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Finite groups admitting an oriented regular representation

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ABSTRACT

In this paper, we investigate finite groups admitting an oriented regular representation and we give a partial answer to a 1980 question of Lazslo Babai: "Which [finite] groups admit an oriented graph as a DRR?" It is easy to see and well-understood that generalised dihedral groups do not admit ORRs. We prove that, apart from C_3^2 and $C_3 \times C_2^3$, every finite group, which is neither a generalised dihedral group nor a 2-group, has an ORR. In particular, the classification of the finite groups admitting an ORR is reduced to the class of 2-groups.

We also give strong structural conditions on finite 2-groups not admitting an ORR. Finally, based on these results and on some extensive computer computations, we state a conjecture aiming to give a complete classification of the finite groups admitting an ORR.

1. Introduction

All groups and graphs in this paper are finite. Let G be a group and let S be a subset of G. The ***Cayley digraph***, denoted by $\mathrm{Cay}(G, S)$, over G with connection set S is the

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digraph with vertex set G and with (x, y) being an arc if $yx^{-1} \in S$. (An ***arc*** is an ordered pair of adjacent vertices.) Since the group G acts faithfully as a group of automorphisms of $\mathrm{Cay}(G, S)$ via the right regular representation, Cayley digraphs represent groups geometrically and combinatorially as groups of automorphisms of digraphs. Naively, the closer G is to the full automorphism group of $\mathrm{Cay}(G, S)$, the closer this representation is from encoding G graphically.

Following this line of thoughts, it is natural to ask which groups G admit a subset S with G being the automorphism group of $\mathrm{Cay}(G, S)$; that is, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. We say that G admits a ***digraphical regular representation*** (or DRR for short) if there exists a subset S of G with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. Babai [1, Theorem 2.1] has given a complete classification of the groups admitting a DRR: except for

$$Q_8,\ C_2^2,\ C_2^3,\ C_2^4 \text{ and } C_3^2, \tag{1}$$

every group admits a DRR.

In light of Babai's result, it is natural to try to combinatorially represent groups as automorphism groups of special classes of Cayley digraphs. Observe that, if S is inverse-closed (that is, $S = \{s^{-1} \mid s \in S\} := S^{-1}$), then $\mathrm{Cay}(G, S)$ is undirected. Now, we say that G admits a ***graphical regular representation*** (or GRR for short) if there exists an inverse-closed subset S of G with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. With a considerable amount of work culminating in [9,11], the groups admitting a GRR have been completely classified. (The pioneer work of Imrich [12–14] was an important step towards this classification.) It is interesting to observe that, although the classification of the groups admitting a DRR is easier than the classification of the groups admitting a GRR, research and interest first focused on finding GRRs. (In some sense this is natural, occasionally graphs draw more interest than digraphs.) It is also worth noting that various researchers have shown that, for certain families of groups, almost all Cayley graphs are GRRs, or almost all Cayley digraphs are DRRs [3,5,9]. The precise definition of "almost all" is slightly technical and it would take us too far astray to include it in this discussion.

We recall that a ***tournament*** is a digraph $\Gamma = (V, A)$ with vertex set V and arc set A such that, for every two distinct vertices $x, y \in V$, exactly one of (x, y) and (y, x) is in A. After the completion of the classification of DRRs and GRRs, Babai and Imrich [2] proved that every group of odd order except for C_3^2 admits a ***tournament regular representation*** (or TRR for short). That is, each finite odd-order group G different from C_3^2 contains a subset S with $\mathrm{Cay}(G, S)$ being a tournament and with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. In terms of the connection set S, the Cayley digraph $\mathrm{Cay}(G, S)$ is a tournament if and only if $S \cap S^{-1} = \emptyset$ and $G \setminus \{1\} = S \cup S^{-1}$. This observation makes it clear that a Cayley digraph on G cannot be a tournament if G contains an element of order 2, so only groups of odd order can admit TRRs.

In [1, Problem 2.7], Babai observed that there is one class of Cayley digraphs that is rather interesting and that has not been investigated in the context of regular representations; that is, the class of oriented Cayley digraphs (or as Babai called them, oriented

Cayley graphs). An ***oriented Cayley digraph*** is in some sense a "proper" digraph. More formally, it is a Cayley digraph $\mathrm{Cay}(G, S)$ whose connection set S has the property that $S \cap S^{-1} = \emptyset$. Equivalently, in graph-theoretic terms, it is a digraph with no digons.

Definition 1.1. The group G admits an ***oriented regular representation*** (or ORR for short) if there exists a subset S of G with $S \cap S^{-1} = \emptyset$ and $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$.

Babai asked in [1] which (finite) groups admit an ORR. Since a TRR is a special type of ORR, and C_3^2 is one of the five groups in Eq. (1) that do not admit a DRR (so cannot admit an ORR), the answer to this question for groups of odd order was already known when Babai published his question. In this paper we give another important contribution towards the classification of groups admitting an ORR.

Theorem 1.2. *Let G be a finite group. Then one of the following holds:*

- **(i):** *G admits an* ORR*;*
- **(ii):** *G has an abelian 2-subgroup A, a normal subgroup N and two elements $g \in G \setminus N$ and $n \in N \setminus A$ with $A < N < G$, $|G : N| = |N : A| = 2$, $g^2 = 1$, $n^g = n^{-1}$ and $a^n = a^{-1}$ for each $a \in A$;*
- **(iii):** *there exists a normal subgroup N of G and $g \in G$ with $|G : N| = 2$, $G = \langle N, g\rangle$, $g^2 = 1$, N is a 2-group and the action of g by conjugation on N inverts precisely half of the elements of N; (Such groups N are classified by Hegarty and MacHale in [10].)*
- **(iv):** *G is isomorphic to Q_8, to $C_3 \times C_3$ or to $C_3 \times C_2^3$;*
- **(v):** *G is generalised dihedral.*

In [16], the authors prove that each non-solvable group admits an ORR. One of the main tools developed in [16] that will be crucial also in the arguments in this paper (including the proof of Theorem 1.2) is the following.

Theorem 1.3. *[16, Theorem 1.8] Let G be a finite group that admits a five-product-avoiding generating set $\{a_1, \ldots, a_\ell\}$ with the following properties:*

- **(i):** *$|a_i| > 2$ for every $i \in \{1, \ldots, \ell\}$; and*
- **(ii):** *$|a_{i+1}a_i^{-1}| > 2$ for every $i \in \{1, \ldots, \ell - 1\}$.*

Then G admits an ORR *if and only if $G \not\cong Q_8$, $G \not\cong C_3 \times C_2^3$, and $G \not\cong C_3^2$.*

We refer to [16, Definition 5.1] for the concept of a five-product-avoiding generating set. Here we observe that, in this paper, we do not need Theorem 1.3 in its full strength and generality, but we simply apply it to generating sets of minimum cardinality or to irredundant generating sets. Every generating set of minimum cardinality, or more generally every irredundant generating set, is five-product-avoiding, see Theorem 2.1 for a version of Theorem 1.3 tailored to our needs.

The classification of Hegarty and MacHale of the 2-groups admitting an automorphism inverting half of their elements is very satisfactory. These groups fall into ten isoclinism classes and, for each class, the authors give a very explicit description of a stem group in the class. Nevertheless, with our current methods we were not able to use this information and deal with the groups in Theorem 1.2 **(iii)**. (One major obstacle in using this classification is that it is conceivable that there are two non-isomorphic stem groups in the same isoclinism class, one admitting an ORR and the other not.)

We believe that no more major breakthrough can be obtained using the methods developed in [16] and in this paper; to obtain a complete classification of the groups admitting an ORR and hence dealing with the few families remaining, we believe that one has to use brute force, that is, analysing each family at the time. In fact, we plan in the near future to adapt the group-theoretic methods in [6,17] for dealing with the groups in Theorem 1.2 **(ii)**.

Finally, we observe that based on some computer computations and on the work in this paper we are one step closer to prove Conjecture 1.5 in [16], which dares to list the groups not admitting an ORR. For the benefit of the reader, we include [16, Conjecture 1.5] here.

Conjecture 1.4. *Every finite group G admits an* ORR, *except when:*

(i): *G is generalised dihedral with $|G| > 2$;*
(ii): *G is isomorphic to one of the following eleven groups*

$$
\begin{aligned}
&Q_8\ (\textit{quaternion of order } 8),\ C_4 \times C_2,\ C_4 \times C_2^2,\ C_4 \times C_2^3,\ C_4 \times C_2^4,\\
&C_3 \times C_3,\ C_3 \times C_2^3,\\
&\langle a, b \mid a^4 = b^4 = (ab)^2 = (ab^{-1})^2 = 1\rangle,\\
&\langle a, b, c \mid a^4 = b^4 = c^4 = (ba)^2 = (ba^{-1})^2 = (bc)^2 = (bc^{-1})^2 = 1,\\
&\qquad a^2 = c^2, a^c = a^{-1}, a^2 = b^2\rangle,\\
&\langle a, b, c \mid a^4 = b^4 = c^4 = (ab)^2 = (ab^{-1})^2 = 1,\\
&\qquad (ac)^2 = (ac^{-1})^2 = (bc)^2 = (bc^{-1})^2 = a^2b^2c^2 = 1\rangle,\\
&\langle a, b, c \mid a^4 = b^4 = c^4 = (ba)^2 = (ba^{-1})^2 = (bc)^2 = (bc^{-1})^2 = 1,\\
&\qquad a^2 = c^2, a^c = a^{-1}, a^2 = b^2\rangle.
\end{aligned}
$$

Recently, combinatorial representations of groups has developed some new vitality and we refer to [6,7,17,19–21] for some recent work on similar problems.

2. Preliminaries

2.1. Beautiful generating tuples

Let G be a finite group, as customary, we denote by $d(G)$ the minimum number of generators for G. Moreover, given $g \in G$, we denote by $o(g)$ the order of the element g.

A generating set $\{g_1, \ldots, g_d\}$ for G is said to be ***irredundant*** if, for each $i \in \{1, \ldots, d\}$, the $d-1$ elements

$$g_1, g_2, \ldots, g_{i-1}, g_{i+1}, \ldots, g_d$$

do not generate G. Observe that each generating set for G of cardinality $d(G)$ is irredundant.

We say that the d-tuple $(g_1, \ldots, g_d)$ of elements of G is ***beautiful*** if the following conditions hold:

(i): $\{g_1, \ldots, g_d\}$ is an irredundant generating set for G,
(ii): $o(g_i) > 2$ for every $i \in \{1, \ldots, d\}$,
(iii): $o(g_{i+1}g_i^{-1}) > 2$ for every $i \in \{1, \ldots, d-1\}$.

Observe that being beautiful is a property of ordered d-tuples and not of sets, that is, it depends upon the ordering of the generating set $\{g_1, \ldots, g_d\}$ for G.

An important connection between beautiful generating tuples and ORRs is given in the next theorem.

Theorem 2.1. *Let G be a finite group admitting a beautiful generating tuple. Then G admits an* ORR *if and only if $G \not\cong Q_8$, $G \not\cong C_3 \times C_2^3$, and $G \not\cong C_3 \times C_3$.*

Proof. From Definition 5.1 in [16], each irredundant generating set for G is five-product-avoiding. Hence every beautiful generating tuple satisfies the hypothesis of Theorem 1.3, and the proof follows. □

Let G be a finite group; we say that G is a ***generalised dihedral*** group (or more specifically, a generalised dihedral group on A) if G contains an abelian subgroup A and $\iota \in G \setminus A$ with $|G : A| = 2$, $\iota^2 = 1$ and $a^\iota = a^{-1}$, for each $a \in A$. Clearly, G is the semidirect product $G = A \rtimes \langle \iota \rangle$. We point out two straightforward facts that can possibly avoid some confusion in our definition of generalised dihedral groups. First, every elementary abelian 2-group of order at least 2 is a generalised dihedral group. Second, if G is a generalised dihedral group on A and A has exponent at least 3, then A is characteristic in G, see [17, Lemma 3.2 (d)]; however, when A has exponent at most 2, G has itself exponent 2 and hence A is not characteristic in G (except when $A = 1$).

We start by recalling the following basic fact.

Lemma 2.2. [16, Lemma 2.6] *Let G be a finite group. Every generating set for G of cardinality $d(G)$ contains at least one involution if and only if G is a generalised dihedral group.*

Lemma 2.3. *Let G be a finite soluble group and let N be a minimal normal proper subgroup of G. If G/N has a beautiful generating tuple, then either so does G, or $G/N \cong C_4$, $N \cong C_2$ and $G \cong C_4 \times C_2$.*

Proof. By hypothesis, G/N has a beautiful generating tuple $g_1'N, \dots, g_\ell'N$. Among all elements of

$$\underbrace{N \times N \times \cdots \times N}_{\ell \text{ times}},$$

choose $(n_1, \dots, n_\ell)$ such that $H := \langle g_1'n_1, \dots, g_\ell'n_\ell\rangle$ has cardinality as large as possible. Set $g_1 := g_1'n_1$, $g_2 := g_2'n_2, \dots, g_\ell := g_\ell'n_\ell$.

If $H = G$, then $g_1, \dots, g_\ell$ is a beautiful generating tuple for G. Assume then $H < G$. As G is generated by $g_1, \dots, g_\ell$ modulo N, we get $G = \langle H, N\rangle$. As $N \trianglelefteq G$ and N is abelian, $N \cap H$ is normal in both N and H, and hence in G. By the minimality of N, we get $H \cap N = 1$ and hence $G = N \rtimes H$. Let $n \in N \setminus \{1\}$.

We now divide the proof in two cases. First we suppose that $\ell > 1$.

Claim. *The $(\ell + 1)$-tuple*

$$g_1,\ \dots,\ g_{\ell-1},\ g_\ell,\ g_{\ell-1}n$$

is a beautiful generating tuple for G.

(Observe that this generating tuple is well-defined because $\ell > 1$.) Set $X := \langle g_1, \dots, g_{\ell-1}, g_\ell, g_{\ell-1}n\rangle$. Now,

$$X = \langle g_1, \dots, g_{\ell-1}, g_\ell, g_{\ell-1}n\rangle = \langle g_1, \dots, g_{\ell-1}, g_\ell, n\rangle = \langle H, n\rangle.$$

In particular, X contains H and a non-identity element of N. As $G = N \rtimes H$ and H acts by conjugation irreducibly as a linear group on N, we deduce that $X = G$ and $g_1, \dots, g_{\ell-1}, g_\ell, g_{\ell-1}n$ is a generating set for G. For every $i \in \{1, \dots, \ell - 2\}$, the elements $g_1, g_2, \dots, g_{i-1}, g_{i+1}, \dots, g_{\ell-1}, g_\ell, g_{\ell-1}n$ do not generate G because $g_1N, g_2N, \dots, g_{i-1}N, g_{i+1}N, \dots, g_{\ell-1}N, g_\ell N$ do not generate G/N being $g_1N = g_1'N, \dots, g_\ell N = g_\ell'N$ an irredundant generating set for G/N. The same argument applies if we remove the generator g_ℓ; namely, $g_1, \dots, g_{\ell-1}, g_{\ell-1}n$ do not generate G because they do not generate G modulo N. Now, if we remove the generator $g_{\ell-1}n$, then

$g_1, \ldots, g_{\ell-1}, g_\ell$ generate H, which by hypothesis is a proper subgroup of G. Finally, from the way that the subgroup H was defined, we get that, if we remove the generator $g_{\ell-1}$, then $g_1, \ldots, g_{\ell-2}, g_\ell, g_{\ell-1}n$ generate a subgroup of G of cardinality at most $|H| < |G|$. Therefore the condition **(i)** in the definition of beautiful generating tuple is satisfied.

Using the fact that $g_1N = g_1'N, \ldots, g_\ell N = g_\ell' N$ is a beautiful generating tuple for G/N, we have $o(g_i) \geq o(g_iN) > 2$, for each $i \in \{1, \ldots, \ell\}$. Moreover, $o(g_{\ell-1}n) \geq o(g_{\ell-1}N) > 2$; hence the condition **(ii)** in the definition of beautiful generating tuple is satisfied.

Using the fact that $g_1N = g_1'N, \ldots, g_\ell N = g_\ell' N$ is a beautiful generating tuple for G/N, we get

$$o(g_{i+1}g_i^{-1}) \geq o(g_{i+1}g_i^{-1}N) = o((g_{i+1}N)(g_iN)^{-1}) = o((g_{i+1}'N)(g_i'N)^{-1}) > 2,$$

for each $i \in \{1, \ldots, \ell - 1\}$. Moreover, we also deduce

$$\begin{aligned} o((g_{\ell-1}n)g_\ell^{-1}) &= o(g_\ell(g_{\ell-1}n)^{-1}) \geq o(g_\ell g_{\ell-1}^{-1}N) \\ &= o((g_\ell N)(g_{\ell-1}N)^{-1}) = o((g_\ell' N)(g_{\ell-1}N)^{-1}) > 2. \end{aligned}$$

Therefore the condition **(iii)** in the definition of beautiful generating tuple is satisfied. □

The proof of this lemma immediately follows from the previous claim when $\ell > 1$.

Suppose then $\ell = 1$, that is, $G/N \cong H$ is cyclic. By definition, $H = \langle g_1 \rangle$ and hence $o(g_1) = |H|$. Moreover, as $G = N \rtimes H$, we get $|H| = |G/N| = o(g_1N)$. As g_1N is a beautiful generating tuple for G/N, we have $o(g_1) = |H| = o(g_1N) > 2$. Assume that $o(g_1) \neq 4$. Now g_1^{-1} together with g_1n generate G, $o(g_1^{-1}) > 2$, $o(g_1n) \geq o(g_1N) = o(g_1) > 2$ and $o(g_1n(g_1^{-1})^{-1}) = o(g_1^2n^{g_1}) \geq o(g_1^2N) = o(g_1^2) > 2$. Therefore g_1^{-1}, g_1n is a beautiful generating tuple for G. Assume that $o(g_1) = 4$. Now, g_1, n is a beautiful generating tuple for G unless $o(n) = 2$. Assume then $o(n) = 2$. Since H and N are both 2-groups, so is G, and hence $|N| = 2$ because N is a minimal normal subgroup of G. Thus $|G| = 8$. Since G/N is cyclic of order 4, we get $G/N \cong C_4$, $N \cong C_2$ and $G \cong C_4 \times C_2$. □

Proposition 2.4. *Let G be a finite soluble group and let N be a normal proper subgroup of G. If G/N has a beautiful generating tuple, then either G has a beautiful generating tuple or $G/N \cong C_4$.*

Proof. We argue by induction on $|G|$. When $N = 1$, there is nothing to prove. Suppose then that $N \neq 1$ and let N_1 be a minimal normal subgroup of G contained in N. We have $|G/N_1| < |G|$ and $(G/N_1)/(N/N_1) \cong G/N$ has a beautiful generating tuple. Therefore we may apply the inductive hypothesis to G/N_1 with proper normal subgroup N/N_1: either G/N_1 has a beautiful generating tuple or $(G/N_1)/(N/N_1) \cong C_4$. In the latter case, $G/N \cong C_4$ and the lemma is proven. In the former case, as N_1 is a minimal normal subgroup of G, we are in the position to apply Lemma 2.3; we get that either G has a beautiful generating tuple, or $N_1 = N$ and $G/N \cong C_4$. □

2.2. An auxiliary result

We denote by $\mathbb{F}_2^k$ the k-dimensional vector space of row vectors over the finite field $\mathbb{F}_2$ of size 2. The $\mathbb{F}_2$-vector space $\mathbb{F}_2^k$ is equipped with a non-degenerate scalar product: for each $x := (x_1, \dots, x_k), y := (y_1, \dots, y_k) \in \mathbb{F}_2^k$, the scalar product $\cdot : \mathbb{F}_2^k \times \mathbb{F}_2^k \to \mathbb{F}_2$ is defined by $x \cdot y := \sum_{i=1}^k x_i y_i$.

Lemma 2.5. *Let k be a positive integer with $k \geq 2$ and let $\bar{\varepsilon}, \bar{\eta} \in \mathbb{F}_2^k$. There exist $\varepsilon, \eta \in \mathbb{F}_2^k$ with $\varepsilon \cdot \eta = 1$ and $(\bar{\varepsilon} + \varepsilon) \cdot (\bar{\eta} + \eta) = 1$.*

Proof. If $\bar{\varepsilon} = \bar{\eta} = 0$, it suffices to take $\varepsilon := (1, 0, \dots, 0)$ and $\eta := (1, 0, \dots, 0)$. Suppose that $\bar{\varepsilon}$ and $\bar{\eta}$ are not both the zero vector. Replacing $\bar{\varepsilon}$ with $\bar{\eta}$ if necessary, we may assume that $\bar{\eta} \neq 0$. Fix $\eta \in \mathbb{F}_2^k \setminus \{0, \bar{\eta}\}$ and observe that this is possible because $k \geq 2$. Now, $\eta^\perp$ and $(\bar{\eta} + \eta)^\perp$ are both $(k-1)$-dimensional subspaces of $\mathbb{F}_2^k$ with $\eta^\perp \neq (\bar{\eta} + \eta)^\perp$ because $\eta \neq \bar{\eta} + \eta$. Therefore, $\eta^\perp$ and $\bar{\varepsilon} + (\bar{\eta} + \eta)^\perp$ are affine hyperplanes of $\mathbb{F}_2^k$ having a non-empty intersection; thus $\eta^\perp \cup (\bar{\varepsilon} + (\bar{\eta} + \eta)^\perp)$ is a proper subset of $\mathbb{F}_2^k$. Fix $\varepsilon \in \mathbb{F}_2^k$ with $\varepsilon \notin \eta^\perp \cup (\bar{\varepsilon} + (\bar{\eta} + \eta)^\perp)$. Thus $\varepsilon \cdot \eta = 1$ because $\varepsilon \notin \eta^\perp$, and $(\bar{\varepsilon} + \varepsilon) \cdot (\bar{\eta} + \eta) = 1$ because $\varepsilon \notin \bar{\varepsilon} + (\bar{\eta} + \eta)^\perp$. □

2.3. Reduction results

We begin with some notation we require from graph theory. For a graph Γ and a subset S of the vertices of Γ, $\Gamma[S]$ denotes the ***induced subgraph*** of Γ on the vertices of S.

Nowitz and Watkins, in their work on the GRR problem, proved a lemma that is very useful in our context also.

Lemma 2.6 *(Nowitz and Watkins [18]). Let G be a group, let S be a subset of G, let $\Gamma = \mathrm{Cay}(G, S)$ and let X be a subset of S. If φ fixes X point-wise for every $\varphi \in \mathrm{Aut}(\Gamma)_1$, then φ fixes $\langle X \rangle$ point-wise for every $\varphi \in \mathrm{Aut}(\Gamma)_1$. In particular, $\mathrm{Aut}(\Gamma)_1 = 1$ if $G = \langle X \rangle$ or if $\Gamma[S]$ is asymmetric.*

If $\Gamma = \mathrm{Cay}(G, S)$ and $\mathrm{Aut}(\Gamma)_1$ is the identity group, then $\mathrm{Aut}(\Gamma) = G$ so that Γ is an ORR for G. We will use this fact repeatedly when we cite the above lemma.

Lemma 2.7 and its proof are inspired by the work of Babai [1] on DRRs and of Babai and Imrich [2] on TRRs.

Lemma 2.7. *Let G be a finite group, let N be a normal subgroup of G with G/N cyclic of order $m \geq 3$, and let $b \in G$ with $G = \langle N, b \rangle$. Suppose that $G \not\cong C_3 \times C_3$ when $m = 3$, and*

$$G \not\cong C_4 \times C_2,\ G \not\cong C_4 \times C_2 \times C_2,\ G \not\cong \langle a, b \mid a^4 = b^4 = (ab)^2 = (ab^{-1})^2 = 1 \rangle$$

when $m = 4$. If N admits an ORR, then G admits an ORR.

Proof. Let T be a subset of N with $\mathrm{Cay}(N,T)$ an ORR for N. As $T \cap T^{-1} = \emptyset$, we get $|T| \leq (|N|-1)/2$ and $1 \notin T$. Set

$$S := \{b\} \cup T \cup \bigcup_{i=\lceil \frac{m+1}{2} \rceil}^{m-2} Nb^i \cup (Nb^{m-1} \setminus \{b^{-1}\}).$$

(For $m \in \{3,4\}$, we have $\lceil (m+1)/2 \rceil > m-2$ and the union $\cup_{i=\lceil (m+1)/2 \rceil}^{m-2} Nb^i$ has to be understood the empty set.)

Write $\Delta := \mathrm{Cay}(N,T)$, $\Gamma := \mathrm{Cay}(G,S)$ and $m_0 := \lceil \frac{m+1}{2} \rceil$. Since Δ is an oriented Cayley graph, we have $T \cap T^{-1} = \emptyset$; thus $S \cap T^{-1} = \emptyset$. Since G/N is cyclic of order $m \geq 3$, for every $s \in S \setminus T$, we have $s^{-1} \notin S \setminus T$. Therefore Γ is an oriented Cayley graph.

For every $s \in S$, write

$$d(s) := |\{(g,s) \text{ arc of } \Gamma \mid g \in G \setminus S\}|.$$

Observe that, if (g,s) is an arc of Γ, then by definition $gs^{-1} \in S$ and hence $g \in Ss$. Thus $d(s) = |Ss \cap (G \setminus S)| = |Ss \setminus S|$.

Let $t \in T$. We have

$$St = \{bt\} \cup Tt \cup \bigcup_{i=m_0}^{m-2} Nb^i \cup \left(Nb^{m-1} \setminus \{b^{-1}t\}\right)$$

and $St \setminus S = \{bt\} \cup (Tt \setminus T) \cup \{b^{-1}\}$. From this it follows that $d(t) \leq 2 + |T|$.

We also have

$$Sb = \{b^2\} \cup Tb \cup \bigcup_{i=m_0+1}^{m-1} Nb^i \cup (N \setminus \{1\}).$$

For $m \geq 5$, it follows that

$$Sb \setminus S = \{b^2\} \cup Tb \cup (N \setminus (\{1\} \cup T)) \cup \{b^{-1}\}$$

and hence $d(b) = 1 + |Tb| + (|N| - 1 - |T|) + 1 = |N| + 1$. When $m = 4$, with a similar computation, we get $Sb = \{b^2\} \cup Tb \cup (N \setminus \{1\})$, $Ss \setminus S = \{b^2\} \cup Tb \cup (N \setminus (\{1\} \cup T))$ and $d(b) = |N|$. When $m = 3$, we have $Sb = \{b^2\} \cup Tb \cup (N \setminus \{1\})$, $Ss \setminus S = Tb \cup (N \setminus (\{1\} \cup T)) \cup (\{b^2\} \setminus (Nb^{-1} \setminus \{b^{-1}\}))$, and $d(b) = |N|$ when $b^2 = b^{-1}$ and $d(b) = |N| - 1$ when $b^2 \neq b^{-1}$. In all cases, $d(b) \geq |N| - 1$.

Let $y \in \bigcup_{i=m_0}^{m-2} Nb^i$ and write $y := nb^j$, with $n \in N$ and $j \in \{m_0, \ldots, m-2\}$. Observe that $m \geq 5$: otherwise the set $\bigcup_{i=m_0}^{m-2} Nb^i$ is empty. We have

$$Sy = \{bnb^j\} \cup Tnb^j \cup \bigcup_{i=m_0}^{m-2} Nb^{i+j} \cup (Nb^{j-1} \setminus \{b^{-1}nb^j\}).$$

We see that $m_0+j \equiv k \pmod m$, with $1 \leq k \leq m/2$. Thus $Nb^{m_0+j} = Nb^k$, $Sy \backslash S \subseteq Nb^k$ and $d(y) \geq |N|$.

Let $y \in Nb^{m-1} \setminus \{b^{-1}\}$. As $Nb^{m-1} = Nb^{-1}$, we may write $y = nb^{-1}$, for some $n \in N \setminus \{1\}$. We have

$$Sy = \{bnb^{-1}\} \cup Tnb^{-1} \cup \bigcup_{i=m_0-1}^{m-3} Nb^i \cup Nb^{m-2} \setminus \{b^{-1}nb^{-1}\}.$$

If $m \geq 5$, then $2 < m_0 \leq m-2$ and hence $Sy \setminus S \supseteq Nb^{m_0-1}$. Thus $d(y) \geq |N|$. When $m = 4$, we see that $Sy \setminus S \supseteq Nb^2 \setminus \{b^{-1}nb^{-1}\}$ and hence $d(y) \geq |N| - 1$. When $m = 3$, we see that $Sy \setminus S \supseteq Nb \setminus \{b, b^{-1}nb^{-1}\}$ and hence $d(y) \geq |N| - 2$.

For the time being, assume

$$|N| \geq \begin{cases} 4 & \text{when } m \geq 5, \\ 6 & \text{when } m = 4, \\ 8 & \text{when } m = 3. \end{cases}$$

Summing up, we have shown that,

(i): if $m \geq 5$, then $d(t) \leq 2 + |T|$ for every $t \in T$, and $d(y) \geq |N|$ for every $y \in S \setminus T$,
(ii): if $m = 4$, then $d(t) \leq 2 + |T|$ for every $t \in T$, and $d(y) \geq |N| - 1$ for every $y \in S \setminus T$,
(iii): if $m = 3$, then $d(t) \leq 2 + |T|$ for every $t \in T$, and $d(y) \geq |N| - 2$ for every $y \in S \setminus T$.

Recall that $|T| \leq (|N| - 1)/2$. As $|N| \geq 4$ when $m \geq 5$, we get $d(t) < d(y)$ for every $t \in T$ and for every $y \in S \setminus T$. Similarly, as $|N| \geq 6$ when $m = 4$, we get $d(t) < d(y)$ for every $t \in T$ and for every $y \in S \setminus T$. Analogously, as $|N| \geq 8$ when $m = 3$, we get $d(t) < d(y)$ for every $t \in T$ and for every $y \in S \setminus T$.

Let A be the automorphism group of Γ. The definition of $d(s)$ gives that $d(s)$ is constant on the A_1-orbits on S, that is, $d(s) = d(s')$ whenever s and s' are in the same A_1-orbit. Now, from above, we deduce that A_1 leaves the sets T and $S \setminus T$ invariant. Thus $N = \langle T \rangle$ is also A_1-invariant and hence A_1 acts as a group of automorphisms of Δ. As Δ is an ORR, we obtain that A_1 fixes point-wise T and hence A_1 fixes point-wise $N = \langle T \rangle$ by Lemma 2.6.

As A_1 fixes point-wise N, from [2, Corollary 3.6], we get that A_1 permutes the N-cosets of G, that is, $\varphi(Nb^k) = N\varphi(b^k)$ for every $\varphi \in A_1$ and $k \in \mathbb{N}$. Now, Nb is the only N-coset having exactly one element in common with S. Thus A_1 fixes setwise the N-coset Nb. Therefore A_1 fixes $S \cap Nb = \{b\}$. Since $G = \langle N, b \rangle$, the group A_1 fixes point-wise G by Lemma 2.6. Therefore $A_1 = 1$ and Γ is an ORR.

Assume now that $m \geq 5$ and $|N| \leq 3$, or $m = 3$ and $|N| \leq 7$. Then N is a proper normal subgroup of G, G/N admits a beautiful generating tuple and $G/N \not\cong C_4$; hence G admits a beautiful generating tuple by Proposition 2.4. Thus, by Theorem 2.1, either G admits an ORR or $G \cong C_3 \times C_3$. Finally, assume that $m = 4$ and $|N| \leq 5$. A careful computer computation with the invaluable help of the computer algebra system `magma` [4] on the finite groups G of order 4κ, with $\kappa \in \{1,2,3,4,5\}$, yields that G admits a beautiful generating tuple, unless G is isomorphic to

$$C_4 \times C_2, \quad C_4 \times C_2 \times C_2, \quad \text{or} \quad \langle a, b \mid a^4 = b^4 = (ab)^2 = (ab^{-1})^2 = 1\rangle.$$

In the former case, G admits an ORR by Theorem 2.1; the remaining three groups are exceptions listed in the statement of the lemma. □

Lemma 2.7 offers (or better, seems to offer) the opportunity to classify groups admitting an ORR by induction on $|G|$; however Lemma 2.7 and its proof give no information in the case that a group G contains a subgroup N with $|G : N| = 2$ and with N admitting an ORR. This makes the classification of groups admitting an ORR rather difficult.

Moreover, the careful reader might have observed that most of the proof of Lemma 2.7 is unnecessarily long; indeed, when $m \neq 4$, Lemma 2.7 can be proved more easily combining Proposition 2.4 with Theorem 2.1. However, because of the lack of information in Proposition 2.4 when $m = 4$, this method does not work when $m = 4$. Hence we have preferred to include a slightly longer proof which deals with each $m \geq 3$ and which does not rely upon the rather difficult Theorem 2.1.

Proposition 2.8. *Let G be a finite soluble group with no beautiful generating tuple. Then there exist a subgroup N of G, $g \in G \setminus N$ and $n_0 \in N$ with*

(i): $N \trianglelefteq G$,
(ii): $g^2 = 1$,
(iii): $G = \langle N, g\rangle$, *and*
(iv): $N = H \cup n_0 H$, *where* $H = \{n \in N \mid n^g = n^{-1}\}$.

Proof. If G is generalised dihedral then the statement is clear, there exists a normal (abelian) subgroup N of G having index 2 and an involution $g \in G \setminus N$ inverting each element of N: thus the conclusion of the proposition holds with $H = N$ and $n_0 = 1$. Assume then that G is not generalised dihedral and let $d := d(G)$. In view of Lemma 2.2, among all generating tuples $g_1, \ldots, g_d$ for G with $o(g_i) > 2$ for each $i \in \{1, \ldots, d\}$, choose one such that $\ell \in \mathbb{N}$ is maximum with the property that

$$o(g_2 g_1^{-1}),\ o(g_3 g_2^{-1}),\ \ldots, o(g_\ell g_{\ell-1}^{-1}) > 2.$$

Observe that $\ell < d$ otherwise $g_1, \ldots, g_d$ is a beautiful generating tuple: against our assumption.

Consider $N := \langle g_1, \ldots, g_\ell, g_{\ell+2}, g_{\ell+3}, \ldots, g_d\rangle$. As both $g_1, \ldots, g_\ell, g_{\ell+1}, g_{\ell+2}, \ldots, g_d$ and $g_1, \ldots, g_\ell, g_{\ell+1}^{-1}, g_{\ell+2}, \ldots, g_d$ are generating sets for G, from the maximality of ℓ, we deduce that $o(g_{\ell+1}g_\ell^{-1}) = 2$ and $o(g_{\ell+1}^{-1}g_\ell^{-1}) = 2$. Now, we infer

$$g_{\ell+1}g_\ell^{-1} = g_\ell g_{\ell+1}^{-1}, \quad g_\ell g_{\ell+1} = g_{\ell+1}^{-1}g_\ell^{-1}, \tag{2}$$

and hence $(g_\ell^2)^{g_{\ell+1}} = g_\ell^{-2}$ and $(g_{\ell+1}^2)^{g_\ell} = g_{\ell+1}^{-2}$.

For each $h \in N$, $g_1, \ldots, g_\ell, hg_{\ell+1}, g_{\ell+2}, \ldots, g_d$ is a generating set for G and hence, from the maximality of ℓ, we deduce that either $o(hg_{\ell+1}) = 2$ or $o(hg_{\ell+1}g_\ell^{-1}) = 2$. Write $K := \{h \in N \mid o(hg_{\ell+1}) = 2\}$ and $H := \{h \in N \mid o(hg_{\ell+1}g_\ell^{-1}) = 2\}$. So far we have observed that

$$N = K \cup H.$$

Write $g := g_{\ell+1}g_\ell^{-1}$ and $n_0 := g_\ell^{-1}$, and notice that $g \in G \setminus N$, $g^2 = 1$ and $G = \langle N, g\rangle$. Let $h \in H$. Then $2 = o(hg_{\ell+1}g_\ell^{-1}) = o(hg)$ and, as $o(g) = 2$, we get $h^g = h^{-1}$. Therefore $H \subseteq \{h \in N \mid h^g = h^{-1}\}$. As the inclusion $\{h \in N \mid h^g = h^{-1}\} \subseteq H$ is clear, we get $H = \{h \in N \mid h^g = h^{-1}\}$. Let $h \in K$. Then $2 = o(hg_{\ell+1}) = o(hg_{\ell+1}g_\ell^{-1}g_\ell) = o(hgn_0^{-1})$, that is, $hgn_0^{-1}hgn_0^{-1} = 1$ and $(n_0^{-1}h)^g = (n_0^{-1}h)^{-1}$ because $o(g) = 2$. Thus $n_0^{-1}h \in H$ and $h \in n_0H$. This shows that $K \subseteq n_0H$ and hence $N = n_0H \cup H$.

Observe that, from $N = H \cup n_0H$, we deduce $|H| \geq |N|/2$. Assume that H is not a subgroup of N. Then $N = \langle H\rangle$. Moreover, since H is g-invariant by conjugation, we obtain that N is normalised by g. As $G = \langle N, g\rangle$, we get $N \trianglelefteq G$ and hence the proposition is proven.

Assume that H is a subgroup of N. As $|H| \geq |N|/2$, we have $|N : H| \leq 2$. If $N = H$, then g acts by conjugation inverting each element of N and hence $G = \langle N, g\rangle$ is a dihedral group, contrary to our assumption.

Assume that H is a subgroup of N with $|N : H| = 2$. In particular, $H \trianglelefteq N$ and hence $H \trianglelefteq \langle N, g\rangle = G$ because g acts by conjugation inverting each element of H. Moreover, H is abelian. Observe that $N = \langle H, g_\ell\rangle$ (because $N = H \cup n_0H$ and $n_0 = g_\ell^{-1}$) and $G = \langle g, g_\ell, H\rangle$. We write $\bar{G} := G/H$ and adopt the "bar notation", that is, for $x \in G$, we denote by $\bar{x}$ the element xH. Now, $\bar{g}_\ell$ has order 2 because $|\bar{N}| = |N : H| = 2$, and $\bar{g}$ has also order 2 because $g^2 = 1$. Therefore $\bar{G} = \langle \bar{g}_\ell, \bar{g}\rangle$ is a dihedral group. Now g acts by conjugation inverting each element of H and hence so does $g^{g_\ell} = g_\ell^{-1}g_{\ell+1}$. Therefore $g_\ell^{-1}g_{\ell+1}g = g_\ell^{-1}g_{\ell+1}^2g_\ell^{-1}$ centralises H. From Eq. (2), we obtain $g_\ell^{-1}g_{\ell+1}^2g_\ell^{-1} = g_{\ell+1}^{-2}g_\ell^{-2}$ and hence $g_{\ell+1}^2$ centralises H.

Write $A := \langle H, g_{\ell+1}^2\rangle$. As $\bar{G} = \langle \bar{g}, \bar{g}_\ell\rangle$ is dihedral and $o(\bar{g}) = o(\bar{g}_\ell) = 2$, we deduce that $\langle \bar{g}\bar{g}^{\bar{g}_\ell}\rangle$ has index 4 in $\bar{G}$. As $\langle \bar{g}\bar{g}^{\bar{g}_\ell}\rangle = \langle \bar{g}_{\ell+1}^2\rangle = \bar{A}$, we deduce $|G : A| = 4$.

Observe that A is abelian and that g acts by conjugation inverting each element of A (recall that $(g_{\ell+1}^2)^{g_\ell} = g_{\ell+1}^{-2}$ and hence $(g_{\ell+1}^2)^g = g_{\ell+1}^{-2}$). Now, $G = \langle A, g_{\ell+1}, g\rangle$, $|G : A| = 4$, and $D_1 := \langle A, g\rangle$, $D_2 := \langle A, g_{\ell+1}\rangle$ and $D_3 := \langle A, g_\ell\rangle$ are the three subgroups of G containing A and having index 2 in G. Now, $D_2 \trianglelefteq G$, $G = \langle D_2, g\rangle$, $g^2 = 1$ and

$D_2 = A \cup g_{\ell+1}A$ and g acts by conjugation inverting each element of A; thus the proof follows by taking $N := D_2$ and $n_0 := g_{\ell+1}$. □

We conclude this section with our first main reduction.

Theorem 2.9. *Let G be a finite soluble group with no beautiful generating tuple. Then one of the following holds.*

(A) *There exist two subgroups A and N of G and two elements $g \in G \setminus N$ and $n \in N \setminus A$ with*

(i): $|G : N| = |N : A| = 2$,

(ii): $g^2 = 1$,

(iii): $n^g = n^{-1}$ *and* $a^g = a^{-1}$ *for each* $a \in A$. *(In particular A is abelian.)*

(B): *There exists a 2-subgroup N of G and $g \in G$ with $|G : N| = 2$, $G = \langle N, g\rangle$ and $g^2 = 1$. Moreover, the action of g by conjugation on N inverts precisely half of the elements of N. The isomorphism class of N is determined in [10].*

Proof. From Proposition 2.8, there exist a subgroup N of G, $g \in G \setminus N$ and $n_0 \in N$ with $N \trianglelefteq G$, $g^2 = 1$, $G = \langle N, g\rangle$ and $N = H \cup n_0 H$ where $H = \{n \in N \mid n^g = n^{-1}\}$. In particular, $|H| \geq |N|/2$ and the element g acts by conjugation on N as an automorphism inverting at least half of its elements.

Finite groups admitting an automorphism inverting at least half of their elements have been classified in [8,10,15]. We use this classification to pin down further the algebraic structure of G. We recall that finite groups admitting an automorphism inverting more than half of their elements are classified by Liebeck and MacHale in [15, Structure Theorem 4.13]; these are divided into three types: **Type I***, **Type II*** and **Type III***. Then, Fitzpatrick [8] considered the finite groups, not having order a power of 2, admitting an automorphism inverting exactly half of their elements. Then, the complete classification was achieved by Hegarty and MacHale [10] who classified the finite 2-groups admitting an automorphism inverting exactly half of their elements. We use these three papers in what follows.

Suppose first that $|H| > |N|/2$. Hence N admits an automorphism (conjugation via the element g) which inverts more than half of its elements. As we mentioned above, the structure of N and the automorphism g are described in [15, Structure Theorem 4.13]. Suppose that N is of **Type I*** as defined in [15, Structure Theorem 4.13]. Then N has an abelian normal subgroup A and an element $n \in N$ with $|N : A| = 2$, $N = A \cup nA$, $n^g = n^{-1}$ and $a^g = a^{-1}$ for each $a \in A$. In particular, in this case G satisfies the conclusion **(A)** of this theorem.

Suppose next that N is of **Type II***. According to [15], we have a fairly explicit description of N and of the action of g on N by conjugation. Here, N is a finite nilpotent group of nilpotency class 2, the commutator subgroup $\langle z\rangle$ of N has order 2, and the centre Z of N has index 2^{2k} in N (for some $k \in \mathbb{N}$ with $k \geq 2$). Moreover, N/Z is an elementary abelian 2-group generated by $x_1Z, x_2Z, \dots, x_kZ, a_1Z, a_2Z, \dots, a_kZ$ subject to the following defining relations:

$$\begin{aligned}
&[x_i, x_j] = [a_i, a_j] = 1 && \text{for every } i, j \in \{1, \dots, k\},\\
&[a_i, x_j] = 1 && \text{for every } i, j \in \{1, \dots, k\} \text{ with } i \neq j,\\
&[a_i, x_i] = z && \text{for every } i \in \{1, \dots, k\}.
\end{aligned}$$

Observe that each element of N can be written as a product $ax_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$, for some $a \in \langle a_1, \dots, a_k, Z\rangle$ and for some $\varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$.

Fix $\varepsilon_i \in \{0, 1\}$ for each $i \in \{1, \dots, k\}$ and let $\varphi : N \to N$ be the automorphism (depending on $\varepsilon_1, \dots, \varepsilon_k$) of N defined by the mapping

$$ax_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k} \mapsto a^{-1}x_1^{-\varepsilon_1} \cdots x_k^{-\varepsilon_k}.$$

From [15], we have $n^g = n^\varphi$ for each $n \in N$, for some automorphism φ of N as above. We determine the set $H := \{n \in N \mid n^g = n^{-1}\}$ explicitly. Let $h \in H$ and write $h = a_1^{\varepsilon_1}a_2^{\varepsilon_2} \cdots a_k^{\varepsilon_k}tx_1^{\eta_1}x_2^{\eta_2} \cdots x_k^{\eta_k}$, with $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \eta_1, \eta_2, \dots, \eta_k \in \{0, 1\}$ and $t \in Z$. Using the relations defining N and the definition of φ, we obtain

$$\begin{aligned}
h^{-1} &= x_1^{-\eta_1}x_2^{-\eta_2} \cdots x_k^{-\eta_k}a_1^{-\varepsilon_1}a_2^{-\varepsilon_2} \cdots a_k^{-\varepsilon_k}t^{-1},\\
h^g &= a_1^{-\varepsilon_1}a_2^{-\varepsilon_2} \cdots a_k^{-\varepsilon_k}t^{-1}x_1^{-\eta_1}x_2^{-\eta_2} \cdots x_k^{-\eta_k}\\
&= x_1^{-\eta_1}x_2^{-\eta_2} \cdots x_k^{-\eta_k}a_1^{-\varepsilon_1}a_2^{-\varepsilon_2} \cdots a_k^{-\varepsilon_k}z^{\varepsilon_1\eta_1+\varepsilon_2\eta_2+\cdots+\varepsilon_k\eta_k}t^{-1}.
\end{aligned}$$

Thus $h^g = h^{-1}$ if and only if $\varepsilon_1\eta_1 + \varepsilon_2\eta_2 + \cdots + \varepsilon_k\eta_k \equiv 0 \pmod 2$. This proves that

$$H = \{a_1^{\varepsilon_1}a_2^{\varepsilon_2} \cdots a_k^{\varepsilon_k}tx_1^{\eta_1}x_2^{\eta_2} \cdots x_k^{\eta_k} \mid \sum_{i=1}^{k} \varepsilon_i\eta_i \equiv 0 \pmod 2\}.$$

Write $n_0 = a_1^{\bar\varepsilon_1}a_2^{\bar\varepsilon_2} \cdots a_k^{\bar\varepsilon_k}\bar t x_1^{\bar\eta_1}x_2^{\bar\eta_2} \cdots x_k^{\bar\eta_k}$, for some $\bar\varepsilon_1, \bar\varepsilon_2, \dots, \bar\varepsilon_k, \bar\eta_1, \bar\eta_2, \dots, \bar\eta_k \in \{0, 1\}$ and $\bar t \in Z$. (Recall that we have $N = H \cup n_0H$.) From Lemma 2.5, there exist $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\, \eta_1, \eta_2, \dots, \eta_k \in \{0, 1\}$ such that $\sum_{i=1}^k \varepsilon_i\eta_i \equiv 1 \pmod 2$ and $\sum_{i=1}^i(\bar\varepsilon_i + \varepsilon_i)(\bar\eta_i + \eta_i) = 1$. Consider now $n := a_1^{\varepsilon_1}a_2^{\varepsilon_2} \cdots a_k^{\varepsilon_k}x_1^{\eta_1}x_2^{\eta_2} \cdots x_k^{\eta_k} \in N$. As $\sum_{i=1}^k \varepsilon_i\eta_i \equiv 1 \pmod 2$, we have $n \notin H$; as $\sum_{i=1}^k(\bar\varepsilon_i + \varepsilon_i)(\bar\eta_i + \eta_i) = 1$, we also have $n \notin n_0H$. However this contradicts $N = H \cup n_0H$.

Suppose next that N is of **Type III***. According to [15], we have again a fairly explicit description of N and of the action of g on N by conjugation. Here, N is a finite nilpotent group of nilpotency class 2, the commutator subgroup $\langle z_1, z_2\rangle$ of N has order 4, and the centre Z of N has index 16 in N. Moreover, N/Z is an elementary abelian 2-group generated by x_1Z, x_2Z, a_1Z, a_2Z subject to the following defining relations:

$$
\begin{aligned}
&[x_1, x_2] = [a_1, a_2] = [x_1, a_2] = [x_2, a_1] = 1,\\
&[a_1, x_1] = z_1,\\
&[a_2, x_2] = z_2.
\end{aligned}
$$

Observe that each element of N can be written as a product $ax_1^{\varepsilon_1}x_2^{\varepsilon_2}$, for some $a \in \langle a_1, a_2, Z\rangle$ and for some $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

Fix $\varepsilon_i \in \{0, 1\}$ for each $i \in \{1, 2\}$ and let $\varepsilon : N \to N$ be the automorphism (depending on $\varepsilon_1, \varepsilon_2$) of N defined by the map

$$ax_1^{\varepsilon_1}x_2^{\varepsilon_2} \mapsto a^{-1}x_1^{-\varepsilon_1}x_2^{-\varepsilon_2}.$$

From [15], we have $n^g = n^\varepsilon$ for each $n \in N$, for some automorphism ε of N as above. We determine the set $H := \{n \in N \mid n^g = n^{-1}\}$ explicitly. Let $h \in H$ and write $h = a_1^{\varepsilon_1}a_2^{\varepsilon_2}x_1^{\eta_1}x_2^{\eta_2}z$, with $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 \in \{0, 1\}$ and $z \in Z$. Using the relations defining N, we obtain

$$
\begin{aligned}
h^{-1} &= x_1^{-\eta_1}x_2^{-\eta_2}a_1^{-\varepsilon_1}a_2^{-\varepsilon_2}z^{-1},\\
h^g &= a_1^{-\varepsilon_1}a_2^{-\varepsilon_2}x_1^{-\eta_1}x_2^{-\eta_2}z^{-1}\\
&= x_1^{-\eta_1}x_2^{-\eta_2}a_1^{-\varepsilon_1}a_2^{-\varepsilon_2}z_1^{\varepsilon_1\eta_1}z_2^{\varepsilon_2\eta_2}z^{-1}.
\end{aligned}
$$

Thus $h^g = h^{-1}$ if and only if $\varepsilon_1\eta_1 = 0$ and $\varepsilon_2\eta_2 = 0$. This proves

$$H = \{a_1^{\varepsilon_1}a_2^{\varepsilon_2}x_1^{\eta_1}x_2^{\eta_2}z \in N \mid \varepsilon_1\eta_1 = 0 \text{ and } \varepsilon_2\eta_2 = 0\}.$$

Write $n_0 = a_1^{\bar\varepsilon_1}a_2^{\bar\varepsilon_2}x_1^{\bar\eta_1}x_2^{\bar\eta_2}\bar z$, for some $\bar\varepsilon_1, \bar\varepsilon_2, \bar\eta_1, \bar\eta_2 \in \{0, 1\}$ and $\bar z \in Z$. (Recall that we have $N = H \cup n_0H$.) Consider $n := a_1a_2^{\bar\varepsilon_2+1}x_1x_2^{\bar\eta_2+1} \in N$. From the characterisation of the elements of H, $n \notin H$ and hence $n_0n^{-1} \in H$. However,

$$n_0n^{-1} = a_1^{\bar\varepsilon_1-1}a_2^{-1}x_1^{\bar\eta_1-1}x_2^{-1}z_1^{\bar\eta_1+1}z_2^{\bar\varepsilon_2+1}\bar z^{-1} \notin H,$$

and this is a contradiction.

This concludes the proof when $|H| > |N|/2$, that is, N admits an automorphism inverting more than half of its elements. For the rest of the proof we assume then that $|H| = |N|/2$. Here, the structure of N is described in [8] when N is not a 2-group and in [10] when N is a 2-group.

We start our analysis with a basic observation on $N/\mathbf{Z}(N)$ (the argument here is potentially much more general, but we only need its application to $N/\mathbf{Z}(N)$ in what follows). Consider, for a moment, the action of g by conjugation on $N/\mathbf{Z}(N)$ and assume that g inverts x elements of $N/\mathbf{Z}(N)$. Let $n_1\mathbf{Z}(N), \ldots, n_x\mathbf{Z}(N)$ be the elements of $N/\mathbf{Z}(N)$ inverted by g by conjugation. If $n \in H$, that is, $n^g = n^{-1}$, then $(n\mathbf{Z}(N))^g = n^{-1}\mathbf{Z}(N) = (n\mathbf{Z}(N))^{-1}$ and hence $n \in n_i\mathbf{Z}(N)$, for some $i \in \{1, \ldots, x\}$.

Therefore, $H \subseteq \cup_i n_i \mathbf{Z}(N)$, $|N|/2 = |H| \leq |\cup_i n_i \mathbf{Z}(N)| = x|\mathbf{Z}(N)|$ and $x \geq |N/\mathbf{Z}(N)|/2$. This shows that the action of g by conjugation on $N/\mathbf{Z}(N)$ inverts at least half of its elements. Of course, the action of g by conjugation on $N/\mathbf{Z}(N)$ may (in principal) invert more than half of its elements.

Assume that N is not a 2-group. Suppose that N is of **Type I**, as defined in [8]. (Despite the detailed description of N in [8], we only need some partial information on the algebraic structure of N.) Here $N/\mathbf{Z}(N) \cong \mathrm{Alt}(4)$, where $\mathrm{Alt}(4)$ is the alternating group on 4 symbols. A computation in $\mathrm{Alt}(4)$ reveals that an automorphism of $\mathrm{Alt}(4)$ that inverts at least half of its elements is the conjugation by a transposition of $\mathrm{Sym}(4)$ and that this automorphism actually inverts precisely half of the elements of $\mathrm{Alt}(4)$. In particular, we may think that the action of g by conjugation on $N/\mathbf{Z}(N)$ is induced by conjugation via the transposition $(1, 2)$. Therefore $G/\mathbf{Z}(N) \cong \mathrm{Sym}(4)$. Now H modulo $\mathbf{Z}(N)$ is the set $\bar{H} = \{1, (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,2)(3\,4)\}$. A computation in $\mathrm{Sym}(4)$ reveals that there exists no $\bar{n}_0 \in \mathrm{Alt}(4)$ with $\mathrm{Alt}(4) = \bar{H} \cup \bar{n}_0\bar{H}$, however this contradicts $N = H \cup n_0 H$. Suppose that N is of **Type II**, as defined in [8]: here the argument requires some careful computations and much more care. We have $N = K \times Z$, where K is a finite group which is not a 2-group and Z is abelian. Moreover, $K/\mathbf{Z}(K) \cong C_2 \times C_2 \times \mathrm{Sym}(3)$ and K is generated modulo $\mathbf{Z}(K)$ by w, v, u, t satisfying the conditions:

$$\begin{aligned}&[w, u] = [v, u] = [w, t] = [v, t] = t^3 = 1,\ [t, u] = t,\ [v, w] \neq 1, \\ &[v, w], u^2, v^2, w^2 \in \mathbf{Z}(K).\end{aligned} \tag{3}$$

As $v^2, [v, w] \in \mathbf{Z}(N)$, we get

$$1 = [v^2, w] = [v, w]^v [v, w] = [v, w]^2,$$

that is, $[v, w]$ has order 2. As usual, $G = \langle N, g\rangle$, where $g^2 = 1$, g normalises N and the action of g by conjugation on N induces an automorphism inverting at least half of its elements. We prove that G has a beautiful generating tuple: here, we argue by induction on the order $|G|$ of G (for all groups $G = \langle N, g\rangle$ where N is of **Type II**, that is, it satisfies all the conditions mentioned above including Eq. (3)). If $|Z| > 1$, then $|G/Z| < |G|$ and N/Z is still of **Type II**. Hence, by induction, G/Z has a beautiful generating tuple, and so does G by Proposition 2.4. We may thus assume that $Z = 1$ and hence $N = K$. Assume that $\mathbf{Z}(N)$ contains a minimal normal subgroup C with $[v, w] \notin C$. Then $|G/C| < |G|$, N/C is still of **Type II** and gN acts by conjugation on N/C inverting at least half of its elements. Therefore, by induction, G/C has a beautiful generating tuple, and so does G again by Proposition 2.4. Thus, we may assume that $\langle [v, w]\rangle$ is the unique minimal normal subgroup of $\mathbf{Z}(N)$. This yields that $\mathbf{Z}(N)$ is cyclic of order 2^ℓ, for some $\ell \in \mathbb{N}$ with $\ell \geq 1$, and $[v, w]$ is the unique involution of $\mathbf{Z}(N)$. If $\ell = 1$, then $|N| = |N : \mathbf{Z}(N)||\mathbf{Z}(N)| = 24 \cdot 2 = 48$. Routine computations with the invaluable help of the computer algebra system `magma` [4] yield that there are four isomorphism classes for the group N. Another computation reveals that each N is contained in a unique (up to isomorphism) group G of order $2|N| = 96$ and admitting an element g with $g^2 = 1$ and with g inverting at least half of the elements of N. We check, again with `magma`, that each of these four groups has a beautiful generating tuple. Suppose next that $\ell \geq 2$.

Let z be a generator of $\mathbf{Z}(N)$. Using the relations in Eq. (3) and the fact that $\mathbf{Z}(N)$ is cyclic, it is not difficult to see that, when $\ell \geq 2$, N is isomorphic to one of the following four groups (this is indeed not difficult, but it requires some detailed computations and the classification of the 2-groups containing a maximal cyclic subgroup):

$$\begin{aligned}
N_1 := \langle t, u, v, w, z \quad & \mid [w,u] = [v,u] = [w,t] = [v,t] = t^3 = u^2 = v^2 = w^2 = (vw)^4 = 1, \\
& [t,u] = t, [t,z] = [u,z] = [v,z] = [w,z] = 1, z^{2^{\ell-1}} = [v,w]\rangle, \\
N_2 := \langle t, u, v, w \quad & \mid [w,u] = [v,u] = [w,t] = [v,t] = t^3 = u^2 = v^{2^{\ell+1}} = w^2 = 1, \\
& [t,u] = t, v^w = v^{1+2^\ell}\rangle, \\
N_3 := \langle t, u, v, w \quad & \mid [w,u] = [v,u] = [w,t] = [v,t] = t^3 = u^{2^{\ell+1}} = v^2 = w^2 = (vw)^4 = 1, \\
& [t,u] = t, u^{2^\ell} = [v,w]\rangle, \\
N_4 := \langle t, u, v, w \quad & \mid [w,u] = [v,u] = [w,t] = [v,t] = t^3 = v^{2^{\ell+1}} = w^2 = 1, \\
& [t,u] = t, v^w = v^{1+2^\ell}, u^2 = v^2\rangle.
\end{aligned}$$

(The group N_1 is isomorphic to $\mathrm{Sym}(3) \times (D_4 \circ C_{2^\ell})$ where the central product $D_4 \circ C_{2^\ell}$ is amalgamated over the centre of the dihedral group D_4, the group N_2 is isomorphic to $\mathrm{Sym}(3) \times \langle v, w \mid v^{2^{\ell+1}} = w^2 = 1, v^w = v^{1+2^\ell}\rangle$. Observe that $\mathbf{Z}(N_1) = \langle z\rangle$, $\mathbf{Z}(N_2) = \langle v^2\rangle$, $\mathbf{Z}(N_3) = \langle u^2\rangle$ and $\mathbf{Z}(N_4) = \langle v^2\rangle = \langle u^2\rangle$.)

The group N_1 has, up to conjugacy in $\mathrm{Aut}(N_1)$, two automorphisms $\varphi_{1,1}$ and $\varphi_{1,2}$ inverting at least half of its elements. These two automorphisms are defined on the generators t, u, v, w, z of N_1 by:

$$\begin{array}{lllll}
\varphi_{1,1}(t) = t, & \varphi_{1,1}(u) = u, & \varphi_{1,1}(v) = v, & \varphi_{1,1}(w) = w, & \varphi_{1,1}(z) = z^{-1}, \\
\varphi_{1,2}(t) = t^{-1}, & \varphi_{1,2}(u) = u, & \varphi_{1,2}(v) = v, & \varphi_{1,2}(w) = w, & \varphi_{1,2}(z) = z^{-1}.
\end{array}$$

Observe that the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ z \mapsto z,\ \varphi_{1,1} \mapsto \varphi_{1,2}u$$

induces a group isomorphism from $N_1 \rtimes \langle \varphi_{1,1}\rangle$ to $N_1 \rtimes \langle \varphi_{1,2}\rangle$. This means that (although we have two $\mathrm{Aut}(N_1)$-conjugacy classes of automorphisms of N_1 inverting at least half of the elements of N_1) we have only one isomorphism class for G when $N \cong N_1$. Thus, we may think that $G = N_1 \rtimes \langle \varphi_{1,1}\rangle$ where g acts on N as the automorphism $\varphi_{1,1}$ of N_1. Now, it can be checked directly that

$$tv,\ tw,\ z,\ gvw,\ uz$$

is a beautiful generating tuple for G. The argument for the remaining groups is similar and requires only a few adjustments.

The group N_2 has, up to conjugacy in $\mathrm{Aut}(N_2)$, four automorphisms $\varphi_{2,1}$, $\varphi_{2,2}$, $\varphi_{2,3}$ and $\varphi_{2,4}$ inverting at least half of its elements. These four automorphisms are defined on the generators t, u, v, w of N_2 by:

$$\begin{array}{llll}
\varphi_{2,1}(t) = t, & \varphi_{2,1}(u) = u, & \varphi_{2,1}(v) = v^{-1+2^\ell}, & \varphi_{2,1}(w) = w, \\
\varphi_{2,2}(t) = t^{-1}, & \varphi_{2,2}(u) = u, & \varphi_{2,2}(v) = v^{-1+2^\ell}, & \varphi_{2,2}(w) = w, \\
\varphi_{2,3}(t) = t, & \varphi_{2,3}(u) = u, & \varphi_{2,3}(v) = v^{-1}, & \varphi_{2,3}(w) = v^{2^\ell} w, \\
\varphi_{2,4}(t) = t^{-1}, & \varphi_{2,4}(u) = u, & \varphi_{2,4}(v) = v^{-1}, & \varphi_{2,4}(w) = v^{2^\ell} w.
\end{array}$$

Observe that the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ \varphi_{2,1} \mapsto \varphi_{2,2}u$$

induces a group isomorphism from $N_2 \rtimes \langle \varphi_{2,1} \rangle$ to $N_2 \rtimes \langle \varphi_{2,2} \rangle$. Similarly, the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ \varphi_{2,3} \mapsto \varphi_{2,4}u$$

induces a group isomorphism from $N_2 \rtimes \langle \varphi_{2,3} \rangle$ to $N_2 \rtimes \langle \varphi_{2,4} \rangle$. Analogously, the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto vw,\ w \mapsto v^{2^\ell} w,\ \varphi_{2,1} \mapsto \varphi_{2,3}$$

induces a group isomorphism from $N_2 \rtimes \langle \varphi_{2,1} \rangle$ to $N_2 \rtimes \langle \varphi_{2,3} \rangle$. Therefore $N_2 \rtimes \langle \varphi_{2,1} \rangle \cong N_2 \rtimes \langle \varphi_{2,2} \rangle \cong N_2 \rtimes \langle \varphi_{2,3} \rangle \cong N_2 \rtimes \langle \varphi_{2,4} \rangle$. This means that (although we have four $\mathrm{Aut}(N_2)$-conjugacy classes of automorphisms of N_2 inverting at least half of its elements) we have only one isomorphism class for G when $N \cong N_2$. Thus, we may think that $G = N_2 \rtimes \langle \varphi_{2,1} \rangle$ where g acts on N as the automorphism $\varphi_{2,1}$ of N_2. Now, it can be checked directly that

$$gv,\ tv,\ t^{-1}w,\ uv^{-1}$$

is a beautiful generating tuple for G.

The group N_3 has, up to conjugacy in $\mathrm{Aut}(N_3)$, two automorphisms $\varphi_{3,1}$ and $\varphi_{3,2}$ inverting at least half of its elements. These two automorphisms are defined on the generators t, u, v, w of N_3 by:

$$\begin{array}{llll}
\varphi_{3,1}(t) = t, & \varphi_{3,1}(u) = u^{-1}, & \varphi_{3,1}(v) = v^{-1}, & \varphi_{3,1}(w) = w, \\
\varphi_{3,2}(t) = t^{-1}, & \varphi_{3,2}(u) = u^{-1}, & \varphi_{3,2}(v) = v^{-1}, & \varphi_{3,2}(w) = w.
\end{array}$$

Observe that the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ \varphi_{3,1} \mapsto \varphi_{3,2}u$$

induces a group isomorphism from $N_3 \rtimes \langle \varphi_{3,1}\rangle$ to $N_3 \rtimes \langle \varphi_{3,2}\rangle$. This means that (although we have two $\mathrm{Aut}(N_3)$-conjugacy classes of automorphisms of N_3 inverting at least half of its elements) we have only one isomorphism class for G when $N \cong N_3$. Thus, we may think that $G = N_3 \rtimes \langle \varphi_{3,1}\rangle$ where g acts on N as the automorphism $\varphi_{3,1}$ of N_3. Now, it can be checked directly that

$$gt,\ vw,\ u,\ u^{-1}v$$

is a beautiful generating tuple for G.

The group N_4 has, up to conjugacy in $\mathrm{Aut}(N_4)$, four automorphisms $\varphi_{4,1}$, $\varphi_{4,2}$, $\varphi_{4,3}$ and $\varphi_{4,4}$ inverting at least half of its elements. These four automorphisms are defined on the generators t, u, v, w of N_4 by:

$$\begin{array}{llll}
\varphi_{4,1}(t) = t, & \varphi_{4,1}(u) = u^{-1}, & \varphi_{4,1}(v) = v^{-1+2^\ell}, & \varphi_{4,1}(w) = w,\\
\varphi_{4,2}(t) = t^{-1}, & \varphi_{4,2}(u) = u^{-1}, & \varphi_{4,2}(v) = v^{-1+2^\ell}, & \varphi_{4,2}(w) = w,\\
\varphi_{4,3}(t) = t, & \varphi_{4,3}(u) = u^{-1}, & \varphi_{4,3}(v) = v^{-1}, & \varphi_{4,3}(w) = v^{2^\ell}w,\\
\varphi_{4,4}(t) = t^{-1}, & \varphi_{4,4}(u) = u^{-1}, & \varphi_{4,4}(v) = v^{-1}, & \varphi_{4,4}(w) = v^{2^\ell}w.
\end{array}$$

Observe that the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ \varphi_{4,1} \mapsto \varphi_{4,2}u$$

induces a group isomorphism from $N_4 \rtimes \langle \varphi_{4,1}\rangle$ to $N_4 \rtimes \langle \varphi_{4,2}\rangle$. Similarly, the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v,\ w \mapsto w,\ \varphi_{4,3} \mapsto \varphi_{4,4}u$$

induces a group isomorphism from $N_4 \rtimes \langle \varphi_{4,3}\rangle$ to $N_4 \rtimes \langle \varphi_{4,4}\rangle$. Analogously, the mapping

$$t \mapsto t,\ u \mapsto u,\ v \mapsto v^{1-2^{\ell-1}}w,\ w \mapsto v^{2^\ell}w,\ \varphi_{4,1} \mapsto \varphi_{4,3}$$

induces a group isomorphism from $N_4 \rtimes \langle \varphi_{4,1}\rangle$ to $N_4 \rtimes \langle \varphi_{4,3}\rangle$. Therefore $N_4 \rtimes \langle \varphi_{4,1}\rangle \cong N_4 \rtimes \langle \varphi_{4,2}\rangle \cong N_4 \rtimes \langle \varphi_{4,3}\rangle \cong N_4 \rtimes \langle \varphi_{4,4}\rangle \cong$. This means that (although we have four $\mathrm{Aut}(N_4)$-conjugacy classes of automorphisms of N_4 inverting at least half of its elements) we have only one isomorphism class for G when $N \cong N_4$. Thus, we may think that $G = N_4 \rtimes \langle \varphi_{4,1}\rangle$ where g acts on N as the automorphism $\varphi_{4,1}$ of N_4. Now, it can be checked directly that

$$gt,\ vw,\ u,\ vu$$

is a beautiful generating tuple for G. This concludes the proof of this case.

Now the proof of Theorem 2.9 is completed: we have shown that $|H| = |N|/2$, that is, the action of g by conjugation on N inverts precisely half of its elements, and that N is one of the 2-groups classified in [10]. □

3. Proof of Theorem 1.2

*3.1. Dealing with the groups in Theorem 2.9 part **(A)***

We set some notation that we use throughout the whole of this section. Let G be a finite group, let A and N be subgroups of G and let $g \in G$ and $n \in N$ with

(i): $|G : N| = |N : A| = 2$,
(ii): $g^2 = 1$, $N = A \cup nA$,
(iii): $n^g = n^{-1}$ and $a^g = a^{-1}$ for each $a \in A$.

Lemma 3.1. *If $\mathbf{C}_A(n) = A$ (that is, N is abelian), or $o(n) = 2$ and $a^n = a^{-1}$ for each $a \in A$, then G is a generalised dihedral group on N or on $\langle A, gn\rangle$. In particular, G has no* ORR*.*

Proof. If $\mathbf{C}_A(n) = A$, then $\langle A, n\rangle = N$ is abelian and g acts by conjugation inverting each element of N. Thus G is generalised dihedral on N.

If $a^n = a^{-1}$ for each $a \in A$, then $\langle A, gn\rangle$ is abelian. Moreover, if $o(n) = 2$, then g acts by conjugation inverting each element of $\langle A, gn\rangle$. Thus G is generalised dihedral on $\langle A, gn\rangle$. □

In view of Lemma 3.1, we also assume:

(iv): $\mathbf{C}_A(n) < A$, and either $o(n) > 2$ or $a_0^n \neq a_0^{-1}$ for some $a_0 \in A$.

Proposition 3.2. *If $|A|$ is odd, then G admits a beautiful generating tuple. In particular, G has a* ORR*.*

Proof. We argue by induction on $|A|$. Since $|G : A| = 4$ and $|A|$ is odd, replacing n by a suitable G-conjugate, we may assume that $\langle g, n\rangle$ is a Sylow 2-subgroup of G. Therefore $G = A \rtimes \langle g, n\rangle$ and $|\langle g, n\rangle| = |G/A| = 4$.

Assume first that $\langle g, n\rangle$ is cyclic, that is, G/A is cyclic of order 4. We prove that G has a beautiful generating tuple by induction on $|A|$. Suppose that A is a minimal normal subgroup of G. Then, by Lemma 2.3, G has a beautiful generating tuple. Suppose that A is not a minimal normal subgroup of G and let B be a minimal normal subgroup of G with $B \leq A$. By induction, G/B has a beautiful generating tuple and hence, by Lemma 2.3, so does G.

Assume then that $\langle g, n\rangle$ is an elementary abelian 2-group. In particular, $o(n) = 2$. From the coprime action, $A = \mathbf{C}_A(n) \times [n, A]$ and, as $o(n) = 2$, the element n acts by conjugation on $[n, A]$ inverting each of its elements. Observe that **(iv)** is equivalent to $\mathbf{C}_A(n) \neq 1$ and $[n, A] \neq 1$.

Let C be a minimal normal subgroup of $\mathbf{C}_A(n)$ and adopt the "bar notation" for the projection of G onto G/C. Clearly, **(i), (ii), (iii)** are satisfied by $\bar{G}, \bar{N}, \bar{A}$ and $\bar{g}, \bar{n}$. Moreover, if $C < \mathbf{C}_A(n)$, then $\bar{A} = \overline{\mathbf{C}_A(n)} \times \overline{[n, A]} = \mathbf{C}_{\bar{A}}(\bar{n}) \times [\bar{n}, \bar{A}]$ and $\mathbf{C}_{\bar{A}}(\bar{n}) \neq 1 \neq [\bar{n}, \bar{A}]$, that is, **(iv)** is also satisfied by $\bar{G}$. Therefore, by induction, $\bar{G}$ has a beautiful generating tuple, that hence so does G by Proposition 2.4. This shows that we may assume that $\mathbf{C}_A(n)$ is a minimal normal subgroup of G, that is, $\mathbf{C}_A(n) = \langle c \rangle$ is cyclic of prime order.

Let D be a minimal normal subgroup of $[n, A]$ and adopt the "bar notation" for the projection of G onto G/D. Clearly, **(i), (ii), (iii)** are satisfied by $\bar{G}, \bar{N}, \bar{A}$ and $\bar{g}, \bar{n}$. Moreover, if $D < [n, A]$, then $\bar{A} = \overline{\mathbf{C}_A(n)} \times \overline{[n, A]} = \mathbf{C}_{\bar{A}}(\bar{n}) \times [\bar{n}, \bar{A}]$ and $\mathbf{C}_{\bar{A}}(\bar{n}) \neq 1 \neq [\bar{n}, \bar{A}]$, that is, **(iv)** is also satisfied by $\bar{G}$. Therefore, by induction, $\bar{G}$ has a beautiful generating tuple, that hence so does G by Proposition 2.4. This shows that we may assume that $[n, A]$ is a minimal normal subgroup of G, that is, $[n, A] = \langle d \rangle$ is cyclic of prime order and $d^n = d^{-1}$.

A direct computation shows that gnd, c^{-1}, nc is a beautiful generating tuple for G. □

3.2. Proof of Theorem 1.2

Let G be a finite group and suppose that G admits no ORR, that is, part **(i)** of Theorem 1.2 is not satisfied. In particular, the group G is solvable by the main result in [16].

Suppose that G admits a beautiful generating tuple. Then, by Theorem 2.1, part **(iv)** of Theorem 1.2 holds. Suppose that G has no beautiful generating tuple. In particular, we may now apply Theorem 2.9. If part **(B)** of Theorem 2.9 holds, then part **(iii)** of Theorem 1.2 holds. Suppose that part **(A)** of Theorem 2.9 holds. If the hypothesis of Lemma 3.1 are satisfied, then G is generalised dihedral and part **(v)** of Theorem 1.2 holds. Suppose then that the hypothesis of Lemma 3.1 are not satisfied and let B be the Sylow 2-subgroup of A. If $B < A$, then, combining Proposition 3.2 (applied to G/B) with Lemma 2.3, we get that part **(ii)** of Theorem 1.2 holds. If $B = A$, then A is a 2-group and part **(i)** of Theorem 1.2 holds.

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