AUTONOMOUS VEHICLES DRIVING TRAFFIC: THE CAUCHY PROBLEM*

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Abstract. This paper deals with the Cauchy Problem for a PDE-ODE model, where a system of two conservation laws, namely the Two-Phase macroscopic model proposed in [13], is coupled with an ordinary differential equation describing the trajectory of an autonomous vehicle (AV), which aims to control the traffic flow. Under suitable assumptions, we prove a global in time existence result.

Keywords. 2×2 Hyperbolic Systems of Conservation Laws; Mixed PDE-ODE System; Continuum Traffic Models; Autonomous Vehicles; Control Problems

AMS subject classifications. 35L65; 90B20

1. Introduction

We consider a coupled PDE-ODE system that describes the mutual interaction between bulk traffic and an autonomous vehicle (AV) responsible to control the traffic flow. In recent years, the study of autonomous vehicles (AVs) to regulate road traffic developed a lot. The AVs acting as controllers, appear to be the most innovative technology for traffic monitoring and management, also in order to reduce congested traffic and pollution; see [15,30,32,43,45]. The main results in this direction are based on ODEs that describe the trajectories of both vehicles driven by humans and the AVs, see for instance [16]. More recently, PDE models were coupled with ODEs via a moving flux constraint, see [17,18] or via AVs see [19,24]. The novelty of this paper is the use of a Two-Phase model for modeling the bulk traffic in the road. Up to now, only the scalar case has been treated in the literature [24].

The PDE system considered in this paper is the Two-Phase traffic model recently proposed in [13]. This is a possibly degenerate system of two conservation laws with Lipschitz continuous flow. It is coupled with an ordinary differential equation describing the trajectory of the AV. More precisely, the bulk traffic is governed by the system

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho,w)) = 0 \\ \partial_t(\rho w) + \partial_x(\rho w v(\rho,w)) = 0, \end{cases} \tag{1.1}$$

where $\rho=\rho(t,x)\in[0,R]$ is the car density, $w=w(t,x)\in[w_{\min},w_{\max}]$ denotes the maximal speed of drivers, and $v=v(\rho,w)=\min\{V_{\max}, w\psi(\rho)\}$ is the average speed. Here $V_{\max}$ is the maximal velocity of all the drivers and $\psi=\psi(\rho)$ a decreasing function. The AV dynamics is described by the control equation

$$\dot y(t) = \min\{u(t), v(\rho(t,y(t)+), w(t,y(t)+))\}, \tag{1.2}$$

where $y(t)$ denotes the position of the AV at time t and the control $u=u(t)\in[0,V_{\max}]$ represents the desired speed. The minimum in (1.2) takes care of the fact that the AV can not go faster than the cars immediately in front. Following [17, 24] the complete

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system is

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) = 0 \\ \dot{y}(t) = \min\{u(t), v(\rho(t, y(t)+), w(t, y(t)+))\} \\ \rho(t, y(t))(v(\rho(t, y(t)), w(t, y(t))) - \dot{y}(t)) \le F_\alpha(w, \dot{y}(t)), \end{cases} \tag{1.3}$$

where the last inequality imposes a flux constraint at the position of the AV through the function F_α, which describes the reduced capacity of the road at the AV position. Finally, we couple (1.3) with the initial conditions

$$\begin{cases} (\rho(0, x), w(0, x)) = (\rho_0(x), w_0(x)) \\ y(0) = y_0. \end{cases} \tag{1.4}$$

In this paper, we prove a global in time existence of solutions to the Cauchy Problem for the PDE-ODE model (1.3)-(1.4). Differently from classical results for hyperbolic conservation laws, we are able to consider initial conditions with finite total variation, not necessarily small. A consequence is that system (1.3) can be easily generalized to the case of several autonomous vehicles provided they do not interact each other. Existence of solutions is obtained through compactness: we use, for the PDE system, the Helly's Theorem together with the wave-front tracking method, while, for the ODE control equation, the classical Ascoli-Arzelà Theorem. Here, different from the scalar case where fine estimates on traces are necessary, non-characteristic conditions at the AV location guarantee that the limit is a solution to the Cauchy problem.

The literature on the modeling of vehicular traffic offers a lot of different approaches as macroscopic, microscopic and kinetic models possibly coupled between them. In the context of macroscopic models, based on partial differential equations, the basic one is the classical Lighthill–Whitham [37] and Richards [41] (LWR) model, given by a single conservation law. Then, the so called *second order* ones are based on two equations, as the Aw-Rascle-Zhang (ARZ) [3, 46], the GARZ [21] and the collapsed GARZ (CGARZ) model [22]. A further class, again based on two equations, is that of Two-Phase or Phase Transition models as (1.1); they are characterized by two different phases: the *Free* and the *Congested one*. A peculiarity of two-phase models is the existence of a free regime where only the density characterizes the state of the system, while in congested regime it is necessary the use of an additional quantity. Thus, in the free phase the model reduces to a single conservation law, while in the congested phase it is a hyperbolic system of two conservation laws. For other Two-Phase and Phase Transition models see [2, 5, 6, 10, 29, 36]. For other kinetic, microscopic or coupled PDE-ODE descriptions see [4, 9, 12, 20, 26, 28, 34, 40].

Up to now, existence of solution to the Cauchy problem for AV coupled with PDE has been obtained only in the scalar case, specifically for the LWR model. A similar, but different, approach consists in the study of conservation laws with pointwise unilateral constraints on the flow. Their peculiarity is the possible presence of a non-classical shock, violating the classical Kružkov [33] or Lax [35] entropy admissibility conditions, at the constraint position. Scalar conservation laws with fixed flux constraints have been introduced in [11]; here, the problem is to provide a mathematical framework to model local constraints in traffic flow, such as traffic lights or tool gates. Results for the second order models as the Aw-Rascle-Zhang (ARZ) model with fixed constraints are provided in [1, 23, 27]. Problems with moving constraints have been considered in [17] for the scalar case and [44] for the ARZ model.

The article is organized as follows. Section 2 gives a description of the model from an analytic point of view and describes the solution of the constrained Riemann problem. Section 3 contains the proof of the existence result for the Cauchy problem; the proof is divided into four different subsections. Finally an appendix with technical lemmas concludes the paper.

2. Basic Properties and the Riemann Problem In this section we recall basic properties of the Two-Phase model and the Riemann problem both in the classical case and in presence of flux constraints. For a detailed description of (1.1) we refer to [13]. The free and congested phases are described by the sets

$$F = \{(\rho,w) \in [0,R] \times [w_{\min}, w_{\max}] \colon v(\rho,\rho w) = V_{\max}\},$$
$$C = \{(\rho,w) \in [0,R] \times [w_{\min}, w_{\max}] \colon v(\rho,\rho w) = w\psi(\rho)\},$$

represented in Figure 2.1. Here we assume the following conditions.

(H-1) $R, w_{\min}, w_{\max}, V_{\max}$ are positive constants, with $V_{\max} < w_{\min} < w_{\max}$; R is the maximal possible density, typically $R = 1$.

(H-2) $\psi \in \mathbf{C^2}([0,R];[0,1])$ is such that $\psi(0) = 1$, $\psi(R) = 0$, and, for every $\rho \in [0,R]$, $c_\psi \le -\psi'(\rho) \le C_\psi$, $\frac{d^2}{d\rho^2}(\rho\psi(\rho)) \le 0$ for suitable constants $0 < c_\psi < C_\psi$.

(H-3) Waves of the first family in the congested phase C have negative speed. More precisely, we assume that there exists a positive constant $\bar\lambda$ such that $\lambda_1(\rho,w) \le -\bar\lambda$, where $\lambda_1(\rho,w) = w(\rho\psi'(\rho) + \psi(\rho))$ is the first eigenvalue of the Jacobian matrix of the flux.

(H-4) There exist $L_F > 0$ and $F_{\alpha,1} \in \mathbf{C^1}([w_{\min}, w_{\max}];\mathbb{R}^+)$ satisfying:
  (a) $F_\alpha(w,\sigma) = F_{\alpha,1}(w)(V_{\max} - \sigma)$ for every $w \in [w_{\min}, w_{\max}]$ and $\sigma \in [0, V_{\max}]$, where F_α is the function appearing in (1.3);
  (b) the inequality
$$|F_{\alpha,1}(w_1) - F_{\alpha,1}(w_2)| \le L_F |w_1 - w_2|$$
  holds for every $w_1, w_2 \in [w_{\min}, w_{\max}]$;
  (c) $\psi(F_{\alpha,1}(w_{\max})) > \frac{V_{\max}}{w_{\min}}$;
  (d) $F'_{\alpha,1}(w) \ge 0$ for every $w \in [w_{\min}, w_{\max}]$.

Note that assumptions **(H-1)-(H-2)** imply that there exists a unique value $\rho_c \in (0,R)$ such that $\psi(\rho_c) = \frac{V_{\max}}{w_{\min}}$; see Figure 2.1 right.

Remark 2.1. *In the congested phase, the variable w is constant along the curves of the first family. Similarly, the variable v is constant along the curves of the second family. In particular, this implies that v and w are Riemann invariants. Thus, system (1.1) can be represented alternatively through the couple (v,w).*

For $w \in [w_{\min}, w_{\max}]$ and $\sigma \in [0, V_{\max}]$, define the function

$$\begin{aligned} \varphi_{w,\sigma} \colon [0,R] &\longrightarrow \mathbb{R} \\ \rho &\longmapsto F_\alpha(w,\sigma) + \rho\sigma \end{aligned}$$

and the unique densities $\check\rho = \check\rho(w,\sigma)$ and $\hat\rho = \hat\rho(w,\sigma)$ respectively as the solutions to

$$\varphi_{w,\sigma}(\check\rho) = \check\rho V_{\max}$$

and to

$$\varphi_{w,\sigma}(\hat\rho) = w\hat\rho\psi(\hat\rho).$$

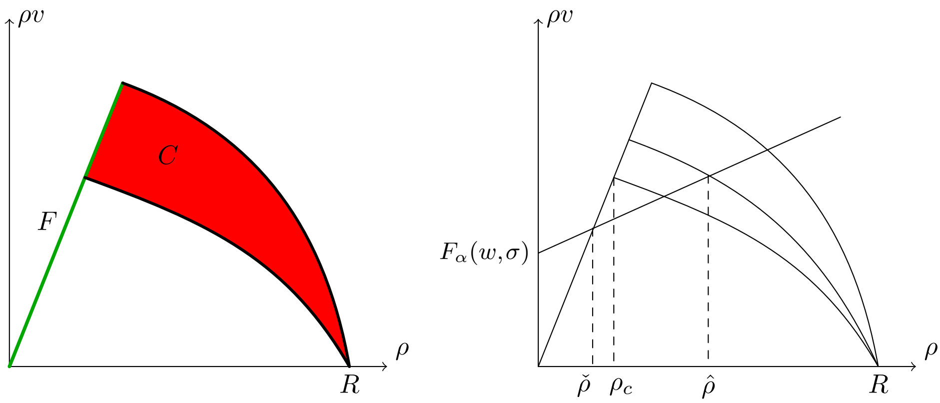

FIG. 2.1. *Left, the free phase F (in green) and the congested phase C (in red) in the coordinates $(\rho,\rho v)$. Right, the construction of $\check\rho$, $\hat\rho$, and ρ_c.*

Moreover

$$\check\rho(w,\sigma) = \frac{F_\alpha(w,\sigma)}{V_{\max}-\sigma} = F_{\alpha,1}(w) \tag{2.1}$$

and so, if $w_1 < w_2$ and $\sigma \in [0, V_{\max}]$, then by **(H-4)**

$$0 \le \check\rho(w_2,\sigma) - \check\rho(w_1,\sigma) \le L_F(w_2 - w_1). \tag{2.2}$$

By **(H-4)**, the function $\check\rho(w,\sigma)$ is increasing with respect to w and constant in σ. Note also that, by assumption (c) in **(H-4)**, $(\check\rho, w) \in F \setminus C$ for every $w \in [w_{\min}, w_{\max}]$. In principle the point $(\hat\rho, w)$ not necessarily belongs to C; however, in the following, when the flux constraint is effective, then $(\hat\rho, w)$ is in C; see Figure 2.1, right.

2.1. Classical and Constrained Riemann Problems

First, we denote by

$$\begin{array}{rccc} \mathcal{RS}: & (F\cup C)^2 & \longrightarrow & \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; F\cup C) \\ & ((\rho_l,w_l),(\rho_r,w_r)) & \longmapsto & (\mathcal{RS}_\rho, \mathcal{RS}_w) \end{array}$$

the Riemann solver for the classical Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho,w)) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v(\rho,w)) = 0, \\ (\rho,w)(0,x) = \begin{cases} (\rho_l,w_l), & x<0, \\ (\rho_r,w_r), & x>0, \end{cases} \end{cases} \tag{2.3}$$

in the sense that the functions

$$\rho(t,x) = \mathcal{RS}_\rho\left(\frac{x}{t}\right), \qquad w(t,x) = \mathcal{RS}_w\left(\frac{x}{t}\right),$$

provide the solution to the Riemann problem (2.3).

At the location of the AV, we have to consider a constrained Riemann problem, which depends also on the speed of the AV; see also [39].

Definition 2.1. *Fix a constant* $\bar{u}\in[0,V_{\max}]$ *and two states* $(\rho_l,w_l),(\rho_r,w_r)\in F\cup C$. *The Riemann problem for (1.3), related to* $\bar{u}$ *and to* (ρ_l,w_l) *and* (ρ_r,w_r)*, is the following Cauchy problem*

$$\begin{cases} \partial_t\rho+\partial_x(\rho v(\rho,w))=0, \\ \partial_t(\rho w)+\partial_x(\rho w v(\rho,w))=0, \\ \dot{y}(t)=\min\{\bar{u},v(\rho(t,y(t)+),w(t,y(t)+))\}, \\ \rho(t,y(t))(v(\rho(t,y(t)),w(t,y(t)))-\dot{y}(t))\leq F_\alpha(w,\dot{y}(t)), \\ (\rho,w)(0,x)=\begin{cases}(\rho_l,w_l), & x<0, \\ (\rho_r,w_r), & x>0,\end{cases} \\ y(0)=0. \end{cases} \tag{2.4}$$

The solution to the constrained Riemann problem (2.4) is given through the constrained Riemann solver, introduced in the following definition.

Definition 2.2. *The constrained Riemann solver*

$$\begin{array}{rcl} \mathcal{RS}^c:\ (F\cup C)^2\times[0,V_{\max}] & \longrightarrow & \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};F\cup C)\times[0,V_{\max}] \\ ((\rho_l,w_l),(\rho_r,w_r),\bar{u}) & \longmapsto & \left((\mathcal{RS}^c_\rho,\mathcal{RS}^c_w),\boldsymbol{u}\right) \end{array}$$

is a function representing a solution to the constrained Riemann problem (2.4), in the sense that the functions

$$\rho(t,x)=\mathcal{RS}^c_\rho\left(\frac{x}{t}\right),\qquad w(t,x)=\mathcal{RS}^c_w\left(\frac{x}{t}\right),\qquad y(t)=\boldsymbol{u}t$$

are a solution to (2.4).

Denoting by $f_1(\rho,w)=\rho v(\rho,w)$, the construction of $\mathcal{RS}^c$ is as follows:

1. if $f_1(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))\leq\varphi_{w_l,\bar{u}}(\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))$, then

$$\begin{aligned} &\mathcal{RS}^c_\rho((\rho_l,w_l),(\rho_r,w_r),\bar{u})\left(\frac{x}{t}\right)=\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))\left(\frac{x}{t}\right), \\ &\mathcal{RS}^c_w((\rho_l,w_l),(\rho_r,w_r),\bar{u})\left(\frac{x}{t}\right)=\mathcal{RS}_w((\rho_l,w_l),(\rho_r,w_r))\left(\frac{x}{t}\right), \\ &\boldsymbol{u}=\min\{\bar{u},v(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}+))\}; \end{aligned}$$

2. if $f_1(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))>\varphi_{w_l,\bar{u}}(\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))$, then

$$\begin{aligned} &\mathcal{RS}^c_\rho((\rho_l,w_l),(\rho_r,w_r),\bar{u})\left(\frac{x}{t}\right)=\begin{cases}\mathcal{RS}_\rho((\rho_l,w_l),(\hat{\rho},w_l))\left(\frac{x}{t}\right), & \frac{x}{t}<\bar{u}, \\ \mathcal{RS}_\rho((\check{\rho},w_l),(\rho_r,w_r))\left(\frac{x}{t}\right), & \frac{x}{t}>\bar{u},\end{cases} \\ &\mathcal{RS}^c_w((\rho_l,w_l),(\rho_r,w_r),\bar{u})\left(\frac{x}{t}\right)=\begin{cases}\mathcal{RS}_w((\rho_l,w_l),(\hat{\rho},w_l))\left(\frac{x}{t}\right), & \frac{x}{t}<\bar{u}, \\ \mathcal{RS}_w((\check{\rho},w_l),(\rho_r,w_r))\left(\frac{x}{t}\right), & \frac{x}{t}>\bar{u},\end{cases} \\ &\boldsymbol{u}=\bar{u}. \end{aligned}$$

Remark 2.2. *The Riemann solver* $\mathcal{RS}^c$ *produces a solution to the constrained Riemann problem* (2.4) *such that the density flux at the AV location is below the constraint imposed by such vehicle. More precisely, it returns the "classical" solution in the case its flux is below the threshold, otherwise it produces a non classical wave connecting two states, which satisfy the flux constraint, traveling with speed of the AV.*

It is interesting to note that the constrained Riemann solver, introduced in Definition 2.2, satisfies a consistency property.

Proposition 2.1. *Fix* $\bar{u}\in[0,V_{\max}]$ *and two states* $(\rho_l,w_l),(\rho_r,w_r)\in F\cup C$ *and define* $\boldsymbol{u}$ *as the velocity component of* $\mathcal{RS}^c((\rho_l,w_l),(\rho_r,w_r),\bar{u})$*. Then*

$$\mathcal{RS}^c((\rho_l,w_l),(\rho_r,w_r),\bar{u})=\mathcal{RS}^c((\rho_l,w_l),(\rho_r,w_r),\boldsymbol{u}). \tag{2.5}$$

Proof. If $f_1(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))>\varphi_{w_l,\bar{u}}(\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))$, then $\boldsymbol{u}=\bar{u}$ and so (2.5) clearly holds.

If $f_1(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))\le\varphi_{w_l,\bar{u}}(\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))(\bar{u}))$, then

$$\boldsymbol{u}=\min\{\bar{u},v(\mathcal{RS}((\rho_l,w_l),(\rho_r,w_r))(\bar{u}+))\}\le\bar{u}.$$

Clearly in the case $\boldsymbol{u}=\bar{u}$, the conclusion easily follows. Assume therefore that $\boldsymbol{u}<\bar{u}$. Define by (ρ_m,w_m) as the middle state (if it exists) in the solution of the classical Riemann problem with left state (ρ_l,w_l) and right state (ρ_r,w_r), otherwise put $(\rho_m,w_m)=(\rho_l,w_l)$. Then define

$$(\bar{\rho},\bar{w})\in\{(\rho_l,w_l),(\rho_m,w_m),(\rho_r,w_r)\}$$

such that $\boldsymbol{u}=v(\bar{\rho},\bar{w})$. Note that $(\bar{\rho},\bar{w})$ exists by the construction in Definition 2.2. Since $\boldsymbol{u}<\bar{u}\le V_{\max}$, then $(\bar{\rho},\bar{w})\in C$ and so

$$f_1(\bar{\rho},\bar{w})=\boldsymbol{u}\bar{\rho}<F_\alpha(w_l,\boldsymbol{u})+\boldsymbol{u}\bar{\rho}=\varphi_{w_l,\boldsymbol{u}}(\mathcal{RS}_\rho((\rho_l,w_l),(\rho_r,w_r))(\bar{u})).$$

This permits to conclude. □

2.2. Wave Notation Below, we list the waves and the notations that we will use in the present paper.

- *First Family Wave* 1: a wave connecting a left state $(\rho_l,w_l)\in C$ with a right state $(\rho_r,w_r)\in C$ such that $w_l=w_r$.
- *Second Family Wave* 2: a wave connecting a left state $(\rho_l,w_l)\in C$ with a right state $(\rho_r,w_r)\in C$ such that $v(\rho_l,w_l)=v(\rho_r,w_r)$.
- *Linear Wave* $\mathcal{LW}$: a wave connecting two states in the free phase.
- *Phase Transition Wave* $\mathcal{PT}$: a wave connecting a left state $(\rho_l,w_l)\in F$ with a right state $(\rho_r,w_r)\in C$ satisfying $w_l=w_r$.
- *Fictitious Wave* $\mathcal{FW}$: a wave denoting the AV trajectory without discontinuity in (ρ,w). The notation stands for a fictitious wave.
- *Non Fictitious Wave* $\mathcal{NFW}$: a wave denoting the AV trajectory with discontinuity in (ρ,w) and connecting the left state (ρ_l,w_l) with the right state (ρ_r,w_r) such that $w_l=w_r$, $\rho_l=\hat{\rho}(w_l,\sigma)$, and $\rho_r=\check{\rho}(w_l,\sigma)$ for some $\sigma\in[0,V_{\max}[$. The notation stands for a non fictitious wave.
- *Special Non Fictitious Wave* $\mathcal{SNFW}$: a wave denoting the AV trajectory with discontinuity in (ρ,w), which is not a $\mathcal{NFW}$.

Remark 2.3. *We note that the states* (ρ_l,w_l) *and* $(\hat{\rho},w_l)$ *are connected by a possible combination of waves of the first family and phase transition waves with speed less than* $\bar{u}$*. The states* $(\check{\rho},w_l)$ *and* (ρ_r,w_r) *are connected either by a linear wave or by a possible combination of a phase transition and a second family wave with speed greater than* $\bar{u}$*.*

Remark 2.4. $\mathcal{NFW}$ *and* $\mathcal{SNFW}$ *are both waves denoting the AV trajectory. In the former case, it is a wave where the flux constraints holds with equality, i.e.* $\rho(v-\dot{y})=F_\alpha$ *with* $\dot{y}<V_{\max}$*. In the latter case, the AV trajectory coincides with a classical wave of the Two-Phase model.*

3. The Cauchy Problem

In this section we consider the Cauchy problem for the control problem (1.3) with moving constraint, that is

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) = 0 \\ \dot{y}(t) = \min\{u(t), v(\rho(t, y(t)+), w(t, y(t)+))\} \\ \rho(t, y(t)) (v(\rho(t, y(t)), w(t, y(t))) - \dot{y}(t)) \le F_\alpha(w, \dot{y}(t)) \\ (\rho, w)(0, x) = (\rho_0(x), w_0(x)) \\ y(0) = y_0, \end{cases} \tag{3.1}$$

with control function $u \in \mathbf{BV}(\mathbb{R}^+; [0, V_{\max}])$, initial data $(\rho_0, w_0) : \mathbb{R} \to F \cup C$, and $y_0 \in \mathbb{R}$. Before stating the main result, we introduce the definition of solution to the constrained Cauchy problem (3.1).

Definition 3.1. *The couple*

$$((\rho^*, w^*), y^*) \in \mathbf{C^0}\left([0, +\infty[; \mathbf{L^1}((\mathbb{R}; F \cup C)\right) \times \mathbf{W^{1,\infty}_{loc}}([0, +\infty[; \mathbb{R})$$

is a solution to (3.1) with control $u = u(t)$, if

1. *the function (ρ^*, w^*) is a weak solution (see Remark 3.2) to the PDE in (3.1), for $(t, x) \in \Omega_-$ and for $(t, x) \in \Omega_+$, where*

$$\Omega_- = \{(t, x) \in (0, +\infty) \times \mathbb{R} : x < y^*(t)\}$$
$$\Omega_+ = \{(t, x) \in (0, +\infty) \times \mathbb{R} : x > y^*(t)\};$$

2. *for a.e. $t > 0$, the function $x \mapsto (\rho^*(t, x), w^*(t, x))$ has bounded total variation;*
3. *$(\rho^*(0, x), w^*(0, x)) = (\rho_0(x), w_0(x))$, for a.e. $x \in \mathbb{R}$;*
4. *the function y is a Caratheodory solution to the ODE in (3.1), i.e. for a.e. $t \in \mathbb{R}^+$*

$$y(t) = y_0 + \int_0^t \min\{u(s), v(s, y(s)+)\}\, ds;$$

5. *the constraint is satisfied, in the sense that for a.e. $t \in \mathbb{R}^+$*

$$\lim_{x \to y(t)\pm} (\rho(t, x) (v(\rho(t, x), w(t, x)) - \dot{y}(t)) - F_\alpha(w(t, x), \dot{y}(t))) \le 0.$$

Remark 3.1. *Note that the point 4. of Definition 3.1 is formulated directly using the variable v, accordingly with Remark 2.1.*

Remark 3.2. *By [13, Remark 5.3], a couple (ρ^*, w^*) is a weak solution to the PDE in* (3.1) *if and only if the couple (ρ^*, η^*), with $\eta^* = \rho^* w^*$, is a weak solution to*

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v(\rho, \frac{\eta}{\rho})\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta v(\rho, \frac{\eta}{\rho})\right) = 0. \end{cases}$$

We can now state the main result of the paper:

Theorem 3.1. *Let assumptions* **(H-1)**, **(H-2)**, **(H-3)**, *and* **(H-4)** *hold. Fix the control function $u \in \mathbf{BV}(\mathbb{R}^+; [0, V_{\max}])$ and the initial conditions $(\rho_0, w_0) \in \mathbf{BV}(\mathbb{R}; F \cup C)$ and $y_0 \in \mathbb{R}$. Then there exists $((\rho^*, w^*), y^*)$, a solution to (3.1) in the sense of Definition 3.1.* The proof is contained in the following subsections. In particular, we construct a sequence of approximate solutions by using the wave-front tracking method, and we prove its convergence. Throughout the section, we implicitly assume that hypotheses **(H-1)**–**(H-4)** hold.

3.1. Wave-Front Tracking Approximate Solution In this subsection we construct piecewise constant approximations via the wave-front tracking method, which is a set of techniques to obtain approximate solutions to hyperbolic conservation laws in one space dimension. It was first introduced by Dafermos [14], see also [7,31] for the general theory.

At first, we give the following definition of an ε-approximate wave-front tracking solution to (3.1).

DEFINITION 3.2. *Given $\varepsilon > 0$, the map $(z_\varepsilon, y_\varepsilon, u_\varepsilon)$ is an ε-approximate wave-front tracking solution to (3.1) if the following conditions hold.*

1. $z_\varepsilon = (\rho_\varepsilon, w_\varepsilon) \in \mathbf{C^0}\left([0,+\infty[; \mathbf{L^1}(\mathbb{R}; F \cup C)\right)$;
2. $y_\varepsilon \in \mathbf{W^{1,\infty}}([0,+\infty[;\mathbb{R})$;
3. $u_\varepsilon \in \mathbf{BV}([0,+\infty[;[0,V_{\max}])$ *is piecewise constant;*
4. $(\rho_\varepsilon, w_\varepsilon)$ *is piecewise constant, with discontinuities along finitely many straight lines in* $(0,+\infty) \times \mathbb{R}$. *Moreover the jumps can be of the first family, of the second family, linear waves or phase transition waves;*
5. *it holds that*

$$\begin{cases} \|(\rho_\varepsilon(0,\cdot), w_\varepsilon(0,\cdot)) - (\rho_0(\cdot), w_0(\cdot))\|_{\mathbf{L^1}(\mathbb{R})} < \varepsilon \\ \mathrm{TV}\,(\rho_\varepsilon(0,\cdot), w_\varepsilon(0,\cdot)) \le \mathrm{TV}\,(\rho_0(\cdot), w_0(\cdot)) \\ \|u_\varepsilon - u\|_{\mathbf{L^1}(\mathbb{R}^+)} < \varepsilon \\ \mathrm{TV}\,(u_\varepsilon) \le \mathrm{TV}\,(u)\,; \end{cases}$$

6. *for a.e.* $t \in \mathbb{R}^+$,

$$y_\varepsilon(t) = y_0 + \int_0^t \min\left\{u_\varepsilon(s), v\left(\rho_\varepsilon(s, y_\varepsilon(s)+), w_\varepsilon(s, y_\varepsilon(s)+)\right)\right\} ds; \tag{3.2}$$

7. *the constraint is satisfied, in the sense that for a.e.* $t \in \mathbb{R}^+$

$$\lim_{x \to y_\varepsilon(t)\pm} \left(\rho_\varepsilon\left(t, y_\varepsilon(t)+\right)\left(v(\rho_\varepsilon(t, y_\varepsilon(t)+), w_\varepsilon(t, y_\varepsilon(t)+)) - \dot{y}_\varepsilon(t)\right)\right)(t,x) \\ \le F_\alpha\left(w_\varepsilon\left(t, y_\varepsilon(t)+\right), \dot{y}_\varepsilon(t)\right).$$

We describe here a procedure for constructing a sequence of wave-front approximate solutions. For every $\nu \in \mathbb{N} \setminus \{0\}$, we consider the triple $(\rho_{0,\nu}, w_{0,\nu}, u_\nu)$ of piecewise constant functions with a finite number of discontinuities such that the following conditions hold.

1. $(\rho_{0,\nu}, w_{0,\nu}) : \mathbb{R} \to F \cup C$ and $u_\nu : \mathbb{R}^+ \to [0, V_{\max}]$;
2. the following limits hold

$$\lim_{\nu \to +\infty} (\rho_{0,\nu}, w_{0,\nu}) = (\rho_0, w_0) \qquad \text{in } \mathbf{L^1}(\mathbb{R}; F \cup C)$$
$$\lim_{\nu \to +\infty} u_\nu = u \qquad \text{in } \mathbf{L^1}(\mathbb{R}^+; [0, V_{\max}]);$$

3. the following inequalities hold

$$\mathrm{TV}(\rho_{0,\nu}, w_{0,\nu}) \le \mathrm{TV}(\rho_0, w_0)$$
$$\mathrm{TV}\,(u_\nu) \le \mathrm{TV}\,(u).$$

Next, for every $\nu \in \mathbb{N} \setminus \{0\}$, we proceed with the following method. At time $t = 0$, we solve all the classical Riemann problems for $x \in \mathbb{R}$, $x \neq y_0$ and the constrained Riemann problem, located at y_0. We approximate every rarefaction wave of the first family with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ traveling with the Rankine-Hugoniot speed. In this way we construct a piecewise approximate solution (ρ_ν, w_ν, y_ν) until the first time at which two waves interact together, or a wave interacts with the AV, or u_ν has a discontinuity. In the first case, we solve the classical Riemann problem and we prolong the approximate solution beyond this time. In the second case or in the third one, we solve the constrained Riemann problem and we prolong the approximate solution beyond this time. We repeat this procedure at every interaction times.

REMARK 3.3. *Slightly changing the velocity of waves, as described in [7, Remark 7.1], or the discontinuity times of u_ν, we may assume that, at every positive time t, at most one of the following possibilities happens:*

1. *two classical waves (first family wave, second family wave, linear wave, and phase transition wave) interact together at a point $x \in \mathbb{R}$, with $x \neq y_\nu(t)$;*
2. *a classical wave hits the AV trajectory y_ν;*
3. $u_\nu(t-) \neq u_\nu(t+)$.

REMARK 3.4. *The $\mathcal{NFW}$ wave connects two states through a non-classical shock with positive speed.*

Instead, the $\mathcal{SNFW}$ connects two states through a wave with positive speed, which can be either a second family wave or a phase transition wave $\mathcal{PT}$, or a linear wave $\mathcal{LW}$.

It is always possible to construct a wave-front tracking approximate solution such that, for t in a right neighborhood of 0, the $\mathcal{SNFW}$ wave is not present.

Moreover, the interaction estimates considered in Section 3.2 imply that, at positive times, the $\mathcal{SNFW}$ wave can be only superimposed to a linear wave.

Given an ε-approximate wave-front tracking solution $(z_\varepsilon, y_\varepsilon, u_\varepsilon)$, define, for a.e. $t > 0$, the following functionals

$$\mathcal{F}_w(t) = \sum_{x \in \mathbb{R}} |w(z_\varepsilon(t,x+)) - w(z_\varepsilon(t,x-))| \tag{3.3}$$

$$\mathcal{F}_{\tilde{v}}(t) = \sum_{x \in \mathbb{R}} |\tilde{v}(z_\varepsilon(t,x+)) - \tilde{v}(z_\varepsilon(t,x-))| - 2|\tilde{v}(z_\varepsilon(t,y_\varepsilon(t)+)) - \tilde{v}(z_\varepsilon(t,y_\varepsilon(t)-))| \tag{3.4}$$

$$\mathcal{F}(t) = \mathcal{F}_w(t) + \mathcal{F}_{\tilde{v}}(t), \tag{3.5}$$

$$\mathcal{N}(t) = \#\{x \in \mathbb{R} : z_\varepsilon(t,x-) \neq z_\varepsilon(t,x+)\}, \tag{3.6}$$

where, we denote by $\tilde{v}$ the function

$$\tilde{v}(\rho, w) = w\psi(\rho).$$

Moreover the functionals $\mathcal{N}_1^-(t)$ and $\mathcal{N}_1^+(t)$ denote respectively the number of discontinuities given by a wave of the first family at the left, resp. at the right, of $y_\varepsilon(t)$. Finally, $\mathcal{N}_2^-(t)$, $\mathcal{N}_2^-(t)$, $\mathcal{N}_{\mathcal{PT}}^-(t)$, $\mathcal{N}_{\mathcal{PT}}^-(t)$, $\mathcal{N}_{\mathcal{LW}}^-(t)$, $\mathcal{N}_{\mathcal{LW}}^-(t)$ are defined similarly. Note that the previous functionals may vary only at times $\bar{t}$, described in Remark 3.3.

The functional $\mathcal{F}(t)$ is composed by 2 terms. The first term measures the strength of waves of second family. The second term measures the strength of waves of first family and of phase transition waves. Moreover both of the first two terms measure the strength of linear waves. Note that the functional $\mathcal{F}_{\tilde{v}}$ contains a term which depends on the position of the AV. This is just a technical tool for having better interaction estimates; see [1] for a similar approach. In general this functional may assume negative values; however it is bounded from below, since its second term is uniform bounded.

3.2. Interaction estimates

We consider interactions estimates between waves. We will consider different types of interactions separately. It is not restrictive to assume that, at any interaction time $t=\bar{t}$, exactly one possibility enumerated in Remark 3.3 happens: either two waves interact, or a wave hits the AV trajectory, or the control changes.

We describe wave interactions by the nature of the involved waves, see [25,26]. For example, if a wave of the second family hits a wave of the first family producing a phase-transition wave, we write 2-1/ $\mathcal{PT}$. Here the symbol "/" divides the waves before and after the interaction.

3.2.1. Collisions between classical waves

For the classical collision between two waves, we have the following result.

Lemma 3.1. *Assume that the wave* $((\rho^l,w^l),(\rho^m,w^m))$ *interacts with the wave* $((\rho^m,w^m),(\rho^r,w^r))$ *at the point* $(\bar{t},\bar{x})$ *with* $\bar{t}>0$ *and* $\bar{x}\in\mathbb{R}$. *Then* $\mathcal{F}(\bar{t}+)\leq\mathcal{F}(\bar{t}-)$. *The possible interactions are:* 2-1/1-2, $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2, 1-1/1, $\mathcal{PT}$-1/$\mathcal{PT}$. *Hence* $\mathcal{N}(\bar{t}+)\leq\mathcal{N}(\bar{t}-)$.

See [38, Lemma 3.4.] for the proof.

3.2.2. Collisions with a fictitious wave In this part, we assume that a wave $((\rho^l,w^l),(\rho^r,w^r))$ interacts at a time $\bar{t}$ with the AV and that there is no discontinuity at the position of the AV before time $\bar{t}$. For simplicity, we introduce also the following notations:

$$\begin{aligned} &u^-=u(\bar{t}), && u^+=u^-, && \check{\rho}=\check{\rho}(u^-,w^l), && \hat{\rho}=\hat{\rho}(u^-,w^l),\\ &v^l=\tilde{v}(\rho^l,w^l), && v^r=\tilde{v}(\rho^r,w^r), && \check{v}=\tilde{v}(\check{\rho},w^l), && \hat{v}=\tilde{v}(\hat{\rho},w^l), \end{aligned} \tag{3.7}$$

and we denote with ξ^- and ξ^+ respectively the speed of the AV before and after the interaction (see (3.2)), and define $\Delta\xi=\xi^+-\xi^-$.

We have the following results.

Lemma 3.2. *Assume that the second family wave* $((\rho^l,w^l),(\rho^r,w^r))$ *interacts from the left with the* $\mathcal{FW}$ *wave at the point* $(\bar{t},\bar{x})$ *with* $\bar{t}>0$ *and* $\bar{x}\in\mathbb{R}$. *We have the following cases:*

1. *No new wave is produced. Then* $\Delta\mathcal{F}_w(\bar{t})=\Delta\mathcal{F}_{\tilde{v}}(\bar{t})=\Delta\mathcal{N}(\bar{t})=0$, *so that* $\Delta\mathcal{F}(\bar{t})=0$. *Finally* $|\Delta\xi|\leq\left|v^l-v^r\right|$.
2. *The interaction is of type* 2-$\mathcal{FW}$/1-$\mathcal{NFW}$-$\mathcal{PT}$-2. *Then we deduce that* $\Delta\mathcal{F}_w(\bar{t})=\Delta\mathcal{F}_{\tilde{v}}(\bar{t})=\Delta\mathcal{F}(\bar{t})=0$, *and* $\Delta\mathcal{N}(\bar{t})=3$. *Finally* $|\Delta\xi|=0$.

Proof. We use the notation in (3.7). Since before $\bar{t}$ there is no discontinuity at the position of the AV, then $\rho^r v^r\leq\varphi_{w^r,u^-}(\rho^r)$. At time $\bar{t}$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^r,w^r),u^-\right)$$

and the AV enters the region with state (ρ^l,w^l). We have two cases.

1. $\rho^l v^l\leq\varphi_{w^l,u^-}(\rho^l)$. In this case, the second family wave crosses the AV trajectory and no new wave is created. Thus, for the functionals (3.3)- (3.6), we have:

$$\Delta\mathcal{F}_w(\bar{t})=\Delta\mathcal{F}_{\tilde{v}}(\bar{t})=\Delta\mathcal{F}(\bar{t})=\Delta\mathcal{N}(\bar{t})=0.$$

Moreover $\xi^-=\min\{v^r,u^-\}$, $\xi^+=\min\{v^l,u^-\}$, and so $|\Delta\xi|\leq\left|v^l-v^r\right|$.

2. $\rho^l v^l>\varphi_{w^l,u^-}(\rho^l)$. In this case, the second family wave crosses the AV trajectory producing a first family shock wave $((\rho^l,w^l),(\hat{\rho},w^l))$, a $\mathcal{NFW}$ wave $((\hat{\rho},w^l),(\check{\rho},w^l))$, a phase transition wave with positive speed $((\check{\rho},w^l),(\rho^l,w^l))$,

and a second family wave $((\rho^l,w^l),(\rho^r,w^r))$, so that $\Delta\mathcal{N}(\bar t)=3$. For the functional (3.3) we have $\Delta\mathcal{F}_w(\bar t)=0$.
For the functional (3.4), since $v^l=v^r$ because the interacting wave is a second family wave and since $v^l>\hat v$, $\check v>\hat v$ and $\check v>v^l$, we have that

$$\begin{aligned}\Delta\mathcal{F}_{\check v}(\bar t) &= \left|v^l-\hat v\right|+|\check v-\hat v|+\left|\check v-v^l\right|+\left|v^l-v^r\right|-2|\check v-\hat v|-\left|v^l-v^r\right|\\ &=0.\end{aligned}$$

Here we have $\xi^-=\min\{v^r,u^-\}=u^-$, $\xi^+=u^-$, and so $\Delta\xi=0$.

□

LEMMA 3.3. *Assume that the phase transition wave $((\rho^l,w^l),(\rho^r,w^r))$ interacts from the left with the $\mathcal{FW}$ wave at the point $(\bar t,\bar x)$ with $\bar t>0$ and $\bar x\in\mathbb{R}$. Then the interacting phase transition wave has positive speed and no new wave is produced. Moreover $\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=0$ and $\Delta\mathcal{N}(\bar t)=0$. Finally $|\Delta\xi|\le\left|v^l-v^r\right|$.*

Proof. We use the notation in (3.7). Since the interacting wave is a phase transition, then $w^l=w^r$. Before $\bar t$ there is no discontinuity at the position of the AV, then $\rho^r v^r\le\varphi_{w^r,u^-}(\rho^r)$.

The speed of the phase transition is bigger than that of the fictitious wave. In particular it is positive and $\rho^l v^l\le\varphi_{w^l,u^-}(\rho^l)$. Thus, at time $\bar t$, the solution of

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^r,w^r),u^-\right)$$

is classical, in the sense that the phase transition wave crosses the AV trajectory and no new wave is created. Thus, for the functionals (3.3)-(3.6), we have:

$$\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=\Delta\mathcal{N}(\bar t)=0.$$

Here $\xi^-=\min\{v^r,u^-\}$, $\xi^+=\min\{V_{\max},u^-\}=u^-$, and so either $|\Delta\xi|=0$ or $|\Delta\xi|=u^- - v^r\le\left|v^l-v^r\right|$. □

LEMMA 3.4. *Assume that the $\mathcal{FW}$ wave interacts from the left with the phase transition wave $((\rho^l,w^l),(\rho^r,w^r))$ at the point $(\bar t,\bar x)$ with $\bar t>0$ and $\bar x\in\mathbb{R}$. Then no new wave is produced. Moreover $\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=0$ and $\Delta\mathcal{N}(\bar t)=0$. Finally $|\Delta\xi|\le\left|v^l-v^r\right|$.*

Proof. We use the notation in (3.7). Since the interacting wave is a phase transition, then $w^l=w^r$. Before $\bar t$ there is no discontinuity at the position of the AV, then $\rho^l v^l\le\varphi_{w^l,u^-}(\rho^l)$.

The speed of the phase transition is smaller than that of the fictitious wave. In particular $\rho^r v^r\le\varphi_{w^l,u^-}(\rho^r)$. Thus, at time $\bar t$, the solution of

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^r,w^r),u^-\right)$$

is classical, in the sense that the phase transition wave crosses the AV trajectory and no new wave is created. Thus, for the functionals (3.3)-(3.6), we have:

$$\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=\Delta\mathcal{N}(\bar t)=0.$$

Here $\xi^-=\min\{V_{\max},u^-\}=u^-$, $\xi^+=\min\{v^r,u^-\}$, and so either $|\Delta\xi|=0$ or $|\Delta\xi|=u^- - v^r\le\left|v^l-v^r\right|$. □

LEMMA 3.5. *Assume that the linear wave $((\rho^l,w^l),(\rho^r,w^r))$ interacts with the $\mathcal{FW}$ wave at the point $(\bar t,\bar x)$ with $\bar t>0$ and $\bar x\in\mathbb{R}$. We have the two different cases:*

1. *No new wave is produced. Then $\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=\Delta\mathcal{N}(\bar t)=0$. Finally $\Delta\xi=0$.*

2. *The interaction is $\mathcal{LW}$-$\mathcal{FW}$ /$\mathcal{PT}$-$\mathcal{NFW}$-$\mathcal{LW}$. Then we get $\Delta\mathcal{F}_w(\bar{t}) = 0$, $\Delta\mathcal{F}(\bar{t}) = \Delta\mathcal{F}_{\check{v}}(\bar{t}) \leq 0$, and $\Delta\mathcal{N}(\bar{t}) = 2$. Finally $\Delta\xi = 0$.*

Proof. We use the notation in (3.7). Before $\bar{t}$ there is no discontinuity at the position of the AV, then $\rho^r v^r \leq \varphi_{w^r,u^-}(\rho^r)$. At time $\bar{t}$, we need to consider

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^r,w^r),u^-\right).$$

We have two different cases.

1. $\rho^l v^l \leq \varphi_{w^l,u^-}(\rho^l)$. In this case, the linear wave crosses the bus trajectory and no new wave is created. Thus, for the functionals (3.3)-(3.6), we have:

$$\Delta\mathcal{F}_w(\bar{t}) = \Delta\mathcal{F}_{\check{v}}(\bar{t}) = \Delta\mathcal{F}(\bar{t}) = \Delta\mathcal{N}(\bar{t}) = 0.$$

   Here $\xi^- = \xi^+ = \min\{V_{\max}, u^-\} = u^-$ and so $\Delta\xi = 0$.
2. $\rho^l v^l > \varphi_{w^l,u^-}(\rho^l)$. In this case the linear wave crosses the AV trajectory producing a phase transition wave $((\rho^l,w^l),(\hat{\rho},w^l))$, a $\mathcal{NFW}$ wave $((\hat{\rho},w^l),(\check{\rho},w^l))$, and a linear wave $((\check{\rho},w^l),(\rho^r,w^r))$. For the functionals (3.3)-(3.6), since $\check{v} > v^l > \hat{v}$, we have:

$$\begin{aligned}\Delta\mathcal{F}_w(\bar{t}) &= \left|w^l - w^r\right| - \left|w^l - w^r\right| = 0,\\ \Delta\mathcal{F}_{\check{v}}(\bar{t}) &= \left|v^l - \hat{v}\right| + |\check{v} - \hat{v}| + |\check{v} - v^r| - 2|\check{v} - \hat{v}| - \left|v^l - v^r\right|\\ &= \left(v^l - \hat{v}\right) - (\check{v} - \hat{v}) + |\check{v} - v^r| - \left|v^l - v^r\right|\\ &\leq 0\\ \Delta\mathcal{N}(\bar{t}) &= 2.\end{aligned}$$

   Here $\xi^- = \min\{V_{\max}, u^-\} = u^-$, $\xi^+ = u^-$ and so $\Delta\xi = 0$.

□

Lemma 3.6. *Assume that the $\mathcal{FW}$ wave interacts with the first family wave $((\rho^l,w^l),(\rho^r,w^r))$ at the point $(\bar{t},\bar{x})$ with $\bar{t} > 0$ and $\bar{x} \in \mathbb{R}$. We have the following cases:*

1. *No new wave is produced. Then $\Delta\mathcal{F}_w(\bar{t}) = \Delta\mathcal{F}_{\check{v}}(\bar{t}) = \Delta\mathcal{F}(\bar{t}) = \Delta\mathcal{N}(\bar{t}) = 0$. Finally $|\Delta\xi| \leq \left|v^l - v^r\right|$.*
2. *The interaction is $\mathcal{FW}$-1/1-$\mathcal{NFW}$-$\mathcal{PT}$. Then $\Delta\mathcal{F}_w(\bar{t}) = 0$, $\Delta\mathcal{F}(\bar{t}) = \Delta\mathcal{F}_{\check{v}}(\bar{t}) < 0$, and $\Delta\mathcal{N}(\bar{t}) = 2$. Finally $|\Delta\xi| \leq v^r - v^l$.*

Proof. We use the notation in (3.7). Before $\bar{t}$ there is no discontinuity at the position of the AV, then $\rho^l v^l \leq \varphi_{w^l,u^-}(\rho^l)$. Moreover $w^l = w^r$ and at time $\bar{t}$, we need to consider

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^r,w^r),u^-\right).$$

We have two different cases.

1. $\rho^r v^r \leq \varphi_{w^l,u^-}(\rho^r)$. In this case, the first family wave crosses the AV trajectory and no new wave is created. Thus, for the functionals (3.3)-(3.6), we have:

$$\Delta\mathcal{F}_w(\bar{t}) = \Delta\mathcal{F}_{\check{v}}(\bar{t}) = \Delta\mathcal{F}(\bar{t}) = \Delta\mathcal{N}(\bar{t}) = 0.$$

   Here $\xi^- = \min\{v^l, u^-\}$, $\xi^+ = \min\{v^r, u^-\}$, and so $|\Delta\xi| \leq \left|v^l - v^r\right|$.
2. $\rho^r v^r > \varphi_{w^l,u^-}(\rho^r)$. In this case, the interaction produces a first family wave $((\rho^l,w^l),(\hat{\rho},w^l))$, a $\mathcal{NFW}$ wave $((\hat{\rho},w^l),(\check{\rho},w^l))$, and a phase transition wave $((\check{\rho},w^l),(\rho^r,w^l))$, with positive speed. Thus, for the functionals (3.3)-(3.6), since $\check{v} > v^r > \hat{v} > v^l$, we have:

$$\Delta\mathcal{F}_w(\bar{t}) = \left|w^l - w^r\right| - \left|w^l - w^r\right| = 0,$$

$$\begin{aligned}\Delta\mathcal{F}_{\tilde v}(\bar t) &= \left|v^l-\hat v\right|+\left|\check v-\hat v\right|+\left|\check v-v^r\right|-2\left|\check v-\hat v\right|-\left|v^l-v^r\right|\\ &=\left(\hat v-v^l\right)-\left(\check v-\hat v\right)+\left(\check v-v^r\right)-\left(v^r-v^l\right)\\ &=2\left(\hat v-v^r\right)<0,\\ \Delta\mathcal{N}(\bar t) &= 2.\end{aligned}$$

Here $\xi^-=\min\{v^l,u^-\}$, $\xi^+=u^-$. Since $v^l<u^-<v^r$, then $|\Delta\xi|=\left|v^l-u^-\right|=u^--v^l\le v^r-v^l$.

□

REMARK 3.5. *We observe that the following interaction can not happen:*

- *A $\mathcal{FW}$ wave can not interact from the left with a second family wave. Indeed the velocity of cars coincides with that of the second family and so equation (1.2) prevents such interaction.*

3.2.3. Collisions from the left with a non fictitious wave

In this part, we assume that a wave $((\rho^l,w^l),(\rho^r,w^r))$ interacts from the left at a time $\bar t$ with the AV and that there is a discontinuity at the position of the AV before time $\bar t$. For simplicity, we introduce also the following notations:

$$\begin{array}{llll} u^-=u(\bar t), & u^+=u^-, & v^l=\tilde v(\rho^l,w^l), & v^r=\tilde v(\rho^r,w^r),\\ \check\rho^l=\check\rho(u^-,w^l), & \check\rho^r=\check\rho(u^-,w^r), & \hat\rho^l=\hat\rho(u^-,w^l), & \hat\rho^r=\hat\rho(u^-,w^r),\\ \check v^l=\tilde v(\check\rho^l,w^l), & \check v^r=\tilde v(\check\rho^r,w^r), & \hat v^l=\hat\rho(\hat\rho^l,w^l), & \hat v^r=\hat\rho(\hat\rho^r,w^r). \end{array} \tag{3.8}$$

and we denote with ξ^- and ξ^+ respectively the speed of the AV before and after the interaction (see (3.2)), and define $\Delta\xi=\xi^+-\xi^-$.

LEMMA 3.7. *Assume that the second family wave $\left((\rho^l,w^l),(\rho^r,w^r)\right)$ interacts from the left with the $\mathcal{NFW}$ wave at the point $(\bar t,\bar x)$ with $\bar t>0$ and $\bar x\in\mathbb{R}$. The interaction is 2-$\mathcal{NFW}$/1-$\mathcal{NFW}$-$\mathcal{LW}$. Moreover $\Delta\mathcal{F}_w(\bar t)=0$, $\Delta\mathcal{F}_{\tilde v}(\bar t)\le 2\left(C_\psi w^l L_F\frac{1}{w_{\min}\lambda}+w^l C_\psi L_F+1\right)\left|w^l-w^r\right|$, $\Delta\mathcal{F}(\bar t)=\Delta\mathcal{F}_{\tilde v}(\bar t)$, and $\Delta\mathcal{N}(\bar t)\le\frac{V_{\max}}{\nu}$, where $\tilde C_F$ is a suitable positive constant depending only on* **(H-4)**. *Finally $\Delta\xi=0$.*

Proof. We use the notation in (3.8). Note that in this situation $u^-<V_{\max}$, otherwise the interaction wave can not happen. The left and right states at the position of the AV before $\bar t$ are given respectively by

$$(\rho^r,w^r)=(\hat\rho^r,w^r)\qquad\text{and}\qquad(\check\rho^r,w^r).$$

At time $\bar t$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\rho^l,w^l),(\check\rho^r,w^r),u^-\right).$$

Let (ρ^m,w^m) be in the intersection between the free and the congested phase, with $w^m=w^l$. We have that $\rho^m v^m>\varphi_{w^m,u^-}(\rho^m)$. In this case, there is a production of a first family rarefaction wave $((\rho^l,w^l),(\hat\rho^l,w^l))$, of a $\mathcal{NFW}$ wave $((\hat\rho^l,w^l),(\check\rho^l,w^l))$ and of a linear wave connecting $(\check\rho^l,w^l)$ with $(\check\rho^r,w^r)$. For the functionals (3.3)-(3.6), since $\check v^l>\hat v^l$, $\check v^r>v^r$ and $v^l=v^r$, we have

$$\begin{aligned}\Delta\mathcal{F}_w(\bar t)&=0,\\ \Delta\mathcal{F}_{\tilde v}(\bar t)&=\left|v^l-\hat v^l\right|+\left|\hat v^l-\check v^l\right|+\left|\check v^l-\check v^r\right|-2\left|\check v^l-\hat v^l\right|\\ &\quad-\left|v^l-v^r\right|-\left|v^r-\check v^r\right|+2\left|v^r-\check v^r\right|\end{aligned}$$

$$
\begin{aligned}
&= \left|v^l - \hat{v}^l\right| - \left(\check{v}^l - \hat{v}^l\right) + \left|\check{v}^l - \check{v}^r\right| \\
&\quad + \left(\check{v}^r - v^l\right), \\
\Delta\mathcal{N}(\bar{t}) &\le \frac{V_{\max}}{\nu}.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) &= \left|v^l - \hat{v}^l\right| - \left(\check{v}^l - \hat{v}^l\right) + \left|\check{v}^l - \check{v}^r\right| + \left(\check{v}^r - v^l\right) \\
&= \left|v^l - \hat{v}^l\right| + \left(\hat{v}^l - v^l\right) + \left|\check{v}^l - \check{v}^r\right| + \left(\check{v}^r - \check{v}^l\right)
\end{aligned}
$$

and so

$$
\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) \le 2\left|v^l - \hat{v}^l\right| + 2\left|\check{v}^l - \check{v}^r\right|.
$$

By **(H-2)**, **(H-4)**, (2.2) and Lemma 4.3, there exists $\tilde{C}_F > 0$ such that

$$
\begin{aligned}
\left|v^l - \hat{v}^l\right| &= w^l\left|\psi\left(\rho^l\right) - \psi\left(\hat{\rho}^l\right)\right| \le C_\psi w^l \left|\rho^l - \hat{\rho}^l\right| \\
&\le C_\psi w^l \frac{1}{w^l\bar{\lambda} + u^-}\left|F_\alpha\left(w^l, u^-\right) - F_\alpha\left(w^r, u^-\right)\right| \\
&\le C_\psi w^l L_F \frac{1}{w_{\min}\bar{\lambda}}\left|w^l - w^r\right|.
\end{aligned}
$$

Moreover, by **(H-2)**, **(H-4)**, (2.1), and (2.2),

$$
\begin{aligned}
\left|\check{v}^l - \check{v}^r\right| &= \left|w^l\psi\left(\check{\rho}^l\right) - w^r\psi\left(\check{\rho}^r\right)\right| \\
&\le w^l C_\psi \left|\check{\rho}^l - \check{\rho}^r\right| + \psi\left(\check{\rho}^r\right)\left|w^l - w^r\right| \\
&= w^l C_\psi \frac{\left|F_\alpha\left(w^l, u^-\right) - F_\alpha\left(w^r, u^-\right)\right|}{V_{\max} - u^-} + \psi\left(\check{\rho}^r\right)\left|w^l - w^r\right| \\
&\le \left(w^l C_\psi L_F + 1\right)\left|w^l - w^r\right|.
\end{aligned}
$$

Therefore we conclude that

$$
\begin{aligned}
\Delta\mathcal{F}_{\tilde{v}}(\bar{t}) &\le 2\left|v^l - \hat{v}^l\right| + 2\left|\check{v}^l - \check{v}^r\right| \\
&\le 2\left(C_\psi w^l L_F \frac{1}{w_{\min}\bar{\lambda}} + w^l C_\psi L_F + 1\right)\left|w^l - w^r\right|.
\end{aligned}
$$

Here $\xi^- = \xi^+ = u^-$ and so $\Delta\xi = 0$. This completes the proof. □

Lemma 3.8. *Assume that the phase transition wave $((\rho^l, w^l), (\rho^r, w^r))$, with positive speed, interacts from the left with the $\mathcal{NFW}$ wave at the point $(\bar{t}, \bar{x})$ with $\bar{t} > 0$ and $\bar{x} \in \mathbb{R}$. The only interaction is $\mathcal{PT}$-$\mathcal{NFW}$/$\mathcal{FW}$-$\mathcal{LW}$. Then $\Delta\mathcal{F}_w(\bar{t}) = \Delta\mathcal{F}_{\tilde{v}}(\bar{t}) = \Delta\mathcal{F}(\bar{t}) = 0$ and $\Delta\mathcal{N}(\bar{t}) = -1$. Finally $\Delta\xi = 0$.*

Proof. We use the notation in (3.8). Note that in this case $w^l = w^r$ and $\check{\rho}^l = \check{\rho}^r, \hat{\rho}^l = \hat{\rho}^r, \check{v}^l = \check{v}^r, \hat{v}^l = \hat{v}^r$. At time $\bar{t}$, we need to consider the Riemann solver

$$
\mathcal{RS}^c\left((\rho^l, w^l), (\check{\rho}^l, w^l), u^-\right).
$$

In this case $\rho^l v(\rho^l, w^l) \le \varphi_{w^l, u^-}(\rho^l)$. Thus, we apply the classical Riemann Problem between the states (ρ^l, w^l) and $(\check{\rho}^l, \check{w})$, see [13, Theorem 2.1]. That is, the phase transition crosses the AV producing a linear wave $((\rho^l, w^l), (\check{\rho}^l, \check{w}))$. For the functionals (3.3)-(3.6), since $v^r < \check{v}^l < v^l$, we have

$$
\Delta\mathcal{F}_w(\bar{t}) = 0,
$$

$$\begin{aligned}\Delta\mathcal{F}_{\check v}(\bar t) &= \left|v^l-\check v^l\right|-\left|v^l-v^r\right|-\left|v^r-\check v^l\right|+2\left|v^r-\check v^l\right| \\ &= v^l-\check v^l-v^l+v^r+\check v^l-v^r=0, \\ \Delta\mathcal{N}(\bar t) &= -1.\end{aligned}$$

Here $\xi^-=u^-$, $\xi^+=\min\{V_{\max},u^-\}=u^-$, and so $\Delta\xi=0$. □

REMARK 3.6. *We note that the following interaction can not happen:*

- *A linear wave can not interact with a $\mathcal{NFW}$ wave $((\hat\rho,\hat w),(\check\rho,\check w))$ from the left, since $(\hat\rho,\hat w)$ is in the congested phase and a linear wave connects two states in the free phase.*

3.2.4. Collisions from the right with a non fictitious wave

In this part, we assume that a wave interacts from the right at a time $\bar t$ with the AV and that there is a discontinuity at the position of the AV before time $\bar t$. Again denote with ξ^- and ξ^+ respectively the speed of the AV before and after the interaction (see (3.2)), and define $\Delta\xi=\xi^+-\xi^-$.

LEMMA 3.9. *Assume that the $\mathcal{NFW}$ $((\hat\rho,\hat w),(\check\rho,\check w))$ interacts with the phase transition wave $((\check\rho,\check w),(\rho^r,w^r))$ at the point $(\bar t,\bar x)$ with $\bar t>0$ and $\bar x\in\mathbb{R}$. The only interaction is $\mathcal{NFW}$-$\mathcal{PT}$/1-$\mathcal{FW}$. Then $\Delta\mathcal{F}_w(\bar t)=\Delta\mathcal{F}_{\check v}(\bar t)=\Delta\mathcal{F}(\bar t)=0$ and $\Delta\mathcal{N}(\bar t)=-1$. Finally $|\Delta\xi|\le v^l-v^r$.*

Proof. We use the notation in (3.7). At time $\bar t$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\hat\rho,\hat w),(\rho^r,w^r),u^-\right).$$

In this case $\rho^r v(\rho^r,w^r)\le\varphi_{w^r,u^-}(\rho^r)$. Thus, we apply the classical Riemann Problem between the states $(\hat\rho,\hat w)$ and (ρ^r,w^r), see [13, Theorem 2.1]. That is, the phase transition wave crosses the AV producing a first family shock wave $((\hat\rho,\hat w),(\rho^r,w^r))$. For the functionals (3.3)-(3.6), since $\hat v>v^r$, $\check v>\hat v$ and $\check v>v^r$, we have

$$\begin{aligned}\Delta\mathcal{F}_w(\bar t) &= 0, \\ \Delta\mathcal{F}_{\check v}(\bar t) &= |\hat v-v^r|-|\hat v-\check v|-|\check v-v^r|+2|\check v-\hat v| \\ &= \hat v-v^r+\check v-\hat v-\check v+v^r=0, \\ \Delta\mathcal{N}(\bar t) &= -1.\end{aligned}$$

Here $\xi^-=u^-$, $\xi^+=\min\{v^r,u^-\}$. If $u^-\le v^r$, then $\Delta\xi=0$. If $u^->v^r$, then $|\Delta\xi|=u^--v^r\le V_{\max}-v^r\le v^l-v^r$. □

REMARK 3.7. *We note that the following interactions can not happen:*

- *The $\mathcal{NFW}$ wave $((\hat\rho,\hat w),(\check\rho,\check w))$ can not interact from the left with a second family wave. Indeed if $(\check\rho,\check w)\in F\setminus C$, then the conclusion easily follows. If $(\check\rho,\check w)\in F\cap C$, then a wave of the second family coincides with a linear wave.*
- *The $\mathcal{NFW}$ wave $((\hat\rho,\hat w),(\check\rho,\check w))$ can not interact from the left with a first family wave.*

3.2.5. Collision with a special non fictitious wave In this part we focus on the possible interactions of a $\mathcal{SNFW}$ with other waves.

LEMMA 3.10. *Assume that, at time $\bar t>0$, the $\mathcal{SNFW}$ connecting $\left(\rho^l,w^l\right)$ with (ρ^r,w^r) interacts with another wave. Suppose that $w^l=w^r$, $(\rho^l,w^l)\in F\cap C$, $(\rho^r,w^l)\in F\setminus C$ and such that $\rho^r=\check\rho(w^l,0)$.*

1. *The interaction with a first family wave is not possible.*
2. *The interaction with a second family wave is not possible.*

3. *The interaction with a linear wave is not possible.*
4. *The interaction with a phase transition wave generates a (shock) wave of the first family and a $\mathcal{FW}$. More precisely the interaction is $\mathcal{SNFW}$-$\mathcal{PT}$/1-$\mathcal{FW}$. In this case $\Delta\mathcal{F}_w(\bar t) = \Delta\mathcal{F}_{\tilde v}(\bar t) = \Delta\mathcal{F}(\bar t) = 0$ and $\Delta\mathcal{N}(\bar t) = -1$. Finally $|\Delta\xi| = v(\rho^l, w^l) - v(\bar\rho^r, w^l)$, where $(\bar\rho^r, w^l)$ is the right state of the $\mathcal{PT}$.*

Proof. By assumption the $\mathcal{SNFW}$ is also a linear wave, so that its velocity is equal to $V_{\max}$. This implies that it can not interact with another linear wave or with a second family wave.

In principle the $\mathcal{SNFW}$ can interact with a first family wave coming from the right. In this situation the state (ρ^r, w^r) is the left state of the wave of the first family, but, by hypothesis, $(\rho^r, w^r) \in F \setminus C$. This is not possible.

Consider the case of the interaction with a phase transition wave connecting $(\rho^r, w^r) \in F \setminus C$ with $(\bar\rho^r, \bar w^r) \in C \setminus F$. Note that $(\bar\rho^r, \bar w^r) \notin F$, otherwise the phase transition has speed $V_{\max}$ and the interaction is not possible. Clearly $\bar w^r = w^r = w^l$. At time $\bar t$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\rho^l, w^l), (\bar\rho^r, w^l), V_{\max}\right).$$

After the interaction, there is a (shock) wave of the first family, connecting (ρ^l, w^l) with $(\bar\rho^r, w^l)$ and a $\mathcal{FW}$ traveling at speed $\min\{V_{\max}, v(\bar\rho^r, w^l)\} = v(\bar\rho^r, w^l)$, so that the interaction is $\mathcal{SNFW}$-$\mathcal{PT}$/1-$\mathcal{FW}$.

For the functionals (3.3)-(3.6), since $(\rho^l, w^l) \in F \cap C$, $(\rho^r, w^l) \in F \setminus C$, and $(\bar\rho^r, w^l) \in C \setminus F$, then $\tilde v(\bar\rho^r, w^l) < \tilde v(\rho^l, w^l) = V_{\max} < \tilde v(\rho^r, w^l)$; thus

$$\begin{aligned}
\Delta\mathcal{F}_w(\bar t) &= 0,\\
\Delta\mathcal{F}_{\tilde v}(\bar t) &= \left|\tilde v(\rho^l, w^l) - \tilde v(\bar\rho^r, w^l)\right| + \left|\tilde v(\rho^l, w^l) - \tilde v(\rho^r, w^l)\right| - \left|\tilde v(\rho^r, w^l) - \tilde v(\bar\rho^r, w^l)\right|\\
&= V_{\max} - \tilde v(\bar\rho^r, w^l) + \tilde v(\rho^r, w^l) - V_{\max} - \tilde v(\rho^r, w^l) + \tilde v(\bar\rho^r, w^l) = 0,\\
\Delta\mathcal{N}(\bar t) &= -1.
\end{aligned}$$

Here $\xi^- = V_{\max}$, $\xi^+ = v(\bar\rho^r, w^l)$. Thus $|\Delta\xi| = v(\rho^l, w^l) - v(\bar\rho^r, w^l)$. □

We collect all the interaction estimates between two waves in Table 3.1.

3.2.6. Control changes We focus here on the situations in which a jump in the control function u occurs. and denote with ξ^- and ξ^+ respectively the speed of the AV before and after the interaction (see (3.2)), and define $\Delta\xi = \xi^+ - \xi^-$.

LEMMA 3.11. *Assume that, at time $\bar t > 0$, the control function u jumps from $u^- = u(\bar t-)$ to $u^+ = u(\bar t+)$ and that we have a $\mathcal{FW}$ at time $\bar t-$. We have the following cases.*

1. *At time $\bar t+$ we have a $\mathcal{FW}$ and no new wave is produced. Then $\Delta\mathcal{F}_w(\bar t) = \Delta\mathcal{F}_{\tilde v}(\bar t) = \Delta\mathcal{F}(\bar t) = 0$ and $\Delta\mathcal{N}(\bar t) = 0$. Finally $|\Delta\xi| \le |u^- - u^+|$.*
2. *At time $\bar t+$ we have a $\mathcal{NFW}$ and the number of waves increases. Then $(\rho^l, w^l) \in C$, $\mathcal{FW}$/1-$\mathcal{NFW}$-$\mathcal{PT}$, and $u^+ < u^-$. The phase transition wave coincides with a linear wave in the case $(\rho^l, w^l) \in F \cap C$. Moreover $\Delta\mathcal{F}_w(\bar t) = \Delta\mathcal{F}_{\tilde v}(\bar t) = \Delta\mathcal{F}(\bar t) = 0$ and $\Delta\mathcal{N}(\bar t) = 3$. Finally $|\Delta\xi| \le |u^- - u^+|$.*

Proof. Here we use the following notations:

$$\begin{aligned}
&u^- = u(\bar t-), && u^+ = u(\bar t+), && v^l = \tilde v(\rho^l, w^l), && v^r = \tilde v(\rho^r, w^r),\\
&\check\rho^l = \check\rho(u^-, w^l), && \check\rho^r = \check\rho(u^-, w^r), && \hat\rho^l = \hat\rho(u^-, w^l), && \hat\rho^r = \hat\rho(u^-, w^r),\\
&\check v^l = \tilde v(\check\rho^l, w^l), && \check v^r = \tilde v(\check\rho^r, w^r), && \hat v^l = \hat\rho(\hat\rho^l, w^l), && \hat v^r = \hat\rho(\hat\rho^r, w^r),
\end{aligned}$$

Interaction type	$\Delta\mathcal{F}_w$	$\Delta\mathcal{F}_v$	$\Delta\mathcal{N}$	Lemma
2-1/1-2 $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2 1-1/1 $\mathcal{PT}$-1/$\mathcal{PT}$	$=0$	≤ 0	≤ 0	Lemma 3.1
2-$\mathcal{FW}$/$\mathcal{FW}$-2 2-$\mathcal{FW}$/1-$\mathcal{NFW}$-$\mathcal{PT}$-2	$=0$	$=0$	$=0$ $=3$	Lemma 3.2
$\mathcal{PT}$-$\mathcal{FW}$/$\mathcal{FW}$-$\mathcal{PT}$	$=0$	$=0$	$=0$	Lemma 3.3
$\mathcal{FW}$-$\mathcal{PT}$/$\mathcal{PT}$-$\mathcal{FW}$	$=0$	$=0$	$=0$	Lemma 3.4
$\mathcal{LW}$-$\mathcal{FW}$ /$\mathcal{FW}$-$\mathcal{LW}$ $\mathcal{LW}$-$\mathcal{FW}$ /$\mathcal{PT}$-$\mathcal{NFW}$-$\mathcal{LW}$	$=0$	$=0$ ≤ 0	$=0$ $=2$	Lemma 3.5
$\mathcal{FW}$-1/1-$\mathcal{FW}$ $\mathcal{FW}$-1/1-$\mathcal{NFW}$-$\mathcal{PT}$	$=0$	$=0$ <0	$=0$ $=2$	Lemma 3.6
2-$\mathcal{NFW}$ /1-$\mathcal{NFW}$-$\mathcal{LW}$	$=0$	$\leq O(1)\left\|w^l-w^r\right\|$	$\leq \frac{V_{\max}}{\nu}$	Lemma 3.7
$\mathcal{PT}$-$\mathcal{NFW}$ /$\mathcal{FW}$-$\mathcal{LW}$	$=0$	$=0$	$=-1$	Lemma 3.8
$\mathcal{NFW}$-$\mathcal{PT}$/1-$\mathcal{FW}$	$=0$	$=0$	$=-1$	Lemma 3.9
$\mathcal{SNFW}$-$\mathcal{PT}$/1-$\mathcal{FW}$	$=0$	$=0$	$=-1$	Lemma 3.10

TABLE 3.1. *The variation of the functionals $\mathcal{F}_w$, $\mathcal{F}_v$, and $\mathcal{N}$ due to interactions between waves. The Landau symbol $O(1)$ denotes a constant; see Lemma 3.7 for the precise expression.*

At time $\bar t$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\rho^l,w^l),(\rho^l,w^l),u^+\right).$$

We have two different cases.

1. $\rho^l v^l \leq \varphi_{w^l,u^+}(\rho^l)$. In this case, at time $\bar t+$, no new wave is produced and for the functionals (3.3)-(3.6), we have

$$\Delta\mathcal{F}_w(\bar t)=\mathcal{F}_{\tilde v}(\bar t)=\Delta\mathcal{F}(\bar t)=\Delta\mathcal{N}(\bar t)=0.$$

   Here $\xi^-=\min\{v^l,u^-\}$, $\xi^+=\min\{v^l,u^+\}$, and so $|\Delta\xi|\leq|u^--u^+|$.
2. $\rho^l v^l > \varphi_{w^l,u^+}(\rho^l)$. In this case $(\rho^l,w^l)\in C$, $u^+<u^-$, and there is a production of a first family wave $((\rho^l,w^l),(\hat\rho,w^l))$, of a $\mathcal{NFW}$ wave $((\hat\rho,w^l),(\check\rho,w^l))$ and of a phase transition wave connecting $(\check\rho,w^l)$ with (ρ^l,w^l). Note that, if $(\rho^l,w^l)\in F\cap C$, then the phase transition wave is indeed a liner wave. For the functionals (3.3)-(3.6), since $\check v>v^l>\hat v$, we have

$$\begin{aligned}\Delta\mathcal{F}_w(\bar t)&=0,\\ \Delta\mathcal{F}_{\tilde v}(\bar t)&=\left|v^l-\hat v\right|+\left|\hat v-\check v\right|+\left|\check v-v^l\right|-2\left|\check v-\hat v\right|\\ &=v^l-\hat v-\check v+\hat v+\check v-v^l=0,\\ \Delta\mathcal{N}(\bar t)&=3.\end{aligned}$$

   Here $\xi^-=\min\{v^l,u^-\}$, $\xi^+=u^+$, and $v^l>u^+$. Thus $|\Delta\xi|\leq|u^--u^+|$.

□

LEMMA 3.12. *Assume that, at time $\bar t>0$, the control function u jumps from $u^-=u(\bar t-)$ to $u^+=u(\bar t+)$ and that we have a $\mathcal{NFW}$ at time $\bar t-$, connecting (ρ^l,w^l) with (ρ^r,w^l). We have the following cases.*

1. *At time $\bar{t}+$ we have a $\mathcal{SNFW}$ and the number of waves increases. The production is $\mathcal{NFW}$ /1-$\mathcal{SNFW}$ ($\mathcal{LW}$). In this case, the left state of the $\mathcal{SNFW}$ belongs to $F \cap C$, the right state belongs to $F \setminus C$, and they have the same w. Moreover $\Delta \mathcal{F}_w(\bar{t}) = 0$, $\Delta \mathcal{F}_{\tilde{v}}(\bar{t}) \leq 2w^l C_\psi \frac{F_{\alpha,1}(w_{\max})+R}{w_{\min}\bar{\lambda}}(u^+ - u^-)$, and $\Delta \mathcal{N}(\bar{t}) \leq 1 + \frac{V_{\max}}{\nu}$. Finally $|\Delta \xi| \leq |u^- - u^+|$.*
2. *At time $\bar{t}+$ we have a $\mathcal{NFW}$ and the number of waves increases. The production is $\mathcal{NFW}$ /1-$\mathcal{NFW}$ -$\mathcal{LW}$. Moreover $\Delta \mathcal{F}_w(\bar{t}) = 0$, $\Delta \mathcal{F}_{\tilde{v}}(\bar{t}) \leq 2w^l C_\psi \frac{F_{\alpha,1}(w^l)+R}{w^l \bar{\lambda}}|u^+ - u^-|$ and $\Delta \mathcal{N}(\bar{t}) \leq 1 + \frac{V_{\max}}{\nu}$. Finally it holds $|\Delta \xi| \leq |u^- - u^+|$.*

Proof. We use the following notations:

$$\begin{aligned} &u^- = u(\bar{t}-), && u^+ = u(\bar{t}+), && \check{\rho} = \check{\rho}(u^-, w^l), && \hat{\rho} = \hat{\rho}(u^-, w^l), \\ &v^l = \tilde{v}(\rho^l, w^l), && v^r = \tilde{v}(\rho^r, w^l), && \check{v} = \tilde{v}(\check{\rho}, w^l), && \hat{v} = \tilde{v}(\hat{\rho}, w^l), \\ &\check{\rho}^+ = \check{\rho}(u^+, w^l), && \hat{\rho}^+ = \hat{\rho}(u^+, w^l), && \check{v}^+ = \check{v}(u^+, w^l), && \hat{v}^+ = \hat{v}(u^+, w^l). \end{aligned}$$

Here in particular we have that $v^l = \hat{v}$ and $v^r = \check{v}$. At time $\bar{t}$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\hat{\rho}, w^l), (\check{\rho}, w^l), u^+\right).$$

Let (ρ^m, w^m) be in the intersection between the free and the congested phase, with $w^m = w^l$. We have two different cases.

1. $\rho^m v^m \leq \varphi_{w^l,u^+}(\rho^m)$. By (2.1), $\check{\rho}(w^l, \sigma)$ depends only on w^l for every $\sigma \in [0, V_{\max}]$. Moreover $\rho^m \geq \check{\rho}(w^l, \sigma)$ for $\sigma \in [0, V_{\max}]$. Therefore we deduce that $u^+ = V_{\max}$ and $\rho^m v^m = \varphi_{w^l,u^+}(\rho^m)$. In this case there is a production of a first family wave $((\hat{\rho}, w^l), (\rho^m, w^l))$ and of a linear wave connecting (ρ^m, w^l) with $(\check{\rho}, w^l)$, which is also a $\mathcal{SNFW}$ wave. Note that the left state of the $\mathcal{SNFW}$ belongs to $F \cap C$, while the right state belongs to $F \setminus C$. For the functionals (3.3)-(3.6), using **(H-2)** and Lemma 4.3, we have

$$\begin{aligned} \Delta \mathcal{F}_w(\bar{t}) &= 0, \\ \Delta \mathcal{F}_{\tilde{v}}(\bar{t}) &= |\hat{v} - v^m| + |v^m - \check{v}| - 2|v^m - \check{v}| \\ &\quad - |\hat{v} - \check{v}| + 2|\hat{v} - \check{v}| \\ &= (v^m - \hat{v}) - (\check{v} - v^m) + (\check{v} - \hat{v}) \\ &= 2(v^m - \hat{v}) = 2(v^m - v^l) \\ &= 2w^l \left(\psi(\rho^m) - \psi(\rho^l)\right) \\ &\leq 2w^l C_\psi \left|\rho^l - \rho^m\right|, \\ &\leq 2w^l C_\psi \frac{F_{\alpha,1}(w_{\max}) + R}{w_{\min}\bar{\lambda}}(u^+ - u^-), \\ \Delta \mathcal{N}(\bar{t}) &\leq 1 + \frac{V_{\max}}{\nu}. \end{aligned}$$

   Here $\xi^- = u^-$, $\xi^+ = u^+$, and so $|\Delta \xi| \leq |u^- - u^+|$.
2. $\rho^m v^m > \varphi_{w^l,u^+}(\rho^m)$. In this case there is a production of a first family wave $((\hat{\rho}, w^l), (\hat{\rho}^+, w^l))$, of a $\mathcal{NFW}$ wave $((\hat{\rho}^+, w^l), (\check{\rho}^+, w^l))$ and of a linear wave connecting $(\check{\rho}^+, w^l)$ with $(\check{\rho}, w^l)$. For the functionals (3.3)-(3.6), since $\check{v}^+ > \hat{v}^+$ and $\check{v} > \hat{v}$, by Lemma 4.1, we have

$$\Delta \mathcal{F}_w(\bar{t}) = 0,$$

$$\begin{aligned}\Delta\mathcal{F}_{\tilde v}(\bar t) &= \left|\hat v - \hat v^+\right| + \left|\hat v^+ - \check v^+\right| + \left|\check v^+ - \check v\right| - 2\left|\hat v^+ - \check v^+\right| \\ &\quad - \left|\hat v - \check v\right| + 2\left|\hat v - \check v\right| \\ &= \left|\hat v^+ - \hat v\right| - \check v^+ + \hat v^+ + \left|\check v^+ - \check v\right| + \check v - \hat v \\ &\le 2\left|\hat v^+ - \hat v\right| + 2\left|\check v^+ - \check v\right| \\ &\le 2w^l C_\psi \left(\left|\hat\rho^+ - \hat\rho\right| + \left|\check\rho^+ - \check\rho\right|\right), \\ \Delta\mathcal{N}(\bar t) &\le 1 + \frac{V_{\max}}{\nu}.\end{aligned}$$

By (2.2), we have that $|\check\rho^+ - \check\rho| = 0$.
By Lemma 4.3,

$$\left|\hat\rho^+ - \hat\rho\right| \le \frac{F_{\alpha,1}(w^l) + R}{w^l \bar\lambda}\left|u^+ - u^-\right|.$$

Hence

$$\Delta\mathcal{F}_{\tilde v}(\bar t) \le 2w^l C_\psi \frac{F_{\alpha,1}(w^l) + R}{w^l \bar\lambda}\left|u^+ - u^-\right|.$$

Here $\xi^- = u^-$, $\xi^+ = u^+$, and so $|\Delta\xi| \le |u^- - u^+|$.
This completes the proof. □

Lemma 3.13. *Assume that, at time $\bar t > 0$, the control function u jumps from $u^- = V_{\max}$ to $u^+ = u(\bar t+)$ and that we have a $\mathcal{SNFW}$ at time $\bar t-$, connecting $(\rho^l, w^l) \in F \cap C$ with $(\rho^r, w^l) \in F \setminus C$ and such that $\rho^r = \check\rho(w^l, \sigma)$ for every $\sigma \in [0, V_{\max}[$.*

Then, at time $\bar t+$ we have a $\mathcal{NFW}$ and the number of waves increases. The production is $\mathcal{SNFW}$ /1-$\mathcal{NFW}$. Moreover $\Delta\mathcal{F}_w(\bar t) = 0$, $\Delta\mathcal{F}_{\tilde v}(\bar t) = 0$, and $\Delta\mathcal{N}(\bar t) = 1$. Finally $|\Delta\xi| = |u^- - u^+|$.

Proof. We use the following notations:

$$\begin{aligned} &u^- = u(\bar t-), && u^+ = u(\bar t+), && \check\rho = \check\rho(w^l, u^+), && \hat\rho = \hat\rho(w^l, u^+), \\ &v^l = \tilde v(\rho^l, w^l), && v^r = \tilde v(\rho^r, w^l), && \check v = \tilde v(\check\rho, w^l), && \hat v = \tilde v(\hat\rho, w^l).\end{aligned}$$

At time $\bar t$, we need to consider the Riemann solver

$$\mathcal{RS}^c\left((\rho^l, w^l), (\rho^r, w^l), u^+\right).$$

Since, by assumption, $\rho^r = \check\rho$, we deduce that there is a production of a first family (shock) wave $((\rho^l, w^l), (\hat\rho, w^l))$ and of a $\mathcal{NFW}$ connecting $(\hat\rho, w^l)$ with (ρ^r, w^l).

For the functionals (3.3)-(3.6), since $\hat v < v^\ell = V_{\max} \le \check v = v^r$ and the wave of the first family is a shock, we have

$$\begin{aligned}\Delta\mathcal{F}_w(\bar t) &= 0, \\ \Delta\mathcal{F}_{\tilde v}(\bar t) &= \left|\hat v - v^l\right| + \left|\hat v - v^r\right| - 2\left|\hat v - v^r\right| - \left|v^l - v^r\right| + 2\left|v^l - v^r\right| \\ &= \left(v^l - \hat v\right) - \left(v^r - \hat v\right) + \left(v^r - v^l\right) = 0 \\ \Delta\mathcal{N}(\bar t) &= 1.\end{aligned}$$

Here $\xi^- = u^- = V_{\max}$, $\xi^+ = u^+$, and so $|\Delta\xi| = |u^- - u^+|$, concluding the proof. □

We collect all the estimates due to the change of the control in Table 3.2.

Waves' type	$\Delta\mathcal{F}_w$	$\Delta\mathcal{F}_v$	$\Delta\mathcal{N}$	Lemma
$\mathcal{FW}/\mathcal{FW}$ $\mathcal{FW}$ /1-$\mathcal{NFW}$-$\mathcal{PT}$	$=0$	$=0$	$=0$ $=2$	Lemma 3.11
$\mathcal{NFW}$ /1–$\mathcal{SNFW}$ ($\mathcal{LW}$) $\mathcal{NFW}$ /1-$\mathcal{NFW}$-$\mathcal{LW}$	$=0$	$\leq O(1)\|u^+ - u^+\|$	$\leq 1+\frac{V_{\max}}{\nu}$	Lemma 3.12
$\mathcal{SNFW}$ /1–$\mathcal{NFW}$	$=0$	$=0$	$=1$	Lemma 3.13

TABLE 3.2. *The variation of the functionals $\mathcal{F}_w$, $\mathcal{F}_v$, and $\mathcal{N}$ due to control changes. The Landau symbol $O(1)$ denotes a constant; see Lemma 3.12 for the precise expression.*

3.3. Existence of a wave-front tracking approximate solution In this part we deal with the existence of a wave-front tracking approximate solution, in the sense of Definition 3.2.

PROPOSITION 3.1. *For every $\nu \in \mathbb{N}\setminus\{0\}$, the construction illustrated in Section 3.1 produces a wave-front tracking approximate solution, defined for every time $t \geq 0$.*

Proof. Fix $\nu \in \mathbb{N}\setminus\{0\}$. We need to prove that the total number of waves and interactions remain finite. By construction u_ν is piecewise constant with a finite number of discontinuities. Note also that, due to Lemma 3.10 and to Lemma 3.13, the $\mathcal{SNFW}$ is superimposed only to a linear wave with left state in $F\cap C$ and right state in $F\setminus C$ with the same w coordinate as described in Lemma 3.13. Hence the interactions, described in Lemma 3.11, in Lemma 3.12, and in Lemma 3.13, can happen at most a finite number of times and so the number of new waves, generated by the changes of the control, is also finite. Thus, without loss of generality, we may assume that the control u_ν is constant. Therefore the interactions of Table 3.2 do not happen. Moreover the interactions in Table 3.1 do not produce the $\mathcal{SNFW}$ wave. This implies that the $\mathcal{SNFW}$ wave can be generated only at time $t=0$, but, under small perturbations in the AV initial position, we can assume that $\mathcal{SNFW}$ wave is not present.

New waves of the first family can not be generated to the right of the AV; see Table 3.1. Therefore the functional $t\mapsto\mathcal{N}_1^+(t)$ does not increase and so, for $t\geq 0$,

$$\mathcal{N}_1^+(t) \leq \mathcal{N}_1^+(0+), \tag{3.9}$$

which implies that also the interactions, studied in Lemma 3.6, can happen at most a finite number of times.

New $\mathcal{LW}$ waves can not be generated to the left of the AV; see Table 3.1. Therefore the functional $t\mapsto\mathcal{N}_{\mathcal{LW}}^-(t)$ does not increase and so, for $t\geq 0$,

$$\mathcal{N}_{\mathcal{LW}}^-(t) \leq \mathcal{N}_{\mathcal{LW}}^-(0+), \tag{3.10}$$

which implies that also the interactions, studied in Lemma 3.5, can happen at most a finite number of times.

If, at time $t>0$, there are two $\mathcal{PT}$ waves at the left (or at the right) of the AV, then there exists at least one $\mathcal{LW}$ and one wave of the first family in between. Thus, for $t>0$, using (3.9) and (3.10),

$$\begin{aligned}\mathcal{N}_{\mathcal{PT}}^-(t) &\leq \mathcal{N}_{\mathcal{LW}}^-(t)+1 \leq \mathcal{N}_{\mathcal{LW}}^-(0+)+1\\ \mathcal{N}_{\mathcal{PT}}^+(t) &\leq \mathcal{N}_1^+(t)+1 \leq \mathcal{N}_1^+(0+)+1\end{aligned}$$

and so

$$\mathcal{N}_{\mathcal{PT}}(t) = \mathcal{N}_{\mathcal{PT}}^-(t)+\mathcal{N}_{\mathcal{PT}}^+(t) \leq \mathcal{N}_{\mathcal{LW}}^-(0+)+\mathcal{N}_1^+(0+)+2.$$

We now focus on the number of interactions of type 2-$\mathcal{NFW}$/1-$\mathcal{NFW}$-$\mathcal{LW}$, described in Lemma 3.7. New waves of the second family can be created at a positive time only with the interaction $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2. Then, the number of times the interactions 2-$\mathcal{NFW}$/1-$\mathcal{NFW}$-$\mathcal{LW}$ may happen is bounded by $\mathcal{N}_2^-(0+)+\mathcal{N}_{\mathcal{LW}}^-(0+)$, since (3.10) holds.

With the same reasoning, the interactions, described in Lemma 3.2, can happen at most $\mathcal{N}_2^-(0+)+\mathcal{N}_{\mathcal{LW}}^-(0+)$ times.

Since the functional $\mathcal{N}$ strictly increases only in the interactions considered in Lemma 3.2, Lemma 3.5, Lemma 3.6 and of Lemma 3.7, then

$$\begin{aligned}
\mathcal{N}(t) \leq& \mathcal{N}(0+)+3\left(\mathcal{N}_2^-(0+)+\mathcal{N}_{\mathcal{LW}}^-(0+)\right)+2\mathcal{N}_{\mathcal{LW}}^-(0+)\\
&+2\mathcal{N}_1^+(0+)+\frac{V_{\max}}{\nu}\left(\mathcal{N}_2^-(0+)+\mathcal{N}_{\mathcal{LW}}^-(0+)\right)\\
\leq& \mathcal{N}(0+)+2\mathcal{N}_1^+(0+)+\left(3+\frac{V_{\max}}{\nu}\right)\mathcal{N}_2^-(0+)\\
&+\left(5+\frac{V_{\max}}{\nu}\right)\mathcal{N}_{\mathcal{LW}}^-(0+)\\
\leq& \left(6+\frac{V_{\max}}{\nu}\right)\mathcal{N}(0+).
\end{aligned} \tag{3.11}$$

Since in the interactions described in Lemma 3.8 and in Lemma 3.9 the number of waves strictly decreases and since (3.11) holds, then these interactions can happen at most a finite number of times. Similarly also the interactions 1-1/1 and $\mathcal{PT}$-1/$\mathcal{PT}$ can happen at most a finite number of times.

We claim now that the number of interactions 2-1/1-2 is finite. Indeed by (3.9) the number of interactions 2-1/1-2 at the right of AV is finite. Moreover, as already proved, the number of waves of the first family generated at a positive time at the AV location is finite; thus the number of interactions 2-1/1-2 at the left of AV is also finite, proving the claim.

Symmetrically also the number of interactions $\mathcal{LW}$-$\mathcal{PT}$/$\mathcal{PT}$-2 is finite. This is a consequence of the fact that the number of new linear waves generated at the location of the AV is finite. Therefore all the interactions described in Lemma 3.1 are finite.

It remains to prove that the interactions in Lemma 3.3 and in Lemma 3.4 can happen at most a finite number of times. Indeed when a phase-transition wave interacts with the fictitious wave, the waves cross each other and they can not interact anymore without other interactions before. This prevents the possibility of a combination of the two interactions happens an infinite number of times.

The proof is so concluded. □

3.4. Existence of a Solution This section deals with the proof of the main result.

Proof. **(Proof of Theorem 3.1)** Fix $(z_\varepsilon, y_\varepsilon, u_\varepsilon)$, an ε-approximate wave-front tracking solution to (3.1), in the sense of Definition 3.2, which exists by Proposition 3.1. By assumptions there exists $M>0$ such that

$$\mathcal{F}_w(0+)\leq M, \qquad \mathcal{F}_v(0+)\leq M, \qquad \text{and} \qquad \mathrm{TV}(u_\varepsilon)\leq M.$$

In particular, at least passing to a subsequence, there exists a control $u^*\in \mathbf{BV}\left([0,+\infty[;[0,\overline{V}]\right)$ such that

$$u_\varepsilon(t)\to u^*(t)$$

for a.e. $t > 0$ as $\varepsilon \to 0$.

By the interaction estimates in Section 3.2, see also Table 3.1 and Table 3.2, we deduce that

$$\mathcal{F}_w(t) = \mathcal{F}_w(0+) \leq M$$

for every $t > 0$. Moreover the possible increments of the functional $\mathcal{F}_v$ are described in Lemma 3.7 and in the two cases of Lemma 3.12. In particular, the maximum possible increment due to the control's change is proportional to the total variation of u_ε; see Lemma 3.12. Instead the maximum possible increment due to the interactions described in Lemma 3.7 is proportional to $\mathcal{F}_w(0+)$, since the waves of the second family interacting with the AV either are original waves, i.e. generated at time $t = 0$, or are generated by the interaction $\mathcal{LW}$-$\mathcal{PT}/\mathcal{PT}$-2 at the left of the AV. The $\mathcal{LW}$ waves can not be generated at positive time at the left of the AV and the strength $\mathcal{F}_v$ of $\mathcal{LW}$ is in the previous interaction is transferred to the wave of the second family. This permits to conclude that

$$\mathcal{F}_{\tilde{v}}(t) \leq O(1)M$$

for every $t > 0$, where the Landau symbol $O(1)$ denotes a constant not depending on t, ε, and M.

Hence, at least passing to a subsequence, there exist w^* and $\tilde{v}^*$ such that

$$w(z_\varepsilon(t,\cdot)) \to w^*(t,\cdot) \qquad \text{and} \qquad \tilde{v}(z_\varepsilon(t,\cdot)) \to \tilde{v}^*(t,\cdot)$$

pointwise for a.e. $t > 0$. Finally we deduce the existence of $\rho^*(t,\cdot)$ such that

$$z_\varepsilon(t,\cdot) \to z^*(t,\cdot) := (\rho^*(t,\cdot), w^*(t,\cdot))$$

pointwise for a.e. $t > 0$.

Since $|\dot{y}_\varepsilon(t)| \leq [0, V_{\max}]$ for a.e. $t > 0$, Ascoli-Arzelà Theorem [42, Theorem 7.25] implies that, at least passing to a subsequence, there exist a continuous function y^* such that

$$y_\varepsilon \to y^*$$

uniformly. Moreover by the estimates in Section 3.2, we deduce that

$$\mathrm{TV}(\dot{y}_\varepsilon) \leq \sup_{t>0} \mathcal{F}_v(t) + \mathrm{TV}(u_\varepsilon) \leq O(1)M.$$

Therefore we deduce that, at least passing to a subsequence,

$$y_\varepsilon \to y^*$$

in $\mathbf{W}^{1,1}_{\mathbf{loc}}([0,+\infty);\mathbb{R})$.

It remains to prove that (z^*, y^*) satisfies Definition 3.1. The points 1., 2., and 3. are straightforward.

Point 4. of Definition 3.1. We need to prove, for a.e. $t > 0$,

$$y^*(t) = y_0 + \int_0^t \min\{u^*(s), v(s, y^*(s)+)\}\, ds. \tag{3.12}$$

By construction we have, for a.e. $t>0$,

$$y_\varepsilon(t) = y_0 + \int_0^t \min\{u_\varepsilon(s), v(s, y_\varepsilon(s)+)\}\, ds. \tag{3.13}$$

We have $u_\varepsilon \to u^*$ and $y_\varepsilon \to y^*$ a.e. as $\varepsilon \to 0$. Moreover the curve $t \mapsto y^*(t)$ is non characteristic for the quantity v, since the Riemann coordinate v travels with velocity given by $\lambda_1 < 0$; see **(H-3)**. Hence $v(s, y_\varepsilon(s)) \to v(s, y^*(s))$ for a.e. $s>0$. Thus, passing to the limit in (3.13) as $\varepsilon \to 0$, we deduce (3.12).

Point 5. of Definition 3.1. We follow the same idea as in [24], based on the fact that z_ε and z^* are weak solutions to the PDE in (3.1). Without loss of generality, we assume that z_ε is a sequence with all the limit properties highlighted in the previous part of the proof. Fix $T>0$, $\varphi \in \mathbf{C}^1_\mathbf{c}(]0,T[\times\mathbb{R};\mathbb{R}^+)$, and consider the sets

$$\begin{aligned} D_l &= \{(t,x) \in [0,T]\times\mathbb{R} : x < y^*(t)\} \\ D_l^\varepsilon &= \{(t,x) \in [0,T]\times\mathbb{R} : x < y_\varepsilon(t)\} \\ I &= \{t>0 : \dot{y}^*(t) \text{ exists, } y_\varepsilon(t) \to y^*(t) \text{ and } \dot{y}_\varepsilon(t) \to \dot{y}^*(t) \text{ as } \varepsilon \to 0\}. \end{aligned}$$

Note that the Lebesgue measure of I is T. By [8, Theorem 2.2], for every ε, we deduce that

$$\begin{aligned} &\int_{D_l^\varepsilon} (\rho_\varepsilon \partial_t \varphi + (\rho_\varepsilon v(\rho_\varepsilon, w_\varepsilon))\partial_x \varphi)\, dt\, dx \\ &= \int_0^T \rho_\varepsilon(t, y_\varepsilon(t)-)\left[v_\varepsilon(t, y_\varepsilon(t)-)) - \dot{y}_\varepsilon(t)\right]\varphi(t, y_\varepsilon(t))dt \end{aligned}$$

and

$$\begin{aligned} &\int_{D_l} (\rho^* \partial_t \varphi + (\rho^* v(\rho^*, w^*))\partial_x \varphi)\, dt\, dx \\ &= \int_0^T \rho^*(t, y^*(t)-)\left[v^*(t, y^*(t)-)) - \dot{y}^*(t)\right]\varphi(t, y^*(t))dt. \end{aligned}$$

The Dominated Convergence Theorem implies

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{D_l^\varepsilon} (\rho_\varepsilon \partial_t \varphi + (\rho_\varepsilon v(\rho_\varepsilon, w_\varepsilon))\partial_x \varphi)\, dt\, dx \\ &= \int_{D_l} (\rho^* \partial_t \varphi + (\rho^* v(\rho^*, w^*))\partial_x \varphi)\, dt\, dx, \end{aligned}$$

so that

$$\begin{aligned} &\int_0^T \rho^*(t, y^*(t)-)\left[v^*(t, y^*(t)-)) - \dot{y}^*(t)\right]\varphi(t, y^*(t))dt \\ &= \lim_{\varepsilon\to 0} \int_0^T \rho_\varepsilon(t, y_\varepsilon(t)-)\left[v_\varepsilon(t, y_\varepsilon(t)-)) - \dot{y}_\varepsilon(t)\right]\varphi(t, y_\varepsilon(t))dt \\ &= \lim_{\varepsilon\to 0} \int_I \rho_\varepsilon(t, y_\varepsilon(t)-)\left[v_\varepsilon(t, y_\varepsilon(t)-)) - \dot{y}_\varepsilon(t)\right]\varphi(t, y_\varepsilon(t))dt. \end{aligned}$$

Define

$$I_1 = \{t \in I : w_\varepsilon(t, y_\varepsilon(t)-) \to w^*(t, y^*(t)) \text{ as } \varepsilon \to 0\}.$$

The construction of approximate solutions implies that

$$\begin{aligned}&\int_{I_1}\rho_\varepsilon(t,y_\varepsilon(t)-)[v_\varepsilon(t,y_\varepsilon(t)-))-\dot y_\varepsilon(t)]\varphi(t,y_\varepsilon(t))dt\\ \le&\int_{I_1}F_\alpha(w_\varepsilon(t,y_\varepsilon(t)-),\dot y_\varepsilon(t))\varphi(t,y_\varepsilon(t))dt\end{aligned}$$

and so, using the Dominated Convergence Theorem,

$$\begin{aligned}&\liminf_{\varepsilon\to0}\int_{I_1}\rho_\varepsilon(t,y_\varepsilon(t)-)[v_\varepsilon(t,y_\varepsilon(t)-))-\dot y_\varepsilon(t)]\varphi(t,y_\varepsilon(t))dt\\ \le&\int_{I_1}F_\alpha(w^*(t,y^*(t)-),\dot y^*(t))\varphi(t,y^*(t))dt.\end{aligned}$$

Consider now $\bar t\in I\setminus I_1$. At least passing to a subsequence, we may assume that $w_\varepsilon(\bar t,y_\varepsilon(\bar t))$ is uniformly far from $w^*(\bar t,y^*(\bar t))$, which implies that the boundary at $\bar t$ is characteristic, so that

$$v_\varepsilon(\bar t,y_\varepsilon(\bar t))\to\dot y^*(\bar t)$$

as $\varepsilon\to0$. The Dominated Convergence Theorem implies that

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{I\setminus I_1}\rho_\varepsilon(t,y_\varepsilon(t)-)[v_\varepsilon(t,y_\varepsilon(t)-))-\dot y_\varepsilon(t)]\varphi(t,y_\varepsilon(t))dt\\ =&0\le\int_{I\infty I_1}F_\alpha(w^*(t,y^*(t)-),\dot y^*(t))\varphi(t,y^*(t))dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^T\rho^*(t,y^*(t)-)[v^*(t,y^*(t)-))-\dot y^*(t)]\varphi(t,y^*(t))dt\\ =&\lim_{\varepsilon\to0}\int_I\rho_\varepsilon(t,y_\varepsilon(t)-)[v_\varepsilon(t,y_\varepsilon(t)-))-\dot y_\varepsilon(t)]\varphi(t,y_\varepsilon(t))dt\\ =&\liminf_{\varepsilon\to0}\int_{I_1}\rho_\varepsilon(t,y_\varepsilon(t)-)[v_\varepsilon(t,y_\varepsilon(t)-))-\dot y_\varepsilon(t)]\varphi(t,y_\varepsilon(t))dt\\ \le&\int_{I_1}F_\alpha(w^*(t,y^*(t)-),\dot y^*(t))\varphi(t,y^*(t))dt\\ \le&\int_IF_\alpha(w^*(t,y^*(t)-),\dot y^*(t))\varphi(t,y^*(t))dt.\end{aligned}$$

Since $\varphi\ge0$, we deduce that

$$\begin{aligned}&\rho^*(t,y^*(t)-)[v^*(t,y^*(t)-))-\dot y^*(t)]\varphi(t,y^*(t))\\ \le&F_\alpha(w^*(t,y^*(t)-),\dot y^*(t))\varphi(t,y^*(t))\end{aligned}$$

for a.e. $t\in[0,T]$, as required. A similar estimate holds for the right traces and so the point 5. of Definition 3.1 is satisfied. The proof is concluded. □

4. Appendix: Technical Lemmas

We state the following results about $\tilde v$.

LEMMA 4.1. *Assume* **(H-2)** *and fix* $w\in[w_{\min},w_{\max}]$. *Then* $\tilde v$ *is Lipschitz continuous with respect to* ρ *with Lipschitz constant* wC_ψ.

Proof. We have

$$|\partial_\rho \tilde{v}| = |w\psi'(\rho)| \leq w C_\psi,$$

completing the proof. □

LEMMA 4.2. *Assume* **(H-2)** *and fix* $\tilde{v} > 0$. *Then the function* $w(\rho) = \frac{\tilde{v}}{\psi(\rho)}$ *is invertible and the inverse function is Lipschitz continuous with Lipschitz constant* $\frac{1}{\tilde{v} c_\psi}$.

Proof. We have

$$|w'(\rho)| = \frac{\tilde{v}}{\psi^2(\rho)} |\psi'(\rho)| \geq \frac{\tilde{v}}{\psi^2(0)} c_\psi = \tilde{v} c_\psi,$$

concluding the proof. □

LEMMA 4.3. *Assume* **(H-2)** *and* **(H-3)**. *Fix* $\bar{w} \in [w_{\min}, w_{\max}]$ *and consider, in the plane* $(\rho, \rho v)$, $\rho \mapsto \mathcal{L}_1(\rho; \bar{w}) = \rho \bar{w} \psi(\rho)$ *the Lax curve of the first family with* $w = \bar{w}$. *Let* $\rho_1 < \rho_2$ *such that* $(\rho_1, \mathcal{L}_1(\rho_1; \bar{w})) \in C$ *and* $(\rho_2, \mathcal{L}_1(\rho_2; \bar{w})) \in C$.

1. *Fix* $u \in [0, V_{\max}[$. *Define* $F_1 > F_2$ *such that*

$$\mathcal{L}_1(\rho_1; \bar{w}) = F_1 + u\rho_1 \qquad \text{and} \qquad \mathcal{L}_1(\rho_2; \bar{w}) = F_2 + u\rho_2. \tag{4.1}$$

Then

$$0 < \rho_2 - \rho_1 \leq \frac{1}{\bar{w}\bar{\lambda} + u}(F_1 - F_2),$$

where $\bar{\lambda}$ *is defined in* **(H-3)**.

2. *Define* $u_1 > u_2$ *such that*

$$\begin{aligned} \mathcal{L}_1(\rho_1; \bar{w}) &= F_\alpha(\bar{w}, u_1) + u_1 \rho_1 \\ \mathcal{L}_1(\rho_2; \bar{w}) &= F_\alpha(\bar{w}, u_2) + u_2 \rho_2. \end{aligned} \tag{4.2}$$

Then

$$|\rho_2 - \rho_1| \leq \frac{R + F_{\alpha,1}(\bar{w})}{\bar{w}\bar{\lambda} + u_2}(u_1 - u_2),$$

where $\bar{\lambda}$ *is defined in* **(H-3)**.

Proof. For case 1, consider the function

$$G(\rho; \bar{w}, u) = \mathcal{L}_1(\rho; \bar{w}) - u\rho = \rho \bar{w} \psi(\rho) - u\rho.$$

Equation (4.1) implies that

$$G(\rho_1; \bar{w}, u) = F_1 \qquad \text{and} \qquad G(\rho_2; \bar{w}, u) = F_2,$$

so that we need to prove that G is invertible with respect to ρ and that the inverse is Lipschitz continuous with Lipschitz constant $\frac{1}{\bar{w}\bar{\lambda} + u}$. We have

$$\partial_\rho G(\rho; \bar{w}, u) = \bar{w} \frac{d}{d\rho}(\rho \psi(\rho)) - u \leq -\bar{w}\bar{\lambda} - u < 0,$$

by **(H-2)** and **(H-3)**. Hence G is invertible with respect to ρ. Moreover

$$\partial_\rho G(\rho; \bar{w}, u) = \bar{w} \frac{d}{d\rho}(\rho \psi(\rho)) - u \geq \bar{w}(\psi(R) + R\psi'(R)) - u.$$

Therefore the inverse function satisfies

$$-\frac{1}{\bar{w}\bar{\lambda}+u}\leq \partial_F G^{-1}(F;\bar{w},u)\leq \frac{1}{\bar{w}\left(\psi(R)+R\psi'(R)\right)-u}<0$$

and so

$$\left|\partial_F G^{-1}(F;\bar{w},u)\right|\leq \frac{1}{\bar{w}\bar{\lambda}+u}$$

concluding the proof.

In case 2, consider the function consider the function

$$G(\rho;\bar{w},u)=\mathcal{L}_1(\rho;\bar{w})-u\rho-F_\alpha(\bar{w},u).$$

Equation (4.2) implies that

$$G(\rho_1;\bar{w},u_1)=0 \qquad \text{and} \qquad G(\rho_2;\bar{w},u_2)=0.$$

Using the Implicit Function Theorem and analogous estimates contained in the previous case, we easily conclude. □

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REFERENCES

[1] B. Andreianov, C. Donadello, and M. D. Rosini. A second-order model for vehicular traffics with local point constraints on the flow. *Math. Models Methods Appl. Sci.*, 26(4):751–802, 2016. 1, 3.1

[2] B. Andreianov, C. Donadello, and M. D. Rosini. Entropy solutions for a two-phase transition model for vehicular traffic with metastable phase and time depending point constraint on the density flow. *NoDEA Nonlinear Differential Equations Appl.*, 28(3):Paper No. 32, 37, 2021. 1

[3] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000. 1

[4] N. Bellomo, A. Bellouquid, J. Nieto, and J. Soler. On the multiscale modeling of vehicular traffic: from kinetic to hydrodynamics. *Discrete Contin. Dyn. Syst. Ser. B*, 19(7):1869–1888, 2014. 1

[5] M. Benyahia and M. D. Rosini. Lack of BV bounds for approximate solutions to a two-phase transition model arising from vehicular traffic. *Math. Methods Appl. Sci.*, 43(18):10381–10390, 2020. 1

[6] S. Blandin, D. Work, P. Goatin, B. Piccoli, and A. Bayen. A general phase transition model for vehicular traffic. *SIAM J. Appl. Math.*, 71(1):107–127, 2011. 1

[7] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem. 3.1, 3.3

[8] G.-Q. Chen and H. Frid. Divergence-measure fields and hyperbolic conservation laws. *Arch. Ration. Mech. Anal.*, 147(2):89–118, 1999. 3.4

[9] G. M. Coclite and M. Garavello. Vanishing viscosity for mixed systems with moving boundaries. *J. Funct. Anal.*, 264(7):1664–1710, 2013. 1

[10] R. M. Colombo. Hyperbolic phase transitions in traffic flow. *SIAM J. Appl. Math.*, 63(2):708–721, 2002. 1

[11] R. M. Colombo and P. Goatin. A well posed conservation law with a variable unilateral constraint. *J. Differential Equations*, 234(2):654–675, 2007. 1

[12] R. M. Colombo and F. Marcellini. A mixed ODE—PDE model for vehicular traffic. *Mathematical Methods in the Applied Sciences*, 38:1292–1302, 2015. 1

[13] R. M. Colombo, F. Marcellini, and M. Rascle. A 2-phase traffic model based on a speed bound. *SIAM J. Appl. Math.*, 70(7):2652–2666, 2010. (document), 1, 2, 3.2, 3.2.3, 3.2.4

[14] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010. 3.1

[15] L. C. Davis. Effect of adaptive cruise control systems on traffic flow. *Phys. Rev. E*, 69:066110, Jun 2004. 1

[16] M. Delle Monache, T. Liard, A. Rat, R. Stern, R. Bhadani, B. Seibold, J. Sprinkle, D. Work, and B. Piccoli. Feedback control algorithms for the dissipation of traffic waves with autonomous vehicles. *Springer Optimization and Its Applications*, 150:275–299, 2019. cited By 3. 1

[17] M. L. Delle Monache and P. Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *J. Differential Equations*, 257(11):4015–4029, 2014. 1, 1, 1

[18] M. L. Delle Monache and P. Goatin. Stability estimates for scalar conservation laws with moving flux constraints. *Netw. Heterog. Media*, 12(2):245–258, 2017. 1

[19] M. L. Delle Monache, T. Liard, B. Piccoli, R. Stern, and D. Work. Traffic reconstruction using autonomous vehicles. *SIAM J. Appl. Math.*, 79(5):1748–1767, 2019. 1

[20] G. Dimarco, A. Tosin, and M. Zanella. Kinetic derivation of Aw-Rascle-Zhang-type traffic models with driver-assist vehicles. *J. Stat. Phys.*, 186(1):Paper No. 17, 26, 2022. 1

[21] S. Fan, M. Herty, and B. Seibold. Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model. *Netw. Heterog. Media*, 9(2):239–268, 2014. 1

[22] S. Fan, Y. Sun, B. Piccoli, B. Seibold, and D. B. Work. A Collapsed Generalized Aw-Rascle-Zhang Model and Its Model Accuracy. *ArXiv e-prints*, Feb. 2017. 1

[23] M. Garavello and P. Goatin. The Aw-Rascle traffic model with locally constrained flow. *J. Math. Anal. Appl.*, 378(2):634–648, 2011. 1

[24] M. Garavello, P. Goatin, T. Liard, and B. Piccoli. A multiscale model for traffic regulation via autonomous vehicles. *J. Differential Equations*, 269(7):6088–6124, 2020. 1, 1, 3.4

[25] M. Garavello and B. Piccoli. *Traffic flow on networks*, volume 1 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006. Conservation laws models. 3.2

[26] M. Garavello and B. Piccoli. Coupling of Lighthill-Whitham-Richards and phase transition models. *J. Hyperbolic Differ. Equ.*, 10(3):577–636, 2013. 1, 3.2

[27] M. Garavello and S. Villa. The Cauchy problem for the Aw-Rascle-Zhang traffic model with locally constrained flow. *Journal of Hyperbolic Differential Equations*, 14(03):393–414, 2017. 1

[28] D. Gazis, R. Herman, and R. Rothery. Nonlinear follow-the-leader models of traffic flow. *Oper. Res.*, 9:545–567, 1961. 1

[29] P. Goatin. The Aw-Rascle vehicular traffic flow model with phase transitions. *Math. Comput. Modelling*, 44(3-4):287–303, 2006. 1

[30] M. Guériau, R. Billot, N.-E. El Faouzi, J. Monteil, F. Armetta, and S. Hassas. How to assess the benefits of connected vehicles? A simulation framework for the design of cooperative traffic management strategies. *Transportation Research Part C Emerging Technologies*, 67, 04 2016. 1

[31] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied Mathematical Sciences*. Springer, Heidelberg, second edition, 2015. 3.1

[32] F. Knorr, D. Baselt, M. Schreckenberg, and M. Mauve. Reducing traffic jams via vanets. *Vehicular Technology, IEEE Transactions on*, 61:3490–3498, 10 2012. 1

[33] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970. 1

[34] C. Lattanzio and B. Piccoli. Coupling of microscopic and macroscopic traffic models at boundaries. *Math. Models Methods Appl. Sci.*, 20(12):2349–2370, 2010. 1

[35] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.*

Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. 1

[36] J. P. Lebacque, X. Louis, S. Mammar, B. Schnetzlera, and H. Haj-Salem. Modélisation du trafic autoroutier au second ordre. *Comptes Rendus Mathematique*, 346(21–22):1203–1206, November 2008. 1

[37] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955. 1

[38] F. Marcellini. Existence of solutions to a boundary value problem for a phase transition traffic model. *Networks and Heterogeneous Media*, 12(2):259–275, 2017. 3.2.1

[39] F. Marcellini. The Riemann problem for a two-phase model for road traffic with fixed or moving constraints. *Math. Biosci. Eng.*, 17(2):1218–1232, 2020. 2.1

[40] F. Marcellini. The follow-the-leader model without a leader: an infinite-dimensional Cauchy problem. *J. Math. Anal. Appl.*, 495(1):Paper No. 124664, 21, 2021. 1

[41] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956. 1

[42] W. Rudin. *Principles of mathematical analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. 3.4

[43] A. Talebpour and H. Mahmassani. Influence of connected and autonomous vehicles on traffic flow stability and throughput. *Transportation Research Part C Emerging Technologies*, 71:143–163, 10 2016. 1

[44] S. Villa, P. Goatin, and C. Chalons. Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model. *Discrete and Continuous Dynamical Systems - Series B*, 22(10):3921–3952, 2017. 1

[45] M. Wang, W. Daamen, S. P. Hoogendoorn, and B. van Arem. Cooperative car-following control: Distributed algorithm and impact on moving jam features. *IEEE Transactions on Intelligent Transportation Systems*, 17(5):1459–1471, 2016. 1

[46] H. Zhang. A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, 36(3):275 – 290, 2002. 1