A variational formulation of network games with random utility functions

Mauro Passacantando and Fabio Raciti

Abstract We consider a class of games played on networks in which the utility functions consist of both deterministic and random terms. In order to find the Nash equilibrium of the game we formulate the problem as a variational inequality in a probabilistic Lebesgue space which is solved numerically to provide approximations for the mean value of the random equilibrium. We also numerically compare the solution thus obtained, with the solution computed by solving the deterministic variational inequality derived by taking the expectation of the pseudogradient of the game with respect to the random parameters.

Key words: Network Game; Nash equilibrium; Random utility function; Variational inequality.

1 Introduction

Games played on networks, are a class of non-cooperative games where players are considered as nodes of a graph, and direct connections between any two players are represented by arcs connecting them. A basic assumption is that the utility function of a given, arbitrary, player depends on his/her strategy, as well as on the strategies of his/her neighbors in the graph. Therefore, it seems natural that this setting has proved to be very useful in describing social or economic interactions among various types of agents. In this regard it is interesting to investigate the two classes of *games with strategic complements and substitutes*. Roughly speaking, in

Mauro Passacantando
Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy, e-mail: mauro.passacantando@unipi.it

Fabio Raciti
Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy, e-mail: fabio.raciti@unict.it

the first case, the incentive for a player to take an action increases when the number of his/her social contacts who take the action increases, while in the second case this monotonic relation in reversed. As is usual in game theory, equilibrium concepts are considered of paramount importance, and in this context, the study of Nash equilibria is investigated with respect to the algebraic or graph-theoretic properties of the network structure. This line of research has been initiated with the seminal paper by Ballester et al. [1], who also used some centrality measures to assess the importance of the various players, along the same lines of Katz and Bonacich (see, e.g., [3]). The interested reader can find in the beautiful survey by Jackson and Zenou [7] an account of the main concepts about network games, along with a wealth of social and economic applications. Most of the scholars dealing with this topic tackle the corresponding problems with classic game-theoretical methods, such as best response analysis and fixed point theory. However, quite recently some authors utilized the variational inequality approach to provide a deep analysis of many aspects of these games and the interesting paper by Parise and Ozdaglar [12] provides a self-consistent treatment of many interesting developments. The fact that Nash equilibrium problems admit, under suitable hypotheses, an equivalent variational inequality formulation was recognized long time ago by Gabay and Moulin [6]. It seems, though, that this powerful tool has not been fully applied to the topic of network games. In this note we allow for the possibility that the utility functions also depend on a random parameter ω of an abstract sample space Ω, and then derive the corresponding parametric variational inequality. However, our objective here is to compute the mean value of the equilibrium, hence, we wish that the solution admits finite first and (possibly) second moments. In this regard, an integral variational inequality in the probabilistic Lebesgue space $L^2(\Omega, P)$ fits our requirements. This variational inequality is then transformed to the image space of the random variables involved so as to be numerically approximated. The theory of random (or stochastic) variational inequalities has been developed by various authors in the last fifteen years, with different methodologies. We follow here the so called *L^p approach* and refer the interested reader to [5, 8, 9] for a detailed account of the theoretical framework and for several applications. For a description of different approaches, as well as for other interesting developments, the reader can see [15], where the authors also describe the so called *expected value approach* which we compare with our approach by means of a worked out example.

The paper is organized as follows. In the following Section 2 we introduce the notation, and briefly outline the basic network game classes. Moreover, we define the random Nash equilibrium, and the associated variational inequality. In Section 3, we describe in detail the linear-quadratic model, investigate the monotonicity property of the relevant operator, and introduce the associated integral variational inequality. In Section 4, we numerically solve a test problem. A short concluding section ends the paper.

2 Network game classes and variational inequality approach

We begin this section by recalling a few concepts and definitions of graph theory that will be used in the sequel. We warn the reader that the terminology is not uniform in the related literature. Formally, a graph g is a pair of sets (V,E), where V is the set of nodes and E is the set of arcs, formed by pairs of nodes (v,w). Arcs which have the same end nodes are called parallel, while arcs of the form (v,v) are called loops. We consider here *simple* graphs, that is graphs with no parallel arcs or loops. In our setting, the players will be represented by the n nodes in the graph. Moreover, we consider here indirect graphs: the arcs (v,w) and (w,v) are the same. Two nodes v and w are adjacent if they are connected by an arc, i.e., if (v,w) is an arc. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i,v_j) is an arc, 0 otherwise. G is thus a symmetric and zero diagonal matrix. Given a node v, the nodes connected to v with an arc are called the *neighbors* of v and are grouped in the set $N_v(g)$. The number of elements of $N_v(g)$ is the *degree* of v.

We now proceed to specify the game that we will consider. For simplicity, the set of players will be denoted by $\{1,2,\ldots,n\}$ instead of $\{v_1,v_2,\ldots,v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player i from the action of all the other players. Let (Ω, P) be a probability space. Each player i is endowed with a payoff function

$$u_i : \Omega \times A \to \mathbb{R}$$

that he/she wishes to maximize for almost every elementary event $\omega \in \Omega$, that is P-almost surely.

The notation $u_i(\omega, a, g)$ is often utilized when one wants to emphasize the influence of the graph structure. The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek a random vector $a^* : \Omega \to A$ such that for each $i \in \{1,\ldots,n\}$, and, P-a.s.:

$$u_i(a_i^*(\omega), a_{-i}^*(\omega)) \geq u_i(a_i, a_{-i}^*(\omega)), \qquad \forall\, a_i \in A_i. \tag{1}$$

A peculiarity of network games is that the vector a_{-i} is only made up of components a_j such that $j \in N_i(g)$, that is, j is a neighbor of i.

We mentioned in the introduction that it is convenient to consider two specific classes of games which allow a deeper investigation of the patterns of interactions among players. For any given player i it is interesting to distinguish how variations of the actions of player's i neighbors affect his/her marginal utility. In the case where the utility functions are twice continuously differentiable the following definitions clarify this point.

Definition 1 We say that the network game has the property of strategic substitutes if for each player i and P-a.s. the following condition holds:

$$\frac{\partial^2 u_i(\omega, a_i, a_{-i})}{\partial a_j \partial a_i} < 0, \qquad \forall (i,j) : g_{ij} = 1, \, \forall \, a \in A.$$

Definition 2 We say that the network game has the property of strategic complements if for each player i and P-a.s. the following condition holds:

$$\frac{\partial^2 u_i(\omega, a_i, a_{-i})}{\partial a_j \partial a_i} > 0, \qquad \forall (i,j) : g_{ij} = 1, \forall \, a \in A.$$

Let us notice that we are requiring that each of the two properties specified above holds for almost every $\omega \in \Omega$, i.e., we assume that the game class does not change according to the random variable.

For the subsequent development it is important to recall that if the u_i are continuously differentiable functions on A, the Nash equilibrium problem is equivalent to the variational inequality $VI(F,A)$: find $a^* \in A$ such that, P-a.s.

$$F(\omega, a^*(\omega))^\top (a - a^*(\omega)) \geq 0, \qquad \forall \, a \in A, \tag{2}$$

where

$$[F(\omega, a)]^\top := -\left(\frac{\partial u_1}{\partial a_1}(\omega, a), \ldots, \frac{\partial u_n}{\partial a_n}(\omega, a) \right) \tag{3}$$

is also called the pseudo-gradient of the game, according to the terminology introduced by Rosen [14]. For an account of variational inequalities the interested reader can refer to [4, 10, 11]. We recall here some useful monotonicity properties.

Definition 3 $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[F(\omega, x) - F(\omega, y)]^\top (x - y) \geq 0, \qquad \forall \, x, y \in A, \, \forall \, \omega \in \Omega.$$

If the equality holds only when $x = y$, F is said to be strictly monotone.

A stronger type of monotonicity is given by the following

Definition 4 $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be β-strongly monotone on A iff, for every ω, we can find $\beta(\omega) > 0$:

$$[F(\omega, x) - F(\omega, y)]^\top (x - y) \geq \beta(\omega) \|x - y\|^2, \qquad \forall \, x, y \in A.$$

If we can find a β which does not depend on ω in the above definition, we say that F is strongly monotone, *uniformly* with respect to ω.

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the Jacobian matrix of the operator.

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [4, 10, 11]), which can be applied to our framework for each (or almost each) fixed ω.

Theorem 1 *If $K \subset \mathbb{R}^n$ is a compact convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(F,K)$ admits at least one solution. In the case that K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[F(x) - F(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$. Furtheremore, if F is strictly monotone on K, then the solution is unique.

3 The random linear-quadratic model

In what follows A_i can be either $\mathbb{R}_+$ for any $i \in \{1,\dots,n\}$, or $[0,L_i]$, hence $A = \mathbb{R}^n_+$ or $[0,L_1] \times ... \times [0,L_n]$. The payoff of player i is given by:

$$u_i(\omega,a,g) = \alpha(\omega)a_i - \frac{1}{2}a_i^2 + \varphi(\omega)a_i \sum_{i=1}^{n} g_{ij}a_j - \gamma a_i \sum_{i=1}^{n} a_j, \tag{4}$$

where $\alpha(\omega), \varphi(\omega) > 0$, P-a.s. and γ is a positive real number. The term involving the adjacency matrix describes the local complementarities ($\varphi(\omega) > 0$), which means that the neighbors of each player contribute to positively enhance his/her strategy. On the other hand, the term involving γ has opposite sign, thus describing strategic substitutes and it is of global nature.

The pseudo-gradient's components of this game are easily computed as:

$$F_i(\omega,a,g) = (1+\gamma)a_i - \varphi(\omega) \sum_{j=1}^{n} g_{ij}a_j - \gamma \sum_{j=1}^{n} a_j - \alpha(\omega) \qquad i \in \{1,\dots,n\},$$

which can be written in compact form as:

$$F(\omega,a,g) = [(1+\gamma)I - \varphi(\omega)G + \gamma U]\, a - \alpha(\omega)\mathbf{1}, \tag{5}$$

where U is the $n \times n$ matrix whose entries are all equal to one and $\mathbf{1} = (1,\dots,1)^\top \in \mathbb{R}^n$.

We will seek random Nash equilibrium points by solving the following variational inequality: for each ω, find $a^*(\omega) \in A$, such that for all $a \in A$ and P-a.s. we have:

$$\begin{aligned} &[(1+\gamma)Ia^*(\omega) - \varphi(\omega)Ga^*(\omega)]^\top (a - a^*(\omega)) + [\gamma U\, a^*(\omega)]^\top (a - a^*(\omega)) \\ &\geq \alpha(\omega)\mathbf{1}^\top (a - a^*(\omega)). \end{aligned} \tag{6}$$

For the subsequent developments it is important to study the monotonicity properties of the operator F in the above variational inequality.

Lemma 1 *Let F as in (5) and $\rho(G)$ be the spectral radius of G. For all ω such that $\varphi(\omega) < (1+\gamma)/\rho(G)$, F is strictly monotone. Moreover, if $\overline{\varphi}$ is a real number such that $0 < \overline{\varphi} < (1+\gamma)/\rho(G)$, then F is strongly monotone uniformly in the set $\{\omega : 0 < \varphi(\omega) \leq \overline{\varphi}\}$, in the sense that it exists $\beta > 0$ such that*

$$[F(\omega,a) - F(\omega,a')]^\top (a-a') \geq \beta \|a-a'\|^2,$$

for all $a,a' \in \mathbb{R}^n$ and for all ω such that $\varphi(\omega) \in (0,\overline{\varphi}]$.

Proof It is sufficient to study the linear part of F. Thus, let us consider the expression:

$$(1+\gamma)Ia - \varphi(\omega)Ga + \gamma U\, a$$

and notice that for every $\gamma > 0$ the matrix γU is positive semidefinite, thus defining a monotone operator. Because the sum of a strongly (strictly) monotone and a monotone operator gives a strongly (strictly) monotone operator, we seek conditions which ensure the strong monotonicity of $(1+\gamma)I - \varphi(\omega)G$. To this end, let us notice that G is a zero trace matrix, hence its largest eigenvalue is positive. Moreover, it can be proved that the largest eigenvalue of G coincides with its spectral radius $\rho(G)$. It follows that, for each ω, the minimum eigenvalue of $(1+\gamma)I - \varphi(\omega)G$ is given by $1+\gamma-\varphi(\omega)\rho(G)$, which is positive whenever $\varphi(\omega)\rho(G) < 1+\gamma$. Thus, for each ω such that $\varphi(\omega) \in (0,(1+\gamma)/\rho(G)]$, we get:

$$a^\top [(1+\gamma)I - \varphi(\omega)G]a \geq [1+\gamma-\varphi(\omega)\rho(G)]\,\|a\|^2.$$

Furthermore, let $\overline{\varphi}$ be a real number such that $0 < \overline{\varphi} < (1+\gamma)/\rho(G)$, and $\beta = 1+\gamma-\overline{\varphi}\rho(G)$. We then obtain that:

$$a^\top [(1+\gamma)I - \varphi(\omega)G]a \geq \beta \|a\|^2$$

holds for any ω such that $0 < \varphi(\omega) \leq \overline{\varphi}$. □

We now proceed to provide an integral formulation of the variational inequality (6). Thus, we make the additional assumptions that the random variable α has finite second order moment, that is, $\alpha \in L^2(\Omega,P)$, while $\varphi \in L^\infty(\Omega,P)$, with $0 < \underline{\varphi} \leq \varphi(\omega) \leq \overline{\varphi}$. We can now consider the variational inequality problem of finding $a^* \in L^2(\Omega,P)$, such that $a^*(\omega) \in A$, and $\forall a \in L^2(\Omega,P)$ such that $a(\omega) \in A$:

$$\int_\Omega \Big\{ [(1+\gamma)Ia^*(\omega) - \varphi(\omega)Ga^*(\omega)]^\top (a - a^*(\omega)) + [\gamma U\, a^*(\omega)]^\top (a - a^*(\omega)) \Big\} dP(\omega) \geq \int_\Omega \alpha(\omega)\mathbf{1}^\top (a - a^*(\omega))dP(\omega). \tag{7}$$

Remark 1 The theoretical investigation of the above variational inequality requires tools from infinite dimensional functional analysis that are beyond the scope of this paper. The interested reader can see [10] or the papers cited in the introduction for more details. Here, we only mention that under the relevant assumption of uniform strong monotonicity of F we get the existence and uniqueness of the solution a^*.

We now transform the variational inequality (7) in the image space of the two random variables involved. To this end, let $y = \alpha(\omega)$, $z = \varphi(\omega)$, and $\mathbb{P}$ the probability induced by P on the image space of the two random variables. We thus have to consider the variational inequality problem of finding $a^* \in L^2(\mathbb{R}^2, \mathbb{P})$ such that $a^*(y,z) \in A$, and for each $a \in L^2(\mathbb{R}^2, \mathbb{P})$ with $a(y,z) \in A$, we get:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\underline{\varphi}}^{\overline{\varphi}} \Big\{ & [(1+\gamma)Ia^*(y,z) - zGa^*(y,z)]^\top [a(y,z) - a^*(y,z)] \\ & + [\gamma U a^*(y,z)]^\top [a(y,z) - a^*(y,z)] \Big\} d\mathbb{P}(y,z) \\ \geq \int_{-\infty}^{\infty} \int_{\underline{\varphi}}^{\overline{\varphi}} & y\mathbf{1}^\top [a(y,z) - a^*(y,z)] \} d\mathbb{P}(y,z). \end{aligned} \tag{8}$$

We denote by $E_{\mathbb{P}}[a^*(y,z)]$ the expected value of the solution with respect to the probability measure $\mathbb{P}$ on the image space of the random variables. The L^p theory of random variational inequalities provides an approximation procedure for the expected values and we refer again the interested reader to the references mentioned in the introduction for a thorough treatment of this matter. In the subsequent section we apply this approximation procedure to a worked out example. Moreover, we compare our results with the ones obtained by solving the deterministic variational inequality obtained by taking the expectation $E_{\mathbb{P}}[F(y,z)]$ of the pseudogradient with respect to the random variables involved. This second solution concept is know as the expected value approach and, in this case, leads in a straightforward manner to solving a finite dimensional variational inequality, since the expectation of the pseudogradient can be computed exactly. Nevertheless, as it will be illustrated by the numerical examples of the following section, the two approaches can give quite different results for certain parameter ranges.

4 Numerical experiments

In this section, we show some preliminary numerical experiments for the random linear-quadratic network game described in Section 3.

Example 1. We consider the network shown in Fig. 1 (see also [2]) with 8 nodes (players). The spectral radius of the adjacency matrix G is $\rho(G) \simeq 3.1019$. We set the congestion parameter $\gamma = 0.1$ and the upper bounds $L_i = 5$ for any player $i = 1, \ldots, 8$. We assume that the random variable $y = \alpha(\omega)$ varies in the interval $[1, 10]$ with either uniform distribution (denoted by $y \sim \mathscr{U}(1,10)$) or truncated normal distribution with mean 5.5 and standard deviation 0.9 ($y \sim \mathscr{N}(5.5, 0.9)$), while the random variable $z = \varphi(\omega)$ varies in the interval $[0.01, 0.34]$ with either uniform distribution ($z \sim \mathscr{U}(0.01, 0.34)$) or truncated normal distribution with mean 0.175

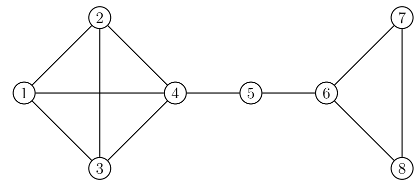

Fig. 1 Network topology of Example 1.

Table 1 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{U}(1,10)$ and $z \sim \mathscr{U}(0.01,0.34)$ and comparison with the solution given by the expected value approach (col. 7–8).

	N					Expected value	
Variables	32	64	128	256	512	approach sol.	Diff.
x_1	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_2	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_3	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_4	3.788	3.835	3.858	3.869	3.875	4.744	−18.33%
x_5	3.270	3.311	3.332	3.342	3.348	3.504	−4.45%
x_6	3.368	3.409	3.429	3.439	3.444	3.750	−8.14%
x_7	3.140	3.179	3.198	3.208	3.213	3.269	−1.71%
x_8	3.140	3.179	3.198	3.208	3.213	3.269	−1.71%

and standard deviation 0.033 ($z \sim \mathscr{N}(0.175, 0.033)$). Notice that $(1+\gamma)/\rho(G) \simeq 0.3546$, hence the assumption of Lemma 1 is satisfied and the operator F is uniformly strongly monotone.

The approximation procedure considers a uniform partition of both intervals $[1,10]$ and $[0.01,0.34]$ into N subintervals and solves N^2 finite dimensional variational inequalities for each N.

Table 1 reports in columns 2–6 the convergence of the mean values of the approximate solution obtained for different values of N, when the random variables y and z vary in the corresponding intervals with uniform distribution. Moreover, column 7 shows the solution given by the expected value approach, while the last column shows the percentage difference between columns 6 and 7. Notice that the difference between the approximate solution found by the L^p approach and the solution given by the expected value approach is significant, especially for the first 4 components.

Tables 2, 3 and 4 reports the convergence of the mean values of the approximate solution and its comparison with the solution given by the expected value approach when y and z vary with different distributions. We remark that the difference between the approximate solution found by the L^p approach and the solution given by the expected value approach is rather small when both random variables y and z vary with truncated normal distribution.

Table 2 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{N}(5.5, 0.9)$ and $z \sim \mathscr{U}(0.01, 0.34)$ and comparison with the solution given by the expected value approach (col. 7–8).

Variables	N 32	64	128	256	512	Expected value approach sol.	Diff.
x_1	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_2	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_3	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_4	4.220	4.267	4.289	4.300	4.305	4.744	−9.25%
x_5	3.457	3.513	3.540	3.554	3.561	3.504	1.65%
x_6	3.678	3.739	3.770	3.785	3.792	3.750	1.14%
x_7	3.278	3.337	3.366	3.381	3.389	3.269	3.67%
x_8	3.278	3.337	3.366	3.381	3.389	3.269	3.67%

Table 3 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{U}(1, 10)$ and $z \sim \mathscr{N}(0.175, 0.033)$ and comparison with the solution given by the expected value approach (col. 7–8).

Variables	N 32	64	128	256	512	expected value approach sol.	Diff.
x_1	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_2	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_3	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_4	3.803	3.850	3.873	3.884	3.890	4.744	−18.01%
x_5	3.259	3.301	3.322	3.333	3.338	3.504	−4.72%
x_6	3.408	3.450	3.470	3.480	3.485	3.750	−7.05%
x_7	3.156	3.195	3.215	3.225	3.230	3.269	−1.19%
x_8	3.156	3.195	3.215	3.225	3.230	3.269	−1.19%

Table 4 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{N}(5.5, 0.9)$ and $z \sim \mathscr{N}(0.175, 0.033)$ and comparison with the solution given by the expected value approach (col. 7–8).

Variables	N 32	64	128	256	512	expected value approach sol.	Diff.
x_1	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_2	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_3	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_4	4.385	4.448	4.478	4.493	4.500	4.744	−5.15%
x_5	3.395	3.452	3.480	3.494	3.501	3.504	−0.07%
x_6	3.648	3.710	3.741	3.757	3.765	3.750	0.41%
x_7	3.207	3.259	3.286	3.299	3.306	3.269	1.15%
x_8	3.207	3.259	3.286	3.299	3.306	3.269	1.15%

5 Conclusions and future research directions

In this chapter we investigated a model of network games with random utility functions by means of its reformulation as a variational inequality in a probabilistic Lebesgue space. We illustrated our methodology through a worked out example which was numerically solved in order to approximate the mean value of the unique random Nash equilibrium of the game. Furthermore the approximated mean value thus computed was compared with the Nash equilibrium which is obtained by solving a deterministic variational inequality derived by taking the expectation of the pseudogradient of the game. Future research work could be performed with nonlinear random utility functions. Another promising research perspective is the variational inequality formulation of generalized network games (with shared constraints), which has been initiated in [13] and offers a wealth of potential theoretical developments and possible applications.

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