}{V_{a,\beta}(x)} e^{\beta}.$$

In particular, at criticality $q^{a,\beta_c(a)} = q^a$. We then have

$$Z^c_{a,\beta,N}(x,y)\mathrm{d}y = \mathcal{P}^{\beta}(N \in \tau, j_0 = x, j_N \in \mathrm{d}y) e^{F^a(\beta)N} \frac{V_{a,\beta}(x)}{V_{a,\beta}(y)}.$$

And in particular

$$Z^c_{a,\beta_c(a),N}(x,y)\mathrm{d}y = \mathcal{P}^{\beta}(N \in \tau, j_0 = x, j_N \in \mathrm{d}y) \frac{V_a(x)}{V_a(y)}.$$

Therefore, under our initial measure the density of the zero-set A in $[0, N]$ together with the corresponding points $J(A) \subset [0, a]^{|A|}$ is

$$\mathbb{P}^c_{a,\beta,N}((A, J(A))) = \mathcal{P}^{\beta}((A, J(A)) | N \in \tau),$$

and more generally

$$\mathbb{P}^c_{a,\beta,N}(x,y)((A, J(A))) = \mathcal{P}^{\beta}((A, J(A)) | N \in \tau, j_0 = x, j_N = y).$$

5.2. *Strip wetting with critical pinning satisfies Condition (a) - proof of Theorem 1.7*

Choose an eigenfunction V_a so that $\int_0^a V_a(x)^2 \mathrm{d}x = 1$. Remember the eigenvalue equation

$$V_a(x) = e^{\beta_c(a)} \int_0^a \sum_{n \geq 1} f_n^a(x,y) V_a(y) \mathrm{d}y,$$

$x \in [0, a]$. Note that for a fixed $a > 0$, V_a is continuous and strictly positive on $[0, a]$ since so is $f_n^a(x, y)$. Also since $f_n^a(\cdot, \cdot)$ is continuous and is dominated by a summable series (of the form $c(a)n^{-3/2}$), then so is V_a, and moreover its derivatives, whenever defined, are given by

$$\frac{\partial^m}{\partial x^m} V_a(x) = e^{\beta_c(a)} \int_0^a \sum_{n\geq 1} \frac{\partial^m}{\partial x^m} f_n^a(x, y) V_a(y) \mathrm{d}y,$$

$m \geq 1$. Therefore, the simple estimate $\frac{\partial}{\partial x} f_n^a(x, y) \geq (a - x) f_n^a(x, y)$ implies that also

$$\frac{\partial}{\partial x} V_a(x) \geq (a - x) V_a(x). \tag{32}$$

Integrating, we get

$$\frac{V_a(z)}{V_a(x)} \geq e^{a(z-x)-\frac{1}{2}(z^2-x^2)} \tag{33}$$

whenever $0 \leq x \leq z \leq a$. Using it for $z = a, x = y$, we have

$$\begin{aligned} e^{-\beta_c(a)} &= \int_0^a \sum_{n\geq 1} f_n^a(a, y) \frac{V_a(y)}{V_a(a)} \mathrm{d}y \\ &\leq \int_0^a \sum_{n\geq 1} f_n^a(a, y) e^{-\frac{1}{2}a^2+ay-\frac{1}{2}y^2} \mathrm{d}y \\ &\leq \int_0^a e^{-\frac{1}{2}a^2+ay-\frac{1}{2}y^2} \mathrm{d}y \cdot \sum_{n\geq 1} f_n \\ &= e^{-\beta_c} \int_0^a e^{-\frac{1}{2}(a-y)^2} \mathrm{d}y \\ &= e^{-\beta_c} \int_0^a e^{-\frac{1}{2}y^2} \mathrm{d}y \\ &\leq a e^{-Da^2} e^{-\beta_c}. \end{aligned}$$

(Indeed, $e^{-x} = 1 - x + o(x)$, so $\int_0^a e^{-\frac{1}{2}y^2} \mathrm{d}y - ae^{-Da^2} = -\frac{1}{6}a^3 + Da^3 + o(a^3)$ and thus for $D < \frac{1}{6}$ the last expression is negative whenever $a > 0$ is small enough.) Therefore the lower bound

$$a e^{\beta_c(a)-\beta_c} \geq e^{Da^2}$$

is achieved. For the upper bound, note first that since V_a is strictly positive (32) implies that it is also (strictly) increasing on $[0, a]$. In particular, $V_a(y) \geq V_a(0)$ for all $y \in [0, a]$, and, using the lower bound (13), we get

$$\begin{aligned} e^{-\beta_c(a)} &= \int_0^a \sum_{n\geq 1} f_n^a(0, y) \frac{V_a(y)}{V_a(0)} \mathrm{d}y \\ &\geq \int_0^a \sum_{n\geq 1} f_n^a(a, y) \mathrm{d}y \\ &\geq \int_0^a \mathrm{d}y \sum_{n\geq 1} f_n e^{-C_0 a} \\ &= a e^{-C_0 a - \beta_c}. \end{aligned}$$

Therefore, the upper bound

$$ae^{\beta_c(a)-\beta_c} \le e^{C_0 a}$$

is also achieved.

Remark 5.2. Following the line of the last proof one gets a stronger statement. Indeed, under $\tilde{\mathbf{E}}^{\alpha}_{\frac{b}{\sqrt{N}},\beta_c(\frac{b}{\sqrt{N}}),N}$ (or generally, under $\tilde{\mathbf{E}}^{\alpha}_{\varphi_{\frac{b}{\sqrt{N}}},N}$) $\mathcal{A}^{\alpha}_N$ is tight and every limit set $\mathcal{B}^{\alpha}$ is absolutely continuous with respect to $\mathcal{A}^{\alpha}_{1/2}$. Moreover, denoting the Radon–Nikodym density by D_b, then for every $\epsilon > 0$

$$(1-\epsilon)e^{-\epsilon L_1} \le D_b \le (1+\epsilon)e^{\epsilon L_1}$$

whenever $b > 0$ is small enough.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Lemma 2.2

In this section we prove Lemma 2.2. First, let

$$P^a_x(n) := \mathbb{P}_x[S_1 > a, \dots, S_n > a], \text{ and } P(n) := P^0_0(n). \tag{34}$$

Note that $P^a_x(n)$ is (continuously) increasing in $x \in [0, a]$. In particular, $P^a_0(n) \le P^a_x(n) \le P^a_a(n) = P(n)$ for $x \in [0, a]$. For the right part a classical result is

$$P(n) \sim \frac{1}{\sqrt{2\pi}} n^{-1/2}.$$

The following is a weak version of Sohier [21, Lemma 2.2.].

Lemma A.1. *There is a monotonously decreasing function $C^a(x) : [0, a] \to \mathbb{R}+$ so that $C^a(a) = 1$ and*

$$P^a_x(n) \sim \frac{C^a(x)}{\sqrt{2\pi}} n^{-1/2}.$$

Proof. If we set $C^a(x) := \mathbf{P}[H_1 \ge a - x]$, the asymptotic equivalence in the line above is the content of [21, Lemma 2.2.], where H_1 is the so called first ascending ladder point. The proof is done by noticing that H_1 is defined to be a non-negative random variable. □

Putting the last statements together, we get that there is a monotonously decreasing function $C^a(x) : [0, a] \to \mathbb{R}+$ so that $C^a(a) = 1$ (and hence also $C^a(0) > 0$) and

$$C^a(0) \sim \sqrt{2\pi} n^{1/2} P_0^a(n) \le \sqrt{2\pi} n^{1/2} P_x^a(n) \le \sqrt{2\pi} n^{1/2} P(n) \sim 1 \tag{35}$$

for $x \in [0, a]$.

As a corollary we have

Corollary A.2. *Assume that $a = a_n \to 0$. Then uniformly in $x_n \in [0, a_n]$*

$$\sqrt{2\pi} n^{1/2} P_{x_n}^{a_n}(n) \to 1 \text{ as } n \to \infty,$$

or equivalently $P_{x_n}^{a_n}(\cdot) \sim P(\cdot)$.

Proof. Indeed,

$$1 = \liminf_{n\to\infty} C^{a_n}(0) \le \liminf_{n\to\infty} \sqrt{2\pi} n^{1/2} P_{x_n}^{a_n}(n) \le \limsup_{n\to\infty} \sqrt{2\pi} n^{1/2} P(n) = 1 \quad \square$$

Remember we assumed in the introduction that $\rho(x) = \frac{1}{\kappa} e^{-V(x)}$, where $V \in C^2$ is symmetric and $V''(x) \in [1/c, c]$ for some $c > 1$. It follows that V' is antisymmetric so that $V'(0) = 0$, and moreover,

$$V' \text{ is positive on the positive real half line and (strictly) increasing on the real line.} \tag{36}$$

Proof of Lemma 2.2. We shall show the following sufficient condition: there are constants $0 < c_0, \tilde{c}_0, c_1, \tilde{c}_1$ so that for all $0 \le a \le 1$ and $n \ge 1$

$$\exp(-c_0 a - \tilde{c}_0 a^2) \le f_n^a / f_n \le \exp(-c_1 a + \tilde{c}_1 a^2).$$

Denote by $A_n(y)$ the event $\{S_1 > 0, \dots, S_{n-1} > 0, S_n = y\}$, so that $f_n^0(x, y) = \mathbb{P}_x[A_n(y)]$ (with the convention in (2)). We first note that $f_n^a = f_n^a(0, 0) = f_n^0(-a, -a) = \mathbb{P}_{-a}[A_n(-a)]$, by stationarity. Taking a derivative from the right-most expression we get

$$\frac{\partial}{\partial a} f_n^a = -\mathbb{E}_{-a}[V'(S_1 + a)\mathbb{1}_{A_n(-a)}] - \mathbb{E}_{-a}[V'(S_{n-1} + a)\mathbb{1}_{A_n(-a)}].$$

On the event $A_n(-a)$ the random variables $S_1 + a$ and $S_{n-1} + a$ have the same distribution under $\mathbb{P}_{-a}$ and therefore

$$\frac{\partial}{\partial a} f_n^a = -2\mathbb{E}_{-a}[V'(S_1 + a)\mathbb{1}_{A_n(-a)}].$$

In particular,

$$\frac{\partial}{\partial a} f^a_{n\,|a=0} = -2\mathbb{E}_0[V'(S_1)\mathbb{1}_{A_n(0)}].$$

A direct calculation for the second derivative yields

$$\begin{aligned}\frac{\partial^2}{\partial a^2} f_n^a &= -2\frac{\partial}{\partial a}\mathbb{E}_{-a}[V'(S_1 + a)\mathbb{1}_{A_n(-a)}] \\ &= 2\mathbb{E}_{-a}[(V'(S_1 + a)^2 - V''(S_1 + a) + V'(S_1 + a)V'(S_{n-1} + a))\mathbb{1}_{A_n(-a)}].\end{aligned}$$

A second order Taylor expansion yields

$$\frac{f_n^a}{f_n} - 1 = -\frac{2a\mathbb{E}_0[V'(S_1)\mathbb{1}_{A_n(0)}]}{\mathbb{P}_0[A_n(0)]} + \frac{2a^2\mathbb{E}_{-a'}[(V'(S_1+a')^2 - V''(S_1+a') + V'(S_1+a')V'(S_{n-1}+a'))\mathbb{1}_{A_n(-a')}]}{\mathbb{P}_0[A_n(0)]},$$

where $0 < a' < a$ (allowed to depend on n). Therefore, the proof is finished once we show that both

$$c_0 \leq \mathbb{E}_0[V'(S_1)\mathbb{1}_{A_n(0)}]/\mathbb{P}_0[A_n(0)] \leq c_1 \tag{37}$$

and

$$-\tilde{c}_0 \leq \mathbb{E}_{-a'}[(V'(S_1+a')^2 - V''(S_1+a') + V'(S_1+a')V'(S_{n-1}+a'))\mathbb{1}_{A_n(-a')}]/\mathbb{P}_0[A_n(0)] \leq \tilde{c}_1 \tag{38}$$

hold for all $0 \leq a' \leq a$ and $n \geq 1$.

To prove (38) it is enough to show that

$$\mathbb{E}_{-a'}[(V'(S_1+a')^2 + V'(S_1+a')V'(S_{n-1}+a'))\mathbb{1}_{A_n(-a')}]/\mathbb{P}_0[A_n(0)] \leq \tilde{c}_1 \tag{39}$$

and

$$\mathbb{P}_{-a'}[A_n(-a')]/\mathbb{P}_0[A_n(0)] = f^{(a')}(n)/f_n \leq \tilde{c}_0. \tag{40}$$

Let us first show (37). By reversibility of the walk (due to symmetry of V) $\mathbb{P}_x[A_n(y)] = \mathbb{P}_y[A_n(x)]$ for all $x, y \geq 0$. In particular,

$$\begin{aligned}\mathbb{P}_0[S_1 > 0, \ldots, S_{n-1} > 0, S_n \in [k, k+1]] &= \int_k^{k+1} \mathbb{P}_0[S_1 > 0, \ldots, S_{n-1} > 0, S_n = x]\mathrm{d}x \\ &= \int_k^{k+1} \mathbb{P}_x[A_n(0)]\mathrm{d}x.\end{aligned}$$

The Ballot theorem [1, Theorem 1] (and the form we shall use [24, Theorem 2.12]) therefore yields that for $k \leq \sqrt{n}$

$$c_2\frac{k+1}{n^{3/2}} \leq \int_k^{k+1} \mathbb{P}_x[A_n(0)]\mathrm{d}x \leq c_3\frac{k+1}{n^{3/2}}, \tag{41}$$

where the upper bound holds for all k.

For the upper bound we get from (1), (36), and the right inequality of (41)

$$\begin{aligned}\mathbb{E}_0[V'(S_1)\mathbb{1}_{A_n(0)}] &= \int_0^\infty V'(x)\rho(x)\mathbb{P}_x[A_{n-1}(0)]\mathrm{d}x \\ &= \sum_{k=0}^\infty \int_k^{k+1} V'(x)\rho(x)\mathbb{P}_x[A_{n-1}(0)]\mathrm{d}x \\ &\leq \sum_{k=0}^\infty V'(k+1)\rho(k) \int_k^{k+1} \mathbb{P}_x[A_{n-1}(0)]\mathrm{d}x \\ &\leq \frac{2c_3}{n^{3/2}} \sum_{k=0}^\infty V'(k+1)(k+1)\rho(k) =: \frac{c_1}{\sqrt{2\pi} n^{3/2}}.\end{aligned}$$

For the lower bound we get from (1), (36), and the left inequality of (41) that

$$\mathbb{E}_0[S_1 \mathbb{1}_{A_n(0)}] \geq \int_1^2 V'(x)\rho(x)\mathbb{P}_x[A_{n-1}(0)]\mathrm{d}x$$
$$\geq \max_{[1,2]}\{V'(x)\rho(x)\}\frac{2c_2}{n^{3/2}} =: \frac{c_0}{\sqrt{2\pi}n^{3/2}}.$$

Using (6) and the fact that $\mathbb{P}_0[A_n(0)] = f_n$, (37) is now proved.

We shall now prove (39). We have

$$\begin{aligned}&\mathbb{E}_{-a}[(V'(S_1+a)^2 + V'(S_1+a)V'(S_{n-1}+a))\mathbb{1}_{A_n(-a)}]\\ &\quad= \mathbb{E}_0[(V'(S_1)^2 + V'(S_1)V'(S_{n-1}))\mathbb{1}_{S_1>a,\dots,S_{n-1}>a,S_n=0}]\\ &\quad\leq \mathbb{E}_0[(V'(S_1)^2 + V'(S_1)V'(S_{n-1}))\mathbb{1}_{S_1>0,\dots,S_{n-1}>0,S_n=0}]\end{aligned}$$

by writing the terms in the explicit integral form. Now, as in the proof of (37)

$$\begin{aligned}\mathbb{E}_0[V'(S_1)^2\mathbb{1}_{S_1>0,\dots,S_{n-1}>0,S_0=0}] &\leq \sum_{k=0}^{\infty} V'(k+1)^2\rho(k)\int_k^{k+1}\mathbb{P}_x[A_{n-1}(0)]\mathrm{d}x\\ &\leq \frac{2c_3}{n^{3/2}}\sum_{k=0}^{\infty}(k+1)V'(k+1)^2\rho(k) =: \frac{c_5}{\sqrt{2\pi}n^{3/2}}.\end{aligned}$$

For the term $\mathbb{E}_0[V'(S_1)V'(S_{n-1})\mathbb{1}_{S_1>0,\dots,S_{n-1}>0,S_n=0}]$, note that

$$\begin{aligned}&\mathbb{P}_0[S_1+y>0,\dots,S_{n-1}+y>0,\, S_n+y\in[k,k+1]]\\ &\quad=\mathbb{P}_y[S_1>0,\dots,S_{n-1}>0,\, S_n\in[k,k+1]]\\ &\quad=\int_k^{k+1}\mathbb{P}_y[S_1>0,\dots,S_{n-1}>0,\, S_n=x]\mathrm{d}x\\ &\quad=\int_k^{k+1}\mathbb{P}_x[A_n(y)]\mathrm{d}x.\end{aligned}$$

We shall use a general variation of The Ballot Theorem: for $0\leq y\leq k+1\leq\sqrt{n}/2$,

$$\int_k^{k+1}\mathbb{P}_x[A_n(y)]\mathrm{d}x \leq c_5\frac{(k+1)(y+1)^2}{n^{3/2}} \tag{42}$$

(see [24, Corollary 2.13]). Now, by the symmetric roles of x and y in the integrand we have

$$\begin{aligned}\mathbb{E}_0[V'(S_1)V'(S_{n-1})\mathbb{1}_{A_n(0)}] &= \int_0^\infty\int_0^\infty V'(x)\rho(x)V'(y)\rho(y)\mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y\\ &\leq 2\int_0^\infty\int_{\sqrt{n}/2}^\infty V'(x)\rho(x)V'(y)\rho(y)\mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y\\ &\quad+\int_0^{\sqrt{n}/2}\int_0^{\sqrt{n}/2} V'(x)\rho(x)V'(y)\rho(y)\mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y\\ &= (I)+(II).\end{aligned}$$

To bound (I) we note first that by the local limit theorem $\mathbb{P}_x[A_{n-2}(y)] \leq \mathbb{P}_x[S_{n-2}=y] \leq C/\sqrt{n}$ for some constant C, uniformly on $x, y\in\mathbb{R}$ and $n\geq 1$. In particular $\mathbb{P}_x[S_{n-2}=y]$ is uniformly bounded from above by C. Therefore,

$$\begin{aligned}
(I) &\leq 2C \int_0^\infty \int_{\sqrt{n}/2}^\infty V'(x)\rho(x)V'(y)\rho(y)\mathrm{d}x\mathrm{d}y \\
&\leq \int_0^\infty V'(y)\rho(y) \left(\frac{2C}{\kappa} \int_{\sqrt{n}/2}^\infty V'(x)e^{-V(x)}\mathrm{d}x \right) \mathrm{d}y \\
&= \frac{2C}{\kappa} e^{-V(\sqrt{n}/2)} \int_0^\infty V'(y)\rho(y)\mathrm{d}y \\
&= \frac{2C}{\kappa} e^{-V(\sqrt{n}/2)} \frac{1}{\kappa} e^{-V(0)} \\
&= \frac{2C}{\kappa^2} e^{-V(\sqrt{n}/2)} \\
&= o(n^{-3/2}),
\end{aligned}$$

here we used the symmetry of V to get $\int_0^\infty V'(y)\rho(y)\mathrm{d}y = e^{-V(0)} = 1$ and we used the strict convexity of V to conclude that $e^{-V(\sqrt{n}/2)}$ is decaying faster than any polynomial. To bound (II), we first have that

$$(II) \leq \sum_{k,l=0}^{\lfloor \sqrt{n}/2 \rfloor} \int_l^{l+1} \int_k^{k+1} V'(x)\rho(x)V'(y)\rho(y)\mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y.$$

By symmetry of $\mathbb{P}_x[A_{n-2}(y)]$ the right hand side equals

$$2 \sum_{k=0}^{\lfloor \sqrt{n}/2 \rfloor} \sum_{l=0}^{k} \int_l^{l+1} \int_k^{k+1} V'(x)\rho(x)V'(y)\rho(y)\mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y,$$

which is not larger than

$$2 \sum_{k=0}^{\lfloor \sqrt{n}/2 \rfloor} \sum_{l=0}^{k} V'(k+1)\rho(k)V'(l+1)\rho(l) \int_k^{k+1} \int_l^{l+1} \mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y.$$

Using (42), if $l \leq k$ then

$$\int_l^{l+1} \int_k^{k+1} \mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y \leq \int_l^{l+1} c_5 \frac{(k+1)(y+1)^2}{n^{3/2}}\mathrm{d}y \leq c_5 \frac{(k+1)(l+2)^2}{n^{3/2}}.$$

$$\begin{aligned}
(II) &\leq 2 \sum_{k=0}^{\lfloor \sqrt{n}/2 \rfloor} \sum_{l=0}^{k} V'(k+1)\rho(k)V'(l+1)\rho(l) \int_k^{k+1} \int_l^{l+1} \mathbb{P}_x[A_{n-2}(y)]\mathrm{d}x\mathrm{d}y \\
&\leq \frac{c_5}{n^{3/2}} \sum_{k=0}^{\sqrt{n}/2} \sum_{l=0}^{k} (l+2)^2 V'(l+1)\rho(l)(k+1)V'(k+1)\rho(k) \\
&\leq \frac{c_5}{n^{3/2}} \sum_{k=0}^{\infty} (k+1)^2 V'(k+1)^2 \rho(k) \\
&=: \frac{c_6}{n^{3/2}}. \quad \square
\end{aligned}$$

Appendix B

Proof of Lemma 5.1. By Lemma 2.1

$$\begin{aligned}\int_0^\infty \int_0^\infty b_\lambda^a(x,y)^2 \mathrm{d}x\mathrm{d}y &= \int_0^a \int_0^a \left(\sum_{n=0}^\infty e^{-\lambda n} f_n^a(x,y)\right)\left(\sum_{m=0}^\infty e^{-\lambda m} f_m^a(x,y)\right)\mathrm{d}x\mathrm{d}y \\ &= \int_0^a \int_0^a \sum_{n,m=0}^\infty e^{-\lambda(n+m)} f_n^a(x,y) f_m^a(x,y)\mathrm{d}x\mathrm{d}y \\ &= \sum_{n,m=0}^\infty e^{-\lambda(n+m)} \int_0^a \int_0^a f_m^a(x,y) f_n^a(x,y)\mathrm{d}x\mathrm{d}y \\ &\le \sum_{n,m=0}^\infty e^{-\lambda(n+m)} f_n f_m \int_0^a \int_0^a \mathrm{d}x\mathrm{d}y \\ &\le c^2 a^2 \sum_{n,m=0}^\infty e^{-\lambda(n+m)} (nm)^{-3/2} \\ &= \left(ca \sum_{n=0}^\infty e^{-\lambda n} n^{-3/2}\right)^2 < \infty,\end{aligned}$$

for every $\lambda \ge 0$. □

Appendix C

Lemma C.1. *Let $(R_N)_{N\ge 1}$ be a sequence of non-negative random variables. Assume that there exist some $\epsilon_0 > 0$ and $C < \infty$ so that $\mathbb{E}[e^{\epsilon_0 R_N}] \le C$ for all N. Then $\mathbb{E}[e^{\epsilon_N R_N}] \to 1$ for every sequence $\epsilon_N \to 0$.*

Proof. We first assume that $\epsilon_N > 0$. Let $\delta > 0$. It is enough to show that $\mathbb{E}[e^{\epsilon_N R_N}] \le 1 + \delta$ for all N large enough. By Chebyshev's Inequality $\mathbb{P}[R_N > r] \le Ce^{-\epsilon_0 r}$ for all r. Take r_0 so that $Ce^{-\epsilon_0 r_0/2} < \delta/2$. It holds that

$$\begin{aligned}\mathbb{E}[e^{\epsilon_N R_N}] &= \mathbb{E}[e^{\epsilon_N R_N}\mathbb{1}_{R_N\le r_0}] + \mathbb{E}[e^{\epsilon_N R_N}\mathbb{1}_{R_N> r_0}] \\ &\le e^{\epsilon_N r_0} + \mathbb{E}[e^{2\epsilon_N R_N}]^{1/2}\mathbb{P}[R_N > r_0]^{1/2} \\ &\le 1 + \delta/2 + C^{1/2}C^{1/2}e^{-\epsilon_0 r_0/2} \\ &\le 1 + \delta\end{aligned}$$

whenever N is so large so that both $e^{\epsilon_N r_0} < 1 + \delta/2$ and $2\epsilon_N \le \epsilon_0$ hold. Here we used Cauchy–Schwarz in the first inequality and the fact that $\mathbb{E}[e^{\epsilon R_N}]$ is increasing in ϵ in the second one. The proof for $-\epsilon_N$ is similar. Indeed,

$$\begin{aligned}\mathbb{E}[e^{-\epsilon_N R_N}] &\ge \mathbb{E}[e^{-\epsilon_N R_N}\mathbb{1}_{R_N\le r_0}] \\ &\ge e^{-\epsilon_N r_0}(1 - \mathbb{P}[R_N > r_0]) \\ &\ge (1 - \frac{\delta}{2})(1 - \frac{\delta}{2}) \\ &\ge 1 - \delta\end{aligned}$$

whenever r_0 is chosen so that $Ce^{-\epsilon_0 r_0} \leq \delta/2$ and then N is so large so that $e^{-\epsilon_N r_0} \leq 1 - \delta/2$. For general ϵ_N's, the lemma follows once we write them as $\epsilon_N = \epsilon_N^+ - \epsilon_N^-$, the negative part subtracted from the positive part, and use the above on each part separately. □

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