Erratum to: A $\mathbb{Q}$–factorial complete toric variety is a quotient of a poly weighted space

Michele Rossi · Lea Terracini

After the publication of [2], we realized that Proposition 3.1, in that paper, contains an error, whose consequences are rather pervasive along the whole section 3 and for some aspects of examples 5.1 and 5.2. Here we give a complete account of needed corrections.

First of all [2, Prop. 3.1] has to be replaced by the following:

PROPOSITION 3.1 *Let $X(\Sigma)$ be a Q–factorial complete toric variety and $Y(\widehat{\Sigma})$ be its universal 1-covering. Let $\{D_\rho\}_{\rho\in\Sigma(1)}$ and $\{\widehat{D}_\rho\}_{\rho\in\widehat{\Sigma}(1)}$ be the standard bases of $\mathcal{W}_T(X)$ and $\mathcal{W}_T(Y)$, respectively, given by the torus orbit closures of the rays. Then*

$$D=\sum_{\rho\in\Sigma(1)} a_\rho D_\rho\in\mathcal{C}_T(X) \quad\Longrightarrow\quad \widehat{D}=\sum_{\rho\in\widehat{\Sigma}(1)} a_\rho \widehat{D}_\rho\in\mathcal{C}_T(Y)\,.$$

Therefore, under the identification $\mathbb{Z}^{|\Sigma(1)|}\cong\mathcal{W}_T(X)\overset{\alpha}{\cong}\mathcal{W}_T(Y)\cong\mathbb{Z}^{|\widehat{\Sigma}(1)|}$ realized by the isomorphism $D_\rho\overset{\alpha}{\mapsto}\widehat{D}_\rho$,

$$\mathcal{C}_T(X)\cong\alpha(\mathcal{C}_T(X))\leq\mathcal{C}_T(Y)\leq\mathcal{W}_T(Y)$$

is a chain of subgroup inclusions. Moreover the induced morphism $\overline{\alpha}:\mathrm{Cl}(X)\to\mathrm{Cl}(Y)$ is injective when restricted to $Pic(X)$, realizing the following further chain of subgroup inclusions

$$\mathrm{Pic}(X)\cong\overline{\alpha}(\mathrm{Pic}(X))\leq\mathrm{Pic}(Y)\leq\mathrm{Cl}(Y)$$

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Michele Rossi
Dipartimento di Matematica - Università di Torino - Via Carlo Alberto 10 - 10123 Torino (Italy)
Tel.: +39-011-6702813
Fax: +39-011-6702878
E-mail: michele.rossi@unito.it

Lea Terracini
Dipartimento di Matematica - Università di Torino - Via Carlo Alberto 10 - 10123 Torino (Italy)
Tel.: +39-011-6702813
Fax: +39-011-6702878
E-mail: lea.terracini@unito.it

Proof: Let us fix a basis $\mathcal{B}$ of the $\mathbb{Z}$-module $M \cong \mathbb{Z}^n$ and let V and $\widehat{V}$ be fan matrices representing the standard morphisms

$$div_X : M \cong \mathbb{Z}^n \stackrel{V^T}{\longrightarrow} \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X) \quad , \quad div_Y : M \cong \mathbb{Z}^r \stackrel{\widehat{V}^T}{\longrightarrow} \mathbb{Z}^{|\widehat{\Sigma}(1)|} \cong \mathcal{W}_T(Y)$$

Let $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ be such that $V = \beta\widehat{V}$ and so realizing an injective endomorphism of the $\mathbb{Z}$-module M. The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

$$\begin{aligned} \mathcal{I}^{\Sigma} &= \{I \subseteq \{1,\dots,n+r\} : \left\langle V^I \right\rangle \in \Sigma(n)\} \\ &= \{I \subseteq \{1,\dots,n+r\} : \left\langle \widehat{V}^I \right\rangle \in \widehat{\Sigma}(n)\} = \mathcal{I}^{\widehat{\Sigma}} \end{aligned} \tag{1}$$

by the construction of $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, given the choice of $\Sigma \in \mathcal{SF}(V)$. Notice that $\mathcal{I}^{\Sigma}$ describes the complements of those sets described by $\mathcal{I}_{\Sigma}$, as defined in [2, Rem. 2.4]. In particular the Weil divisor $\sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X)$ is Cartier if and only if

$$\forall\, I \in \mathcal{I}^{\Sigma} \quad \exists\, \mathbf{m}_I \in M : \forall\, j \notin I \ \mathbf{v}_j^T \mathbf{m}_I = a_j\,, \tag{2}$$

where $\mathbf{v}_j$ is the j-th column of V. Then $\alpha(\sum_{j=1}^{n+r} a_j D_j) = \sum_{j=1}^{n+r} a_j \widehat{D}_j$ is a Cartier divisor since

$$\forall\, I \in \mathcal{I}^{\Sigma} \quad \forall\, j \notin I \quad \widehat{\mathbf{v}}_j^T(\beta^T \mathbf{m}_I) = a_j$$

where $\widehat{\mathbf{v}}_j$ is the j-th column of $\widehat{V}$.
The injectivity of $\overline{\alpha}$ follows from the well-known freeness of $\mathrm{Pic}(X)$. □

As a consequence, parts 1, 4, 5 of [2, Thm. 3.2] still hold, while parts 2, 3, 6, 7 have to be replaced by the following:

Theorem 3.2 *Let $X = X(\Sigma)$ be a n–dimensional $\mathbb{Q}$–factorial complete toric variety of rank r and $Y = Y(\widehat{\Sigma})$ be its universal 1–covering. Let V be a reduced fan matrix of X, $Q = \mathcal{G}(V)$ a weight matrix of X and $\widehat{V} = \mathcal{G}(Q)$ be a CF–matrix giving a fan matrix of Y.*

2. *Define $\mathcal{I}^{\Sigma}$ as in (1). For any $I \in \mathcal{I}^{\Sigma}$ let E_I be the $r \times (n+r)$ matrix admitting as rows the standard basis vectors $e_i = (0,\dots,0,\underset{i}{1},0,\dots,0)$, for $i \in I$, representing the i-th basis divisor $D_i \in \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Set $\widetilde{V}_I := \left(V^T \mid E_I^T\right) \in \mathbf{M}_{n+r}(\mathbb{Z})$. Then Cartier divisors give rise to the following maximal rank subgroup of $\mathcal{W}_T(X)$*
$$\mathcal{C}_T(X) \cong \bigcap_{I \in \mathcal{I}^{\Sigma}} \mathcal{L}_c\left(\widetilde{V}_I\right) \leq \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X)$$
and a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X)$ can be explicitly computed by applying the procedure described in [1, § 1.2.3].
3. *Let $C_X \in \mathrm{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z})$ be a matrix whose rows give a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$, as obtained in the previous part 2. Identify $\mathrm{Cl}(X)$ with $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$ by item (c) of part 4 in [2, Thm. 3.2], and represent the morphism d_X by $Q \oplus \Gamma$, according to parts 1 and 5. Let $A \in \mathrm{GL}_{n+r}(\mathbb{Z})$ be a matrix such that $A \cdot C_X \cdot Q^T$ is in* HNF. *Let $\mathbf{c}_1,\dots,\mathbf{c}_r$ be the first r rows of the matrix $A \cdot C_X$ and for $i = 1,\dots r$ put $\mathbf{b}_i = Q \cdot \mathbf{c}_i^T + \Gamma \cdot \mathbf{c}_i^T$. Then $\mathbf{b}_1,\dots \mathbf{b}_r$ is a basis of the free group $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$.*
6. *Given the choice of $\widehat{V}$ and V as in the previous parts 4 and 5 of [2, Thm. 3.2], consider*
$$\begin{aligned} U &:= \begin{pmatrix} {}^rU_Q \\ \widehat{V} \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Z}) \\ W &\in \mathrm{GL}_{n+r}(\mathbb{Z}) \ : \ W \cdot ({}^{n+r-s}U)^T = \mathrm{HNF}\left(({}^{n+r-s}U)^T\right) \\ G &:= {}_s\widehat{V} \cdot \ ({}_sW)^T \in \mathbf{M}_s(\mathbb{Z}) \\ U_G &\in \mathrm{GL}_s(\mathbb{Z}) \ : \ U_G \cdot G^T = \mathrm{HNF}(G^T)\,. \end{aligned}$$

Then a "torsion matrix" representing the "torsion part" of the morphism d_X, that is, $\tau_X : \mathcal{W}_T(X) \to \mathrm{Tors}(\mathrm{Cl}(X))$, is given by

$$\Gamma = U_G \cdot\ {}_sW \mod \boldsymbol{\tau} \tag{3}$$

where this notation means that the (k, j)–entry of Γ is given by the class in $\mathbb{Z}/\tau_k\mathbb{Z}$ represented by the corresponding (k, j)–entry of ${}^sU_G \cdot\ {}_sW$, for every $1 \le k \le s$, $1 \le j \le n + r$.

7. *Setting $\delta_\Sigma := \mathrm{lcm}\left(\det(Q_I) : I \in \mathcal{I}^\Sigma\right)$ then*

$$\delta_\Sigma \mathcal{W}_T(X) \subseteq \mathcal{C}_T(X) \quad \textit{and} \quad \delta_\Sigma \mathcal{W}_T(Y) \subseteq \mathcal{C}_T(Y)$$

and there are the following divisibility relations

$$\delta_\Sigma \mid [\mathrm{Cl}(Y) : \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\mathrm{Cl}(X) : \mathrm{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)]\,.$$

Proof: (2): Recalling relation (2) in the proof of Proposition 3.1, set

$$\forall\, I \in \mathcal{I}^\Sigma \quad \mathcal{P}^I = \{L = \sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X) \mid \exists\, \mathbf{m} \in M : \forall\, j \notin I\ \mathbf{m} \cdot \mathbf{v}_j = a_j\}.$$

Then $\mathcal{P}^I$ contains $\mathrm{Im}(div_X : M \to \mathcal{W}_T(X)) = \mathcal{L}_c\left(V^T\right)$ and a $\mathbb{Z}$-basis of $\mathcal{P}^I$ is given by

$$\{D_j, j \in I\} \cup \{\sum_{k=1}^{n+r} v_{ik} D_k, i = 1, \dots, n\},$$

where $\{v_{ik}\}$ is the i-th entry of $\mathbf{v}_k$, so giving the rows of the matrix $\widetilde{V}_I$ defined in the statement.
(3): By definition

$$\mathrm{Pic}(X) = \mathrm{Im}(\mathcal{C}_T(X) \hookrightarrow \mathcal{W}_T(X) \stackrel{d_X}{\to} \mathrm{Cl}(X))$$

so that $\mathrm{Pic}(X)$ is generated by the image under $Q \oplus \Gamma$ of the transposed of the rows of C_X. Since $\mathrm{rk}(C_X) = n + r$ and $\mathrm{rk}(Q) = r$, the matrix $C_X \cdot Q^T$ has rank r and therefore its HNF has the last $n - r$ rows equal to zero. Therefore the rows of the matrix $A \cdot C_X$ provide a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$ such that its last n rows are a basis of $\mathcal{L}_r(\widehat{V}) \cap \mathcal{C}_T(X) = \mathcal{L}_r(V)$. Since $\mathrm{Pic}(X)$ is free of rank r it is freely generated by the images under d_X of the first r rows.
(6): A representative matrix of the torsion part $\tau_X : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$ of the morphism d_X is any matrix satisfying the following properties:

(*i*) $\Gamma = (\gamma_{kj})$ with $\gamma_{kj} \in \mathbb{Z}/\tau_k\mathbb{Z}$,
(*ii*) $\Gamma \cdot ({}^rU_Q)^T = \mathbf{0}_{s,r} \mod \boldsymbol{\tau}$, meaning that Γ kills the generators of the free part $F \le \mathrm{Cl}(X)$ defined in display (4) of part 1 of [2, Thm. 3.2],
(*iii*) $\Gamma \cdot V^T = \mathbf{0}_{s,n} \mod \boldsymbol{\tau}$, where V is a fan matrix satisfying condition 4.(b) in [2, Thm. 3.2]: this is due to the fact that the rows of V span $\ker(d_X)$,
(*iv*) $\Gamma \cdot ({}_s\widehat{V})^T = \mathbf{I}_s \mod \boldsymbol{\tau}$, since the rows of ${}_s\widehat{V}$ give the generators of $\mathrm{Tors}(\mathrm{Cl}(X))$, as in display (6) of part 5 of [2, Thm. 3.2].

Therefore it suffices to show that the matrix $U_G \cdot\ {}_sW$ in (3) satisfies the previous conditions (*ii*), (*iii*) and (*iv*) without any reduction mod $\boldsymbol{\tau}$, that is,

$$U_G \cdot\ {}_sW \cdot\ ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s} \quad , \quad U_G \cdot\ {}_sW \cdot\ ({}_s\widehat{V})^T = \mathbf{I}_s\,.$$

The first equation follows by the definition of W, in fact

$$W \cdot ({}^{n+r-s}U)^T = \mathrm{HNF}\left(({}^{n+r-s}U)^T\right) = \begin{pmatrix} \mathbf{I}_{n+r-s} \\ \mathbf{0}_{s,n+r-s} \end{pmatrix} \Rightarrow {}_sW \cdot\ ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s}$$

The second equation follows by the definition of U_G, in fact

$$U_G \cdot {}_sW \cdot \ ({}_s\widehat{V})^T = U_G \cdot G^T = \mathrm{HNF}(G^T) = \mathbf{I}_s\,.$$

(7): Part (4) of [1, Thm. 2.9] gives that $\delta_\Sigma \mid [\mathrm{Cl}(Y) : \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)]$. On the other hand Proposition 3.1 gives that $[\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\mathcal{W}_T(X) : \mathcal{C}_T(X)] = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. □

Considerations i, ii, iii, iv, v of [2, Rem. 3.3] still holds, while vi, vii and the remaining part of Remark 3.3 have to be replaced by the following

Remark 3.3

vi. apply procedure [1, § 1.2.3], based on the HNF algorithm, to get a $(n+r)\times(n+r)$ matrix C_X whose rows give a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$;

vii. apply procedure described in part 6 of Theorem 3.2 to get a system of generators of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$. Precisely, let $A \in \mathrm{GL}_{n+r}(\mathbb{Z})$ be a switching matrix such that $\mathrm{HNF}(C_X \cdot Q^T) = A \cdot C_X \cdot Q^T$, and put

$$B_X = \ {}^r(A \cdot C_X \cdot Q^T), \quad \Theta_X = \ {}^r(A \cdot C_X \cdot \Gamma^T) \tag{4}$$

Then the rows of the matrices B_X and Θ_X represent respectively the free part and the torsion part of a basis of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$, where the latter is identified to $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$.
Moreover:

- recall that, for the universal 1–covering Y of X, once fixed the basis $\{\widehat{D}_j\}_{j=1}^{n+r}$ of $\mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_Y(\widehat{L}_i)\}_{i=1}^r$ of $\mathrm{Cl}(Y) \cong \mathbb{Z}^r$, (see (11) in [1, Thm. 2.9]), one gets the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)} & \mathcal{C}_T(Y) \cong \mathrm{Pic}(Y) \oplus M & \xrightarrow{\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)} & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle C_Y^T} & & \downarrow{\scriptstyle B_Y^T} & & \\
0 & \longrightarrow & M & \xrightarrow[\widehat{V}^T]{div_Y} & \mathcal{W}_T(Y) = \bigoplus_{j=1}^{n+r} \mathbb{Z}\cdot D_j & \xrightarrow[Q]{d_Y} & \mathrm{Cl}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{T}_Y & \xrightarrow{\cong} & \mathcal{T}_Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where B_Y is the $r\times r$ matrix constructed in [1, Thm. 2.9(3)] and

$$C_Y = \begin{pmatrix} B_Y & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B_Y \cdot {}^rU_Q \\ \widehat{V} \end{pmatrix},$$

– once fixed the basis $\{D_j\}_{j=1}^{n+r}$ for $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_X(L_i)\}_{i=1}^{r}$ of the free part $F \cong \mathbb{Z}^r$ of $\mathrm{Cl}(X)$, constructed in part 1 of [2, Thm. 3.2], one gets the following commutative diagram

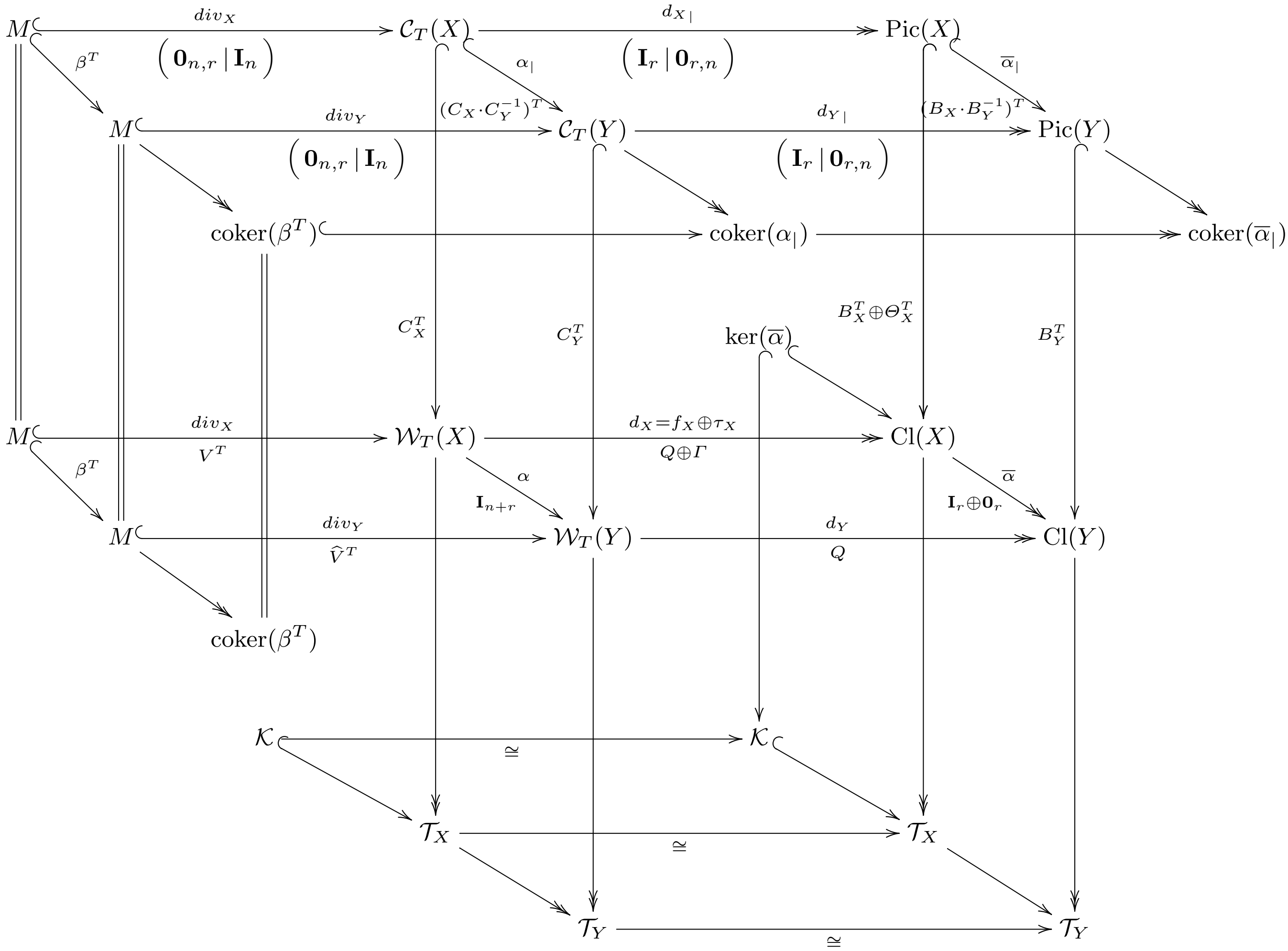
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)} & \mathcal{C}_T(X) \cong \mathrm{Pic}(X) \oplus M & \xrightarrow{\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)} & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle C_X^T} & & \downarrow{\scriptstyle B_X^T \oplus \Theta_X^T} & & \\
0 & \longrightarrow & M & \xrightarrow[V^T]{div_X} & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z}\cdot D_j & \xrightarrow[Q\oplus\Gamma]{d_X = f_X \oplus \tau_X} & \mathrm{Cl}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{T}_X & \xrightarrow{\cong} & \mathcal{T}_X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Moreover:

– recall the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \ker(\overline{\alpha}) = \mathrm{Tors}(\mathrm{Cl}(X)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow[V^T]{div_X} & \mathcal{W}_T(X) = \mathbb{Z}^{|\Sigma(1)|} & \xrightarrow{d_X} & \mathrm{Cl}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta^T} & & {\scriptstyle \mathbf{I}_{n+r}}\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \\
0 & \longrightarrow & M & \xrightarrow[\widehat{V}^T]{div_Y} & \mathcal{W}_T(Y) = \mathbb{Z}^{|\widehat{\Sigma}(1)|} & \xrightarrow{d_Y} & \mathrm{Cl}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\beta^T) \cong \mathrm{Tors}(\mathrm{Cl}(X)) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{5}$$

then, putting all together, one gets the following 3–dimensional commutative diagram



(6)

The Snake Lemma implies

$$\begin{aligned} \mathrm{coker}(\beta^T) &\cong \ker(\overline{\alpha}) \cong \mathrm{Tors}(\mathrm{Cl}(X)) \\ \mathcal{K} &\cong \mathrm{coker}(\alpha_|) \cong \mathcal{C}_T(Y)/\mathcal{C}_T(X) \end{aligned}$$

so giving the following short exact sequences on torsion subgroups

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Tors}(\mathrm{Cl}(X)) & \longrightarrow & \mathcal{C}_T(Y)/\mathcal{C}_T(X) & \longrightarrow & \mathrm{Pic}(Y)/\,\mathrm{Pic}(X) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathrm{Cl}(X)/\,\mathrm{Pic}(X) & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathrm{Cl}(Y)/\,\mathrm{Pic}(Y) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{7}$$

For what concerns the examples given in section 5, considerations related with parts v, vi and vii of Remark 3.3 have to be replaced as follows

EXAMPLE 5.1
v. A matrix $W \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}\left(({}^3U)^T\right) = W \cdot ({}^3U)^T$ is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

giving

$$G := {}_1\widehat{V} \cdot \ ({}_1W)^T = \begin{pmatrix} 1 \end{pmatrix}$$

Therefore

$$\Gamma = {}_1W \mod 5 = \begin{pmatrix} [0]_5 & [4]_5 & [2]_5 & [1]_5 \end{pmatrix} .$$

Consequently display (16) in [2], giving the action of $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_5$ on $Y = \mathbb{P}^3$, should be replaced by the following (equivalent) one:

$$\begin{array}{ccc} \mu_5 \times \mathbb{P}^3 & \longrightarrow & \mathbb{P}^3 \\ (\varepsilon, [x_1 : \ldots : x_4]) & \mapsto & \left[x_1 : \varepsilon^4 x_2 : \varepsilon^2 x_3 : \varepsilon x_4\right] \end{array} . \tag{8}$$

vi. Applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a 4×4 matrix C_X whose rows give a basis of $\mathcal{C}_T(X)$ inside $\mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Namely

$$C_X = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{pmatrix}$$

meaning that

$$\mathcal{C}_T(X) = \mathcal{L}\left(5D_1, 5D_2, -3D_1 - 3D_2 + D_3, -2D_1 - 4D_2 + D_4\right) .$$

On the other hand, by part (3) of [1, Thm. 2.9], a basis of $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$ is given by the rows of

$$C_Y = \mathbf{I}_4 \cdot U_Q = U_Q \in \mathrm{GL}_n(\mathbb{Z})$$

giving $\mathcal{C}_T(Y) = \mathcal{W}_T(Y)$, as expected for $Y = \mathbb{P}^3$.
vii. A basis of $\mathrm{Pic}(X)$ inside $\mathrm{Cl}(X)$ is then obtained by applying part 6 of Theorem 3.2. With the notation of Remark 3.3 vii, a switching matrix A such that $A \cdot C_X \cdot Q^T$ is in HNF is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$B_X = {}^1(A \cdot C_X \cdot Q^T) = \begin{pmatrix} 5 \end{pmatrix}$$
$$\Theta_X = {}^1(A \cdot C_X \cdot \Gamma^T) = \begin{pmatrix} 0 \end{pmatrix}$$

Then

$$\mathrm{Pic}(X) \cong \mathbb{Z}[5d_X(D_1)] \leq \mathbb{Z}[d_X(D_1)] \oplus \mathbb{Z}/5\mathbb{Z}[d_X(D_3 - D_4)] \cong \mathrm{Cl}(X) \ \Rightarrow \ \mathrm{Cl}(X)/\,\mathrm{Pic}(X) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}\,.$$

Example 5.2
v. A matrix U as defined in part 6 of Theorem 3.2 is given by

$$U = \begin{pmatrix} {}^2U_Q \\ \widehat{V}' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -6 & 3 & 1 & 0 & 0 & 0 \\ 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -184 & 105 & 53 & -2 & -23 & -1 \\ 191 & -109 & -55 & 2 & 24 & 1 \end{pmatrix}$$

A matrix $W \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}(({}^4U)^T) = W \cdot (({}^4U)^T)$ is given by

$$W = \begin{pmatrix} -57 & -115 & 3 & -549 & 17 & 0 \\ 4 & 8 & 1 & 3 & 7 & 0 \\ -125 & -250 & 0 & -1090 & 14 & 0 \\ -170 & -340 & 0 & -1482 & 19 & 0 \\ -188 & -376 & 0 & -1639 & 21 & 0 \\ -126 & -252 & 0 & -1092 & 13 & 1 \end{pmatrix}$$

then

$$G = {}_2\widehat{V}' \cdot ({}_2W)^T = \begin{pmatrix} -2093 & -1392 \\ 2302 & 1531 \end{pmatrix}$$

A matrix $U_G \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathrm{HNF}(G^T) = U_G \cdot G^T$ is given by

$$U_G = \begin{pmatrix} 1531 & -2302 \\ 1392 & -2093 \end{pmatrix}$$

hence giving

$$\begin{aligned} \Gamma &= U_G \cdot {}_2W \mod \boldsymbol{\tau} \\ &= \begin{pmatrix} 2224 & 4448 & 0 & 4475 & 2225 & -2302 \\ 2022 & 4044 & 0 & 4068 & 2023 & -2093 \end{pmatrix} \mod \begin{pmatrix} 3 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} [1]_3 & [2]_3 & [0]_3 & [2]_3 & [2]_3 & [2]_3 \\ [12]_{15} & [9]_{15} & [0]_{15} & [3]_{15} & [13]_{15} & [7]_{15} \end{pmatrix} \end{aligned}$$

Consequently display (20) in [2] should be replaced by the following (equivalent) one

$$\begin{aligned} & g\left(((t_1,t_2),\varepsilon,\eta),(x_1,\dots:x_6)\right) := \\ & \left(t_1^2t_2\varepsilon\eta^{12}\ x_1, t_1^4t_2\varepsilon^2\eta^9\ x_2, t_1t_2^3\ x_3, t_1^5t_2^2\varepsilon^2\eta^3\ x_4, t_1^4t_2^3\varepsilon^2\eta^{13}\ x_5, t_1^3t_2^7\varepsilon^2\eta^7\ x_6\right) \end{aligned} \tag{9}$$

vi. Depending on the choice of the fan $\Sigma_i \in \mathcal{SF}(V)$, by applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a 6×6 matrix $C_{X,i}$ whose rows give a basis of $\mathcal{C}_T(X_i)$ inside $\mathcal{W}_T(X_i) \cong \mathbb{Z}^{|\Sigma_i(1)|}$. Namely

$$C_{X,1} = \begin{pmatrix} 265926375 & 0 & 0 & 0 & 0 & 0 \\ -148978500 & 825 & 0 & 0 & 0 & 0 \\ -58474020 & -375 & 15 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ -58473933 & -417 & -3 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,2} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -34716000 & 15 & 0 & 0 & 0 & 0 \\ -594165 & 0 & 30 & 0 & 0 & 0 \\ -34715963 & -3 & -7 & 1 & 0 & 0 \\ 17655087 & -12 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,3} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -37009500 & 825 & 0 & 0 & 0 & 0 \\ -6534165 & -750 & 30 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ 87 & -42 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

vii. A basis of $\mathrm{Pic}(X_i)$ inside $\mathrm{Cl}(X_i)$ is then obtained by applying part 6 of Theorem 3.2. For $i = 1, 2, 3$, matrices A_i switching $C_{X_i} \cdot Q^T$ in Hermite normal form are respectively

$$A_1 = \begin{pmatrix} -351039 & -449987 & -449987 & 0 & 0 & 0 \\ -502913 & -644670 & -644670 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -93838 & -117699 & 0 & 0 & 0 & 0 \\ -1157199 & -1451450 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -10317 & -12139 & 0 & 0 & 0 & 0 \\ -22429 & -26390 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$B_{X_1} = \ {}^2(A_1 \cdot C_{X_1} \cdot Q^T) = \begin{pmatrix} 825 & 185620050 \\ 0 & 265926375 \end{pmatrix}$$

$$B_{X_2} = \ {}^2(A_2 \cdot C_{X_2} \cdot Q^T) = \begin{pmatrix} 60 & 1765515 \\ 0 & 21771750 \end{pmatrix}$$

$$B_{X_3} = \ {}^2(A_3 \cdot C_{X_3} \cdot Q^T) = \begin{pmatrix} 3300 & 10016325 \\ 0 & 21771750 \end{pmatrix}$$

$$\Theta_{X_i} = \ {}^2(A_i \cdot C_{X_i} \cdot \Gamma^T) = \begin{pmatrix} [0]_3 & [0]_{15} \\ [0]_3 & [0]_{15} \end{pmatrix} \ , \quad \text{for } i = 1, 2, 3\,.$$

References

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2. Rossi, M., and Terracini, L. *A $\mathbb{Q}$–factorial complete toric variety is a quotient of a poly weighted space* Ann. Mat. Pur. Appl. **196** (2017), 325–347; DOI:10.1007/s10231-016-0574-7; `arXiv:1502.00879`.